Lifshitz black holes and vacuum polarization

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In this work we summarize our computation of the vacuum polarization for a massive non-minimally coupled scalar field on a Lifshitz black hole background. The general method for computing the Green function is outlined and a procedure to renormalize is described We also provide numerical results for some specific values of mass, nonminimal coupling, and dynamical exponent.

Keywords: Vacuum polarization; Lifshitz spacetime; black holes.

1. Introduction

Classically, the effect of matter over the curvature of spacetime is characterized by the stress-energy tensor $T_{\mu\nu}$. If one wishes to introduce quantum effects, it is expected that, in the semiclassical approximation, the Einstein equations become

$$G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle \, . \tag{1}$$

Unfortunately, calculating $\langle T_{\mu\nu} \rangle$ is, most of the times, a formidable task, even numerically. A quantity that incorporates information about quantum effects, particle production, symmetry breaking, and that is slightly simpler to compute is the vacuum polarization. For a scalar field it is indicated as $\langle \phi^2 \rangle$ and corresponds to the coincidence limit of the euclideanized scalar Green function

$$\langle \phi^2(x) \rangle = \lim_{x' \to x} G_E(x, x') \, . \tag{2}$$

Calculations of such kind in black hole spacetimes were initiated by Candelas[1,2]. In this proceedings we summarize our results regarding the computation of the scalar vacuum polarization for a class of black hole spacetimes with Lifshitz scaling[3]. We will address the method, main difficulties and present some numerical results.

2. The Green function in a Lifshitz spacetime

A (euclideanized) Lifshitz spacetime is specified by the general line element

$$ds^2 = r^{2z} f(r) d\tau^2 + \frac{u(r)}{r^2} dr^2 + r^2 d\Omega_2^2 \, , \tag{3}$$

with z being the dynamical exponent. Details for these kind of spacetimes can be found, for instance, in Refs.[4,5].

With the aim of computing the vacuum polarization, we will follow the details of the method outlined in Refs.[6,7] and Ref.[8] and start from the the Green function of a scalar field in a curved spacetime

$$\left(\Box_x - m^2 - \xi R\right) G_E(x, x') = -\frac{\delta^{(4)}(x - x')}{\sqrt{g}} \,. \tag{4}$$

For spherically symmetric spacetimes, we can expand the Green function in terms of spherical harmonics

$$G_E(x, x') = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} e^{in\alpha(\tau - \tau')} P_l(\cos\gamma) G_{nl}(r, r') \,. \tag{5}$$

where $\beta = 1/T$ and $\alpha = 2\pi/\beta$. Inserting this expansion in Eq. (4) leads to the differential equation for the radial Green function

$$\left[\frac{d}{dr}\left(r^{z+3}\sqrt{\frac{f}{u}}\frac{d}{dr}\right) - r^{z+1}\sqrt{fu}\left(\frac{n^2\alpha^2}{r^{2z}f} + \frac{l(l+1)}{r^2} + M^2\right)\right] G_{nl}(r, r') = -\delta(r' - r) \,, \tag{6}$$

where $M^2 = m^2 + \xi R$. In general, we can express a solution to the above equation in terms of solutions regular at the horizon or at infinity, so that the Green function is valid everywhere outside the Lifshitz black hole.

3. The WKB approximation of the solution

To evaluate the Green function we are faced with the task of determining the solutions of the homogeneous equation associated to Eq. (6). In that effort, we shall use a WKB ansatz

$$\chi_{nl}^{(\pm)}(r) = r^{-(z+2)/2} W_{nl}^{-1/2} \exp\left(\pm \int_{r_h}^{r} \frac{W_{nl}(s)}{s}\sqrt{\frac{u}{f}}\, ds\right) \,, \tag{7}$$

which is chosen in such a way that we get the same differential equation for $W(r)$ in the case of both solutions. The resulting equation is of the form

$$W^2 = \varpi(n, l, r) + \sigma + a_1(r)\frac{W'}{W} + a_2(r)\frac{W'^2}{W^2} + a_3(r)\frac{W''}{W} \,. \tag{8}$$

We take an iterative solution of the form $W = W_0 + W_1 + \ldots$ and retain terms up to next-to-leading order. The Green function in the coincidence limit can thus be expressed solely in terms of W_{nl} for each mode n and l, having the form

$$G_E(x, x) = \frac{\alpha}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \frac{(l + 1/2)}{r^{z+2}\tilde{W}_{nl}} \,, \tag{9}$$

with $\tilde{W}_{nl}$ denoting the truncated solution. The above result requires renormalization due to the presence of divergences. One type of divergence is linked, like in the Schwarzschild case, to the l summation (for large l, $\tilde{W}_{nl} \sim (l + 1/2)\sqrt{f}/r$ leads to an infinity the summation over l). This is not a harmful divergence and it can be easily removed by adding a multiple of $\delta(\tau - \tau')$. The other infinity comes from the summation over n and cannot be removed by any mathematical trick. It originates from the fact that high energies (large n) are not bounded in the theory, and to regulate this behavior, one needs to add appropriate counter-terms.

The specific expression for the divergent part in any four dimensional spacetime has been thoroughly studied in [9], where a general result for the divergent part of the coincident limit of the Green function and a procedure to compute the counter-terms was given. In fact, the passage from the result of Ref. [9] to the fully renormalized result for the vacuum polarization involves some nontrivial transformations allowing appropriate combination with the divergent result leading to a cancellation of the infinities, that are all contained in the first two orders of the WKB approximation. Details will be given elsewhere.

4. The renormalized vacuum polarization

The exact result for the renormalized vacuum polarization $\langle\phi^2(x)\rangle$ is obtained by adding and subtracting the vacuum polarization obtained through the truncated WKB solution $\langle\phi^2(x)\rangle_{WKB}$. The final expression can therefore be expressed as a term which is calculated analytically plus a reminder of the WKB approximation $\delta\langle\phi^2(x)\rangle = \langle\phi^2(x)\rangle - \langle\phi^2(x)\rangle_{WKB}$, which is regular and can be calculated numerically. We organize this in the following expression

$$\langle\phi^2(x)\rangle = \frac{\alpha}{8\pi^2}\{\Upsilon_0 + \Sigma_1 + \Sigma_2 - \Delta\} \tag{10}$$

where we have defined

$$\Upsilon_n = \sum_{l=0}^{\infty}\left(\frac{l+1/2}{r^{z+2}\tilde{W}_n(l)} - \frac{1}{r^{z+1}\sqrt{f}}\right),$$

$$\Sigma_1 = \sum_{n=0}^{\infty}(2-\delta_{n0})\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\left(\frac{1}{r^{z+2}W_n(l)} - \frac{1}{r^{z+2}\tilde{W}_n(l)}\right),$$

$$\Sigma_2 = \sum_{n=1}^{\infty}\left[2\Upsilon_n + \frac{2n\alpha}{r^{2z}f} + \left[m^2 - \left(\xi - \frac{1}{6}\right)R\right]\frac{1}{n\alpha}\right]$$

and Δ is a somewhat complicated function of the radial coordinate r, metric components, mass and curvature. The term Σ_1 is the reminder of the WKB approximation, which is numerically computed and Σ_2 contains the cancellation of the divergences. Both Υ_0 and Δ are explicitly finite. Some examples of numerical calculations are shown in Figs. 1 and 2.

For $z = 1$ the metric is asymptotically AdS, allowing us to compare our results with those of Ref. [8]. We have also checked the limiting value of the vacuum polarization for large r. From Fig. 1, one can fit the data in the asymptotic limit as

$$\langle\phi^2(x)\rangle \approx a + \frac{b}{f} \tag{11}$$

with $b = -1/48\pi^2$, which is the correct value [10].

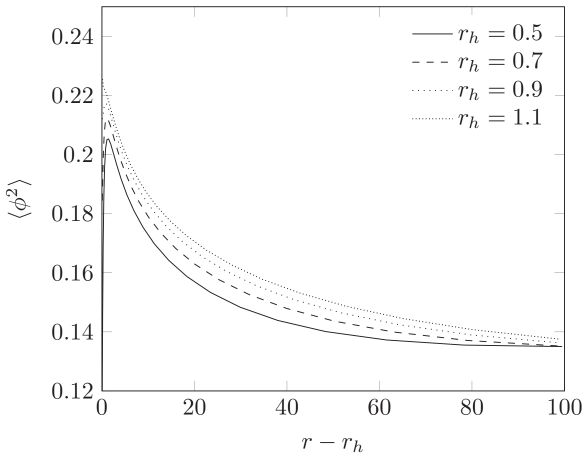

Fig. 1. Vacuum polarization for $\xi = 0$, $m = 0.01$ and $z = 1$. The metric functions are $f = 1/u = 1 - (r_h/r)^{z+3}$.

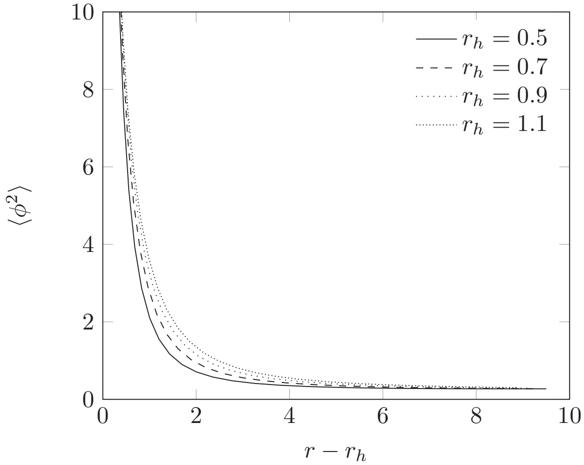

Fig. 2. Vacuum polarization for $\xi = 0$, $m = 0.01$ and $z = 2$. The metric functions are $f = 1/u = 1 - (r_h/r)^{z+3}$.

5. Conclusions

The general procedure for calculating the vacuum polarization in a Lifshitz spacetime was presented. We started by expanding the Green function of a general scalar field in curved space in spherical harmonics, thus reducing the problem to solving a second order differential equation for the radial component. Using a WKB ansatz for the solution, we arrived at a result expressed in terms of sums in the energy and angular momentum modes. Two divergences appeared: an apparent one in the angular modes and a physical one in the energy modes. The former was removed using a mathematical trick involving multiples of a delta function and the latter was removed by subtracting the Christensen counter-terms for the coincidence limit of a Green function in a curved spacetime. A fully renormalized result was thus attainable and some numerical examples were presented, which were seen to be in accordance with previous results. A more detailed version of this work can also be consulted[3].

Acknowledgments

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