

# Group averaging, positive definiteness and superselection sectors

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**Abstract.** I discuss group averaging as a method for quantising constrained systems whose gauge group is a noncompact Lie group. Focussing on three case studies, I address the convergence of the averaging, possible indefiniteness of the prospective physical inner product and the emergence of superselection sectors.

## 1. Introduction

In quantisation of constrained systems, one approach to finding gauge invariant states is to first build an unconstrained quantum theory and then to average states in this theory over the action of the gauge group [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. When the gauge group is a compact Lie group with a unitary action on the unconstrained Hilbert space, the mathematical setting is well-understood: the averaging converges and yields a projection operator to the Hilbert subspace of gauge invariant states. When the gauge group is a *noncompact* Lie group, the averaging can in certain circumstances be interpreted in terms of ‘distributional’ states [6, 8, 11, 12, 13, 14, 15], but at present few general results [10] are known as to when such circumstances can be expected to occur. When the gauge group is not a Lie group, the situation is even more open, although some case studies are known [7, 18] and a formalism relating group averaging to BRST quantisation has been developed [19]. There are also close connections between group averaging, projection operator quantisation [20] and the master constraint programme [21, 22].

In this contribution I will discuss averaging over a noncompact Lie group, focussing on three systems whose classical phase space is finite dimensional [12, 13, 14, 15]. The conclusions will come in two parts. First, if something can go wrong [23] in the mathematics of group averaging, it tends indeed to do so. In particular, in section 2 we will discuss a system in which the would-be physical inner product produced by the averaging turns out to be indefinite. Second, when the mathematics of the averaging goes wrong, there tends to be something subtle already in the classical properties of the system. For example, in the system with the indefinite would-be inner product, the classical solution space consists of just three points. In section 4 we will discuss a system in which isolated singular subsets in the classical solution space make it necessary to renormalise the averaging, and this renormalisation will result into superselection sectors in the quantum theory. Finding a more general setting for such pathologies presents an interesting challenge for the future.

The status of group averaging as of approximately 2000 has been comprehensively reviewed in [16, 17]. I shall aim at a qualitatively self-contained discussion of the systems in [12, 13, 14, 15].

## 2. $\text{SL}(2, \mathbb{R})$ system

### 2.1. Classical system

We consider a system [12, 13, 24] whose action reads

$$S = \int dt (\mathbf{p} \cdot \dot{\mathbf{u}} + \boldsymbol{\pi} \cdot \dot{\mathbf{v}} - NH_1 - MH_2 - \lambda D) \quad , \quad (1)$$

where  $\mathbf{u}$  and  $\mathbf{p}$  are real vectors of dimension  $p \geq 1$ ,  $\mathbf{v}$  and  $\boldsymbol{\pi}$  are real vectors of dimension  $q \geq 1$  and the overdot denotes differentiation with respect to  $t$ .  $\mathbf{p}$  and  $\boldsymbol{\pi}$  are respectively the momenta conjugate to  $\mathbf{u}$  and  $\mathbf{v}$  and the phase space is  $\Gamma \simeq T^*\mathbb{R}^{p+q}$ .  $N$ ,  $M$  and  $\lambda$  are Lagrange multipliers associated with the constraints

$$\begin{aligned} H_1 &:= \frac{1}{2}(\mathbf{p}^2 - \mathbf{v}^2) \quad , \\ H_2 &:= \frac{1}{2}(\boldsymbol{\pi}^2 - \mathbf{u}^2) \quad , \\ D &:= \mathbf{u} \cdot \mathbf{p} - \mathbf{v} \cdot \boldsymbol{\pi} \quad , \end{aligned} \quad (2)$$

whose Poisson bracket algebra is the  $\mathfrak{sl}(2, \mathbb{R})$  Lie algebra,

$$\begin{aligned} \{H_1, H_2\} &= D \quad , \\ \{H_1, D\} &= -2H_1 \quad , \\ \{H_2, D\} &= 2H_2 \quad . \end{aligned} \quad (3)$$

The constraint algebra (3) can be regarded as a (much) simplified version of that of general relativity,  $H_1$  and  $H_2$  being two ‘scalar’ constraints and  $D$  a single ‘vector’ constraint. The gauge transformations generated by the constraints are

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} \mapsto g \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{\pi} \\ \mathbf{v} \end{pmatrix} \mapsto g \begin{pmatrix} \boldsymbol{\pi} \\ \mathbf{v} \end{pmatrix}, \quad (4)$$

where  $g$  is an  $\text{SL}(2, \mathbb{R})$  matrix. The classical gauge group is thus  $\text{SL}(2, \mathbb{R})$ .

When  $\min(p, q) > 1$ , the reduced phase space contains a regular part that is a symplectic manifold of dimension  $2(p + q - 3)$ , with a symplectic form induced from  $\Gamma$ , and certain lower-dimensional, singular subsets. When  $\min(p, q) = 1$ , the regular part is absent and all that remains is singular in the sense that the symplectic form of  $\Gamma$  is pulled back to a vanishing 2-form.

$\Gamma$  admits an  $\text{O}(p, q)$ -action that commutes with the gauge transformations (4). The generators of this action form an  $\mathfrak{o}(p, q)$  algebra of functions on  $\Gamma$ , quadratic in the phase space coordinates. These functions are observables [25] (or perennials [26]) in the sense that their Poisson brackets with the constraints (2) vanish. It can be verified that this  $\mathfrak{o}(p, q)$  algebra contains all the information about generic classical solutions in the sense that it separates the regular part of the reduced phase space. It will therefore be of interest to keep track of what happens to these  $\mathfrak{o}(p, q)$  observables in the quantum theory.

### 2.2. Group averaging

There exists a reasonably straightforward way to quantise the above classical structure without imposing the constraints. The unconstrained, or auxiliary, Hilbert space  $\mathcal{H}_{\text{aux}}$  is taken to be the space of square integrable functions  $\Psi(\mathbf{u}, \mathbf{v})$  in the inner product

$$(\Psi_1, \Psi_2)_{\text{aux}} := \int d^p \mathbf{u} d^q \mathbf{v} \overline{\Psi_1} \Psi_2 \quad , \quad (5)$$

where the overline denotes complex conjugation. The classical constraints (2) can be promoted into essentially self-adjoint quantum constraints by the usual operator substitution  $\mathbf{p} \rightarrow -i\partial_{\mathbf{u}}$ ,  $\boldsymbol{\pi} \rightarrow -i\partial_{\mathbf{v}}$ , and choosing in  $D$  the symmetric ordering. The commutator algebra of the quantum constraints is  $i$  times the Poisson bracket algebra (3), and the quantum constraints exponentiate into a unitary representation  $U$  of  $\text{SL}(2, \mathbb{R})$  when  $p+q$  is even and into a unitary representation of the double cover of  $\text{SL}(2, \mathbb{R})$  when  $p+q$  is odd.  $U$  is isomorphic to the  $(p, q)$  oscillator representation of the double cover of  $\text{SL}(2, \mathbb{R})$  [27] via the Fourier transform in  $\mathbf{v}$ . Finally, the operator substitution promotes the classical  $\mathfrak{o}(p, q)$  observables into an  $\mathfrak{o}(p, q)$  algebra of densely-defined self-adjoint operators that commute with  $U$ .

We now wish to implement group averaging via the map

$$\eta : \phi \mapsto \int dg \phi^\dagger U(g) \ , \quad (6)$$

where  $dg$  is the bi-invariant Haar measure and the right-hand side is to be interpreted in terms of its action on suitable states in  $\mathcal{H}_{\text{aux}}$ . (That this map is chosen antilinear rather than linear is just a convention.) The task is to give (6) a meaning that makes  $\eta$  into a refined algebraic quantisation *rigging map* [7, 10]: a map satisfying certain technical postulates that enable the image of  $\eta$  to be interpreted, in the infinite-dimensional case after Cauchy completion, as the physical Hilbert space of the constrained theory. The physical inner product  $(\cdot, \cdot)_{\text{RAQ}}$  on the image of  $\eta$  will then be given by

$$(\eta(\phi_1), \eta(\phi_2))_{\text{RAQ}} := \eta(\phi_2)[\phi_1] \ . \quad (7)$$

We seek to define  $\eta$  via the integral

$$(\phi_1, \phi_2)_{\text{ga}} := \int dg (\phi_1, U(g)\phi_2)_{\text{aux}} \ . \quad (8)$$

While (8) is not well defined on all of  $\mathcal{H}_{\text{aux}}$ , it is possible to find a dense linear subspace  $\Phi \subset \mathcal{H}_{\text{aux}}$ , called the test space, on which the integral in (8) converges in absolute value. Equation (8) then defines on  $\Phi$  a Hermitian sesquilinear form  $(\cdot, \cdot)_{\text{ga}}$  that is invariant under  $U$  in each argument. When (6) is interpreted as

$$\eta(\phi_1)[\phi_2] := (\phi_1, \phi_2)_{\text{ga}} \ , \quad (9)$$

where the square brackets denote the action of  $\eta(\phi_1)$  on  $\phi_2 \in \Phi$ ,  $\eta$  is then a map from  $\Phi$  to its algebraic dual,  $\Phi^*$ . The image of  $\eta$  consists of ‘distributional’ states in the sense that  $\Phi^*$  is not contained in the Hilbert dual of  $\mathcal{H}_{\text{aux}}$ . It follows that the image of  $\eta$  consists of states that are gauge invariant, in the sense that for every  $\phi \in \Phi$  and  $g$  in the gauge group, a state in the image of  $\eta$  has the same action on  $U(g)\phi$  and  $\phi$ . It is further possible to choose  $\Phi$  to be invariant under the quantum  $\mathfrak{o}(p, q)$  observables and to satisfy certain technical conditions introduced in [10]. This means that we will obtain a quantum theory if just two more conditions can be shown to hold: The image of  $\eta$  needs to be nontrivial and the sesquilinear form (7) on this image needs to be positive definite.

The outcome depends sensitively on  $p$  and  $q$ :

- From the viewpoint of the classical theory, the most interesting case is  $\min(p, q) > 1$ . For  $p+q$  even, we do recover a quantum theory, and this theory carries a nontrivial representation of the quantum  $\mathfrak{o}(p, q)$  observables. For  $p+q$  odd, by contrast, the image of  $\eta$  is trivial and we obtain no quantum theory. Quantisation for  $p+q$  odd would seem to require some modification that allows the test states to have *noninteger* angular momenta in  $\mathbf{u}$  or  $\mathbf{v}$ .

- From the viewpoint of the classical theory, one would expect something pathological to happen when  $\min(p, q) = 1$ . This turns out to be the case. We obtain either no quantum theory or a quantum theory with a one-dimensional Hilbert space and a trivial representation of the quantum  $\mathfrak{o}(p, q)$  observables.

For  $(p, q) = (1, 1)$ , the sesquilinear forms (7) and (8) are nonvanishing but have *indefinite* signature. The image of  $\eta$  is nontrivial and two-dimensional, but the would-be inner product (7) on this image is indefinite. This is the first example known to the author in which group averaging fails by producing an indefinite would-be inner product.

### 3. Triangular $SL(2, \mathbb{R})$ system

We next consider a system [14] obtained from the  $SL(2, \mathbb{R})$  system of section 2 by dropping  $H_1$  from the action (1). The classical gauge group is then the connected component of the lower triangular subgroup of  $SL(2, \mathbb{R})$ .

The interest in this system arises from the observation that since the gauge group is not unimodular, the left and right Haar measures do not coincide, and neither of these measures is invariant under the group inverse. In order for the group averaging to produce in (8) a Hermitian sesquilinear form, the integration measure needs to be invariant under the group inverse, and a measure with this property is the geometric average  $d_0g$  of the left and right invariant Haar measures. As noted in [10], adopting  $d_0g$  in (8) means that the physical states will *not* be invariant under the gauge group action. Instead, the physical states will satisfy the constraints in a sense that in certain classes of systems has been shown to be equivalent to geometric quantisation in the reduced phase space [28, 29, 30].

The triangular  $SL(2, \mathbb{R})$  system is classically singular for  $(p, q) = (1, 1)$  in the sense that the pull-back of the phase space symplectic form to the reduced phase space vanishes. For  $(p, q) \neq (1, 1)$  the reduced phase space has still singularities, but these singularities form a set of measure zero and the rest is a symplectic manifold, symplectomorphic to  $T^*(S^{p-1} \times S^{q-1})$ . In this sense the system is classically regular for  $(p, q) \neq (1, 1)$ .

The system possesses the same  $\mathfrak{o}(p, q)$  algebra of classical observables as the  $SL(2, \mathbb{R})$  system of section 2. For  $(p, q) = (1, 1)$  these observables vanish on all of reduced phase space, but for  $(p, q) \neq (1, 1)$  they separate the regular part of the reduced phase space up to a certain measure zero subset and an overall twofold degeneracy. As in the  $SL(2, \mathbb{R})$  system, it will therefore be of interest to keep track of what happens to these observables on quantisation.

Group averaging with the measure  $d_0g$  in (8) can now be completed for all  $(p, q)$  and yields in each case a quantum theory that carries a maximally degenerate principal unitary series representation of the quantum  $\mathfrak{o}(p, q)$  observables [31]. The representation is nontrivial iff  $(p, q) \neq (1, 1)$ , that is, precisely when the classical system is regular in the above sense.

### 4. Ashtekar-Horowitz-Boulware system

Our third system [15] was introduced by Boulware [32] as a simplified version of the Ashtekar-Horowitz model [33], as a venue for studying tunnelling phenomena in constrained quantisation [32, 33, 34, 35, 36]. While our results on tunnelling coincide with those in [32], the new issue of interest for us is in the sense of convergence of the group averaging and in the resulting superselection sectors in the quantum observable algebra.

The configuration space of the classical system is  $S^1 \times S^1 = \{(x, y)\}$ , where  $x$  and  $y$  are understood periodic with period  $2\pi$ . The action reads

$$S = \int dt (p_x \dot{x} + p_y \dot{y} - \lambda C) , \quad (10)$$

where  $\lambda$  is a Lagrange multiplier associated with the constraint

$$C := p_x^2 - R(y) \quad (11)$$

and the potential  $R : S^1 \rightarrow \mathbb{R}$  is a smooth function that is positive at least somewhere and satisfies certain genericity conditions. In particular we assume  $R$  to have finitely many zeroes and stationary points, all stationary points to have finite order and no zero to be a stationary point. It follows that the reduced phase space  $\Gamma_{\text{red}}$  is a two-dimensional symplectic manifold with certain singular points and finite symplectic volume. The singularities are located at the stationary points of  $R$ , and they are topological in nature, arising from the periodicity of  $x$ .

We choose the auxiliary Hilbert space  $\mathcal{H}_{\text{aux}}$  to be the space of square integrable functions of the configuration variables in the inner product

$$(\phi_1, \phi_2)_{\text{aux}} := \int dx dy \overline{\phi_1(x, y)} \phi_2(x, y) . \quad (12)$$

The classical constraint (11) is promoted into the essentially self-adjoint quantum constraint

$$\hat{C} := -\frac{\partial^2}{\partial x^2} - \hat{R} , \quad (13)$$

which exponentiates into the one-parameter family of unitary operators

$$U(t) := e^{-it\hat{C}} , \quad t \in \mathbb{R} . \quad (14)$$

The gauge group of the quantum theory is thus  $\mathbb{R}$ .

For the test space  $\Phi \subset \mathcal{H}_{\text{aux}}$ , in which formula (6) is to be given a meaning, we adopt the space of functions of the form  $f(x, y) = \sum_{m \in \mathbb{Z}} e^{imx} f_m(y)$ , where each  $f_m : S^1 \rightarrow \mathbb{C}$  is smooth and only finitely many  $f_m$  are different from zero for each  $f$ . Proceeding for the moment formally, (6) then yields

$$(\eta(f))(x, y) = 2\pi \sum_{mj} \frac{e^{-imx} \overline{f_m(y)}}{|R'(y_{|m|j})|} \delta(y, y_{|m|j}) , \quad (15)$$

where  $y_{|m|j}$  are the solutions to the equation

$$m^2 = R(y) \quad (16)$$

and the delta-distribution on the right-hand side of (15) is that on  $S^1$ . The task is now to examine whether equation (15) in fact defines a rigging map. A necessary condition clearly is that (16) has solutions for some  $m \in \mathbb{Z}$ .

Suppose first that no  $y_{|m|j}$  is a stationary point of  $R$ . Most of the rigging map properties can then be immediately verified from (15). In particular, the image of  $\eta$  is a nontrivial finite-dimensional vector space, states in the image of  $\eta$  are gauge invariant in the sense that  $\eta(f)[U(t)g] = \eta(f)[g]$  for all  $f, g \in \Phi$  and  $t \in \mathbb{R}$ , and the sesquilinear form defined on the image of  $\eta$  by (7) is positive definite.

The rigging map property that is not immediate from (15) is whether  $\eta$  induces a representation of the refined algebraic quantisation observable algebra  $\mathcal{A}_{\text{obs}}$  on the physical Hilbert space. The operators in  $\mathcal{A}_{\text{obs}}$  are densely defined on  $\mathcal{H}_{\text{aux}}$  and required to be gauge invariant in the sense that they commute with  $U(t)$  for all  $t$ , but they are also required to satisfy certain technical conditions, in particular that their domain includes  $\Phi$  and they map  $\Phi$  to itself. The condition for  $\eta$  to induce a representation of  $\mathcal{A}_{\text{obs}}$  on the physical Hilbert space reads

$$\eta(A\phi_1)[\phi_2] = \eta(\phi_1)[A^\dagger\phi_2] \quad (17)$$

for all  $\phi_1, \phi_2 \in \Phi$  and  $A \in \mathcal{A}_{\text{obs}}$ . As the definition of  $\mathcal{A}_{\text{obs}}$  is implicit rather than constructive, verifying (17) involves some subtlety but can be accomplished by considering matrix elements

of the form  $(Af, U(t)g)_{\text{aux}}$ . Hence  $\eta$  is a rigging map. It can be verified that the representation of  $\mathcal{A}_{\text{obs}}$  on the resulting physical Hilbert space is irreducible, and in fact transitive.

Suppose then that some solutions to (16) are stationary points of  $R$ . The corresponding terms in (15) have a zero in the denominator and are hence ill defined. However, under appropriate technical conditions on the stationary point structure of  $R$ , these formally divergent terms can be replaced by finite ones that involve fractional powers of higher derivatives of  $R$  in the denominator. This replacement can be understood as a renormalisation of the averaging by an infinite multiplicative constant whose ‘magnitude’ depends on the order of the stationary point. We find that including in (15) only the non-stationary solutions to (16) yields one rigging map, but there are also others, given by the renormalised terms in which the solutions to (16) are respectively (i) maxima of a given order, (ii) minima of a given order, and (iii) points of inflexion of a given order. Each of these rigging maps leads to a physical Hilbert space that carries a transitive representation of  $\mathcal{A}_{\text{obs}}$ .

As the images of any two of these rigging maps have trivial intersection in  $\Phi^*$ , we can regard the direct sum of the individual Hilbert spaces as the ‘total’ physical Hilbert space  $\mathcal{H}_{\text{RAQ}}^{\text{tot}}$ , with a representation of  $\mathcal{A}_{\text{obs}}$  that decomposes into the representations on the summands. The summands thus constitute superselection sectors in  $\mathcal{H}_{\text{RAQ}}^{\text{tot}}$ .

The emergence of superselection sectors is related to the singularities in  $\Gamma_{\text{red}}$ . The coordinate  $x$  is periodic, and the conjugate momentum  $p_x$  has become quantised in integer values. For generic potentials, these integer values entirely miss the singular, measure zero subsets of  $\Gamma_{\text{red}}$ , and in this case the quantum theory has no superselection sectors. Superselection sectors appear precisely for those potentials for which some of the quantised values of  $p_x$  hit a singular subset of  $\Gamma_{\text{red}}$ .

Finally, consider the semiclassical limit. Let  $\mathcal{H}_{\text{RAQ}}^1$  denote the Hilbert space that comes from the nonstationary solutions to (16). When  $\hbar$  is restored, the dimension of  $\mathcal{H}_{\text{RAQ}}^1$  asymptotes in the  $\hbar \rightarrow 0$  limit to  $(2\pi\hbar)^{-1}$  times the volume of  $\Gamma_{\text{red}}$ , while the dimension of the orthogonal complement of  $\mathcal{H}_{\text{RAQ}}^1$  in  $\mathcal{H}_{\text{RAQ}}^{\text{tot}}$  remains bounded. In this sense, the semiclassical limit in  $\mathcal{H}_{\text{RAQ}}^{\text{tot}}$  comes entirely from the superselection sector  $\mathcal{H}_{\text{RAQ}}^1$ . Note that the semiclassical limit is as might have been expected on comparison with geometric quantisation on compact phase spaces [37, 38].

## 5. Discussion and outlook

Although group averaging has proved a viable method for quantising certain systems whose gauge group is a noncompact Lie group, general principles of when and how group averaging might be expected to work remain largely open. We conclude by discussing some of these open issues in the light of our systems.

A central ingredient in the quantisation is the choice of the test space  $\Phi \subset \mathcal{H}_{\text{aux}}$ . This space has a dual role. On the one hand, the test space is a technical necessity, since the integral over a noncompact gauge group is not expected to converge on all of  $\mathcal{H}_{\text{aux}}$ . On the other hand, the test space has a deep physical significance in that it determines the observables of the quantum theory. Refined algebraic quantisation does not as such require a single classical or quantum observable to be explicitly constructed, but if some classical observables of interest are known, then it is the choice of the test space that determines whether corresponding observables will exist in the quantum theory. This issue arises prominently in our  $\text{SL}(2, \mathbb{R})$  system, where considerable tuning of the test space is required to guarantee that a quantisation of the classical  $\mathfrak{o}(p, q)$  observable algebra is included in the observable algebra of the quantum theory. General conditions under which technically viable and physically appropriate test spaces exist remain an open problem.

The  $\text{SL}(2, \mathbb{R})$  system with  $(p, q) = (1, 1)$  provides an example in which the group averaging sesquilinear form (8) is indefinite. Any general theorems one might wish to prove on group

averaging therefore do need to take this eventuality into account. The Giulini-Marolf uniqueness theorem [10] is a perspicacious case in point, implying that when group averaging converges sufficiently strongly to an indefinite sesquilinear form, the system admits no rigging maps. It would be of interest to understand just how special the  $SL(2, \mathbb{R})$  system with  $(p, q) = (1, 1)$  is. The system is classically quite singular, with a reduced phase space that contains just three points, non-Hausdorff close to each other. Can an indefinite would-be inner product arise in classically well-behaved systems?

The Ashtekar-Horowitz-Boulware system displays a situation in which group averaging does not converge in absolute value, and with some potentials not even in a conditional sense. Nevertheless, the averaging can be consistently renormalised, with the consequence that the representation of the physical observable algebra decomposes into superselection sectors. This phenomenon has been observed previously [11], but what is striking in the Ashtekar-Horowitz-Boulware system is that the superselection sectors appear precisely when some of the discrete eigenvalues of a certain operator take values at which the corresponding classical observable hits a singularity in the reduced phase space. Is there a more general connection between classical singularities and quantum superselection sectors?

From the gravitational viewpoint, systems whose gauge group is a Lie group tend to arise in symmetry reductions of gravity, as is the case with spatially homogeneous cosmologies, or in systems that have been constructed by hand to mimic certain aspects of gravity, as is the case with all the systems discussed in this contribution. In gravity proper, however, the gauge group is infinite dimensional, and the Poisson bracket algebra of the constraints closes not by structure constants but by structure functions. While group averaging with nonunimodular Lie groups may give some insight into structure functions [10], and while a formalism that ties group averaging to BRST techniques has been developed [19], an extension of group averaging to systems with structure functions remains yet to be developed to a level that would allow a precise discussion of convergence properties and the observables in the ensuing quantum theory.

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