Aspects of
Compactifications and Black Holes
in Four-Dimensional Supergravity

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Aspecten van
Compactificaties en Zwarte Gaten
in Vier-Dimensionale Superzwaartekracht

(met een samenvatting in het Nederlands)

Proefschrift

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Hugo Thomas Looijestijn

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Promotor :	Prof. dr. B. Q. P. J. de Wit
Co-promotor:	Dr. S. J. G. Vandoren

Contents

Publications

This thesis is based on the following publications:

- H. Looyestijn and S. Vandoren, *On NS5-brane instantons and volume stabilization.*, *JHEP* **04** (2008) 24.
- K. Hristov, H. Looyestijn and S. Vandoren, *Maximally supersymmetric solutions of* $D = 4, \mathcal{N} = 2$ *gauged supergravity.*, *JHEP* **11** (2009) 115.
- K. Hristov, H. Looyestijn and S. Vandoren, *BPS black holes in* $\mathcal{N} = 2$, $D = 4$ *gauged supergravities.*, *JHEP* **08** (2010) 103.
- H. Looyestijn, E. Plauschinn and S. Vandoren, *New potentials from Scherk-Schwarz reductions.*, arXiv:1008.4286 [hep-th], submitted for publication to *JHEP*.

Chapter 1

Introduction

General Relativity and the Standard Model

The 20th century has seen an astonishing progress in the description of fundamental physics. We can now accurately describe many phenomena with the theory of General Relativity and the Standard Model of elementary particles.

General Relativity is Einstein's theory of gravity, originating from his early 20th century publications. It describes how space-time is curved by the presence of matter, and how matter moves in a curved space-time. Early successes include the prediction of Mercury's perihelion precession and the deflection of starlight. Later indications for its correctness led to the development of cosmology, which uses General Relativity to study the universe as a whole. It has been tested thoroughly, especially in the weak field limit, such as in most astrophysical systems. The 1993 Nobel prize was awarded for the discovery of a binary star system, whose energy loss due to gravitational radiation is in excellent agreement with General Relativity, and provides one of the strongest arguments for its validity.

The Standard Model is the culmination of research on particle physics. In the early 20th century, Max Planck's work on black body radiation gave birth to quantum mechanics. Subsequent work of Einstein on the photo-electric effect and of Bohr on the atomic model led to further confirmation of quantum mechanics as a theory of particle physics. A full quantum treatment of relativistic electrodynamics led to the development of quantum field theories. The Standard Model is a quantum field theory which aims to describe all fundamental physical phenomena, excluding gravity. Many of its predictions are being tested to astonishing precision in particle accelerators. At the time of writing, the Large Hadron Collider at CERN is running experiments, with the hope of detecting the Higgs boson - the last undiscovered Standard Model particle.

Despite these great successes, there are many open questions in theoretical physics. A direct puzzle is the fact that General Relativity cannot consistently be written in the language of quantum mechanics. Besides being theoretically unsatisfying, it makes it impossible to study a regime where both theories are important. One of these areas is the study of black holes, where quantum effects on gravity are important, and the Big Bang, the event which created our universe. Other questions come from cosmological observations that indicate that Standard Model matter is only 4% of all matter in our universe. The remaining matter is believed to consist of dark matter and dark energy, neither of which is satisfactorily explained by the Standard Model or General Relativity. Dark matter consists of particles which are not described by the Standard Model. Dark energy can be described by the cosmological constant, which is a parameter in Einstein's equations. Its value has profound implications for the dynamics of our universe. Current observations predict a small, positive value for this constant. However, a combination of General Relativity and the Standard Model would predict a very large value.

To address some of these open questions, we need a theory that can reconcile the principles of quantum mechanics with those of General Relativity - a theory of quantum gravitation.

String theory

One of the leading candidates for a theory of quantum gravitation is string theory, whose development started around 1970. A main ingredient is the idea that the fundamental degrees are not point-like, as in the Standard Model, but are strings. The particles we observe, such as quarks, electrons and gravitons, are believed to be oscillations of these strings. Using these strings and other extended objects, string theory can incorporate General Relativity in a quantum mechanical theory. Books on string theory include the famous Green, Schwarz and Witten book [1, 2], the books by Polchinsky [3, 4] and the book by Becker and Schwarz [5].

An area where string theory has proven itself a serious candidate is the study of black holes. It extends the General Relativity description with a quantum mechanical interpretation, providing insight into their thermal properties, their evolution and the information loss paradox [6, 7].

Another development which originated from string theory is gauge/gravity duality [8], based on earlier work of 't Hooft [9, 10]. This conjecture states that a system without gravity can be described in terms of a system with gravity, living one dimension higher. Both theories contain a coupling constant, which measures the strength of the interaction. The interesting aspect is that the duality maps a theory with a large coupling constant to a theory with a small coupling constant, which basically maps a difficult problem to an easier problem. Utilizing this, one studies strongly coupled QFTs employing string theory in four dimensions. Examples of these are the studies of quark gluon plasmas and condensed matter systems.

Supersymmetry and supergravity

All fundamental particles in a high-energy quantum theory can be divided into two types: bosons and fermions. They differ in the quantum statistics they obey: only one fermion can occupy a quantum state at a given time, whereas there is no such condition for bosons. In the Standard Model, the quarks and electrons are fermions, whereas the Higgs particle, the photons and the other force-carrying particles are bosons. Supersymmetry relates these two different types of particles. Each boson has a fermionic partner, and vice versa, which are called the superpartners. Although the Standard Model contains both bosons and fermions, they cannot be superpartners of each other, so we would need to add additional particles to make this theory supersymmetric. One such example is the Minimal Supersymmetric Standard Model.

Supersymmetry has many attractive features. The superpartners of the Standard Model particles have not been discovered, they are a natural candidate for dark matter, where supersymmetry explains the stability of these particles. At the time of writing, experiments are carried out at the Large Hadron Collider, which might be able to discover supersymmetry. The amount of supersymmetry is denoted with the letter $\mathcal{N}$. The simplest theories have $\mathcal{N} = 1$, but we will mainly study $\mathcal{N} = 2$ theories, which have more supersymmetry. The higher amount of symmetry of these theories makes them easier to study. Examples of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric field theories will be discussed in chapter 2.

Many of the predictions of string theory concern physics on very high energy scales. The exact region which is considered high differs between models, but a rough indication of these energies is the Planck energy, which is approximately 10^{19} GeV. The highest energy a particle accelerator has reached so far is at the Large Hadron Collider at CERN, which has a maximum of 1.4×10^4 GeV, which is about 15 orders of magnitude below the Planck scale. It is therefore more interesting to study the dynamics of string theory at lower energy scales. These dynamics are described by a quantum field theory. Besides the presence of gravity, another feature of these theories is that they are supersymmetric, just as the string theories they originate from. Due to the presence of gravity and supersymmetry, these theories are called supergravity theories.

Questions in string theory

Although string theory has it successes in explaining puzzles of quantum gravity, it also raises its own questions. A direct issue is that supersymmetric string theories are only well-defined in ten dimensions, which does not directly agree with the four-dimensional appearance of our world. A possible resolution is that we only observe four dimensions, but the world is truly ten dimensional. The remaining dimensions are curled up into an internal space, which is so small that they are not observed. We can then produce a theory that only describes the physics in four dimensions, using a method that is called compactification. A popular choice for the internal space is a so-called Calabi-Yau threefold, which is a six-dimensional, compact space. Compactification of type II string theories on a Calabi-Yau threefold gives a four-dimensional theory with $\mathcal{N} = 2$ supersymmetry.

One of the questions of compactification is that there is a large variety of options for the compact six dimensions, but which options will lead to a description of the world as we know it? So far, it still is a challenge to find a compactification of string theory that yields the Standard Model. A related question is: what do the other options, that yield different four-dimensional physics, represent? How should one think about these? The interpretation of this so-called landscape of possible solutions leads to border of physics and philosophy, and will not be discussed in this thesis.

Nevertheless, even if one chooses a specific compactification, there can be small differences in the six-dimensional space when one moves through space-time. These changes manifest themselves finally as massless particles in the low-energy spectrum, which are called moduli. Such massless particles are not observed in nature. What can we change in such a way that these massless particles will not be present?

A possible mechanism to address the problems of moduli is via flux compactifications (for some reviews, see [11–13]). In such a setup, there is a flux (similar to the well-known electric and magnetic fluxes of Maxwell theory) in the internal space. The small differences in the six-dimensional space, which were the origin of the moduli fields, will now cost some energy. Their vacuum expectation values will be determined by a local energy minimum, and as fluctuations around these values cost energy, this gives the moduli their masses. If the masses are now heavy enough, there is no longer a contradiction with experiment: the particles are simply to heavy to be observed. Furthermore, the expectation values of the moduli are also interesting. One of the moduli is called the dilaton, and its expectation value determines the string coupling constant. What is the expectation value of the moduli?

The flux compactifications eventually lead to scalar potentials for the moduli. At fixed values of the moduli, the potential serves as a cosmological constant, which might make them interesting cosmological models. What will be the value of this cosmological constant? How can we obtain the small, positive value that we presently see?

Another issue concerns the construction of the supergravity theory out of the string theory. Although we are only interested in the four-dimensional, low-energy dynamics, there still is an influence of the higher-dimensional, high-energy dynamics. While string theory was originally formulated as a theory of strings, we now know that there are more objects in the theory. A class of these objects are D-branes, which are extended objects on which strings can end. These manifest themselves as corrections to the four-dimensional supergravity, and their effects can be important. Without these quantum corrections, the moduli are usually not stabilized. How can we compute these quantum corrections? What is their effect on the program of moduli stabilization?

Content of this thesis

In chapter 2 we introduce supersymmetry and supergravity theories. We will show the structure of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ gauged supergravity theories in four dimensions, and illustrate how they can be obtained from string theory.

In chapter 3 we start with the study of $\mathcal{N} = 2$ theories. The advantage of $\mathcal{N} = 2$ theories over $\mathcal{N} = 1$ is that in the latter, many quantum effects arise, which are largely unknown, whereas the extra symmetry of the former keeps these contributions under somewhat better control. In this chapter, we find a classification of the fully supersymmetric solutions of gauged $\mathcal{N} = 2$ theories. There are many other solutions, but not all of them will preserve the full supersymmetry. Finding the supersymmetric configurations is important for several reasons. If one makes a string theory compactification which preserves $\mathcal{N} = 2$ supersymmetry, one must be able to find this vacuum again in the four-dimensional effective theory. Knowledge of the supersymmetric configurations gives therefore insight in the possible $\mathcal{N} = 2$ compactifications. So far, it is not so clear how many compactifications preserve the full amount of supersymmetry, so this is an important issue to study.

Another important property of these configurations is that black holes are often solutions that interpolate between two different $\mathcal{N} = 2$ configurations, located at the horizon and at spatial infinity. Finally, applications such as the gauge/gravity duality mentioned above, also require knowledge of the supersymmetric configurations. We perform a systematic analysis of the supersymmetry transformations and see which conditions they imply. Using these, we are able to classify all the fully supersymmetric solutions. There are basically only three configurations: Minkowski space, a negatively curved space known as AdS_4, and a combination of such a negatively curved space with a sphere, to form $AdS_2 \times S^2$.

In chapter 4 we search for black hole solutions of gauged $\mathcal{N} = 2$ theories. They solve the equations of motion, but they only preserve half of the $\mathcal{N} = 2$ symmetry, or less. One of the motivations is to understand the microscopic entropy of (asymptotically flat) black holes. In ungauged supergravity, arising e.g. from Calabi-Yau compactifications, this is relatively well understood in terms of counting states in a weakly coupled D-brane set-up [6, 14], and then extrapolating from weak to strong string coupling. In flux compactifications, with effective gauged supergravity actions, this picture is expected to be modified. The most dramatic modification is probably when the dilaton is stabilized by the fluxes, such that one cannot extrapolate between strong and weak string coupling.

Another motivation stems from the AdS/CFT correspondence and its applications to strongly coupled field theories. Here, finite temperature black holes that asymptote to anti-de Sitter space-time describe the thermal behavior of the dual field theory. Often, like e.g. in holographic superconductors, see e.g. [15, 16] for some reviews or [17, 18] for more recent work, charged scalar fields are present in this black hole geometry, providing non-trivial scalar hair that can be computed numerically. Therefore, one is in need of finding large classes of asymptotically AdS black holes with charged scalars. This is one of the aims of this chapter. Although we mostly work in the context of supersymmetric black holes, some of our analysis can be carried out for finite temperature black holes as well.

The $\mathcal{N} = 2$ theories contain many particles, which are organized in multiplets, according to their supersymmetry properties. One of these is a hypermultiplet, which contains four scalar fields. In known solutions so far, the hypermultiplet scalars are neutral and take constant values; we initiate the extension to general gauged supergravities in which the hyperscalars are charged. We find black hole solutions that arise after spontaneous symmetry breaking of the gauged theory. Using this method, we can embed known black holes. When searching for new solutions, we only find solutions where the metric has ripples and the vector multiplet scalars become ghost-like.

In chapter 5 we look at compactifications of eleven-dimensional supergravity. Eleven dimensions is the highest possible dimension for a supergravity theory (without particles of spin higher than two) to exist. It is presumed to be the low-energy limit of some (membrane) theory, dubbed M-theory, but this is still largely unknown. One can compactify M-theory on a Calabi-Yau three-fold to yield a five-dimensional theory. This can be further compactified on a circle to give a four-dimensional theory. We want to investigate how one can stabilize moduli in such a scenario. This will be done by performing a Scherk-Schwarz compactification on the circle. Alternatively, this can be seen as a compactification of eleven-dimensional supergravity on a seven-dimensional space, which is made by a non-trivial fibration of the Calabi-Yau manifold over a circle. This procedure leads to a gauged four-dimensional $\mathcal{N} = 2$ supergravity.

When compactifying type II string theory on a Calabi-Yau threefold, one can choose to orientifold the theory. In such a scenario, there is an orientifold operator acting on the string states. The states are then required to have a definite parity under this operator. This projects out half of the states, and reduces the preserved supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. As we do not have a microscopic description of M-theory, we cannot directly perform an orientifold compactification. We can, however, use these as inspiration to define a truncation of the four-dimensional theory. In this chapter we perform this truncation and show that this yields an $\mathcal{N} = 1$ theory. Although we do not have a microscopic understanding of this procedure, it does give a new way to produce $\mathcal{N} = 1$ theories, for which we compute the superpotential.

In chapter 6 we study the stabilization of moduli. Besides the dilaton, mentioned above, another important modulus is the volume of the internal manifold. In the effective supergravity approximation, one assumes that curvatures are small and hence the volume of the internal space is large compared to the scale set by the string length $l_s = 2\pi\sqrt{\alpha'}$. Stabilization by finding the minima of the scalar potential of the low-energy effective action must therefore lead to a large value for the volume for this approach to make sense. This is one of the basic assumptions in the KKLT scenario [19], or the more recent so-called 'Large volume scenarios' (LVS) [20, 21].

In a classical compactification, the volume modulus is usually not stabilized. To change this, we study a class of corrections to the scenario of compactification. Besides the D-branes mentioned earlier, string theory also contains objects known as NS5-branes, which have a six-dimensional world-volume. When they completely live in the internal six dimensions, they are instantons from the four-dimensional perspective. These non-perturbative effects give corrections to the four-dimensional action which depend on the volume $\mathcal{V}$ of the internal six dimensions. They might therefore be useful to stabilize $\mathcal{V}$. In this chapter, we analyze these corrections, and study their effects on the program of moduli stabilization. The main conclusion is that their effects are important in certain regimes of parameter space, but other corrections, arising from loop effects in string theory, become equally important. Not all of these effects are known, which makes it difficult to compare their effects to the ones from the NS5-branes.

Chapter 2

Supergravity

This chapter gives an introduction to supersymmetry and supergravity. Given the number of topics we have to cover, we will only be brief. A thorough introduction to supersymmetry can be found in [22–24], whereas supergravity is covered in [25–28]. The connection between the four- and ten-dimensional supergravities is reviewed in [11–13].

We will start with supersymmetric field theories in four space-time dimensions, and show some examples of $\mathcal{N} = 1$ supersymmetric theories. This symmetry can be made local to give rise to supergravity theories. We discuss $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supergravity theories, and show how one can deform those by gauging isometries. Finally, we indicate how the four-dimensional theories can be obtained by compactifying ten-dimensional theories.

2.1 Supersymmetry

A supersymmetric theory is, by definition, invariant under transformations with a constant, fermionic parameter. In a quantum field theory, such a fermionic parameter is represented by a spinor. In four space-time dimensions, a general Dirac spinor has 8 (real) degrees of freedom. This representation is however reducible. We can impose a reality condition, which leads to Majorana spinors, or a chirality condition, which leads to Weyl spinors. In four dimensions, it is impossible to impose them both, so we can choose to work with Majorana spinors, or Weyl spinors (also called chiral spinors), but not with Majorana-Weyl spinors. Both choices are equivalent, and in this thesis we work with Weyl spinors.

Invariance under supersymmetry gives a number of conserved charges, which are called supercharges. The number of supercharges divided by the dimension of the smallest possible spinor representation is denoted by $\mathcal{N}$. The smallest amount of supersymmetry possible is therefore denoted by $\mathcal{N}=1$, which has four supercharges. We will introduce some notation that is easily extended to the forthcoming case of $\mathcal{N}=2$ supersymmetry and refer to appendix A.1 for our conventions. A chiral fermion is denoted as $\lambda_\bullet$ and has positive chirality ($\gamma_5\lambda_\bullet = \lambda_\bullet$). Its complex conjugate $\lambda^\bullet \equiv \lambda_\bullet^*$ has negative chirality. We use the conjugate $\overline{\lambda^\bullet} \equiv (\lambda^\bullet)^t\gamma_0$.

The simplest example of a supersymmetric action is given by the $\mathcal{N}=1$ chiral multiplet [29]. It consists of a complex scalar z and a chiral fermion $\lambda_\bullet$. The free action is then given by

$$S = \int d^4x \left(\partial_\mu z \partial^\mu \overline{z} - i\overline{\lambda^\bullet}\gamma^\mu\partial_\mu\lambda_\bullet\right) , \tag{2.1}$$

which is invariant under the transformations

$$\begin{aligned} \delta z &= \overline{\lambda_\bullet}\varepsilon_\bullet \ , \\ \delta\lambda_\bullet &= i\partial_\mu z\gamma^\mu\varepsilon^\bullet , \end{aligned} \tag{2.2}$$

where $\varepsilon_\bullet$ is a constant spinor.

The supersymmetry transformations form an algebra, called the super-Poincaré algebra, which contains supersymmetry transformations, Lorentz transformations and translations. The anti-commutator of two such transformations, acting on $\lambda_\bullet$, gives two terms: a translation of the spinor, and a term proportional to the equations of motion of the spinor. The algebra only closes if one imposes the equations of motion. The transformations (2.2) therefore only form an on-shell representation of the supersymmetry algebra. This reflects itself in the counting of degrees of freedom: the complex scalar z has two real degrees of freedom, whereas the chiral spinor $\lambda_\bullet$ has four. However, its equation of motion, which is the massless Dirac equation, halves the degrees for the spinor, so there is a match of degrees of freedom. It is also possible to add extra fields, called auxiliary fields, to obtain a representation which closes off-shell; in this case the degrees of freedom also match off-shell.

The second example of a supersymmetric action is an $\mathcal{N}=1$ vector multiplet, containing a chiral fermion $\chi_\bullet$ and a vector field A_μ. The free action is given by

$$S = \int d^4x \left(F_{\mu\nu}F^{\mu\nu} - i\overline{\chi^\bullet}\gamma^\mu\partial_\mu\chi_\bullet\right) , \tag{2.3}$$

where $F_{\mu\nu} = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)$. This action is invariant under the transformations

$$\begin{aligned} \delta\chi_\bullet &= \frac{1}{2}F^-_{\mu\nu}\gamma^{\mu\nu}\varepsilon_\bullet \ , \\ \delta A_\mu &= i\overline{\chi_\bullet}\gamma_\mu\varepsilon^\bullet - i\overline{\varepsilon_\bullet}\gamma_\mu\chi^\bullet \ , \end{aligned} \tag{2.4}$$

where $F^-_{\mu\nu}$ is the anti-self-dual component of the field strength $F_{\mu\nu}$, which is defined as $F^{\pm}_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} \pm \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma})$, see also appendix A.1.

The final example consists of a theory with n chiral fields. We denote the scalars as z^i, with $i = 1, \ldots, n$. The action is given by [30]

$$S = \int d^4x \left(g_{i\bar{\jmath}}\partial_\mu z^i \partial^\mu z^{\bar{\jmath}} - i g_{i\bar{\jmath}} \overline{\chi^{\bullet j}} \gamma^\mu \nabla_\mu \chi^i_\bullet - \frac{1}{4} R_{ij\bar{k}\bar{\ell}} \overline{\chi^k_\bullet} \chi^i_\bullet \overline{\chi^\ell_\bullet} \chi^j_\bullet \right) , \tag{2.5}$$

where $g_{i\bar{\jmath}}$ is a hermitian matrix, which can depend on z^i and $z^{\bar{\jmath}}$. One can see $g_{i\bar{\jmath}}$ as a (fixed) metric on the space of the fields z^i; this space is called the target space. The covariant derivative is $\nabla_\mu \chi^i_\bullet = \partial_\mu \chi^i_\bullet + \Gamma_{jk}{}^i \partial_\mu z^j \chi^k_\bullet$, and $R_{ij\bar{k}\bar{\ell}}$ is the Riemann tensor, computed from $g_{i\bar{\jmath}}$. The requirements of supersymmetry force $g_{i\bar{\jmath}}$ to be a Kähler metric [30]. This property implies that $g_{i\bar{\jmath}}$ is locally the derivative of a real function $\mathcal{K}$

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} \mathcal{K} , \tag{2.6}$$

where $\partial_i = \frac{\partial}{\partial z^i}$. The function $\mathcal{K}$ is called the Kähler potential. Besides supersymmetry, the action (2.5) is also invariant under the transformations $\delta z^i = k^i$, if the (z^i-dependent) field k^i is an isometry of the Kähler metric. The function $\mathcal{K}$, as used in (2.6), is not unique, as the metric $g_{i\bar{\jmath}}$ remains the same under the Kähler transformation

$$\mathcal{K} \to \mathcal{K} + f(z) + \bar{f}(\bar{z}) . \tag{2.7}$$

The metric $g_{i\bar{\jmath}}$ typically depends on the fields z^i, and could contain regions in field space where it is no longer positive definite. Therefore, one has to restrict the fields to the so-called positivity domain, where the metric is positive definite.

It is possible to add a so-called superpotential to the theory, while keeping the theory supersymmetric. This is a function $W(z)$, holomorphic in the scalar fields z^i. The supersymmetry transformations get modified to

$$\begin{aligned} \delta_\varepsilon z^i &= \overline{\lambda^i_\bullet} \varepsilon_\bullet , \\ \delta_\varepsilon \lambda^i_\bullet &= i\partial_\mu z^i \gamma^\mu \varepsilon^\bullet - g^{i\bar{\jmath}} \partial_{\bar{\jmath}} \overline{W} \varepsilon_\bullet . \end{aligned} \tag{2.8}$$

The bosonic terms in the Lagrangian become

$$\mathcal{L} = g_{i\bar{\jmath}} \partial_\mu z^i \partial^\mu z^{\bar{\jmath}} - V , \tag{2.9}$$

where the scalar potential V is given by

$$V = g^{i\bar{\jmath}} \partial_i W \partial_{\bar{\jmath}} \overline{W} . \tag{2.10}$$

2.2 Local symmetries and supergravity

The theories discussed so far are invariant under constant supersymmetry variations. Such an invariance is often called rigid supersymmetry. In this section, we will discuss theories invariant under local symmetries.

There is a systematic technique to construct an action and transformation rules, invariant under a local symmetry, starting from an action and transformations, invariant under a global symmetry. This technique is called the Noether procedure.

We will not discuss this in detail, but as an example, we consider the free action of a complex scalar field φ

$$S = \int d^4x \; \partial_\mu \varphi \overline{\partial^\mu \varphi} \, . \tag{2.11}$$

This action is invariant under the global transformation $\varphi \to \mathrm{e}^{i\alpha}\varphi$, which has an infinitesimal form $\delta\varphi = i\varphi\alpha$. Going through the Noether procedure, we find the action

$$S = \int d^4x \left(D_\mu \varphi \overline{D^\mu \varphi} - F_{\mu\nu} F^{\mu\nu} \right) \, , \tag{2.12}$$

with a covariant derivative $D_\mu \varphi = \partial_\mu \varphi - i\varphi A_\mu$. The term $F_{\mu\nu}F^{\mu\nu}$ was not needed to make the action invariant under the local symmetry, but it makes the field A_μ a propagating field. This action is now invariant under the local transformations

$$\begin{aligned} \delta\varphi &= i\varphi\alpha \, , \\ \delta A_\mu &= \partial_\mu \alpha \, . \end{aligned} \tag{2.13}$$

One can perform the same procedure with one of the supersymmetric actions in section 2.1. The first step is to specify the gauge field for local supersymmetry transformations, analogous to the gauge field A_μ above. We therefore expect a transformation rule

$$\delta\psi_{\bullet\mu} = \partial_\mu \varepsilon_\bullet \, . \tag{2.14}$$

From this, one sees that the gravitino field $\psi_{\bullet\mu}$ carries a vector index (as it is a gauge field) and a spinor index $\bullet$ (as the right-hand side is spinor-valued). A kinetic term, which is also invariant under the transformation (2.14), is given by the Rarita-Schwinger action

$$S = \int d^4x \; \overline{\psi^\bullet_\mu} \gamma^{\mu\nu\rho} \partial_\nu \psi_{\bullet\rho} \, , \tag{2.15}$$

Now that we have the gauge field, the next step is to embed it into a supersymmetric multiplet. When we have local supersymmetry transformations, the super-Poincaré algebra, mentioned earlier, becomes an algebra of local transformations, and therefore includes local coordinate transformations. The theory we obtain therefore incorporates gravity, and we obtain the theory of pure supergravity [31], defined by

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + i\overline{\psi^\bullet_\mu}\gamma^{\mu\nu\rho}D_\nu\psi_{\bullet\rho}\right). \tag{2.16}$$

The covariant derivative D_ν contains the spin connection and terms quadratic in the gravitino. Up to higher order in the fermions, the derivative reads

$$D_\nu\psi_{\bullet\rho} = \partial_\nu\psi_{\bullet\rho} - \frac{1}{4}\omega_\nu^{ab}\gamma_{ab}\psi_{\bullet\rho}\,. \tag{2.17}$$

Some more details on the spin connection ω_ν^{ab} can be found in (A.7).

The normalization of the gravity term in (2.16) implies that we have chosen units such that Newton's constant κ is fixed as $\kappa^2 = 1$, which is used in the remainder of this thesis. The gravitino $\psi_{\bullet\mu}$ and the metric $g_{\mu\nu}$ form the on-shell content of the graviton multiplet. We will show the supersymmetry transformations in the next chapter, when we discuss $\mathcal{N} = 2$ supergravity.

The final step is to couple the other multiplets, such as the chiral or the vector multiplet, to the graviton multiplet. In principle, this can be done with the Noether method, but in practice, more efficient methods have been developed, such as superconformal multiplet calculus [32] or superspace inspired methods [28]. We will not explain these methods, but we will only use their results.

2.3 $\mathcal{N} = 2$ supergravity

The theories in previous sections were invariant under transformations with a single spinor. It is also possible to construct theories with more supersymmetry, which is called extended supersymmetry. In this thesis, we work with $\mathcal{N} = 2$ supersymmetry. There is a group of transformations that leave the supersymmetry algebra invariant, which is called the R-symmetry group. For $\mathcal{N} = 2$ this is given by $SU(2)$ and the two supersymmetry parameters form a doublet ε_A under $SU(2)$, with $A = 1, 2$. We will now discuss some multiplets of $\mathcal{N} = 2$ supergravity.

2.3.1 The $\mathcal{N} = 2$ graviton and vector multiplets

The $\mathcal{N} = 2$ graviton multiplet [33, 34] consists of the graviton $g_{\mu\nu}$, a doublet of gravitinos $\psi_{A\mu}$ with positive chirality and a gauge field A^0_μ, which is called the graviphoton. As always in this thesis, the negative chirality fermions are given by $\psi^A_\mu \equiv (\psi_{A\mu})^*$. The

$\mathcal{N} = 2$ vector multiplet is a combination of an $\mathcal{N} = 1$ chiral multiplet with an $\mathcal{N} = 1$ vector multiplet, and contains a complex scalar z, a doublet of chiral fermions λ^A with positive chirality, called the gauginos, and a vector field A_μ.

We now couple a number n_V of vector multiplets, with an index $i = 1, \ldots, n_V$, to the graviton multiplet. We use a common index $\Lambda = 0, \ldots, n_V$ to group the gauge fields A^0_μ and A^i_μ together. From this point onward, we write $S = \int d^4x\sqrt{-g}\mathcal{L}$. The bosonic terms in the Lagrangian are given by

$$\mathcal{L} = \frac{1}{2}R(g) + g_{i\bar{\jmath}}\partial^\mu z^i \partial_\mu \bar{z}^{\bar{\jmath}} + I_{\Lambda\Sigma}(z) F^\Lambda_{\mu\nu} F^{\Sigma\,\mu\nu} + \frac{1}{2}R_{\Lambda\Sigma}(z)\epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma}\,. \tag{2.18}$$

The first term is the Einstein-Hilbert term for gravity and the second is a non–linear sigma model for the complex scalars z^i. The third term is the kinetic term for the gauge fields, and the last term is a generalization of the θ-angle term of Maxwell theory. The supersymmetry transformations will be given below; we will first discuss the various objects appearing in (2.18).

Special geometry

Supersymmetry requires that the space, described by the metric $g_{i\bar{\jmath}}$, is a special Kähler space [35]. This means that there are sections X^Λ and F_Λ, which are holomorphic functions of z^i. The Kähler potential $\mathcal{K}$ for the metric $g_{i\bar{\jmath}}$, as in (2.6), is then given by

$$\mathcal{K}(z,\bar{z}) = -\ln\left[i\bar{X}^\Lambda F_\Lambda - iX^\Lambda \bar{F}_\Lambda\right]\,. \tag{2.19}$$

From the sections X^Λ and F_Λ, we can construct

$$L^\Lambda \equiv \mathrm{e}^{\mathcal{K}/2} X^\Lambda\,, \qquad M_\Lambda \equiv e^{\mathcal{K}/2} F_\Lambda\,, \tag{2.20}$$

$$f^\Lambda_i \equiv \mathrm{e}^{\mathcal{K}/2}(\partial_i + \partial_i\mathcal{K})X^\Lambda\,, \qquad h_{\Lambda|i} \equiv \mathrm{e}^{\mathcal{K}/2}(\partial_i + \partial_i\mathcal{K})F_\Lambda\,, \tag{2.21}$$

where $\partial_i\mathcal{K} \equiv \frac{\partial}{\partial z^i}\mathcal{K}$. A further requirement of special Kähler geometry is then

$$X^\Lambda h_{\Lambda|i} - F_\Lambda f^\Lambda_i = 0\,. \tag{2.22}$$

The terms proportional to $\partial_i\mathcal{K}$ make f^Λ_i and $h_{\Lambda|i}$ transform covariantly under Kähler transformations (2.7). These terms define the $U(1)$ Kähler connection

$$A_\mu \equiv -\frac{i}{2}(\partial_i\mathcal{K}\partial_\mu z^i - \partial_{\bar{\imath}}\mathcal{K}\partial_\mu z^{\bar{\imath}})\,. \tag{2.23}$$

The period matrix $\mathcal{N}_{\Lambda\Sigma}$ is defined by the properties

$$F_\Lambda = \mathcal{N}_{\Lambda\Sigma} X^\Sigma, \qquad h_{\Lambda|\bar{\imath}} = \mathcal{N}_{\Lambda\Sigma} f^\Sigma_{\bar{\imath}}. \tag{2.24}$$

It can be shown [36] that the matrix $(L^\Lambda\ f_{\bar{\imath}}^\Lambda)$ is invertible, which gives the expression

$$\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} h_{\Lambda|\bar{\imath}} \\ M_\Lambda \end{pmatrix} \cdot \begin{pmatrix} f_{\bar{\imath}}^\Sigma \\ L^\Sigma \end{pmatrix}^{-1} , \tag{2.25}$$

and one can show that $\mathcal{N}_{\Lambda\Sigma}$ is symmetric. We then define

$$I_{\Lambda\Sigma} = \mathrm{Im}\mathcal{N}_{\Lambda\Sigma} , \qquad R_{\Lambda\Sigma} = \mathrm{Re}\mathcal{N}_{\Lambda\Sigma} , \tag{2.26}$$

and these matrices appear as couplings in the Lagrangian (2.18). It can be shown [36] that $I_{\Lambda\Sigma}$ is invertible and negative definite, and therefore each gauge field has a kinetic term with a positive sign.

Some further identities one can derive are

$$L^\Lambda I_{\Lambda\Sigma} \bar{L}^\Sigma = -\frac{1}{2} , \qquad L^\Lambda I_{\Lambda\Sigma} f_i^\Sigma = 0 , \tag{2.27}$$

$$f_i^\Lambda I_{\Lambda\Sigma} L^\Sigma = 0 , \qquad f_i^\Lambda I_{\Lambda\Sigma} f_{\bar{\jmath}}^\Sigma = -\frac{1}{2} g_{i\bar{\jmath}} , \tag{2.28}$$

and

$$f_i^\Lambda g^{i\bar{\jmath}} f_{\bar{\jmath}}^\Sigma = -\frac{1}{2} I^{\Lambda\Sigma} - \bar{L}^\Lambda L^\Sigma . \tag{2.29}$$

It is sometimes possible to specify the sections X^Λ and F_Λ in terms of a single holomorphic function $F(X^\Lambda)$, called the prepotential. In applications to supersymmetry, F is then given as $F = \frac{1}{2} X^\Lambda F_\Lambda$, and is homogeneous of second degree. We then have $F_\Lambda = \partial_{X^\Lambda} F$ and $z^i = X^i / X^0$.

Using the period matrix, we define the linear combinations

$$\begin{aligned} T_{\mu\nu}^- &= 2i L^\Lambda I_{\Sigma\Lambda} F_{\mu\nu}^{\Sigma -} , \\ G_{\mu\nu}^{i-} &= -g^{i\bar{\jmath}} f_{\bar{\jmath}}^\Lambda I_{\Lambda\Sigma} F_{\mu\nu}^{\Sigma -} , \end{aligned} \tag{2.30}$$

which are called the graviphoton and matter field strengths, respectively. These relations can be inverted to yield

$$F_{\mu\nu}^{\Lambda -} = i \bar{L}^\Lambda T_{\mu\nu}^- + 2 f_i^\Lambda G_{\mu\nu}^{i-} . \tag{2.31}$$

Examples

A simple example of special geometry is given by the prepotential

$$F = -\frac{i}{2}(X^0 X^0 - X^1 X^1) . \tag{2.32}$$

Using a coordinate $z = X^1/X^0$, and choosing the gauge $X^0 = 1$, the sections X^Λ and F_Λ read

$$X^0 = 1\,, \qquad X^1 = z\,, \qquad F_0 = -i\,, \qquad F_1 = iz\,, \tag{2.33}$$

which are clearly holomorphic functions of z. The metric $g_{i\bar{\jmath}}$ can easily be computed from (2.19) and reads $g = (1 - z\bar{z})^{-2}\,\mathrm{d}z\,\mathrm{d}\bar{z}$. We see we have to restrict ourselves to $|z| < 1$, and we recognize this as the metric on the Poincaré disk.

Another important class of examples, which arise in Calabi-Yau compactifications (which will be introduced in section 2.5), is given by

$$F = -\frac{1}{6}\frac{\mathcal{K}_{ijk}X^iX^jX^k}{X^0}\,, \tag{2.34}$$

where the $\mathcal{K}_{ijk}$ are constant, real numbers, determined by the topology of the Calabi-Yau manifold.

2.3.2 Hypermultiplets

A hypermultiplet is formed by combining two chiral $\mathcal{N} = 1$ multiplets. The on-shell contents are four real scalars q^u and two chiral fermions ζ_α. With n_H hypermultiplets, we have bosonic fields q^u, with $u = 1, \ldots, 4n_H$ and fermions ζ_α with $\alpha = 1, \ldots, 2n_H$. The bosonic Lagrangian for the hypermultiplets is a non-linear sigma model

$$\mathcal{L} = h_{uv}\partial_\mu q^u \partial^\mu q^v. \tag{2.35}$$

Supersymmetry [37] now requires the $4n_H$-dimensional metric h_{uv} to be a quaternionic-Kähler space, of negative scalar curvature[1]. This requires that there are three almost complex structures J^x, $x = 1, 2, 3$, that satisfy a quaternionic algebra

$$J^xJ^y = -\delta^{xy} + \epsilon^{xyz}J^z. \tag{2.36}$$

The metric h_{uv} is Hermitian with respect to each J^x, and we can define three quaternionic two-forms $K^x_{uv} = h_{uw}(J^x)^w{}_v$. They are not closed, but they are covariantly constant with respect to an $SU(2)$ connection ω^x:

$$DK^x \equiv dK^x - \epsilon^{xyz}\omega^y \wedge K^z = 0\,. \tag{2.37}$$

The $SU(2)$ connection ω^x defines the $SU(2)$ curvature $\Omega^x \equiv d\omega^x - \frac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z$, and then we have the relation

$$\Omega^x = \lambda K^x\,, \tag{2.38}$$

[1] A quaternionic-Kähler space need not be Kähler, and by a slight abuse of nomenclature, we will refer to them as quaternionic spaces.

where λ is a non-zero[2] constant. Supersymmetry requires this constant to be related to Newton's constant (defined below (2.16)) as $\lambda = -\kappa^2$, and therefore we have $\lambda = -1$. With these units, the Ricci scalar curvature of the quaternionic manifold is given as $R = -8n_H(n_H + 2)$, and is therefore always negative.

We can decompose the metric h_{uv} in quaternionic vielbeine $\mathcal{U}_u^{A\alpha}$ as

$$h_{uv} = \mathcal{U}_u^{A\alpha}\mathcal{U}_v^{B\beta}\mathbb{C}_{\alpha\beta}\epsilon_{AB} \, , \tag{2.39}$$

where $\mathbb{C}_{\alpha\beta}$ and ϵ_{AB} are the antisymmetric symplectic and $SU(2)$ tensors.

The universal hypermultiplet

As an example of a quaternionic space, we discuss the so-called 'universal hypermultiplet'. In a Calabi-Yau compactification of type II string theory (which will be more discussed in section 2.5), one finds a number of hypermultiplets. One of these is always present, and is therefore called the universal hypermultiplet.

It is possible to construct a compactification such that it is also the only hypermultiplet. The metric is then the coset space $SU(2,1)/U(2)$, which can be written in terms of the real coordinates $\{r, \chi, \varphi, \sigma\}$ as

$$\mathrm{d}s^2 = \frac{1}{r^2}\Big(\mathrm{d}r^2 + r\,(\mathrm{d}\chi^2 + \mathrm{d}\varphi^2) + \big(\mathrm{d}\sigma + \chi \mathrm{d}\varphi\big)^2\Big) \, . \tag{2.40}$$

The field r is called the dilaton and the metric is restricted to the region where $r > 0$. Its expectation value $\langle r \rangle$ determines the string coupling constant g_s via $g_s = \mathcal{V}\langle r\rangle^{-1/2}$, where $\mathcal{V}$ is the volume of the Calabi-Yau space.

2.3.3 The $\mathcal{N} = 2$ supersymmetry transformations

The bosonic part of the complete ungauged $\mathcal{N} = 2$ Lagrangian is given by addition of (2.18) and (2.35). The fermionic part is lengthy [28], and we will not give it here. It contains the usual Dirac kinetic terms for the fermions, terms quartic in the fermions (as in (2.5)) and scalar-fermion-fermion couplings. It is invariant under the transformations

$$\begin{aligned} \delta_\varepsilon \psi_{A\mu} &= \nabla_\mu \varepsilon_A + T^-_{\mu\nu}\gamma^\nu \varepsilon_A \, , \\ \delta_\varepsilon \lambda^{iA} &= i\partial_\mu z^i \gamma^\mu \varepsilon^A + G^{i-}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{AB}\varepsilon_B \, , \\ \delta_\varepsilon \zeta_\alpha &= i\,\mathcal{U}_u^{B\beta}\partial_\mu q^u \gamma^\mu \epsilon_{AB}\mathbb{C}_{\alpha\beta}\varepsilon^A \, , \end{aligned} \tag{2.41}$$

up to terms higher in fermions. The covariant derivative reads

$$\nabla_\mu \varepsilon_A = \big(\partial_\mu - \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\big)\varepsilon_A + \frac{i}{2}A_\mu \varepsilon_A + \omega_{\mu A}{}^B \varepsilon_B \, . \tag{2.42}$$

[2]When $\lambda = 0$, we have a hyper-Kähler manifold, which features in the rigid $\mathcal{N} = 2$ hypermultiplet.

The connections A_μ and $\omega_{\mu A}{}^B$ are discussed in (2.23) and (2.37). We used a general rule to convert $SU(2)$ indices for the $SU(2)$ connection

$$\omega_{\mu A}{}^B \equiv \frac{i}{2}\omega_\mu^x \equiv \frac{i}{2}\partial_\mu q^u \omega_u^x \,. \tag{2.43}$$

2.4 Gauging isometries and superpotentials

The $\mathcal{N}=1$ and $\mathcal{N}=2$ supergravity theories are invariant under a set of global transformations. As we have seen, the chiral, the $\mathcal{N}=2$ vector and hypermultiplets contain a non-linear sigma model. If their target space metrics allow for isometries, then they are invariances of the theory. It is possible to make some of these symmetries local, while preserving supersymmetry, if we make some modifications to the Lagrangian and supersymmetry transformations. We start with the situation in $\mathcal{N}=2$ and finish with the $\mathcal{N}=1$ theory.

$\mathcal{N}=2$ vector multiplets

We first consider the $\mathcal{N}=2$ vector multiplets and assume the scalar sector to be invariant under the isometries

$$\delta_G z^i = -g k^i_\Lambda \alpha^\Lambda, \tag{2.44}$$

where α^Λ are the parameters of the transformations, and we have included a coupling constant g. This is the generalization of the first equation in (2.13). To preserve supersymmetry when we gauge these isometries, the Killing vector fields k^i_Λ must be holomorphic.

To close the gauge algebra on the scalars, the Killing vector fields must span a Lie-algebra with commutation relations

$$[k_\Lambda, k_\Sigma] = f_{\Lambda\Sigma}{}^\Gamma k_\Gamma \,, \tag{2.45}$$

and structure constants $f_{\Lambda\Sigma}{}^\Gamma$ of some Lie-group G that one wishes to gauge. Not all holomorphic isometries can be gauged within $\mathcal{N}=2$ supergravity. The induced change on the sections needs to be consistent with the symplectic structure of the theory, and this requires the holomorphic sections to transform as

$$\delta_G \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} = -g\alpha^\Sigma \left[T_\Sigma \cdot \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} + r_\Sigma(z) \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} \right] . \tag{2.46}$$

The first term on the right-hand-side of (2.46) contains a constant matrix T_Σ that acts on the sections as infinitesimal symplectic transformations. For electric gaugings, which we consider in this thesis, we mean, by definition, that the representation is of the form

$$T_\Lambda = \begin{pmatrix} -f_\Lambda & 0 \\ c_\Lambda & f_\Lambda^t \end{pmatrix} , \tag{2.47}$$

where f_Λ denotes the matrix $(f_\Lambda)_\Sigma{}^\Pi = f_{\Lambda\Sigma}{}^\Pi$ and f_Λ^t is the transposed. The tensor $c_{\Lambda,\Sigma\Pi} \equiv (c_\Lambda)_{\Sigma\Pi}$ is required to be symmetric for T_Λ to be a symplectic generator. Moreover, there are some additional constraints on the c_Λ in order for the T_Λ to be symplectically embedded within the same Lie-algebra as in (2.45). One can easily derive them, for explicit formulae see [38], or (2.64). The second term induces a Kähler transformation on the Kähler potential

$$\delta_G \mathcal{K}(z,\bar{z}) = g\alpha^\Lambda \big(r_\Lambda(z) + \bar{r}_\Lambda(\bar{z})\big) , \tag{2.48}$$

for some holomorphic functions $r_\Lambda(z)$. Finally, closure of the gauge transformations on the Kähler potential requires that

$$k_\Lambda^i \partial_i r_\Sigma - k_\Sigma^i \partial_i r_\Lambda = f_{\Lambda\Sigma}{}^\Gamma r_\Gamma . \tag{2.49}$$

We summarize some other identities on vector multiplet gauging in appendix C.

Magnetic gaugings allow also non-zero entries in the upper–right corner of T_Λ, but we will not consider them here. The gauged action, in particular the scalar potential, that we consider below is not invariant under magnetic gauge transformation. To restore this invariance, one needs to introduce massive tensor multiplets, but the most general Lagrangian with both electric and magnetic gauging is not fully understood yet (for some partial results see [39–43]).

Given a choice for the gauge group (2.47), one can reverse the order of logic and determine the form of the Killing vectors, and therefore the gauge transformations of the scalar fields z^i. This analysis was done in [27], and the result is written in the appendix, see (C.6).

In the Lagrangian, one replaces the partial derivatives with the covariant derivatives

$$\nabla_\mu z^i = \partial_\mu z^i + g k_\Lambda^i A_\mu^\Lambda , \tag{2.50}$$

where the gauge fields A_μ^Λ transform as $\delta_G A_\mu^\Lambda = \partial_\mu \alpha^\Lambda$. Furthermore, the Lagrangian contains the full non-Abelian field strengths

$$F_{\mu\nu}^\Lambda = \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \frac{1}{2} f_{\Sigma\Gamma}{}^\Lambda A_\mu^\Sigma A_\nu^\Gamma . \tag{2.51}$$

Finally, to preserve supersymmetry, we have to modify the supersymmetry transformations and add additional terms to the Lagrangian [28], such as mass terms for the fermions. Also, a scalar potential has to be added, which is given by

$$V = g^2 g_{i\bar{\jmath}} k_\Lambda^i k_\Sigma^{\bar{\jmath}} \bar{L}^\Lambda L^\Sigma . \tag{2.52}$$

Hypermultiplets

Similar to the vector multiplet scalars, the hypermultiplet Lagrangian (2.35) has its isometries as global symmetries, and we can gauge them. The transformation of the scalars is denoted as

$$\delta_G q^u = -g\tilde{k}^u_\Lambda \alpha^\Lambda \,, \tag{2.53}$$

and these Killing vectors $\tilde{k}^u_\Lambda$ form a representation of the same gauge algebra as in (2.45):

$$[\tilde{k}_\Lambda, \tilde{k}_\Sigma] = f_{\Lambda\Sigma}{}^\Gamma \tilde{k}_\Gamma \,. \tag{2.54}$$

These gaugings again introduce additional fermionic terms and a scalar potential, which will be given in (2.72).

Moment maps

The Killing vectors k^i_Λ are holomorphic. Using the Killing equation, one finds that they can be written as

$$k^i_\Lambda = -ig^{i\bar{\jmath}}\partial_{\bar{\jmath}} P_\Lambda \,, \tag{2.55}$$

where the real, scalar functions P_Λ are called moment maps. For special Kähler spaces, it is convenient to use instead a definition

$$P_\Lambda \equiv i(k^i_\Lambda \partial_i \mathcal{K} + r_\Lambda) \,, \tag{2.56}$$

where r_Λ was defined in equation (2.46). Since the Kähler potential satisfies (2.48), it is easy to show that P_Λ is real. From this definition, it is easy to verify (2.55). Hence the P_Λ can be called moment maps, but they are *not* subject to arbitrary additive constants. Using (2.49) and (2.56), it is now easy to prove the relation

$$k^i_\Lambda g_{i\bar{\jmath}} k^{\bar{\jmath}}_\Sigma - k^i_\Sigma g_{i\bar{\jmath}} k^{\bar{\jmath}}_\Lambda = if_{\Lambda\Sigma}{}^\Gamma P_\Gamma \,, \tag{2.57}$$

also called the equivariance condition. The $U(1)$ Kähler connection also gets additional terms due to the gauging and reads

$$A_\mu \equiv -\frac{i}{2}\Big(\partial_i \mathcal{K} \nabla_\mu z^i - \partial_{\bar{\imath}} \mathcal{K} \nabla_\mu \bar{z}^{\bar{\imath}}\Big) - \frac{i}{2} g A^\Lambda_\mu (r_\Lambda - \bar{r}_\Lambda) \,. \tag{2.58}$$

Although the quaternionic spaces are not complex, we can still define moment maps for the quaternionic Killing vectors. The moment maps P^x_Λ are defined by

$$K^x_{uv} k^v_\Lambda = D_u P^x_\Lambda \equiv \partial_u P^x_\Lambda - \epsilon^{xyz}\omega^y_u P^x_\Lambda \,. \tag{2.59}$$

Using these, we find the equivariance condition

$$K^x_{uv} k^u_\Lambda k^v_\Sigma + \frac{1}{2}\epsilon^{xyz} P^y_\Lambda P^z_\Sigma = \frac{1}{2} f_{\Lambda\Sigma}{}^\Gamma P^x_\Gamma \,. \tag{2.60}$$

The $SU(2)$ connection gets modified to

$$\omega_{\mu\, A}{}^B \equiv \partial_\mu q^u \omega_{u\, A}{}^B + g A^\Lambda_\mu P_{\Lambda\, A}{}^B \,, \tag{2.61}$$

where $P_{\Lambda\, A}{}^B = \frac{i}{2}\sigma^x{}_A{}^B P^x_\Lambda$. In absence of hypersmultiplets, $n_H = 0$, and for suitable structure constants $f_{\Lambda\Sigma}{}^\Gamma$, it is possible to keep P^x_Λ as non-zero constants. This is only possible if the gauge group contains $SU(2)$ or $U(1)$ factors. Such constants are called Fayet-Iliopoulos (FI) terms.

Gauge invariance

Under the gauge transformations (2.44), the period matrix $\mathcal{N}_{\Lambda\Sigma}$ transforms. From (2.25) one finds

$$\delta_G \mathcal{N}_{\Lambda\Sigma} = -g\alpha^\Pi \left(f_{\Pi\Lambda}{}^\Gamma \mathcal{N}_{\Gamma\Sigma} + f_{\Pi\Sigma}{}^\Gamma \mathcal{N}_{\Gamma\Lambda} + c_{\Pi,\Lambda\Sigma} \right) . \tag{2.62}$$

To compensate for this transformation, we need to add an additional term to the Lagrangian [38], which involves the c_Λ tensor, which will be given in (2.65). There are some additional constraints on this tensor. In the abelian case, the only constraint is that the totally symmetrized c-tensor vanishes, i.e.

$$c_{\Lambda,\Sigma\Pi} + c_{\Pi,\Lambda\Sigma} + c_{\Sigma,\Pi\Lambda} = 0 \,. \tag{2.63}$$

This implies that for a single vector field, the c_Λ tensor term vanishes. The additional constraints for nonabelian gaugings involve the structure constants [38]:

$$f_{\Lambda\Sigma}{}^\Gamma c_{\Gamma,\Pi\Omega} + f_{\Omega\Sigma}{}^\Gamma c_{\Lambda,\Gamma\Pi} + f_{\Pi\Sigma}{}^\Gamma c_{\Lambda,\Gamma\Omega} + f_{\Lambda\Omega}{}^\Gamma c_{\Sigma,\Gamma\Pi} + f_{\Lambda\Pi}{}^\Gamma c_{\Sigma,\Gamma\Omega} = 0 \,. \tag{2.64}$$

2.4.1 Gauged $\mathcal{N} = 2$ supergravity

The bosonic part of the gauged $\mathcal{N} = 2$ Lagrangian, which is the most general $\mathcal{N} = 2$ theory we consider in this thesis, is given by

$$\begin{aligned} \mathcal{L} =& \frac{1}{2}R(g) + g_{i\bar{\jmath}}\nabla^\mu z^i \nabla_\mu \bar{z}^{\bar{\jmath}} + h_{uv}\nabla^\mu q^u \nabla_\mu q^v + I_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\Sigma\,\mu\nu} + \frac{1}{2} R_{\Lambda\Sigma}\epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma} \\ & - \frac{1}{3} g\, c_{\Lambda,\Sigma\Pi}\, \epsilon^{\mu\nu\rho\sigma} A^\Lambda_\mu A^\Sigma_\nu \left(\partial_\rho A^\Pi_\sigma - \frac{3}{8} f_{\Omega\Gamma}{}^\Pi A^\Omega_\rho A^\Gamma_\sigma \right) - V(z, \bar{z}, q) \,. \end{aligned} \tag{2.65}$$

The first term on the second line is the additional term discussed below (2.62).

The supersymmetry transformations are (again, up to higher order in fermions)

$$\delta_\varepsilon \lambda^{iA} = i\nabla_\mu z^i \gamma^\mu \varepsilon^A + G^{-i}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{AB}\varepsilon_B + gW^{iAB}\varepsilon_B \,, \tag{2.66}$$

$$\delta_\varepsilon \zeta_\alpha = i\mathcal{U}^{B\beta}_u \nabla_\mu q^u \gamma^\mu \epsilon_{AB}\mathbb{C}_{\alpha\beta}\varepsilon^A + gN^A_\alpha \varepsilon_A \,, \tag{2.67}$$

$$\delta_\varepsilon \psi_{\mu A} = \nabla_\mu \varepsilon_A + T^-_{\mu\nu}\gamma^\nu \epsilon_{AB}\varepsilon^B + igS_{AB}\gamma_\mu \varepsilon^B \,. \tag{2.68}$$

The matrices W^{iAB}, N^A_α and S_{AB} are called the gaugino, hyperino and gravitino mass matrices respectively, and are given by

$$W^{iAB} = k^i_\Lambda \bar{L}^\Lambda \epsilon^{AB} + ig^{i\bar{\jmath}} f^\Lambda_{\bar{\jmath}} P^x_\Lambda \sigma^{AB}_x \,, \tag{2.69}$$

$$N^A_\alpha = 2\mathcal{U}^A_{\alpha u}\tilde{k}^u_\Lambda \bar{L}^\Lambda \,, \tag{2.70}$$

$$S_{AB} = \frac{i}{2} P^x_\Lambda L^\Lambda \sigma^x_{AB} \,. \tag{2.71}$$

Due to the gauging, a scalar potential has to be added to the Lagrangian, and reads

$$V = g^2 \Big[(g_{i\bar{\jmath}} k^i_\Lambda k^{\bar{\jmath}}_\Sigma + 4h_{uv} k^u_\Lambda k^v_\Sigma)\bar{L}^\Lambda L^\Sigma + (g^{i\bar{\jmath}} f^\Lambda_i \bar{f}^\Sigma_{\bar{\jmath}} - 3\bar{L}^\Lambda L^\Sigma) P^x_\Lambda P^x_\Sigma \Big] \,. \tag{2.72}$$

The first three terms are non-negative, whereas the last one is non-positive. The scalar potential can be written in terms of the mass matrices as

$$V = -6S^{AB}S_{AB} + \frac{1}{2} g_{i\bar{\jmath}} W^{iAB} W^{\bar{\jmath}}_{AB} + N^A_\alpha N^\alpha_A \,. \tag{2.73}$$

2.4.2 Gauged $\mathcal{N} = 1$ supergravity

Similar to the $\mathcal{N} = 2$ situation, the holomorphic isometries of the non-linear sigma model in (2.5) can be gauged. To do this, we first couple these chiral multiplets to gravity and add additional $\mathcal{N} = 1$ vector multiplets to the theory. The bosonic terms in the ungauged Lagrangian read [44]

$$\mathcal{L} = \frac{1}{2}R + g_{i\bar{\jmath}}\partial_\mu z^i \partial^\mu z^{\bar{\jmath}} + \mathrm{Re}\, f_{\Lambda\Sigma}(z) F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{2}\mathrm{Im}\, f_{\Lambda\Sigma}(z)\epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma} - V \,. \tag{2.74}$$

The gauge-coupling matrix $f_{\Lambda\Sigma}(z)$ is holomorphic in the z^i, and its real part should be invertible. The scalar potential V is given by

$$V = \mathrm{e}^{\mathcal{K}}\big(g^{i\bar{\jmath}} D_i W D_{\bar{\jmath}} \overline{W} - 3|W|^2\big) \,, \tag{2.75}$$

which is the supergravity version of (2.10). In supergravity, under the Kähler transformations (2.7), the superpotential W changes as $W \to \mathrm{e}^{-f} W$. The Kähler covariant derivative $D_i W$ is given by $D_i W \equiv (\partial_i + \partial_i \mathcal{K})W$, and the scalar potential (2.75) is invariant under Kähler transformations. Due to these Kähler transformations, it can be seen that the potential only depends on the invariant combination $G \equiv \mathrm{e}^{\mathcal{K}}|W|^2$, but we find it convenient to write it in terms of $\mathcal{K}$ and W. Comparison of this potential with the potential in rigid supersymmetry (2.10) also shows the addition of a negative term, which is a typical feature of supergravity.

Gauging the isometries of $g_{i\bar{\jmath}}$ now leads to the $\mathcal{N}=1$ Lagrangian [45–47]

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}R + g_{i\bar{\jmath}}\nabla_\mu z^i \nabla^\mu z^{\bar{\jmath}} + \mathrm{Re}\, f_{\Lambda\Sigma}(z) F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{2}\mathrm{Im}\, f_{\Lambda\Sigma}(z)\epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma} \\ &- \frac{1}{3} g\, c_{\Lambda,\Sigma\Pi}\, \epsilon^{\mu\nu\rho\sigma} A^\Lambda_\mu A^\Sigma_\nu \left(\partial_\rho A^\Pi_\sigma - \frac{3}{8} f_{\Omega\Gamma}{}^\Pi A^\Omega_\rho A^\Gamma_\sigma\right) - V\,. \end{aligned} \tag{2.76}$$

The term proportional to $c_{\Lambda,\Sigma\Pi}$ is again needed to restore gauge invariance when $f_{\Lambda\Sigma}(z)$ transforms non-trivially under gauge transformations. In $\mathcal{N}=1$, one does not need to impose condition (2.63) (although (2.64) is still required), as the completely symmetric part of $c_{\Lambda,\Sigma\Pi}$ can be canceled by an anomaly in the chiral fermion spectrum. For a modern account on these terms in $\mathcal{N}=1$ we refer to [48]. The $\mathcal{N}=1$ scalar potential is then given by

$$V = \mathrm{e}^{\mathcal{K}}\big(g^{i\bar{\jmath}} D_i W D_{\bar{\jmath}}\overline{W} - 3|W|^2\big) + \frac{1}{2}\mathrm{Re}\, f^{-1|\Lambda\Sigma} D_\Lambda D_\Sigma\,. \tag{2.77}$$

The D_Λ are again moment maps, and hence real solutions to the equation

$$k^i_\Lambda = -i g^{i\bar{\jmath}}\partial_{\bar{\jmath}} D_\Lambda\,. \tag{2.78}$$

The last term in (2.77) is called the D-term, and the first two are called the F-terms.

2.5 Compactifications

The four-dimensional supergravities can sometimes be related to higher-dimensional theories. If one studies the low energy dynamics of string theory, one finds a higher-dimensional supergravity theory. In this thesis, we use the type IIA and IIB superstring theories, which give a ten-dimensional supergravity theory. We also study eleven-dimensional supergravity theory.

There is a way in which these ten- and eleven-dimensional theories could describe our four-dimensional world. One assumes that six or seven dimensions are compact and very small, so they are not observed in our nature. The remaining four dimensions are non-compact, and describe our world. Using this assumption, one can form an effective theory of the four-dimensional dynamics. This procedure is called compactification.

As an example, consider a scalar field ϕ in D+1 dimensions, satisfying the Klein-Gordon equation

$$(\Box_{D+1} - m^2)\phi = 0\,. \tag{2.79}$$

Suppose now that one of these dimensions is a circle of radius R, with a coordinate z. We then expand the field ϕ in Fourier modes as

$$\phi(x^\mu, z) = \sum_n \phi_n(x^\mu) \mathrm{e}^{inz/R}\,, \tag{2.80}$$

where x^μ label the other D dimensions. Using this in the Klein-Gordon equation, we obtain

$$\sum_n \left(\Box_D \phi_n - \left(m^2 + \frac{n^2}{R^2}\right)\phi_n\right) \mathrm{e}^{inz/R} = 0\,. \tag{2.81}$$

The various modes $\phi_n(x^\mu)$ decouple in the equations of motion. The mass of each mode gets an addition term, increasing with $|n|$. When R is very small, the masses of all modes will be very large, except for the mode $\phi_0(x^\mu)$. To study the low-energy dynamics, one can integrate out the other modes, and keep only ϕ_0.

2.5.1 Kaluza-Klein theory

This simple example can be generalized to more complicated fields and more complicated space-times. One such example is Kaluza-Klein theory. Here one starts with pure gravity in five dimensions

$$S = \frac{1}{2}\int d^5x \sqrt{-\hat{g}}\hat{R}(\hat{g})\,, \tag{2.82}$$

where we use hats to indicate five-dimensional quantities. One now assumes that one of the spatial dimensions has the topology of a circle S^1 with radius R. We decompose the five-dimensional metric as

$$\hat{g} = \begin{pmatrix} k^{-1}g_{\mu\nu} + R^2 A_\mu A_\nu & -R^2 A_\mu \\ -R^2 A_\nu & R^2 \end{pmatrix}\,. \tag{2.83}$$

Here we have defined a four-dimensional metric g and a gauge field A_μ, and we assume that none of these fields depend on the circle coordinate. The symmetries of the five-dimensional metric precisely yield the correct symmetries in four-dimensions, such that A_μ has a gauge degree of freedom. Inserting this ansatz into (2.82) one computes

$$S = \frac{1}{2}\int d^4x \int_{S^1} \sqrt{-\hat{g}}\hat{R}(\hat{g}) = \frac{1}{2}\int d^4x\sqrt{-g}\left(R(g) - \frac{3}{2}k^{-2}\partial_\mu R \partial^\mu R - R^3 F_{\mu\nu}F^{\mu\nu}\right)\,, \tag{2.84}$$

where $F_{\mu\nu}$ is the gauge field associated with A_μ. We see that a theory of only gravity in five dimensions gives a theory with gravity, a gauge field and a scalar in four dimensions. This does not capture the full dynamics of the five-dimensional theory, as we have left out the massive modes in the ansatz (2.83).

As supergravity contains gravity, the compactifications of supergravity are generalizations of this Kaluza-Klein reduction. The other fields in the theory are then also expanded, using an ansatz as (2.83). If one reduces the eleven-dimensional supergravity on a small circle, one precisely finds the ten-dimensional IIA supergravity theory.

2.5.2 Scherk-Schwarz compactifications

A related method is given by the Scherk-Schwarz compactifications. Suppose we have a five-dimensional, complex free scalar field $\hat{\phi}$

$$\mathcal{L} = \partial_{\tilde{\mu}}\hat{\phi}\overline{\partial^{\tilde{\mu}}\hat{\phi}}\,. \tag{2.85}$$

We now assume that $\hat{\phi}$ has a non-trivial dependence on the circle coordinate z:

$$\partial_z\hat{\phi} = i\alpha\hat{\phi}\,, \tag{2.86}$$

where α is a constant. This dependence is chosen such that it corresponds to a symmetry of the scalar field. One can combine this ansatz with the complete tower of Kaluza-Klein states of (2.80); this will be discussed in the introduction of section 5. Although $\hat{\phi}$ depends on z, the five-dimensional Lagrangian (2.85) does not, so we can evaluate the action at some fixed $z = z_0$ and perform the integral over the circle. Using the same metric ansatz (2.83), one computes

$$\int d^4x \int_{S^1} dz \sqrt{-\tilde{g}}\partial_{\hat{\mu}}\hat{\phi}\,\overline{\partial^{\hat{\mu}}\hat{\phi}} = \int d^4x\,\sqrt{-g}\left(D_\mu\phi\overline{D^\mu\phi} + R^{-3}\alpha^2|\phi|^2\right)\,, \tag{2.87}$$

where $D_\mu\phi = \partial_\mu\phi + i\alpha A_\mu$. We see that the massless five-dimensional scalar gives a four-dimensional, gauged scalar and we have generated a scalar potential. This construction could therefore give masses to moduli fields and will be studied in section 5.

2.5.3 Calabi-Yau compactifications

A next step is to compactify the ten-dimensional action to four dimensions. One could compactify on six circles, but it is far more interesting to compactify on different spaces. A popular choice is to compactify on a compact Calabi-Yau (CY) three-fold. This is a six-dimensional space which admits a Ricci-flat Kähler metric. The Ricci-flatness implies that this is a solution to the ten-dimensional field equations. For constant gauge fields, with vanishing field strengths, these equations state the vanishing of the ten-dimensional Ricci tensor, $R_{MN} = 0$. The product space $R^{3,1} \times CY$ solves this condition, as both factors are Ricci-flat. Furthermore, they admit covariantly constant spinors, which ensures that they preserve some supersymmetry. Compactification of a maximally supersymmetric theory in ten dimensions, which has 32 supercharges, yields a four-dimensional theory with 8 supercharges, which is an $\mathcal{N} = 2$ theory.

We now turn to some geometrical concepts of Calabi-Yau manifolds; we refer to [49, 50] for more detail and background. As a complex space, a CY manifold has a holomorphic exterior derivative ∂ and an anti-holomorphic $\bar{\partial}$. Their (equal) cohomology groups are denoted by $H^{p,q}$, and their dimensions by the Hodge numbers $h^{p,q}$.

As the CY manifold is Kähler, the Laplacian Δ constructed out of ∂ equals the one constructed from $\bar{\partial}$ and (up to a factor 2) the one from $\mathrm{d} = \partial + \bar{\partial}$. An harmonic form ω satisfies $\Delta\omega = 0$. An important result is that the dimension of harmonic p, q-forms equals $h^{p,q}$; in particular it is finite. The numbers $h^{1,1}$ and $h^{2,1}$ completely specify the number of harmonic forms on a simply-connected CY three-fold: there always is only one zero-form, and the number of p-forms is equal to the number of $(6-p)$-forms. There are no one-forms as it is simply connected. The number of harmonic two-forms is then given by $h^{1,1}$ and the number of harmonic three-forms by $2h^{2,1} + 2$.

Similar to (2.80), we expand our fields as

$$\phi(x^\mu, y^i) = \sum_n \phi_n(x^\mu)\omega^n(y^i)\,, \tag{2.88}$$

where y^i are coordinates on the Calabi-Yau threefold, and ω^n is a basis of harmonic forms on this space. If the field $\phi(x^\mu, y^i)$ has additional indices (such as with a gauge field), these will also appear on ϕ_n and ω^n. As we only keep harmonic forms in this expansion, this will only produce the massless modes. This expansion is then inserted into the Lagrangian, and the integration over the coordinates y^i is performed. The resulting four-dimensional action then only contains fields that depend on x^μ. Although the geometry of a CY is complicated, we do know enough about the harmonic forms ω^n to do the expansions and integration. Such a compactification is done in detail in section 5.2.1.

We will only highlight some important points. Suppose we study the expansion of the ten-dimensional two-form $\hat{B}_2$. We expand this as

$$\hat{B}_2(x, y) = b_2(x) + b^i(x)\omega_i(y)\,, \tag{2.89}$$

where $b_2(x)$ is a two-form in four dimensions, b^i are scalars and ω_i are harmonic two-forms on the CY space. We now look at the field-strength $\hat{H}_3 \equiv \mathrm{d}\hat{B}_2$. This is given by

$$\hat{H}_3 = \mathrm{d}b_2 + \mathrm{d}b^i \wedge \omega_i\,. \tag{2.90}$$

Here we have used that ω_i is closed, $d\omega_i = 0$. As the ten-dimensional action is written in terms of $\hat{H}_3$, we can insert this expansion into the action. Ignoring the terms proportional to b_2 for clarity, we find

$$\int_{\mathbb{R}^{3,1}} \int_{CY} \hat{H}_3 \wedge \star_{10} \hat{H}_3 = \int_{\mathbb{R}^{3,1}} g_{ij} \, db^i \wedge \star_4 \, db^j \,, \tag{2.91}$$

where $g_{ij} \equiv \int_{CY} \omega_i \wedge \star_6 \omega_j$. As there are no mass-terms, the scalars b^i are massless scalars and are part of the moduli.

2.5.4 Flux compactifications

A possible construction to give masses to the moduli is via a flux compactification. We now give the ten-dimensional field strengths a non-trivial expectation value. This is very similar to the electric and magnetic fluxes of Maxwell theory (see e.g. section IID of [12]). The ten-dimensional field equations imply that the internal space cannot be a Calabi-Yau manifold anymore, due to the backreaction of the fluxes on the geometry. We therefore need different spaces to use in the compactification.

A popular choice are manifolds with $SU(3)$ or with $SU(3)\times SU(3)$ structure, as they have a globally defined spinor (for more details on this, see section 3.2 in [11]). A subclass of these models are the conformal Calabi-Yau manifolds. In these models [51], the Calabi-Yau metric is multiplied with an overall scalar function, called the warp factor. It is then shown that for large volume $\mathcal{V}$, the warp factor becomes constant. This construction led to the development of the 'large volume scenario' (LVS) [20], where the volume is stabilized at an exponentially large value. For more information on flux compactifications, we refer to the reviews [11–13].

Chapter 3

Fully supersymmetric vacua

3.1 Introduction

In this chapter, we consider four-dimensional $\mathcal{N} = 2$ gauged supergravities, and study the configurations that preserve maximal supersymmetry, i.e. eight supercharges. We only consider electric gaugings because magnetic gaugings require in addition massive tensor multiplets which have not been fully constructed yet. In the ungauged case, $\mathcal{N} = 2$ models arise e.g. from Calabi-Yau compactifications of type II string theories, or $K3 \times T^2$ compactifications of the heterotic string. Both models are known to have a rich dynamical structure with controllable quantum effects in both vector- and hypermultiplet sectors that are relatively well understood. Gaugings in $\mathcal{N} = 2$ supergravity are well studied and have a long history [28, 35, 38, 52–56]. Their analysis in terms of string compactifications with fluxes started in [57–59], and is an ongoing research topic. For a (partial) list of references, see [60–67].

In the ungauged case, a complete classification of all the supersymmetric solutions already exists [68–71], while there also are solutions in the gauged case for (abelian) vector multiplets [72–74]. We extend this by taking completely general vector- and hypermultiplet sectors. Since we concentrate only on the maximally supersymmetric solutions, we use different methods than the ones in the above references. In fact the space-time conditions we obtain for our solutions closely resemble other maximally supersymmetric solutions in different theories such as [75].

Plan of this chapter

The plan of this chapter is as follows. In section 2, we analyze the supersymmetry rules and derive the conditions for maximally supersymmetric vacua. The possible solutions divide in two classes of space-times, with zero scalar curvature and with negative scalar curvature, and we explicitly list all the possible outcomes. We give the Lagrangian and the scalar potential for the obtained vacua in section 3, paying special attention to the Chern-Simons-like term determined by the c-tensor of the electric gauging. This term generically exists in $\mathcal{N} = 2$ supergravity and string theory compactifications and we show how it influences the maximally supersymmetric vacua. In section 4, we discuss explicit cases from string theory compactifications and general supergravity considerations that exemplify the use of our maximal supersymmetry conditions. Some definitions of our conventions and notations are given in the appendices, where we also present some intermediate and final formulae that are important for our results.

3.2 Supersymmetry transformations

It can be seen by inspection that the maximally supersymmetric configurations[3] are purely bosonic, and the fermions need to be zero. This follows from the supersymmetry variations of the bosonic fields, which can be read off from [28]. Therefore, we can restrict ourselves to the supersymmetry variations of the fermions only.

3.2.1 Gauginos

As seen in section 2.4.1, the transformation of the gauginos is given by

$$\delta_\varepsilon \lambda^{iA} = i\nabla_\mu z^i \gamma^\mu \varepsilon^A + G^{i-}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{AB}\varepsilon_B + gW^{iAB}\varepsilon_B \,, \tag{3.1}$$

up to terms that are higher order in the fermions and which vanish for purely bosonic configurations.

A maximally supersymmetric configuration preserves the full eight supercharges, hence the variation of the fermions should vanish for all choices of the supersymmetry parameters. Since at each point in spacetime they are linearly independent, the first term on the right hand side of (3.1) must vanish separately from the others,

$$\nabla_\mu z^i \equiv \partial_\mu z^i + gA^\Lambda_\mu k^i_\Lambda = 0 \,. \tag{3.2}$$

It implies the integrability condition[4]

$$F^\Lambda_{\mu\nu} k^i_\Lambda = 0 \,, \tag{3.3}$$

and complex conjugate. Here, $F^\Lambda_{\mu\nu}$ is the full non-abelian field strength, given by

$$F^\Lambda_{\mu\nu} = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2} f_{\Sigma\Gamma}{}^\Lambda A^\Sigma_\mu A^\Gamma_\nu \,. \tag{3.4}$$

[3] In this paper we use interchangeably the terms maximally supersymmetric configurations and BPS configurations, meaning the field values that are invariant under all eight supercharges in the theory.

[4] We will assume in the remainder of this chapter that the gauge coupling constant $g \neq 0$. The case of $g = 0$ is treated in the literature in e.g. [70, 71].

The second and third term in the supersymmetry variation of the gauginos, equation (3.1), need also to vanish separately, since they multiply independent spinors of the same chirality. For the second term, this leads to

$$G^{i\,-}_{\mu\nu} = 0\,, \tag{3.5}$$

where $g^{i\bar{\jmath}}$ is the inverse Kähler metric, with Kähler potential $\mathcal{K}$ from (2.19).

Finally, setting the third term in the supersymmetry variation to zero leads to

$$W^{iAB} \equiv k^i_\Lambda \bar{L}^\Lambda \epsilon^{AB} + ig^{i\bar{\jmath}} f^\Lambda_{\bar{\jmath}} P^x_\Lambda \sigma_x^{AB} = 0\,. \tag{3.6}$$

Close inspection of (3.6) shows that both terms are linearly independent in $SU(2)_R$ space, hence they must vanish separately,

$$k^i_\Lambda \bar{L}^\Lambda = 0\,, \qquad P^x_\Lambda f^\Lambda_i = 0\,, \tag{3.7}$$

and their complex conjugates.

3.2.2 Hyperinos

The hyperinos transform as

$$\delta_\varepsilon \zeta_\alpha = i\mathcal{U}^{B\beta}_u \nabla_\mu q^u \gamma^\mu \varepsilon^A \epsilon_{AB} \mathbb{C}_{\alpha\beta} + gN^A_\alpha \varepsilon_A\,, \tag{3.8}$$

again, up to terms that are of higher order in the fermions. The hyperino mass matrix N^A_α is defined by

$$N^A_\alpha \equiv 2\mathcal{U}^A_{\alpha\,u} \tilde{k}^u_\Lambda \bar{L}^\Lambda\,. \tag{3.9}$$

Similarly as for the gauginos, $\mathcal{N}=2$ supersymmetric configurations require the two terms in (3.8) to vanish separately. Since the quaternionic vielbeine are invertible and nowhere vanishing, the scalars need to be covariantly constant,

$$\nabla_\mu q^u \equiv \partial_\mu q^u + gA^\Lambda_\mu \tilde{k}^u_\Lambda = 0\,, \tag{3.10}$$

implying the integrability conditions

$$F^\Lambda_{\mu\nu}\, \tilde{k}^u_\Lambda = 0\,. \tag{3.11}$$

Furthermore, there is a second condition from (3.8) coming from the vanishing of the hyperino mass matrix N^A_α. This leads to

$$\tilde{k}^u_\Lambda L^\Lambda = 0\,, \tag{3.12}$$

and complex conjugate.

In the absence of hypermultiplets, i.e. when $n_H = 0$, the $\mathcal{N}=2$ conditions from the variations of the hyperinos disappear. However, the second condition in (3.7) remains, with the moment maps replaced by FI parameters. Our formalism therefore automatically includes the case $n_H = 0$.

3.2.3 Gravitinos

The supersymmetry transformations of the gravitinos are (up to irrelevant higher order terms in the fermions)

$$\delta_\varepsilon \psi_{\mu A} = \nabla_\mu \varepsilon_A + T^-_{\mu\nu}\gamma^\nu \epsilon_{AB}\varepsilon^B + igS_{AB}\gamma_\mu \varepsilon^B \,. \tag{3.13}$$

Here, $\nabla_\mu \varepsilon_A$ is the gauged supercovariant derivative (specified in equation (2.4.1)).

Notice again that for $n_H = 0$, in fact even also in the absence of vector multiplets when $n_V = 0$, the gravitino mass-matrix S_{AB} can be non-vanishing and constant. In the Lagrangian, which we discuss in the next section, this leads to a (negative) cosmological constant term. The anti-selfdual part of the graviphoton field strength $T_{\mu\nu}$ satisfies the identity (2.31)

$$F^{\Lambda\,-}_{\mu\nu} = i\bar{L}^\Lambda T^-_{\mu\nu} + 2f^\Lambda_i G^{i\,-}_{\mu\nu} \,, \tag{3.14}$$

with $G^{i\,-}_{\mu\nu}$ defined in (2.30). From the vanishing of the gaugino variation, we have that $G^{i\,-}_{\mu\nu} = 0$, so a maximally supersymmetric configuration must satisfy $F^{\Lambda\,-}_{\mu\nu} = i\bar{L}^\Lambda T^-_{\mu\nu}$, or

$$F^\Lambda_{\mu\nu} = i\overline{L}^\Lambda T^-_{\mu\nu} - iL^\Lambda T^+_{\mu\nu} \,. \tag{3.15}$$

Using this, we then see that equation (3.12) implies the integrability conditions (3.11) in the hypermultiplet sector. For the integrability equations in the vector multiplet sector, the situation is more subtle, as the Killing vectors are complex and holomorphic. Now, the BPS condition (3.7) only implies that

$$k^i_\Lambda F^\Lambda_{\mu\nu} = -ik^i_\Lambda L^\Lambda T^+_{\mu\nu} \,. \tag{3.16}$$

In appendix C we show that $k^i_\Lambda L^\Lambda = 0$ is an identity of the theory, and hence the integrability condition is always satisfied. The integrability condition might only locally be sufficient, but this is fine for our purposes. One might however check in addition whether the covariant constancy of the vector multiplet scalars imposes further (global) restrictions.

To solve the constraints from the gravitino variation, we must first look at the gauged supercovariant derivative on the supersymmetry parameter, given by (2.42)

$$\nabla_\mu \varepsilon_A = (\partial_\mu - \frac{1}{4}\omega^{ab}_\mu \gamma_{ab})\varepsilon_A + \frac{i}{2}A_\mu \varepsilon_A + \omega_{\mu\,A}{}^B \varepsilon_B \,. \tag{3.17}$$

Besides the spin connection ω^{ab}_μ, there appear two other connections associated to the special Kähler and quaternion-Kähler manifolds. We need to compute their curvatures since they enter the integrability conditions that follow from the Killing spinor equations. The first one is called the gauged $U(1)$ Kähler-connection, defined by [27, 28]

$$A_\mu \equiv -\frac{i}{2}\Big(\partial_i \mathcal{K}\nabla_\mu z^i - \partial_{\bar\imath}\mathcal{K}\nabla_\mu \bar{z}^{\bar\imath}\Big) - \frac{i}{2}gA^\Lambda_\mu(r_\Lambda - \bar{r}_\Lambda) \,. \tag{3.18}$$

Under a gauge transformation, one finds that

$$\delta_G A_\mu = \frac{i}{2} g\, \partial_\mu \Big[\alpha^\Lambda (r_\Lambda - \bar{r}_\Lambda) \Big] . \tag{3.19}$$

The curvature of this connection can be computed to be

$$F_{\mu\nu} = i g_{i\bar{\jmath}} \nabla_{[\mu} z^i \nabla_{\nu]} \bar{z}^{\bar{\jmath}} - g F^\Lambda_{\mu\nu} P_\Lambda \,, \tag{3.20}$$

where P_Λ is the moment map, defined in (2.56), and we have used the equivariance condition (2.57). For maximally supersymmetric configurations, the scalars are covariantly constant and hence the curvature of the Kähler connections satisfies $F_{\mu\nu} = -g F^\Lambda_{\mu\nu} P_\Lambda$.

The second connection appearing in the gravitino supersymmetry variation is the gauged $Sp(1)$ connection of the quaternion-Kähler manifold (2.61). It reads

$$\omega_{\mu\, A}{}^B \equiv \partial_\mu q^u \omega_{u\, A}{}^B + g A^\Lambda_\mu P_{\Lambda\, A}{}^B \,, \tag{3.21}$$

where $\omega_{u\, A}{}^B$ is the (ungauged) $Sp(1)$ connection of the quaternion-Kähler manifold, whose curvatures are related to the three quaternionic two-forms. The effect of the gauging is to add the second term on the right hand side of (3.21), proportional to the triplet of moment maps of the quaternionic isometries, with $P_{\Lambda\, A}{}^B = \frac{i}{2} P^x_\Lambda (\sigma^x)_A{}^B$. The curvature of (3.21) can then be computed to be

$$\Omega_{\mu\nu\, A}{}^B = 2\Omega_{uv\, A}{}^B \nabla_{[\mu} q^u \nabla_{\nu]} q^v + g F^\Lambda_{\mu\nu} P_{\Lambda\, A}{}^B \,, \tag{3.22}$$

where $\Omega_{uv\, A}{}^B$ is the quaternionic curvature. For fully BPS solutions, we therefore have $\Omega_{\mu\nu\, A}{}^B = g F^\Lambda_{\mu\nu} P_{\Lambda\, A}{}^B$.

We can now investigate the integrability conditions that follow from the vanishing of the gravitino transformation rules (3.13). From the definition of the supercovariant derivative (3.17), we find[5]

$$[\nabla_\mu, \nabla_\nu] \varepsilon_A = -\frac{1}{4} R_{\mu\nu}{}^{ab} \gamma_{ab}\, \varepsilon_A - i g F^\Lambda_{\mu\nu} P_\Lambda \varepsilon_A + 2 g F^\Lambda_{\mu\nu} P_{\Lambda\, A}{}^B \varepsilon_B \,, \tag{3.23}$$

where we have used the covariant constancy of the scalars. We recall that P_Λ are the moment maps on the special Kähler geometry, whereas $P_{\Lambda\, A}{}^B$ are the quaternion-Kähler moment maps. Alternatively, we can compute the commutator from the vanishing of the gravitino variations spelled out in (2.68). By equating this to the result of (3.23), we get a set of constraints. Details of the calculation are given in appendix C, and the

[5] Strictly speaking, we get the supercovariant curvatures appearing in (3.23), which also contain fermion bilinears. Since the fermions are zero on maximally supersymmetric configurations, only the bosonic part of the curvatures remains.

results can be summarized as follows. First of all, we find the covariant constancy of the graviphoton field strength[6]

$$D_\rho T^+_{\mu\nu} = 0\,. \tag{3.24}$$

Secondly, we get that the quaternionic moment maps must satisfy

$$\epsilon^{xyz} P^y \overline{P^z} = 0\,, \qquad P^x \equiv L^\Lambda P^x_\Lambda\,. \tag{3.25}$$

Moreover, there are cross terms between the graviphoton and the moment maps, which enforce the conditions

$$T^+_{\mu\nu}\, P^x = 0\,. \tag{3.26}$$

This equation separates the classification of BPS configurations in two sectors, those with a solution of $P^x = 0$ at a particular point (or locus) in field space, and those with non-vanishing P^x (for at least one index x) but $T_{\mu\nu} = 0$. We will see later on that this distinction corresponds to zero or non-zero (and negative) cosmological constant in the spacetime.

Another requirement that follows from the gravitino integrability conditions is

$$F^\Lambda_{\mu\nu} P_\Lambda = 0\,, \tag{3.27}$$

where P_Λ is defined in (2.56), and is real. Using (3.15), this is equivalent to the condition

$$\bar{L}^\Lambda P_\Lambda T^-_{\mu\nu} = L^\Lambda P_\Lambda T^+_{\mu\nu}\,, \tag{3.28}$$

which is satisfied as $P_\Lambda L^\Lambda = 0$, so (3.27) does not lead to any new constraint.

Finally, there is the condition on the spacetime Riemann curvature. It reads

$$R_{\mu\nu\rho\sigma} = 4T^+_{\mu[\sigma}T^-_{\rho]\nu} + g^2 P^x \overline{P^x} g_{\mu\sigma} g_{\nu\rho} - (\mu \leftrightarrow \nu)\,. \tag{3.29}$$

It can be checked that this leads to a vanishing Weyl tensor, implying conformal flatness. From the curvature, we can compute the value of the Ricci-scalar to be

$$R = -12 g^2 P^x \overline{P^x}\,. \tag{3.30}$$

Hence, the classification of fully supersymmetric configurations separates into negative scalar curvature with $P^x \overline{P^x} \neq 0$, and zero curvature with $P^x = 0$ at the supersymmetric point. In both of these cases there are important simplifications.

[6]Recall that T^+ and T^- are related by complex conjugation, and hence the vanishing of DT^+ implies $DT = 0$.

Negative scalar curvature

The case of negative scalar curvature is characterized by $T_{\mu\nu} = 0$ and $P^x\overline{P^x} \neq 0$ at the supersymmetric point. Since the BPS conditions imply that then both $T_{\mu\nu}$ and $G^{i-}_{\mu\nu} = 0$ (see equation (3.5)), we find that all field strengths should be zero: $F^{\Lambda}_{\mu\nu} = 0$. The gauge fields then are required to be pure gauge, but can still be topologically non-trivial. Furthermore, because of the vanishing field strengths, the integrability conditions on the scalar fields are satisfied, and a solution for the sections $X^{\Lambda}(z)$ is obtained by a gauge transformation on the constant (in spacetime) sections. Finally, the Riemann tensor is given by

$$R_{\mu\nu\rho\sigma} = g^2 P^x\overline{P^x}\left(g_{\mu\sigma}g_{\nu\rho} - g_{\nu\sigma}g_{\mu\rho}\right) \ , \tag{3.31}$$

which shows that the space is maximally symmetric, and therefore locally AdS_4. The scalar curvature is $R = -12g^2 P^x\overline{P^x}$.

Zero scalar curvature

The class of zero curvature is characterized by configurations for which $P^x = 0$ at the supersymmetric point. In this case, we can combine the conditions $P^x_{\Lambda} f^{\Lambda}_i = 0$ and $P^x \equiv P^x_{\Lambda} L^{\Lambda} = 0$ into

$$P^x_{\Lambda}\begin{pmatrix} \bar{L}^{\Lambda} \\ f^{\Lambda}_i \end{pmatrix} = 0\,. \tag{3.32}$$

The matrix appearing here is the invertible matrix of special geometry (as used in (2.25)), hence we conclude that $P^x_{\Lambda} = 0$. The Riemann tensor is then

$$R_{\mu\nu\rho\sigma} = 4T^{+}_{\mu[\sigma}T^{-}_{\rho]\nu} - (\mu \leftrightarrow \nu)\,. \tag{3.33}$$

From the covariant constancy of the graviphoton, condition (3.24), we find $D_{\rho}R_{\mu\nu\sigma\tau} = 0$. Spaces with covariantly constant Riemann tensor are called locally symmetric, and they are classified, see e.g. [75–77]. In our case we also have zero scalar curvature, and then only three spaces are possible:

1. Minkowski space M_4 ($T_{\mu\nu} = 0$)
2. $AdS_2 \times S^2$
3. The pp-wave solution

The explicit metrics and field strengths for the latter two cases (M_4 and AdS_4 are well-known and have vanishing field strengths) are listed in appendix A.2.

3.2.4 Summary

Let us now summarize the results. There are two different classes: negative scalar curvature (leading to AdS_4) and zero scalar curvature solutions (leading to M_4, $AdS_2 \times S^2$ or the pp–wave).

The result of our analysis is that all the conditions on the spacetime dependent part are explicitly solved[7], and the remaining conditions are purely algebraic, and depend only on the geometry of the special Kähler and quaternionic manifolds. The solutions to these algebraic equations define the configuration space of maximally supersymmetric configurations. There are two separate cases:

Negative scalar curvature (AdS_4)

This case is characterized by configurations for which $P^x \overline{P^x} \neq 0$ at the supersymmetric point. The BPS conditions are

$$k^i_\Lambda \overline{L}^\Lambda = 0\,, \qquad \tilde{k}^u_\Lambda L^\Lambda = 0\,,$$
$$P^x_\Lambda f^\Lambda_i = 0\,, \qquad \epsilon^{xyz} P^y \overline{P^z} = 0\,,$$

which should be satisfied at a point (or a locus) in field space. The field strengths are zero, $F^\Lambda_{\mu\nu} = 0$, and the space–time is AdS_4 with scalar curvature $R = -12g^2 P^x \overline{P^x}$.

Zero scalar curvature (M_4, $AdS_2 \times S^2$ or pp–wave)

In this case, the BPS conditions are

$$k^i_\Lambda \overline{L}^\Lambda = 0\,, \qquad \tilde{k}^u_\Lambda L^\Lambda = 0\,,$$
$$P^x_\Lambda = 0\,.$$

We remind that, when $T_{\mu\nu} = 0$ (Minkowski space), all field strengths are vanishing ($F^\Lambda_{\mu\nu} = 0$).

[7]This is apart from the scalar fields and Killing spinors, which are spacetime dependent. The integrability conditions that we have imposed guarantee locally the existence of a solution, although we did not explicitly construct it. Its construction cannot be done in closed form in full generality, but can be worked out in any given example [77].

3.3 Examples

In this section we list some (string theory motivated) examples of $\mathcal{N} = 2, D = 4$ theories, leading to $\mathcal{N} = 2$ supersymmetric configurations. We will first mention briefly some already known and relatively well-understood $\mathcal{N} = 2$ vacua from string theory and then concentrate on our two main examples in subsections 3.3.1 and 3.3.2 that exhibit best the different features discussed above. In the last subsection we include some supergravity models, not necessarily obtained from string compactifications, leading to AdS_4 vacua that can be of interest.

Obtaining gauged $\mathcal{N} = 2, D = 4$ supergravity seems to be important for string theory compactifications since it is an intermediate step between the more realistic $\mathcal{N} = 1$ models and the mathematically controllable theories. Thus in the last decade there has been much literature on the subject. An incomplete list of examples consists of [60, 64–67] and it is straightforward to impose and solve the maximal supersymmetry constraints in each case. In some cases the vacua have been already discussed or must exist from general string theory/M-theory considerations.

For example, it was found that the coset compactifications studied in [66] do not lead to $\mathcal{N} = 2$ supersymmetric configurations. This can also be seen from imposing the constraints in section 3.2.4. In contrast, the compactification on $K3 \times T^2/\mathbb{Z}_2$ presented in [60] does exhibit $\mathcal{N} = 2$ solutions with non-trivial hypermultiplet gaugings. The authors of [60] explicitly found $\mathcal{N} = 2$ Minkowski vacua by satisfying the same susy conditions as in section 3.2.4. From our analysis, it trivially follows that also the pp-wave and the $AdS_2 \times S^2$ backgrounds are maximally supersymmetric.

A similar example is provided by the (twisted) $K3 \times T^2$ compactification of the heterotic string, recently analyzed in [67]. For abelian gaugings, one can verify that the three zero scalar curvature vacua are present in these models.

We now turn to discuss the remaining models in more detail.

3.3.1 M-theory compactification on SU(3) structure manifolds

There is a very interesting model for $\mathcal{N} = 2, D = 4$ supergravity with non-abelian gauging of the vector multiplet sector, arising from compactifications of M-theory on seven-manifolds with $SU(3)$ structure [64]. More precisely, they consider Calabi-Yau (CY) threefolds fibered over a circle. The c-tensor, introduced in (2.62)-(2.63), is non-trivial in these models. These models will be extended in chapter 5. For the precise M-theory set-up, we refer the reader to section 5.4 or the original paper [64]; here we only discuss the relevant data for analyzing the maximal supersymmetry conditions:

- The vector multiplet space can be parametrized by special coordinates, $X^\Lambda = (1, t^i)$, $t^i = b^i + iv^i$, and prepotential

$$F(X) = -\frac{1}{6}\mathcal{K}_{ijk}\frac{X^i X^j X^k}{X^0} \,, \tag{3.34}$$

  with the triple intersection numbers $\mathcal{K}_{ijk}$ that depend on the particular choice of the CY-manifold. This gives the Kähler potential

$$\mathcal{K} = -\log\left[\frac{i}{6}\kappa_{ijk}(t^i - \bar{t}^i)(t^j - \bar{t}^j)(t^k - \bar{t}^k)\right] \equiv -\log\mathcal{V} \,, \tag{3.35}$$

  where $\mathcal{V}$ denotes the volume of the compact manifold. The gauge group is non-abelian with structure constants

$$f_{\Lambda\Sigma}{}^0 = 0 = f_{ij}{}^k, \qquad f_{i0}{}^j = -M_i^j \,, \tag{3.36}$$

  and a c-tensor whose only non-vanishing components are

$$c_{i,jk} = \frac{1}{2}M_i^l\mathcal{K}_{ljk} \,. \tag{3.37}$$

  The constant matrix M_i^j specifies the Killing vectors and moment-maps of the special Kähler manifold:

$$k_0^j = -M_k^j t^k \,, \qquad\qquad k_i^j = M_i^j \,, \tag{3.38}$$

  and

$$P_0 = -M_i^j t^i \partial_j\mathcal{K} \,, \qquad\qquad P_i = M_i^j\partial_j\mathcal{K} \,. \tag{3.39}$$

  Not for any choice of M_i^j is the Killing equation satisfied. As explained in [64], this is only the case when the relation (2.63) holds. This also ensures that (2.64) is satisfied, as one can easily check.

- Generally in this class of compactifications there always appear hypermultiplet scalars, but there is no gauging of this sector, so the Killing vectors and the moment maps P_Λ^x are vanishing.

The scalar potential in this case reduces to the simple formula

$$V = -\frac{8}{\mathcal{V}^2}M_i^k M_j^l \mathcal{K}_{klm} v^i v^j v^m \,, \tag{3.40}$$

which is positive semi-definite.

Analyzing the susy conditions is rather straightforward. Since $P^x = 0$, the only allowed $\mathcal{N} = 2$ vacua are the ones with zero-scalar curvature. What is left for us to check are the conditions $k^i_\Lambda \bar{L}^\Lambda = 0$ and $P_\Lambda L^\Lambda = 0$. The latter is very easy to check and holds as an identity at every point in the special Kähler manifold. Also, it is equivalent to the relation $k^i_\Lambda L^\Lambda = 0$ which is satisfied whenever there exists a prepotential [54]. The condition $k^i_\Lambda \bar{L}^\Lambda = 0$ eventually leads to

$$\frac{M^i_j(t^j - \bar{t}^j)}{\mathcal{V}} = 2i\frac{M^i_j v^j}{\mathcal{V}} = 0 \ , \qquad \forall i \ . \tag{3.41}$$

The solution to the above equation that always exists is the decompactification limit when $\mathcal{V} \to \infty$. The other more interesting solutions depend on the explicit form of the matrix M. In case M_i^j is invertible there are no further solutions to (3.41). On the other hand, when M has zero eigenvalues we can have $\mathcal{N} = 2$ M-theory vacua, given by (a linear combination of) the corresponding zero eigenvectors of M. For the supergravity approximation to hold, one might require that this solution leads to a non-vanishing (and large) volume of the CY three-fold. Each eigenvector will correspond to a flat direction of the scalar potential, and with $V = 0$ along these directions. The case where the full matrix M is zero corresponds to a completely flat potential, the one of a standard M-theory compactification on $CY \times S^1$ without gauging.

Thus it is clear that M_i^j is an important object for this type of M-theory compactifications and we now give a few more details on its geometrical meaning [64]. In the above class of M-theory compactifications we have a very specific fibration of the Calabi-Yau manifold over the circle. It is chosen such that only the second cohomology $H^{(1,1)}(CY)$ is twisted with respect to the circle, while the third cohomology $H^3(CY)$ is unaffected. Thus the hypermultiplet sector remains ungauged as in regular $CY \times S^1$ compactification, while the vector multiplets feel the twisting and are gauged. This twisting is parametrized exactly by the matrix M, as it determines the differential relations of the harmonic (on the CY manifold) two-forms:

$$\mathrm{d}\omega_i = M_i^j \omega_j \wedge \mathrm{d}z \ , \tag{3.42}$$

where z is the circle coordinate.

Let us now zoom in on the interesting case when we have nontrivial zero eigenvectors of M, corresponding to non-vanishing volume of the CY manifold. For a vanishing volume, or a vanishing two-cycle, the effective supergravity description might break down due to additional massless modes appearing in string theory[8]. Therefore the really consistent and relevant examples for $\mathcal{N} = 2$ vacua are only those when the matrix M is non-invertible with corresponding zero eigenvectors that give nonzero value for every v^i.

To illustrate this better, we consider a particular example, given in section 2.5 of [64], of a compactification where the CY three-fold is a $K3$-fibration. In this setting one can

[8]For a detailed analysis of the possibilities in a completely analogous case in five dimensions see [78].

explicitly construct an M-matrix, compatible with the intersection numbers $\mathcal{K}_{ijk}$. Here one can find many explicit cases where all of the above described scenarios happen. As a very simple and suggestive example we consider the 5-scalar case with $\mathcal{K}_{144} = \mathcal{K}_{155} = 2$, $\mathcal{K}_{123} = -1$, and twist-matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & -2 & -2 \\ 0 & 0 & -4 & 2 & 2 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix} . \tag{3.43}$$

The general solution of $M \cdot \vec{v} = 0$ is

$$\vec{v} = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} , \tag{3.44}$$

and the resulting volume is

$$\mathcal{V} = 8\lambda \left(2\mu^2 + 2\nu^2 + (\mu - \nu)^2\right) , \tag{3.45}$$

which is clearly positive semi-definite. In the case when either μ or ν vanishes we have a singular manifold that is still a solution to the maximal supersymmetry conditions. When all three coefficients (that are essentially the remaining unstabilized moduli fields) are non-zero, we have a completely proper solution both from supergravity and string theory point of view, thus providing an example of $SU(3)$ structure compactifications with zero-curvature $\mathcal{N} = 2$ vacua. This example can be straightforwardly generalized to a higher number of vector multiplets, as well as to the lower number of 4 scalars (there cannot be less than 4 vector multiplets in this particular case).

Finally we note that a special case of the general setup described above was already known for more than twenty years in [38] (3.21), where $M_1^1 = -2$, $M_2^2 = 1$, and $\mathcal{K}_{122} = 2$. It was derived purely from 4d supergravity considerations, but it now seems that one can embed it in string theory.

3.3.2 Reduction of M-theory on Sasaki-Einstein$_7$

There has been much advance in the last years in understanding Sasaki-Einstein manifolds and their relevance for M-theory compactifications, both from mathematical and physical perspective. A metric $\mathrm{d}s^2$ is Sasaki-Einstein iff the cone metric, defined as

$\mathrm{ds}^2_{\mathrm{cone}} \equiv \mathrm{d}r^2 + r^2\mathrm{ds}^2$, is Kähler and Ricci-flat. These spaces are good candidates for examples of the AdS_4/CFT_3 correspondence and an explicit reduction to $D = 4$ has been recently obtained in [65]. Originally the effective Lagrangian includes magnetic gauging and a scalar-tensor multiplet, but after a symplectic rotation it can be formulated in the standard $\mathcal{N} = 2$ formalism discussed here. After the dualization of the original tensor to a scalar we have the following data for the multiplets, needed for finding maximally supersymmetric vacua:

- There is one vector multiplet, given by $X^\Lambda = (1, \tau^2)$ and $F(X) = \sqrt{X^0(X^1)^3}$, leading to $F_\Lambda = (\frac{1}{2}\tau^3, \frac{3}{2}\tau^2)$ and Kähler potential

$$\mathcal{K} = -\log \frac{i}{2}(\tau - \bar{\tau})^3 \ . \tag{3.46}$$

  There is no gauging in this sector, i.e. $k^i_\Lambda = 0$ and $P_\Lambda = 0$ for all i, Λ. This also means that both $f_{\Lambda\Sigma}{}^\Pi$ and $c_{\Lambda,\Sigma\Pi}$ vanish.

- The hypermultiplet scalars are $\{r, \chi, \varphi, \sigma\}$ with the universal hypermultiplet metric[9], introduced in (2.40):

$$\mathrm{d}s^2 = \frac{1}{r^2}\left(\mathrm{d}r^2 + r(\mathrm{d}\chi^2 + \mathrm{d}\varphi^2) + (\mathrm{d}\sigma + \chi\mathrm{d}\varphi)^2\right) \ . \tag{3.47}$$

  We have an abelian gauging, given by:

$$\begin{aligned} \tilde{k}_0 &= 24\partial_\sigma + 4\big(\chi\partial_\varphi - \varphi\partial_\chi + \frac{1}{2}(\phi^2 - \chi^2)\partial_\sigma\big) \ , \\ \tilde{k}_1 &= 24\partial_\sigma \ , \end{aligned} \tag{3.48}$$

  and the moment maps, calculated in the appendix (D), are

$$\begin{aligned} P^1_0 &= \frac{4\chi}{\sqrt{r}} \ , & P^2_0 &= \frac{4\varphi}{\sqrt{r}} \ , & P^3_0 &= -\frac{12}{r} + 4 - \frac{\chi^2 + \varphi^2}{r} \ , \\ P^1_1 &= 0 \ , & P^2_1 &= 0 \ , & P^3_1 &= -\frac{12}{r} \ . \end{aligned} \tag{3.49}$$

We can now proceed to solving the maximal supersymmetry constraints. The conditions involving vector multiplet gauging are satisfied trivially, while from $\tilde{k}^u_\Lambda L^\Lambda = 0$ we obtain the conditions $\chi = \varphi = 0$ and $1 + \tau^2 = 0$. Therefore $\tau = i$ (the solution $\tau = -i$

[9] The relation with the coordinates $\{\rho, \sigma, \xi, \bar{\xi}\}$ used in [65, 79] is given by $\rho = r, \sigma_{\mathrm{theirs}} = \sigma_{\mathrm{here}} + \frac{1}{2}uv$ and $\xi = \frac{1}{2}(\chi + i\varphi)$. Furthermore, there is an overall factor $\frac{1}{4}$ in their definition of the universal hypermultiplet. Finally, they use a different $SU(2)$ frame to calculate the moment maps P^x_Λ, which is why they are rotated with respect to the ones displayed here.

makes the Kähler potential ill-defined) and $\mathcal{K} = -\log 4$. However, not all the moment maps at this vacuum can be zero simultaneously, leaving AdS_4 as the only possibility for a $\mathcal{N} = 2$ vacuum solution. One can then see that $\epsilon_{xyz} P^y \overline{P^z} = 0$ is satisfied, so the only remaining condition is $P^3_\Lambda f^\Lambda_\tau = 0$. This fixes $r = 4$. Therefore we have stabilized all (ungauged) directions in moduli space: $\chi = \varphi = 0, \tau = i, r = 4$. The potential is nonzero in this vacuum since $P^3 = 2$, which means the only possibility for the spacetime is to be AdS_4 with vanishing field strengths. This is indeed expected since SE_7 compactifications of M-theory lead to an $\mathcal{N} = 2$ AdS_4 vacuum, the one just described by us in the dimensionally reduced theory.

One can verify that this vacuum is stable under deformations in the hypermultiplet sector of the type discussed in [80, 81]. To show this, first observe that the condition $\tilde{k}^u_\Lambda L^\Lambda = 0$ for $u = \chi$ and $u = \varphi$ always ensures vanishing χ and φ. Secondly, one may verify that the deformations to the quaternionic moment maps are proportional to χ or φ, and hence the remaining $\mathcal{N} = 2$ conditions from section 3.2.4 are satisfied. It would be interesting to understand if this deformation corresponds to a perturbative one-loop correction in this particular type of M-theory compactification.

3.3.3 Other gaugings exhibiting $\mathbf{AdS_4}$ vacua

Another example of an AdS_4 supersymmetric vacuum can be obtained from the universal hypermultiplet. In the same coordinates $\{r, \chi, \varphi, \sigma\}$ as used in the previous example, the metric is again given by (3.47). This space has a rotational isometry acting on χ and φ, given by $\tilde{k}_1 - \tilde{k}_0$ in the notation of (3.48). We leave the vector multiplet sector unspecified for the moment, and gauge the rotation isometry by a linear combination of the gauge fields A^Λ_μ. This can be done by writing the Killing vector as

$$\tilde{k}^u_\Lambda = \alpha_\Lambda \Big(0, \varphi, -\chi, -\frac{1}{2}(\varphi^2 - \chi^2)\Big) \, , \tag{3.50}$$

for some real constant parameters α_Λ. The quaternionic moment maps are given by (see appendix D)

$$P^x_\Lambda = \alpha_\Lambda \left(\frac{4\varphi}{\sqrt{r}}, \frac{4\chi}{\sqrt{r}}, -\frac{12}{r} + 4 - \frac{\varphi^2 + \chi^2}{r} \right) \, . \tag{3.51}$$

It can be seen that there are no points for which $P^x_\Lambda = 0, \forall x$, so this means that only AdS_4 $\mathcal{N} = 2$ vacua are possible. To complete the example, we have to specify the vector multiplet space, and solve the conditions $P^x_\Lambda f^\Lambda_i = 0$ and $\tilde{k}^u_\Lambda L^\Lambda = 0$. The latter can be solved as $\chi = \varphi = 0$, and then also $\epsilon^{xyz} P^y \overline{P^z} = 0$. The first one then reduces to $\alpha_\Lambda f^\Lambda_i = 0$. This condition is trivially satisfied when e.g. $n_V = 0$. A more complicated example is to take the special Kähler space of the previous subsection with no gauging in the vector multiplet sector. There is one complex scalar τ, a section $X^\Lambda = (1, \tau^2)$ and a prepotential $F = \sqrt{X^0(X^1)^3}$. We then find a solution for $\tau = i\sqrt{\frac{-3\alpha_0}{\alpha_1}}$, under the condition that α_0 and α_1 are non-vanishing real constants of opposite sign. More complicated examples with more vector multiplets may be constructed as well. It would be interesting to study if such examples can be embedded into string theory.

A similar situation arises in the absence of hypermultiplets. As mentioned in the end of section 3.2.2, we can have non-vanishing moment maps that can be chosen as $P^x_\Lambda = \alpha_\Lambda \delta^{x3}$. Then we again need to satisfy the same condition $\alpha_\Lambda f_i^\Lambda = 0$ as above, and we already discussed the possible solutions.

Chapter 4

Black holes in gauged supergravity

4.1 Introduction

One of the interesting predictions of general relativity is the existence of black hole solutions. There is strong indication for the presence of black holes in the galaxy. Black holes are objects where gravity becomes strong enough to let not even light escape. There is an event horizon, and particles that move inside the horizon can (classically) never return to the original space-time.

In 1974, Stephen Hawking discovered that black holes do emit radiation [82], due to quantum effects. Due to this radiation, black holes follow the rules of thermodynamics, and one can derive their entropy [82, 83]. It is then a challenge for a theory of quantum gravitation to provide for the microscopic description of this entropy.

The Reissner-Nordström solution

We start our discussion of black holes with the Reissner-Nordström solution. This describes a charged black hole solution in the theory of general relativity with a Maxwell field. The gauge field has non-vanishing components

$$A_t = \frac{2Q}{r}\,, \qquad A_\phi = -2P\cos\theta\,, \tag{4.1}$$

where Q is the electric and P the magnetic charge of the black hole.

The metric can be written as

$$\mathrm{d}s^2 = V\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{V} - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\,, \tag{4.2}$$

with the metric function

$$V(r) = 1 - \frac{2M}{r} + \frac{Z^2}{r^2}\,, \qquad Z^2 \equiv Q^2 + P^2\,. \tag{4.3}$$

In the limit as r goes to infinity the metric approaches flat Minkowski space; also the electric and magnetic fields computed from (4.1) vanish at infinity. Besides the point $r = 0$, which is a true singularity, there are other possible values of r where $V(r) = 0$. Although the metric (4.2) does not behave well at these points, it turns out that these surfaces are coordinate singularities; by a different choice of coordinates the metric remains well-defined.

The event horizons follow from the equations

$$r_{\pm} = M \pm \sqrt{M^2 - Z^2}\,, \tag{4.4}$$

which could have two, one or zero real solutions for $r_{\pm}$.

$M^2 > Z^2$
: In this case, there are two different roots of V, given by $r = r_{\pm}$. There is an inner and an outer horizon.

$M^2 = Z^2$
: If the charge balances the mass, we call the black hole an extremal black hole. The real singularity is shielded by the event horizon at $r = r_+ = r_-$.

$M^2 < Z^2$
: If the charge exceeds the mass, there are no roots of V, and any observer can travel to the real singularity at $r = 0$, which is not shielded by an event horizon. This is called a naked singularity, and is deemed unphysical. Such a configuration will not form under gravitational collapse of a spherical mass shell, see e.g. [84].

We therefore find a mass bound on physical solutions. For the remainder, we will only look at the extremal black hole. We redefine the radial coordinate as $r \to r + M$, and then the metric is given by

$$\mathrm{d}s^2 = \frac{r^2}{(r+M)^2}\mathrm{d}t^2 - \frac{(r+M)^2}{r^2}\Big(\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\Big). \tag{4.5}$$

The true singularity is now at $r = -M$, and the horizon is at $r = 0$. Close to the horizon, we can approximate $r + M \simeq M$, and we find

$$\mathrm{d}s^2 = \frac{r^2}{M^2}\mathrm{d}t^2 - \frac{M^2}{r^2}\mathrm{d}r^2 + M^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{4.6}$$

The metric becomes a product metric: we have an AdS_2 space, parametrized by t and r, and an S^2 space, parametrized by θ and ϕ. The mass M determines the curvature of both spaces, which are equal in magnitude; as AdS_2 has negative curvature and S^2 positive, the total curvature at the horizon is zero.

Black holes in supergravity

The extremal Reissner-Nordström black hole is a solution of $\mathcal{N} = 2$ supergravity theory [85, 86] and preserves $\mathcal{N} = 1$ supersymmetry. For very small and very large values of r, the space-time approaches the fully supersymmetric configurations $AdS_2 \times S^2$ and flat Minkowski space, respectively, as explained in section 3.2.4. The non-extremal solution, with $M^2 > Z^2$ is a solution as well, but breaks all supersymmetry.

Black holes are interesting for a variety of reasons. The solutions one finds in supergravity can often be constructed in the full-fledged string theory. If string theory is a viable theory of quantum gravity, it should provide a complete, microscopic description of a black hole. One can compare this with the macroscopic description in supergravity, which is its low energy effective action. For instance, in the macroscopic picture, one can compute the entropy. In the microscopic picture, one can now count the number of microstates [6], which reproduces the same entropy.

Black holes in AdS spacetime

One can generalize the RN solution above to the AdS-RN solution. One keeps (4.1) and (4.2), where the metric function $V(r)$ is now given by

$$V(r) = 1 - \frac{2M}{r} + \frac{Z^2}{r^2} - \frac{1}{3}\Lambda r^2 \, , \tag{4.7}$$

where Λ is the cosmological constant, which is negative in AdS.

This configuration preserves half the supersymmetry when Λ is negative (for $\Lambda = 0$ one recovers the original RN solution), the magnetic charge vanishes ($P = 0$) and the BPS bound is satisfied ($Q^2 = M^2$). However, one sees that in this case the function $V(r)$ does not have any zeros, so there are no horizons that shield the true singularity at $r = 0$. Recent developments in the AdS/CFT correspondence suggest that holographic superconductors are related to non-extremal static black holes in the presence of a charged scalar, which makes them interesting to study.[15–18].

Black holes with hypermultiplets

The main aim of this chapter is the search for supersymmetric four-dimensional black holes in gauged $\mathcal{N} = 2$ supergravities in the presence of hypermultiplets, charged under an abelian gauge group. In the original references on BPS black holes in $D = 4, \mathcal{N} = 2$ supergravity [68, 86–89], and subsequent literature, see e.g. [69–71, 90, 91], one usually considers ungauged hypermultiplets, which then decouple from the supersymmetry variations and equations of motion for the vector multiplet fields. We want to explore how the story changes when the hypers couple non-trivially to the vector multiplets via gauge couplings and scalar potentials that are allowed within gauged $\mathcal{N} = 2$ supergravity [28, 35, 38, 52–54, 56]. For the simpler case of minimally gauged supergravity, where no hypermultiplets are present but only a cosmological constant or Fayet-Iliopoulos terms, asymptotically anti-de Sitter BPS black holes can be found. This has been discussed in the literature, starting from the early references [92, 93], or more recently in [94, 95]. We initiate here the extension to general $D = 4, \mathcal{N} = 2$ gauged supergravities, including hypermultiplets.

Plan of this chapter

The plan of this chapter is as follows. First, in section 4.2, we give a brief summary of the known black hole solutions in $\mathcal{N} = 2$ supergravity with neutral hypermultiplets, making a clear distinction between the asymptotically flat and asymptotically AdS spacetimes. We then explain the model with gauged hypermultiplets we are interested in and how this fits within the framework of $\mathcal{N} = 2$ gauged supergravity.

In section 4.3 we first explain how one can use a Higgs mechanism for spontaneous gauge symmetry breaking, in order to obtain effective $\mathcal{N} = 2$ ungauged theories from a general gauged $\mathcal{N} = 2$ supergravity. We keep the discussion short since these results follow easily from chapter 3. Then we show how this method can be used to embed already known black hole solutions into gauged supergravities and explain the physical meaning of the new solutions. We illustrate this with an explicit example of a static, asymptotically flat black hole with the well-known STU model and one gauged hypermultiplet (the universal hypermultiplet). We also give examples of AdS black holes with charged scalars, that may have applications in the emerging field of holographic superconductivity [15–18].

In section 4.4 we discuss in more general terms asymptotically flat, stationary spacetimes preserving half of the supersymmetries. We analyze the fermion susy variations in gauged supergravity after choosing a particular ansatz for the Killing spinor. One finds two separate cases, defined by $T^-_{\mu\nu} = 0$ and $P^x_\Lambda = 0$, respectively. Whereas the former case contains only Minkowski and AdS_4 solutions, the latter leads to a class of solutions that generalize the standard black hole solutions of ungauged supergravity. We analyze this in full detail in section 4.4.6 and give the complete set of equations that guarantees a half-BPS solution. We then explain how this fits to the solutions obtained in section 4.3.

Finally, in section 4.5, we study asymptotically flat black holes with scalar hair[10] .

[10] By scalar hair, in this thesis, we mean a scalar field that is zero at the horizon of the black hole, but non-zero outside of the horizon. According to this definition, the vector multiplet scalars subject to the attractor mechanism in $\mathcal{N} = 2$ ungauged supergravity, do not form black holes with scalar hair. The solutions that we discuss in section 4.5, however, will have hair.

We find two separate classes of such solutions. One is a purely bosonic solution with scalar hair, but with the shortcoming of having ghost modes in the theory. The other class of solutions has no ghosts but along with scalar hair we also find fermionic hair, i.e. the fermions are not vanishing in such a vacuum.

Some of the more technical aspects of this chapter, including explicit hypermultiplet gaugings, are presented in the appendices.

4.2 Review of supersymmetric black holes

In the first part of this section, we set our notation and briefly review the BPS black hole solutions in four-dimensional ungauged $\mathcal{N} = 2$ supergravity. In the second part, we review some of the BPS black holes that asymptote to anti-de Sitter spacetime. For a review of $\mathcal{N} = 2$ (gauged) supergravity we refer to [28], which notation we closely follow.

4.2.1 Black holes in asymptotically Minkowski spacetime

Asymptotically flat and stationary BPS black hole solutions of ungauged supergravity have been a very fruitful field of research in the last decades. In absence of vector multiplets ($n_V = 0$), with only the graviphoton present, the supersymmetric solution is just the well-known extremal Reissner-Nordström (RN) black hole. This solution was later generalized to include a number of vector multiplets [86]. The most general classification of the BPS solutions, including multicentered black holes, was given by Behrndt, Lüst and Sabra [68] and we will refer to those as BLS solutions. The hypermultiplet scalars q^u do not mix with the other fields (apart from the graviton) at the level of the equations of motion, and it is therefore consistent to set them to a constant value. We will briefly list the main points of the solutions, as they will play an important role in what follows.

To characterize the black hole solutions, we first denote the imaginary parts of the holomorphic sections by

$$\tilde{H}^\Lambda \equiv i(X^\Lambda - \bar{X}^\Lambda)\,, \qquad H_\Lambda \equiv i(F_\Lambda - \bar{F}_\Lambda)\,. \tag{4.8}$$

We assume stationary solutions with axial symmetry parametrized by an angular coordinate ϕ. The result of the BPS analysis is that the metric takes the form[11]

$$\mathrm{d}s^2 = \mathrm{e}^{\mathcal{K}}(\mathrm{d}t + \omega_\phi \mathrm{d}\phi)^2 - \mathrm{e}^{-\mathcal{K}}\left(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_2^2\right)\,, \tag{4.9}$$

[11] Note that all the results are in spherical coordinates, see [68, 70] for the coordinate independent results.

where $\mathcal{K}$ is the Kähler potential (2.19) of special geometry. The metric components and the symplectic vector $\left(\tilde{H}^\Lambda, H_\Lambda\right)$ only depend on the radial variable r and the second angular coordinate θ, and the BPS conditions imply the differential equations on ω_ϕ

$$\frac{1}{r^2 \sin\theta}\partial_\theta \omega_\phi = H_\Lambda \partial_r \tilde{H}^\Lambda - \tilde{H}^\Lambda \partial_r H_\Lambda \, , \qquad -\frac{1}{\sin\theta}\partial_r \omega_\phi = H_\Lambda \partial_\theta \tilde{H}^\Lambda - \tilde{H}^\Lambda \partial_\theta H_\Lambda \, . \quad (4.10)$$

From this follows the integrability condition $H_\Lambda \Box \tilde{H}^\Lambda - \tilde{H}^\Lambda \Box H_\Lambda = 0$, where $\Box$ is the 3-dimensional Laplacian.

What is left to specify are the gauge field strengths $F^\Lambda_{\mu\nu}$. First we define the magnetic field strengths

$$G_{\Lambda\mu\nu} \equiv R_{\Lambda\Sigma} F^\Sigma_{\mu\nu} - \frac{1}{2} I_{\Lambda\Sigma}\, \epsilon_{\mu\nu\gamma\delta} F^{\Sigma\gamma\delta} \, , \quad (4.11)$$

such that the Maxwell equations and Bianchi identities take the simple form

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu G_{\Lambda\rho\sigma} = 0, \quad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F^\Lambda_{\rho\sigma} = 0 \, , \quad (4.12)$$

such that (F^Λ, G_Λ) transforms as a vector under electric-magnetic duality transformations.

For the full solution it is enough to specify half of the components of F^Λ and G_Λ, since the other half can be found from (4.11). In spherical coordinates, the BPS equations imply the non-vanishing components[12]

$$F^\Lambda_{r\phi} = \frac{-r^2 \sin\theta}{2}\partial_\theta \tilde{H}^\Lambda, \qquad F^\Lambda_{\theta\phi} = \frac{r^2 \sin\theta}{2}\partial_r \tilde{H}^\Lambda, \quad (4.13)$$

and

$$G_{\Lambda r\phi} = \frac{-r^2 \sin\theta}{2}\partial_\theta H_\Lambda, \qquad G_{\Lambda\theta\phi} = \frac{r^2 \sin\theta}{2}\partial_r H_\Lambda. \quad (4.14)$$

From (4.12) it now follows that H_Λ and $\tilde{H}^\Lambda$ are harmonic functions. With the above identities we can always find the vector multiplet scalars z^i, given that we know explicitly how they are defined in terms of the sections X^Λ and F_Λ. The integration constants of the harmonic functions specify the asymptotic behavior of the fields at the black hole horizon(s) (the constants can be seen to be the black hole electric and magnetic charges) and at spatial infinity.

The complete proof that these are indeed all the supersymmetric black hole solutions with abelian vector multiplets and no cosmological constant was given in [70]. Note that the BLS solutions describe half-BPS stationary spacetimes with (only for the multi-centered cases) or without angular momentum. The near-horizon geometry around

[12] The BPS conditions also imply $F^\Lambda_{r\theta} = G_{\Lambda r\theta} = 0$ due to axial symmetry.

each center is always $AdS_2 \times S^2$ with equal radii of the two spaces, determined by the charges of the black hole. All solutions exhibit the so-called attractor mechanism [86]. This means that the (vector multiplet) scalar fields get attracted to constant values at the horizon of the black hole that only depend on the black hole charges. As the scalars can be arbitrary constants at infinity we also find the so-called attractor flow, i.e. the scalars flow from their asymptotic value to the fixed constant at the horizon. This phenomenon seems not to be related with supersymmetry, but rather with extremality, since attractor mechanisms have been discovered also in non-supersymmetric (but extremal) solutions. The full classification of non-BPS solutions and attractors is, however, more involved and is still in progress.

4.2.2 Gauged supergravity

We now turn to the bosonic Lagrangian (2.65) for gauged $\mathcal{N} = 2$ supergravity in presence of n_V abelian vector multiplets and n_H hypermultiplets, charged under the abelian gauge group.

The fully $\mathcal{N} = 2$ supersymmetric configurations obtained from (2.66)-(2.68) were analyzed in chapter 3. Two possibilities arise, namely for zero or nonzero cosmological constant in the vacuum. For zero cosmological constant, the different supersymmetric spacetimes are either Minkowski or $AdS_2 \times S^2$ (or its Penrose limit, the supersymmetric pp-wave), whereas for nonzero cosmological constant only AdS_4 can be fully BPS. In the former case, additional constraints arise on the scalar fields, namely (for abelian gaugings)

$$\tilde{k}^u_\Lambda L^\Lambda = 0 \, , \qquad P^x_\Lambda = 0 \, , \tag{4.15}$$

together with $F^\Lambda_{\mu\nu} = 0$ (Minkowski) and $\tilde{k}^u_\Lambda F^\Lambda_{\mu\nu} = 0$ ($AdS_2 \times S^2$). In the latter case, for AdS_4, one has the conditions

$$\tilde{k}^u_\Lambda L^\Lambda = 0 \, , \qquad P^x_\Lambda f^\Lambda_i = 0 \, , \qquad \epsilon^{xyz} P^y \overline{P^z} = 0 \, , \tag{4.16}$$

with vanishing field strengths, $F^\Lambda_{\mu\nu} = 0$, and negative scalar curvature for AdS_4 spacetime, $R = -12 g^2 P^x \overline{P^x}$, where $P^x \equiv P^x_\Lambda L^\Lambda$. In all these cases, the scalars are constant or covariantly constant. The fully supersymmetric configurations will play an important role in the construction of 1/2 BPS black hole solutions, since both their near horizon and asymptotic region fall into this class. We will discuss this in detail in the following sections.

A particular class of supergravities arises in the absence of hypermultiplets. This situation is interesting since it allows for a bare negative cosmological constant in the Lagrangian, through the moment maps P^x_Λ that appear in the scalar potential. It is well-known that, for $n_H = 0$ and abelian gauge groups, these moment maps can be replaced by constants (similar to Fayet-Iliopoulos terms), giving rise to a potential

$$V = (g^{i\bar{\jmath}} f_i^\Lambda f_{\bar{\jmath}}^\Sigma - 3\bar{L}^\Lambda L^\Sigma) P_\Lambda^x P_\Sigma^x \,, \tag{4.17}$$

with P_Λ^x numerical constants. When also $n_V = 0$, one can take the sections L^Λ to be constants as well, such that the potential is negative and given by $V = -\Lambda$, with $\Lambda = 3P^x\overline{P^x}$.

4.2.3 Asymptotically $\mathbf{AdS_4}$ black holes with $\mathbf{n_H = 0}$

The construction of BPS black holes in AdS_4 spacetimes is technically more involved due to the presence of the gauged hypermultiplets, and at present there is no complete analysis for this case. Until now, only the case with no hypermultiplets, $n_H = 0$, but with a bare cosmological constant or a potential of the type (4.17) has been investigated in the literature [72–74, 93–95]. Static and spherically symmetric (non-rotating) black hole solutions preserving some supersymmetry have been constructed, but they seem to suffer from naked singularities [92, 96, 97] or from scalar ghosts inside the horizon. On the other hand there are proper BPS black holes when one allows for a non-zero angular momentum [93, 98]. The non-BPS and non-extremal solutions, however, do allow for proper horizons also in the non-rotating case.

Let us illustrate some of these issues in the case of static spacetimes in gauged supergravities with no vector multiplets, so there is only a single gauge field, the graviphoton. Here we have the AdS generalization of the Reissner-Nordström black holes (RNAdS). More explicitly, the metric in our signature is

$$\mathrm{d}s^2 = V\mathrm{d}t^2 - \frac{dr^2}{V} - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{4.18}$$

with

$$V(r) = 1 - \frac{2M}{r} + \frac{Q^2+P^2}{r^2} - \frac{\Lambda r^2}{3} \,. \tag{4.19}$$

Here, Λ is the (negative) cosmological constant and Q and P are the electric and magnetic charge respectively. The field strengths are given by

$$F_{tr}^- = \frac{1}{2r^2}\left(Q - iP\right), \qquad F_{\theta\phi}^- = \frac{\sin\theta}{2}\left(P + iQ\right) \,. \tag{4.20}$$

For the $1/2$ BPS solution the magnetic charge is vanishing, $P = 0$ and $M = Q$ [92]. Of course, this example describes naked singularities rather than black holes. This is because $V(r)$ has no zeroes for $\Lambda < 0$, so no horizons, and therefore a naked singularity appears at $r = 0$. For a genuine AdS_4 black hole solution we have to break the full supersymmetry, i.e. the mass has to be free to violate the BPS bound. If M is within a certain range, as explained in detail in e.g. [98], the solution has a proper horizon and describes a thermal AdS_4 black hole. There are some BPS generalizations of these solutions to the case of arbitrary number of vector multiplets [96, 97], but the problem of naked singularities remains. For some further references on four-dimensional AdS black holes, including the non-extremal ones, see e.g. [99, 100].

Interestingly, recent developments in the AdS/CFT correspondence suggest that holographic superconductors are related to non-extremal static black holes in the presence of a charged scalar. Such cases will arise in $\mathcal{N} = 2$ supergravity only when the hypermultiplets are gauged. Thus we will be able to give some statements about this interesting class of black holes, which we leave for section 4.3.2. In the rest of the chapter we will mainly concentrate on the asymptotically flat BPS solutions with gauged hypers.

4.3 Black holes and spontaneous symmetry breaking

In this section we explain how to obtain a class of black hole solutions in gauged supergravity, starting from known solutions in ungauged supergravity. The main idea is simple: In gauged supergravity, one can give expectation values to some of the scalars (from both the vector and hypermultiplets) such that one breaks the gauge symmetry spontaneously in a maximally supersymmetric $\mathcal{N} = 2$ vacua, specified by the conditions (4.15) or (4.16). Let us suppose for simplicity that the vacuum has zero cosmological constant, the argument can be repeated for $\mathcal{N} = 2$ preserving anti-de Sitter vacua. Due to the Higgs mechanism some of the fields become massive, and as a consequence of the $\mathcal{N} = 2$ preserving vacua, the gravitinos remain massless and the heavy modes form massive $\mathcal{N} = 2$ vector multiplets. As a second step, we can set the heavy fields to zero, and the theory gets truncated to an ungauged $\mathcal{N} = 2$ supergravity. These truncations are consistent due to the fact that supersymmetry is unbroken. Black hole solutions can then be found by taking any solution of the ungauged theory and augmenting it with the massive fields that were set to zero. In fact, it is clear from this procedure that one can even implement a non-BPS black hole solution of the ungauged theory into the gauged theory. It is also clear that this procedure works for non-abelian gaugings, as long as it is broken spontaneously to an abelian subgroup with residual $\mathcal{N} = 2$ supersymmetry. But for simplicity, and to streamline with subsequent sections, we will however only consider abelian gaugings. What is perhaps less clear, is to see if this procedure gives the most general black hole solutions. In other words, one can look for other solutions in which the massive scalars are non-trivial (i.e. with scalar hair). This is the subject of section 4.4.6, where we investigate the conditions for which new BPS black holes with scalar hair exist.

Let us now illustrate the above mechanism in some more detail. We restrict ourselves first to spontaneous symmetry breaking in Minkowski vacua, where one has $\langle P^x_\Lambda \rangle = 0$ and $\langle \tilde{k}^u_\Lambda L^\Lambda \rangle = 0$ according to (4.15). At such a point, the resulting potential is zero, see (2.72), as required by a Minkowski vacuum. After the hypermultiplet scalar fields take their vacuum expectation value, the Lagrangian (2.65) contains a mass-term for some of the gauge fields, given by

$$\mathcal{L}^V_{\text{mass}} = M_{\Lambda\Sigma} A^\Lambda_\mu A^{\mu\Sigma} \ , \qquad M_{\Lambda\Sigma} \equiv g^2 \langle h_{uv} \tilde{k}^u_\Lambda \tilde{k}^v_\Sigma \rangle \ . \tag{4.21}$$

There is no contribution to the mass matrix for the vector fields coming from expectation values of the vector multiplet scalars, since the gauging was chosen to be abelian. The number of massive vectors is then given by the rank of $M_{\Lambda\Sigma}$, and as h_{uv} is positive definite, one has $\text{rank}(M_{\Lambda\Sigma}) = \text{rank}(\tilde{k}^u_\Lambda)$. Hence, the massive vector fields are encoded by the linear combinations $\tilde{k}^u_\Lambda A^\Lambda_\mu$. Similarly, some of the vector and hypermultiplet scalars acquire a mass, determined by expanding the scalar potential,

$$V = 4 h_{uv} \tilde{k}^u_\Lambda \tilde{k}^v_\Sigma \bar{L}^\Lambda L^\Sigma + \left(g^{i\bar{\jmath}} f^\Lambda_i f^\Sigma_{\bar{\jmath}} - 3 \bar{L}^\Lambda L^\Sigma \right) P^x_\Lambda P^x_\Sigma \ , \tag{4.22}$$

to quadratic order in the fields. Then one reads off the mass matrix, and in general there can be off-diagonal mass terms between vector and hypermultiplet scalars. Massive vector multiplets can then be formed out of a massive vector, a massive complex scalar from the vector multiplet, and 3 hypermultiplet scalars. The fourth hypermultiplet scalar is the Goldstone mode that is eaten by the vector field. We will illustrate this more explicitly in some concrete examples below.

Upon setting the massive fields to zero (or integrating them out), one obtains a supergravity theory with only massless fields. Because of $\langle P^x_\Lambda \rangle = 0$, the mass matrix for the gravitinos is zero as follows from (2.71). Therefore, the resulting theory is an ungauged supergravity theory of the type discussed in section 2.3.3. Black hole solutions can then be simply copied from the results in section 4.2.1. By going through the Higgs mechanism in reverse order, one can uplift this solution easily to the gauged theory by augmenting it with the necessary expectation values of the scalars. It is then clear that the black hole solution is not charged with respect to the gauge fields that acquired a mass.

The situation for spontaneous symmetry breaking in an AdS vacuum is similar. To generate a negative cosmological constant from the potential (2.72), we must have a $\langle P^x_\Lambda \rangle \neq 0$ in the vacuum. The conditions for unbroken $\mathcal{N} = 2$ supersymmetry are given in (4.16). After expanding the fields around this vacuum, one can truncate the theory further to a Lagrangian with a bare cosmological constant, in which one can construct black hole solutions of the type discussed in section 4.2.3. We will discuss an example at the end of this section.

4.3.1 Solution generating technique

We now elaborate on constructing the black hole solutions more explicitly. As explained above, the general technique is to embed a (BPS) solution in ungauged supergravity into a gauged supergravity. The considerations in this subsection also apply for the more general case of non-abelian gaugings, although we are mainly interested here in the abelian case. First, to illustrate the systematics of our procedure, we analyze a simpler setup in which we embed solutions from pure supergravity into a model with vector multiplets only. Then we extend the models to include both hypermultiplets and vector multiplets, i.e. the most general (electrically) gauged supergravities. We always consider solutions with vanishing fermions, i.e. the discussion concerns only the bosonic fields.

Vector multiplets

We start from pure $\mathcal{N} = 2$ supergravity, i.e. only the gravity multiplet normalized as $\mathcal{L} = \frac{1}{2}R(g) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \Lambda$. Let us assume we have found a solution of this Lagrangian, which we denote by $\mathring{g}_{\mu\nu}, \mathring{F}_{\mu\nu}$. We can embed this into a supergravity theory with only vector multiplets as follows. If we have a theory with (gauged) vector multiplets we can find a corresponding solution to it by satisfying

$$\nabla_\mu z^i = 0\,, \qquad G^i_{\mu\nu} = 0\,, \qquad k^i_\Lambda \bar{L}^\Lambda = 0\,. \tag{4.23}$$

Note that the integrability condition following from $\nabla_\mu z^i = 0$ is always satisfied given the other constraints [13]. We further have the relations

$$g_{\mu\nu} = \mathring{g}_{\mu\nu}\,, \quad \sqrt{2I_{\Lambda\Sigma}\bar{L}^\Lambda\bar{L}^\Sigma}\,T^-_{\mu\nu} = \mathring{F}^-_{\mu\nu}\,. \tag{4.24}$$

The last equality is to be used for determining $T^-_{\mu\nu}$. Then we can find the solution for our new set of gauge field strengths by $F^{\Lambda-}_{\mu\nu} = i\bar{L}^\Lambda T^-_{\mu\nu}$ since we already know that $G^i_{\mu\nu} = 0$.

The new configuration will, by construction, satisfy all equations of motion of the theory and will preserve the same amount of supersymmetry (if any) as the original one. This can be checked explicitly from the supersymmetry transformation rules (2.66) and (2.68) combined with the results from section 3. Indeed (4.23) comes from imposing the vanishing of (2.66), while (4.24) is required by the Einstein equations. We will give a more explicit realization of this procedure in section 4.3.2.

[13] Also note that we have used the Killing vectors k^i_Λ that specify a gauged isometry $\nabla_\mu z^i = \partial_\mu z^i + gk^i_\Lambda A^\Lambda_\mu$ on the vector multiplet scalar manifold. These automatically vanish if the isometry is abelian, and therefore will not be discussed further in this chapter. The formulas here are still valid for any gauged isometry.

Hypermultiplets

Given any solution of $\mathcal{N} = 2$ supergravity with no hypermultiplets, we can obtain a new solution with (gauged) hypermultiplets preserving the same amount of supersymmetry as the original one. We require the theory to remain the same in the other sectors (vector and gravity multiplets with solution $\mathring{g}_{\mu\nu}, \mathring{F}^{\Lambda}_{\mu\nu}, \mathring{z}^i$) and impose some additional constraints that have to be satisfied in addition to the already given solution. We then simply require the fields of our new theory to be

$$g_{\mu\nu} = \mathring{g}_{\mu\nu}, \qquad F^{\Lambda}_{\mu\nu} = \mathring{F}^{\Lambda}_{\mu\nu} \,, \qquad z^i = \mathring{z}^i \,, \tag{4.25}$$

under the following restriction that has to be solved for the hypers. Here we are left with two cases: the original theory was either with or without Fayet-Iliopoulos (FI) terms (cosmological constant). In absence of FI terms, a new solution after adding hypers is given by imposing the constraints:

$$\nabla_\mu q^u = 0 \;\Rightarrow\; \tilde{k}^u_\Lambda F^\Lambda_{\mu\nu} = 0 \,, \qquad P^x_\Lambda = 0 \,, \qquad \tilde{k}^u_\Lambda L^\Lambda = 0 \,, \tag{4.26}$$

while in the case of original solution with FI terms we have a solution after adding hypers (thus no longer allowing for FI terms but keeping $P^x_\Lambda L^\Lambda$ the same) with:

$$\nabla_\mu q^u = 0 \Rightarrow \tilde{k}^u_\Lambda F^\Lambda_{\mu\nu} = 0 \,, \quad P^x_\Lambda f^\Lambda_i = 0 \,, \quad \epsilon^{xyz} P^y_\Lambda P^z_\Sigma L^\Lambda \bar{L}^\Sigma = 0 \,, \quad \tilde{k}^u_\Lambda L^\Lambda = 0 \,. \tag{4.27}$$

The new field configuration (given it can be found from the original data) again satisfies all equations of motion and preserves the same amount of supersymmetry as the original one. This is true because the susy variations of gluinos and gravitinos remain the same as in the original solution, and also the variations for the newly introduced hyperinos are zero.

Vector and hypermultiplets

This case is just combining the two cases above. If we start with no FI terms the new solution will be generated by imposing equations (4.26) and (4.23). If we have a solution with a cosmological constant we need to impose (4.27) and (4.23). Then the integrability condition following from $\nabla_\mu q^u = 0$ is automatically satisfied in both cases, using relations (4.24).

4.3.2 Examples

The STU model with gauged universal hypermultiplet

Here we discuss an example to illustrate explicitly the procedure outlined above. Let us consider an $\mathcal{N} = 2$ theory with the universal hypermultiplet (UHM). Its quaternionic metric and isometries are given in D, and isometry 5 is chosen to be gauged. This allows for asymptotically flat black holes, since we can find solutions of (4.26), as we shall see below[14]. The quaternionic Killing vector and moment maps are given by

$$\tilde{k}_\Lambda = a_\Lambda \left(2r\partial_R + \chi\partial_\chi + \varphi\partial_\varphi + 2\sigma\partial_\sigma\right), \tag{4.28}$$

$$\vec{P}_\Lambda = a_\Lambda \left\{-\frac{\chi}{\sqrt{R}}, \frac{\varphi}{\sqrt{R}}, -\frac{\sigma + \frac{1}{2}\varphi\chi}{R}\right\}, \tag{4.29}$$

with a_Λ arbitrary constants. In this chapter we will use R to denote the coordinate on the UHM, to avoid confusion with the radial, space-time coordinate r.

In the vector multiplet sector we take the so-called STU model, based on the prepotential

$$F = \frac{X^1 X^2 X^3}{X^0}, \tag{4.30}$$

together with $z^i = \frac{X^i}{X^0}; i = 1, 2, 3$. The gauge group is $U(1)^3$, but it will be broken to $U(1)^2$ in the supersymmetric Minkowski vacua, in which we construct the black hole solution. The conditions for a fully BPS Minkowski vacuum require $F_{\mu\nu}^{\text{vev}} = 0$, $z^{i\,\text{vev}} = \langle z^i\rangle = \langle b^i\rangle + i\langle v^i\rangle$, $\chi^{\text{vev}} = \varphi^{\text{vev}} = \sigma^{\text{vev}} = 0$, $R^{\text{vev}} = \langle R\rangle$, with arbitrary constants $\langle z^i\rangle$ and $\langle R\rangle$. Moreover, from (4.15), the vector multiplets scalar vevs must obey $(a_\Lambda L^\Lambda)^{\text{vev}} = 0$ (which is an equation for the $\langle z^i\rangle$'s). Then, after expanding around this vacuum, the mass terms for the scalar fields are given by the quadratic terms in (4.22). Now, if we make the definition $z \equiv a_\Lambda L^\Lambda$, we have $z^{\text{vev}} = 0$. Expanding the first term in (4.22) gives the mass term for z,

$$\left(4h_{uv}\tilde{k}^u_\Lambda \tilde{k}^v_\Sigma \bar{L}^\Lambda L^\Sigma\right)^{\text{quadratic}} = 16 z\bar{z}.$$

Expanding the second term to quadratic order gives the mass for three of the hypers:

$$\left(g^{i\bar{j}} f_i^\Lambda \bar{f}_{\bar{j}}^\Sigma P^x_\Lambda P^x_\Sigma\right)^{\text{quadratic}} = \frac{a_i^2 \langle v^i\rangle^2}{\langle v^1 v^2 v^3\rangle \langle R\rangle}\left(\chi^2 + \varphi^2 + \frac{(\sigma + \frac{1}{2}\chi\varphi)^2}{\langle R\rangle}\right), \tag{4.31}$$

while the third term vanishes at quadratic order and does not contribute to the mass matrix of the scalars.

Therefore two of the six vector multiplet scalars become massive (i.e. the linear combination given by our definition for z), together with three of the hypers. The fourth hyper R remains massless and is eaten up by the massive gauge field $a_\Lambda A^\Lambda_\mu$ (with mass 4 given by (4.21)). Thus we are left with an effective $\mathcal{N} = 2$ supergravity theory of

[14] A suitable combination of isometries 1 and 4 would also do the job. Note that typically in string theory isometry 5 gets broken perturbatively while 1 and 4 remain also at quantum level. For the present discussion it is irrelevant which one we choose since we are not trying to directly obtain the model from string theory.

one massive and two massless vector multiplets and no hypermultiplets, which can be further consistently truncated to only include the massless modes. One can then search for BPS solutions in the remaining theory and the prescription for finding black holes is again the one given by Behrndt, Lüst and Sabra and explained in section 4.2.1.

We now construct the black hole solution more explicitly, following the solution generating technique of section 4.3.1. For this, we need to satisfy (4.25) and (4.26). The condition $P^x_\Lambda = 0$ fixes $\chi = \varphi = \sigma = 0$ and the remaining non-zero Killing vectors are $k^R_\Lambda = 2Ra_\Lambda$. Now we have to satisfy the remaining conditions $\tilde{k}^u_\Lambda X^\Lambda = 0$ and $\tilde{k}^u_\Lambda F^\Lambda_{\mu\nu} = 0$. To do so, we use the BLS solution of the STU model. For simplicity we take the static limit $\omega_m = 0$, discussed in detail in section 4.6 of [68]. The solution is fully expressed in terms of the harmonic functions

$$H_0 = h_0 + \frac{q_0}{r}, \qquad \tilde{H}^i = h^i + \frac{p^i}{r}\,, \qquad i = 1, 2, 3\,, \tag{4.32}$$

under the condition that one of them is negative definite. The sections then read

$$X^0 = \sqrt{-\frac{\tilde{H}^1\tilde{H}^2\tilde{H}^3}{4H_0}}\,, \qquad X^i = -i\frac{\tilde{H}^i}{2}\,, \tag{4.33}$$

with metric function

$$\mathrm{e}^{-\mathcal{K}} = \sqrt{-4H_0\tilde{H}^1\tilde{H}^2\tilde{H}^3}. \tag{4.34}$$

In this case $F^0_{mn} = 0$ and the F^i_{mn} components (here m, n are the spatial indices) are expressed solely in terms of derivatives of $\tilde{H}^i$. After evaluating the period matrix we obtain $F^i_{mt} = 0$ and F^0_{mt} are given in terms of derivatives of $H_0, \tilde{H}^i$. Thus the equations $\tilde{k}^R_\Lambda X^\Lambda = 0$ and $\tilde{k}^R_\Lambda F^\Lambda_{\mu\nu} = 0$ lead to

$$a_0 = 0\,, \qquad a_i h^i = 0\,, \qquad a_i p^i = 0\,. \tag{4.35}$$

The solution is qualitatively the same as the original one, but the charges p^i and the asymptotic constants h^i are now related by (4.35). So effectively, the number of independent scalars and vectors is decreased by one, consistent with the results from spontaneous symmetry breaking. The usual attractor mechanism for the remaining, massless vector multiplet scalars holds while for the hypermultiplet scalars we know that $\chi = \varphi = \sigma = 0$ and R is fixed to an arbitrary constant everywhere in spacetime with no boundary conditions at the horizon. In other words the hypers are not 'attracted'.

Our construction can be generalized for non-BPS solutions as well. In the particular case of the STU model, we can obtain a completely analogous, non-BPS, solution by following the procedure described in [101]. We flip the sign of one of the harmonic functions in (4.8) such that

$$\mathrm{e}^{-\mathcal{K}} = \sqrt{4H_0\tilde{H}^1\tilde{H}^2\tilde{H}^3}\ . \tag{4.36}$$

This solution preserves no supersymmetry, but it is extremal. By following our procedure above, we can embed this solution into the gauged theory.

Asymptotically AdS black holes

Here we give a simple but yet qualitatively very general example of how to apply the procedure outlined above to find asymptotically anti-de Sitter black hole solutions with gauged hypers, starting from already known black hole solutions without hypers. In this case we start from a solution of pure supergravity and add abelian gauged vector multiplets and hypermultiplets. Alternatively, one can think of it as breaking the gauge symmetry such that all hyper- and vector multiplets become massive, and one is left with a gravity multiplet with cosmological constant. Here we already know the full classification of black hole solutions, as described in section 4.2.3.

An already worked out example in section 3.3.2 is the case of the gauged supergravity, arising from a consistent reduction to four dimensions of M-theory on a Sasaki-Einstein$_7$ manifold [65]. The resulting low-energy effective action has a single vector multiplet and a single hypermultiplet (the universal hypermultiplet). The special geometry prepotential is given by

$$F = \sqrt{X^0(X^1)^3},$$

with $X^\Lambda = \{1, \tau^2\}$, where τ is the vector multiplet scalar, and the isometries on the UHM are given by

$$\begin{aligned}\tilde{k}_0 &= 24\partial_\sigma + 4\big(\chi\partial_\varphi - \varphi\partial_\chi + \frac{1}{2}(\varphi^2 - \chi^2)\partial_\sigma\big)\ ,\\ \tilde{k}_1 &= 24\partial_\sigma\ ,\end{aligned} \tag{4.37}$$

which is combination of isometries 1 and 4 from appendix D. The corresponding moment maps, see appendix D, are given by

$$\begin{aligned}&P_0^1 = \frac{4\chi}{\sqrt{R}}\ , \qquad P_0^2 = \frac{4\varphi}{\sqrt{R}}\ , \qquad P_0^3 = -\frac{12}{R} + 4 - \frac{\chi^2+\varphi^2}{R}\ ,\\ &P_1^1 = 0\ , \qquad P_1^2 = 0\ , \qquad P_1^3 = -\frac{12}{R}\ .\end{aligned} \tag{4.38}$$

Maximally supersymmetric AdS_4 vacua were found in section 3.3.2. The condition (4.16) fixes the values of the vector multiplet scalar $\tau^{\rm vev} \equiv (\tau_1 + i\tau_2)^{\rm vev} = i$ and two of the four hypers $\chi^{\text{vev}} = \varphi^{\text{vev}} = 0$. The third ungauged hyper, which is the dilaton, is fixed to the constant non-zero value $R^{\text{vev}} = 4$. The remaining hypermultiplet scalar is an arbitrary constant $\sigma^{\text{vev}} = \langle\sigma\rangle$. All the gauge fields have vanishing expectation values at this fully supersymmetric AdS_4 vacuum. If we now expand the scalar field potential (4.22) up to second order in fields we obtain the following mass terms

$$V^{\text{quadratic}} = -12 + 138(\tau_1^2 + \tau_2^2) + \frac{3}{4}R^2 + 6R\tau_2 + 10(\chi^2 + \varphi^2)\,. \tag{4.39}$$

We can see that three of the hyperscalars and the (complex) vector multiplet scalar acquire mass. There is also a mass term $m^2 = 36$ for the gauge field $A_0 + A_1$, this field thus eats up the remaining massless hyperscalar σ. So we observe the formation of a massive $\mathcal{N} = 2$ vector multiplet consisting of one massive vector and five massive scalars, and we can consistently set all these fields to zero. The resulting Lagrangian is that of pure $\mathcal{N} = 2$ supergravity with a cosmological constant $\Lambda = -12$. Using the static class of black hole solutions of (4.18), it is straightforward to provide a solution of the gauged supergravity theory. All the solutions described in section 4.2.3 will also be solutions in our considered model as they obey the Einstein-Maxwell equations of pure supergravity.

4.4 1/2 BPS solutions

In this section we will take a more systematic approach to studying the supersymmetric solutions of (2.65). We search for a solution where the expectation values of the fermions are zero. This implies that the supersymmetry variations of the bosons should be zero. The vanishing of the supersymmetry variations (2.66)-(2.68) then guarantees some amount of conserved supersymmetry. Depending on the number of independent components of the variation parameters ε_A we will have different amount of conserved supersymmetry. Here we will focus on particular solutions preserving (at least) 4 supercharges, i.e. half-BPS configurations. A BPS configuration has to further satisfy the equations of motion in order to be a real solution of the theory, so we also impose those. The fermionic equations of motion vanish automatically, so we are left with the equations of motion for the graviton $g_{\mu\nu}$, the vector fields A_μ^Λ, and the scalars z^i and q^u. We will come to the relation between the BPS constraints and the field equations in due course, but we first introduce some more relations for the Killing spinors ε_A.

4.4.1 Killing spinor identities

We will make use of the approach [102] where one first assumes the existence of a Killing spinor. From this spinor, various bilinears are defined, whose properties constrain the form of the solution to a degree where a full classification is possible. We use this method in $D = 4, \mathcal{N} = 2$, which is generalizing the main results of [70, 71] to include hypermultiplets in the description. As it later turns out, we cannot completely use this method to classify all the supersymmetric configurations, but the method nevertheless gives useful information.

We define ε_A to be a Killing spinor if it solves the gravitino variation $\delta_\varepsilon \psi_{\mu A} = 0$, defined in (2.68), and assume ε_A to be a Killing spinor in the remainder of this article. Such spinors anti-commute, but we can expand them on a basis of Grassmann variables and only work with the expansion coefficients. This leads to a commuting spinor, which we also denote with ε_A, and we define[15]

$$\begin{aligned} \overline{\varepsilon_A} &\equiv i(\varepsilon^A)^\dagger \gamma_0 \ , \\ X &\equiv \frac{1}{2}\epsilon^{AB}\overline{\varepsilon_A}\varepsilon_B \ , \\ V_\mu{}^A{}_B &\equiv i\overline{\varepsilon^A}\gamma_\mu \varepsilon_B \ , \\ \Phi_{AB\mu\nu} &\equiv \overline{\varepsilon_A}\gamma_{\mu\nu}\varepsilon_B \ . \end{aligned} \tag{4.40}$$

We now show that this implies that $V^\mu \equiv V^{\mu A}{}_A$ is a Killing vector. For its derivatives we find

$$\begin{aligned} \nabla_\mu V_\nu{}^A{}_B = {}& i\delta^A{}_B(T^+_{\mu\nu}X - T^-_{\mu\nu}\bar{X}) - g_{\mu\nu}(S^{AC}\epsilon_{CB}X - S_{BC}\epsilon^{AC}\bar{X}) \\ & - i(\epsilon^{AC}T^{+\rho}_\mu \Phi_{CB\rho\nu} + \epsilon_{BC}T^{-\rho}_\mu \Phi^{AC}{}_{\nu\rho}) - (S^{AC}\Phi_{CB\mu\nu} + S_{BC}\Phi^{AC}{}_{\mu\nu}) \ . \end{aligned} \tag{4.41}$$

The second and third term are traceless, so they vanish when we compute $\nabla_\mu V_\nu$. The other terms are antisymmetric in $\mu\nu$, so this proves

$$\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0 \ , \tag{4.42}$$

thus V_μ is a Killing vector. We make the decomposition $V^A{}_{B\mu} = \frac{1}{2}V_\mu \delta^A{}_C + \frac{1}{\sqrt{2}}\sigma^{xA}{}_B V^x_\mu$ and using Fierz identities one finds

$$V_\mu{}^A{}_B V_\nu{}^B{}_A = V_\mu V_\nu - \frac{1}{2}g_{\mu\nu}V^2 \ . \tag{4.43}$$

One can show that $V_\mu V^\mu = 4|X|^2$, which shows that the Killing vector V_μ is timelike or null. For the remainder of this chapter we restrict ourselves to a timelike Killing spinor Ansatz, defined as one that leads to a timelike Killing vector. We make this choice, as our goal is to find stationary black hole solutions, which always have a timelike isometry. In this case, by definition, $V_\mu V^\mu = 4|X|^2 \neq 0$, so we can solve (4.43) for the metric as

$$g_{\mu\nu} = \frac{1}{4|X|^2}\left(V_\mu V_\nu - 2V^x_\mu V^x_\nu\right) \ . \tag{4.44}$$

[15] We will be brief on some technical points of the discussion, and refer to [70, 71] for more information.

It follows that

$$V_\mu = g_{\mu\nu} V^\nu = V_\mu - \frac{1}{2|X|^2} V^x_\mu (V^x_\nu V^\nu) , \tag{4.45}$$

so $V^x_\mu V^\mu = 0$. We define a time coordinate by $V^\mu \partial_\mu = \sqrt{2}\partial_t$, which implies $V^x_t = 0$. We decompose $V_\mu \mathrm{d}x^\mu = 2\sqrt{2} X\bar{X}(\mathrm{d}t + \omega)$, where the factor in front of $\mathrm{d}t$ follows from $V^2 = 4X\bar{X}$ and ω has no $\mathrm{d}t$ component. The metric is then given by

$$\mathrm{d}s^2 = 2|X|^2 (\mathrm{d}t + \omega)^2 - \frac{1}{2|X|^2} \gamma_{mn} \mathrm{d}x^m \mathrm{d}x^n , \tag{4.46}$$

where $|X|, \omega$ and γ_{mn} are independent of time.

Now we are ready to make a relation between the susy variations (2.66)–(2.68) and the equations of motion, using an elegant and simple argument of Kallosh and Ortin [103] that was later generalized in [70, 71]. Assuming the existence of (any amount of) unbroken supersymmetry, one can derive a set of equations relating the equations of motion for the bosonic fields with derivatives of the bosonic susy variations. For our chosen theory these read:

$$\begin{aligned} \mathcal{E}^\mu_\Lambda i f^\Lambda_i \gamma_\mu \varepsilon^A \epsilon_{AB} + \mathcal{E}_i \varepsilon_B = 0 , \\ \mathcal{E}^\mu_a (-i\gamma^a \varepsilon^A) + \mathcal{E}^\mu_\Lambda \left(2\bar{L}^\Lambda \varepsilon_B \epsilon^{AB}\right) = 0 , \\ \mathcal{E}_u \mathcal{U}^u_{\alpha A} \varepsilon^A = 0 , \end{aligned} \tag{4.47}$$

where $\mathcal{E}$ is the equation of motion for the corresponding field in subscript. More precisely, $\mathcal{E}^\mu_a$ is the equation for the vielbein e^a_μ (the Einstein equations), $\mathcal{E}^\mu_\Lambda$ corresponds to A^Λ_μ (the Maxwell equations), $\mathcal{E}_u$ corresponds to q^u and $\mathcal{E}_i$ to z^i. Now, let us assume that the Maxwell equations are satisfied, $\mathcal{E}^\mu_\Lambda = 0$. If we multiply each of the remaining terms in the three equations by $\overline{\varepsilon^B}$ and $\gamma^\nu \overline{\varepsilon^B}$ and use the fact that the Killing spinor is timelike such that $X \neq 0$ we directly obtain that the remaining field equations are satisfied. So, apart from the BPS conditions, only the Maxwell equations

$$\epsilon^{\mu\nu\rho\sigma} \partial_\nu G_{\Lambda\rho\sigma} = -g h_{uv} \tilde{k}^u_\Lambda \nabla^\mu q^v , \tag{4.48}$$

need to be satisfied.

4.4.2 Killing spinor ansatz

Contracting the gaugino variation (2.66) with ε_A we find the condition

$$0 = -2i\bar{X}\nabla_\mu z^i + 4i G^{-i}_{\rho\mu} V^\rho - igk^i_\Lambda \bar{L}^\Lambda V_\mu - \sqrt{2} g g^{i\bar{\jmath}} \bar{f}^\Lambda_{\bar{\jmath}} P^x_\Lambda V^x_\mu . \tag{4.49}$$

Using this to eliminate $\nabla_\mu z^i$ and plugging back into $\delta\lambda^{iA} = 0$ we find[16]

$$G^{i-}_{\rho\mu}\gamma^\mu \left(2iV^\rho\varepsilon^A - \bar{X}\gamma^\rho\epsilon^{AB}\varepsilon_B\right) + gg^{i\bar{\jmath}}\bar{f}^\Lambda_{\bar{\jmath}}P^x_\Lambda \left(-\frac{1}{\sqrt{2}}V^x_\mu\gamma^\mu\varepsilon^A + i\bar{X}\sigma^{xAB}\varepsilon_B\right) = 0 \ . \quad (4.50)$$

It is here that we find an important difference with the ungauged theories. In the latter case, $g = 0$, and the second term is absent. Then, assuming that the gauge fields $G^{i-}_{\rho\mu}$ are non-zero, one can rewrite equation (4.50) as

$$\varepsilon^A + i\mathrm{e}^{-i\alpha}\gamma_0\epsilon^{AB}\varepsilon_B = 0 \ , \quad (4.51)$$

where $\mathrm{e}^{i\alpha} \equiv \frac{X}{|X|}$. One has thus *derived* the form of the Killing spinor, which is not an ansatz anymore.

In gauged supergravity, $g \neq 0$, so there are various ways to solve equation (4.50). One could, for instance, generalize (4.51) to

$$\varepsilon^A = b\gamma^0\epsilon^{AB}\varepsilon_B + a^x_m\gamma^m\sigma^{xAB}\varepsilon_B \ . \quad (4.52)$$

Plugging this back into (4.50), one obtains BPS conditions on the fields which one can then try to solve. While this is hard in general, it has been done in a specific case. Namely, the ansatz used for the AdS-RN black holes in minimally gauged supergravity (with a bare cosmological constant), as analyzed by Romans [92], fits into (4.52), but not in (4.51). In fact, we will see later that with (4.51) one cannot find AdS black holes.

In the remainder of this article, we will use (4.51) as a particular ansatz, hoping to find new BPS black hole solutions that are asymptotically flat. The reader should keep in mind that more general Killing spinors are possible, even for asymptotically flat black holes, and therefore our procedure will most likely not be the most general. The search for BPS black holes that asymptote to AdS_4, and their Killing spinors, will be postponed for future research.

4.4.3 Metric and gauge field ansatz

We will further make the extra assumption that the solution for the spacetime metric, field strengths and scalars, is axisymmetric, i.e. there is a well-defined axis of rotation, such that $\omega = \omega_\varphi \mathrm{d}\varphi$ lies along the angle of rotation (we choose to call it φ) in (4.46). For a stationary axisymmetric black hole solution the symmetries constrain the metric not to depend on t and φ. These symmetries also constrain the scalars and gauge field strengths to depend only on the remaining coordinates, which we choose to call r and θ. We further assume $F^\Lambda_{r\theta} = 0$, such that (after also using the gauge freedom) we can set $A^\Lambda_r = A^\Lambda_\theta = 0$ for all Λ.

[16] One could, as done in e.g. [70, 71], eliminate the gauge fields $G^{i-}_{\rho\mu}$ to obtain an equivalent relation.

4.4.4 Gaugino variation

Plugging the ansatz (4.51) into the gaugino variation $\delta\lambda^{iA} = 0$ gives

$$P^x_\Lambda f^\Lambda_i = 0\,, \tag{4.53}$$

and

$$\left(\mathrm{e}^{-i\alpha}\partial_\mu z^i \gamma^\mu\gamma^0 + G^{-i}_{\mu\nu}\gamma^{\mu\nu}\right)\varepsilon_A = 0\,. \tag{4.54}$$

The latter condition can be simplified further, but we will see in what follows that it automatically becomes simpler or gets satisfied in certain cases, so we will come back to (4.54) later. We will make use of condition (4.53) when solving the gravitino integrability conditions.

4.4.5 Hyperino variation

With the ansatz (4.51), setting the hyperino variation to zero gives the condition

$$\mathrm{e}^{-i\alpha}\nabla_\mu q^u \gamma^\mu\gamma_0 + 2g\tilde{k}^u_\Lambda \bar{L}^\Lambda = 0\,. \tag{4.55}$$

Using the independence of the gamma matrices, one finds

$$\begin{aligned}
\nabla_r q^u &= \nabla_\theta q^u = 0\,,\\
\nabla_\phi q^u &= \omega_\phi \nabla_t q^u\,,\\
\nabla_t q^u &= -\sqrt{2}g\tilde{k}^u_\Lambda\left(X\bar{L}^\Lambda + \bar{X}L^\Lambda\right),\\
0 &= \tilde{k}^u_\Lambda\left(\bar{X}L^\Lambda - X\bar{L}^\Lambda\right)\,.
\end{aligned} \tag{4.56}$$

Using axial symmetry and the gauge choice for the vector fields, $A^\Lambda_r = A^\Lambda_\theta = 0$, it follows that $\nabla_r q^u = \partial_r q^u$ and $\nabla_\theta q^u = \partial_\theta q^u$, and these both vanish from the BPS conditions. Furthermore, the hypers cannot depend on t and ϕ, because this would induce such dependence also on the vector fields and complex scalars via the Maxwell equations (4.48). Thus the hypers cannot depend on any of the space-time coordinates, so they are constant. This will be important when we analyze the gravitino variation.

4.4.6 Gravitino variation

The gravitino equation reads

$$\nabla_\mu\varepsilon_A = -\mathrm{e}^{-i\alpha}\left(T^-_{\mu\rho}\gamma^\rho\delta_A{}^C + gS_{AB}\epsilon^{BC}\gamma_\mu\right)\gamma_0\varepsilon_C\,. \tag{4.57}$$

We study the integrability condition which follows from this equation. The explicit computation is presented in appendix B.3. The main result that we will first focus on is equation (B.9),

$$T^{-}_{\mu\nu} P^x_\Lambda L^\Lambda = 0 \,, \tag{4.58}$$

so that there are two separate cases: $T^{-}_{\mu\nu} = 0$ or $P^x_\Lambda L^\Lambda = 0$. We will study these two cases in different subsections.

Case 1: $T^{-}_{\mu\nu} = 0$

In this case the integrability conditions imply that the space-time is maximally symmetric with constant scalar curvature $P^x_\Lambda L^\Lambda$, as further explained in appendix B.3.1 . This corresponds either to Minkowski space when $P^x_\Lambda L^\Lambda = 0$, or AdS_4 when the scalar curvature is non-zero. Although there might be interesting half BPS solutions here, they will certainly not describe black holes.

Case 2: $P^x_\Lambda = 0$

The second case is $P^x_\Lambda L^\Lambda = 0$. We combine this identity with $P^x_\Lambda f^\Lambda_i = 0$ from (4.53). We now obtain

$$P^x_\Lambda \begin{pmatrix} \bar{L}^\Lambda \\ f^\Lambda_i \end{pmatrix} = 0 \,. \tag{4.59}$$

The matrix between brackets on the left hand side is invertible. This follows from the properties of special geometry, and we used it also in the characterization of the maximally supersymmetric vacua in [79]. We therefore conclude that $P^x_\Lambda = 0$. Next, we show that in this case we have enough information to solve the gravitino variation and give the metric functions.

From the definition (2.42) for $\nabla_\mu \varepsilon_A$, the quaternionic $Sp(1)$ connection $\omega_{\mu A}{}^B$ vanishes, as the hypers are constant by the arguments in section 4.4.5. Combining this with $P^x_\Lambda = 0$, we see that the gravitino variation (2.68) is precisely the same as in a theory without hypermultiplets and vanishing FI-terms. Thus our problem reduces to finding the most general solution of the gravitino variation in the ungauged theory. The answer, as proven by [70, 71], is that this is the well-known BLS solution [68] for stationary black holes (or naked singularities and monopoles in certain cases). Thus we can use the BLS solution, which in fact also solves the gaugino variation (4.54). We now only have to impose the Maxwell equations, which are not the same as in the BLS setup, due to the gauging of the hypermultiplets.

The sections are again described by functions H_Λ and $\widetilde{H}^\Lambda$, as in (4.8), although not all of them are harmonic. The metric and field strengths are given by (4.9), (4.13) and (4.14). In terms of our original description (4.46), we have that γ_{mn} is three-dimensional flat space and

$$\mathrm{e}^{\mathcal{K}} = 2|X|^2 \,. \tag{4.60}$$

In the ungauged case the Maxwell equations have no source term and the field strengths are thus described by harmonic functions, while now in our case they will be more complicated. We can then directly compare to the original BLS solution described in section 4.2.1 and see how the new equations of motion change it. At this point we have chosen the phase α in (4.51) to vanish, just as it does in the BLS solution. We can do this without any loss of generality since an arbitrary phase just appears in the intermediate results for the symplectic sections (4.8), but drops out of the physical quantities such as the metric and the field strengths.

We repeat that the Maxwell equations are given by (4.48),

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu G_{\Lambda\rho\sigma} = -g h_{uv} \tilde{k}^u_\Lambda \nabla^\mu q^v \,, \tag{4.61}$$

with $G_{\mu\nu}$ defined as in (4.11). Since our Bianchi identities are unmodified, and the same as in BLS, we again solve them by taking the $\tilde{H}^\Lambda$'s to be harmonic functions. The difference is in the Maxwell equations.

We plug in the identities from (4.56), (4.9) and (4.14). The components of (4.61) with $\mu \neq t$ are then automatically satisfied. The only non-trivial equation follows from $\mu = t$, and reads

$$\Box H_\Lambda = -2g^2 \mathrm{e}^{-\mathcal{K}} h_{uv} \tilde{k}^u_\Lambda \tilde{k}^v_\Sigma X^\Sigma \,.$$

Here, $\Box$ is again the three-dimensional Laplacian in flat space. The left hand side is real, and so is the right hand side, as a consequence of the last equation in (4.56) and the fact that we have chosen the phase in $X/|X|$ (see (4.51) to vanish. In other words, X is real, and therefore also $\tilde{k}^u_\Lambda X^\Lambda$ is real.

We furthermore have a consistency condition for the field strengths. The gauge potentials appear in (4.56), but also in (4.14), and these should lead to the same solution. These consistency conditions were not present in the ungauged case, since in that case there are no restrictions on F^Λ from the hyperino variation. The constraints can be easily derived from the integrability conditions of (4.56), and are given by

$$\begin{aligned}
\tilde{k}^u_\Lambda \tilde{H}^\Lambda &= 0 \, , \\
\tilde{k}^u_\Lambda F^\Lambda_{r\phi} &= -\tilde{k}^u_\Lambda \partial_r \left(\omega_\phi \mathrm{e}^{\mathcal{K}} X^\Lambda\right) , \\
\tilde{k}^u_\Lambda F^\Lambda_{\theta\phi} &= -\tilde{k}^u_\Lambda \partial_\theta \left(\omega_\phi \mathrm{e}^{\mathcal{K}} X^\Lambda\right) , \\
\tilde{k}^u_\Lambda F^\Lambda_{rt} &= -\tilde{k}^u_\Lambda \partial_r \left(\mathrm{e}^{\mathcal{K}} X^\Lambda\right) , \\
\tilde{k}^u_\Lambda F^\Lambda_{\theta t} &= -\tilde{k}^u_\Lambda \partial_\theta \left(\mathrm{e}^{\mathcal{K}} X^\Lambda\right) .
\end{aligned} \tag{4.62}$$

The first condition can always be satisfied as it merely implies that some of the harmonic functions $\widetilde{H}^\Lambda$ depend on the others (remember that the hypermultiplet scalars are constant, and therefore also the Killing vectors $\tilde{k}^u_\Lambda$). In more physical terms, this constraint decreases the number of magnetic charges by the rank of $\tilde{k}^u_\Lambda$. The other constraints have to be checked against the explicit form of the field strengths (4.13) and (4.14). This cannot be done generically and has to be checked once an explicit model is taken.

In section 4.3, we explained how the vanishing of $\tilde{k}^u_\Lambda L^\Lambda$ and $\tilde{k}^u_\Lambda A_\mu$ led to a BPS solution using spontaneous symmetry breaking. We can see that also from the equations of this section. When $\tilde{k}^u_\Lambda L^\Lambda = 0$, the right hand side of (4.4.6) is zero. This equation is then solved by harmonic functions H_Λ. Furthermore, as $\tilde{k}^u_\Lambda$ is constant, we can move it inside the derivatives in (4.62), so the right hand sides are zero. The left hand sides are zero as well, as $\tilde{k}^u_\Lambda F^\Lambda_{\mu\nu} = 0$. Finally, the condition $\tilde{k}^u_\Lambda \tilde{H}^\Lambda = 0$ is satisfied as $\tilde{k}^u_\Lambda L^\Lambda$ is already real.

4.5 Solutions with scalar hair

In this section, we search for solutions of the above BPS conditions that do not fall in the class described in section 4.3. They describe asymptotically flat black holes and would have non-trivial profiles for the massive vector and scalar fields, i.e. they would be distinguishable by the scalar hair degrees of freedom outside the black hole horizon. Remarkably, we could not find models with pure scalar hair solutions without the need to introduce some extra features, such as ghost modes or non-vanishing fermions. Below, we describe two examples of solutions that lead to at least one negative eigenvalue of the Kähler metric. We show that if we require strictly positive definite kinetic terms in the considered models, one cannot find scalar hair solutions, but only the ones described in section 4.3. It is of course hard to justify these ghost solutions physically. However, there have been cases in literature where this is not necessarily a problem, e.g. in Seiberg-Witten theory [104, 105] one has to perform duality transformation such that the kinetic terms remain positive definite. Whether a similar story holds in our case remains to be seen. If such duality transformations exist they will have to map the ghost black hole solutions of our abelian electrically gauged supergravity to proper black hole solutions, possibly of magnetically gauged supergravity. However, we cannot present any direct evidence for such a possibility.

4.5.1 Ghost solutions

Before we present our examples, we start with a general comment. We can obtain some more information from the Einstein equations. The trace of the Einstein equations reads

$$R = T^q + T^z + 4V \ , \tag{4.63}$$

where R is the Ricci scalar, and we have defined

$$T^q = -2h_{uv}\nabla_\mu q^u \nabla^\mu q^v \ , \qquad T^z = -2g_{i\bar{\jmath}}\partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}} \ . \tag{4.64}$$

Using the BPS conditions in (4.56), one quickly finds $T^q = -2V$. Furthermore, as $\partial_t z^i = 0$, we find[17] $T^z \geq 0$, and $V \geq 0$ by equations (2.72) and the condition $P^x_\Lambda = 0$. We therefore find

$$R = T^z + 2V \geq 0 \ , \tag{4.65}$$

as long as the metric $g_{i\bar{\jmath}}$ is positive definite. So the BPS conditions forbid the Ricci scalar R to become negative. In our examples below, the metric components will show some oscillatory behavior, as a consequence of the non-linear differential equation (4.4.6). Therefore, their derivatives, and hence the Ricci scalar, will oscillate between positive and negative values. This would contradict the positivity bound (4.65), unless the Kähler metric $g_{i\bar{\jmath}}$ contains regions in which it is not positive definite. We now discuss this in detail with two examples.

Quadratic prepotential

We start with two simple models, which have only one vector multiplet. They are described by the two prepotentials

$$F = -\frac{i}{2}\left(X^0X^0 \pm X^1X^1\right) . \tag{4.66}$$

These lead to the special Kähler metrics

$$g_{z\bar{z}} = \frac{\mp 1}{(1 \pm z\bar{z})^2} \ , \tag{4.67}$$

[17] Recall that our spacetime signature convention is $(+,-,-,-)$.

where $z = X^1/X^0$. With the upper sign, we therefore get a negative definite Kähler metric and the vector multiplet scalar is a ghost field. With the lower sign, we obtain a positive definite metric. We couple this to the universal hypermultiplet, and gauge isometry 5 from appendix D, using A^1_μ as the gauge field. The condition $P^x_\Lambda = 0$ fixes $\chi = \varphi = \sigma = 0$ and the only non-vanishing component of the Killing vectors is then $\tilde{k}^R_1 = 2Ra_1$, where a_1 is a constant.

From the relations (4.8) follows that $X^0 = \frac{1}{2}(H_0 - i\tilde{H}^0)$ and $X^1 = \frac{1}{2}(\pm H_1 - i\tilde{H}^1)$. The Kähler potential (2.19) is then

$$\mathrm{e}^{-\mathcal{K}} = 2\left(X^0\bar{X}^0 \pm X^1\bar{X}^1\right). \tag{4.68}$$

As we do not use A^0_μ for the gauging, X^0 remains harmonic, such that even if the solution for X^1 is considerably different, we still have hope of producing a black hole by having X^1 as a small perturbation of the leading term X^0 in the metric function $\mathrm{e}^{-\mathcal{K}}$. For simplicity, we restrict ourself to the spherically symmetric single-centered case, so now our constraints (4.62) lead to $\widetilde{H}^1 = 0$ and $\tilde{k}^u_\Lambda F^\Lambda_{rt} = -\tilde{k}^u_\Lambda \partial_r\left(\mathrm{e}^{\mathcal{K}} X^\Lambda\right)$. The latter eventually implies that $\widetilde{H}^0$ is constant. Since we can absorb this constant by rescaling H_0, we will set $\tilde{H}^0 = 0$. Thus we are left with $2X^0 = H_0 = \sqrt{2} + \frac{q_0}{r}$ ($q_0 > 0$), where we set the constant of the harmonic function to $\sqrt{2}$ to obtain canonically normalized Minkowski space as $r \to \infty$.

The metric is given by (4.9), where

$$\mathrm{e}^{-\mathcal{K}} = \frac{1}{2}\left(\left(\sqrt{2} + \frac{q_0}{r}\right)^2 \pm H_1^2\right). \tag{4.69}$$

The only undetermined function is H_1, which is subject to the only equation left to be satisfied, (4.4.6), which in this case is given by

$$\Box H_1 = \mp \mathrm{e}^{-\mathcal{K}} H_1 = \mp\frac{1}{2}\left(\left(\sqrt{2} + \frac{q_0}{r}\right)^2 \pm H_1^2\right) H_1, \tag{4.70}$$

after setting $g|\tilde{k}| = 1$. Besides the trivial solution $H_1 = 0$ (belonging to the class solutions from section 4.3), we could not find an analytic solution to these equations. We can analyze the differential equation as $r \to 0$ and $r \to \infty$. As $r \to \infty$, we require $\mathrm{e}^{-\mathcal{K}} \to 1$, to obtain flat space at infinity. Likewise, we require, as $r \to 0$, that $\mathrm{e}^{-\mathcal{K}} \to q^2 r^{-2}$, to obtain $AdS_2 \times S^2$ at the horizon. The constant q (which is not necessarily equal to q_0) determines the (equal) radii of AdS_2 and S^2. If we solve (4.70) for large values of r, we have to solve $\Box H_1 = \mp H_1$; for small values of r we have to solve $\Box H_1 = \mp\frac{1}{2}q^2 r^{-2} H_1$.

- With the upper sign (the ghost model), we find the general solution

$$H_1 = A\frac{\cos(r)}{r} + B\frac{\sin(r)}{r} \,, \qquad r \to \infty \,, \tag{4.71}$$

$$H_1 = Cr^{-\frac{1}{2}-\frac{1}{2}\sqrt{1-4q^2}} + Dr^{-\frac{1}{2}+\frac{1}{2}\sqrt{1-4q^2}} \,, \qquad r \to 0 \,. \tag{4.72}$$

  As long as $4q^2 < 1$, all the asymptotics are fine.

- With the lower sign (the non-ghost model), we find the general solution

$$H_1 = A\frac{\mathrm{e}^{-r}}{r} + B\frac{\mathrm{e}^{r}}{r} \,, \qquad r \to \infty \,, \tag{4.73}$$

$$H_1 = Cr^{-\frac{1}{2}-\frac{1}{2}\sqrt{1+4q^2}} + Dr^{-\frac{1}{2}+\frac{1}{2}\sqrt{1+4q^2}} \,, \qquad r \to 0 \,. \tag{4.74}$$

  When B is nonzero, this violates the boundary condition that $\mathrm{e}^{-\mathcal{K}} \to 1$ as $r \to \infty$, so we have to set $B = 0$. Likewise, we have to set $C = 0$. We will now prove that imposing such boundary conditions implies $H_1 = 0$. To do this, we use the identity

$$\int_0^\infty (rH_1)\partial_r^2(rH_1)\,\mathrm{d}r = -\int_0^\infty \partial_r(rH_1)\partial_r(rH_1)\,\mathrm{d}r + (rH_1)\partial_r(rH_1)\Big|_{r=0}^{r=\infty} \,. \tag{4.75}$$

  Using (4.73) and (4.74) one finds that, for $B = C = 0$, the boundary term vanishes. On the left-hand side, we use (4.70), and we obtain (using $\Box H_1 = r^{-1}\partial_r^2(rH_1)$)

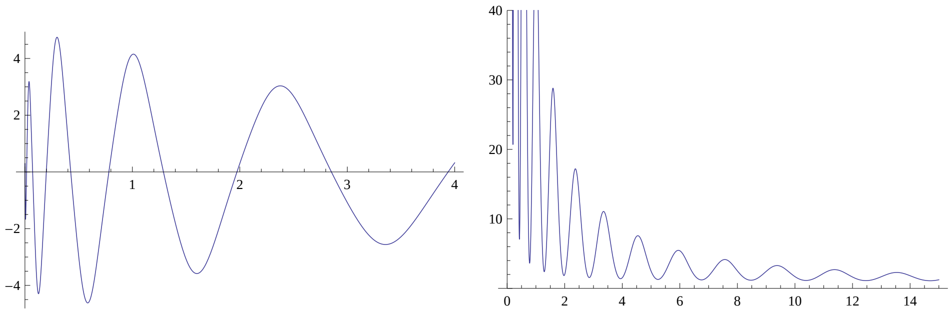
$$\int_0^\infty H_1\mathrm{e}^{-\mathcal{K}}H_1\,\mathrm{d}r = -\int_0^\infty \partial_r(rH_1)\partial_r(rH_1)\,\mathrm{d}r \,. \tag{4.76}$$

  The left-hand side is non-negative, whereas the right-hand side is non-positive, so this proves $H_1 = 0$. This argument can easily be repeated for solutions with only axial symmetry.

We can plot the solution with the upper sign numerically with generic starting conditions, and the result is shown on figure 4.1(a). The metric function gets oscillatory perturbations, while having its endpoints fixed to the desired values as shown on figure 4.1(b).

The function H_1 approaches zero as $r \to \infty$ in an oscillatory fashion, which can be seen in figure 4.1(a). To investigate the behavior near the horizon at $r = 0$, we also checked that rH_1 approaches zero, and hence H_1 diverges slower than $1/r$. Both are in agreement with the asymptotic analysis above.

The numerics further show that the metric function for negative values of r yields the expected singularity at $r = -\frac{q_0}{\sqrt{2}}$. We conclude that this is indeed a black hole spacetime, having one electric charge q_0, and the fluctuations around the usual form of the metric are due to the effect of the abelian gauging of the hypermultiplet.



(a) The function $z = H_1/H_0$. (b) The Kähler potential $\mathrm{e}^{-\mathcal{K}}$.

Figure 4.1: Plots of the solution to the differential equation (4.70) for $q_0 = 1$, using boundary conditions $H_1(1) = 10$ and $H_1'(1) = 1$. The scalar z approaches zero at the horizon at $r = 0$, and the Kähler potential $\mathrm{e}^{-\mathcal{K}}$ approaches 1 as $r \to \infty$.

Let us now try to give a bit more physical interpretation of this new black hole spacetime. After more careful inspection of the solution, we see that at the horizon and asymptotically at infinity we again have supersymmetry enhancement, since the vector multiplet scalars are fixed to a constant value. It is interesting that the electric charge, associated to the broken gauge symmetry vanishes at the horizon, i.e. the black hole itself is not charged with q_1 exactly as in the normal case without ghosts. Yet there is a non-zero charge density for this charge everywhere in the spacetime outside the black hole, which is the qualitatively new feature of the ghost solutions. Clearly the fact that there is non-vanishing charge density everywhere in space-time does not change the asymptotic behavior, but it seems that it is physically responsible for the ripples that can be observed in the metric function on figure 4.1(b) (of course this is all related to the fact that we have propagating ghost fields). We should note that these are not the first rippled black hole solutions, similar behavior is found in the higher derivative ungauged solutions, e.g. in [106], where also one finds ghost modes in the resulting theory. The detailed analysis in section 4 of [106] holds in our case, i.e. the main physical feature of the ripples is that gravitational force changes from attractive to repulsive in some space-time points.

Cubic prepotential

The example above shows already the general qualitatively new features of this class of black holes with ghost fields, but is still not interesting from a string theory point of view, since Calabi-Yau compactifications lead to cubic prepotentials of the form

$$F = -\frac{\mathcal{K}_{ijk} X^i X^j X^k}{6X^0} \ . \tag{4.77}$$

The simplest case one can consider is the STU model of section 4.3.2. We coupled it to the universal hypermultiplet with a single gauged isometry and found it impossible to produce any new solutions. However, other choices of $\mathcal{K}_{ijk}$ allow for interesting numerical solutions of (4.4.6). For this purpose we consider a relatively simple model with three vector multiplets:

$$F = \frac{(X^1)^3 - (X^1)^2 X^2 - X^1 (X^3)^2}{2X^0} \ . \tag{4.78}$$

We again use the universal hypermultiplet and gauge the same isometry as before, but we now use only A^3_μ for our gauging. Again, the condition $P^x_\Lambda = 0$ fixes $\chi = \varphi = \sigma = 0$, and the only non-vanishing component of the Killing vector is $\tilde{k}^R_3 = 2Ra_3$. In parts of moduli space this model exhibits proper Calabi-Yau behavior, i.e. the Kähler metric is positive definite, but there are regions where $g_{i\bar{\jmath}}$ has negative eigenvalues (or $\mathrm{e}^{-\mathcal{K}}$ becomes negative). There is no general expression for this so-called positivity domain; one has to analyze an explicit model to find the conditions.

For simplicity, we set $\tilde{H}^i = H_0 = 0$, so the non-vanishing functions are H_i and $\tilde{H}^0$. Inverting (4.8) we obtain for the Kähler potential

$$\mathrm{e}^{-\mathcal{K}} = \sqrt{2H_2}\sqrt{\tilde{H}^0}\left(H_1 + H_2 + \frac{H_3^2}{4H_2}\right) \ . \tag{4.79}$$

We see that, as is commonly encountered in these models, one has to choose the signs of the functions H_i and $\tilde{H}^0$ such that this gives a real and positive quantity. With these we satisfy all conditions in (4.62) and are left to solve (4.4.6) that explicitly reads:

$$\Box H_3 = -a_3^2 \tilde{H}^0 \left(H_1 + H_2 + \frac{H_3^2}{4H_2}\right) H_3 \ , \tag{4.80}$$

where $\tilde{H}^0$, H_1 and H_2 are harmonic functions, and we have set $g|\tilde{k}| = 1$ for convenience.

We impose the same boundary conditions, so as $r \to \infty$, we require $\mathrm{e}^{-\mathcal{K}} \to 1$, to obtain flat space at infinity. Likewise, we require, as $r \to 0$, that $\mathrm{e}^{-\mathcal{K}} \to q^2 r^{-2}$, to obtain $AdS_2 \times S^2$ at the horizon. Using (4.79), we then find that we have to solve

$$\Box H_3 = -a_3^2 q^2 r^{-2} H_3 \ , \qquad \text{as } r \to 0 \ , \tag{4.81}$$

$$\Box H_3 = -a_3^2 c^2 H_3 \ , \qquad \text{as } r \to \infty \ , \tag{4.82}$$

where c^2 is also a constant, specified by the asymptotics of $\tilde{H}^0$, H_1 and H_2. We therefore again find

$$H_3 = A\frac{\cos(a_3 cr)}{r} + B\frac{\sin(a_3 cr)}{r} \,, \qquad \text{as } r \to \infty \,. \tag{4.83}$$

These functions are oscillating; therefore the Kähler potential (4.79) will also oscillate. This causes the Ricci scalar to become negative, which is in violation of the bound (4.65). Therefore, there is always a negative eigenvalue of the metric, corresponding to a ghost mode.

We could only find a numerical solution to this equation, and the results are qualitatively the same as the ones on figure 4.1, so we will omit them for this model.

It is therefore possible to find black hole solutions in these Calabi-Yau models, but they do contain regions in which scalars become ghost-like.

4.5.2 Fermionic hair

There is a different way of generating scalar hair with properly normalized positive-definite kinetic terms. As such, we can thereby avoid the ghost-like behavior of the previously discussed examples. The idea is simple and works for any solution that breaks some supersymmetry. By acting with the broken susy generators on a bosonic solution, we will turn on the fermionic fields to yield the fermionic zero modes. These fermionic zero modes solve the linearized equations of motion and produce fermionic hair. In turn, the fermionic hair sources the equations of motion for the bosonic field, and in particular, the scalar field equations will have a source term which is bilinear in the fermions. The solution of this equation produces scalar hair and can be found explicitly by iterating again with the broken supersymmetries. This iteration procedure stops after a finite number of steps and produces a new solution to the full non-linear equations of motion. By starting with a BPS black hole solution of the type discussed in section 4.3, one therefore produces new solutions with both fermionic and scalar hair. For a discussion on this for black holes in ungauged supergravity, see [107].

The explicit realization of this idea is fairly complicated since it requires to explicitly find the Killing spinors preserving supersymmetry. This can sometimes be done also just by considering the possible bosonic and fermionic deformations of the theory, as done in e.g. [108, 109] for black holes in ungauged supergravity. The extension of this hair-analysis to gauged supergravities would certainly be an interesting extension of our work.

Chapter 5

New potentials from Scherk-Schwarz reductions

5.1 Introduction

Scherk-Schwarz reductions [110, 111] provide a way to construct gauged supergravities from higher dimensional ungauged ones. They were quickly mentioned in the introduction in section 2.5.2 and we will elaborate on this.

Suppose one studies a (D+1)-dimensional complex scalar field ϕ and assumes that space-time is a D-dimensional space-time times a circle of radius R. One can then expand the field as

$$\phi(x, z) = \sum_{n=-\infty}^{\infty} \phi_n(x) \mathrm{e}^{inz/R} \,, \tag{5.1}$$

where x is the D-dimensional coordinate and z is a coordinate on a circle. Scherk and Schwarz noticed that this is not the most general ansatz possible: if the theory is invariant under the global transformations $\phi \to \mathrm{e}^{i\alpha}\phi$, one can extend this ansatz to

$$\phi(x, z) = \mathrm{e}^{iMz} \sum_{n=-\infty}^{\infty} \phi_n(x) \mathrm{e}^{inz/R} \,. \tag{5.2}$$

This is no longer single-valued on the $(D + 1)$-dimensional spacetime, as we have that $\phi(x, z + 2\pi R) = \mathrm{e}^{2\pi i M R}\phi(x, z)$, but we can remove the offending phase by acting with the global symmetry. If we now keep M fixed and take the limit $R \to 0$, we can ignore all terms but S_0, which will be a D-dimensional field of mass M. In this way we obtain the example of section 2.5.2.

Scherk-Schwarz reductions typically lead to semi-positive definite potentials for the scalar fields with local minima that can describe Minkowski or de Sitter vacua. Such models have been studied intensely over recent times in the context of compactifications of string- and M-theory, with and without fluxes. For some background material and earlier references, see e.g. [112–123].

Actually, there are two classes of Scherk-Schwarz reductions: the case of reductions over a circle with a duality twist along the circle, as outlined above, and the case of twisted tori (or twistings of other manifolds). In the latter setting, one expands in forms that are not closed. For example, if we expand the ten-dimensional form $\hat{B}_2$ as

$$\hat{B}_2(x, y) = b^i(x)\omega_i(y)\,, \tag{5.3}$$

but now (in contrast with the Calabi-Yau compactifications of section 2.5.3) the forms ω_i are not closed. We then find

$$\hat{H}_3 = \mathrm{d}b^i \wedge \omega_i(y) + b^i \mathrm{d}\omega_i\,. \tag{5.4}$$

Integrating now the ten-dimensional kinetic term $\hat{H}^3 \wedge \star\hat{H}^3$ over the internal space $\mathcal{X}$ gives

$$V = b^i b^j \int_{\mathcal{X}} \omega_i \wedge \star\omega_j\ . \tag{5.5}$$

This is a potential for the moduli fields b^i; the problem is now computing it. In suitable models, one can expand $\mathrm{d}\omega_i$ again in a basis of harmonic three-forms, and one can actually perform the integrations.

Sometimes, these two classes are related to each other, and reductions with duality twists can be understood in terms of compactifications on twisted tori. For a discussion on this, see e.g. [121]. This equivalence will also play a role in our investigation, although we focus mostly on the reductions with a duality twist.

5.2 M-theory on Calabi-Yau manifolds

In this section, we review aspects of compactifications of eleven-dimensional supergravity on Calabi-Yau threefolds. Almost all material in this section is known, and collected from various places in the literature, which we refer to below. We give this review to recall some of the duality symmetries in five dimensions, and to set our notation for subsequent sections. The reader who is very familiar with five-dimensional matter coupled to $\mathcal{N} = 2$ supergravity might skip this section and go straight to the next section where we start the Scherk-Schwarz reduction to four dimensions.

The low-energy limit of M-theory can be described in terms of eleven-dimensional supergravity. In form-notation, the bosonic part of this action reads [124]

$$\hat{S} = \frac{1}{2}\int \left(\hat{R}\star 1 - \frac{1}{2}\,\hat{F}_4 \wedge \star\hat{F}_4 - \frac{1}{6}\,\hat{F}_4 \wedge \hat{F}_4 \wedge \hat{C}_3\right)\ . \tag{5.6}$$

Here, $\hat{R}$ denotes the eleven-dimensional Ricci scalar and $\star$ stands for the eleven-dimensional Hodge star operator. Furthermore, $\hat{C}_3$ is a three-form potential, $\hat{F}_4 = \mathrm{d}\hat{C}_3$ denotes the corresponding field strength and we have set the eleven-dimensional Planck constant to one.

In the following, we compactify M-theory on a simply-connected Calabi-Yau threefold $\mathcal{X}$, which leads to a supergravity theory in five dimensions with eight (real) supercharges [125].

5.2.1 Calabi-Yau manifolds and dimensional reduction

Notation

We begin by establishing some notation for the Calabi-Yau three-fold $\mathcal{X}$. Let us denote a basis of harmonic $(1,1)$-forms on $\mathcal{X}$ by

$$\omega_A \ , \qquad\qquad A = 1, \ldots, h^{1,1} \ , \tag{5.7}$$

where here and in the following $h^{p,q}$ denote the Hodge numbers of the Calabi-Yau threefold. The triple intersection numbers for $\mathcal{X}$ are defined by

$$\mathcal{K}_{ABC} = \int_{\mathcal{X}} \omega_A \wedge \omega_B \wedge \omega_C \ . \tag{5.8}$$

For the third co-homology group $H^3(\mathcal{X})$ we denote a real basis by

$$\left\{\alpha_K, \beta^L\right\} \ , \qquad\qquad K, L = 0, \ldots h^{2,1} \ , \tag{5.9}$$

which is chosen such that

$$\int_{\mathcal{X}} \alpha_K \wedge \beta^L = \delta_K{}^L \ , \qquad \int_{\mathcal{X}} \alpha_K \wedge \alpha_L = 0 \ , \qquad \int_{\mathcal{X}} \beta^K \wedge \beta^L = 0 \ . \tag{5.10}$$

The Calabi-Yau threefold is endowed with a Kähler form J and a holomorphic three-form Ω. In terms of the bases (5.7) and (5.9), these can be decomposed in the following way

$$J = v^A \omega_A \ , \qquad\qquad \Omega = Z^K \alpha_K - G_K \beta^K \ , \tag{5.11}$$

where the expansion coefficients v^A are real. The functions (Z^K, G_K) are the holomorphic sections of special geometry and depend on the complex structure moduli z^r of the Calabi-Yau manifold, where $r = 1, \ldots, h^{2,1}$. The volume of $\mathcal{X}$ can be expressed in terms of the Kähler form J as follows

$$\mathcal{V} = \frac{1}{3!} \int_{\mathcal{X}} J \wedge J \wedge J = \frac{1}{3!} \, \mathcal{K}_{ABC} v^A v^B v^C \ . \tag{5.12}$$

Ansatz for the compactification

To perform the dimensional reduction of the action (5.6), we make the following ansatz for the eleven-dimensional metric

$$\hat{G}_{MN} = \begin{pmatrix} \tilde{g}_{\tilde{\mu}\tilde{\nu}} & 0 \\ 0 & G_{mn} \end{pmatrix} , \qquad \begin{aligned} \tilde{\mu}, \tilde{\nu} \ &= 0, \ldots, 4 , \\ m, n \ &= 1, \ldots, 6 , \end{aligned} \tag{5.13}$$

where $\tilde{g}_{\tilde{\mu}\tilde{\nu}}$ denotes a five-dimensional metric and G_{mn} is the metric of a Calabi-Yau threefold. For the three-form potential, we chose the expansion

$$\hat{C}_3 = \tilde{c}_3 + A^A \wedge \omega_A + C_3 , \qquad C_3 = \sqrt{2}\,\xi^K \alpha_K - \sqrt{2}\,\tilde{\xi}_K \beta^K , \tag{5.14}$$

with $\tilde{c}_3(\tilde{x}^{\tilde{\mu}})$ a three-form in five dimensions which depends solely on the five-dimensional coordinates $\tilde{x}^{\tilde{\mu}}$. Similarly, $A^A(\tilde{x}^{\tilde{\mu}})$ are five-dimensional one-forms while $\xi^K(\tilde{x}^{\tilde{\mu}})$ and $\tilde{\xi}_K(\tilde{x}^{\tilde{\mu}})$ are five-dimensional scalars. We have separated the pure Calabi-Yau part C_3 from from $\tilde{c}_3$ and A^A for later convenience.

Dimensional reduction to five-dimensional supergravity

Let us begin with the dimensional reduction of the eleven-dimensional Ricci scalar appearing in the action (5.6). We first decompose (up to total derivatives)

$$\begin{aligned} \frac{1}{2} \int \hat{R} \star 1 = \frac{1}{2} \int d^{11}\hat{x}\, \sqrt{\hat{G}} \Big[R_{(5)} + R_{\mathcal{X}} &- \frac{1}{4} \big(G^{ab}\partial_{\tilde{\mu}} G_{bc}\big)\big(G^{cd}\partial^{\tilde{\mu}} G_{da}\big) \\ &+ \frac{1}{4} \big(G^{ab}\partial_{\tilde{\mu}} G_{ab}\big)\big(G^{cd}\partial^{\tilde{\mu}} G_{cd}\big) \Big], \end{aligned} \tag{5.15}$$

where $R_{(5)}$ denotes the Ricci scalar computed from the five-dimensional metric $\tilde{g}_{\tilde{\mu}\tilde{\nu}}$, $R_{\mathcal{X}} = 0$ is the Ricci scalar of the Calabi-Yau manifold $\mathcal{X}$ and $\partial_{\tilde{\mu}}$ are derivatives with respect to the five-dimensional coordinates $\tilde{x}^{\tilde{\mu}}$. We then split the Calabi-Yau metric G_{mn} into a constant background part $\mathring{G}_{mn}$ and fluctuations around this background

$$G_{mn} = \mathring{G}_{mn} + \delta G_{mn} . \tag{5.16}$$

Following [126–129], the fluctuations (in a complex basis with holomorphic indices a, b and anti-holomorphic indices $\overline{a}, \overline{b}$) can be expressed as

$$\begin{aligned} \delta G_{a\bar{b}} \ &= -i\, \delta v^A \, (\omega_A)_{a\bar{b}} , & A \ &= 1, \ldots, h^{1,1} , \\ \delta G_{ab} \ &= \frac{\mathcal{V}}{\int_{\mathcal{X}} \Omega \wedge \overline{\Omega}} \, \overline{z}^r \, (\overline{\chi}_r)_{a\bar{a}\bar{b}} \, \Omega^{\bar{a}\bar{b}}{}_b , & r \ &= 1, \ldots, h^{2,1} , \end{aligned} \tag{5.17}$$

where δv^A are fluctuations around the background value $\mathring{v}^A$ of the expansion parameters of the Kähler form given in (5.11). In the following, these will be combined into

$$v^A = \mathring{v}^A + \delta v^A \ . \tag{5.18}$$

Furthermore, χ_r denotes a basis of harmonic $(2,1)$-forms on $\mathcal{X}$, and the holomorphic three-form Ω was introduced in (5.11). The volume $\mathcal{V}$ of the Calabi-Yau three-fold was defined in equation (5.12). At lowest order in the fluctuations, χ_r in δG_{ab} does not depend on the five-dimensional coordinates whereas z^r (as well as δv^A in $\delta G_{a\bar{b}}$) are functions of $\tilde{x}^{\tilde{\mu}}$. We also note the relation

$$\mathring{G}^{a\bar{b}}(\omega_A)_{a\bar{b}} = \frac{i}{2}\,\frac{\mathcal{K}_{ABC}v^B v^C}{\mathcal{V}} \ , \tag{5.19}$$

and we define and compute

$$g_{(5)AB} \equiv \frac{1}{4\mathcal{V}}\int_{\mathcal{X}} \omega_A\wedge\star\omega_B = -\frac{1}{4\mathcal{V}}\left(\mathcal{K}_{ABC}v^C - \frac{\mathcal{K}_{ACD}v^C v^D\mathcal{K}_{BEF}v^E v^F}{4\mathcal{V}}\right), \tag{5.20}$$

as well as

$$G_{r\bar{s}} \equiv -\frac{\int_{\mathcal{X}}\chi_r\wedge\overline{\chi}_s}{\int_{\mathcal{X}}\Omega\wedge\overline{\Omega}} \ , \qquad\qquad r,s = 1,\ldots,h^{2,1} \ . \tag{5.21}$$

Up to second order in the fluctuations δG, we then find

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^{4,1}\times\mathcal{X}}\hat{R}\star 1 = \int_{\mathbb{R}^{4,1}}\Bigg[\ &\frac{\mathcal{V}}{2}\,R_{(5)}\star_5 1 - \mathcal{V}\,g_{(5)AB}\mathrm{d}v^A\wedge\star_5\mathrm{d}v^B - \mathcal{V}\,G_{r\bar{s}}\mathrm{d}z^r\wedge\star_5\mathrm{d}z^{\bar{s}} \\ &+\frac{\mathcal{V}}{2}\,\mathrm{d}\log\mathcal{V}\wedge\star_5\mathrm{d}\log\mathcal{V}\ \Bigg] \ . \end{aligned}\tag{5.22}$$

Let us next turn to the kinetic term for the three-form potential $\hat{C}_3$. Using the ansatz (5.14), we compute

$$\begin{aligned}-\frac{1}{4}\int_{\mathbb{R}^{4,1}\times\mathcal{X}}\hat{F}_4\wedge\star\hat{F}_4 = -\frac{1}{4}\int_{\mathbb{R}^{4,1}}\Bigg[\ &\mathcal{V}\,\mathrm{d}\tilde{c}_3\wedge\star_5\mathrm{d}\tilde{c}_3 + 4\mathcal{V}\,g_{(5)AB}\,\mathrm{d}A^A\wedge\star_5\mathrm{d}A^B \\ &-2\left(\mathrm{Im}\,\mathcal{M}\right)^{-1\,KL}\left(\mathrm{d}\tilde{\xi}_K - \mathcal{M}_{KN}\mathrm{d}\xi^N\right)\wedge\star_5\left(\mathrm{d}\tilde{\xi}_L - \overline{\mathcal{M}}_{LM}\mathrm{d}\xi^M\right)\Bigg] \ .\end{aligned}\tag{5.23}$$

Here, we have employed the period matrix $\mathcal{M}_{KL}$ which satisfies [130, 131]

$$\begin{aligned}\int_{\mathcal{X}}\alpha_K\wedge\star_6\alpha_L \ &= \ \left[-\left(\mathrm{Im}\,\mathcal{M}\right) - \left(\mathrm{Re}\,\mathcal{M}\right)\left(\mathrm{Im}\,\mathcal{M}\right)^{-1}\left(\mathrm{Re}\,\mathcal{M}\right)\right]_{KL} \ , \\ \int_{\mathcal{X}}\alpha_K\wedge\star_6\beta^L \ &= \ \left[-\left(\mathrm{Re}\,\mathcal{M}\right)\left(\mathrm{Im}\,\mathcal{M}\right)^{-1}\right]_K^{\ \ L} \ , \\ \int_{\mathcal{X}}\beta^K\wedge\star_6\beta^L \ &= \ \left[-\left(\mathrm{Im}\,\mathcal{M}\right)^{-1}\right]^{KL} \ ,\end{aligned}\tag{5.24}$$

with matrix products understood and $\{\alpha_K, \beta^L\}$ denoting the basis introduced in (5.9). For the topological term in the action (5.6) we compute (up to total derivatives)

$$\begin{aligned} & -\frac{1}{12}\int_{\mathbb{R}^{4,1}\times\mathcal{X}} \hat{F}_4 \wedge \hat{F}_4 \wedge \hat{C}_3 \\ = & -\frac{1}{12}\int_{\mathbb{R}^{4,1}} \Big[\, 6\,\mathrm{d}\tilde{c}_3 \wedge \Big(\xi^K \mathrm{d}\tilde{\xi}_K - \tilde{\xi}_K \mathrm{d}\xi^K\Big) + \mathcal{K}_{ABC}\,\mathrm{d}A^A \wedge \mathrm{d}A^B \wedge A^C \,\Big] \,. \end{aligned} \tag{5.25}$$

To dualize $\tilde{c}_3$ to a scalar field, we introduce a Lagrange multiplier a for $d\tilde{c}_3$ and add this term to the combined action (5.23) and (5.25). After solving the equations of motion for $\tilde{c}_3$ and substituting them back into the action, the terms involving $\tilde{c}_3$ become

$$\begin{aligned} & -\frac{1}{4}\int_{\mathbb{R}^{4,1}} \mathcal{V}\, d\tilde{c}_3 \wedge \star_5 \mathrm{d}\tilde{c}_3 + 2\mathrm{d}\tilde{c}_3 \wedge (\tilde{\xi}_K \mathrm{d}\xi^K - \xi^K \mathrm{d}\tilde{\xi}_K) + 2\mathrm{d}\tilde{c}_3 \wedge \mathrm{d}a \\ = & -\frac{1}{4}\int_{\mathbb{R}^{4,1}} \frac{1}{\mathcal{V}} \Big(\mathrm{d}a + \xi^K \mathrm{d}\tilde{\xi}_K - \tilde{\xi}_K \mathrm{d}\xi^K\Big) \wedge \star_5 \Big(\mathrm{d}a + \xi^L \mathrm{d}\tilde{\xi}_L - \tilde{\xi}_L \mathrm{d}\xi^L\Big) \,. \end{aligned} \tag{5.26}$$

Finally, we combine the above expressions and perform a Weyl rescaling $\tilde{g}_{\tilde{\mu}\tilde{\nu}} \to \mathcal{V}^{-\frac{2}{3}} \tilde{g}_{\tilde{\mu}\tilde{\nu}}$ of the five-dimensional metric to arrive at

$$\begin{aligned} S_5 = \int_{\mathbb{R}^{4,1}} \Big[& + \frac{1}{2} R_{(5)} \star_5 1 - \frac{1}{6}\,\mathrm{d}\log\mathcal{V} \wedge \star_5 \mathrm{d}\log\mathcal{V} - g_{(5)AB} \mathrm{d}v^A \wedge \star_5 \mathrm{d}v^B \\ & - G_{r\bar{s}} \mathrm{d}z^r \wedge \star_5 \mathrm{d}z^{\bar{s}} - \mathcal{V}^{\frac{2}{3}}\, g_{(5)AB}\, \mathrm{d}A^A \wedge \star_5 \mathrm{d}A^B \\ & - \frac{1}{4\mathcal{V}^2} \Big(\mathrm{d}a + \xi^K \mathrm{d}\tilde{\xi}_K - \tilde{\xi}_K \mathrm{d}\xi^K\Big) \wedge \star_5 \Big(\mathrm{d}a + \xi^L \mathrm{d}\tilde{\xi}_L - \tilde{\xi}_L \mathrm{d}\xi^L\Big) \\ & + \frac{1}{2\mathcal{V}} \big(\mathrm{Im}\,\mathcal{M}\big)^{-1\,KL} \Big(\mathrm{d}\tilde{\xi}_K - \mathcal{M}_{KN} \mathrm{d}\xi^N\Big) \wedge \star_5 \Big(\mathrm{d}\tilde{\xi}_L - \overline{\mathcal{M}}_{LM} \mathrm{d}\xi^M\Big) \\ & - \frac{1}{12}\, \mathcal{K}_{ABC}\, \mathrm{d}A^A \wedge \mathrm{d}A^B \wedge A^C \;\Big] \,. \end{aligned} \tag{5.27}$$

As it turns out, the field $\mathcal{V}$ belongs to a hypermultiplet and so (5.27) contains terms mixing hyper- and vector multiplets. To make contact with the standard formulation of $\mathcal{N} = 2$ supergravity in five dimensions, we introduce new fields

$$\nu^A = \mathcal{V}^{-\frac{1}{3}}\, v^A \,. \tag{5.28}$$

By definition, these satisfy $\frac{1}{6}\,\mathcal{K}_{ABC}\nu^A\nu^B\nu^C = 1$ and so there are $h^{1,1}$ scalar fields ν^A subject to one constraint, as well as the independent field $\mathcal{V}$. We then arrive at the following form of the five-dimensional action [132, 133]

$$
\begin{aligned}
S_5 = \int_{\mathbb{R}^{4,1}} \Big[& + \frac{1}{2} R_{(5)} \star_5 1 - \frac{1}{4} \mathrm{d}\log\mathcal{V} \wedge \star_5 \mathrm{d}\log\mathcal{V} + \frac{1}{4} \mathcal{K}_{ABC}\nu^C \mathrm{d}\nu^A \wedge \star_5 \mathrm{d}\nu^B \\
& + \frac{1}{4}\Big(\mathcal{K}_{ABC}\nu^C - \frac{1}{4}\mathcal{K}_{ACD}\nu^C\nu^D\mathcal{K}_{BEF}\nu^E\nu^F\Big)\, \mathrm{d}A^A \wedge \star_5 \mathrm{d}A^B \\
& - \frac{1}{12}\, \mathcal{K}_{ABC}\, \mathrm{d}A^A \wedge \mathrm{d}A^B \wedge A^C - G_{r\bar{s}} \mathrm{d}z^r \wedge \star_5 \mathrm{d}z^{\bar{s}} \\
& - \frac{1}{4\mathcal{V}^2}\Big(\mathrm{d}a + \xi^K \mathrm{d}\tilde{\xi}_K - \tilde{\xi}_K \mathrm{d}\xi^K\Big) \wedge \star_5 \Big(\mathrm{d}a + \xi^L \mathrm{d}\tilde{\xi}_L - \tilde{\xi}_L \mathrm{d}\xi^L\Big) \\
& + \frac{1}{2\mathcal{V}} (\mathrm{Im}\,\mathcal{M})^{-1\,KL} \Big(\mathrm{d}\tilde{\xi}_K - \mathcal{M}_{KN}\mathrm{d}\xi^N\Big) \wedge \star_5 \Big(\mathrm{d}\tilde{\xi}_L - \overline{\mathcal{M}}_{LM}\mathrm{d}\xi^M\Big) \Big] \, .
\end{aligned} \tag{5.29}
$$

The first term in this expression is the five-dimensional Ricci scalar, $\mathcal{V}$ is the volume of the Calabi-Yau manifold and $\mathcal{K}_{ABC}$ denote the triple intersection numbers defined in (5.8). Furthermore, the scalars ν^A are related to the expansion coefficients v^A of the Kähler form J by a rescaling with the volume (see equation (5.28)), such that they satisfy

$$
\frac{1}{6}\, \mathcal{K}_{ABC}\nu^A\nu^B\nu^C = 1 \, . \tag{5.30}
$$

Thus, there are $h^{1,1} - 1$ scalar degrees of freedom in these fields. Accordingly, the vector fields A^A comprise the graviphoton and $h^{1,1} - 1$ additional vector fields to form five-dimensional vector multiplets. The remaining scalar fields $\{\mathcal{V}, a, z^r, \bar{z}^r, \xi^K, \tilde{\xi}_K\}$ form $h^{2,1} + 1$ hypermultiplets that parametrize a quaternion-Kähler manifold [125].

5.2.2 Symmetries of the five-dimensional theory

Symmetries in the vector multiplet sector

We begin our discussion on the symmetries of (5.29) with the vector multiplets. Besides the usual gauge invariances acting on the vector potentials, there are additional symmetries in the scalar sector. In particular, the scalars in the vector multiplets parametrize a so-called real special geometry, whose isometries have been studied in [132]. As explained in [134], not all isometries extend to symmetries of the full Lagrangian, but only transformations

$$
\delta\nu^A = M^A{}_B \nu^B \, , \qquad\qquad \delta A^A = M^A{}_B A^B \, , \tag{5.31}
$$

where the constant, real matrix $M^A{}_B$ is subject to the constraint

$$
0 = \mathcal{K}_{D(AB} M^D{}_{C)} = \mathcal{K}_{DBC} M^D{}_A + \mathcal{K}_{ADC} M^D{}_B + \mathcal{K}_{ABD} M^D{}_C \, , \tag{5.32}
$$

lead to symmetries of the full action, including the Chern-Simons terms.

Generically, the real special manifolds parametrized by the scalars in the vector multiplets need not be homogeneous, and solutions to (5.32) are not known in general. However, for homogeneous spaces a classification can be found in [135, 136]. A special subclass of the latter is given by the manifolds

$$SO(1,1) \times \frac{SO(n+1,1)}{SO(n+1)} , \tag{5.33}$$

for any integer n, with isometry group $SU(1,1) \times SO(n+1,1)$. This case arises in compactifications in which the Calabi-Yau manifold is a $K3$-fibration over a base P^1. In the present context, this situation has been studied in [64].

Symmetries in the hypermultiplet sector

To study the isometries for the hypermultiplets, we first introduce some notation. The hypermultiplet scalars were given by $\{\mathcal{V}, a, z^r, \bar{z}^r, \xi^K, \tilde{\xi}_K\}$, which parametrize a particular type of quaternionic manifolds called 'very special' in [136].

Since we consider M-theory on a Calabi-Yau manifold, the subspace of complex structure deformations z^r is described by special Kähler geometry, for which there exists a prepotential. In the large complex structure limit, it is given by [18]

$$G(Z) = -\frac{1}{3!}\, d_{rst} \frac{Z^r Z^s Z^t}{Z^0} , \qquad r, s, t = 1, \ldots, h^{2,1} . \tag{5.34}$$

Here, d_{rst} is a real symmetric tensor, the Z^K appear in the expansion (5.11) of the holomorphic three-form Ω and $G_K = \partial G(Z)/\partial Z^K$. The connection to the scalars z^r is made by introducing projective coordinates

$$z^r = \frac{Z^r}{Z^0} , \qquad r = 1, \ldots, h^{2,1} . \tag{5.35}$$

The corresponding Kähler potential reads

$$\mathcal{K}^{\rm cs} = -\ln\left(i \int_{\mathcal{X}} \Omega \wedge \overline{\Omega} \right) = -\ln\left(\frac{4}{3} \left|Z^0\right|^2 d \right) , \tag{5.36}$$

where here and in the following we employ the notation

$$d = d_{rst} x^r x^s x^t , \qquad d_r = d_{rst} x^s x^t , \qquad d_{rs} = d_{rst} x^t , \tag{5.37}$$

[18] We reserve the usual notation F and X for the special geometry in the vector multiplets.

with $x^r = \mathrm{Im}\, z^r$. From (5.36), we can then compute the Kähler metric as [19]

$$G_{r\bar{s}} = \frac{\partial^2}{\partial z^r \partial z^{\bar{s}}} \mathcal{K}^{\mathrm{cs}} = -\frac{3}{2}\frac{d_{rs}}{d} + \frac{9}{4}\frac{d_r d_s}{d^2} \ . \tag{5.38}$$

With $G^{r\bar{s}}$ denoting the inverse of (5.38), the curvature for this metric can be computed as follows [136]

$$R^r{}_{st}{}^v = \delta^r_s \delta^v_t + \delta^r_t \delta^v_s - \frac{4}{3} C^{rvu} d_{stu} \ , \qquad \text{where} \qquad C^{rst} = \frac{27}{64}\frac{1}{d^2} G^{r\bar{u}} G^{s\bar{v}} G^{t\bar{w}} d_{uvw} \ . \tag{5.39}$$

Since the scalars z^r appearing in the action (5.29) can be described by a Kähler potential, their kinetic term is invariant provided that (5.36) does not change under the transformations of interest.[20] We then make the following ansatz for the transformation of the sections (Z^K, G_K) appearing in the holomorphic three-form Ω

$$\delta \begin{pmatrix} Z^K \\ G_K \end{pmatrix} = \begin{pmatrix} \mathcal{Q}^K{}_L & \mathcal{R}^{KL} \\ \mathcal{S}_{KL} & \mathcal{T}_K{}^L \end{pmatrix} \begin{pmatrix} Z^L \\ G_L \end{pmatrix} , \tag{5.40}$$

where, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ are constant, real $h^{2,1}+1$ square matrices. Imposing the invariance of the Kähler potential (5.36) under this transformation, i.e.

$$\delta \int_{\mathcal{X}} \Omega \wedge \overline{\Omega} = 0 \ , \tag{5.41}$$

we are lead to the constraints

$$\mathcal{T} = -\mathcal{Q}^T \ , \qquad \mathcal{S} = \mathcal{S}^T \ , \qquad \mathcal{R} = \mathcal{R}^T \ , \tag{5.42}$$

so the isometries are contained in the symplectic group $Sp\big(2(h^{2,1}+1), \mathbb{R}\big)$. However, as we are considering a Calabi-Yau manifold, we know that the sections G_K are related to Z^K through a prepotential $G(Z)$ as $G_K = \partial G(Z)/\partial Z^K$. Therefore, in the ansatz (5.40) the transformation δG_K is not independent of δZ^K, but we have to require

$$\delta G_K = \frac{\partial G_K}{\partial Z^L}\, \delta Z^L \ . \tag{5.43}$$

As G_K is a homogeneous function of degree one, we have $G_K = (\partial G_K / \partial Z^L) Z^L$. We then infer from (5.43) that [136]

$$0 = G^T \mathcal{Q}\, Z + G^T \mathcal{R}\, G - Z^T \mathcal{S}\, Z - Z^T \mathcal{T}\, G \ . \tag{5.44}$$

[19]The identification of (5.38) with the metric (5.21) can be made by noting that $\chi_r = \partial_{z^r}\Omega + \big(\partial_{z^r}\mathcal{K}^{\mathrm{cs}}\big)\Omega$ as well as that $\int_{\mathcal{X}} \partial_{z^r}\Omega \wedge \overline{\Omega} = 0$.

[20]Strictly speaking, (5.36) should be invariant up to Kähler transformations, but we will ignore those in the present analysis.

Furthermore, to leading order in the large z^r-expansion, for Calabi-Yau threefolds the prepotential $G(Z)$ is given by (5.34). The solution to (5.44) in this case can be found in [136] which we briefly recall. In particular, the matrices $\mathcal{Q}, \mathcal{R}, \mathcal{S}$ and $\mathcal{T}$ appearing in (5.40) can be parametrized as

$$\begin{aligned} \mathcal{Q}^K{}_L = -(\mathcal{T}^T)^K{}_L &= \begin{pmatrix} \beta & a_s \\ b^r & B^r{}_s + \frac{1}{3}\,\beta\,\delta^r{}_s \end{pmatrix} , \\ \mathcal{S}_{KL} = -\begin{pmatrix} 0 & 0 \\ 0 & d_{rst}b^t \end{pmatrix} , &\qquad \mathcal{R}^{KL} = -\begin{pmatrix} 0 & 0 \\ 0 & \frac{4}{3}\,C^{rst}a_t \end{pmatrix} , \end{aligned} \tag{5.45}$$

with β, b^r, a_s and $B^r{}_s$ constant parameters. The matrix $B^r{}_s$ is subject to the constraint

$$B^r{}_{(s}d_{tu)v} = 0 \; , \tag{5.46}$$

where $(\cdots)$ denotes symmetrization and the constants a_s are constrained by

$$0 = a_s\, E^s_{tuvw} \; , \qquad \text{where} \qquad E^s_{tuvw} = C^{yzs}\, d_{y(tu}d_{vw)z} - \delta^s{}_{(t}d_{uvw)} \; . \tag{5.47}$$

With this information, we can compute the transformation of the projective coordinates z^r introduced in (5.35). Employing (5.45) as well as (5.35), we find [136]

$$\delta z^r = b^r - \frac{2}{3}\,\beta\, z^r + B^r{}_s z^s - \frac{1}{2}\, R^r{}_{st}{}^v z^s z^t a_v \; , \tag{5.48}$$

and we note that the condition (5.47) implies that $R^r{}_{st}{}^v a_v$ is constant.

To promote the symmetry of the complex structure deformations z^r to a symmetry of the full hypermultiplets, and hence to isometries of the quaternionic space, we follow again [136]. First we note that the period matrix $\mathcal{M}$ appearing in the action (5.29) (as well as in equations (5.24)) satisfies the relation

$$G_K = \mathcal{M}_{KL} Z^L \; . \tag{5.49}$$

From the transformation of (Z^K, G_K) shown in (5.40), we infer that $\mathcal{M}$ transforms as

$$\delta\mathcal{M} = \mathcal{S} + \mathcal{T}\mathcal{M} - \mathcal{M}\mathcal{Q} - \mathcal{M}\mathcal{R}\mathcal{M} \; . \tag{5.50}$$

Requiring the kinetic term of the scalars $(\xi^K, \tilde{\xi}_K)$ in (5.29) to be invariant implies the following transformation

$$\delta \begin{pmatrix} \xi^K \\ \tilde{\xi}_K \end{pmatrix} = \begin{pmatrix} \mathcal{Q}^K{}_L & \mathcal{R}^{KL} \\ \mathcal{S}_{KL} & \mathcal{T}_K{}^L \end{pmatrix} \begin{pmatrix} \xi^L \\ \tilde{\xi}_L \end{pmatrix} , \tag{5.51}$$

which also leads to the invariance of the $\left(\xi^K \mathrm{d}\tilde{\xi}_K - \tilde{\xi}_K \mathrm{d}\xi^K\right)$ terms and agrees with [136]. Hence, just like (Z^K, G_K), the $(\xi^K, \tilde{\xi}_K)$ form a symplectic pair.

Finally, we should add that the hypermultiplet space in general possesses more symmetries than the ones described here, for instance the Heisenberg algebra of isometries (which include the Peccei-Quinn shifts on $(\xi^K, \tilde{\xi}_K)$) that act on the coordinates $(\xi^K, \tilde{\xi}_K)$ and a only. Furthermore, there are additional isometries that act non-trivially on the volume $\mathcal{V}$ and the axion a – for a complete classification see [136]. Including these in a Scherk-Schwarz reduction would be an interesting extension of our work. We will not consider them in our present discussion.

5.3 Scherk-Schwarz reduction to four dimensions

In this section, we compactify the five-dimensional theory given by (5.29) on a circle of radius R. In addition we impose a non-trivial dependence on the coordinate of the circle. Such a setup was studied first in [118] and, without hypermultiplets, further worked out in [64].

5.3.1 Ansatz for the compactification

To perform the compactification from five to four dimensions, we split the five-dimensional coordinates as

$$\{\tilde{x}^{\tilde{\mu}}\} \longrightarrow \{x^\mu, z\} , \qquad \begin{aligned} \tilde{\mu} &= 0, \ldots, 4 , \\ \mu &= 0, \ldots, 3 , \end{aligned} \tag{5.52}$$

where z denotes the coordinate of the circle normalized as $z \sim z+1$. The dependence of the five-dimensional scalars ν^A and the five-dimensional vectors A^A on the coordinate z is chosen in the following way

$$\partial_z \nu^A = M^A{}_B \nu^B , \qquad \partial_z A^A = M^A{}_B A^B , \tag{5.53}$$

where $M^A{}_B$ satisfies (5.32). These expressions can be integrated to obtain

$$\nu^A(z) = \Big[\exp(Mz)\Big]^A{}_B \, \nu^B(0) , \qquad A^A(z) = \Big[\exp(Mz)\Big]^A{}_B \, A^B(0) , \tag{5.54}$$

where the exponential of a matrix is understood as a matrix product and where only the z-dependence of the fields is shown explicitly.

Clearly, the fields are not periodic around the circle, but are related to each other by the duality transformations (5.31) generated by M. These duality transformations form a group G, and therefore one should have

$$\exp(M) \in G . \tag{5.55}$$

Classically, the group G is taken over the real numbers, and hence the entries of M can be taken as arbitrary real constants. They are related to the masses of the fields, and are treated as continuous parameters which we can take arbitrary small values – or at least values smaller than the masses of the Kaluza-Klein modes that we neglected. In the quantized theory, we expect the duality group to be defined over the integers, and hence the masses will be quantized in some units. This could lead to complications in the truncation of the theory to the lightest modes, which we will ignore in this chapter. For discussions on this issue for toroidal compactifications, see for instance [116, 121].

Turning to the hypermultiplets, for the dependence of the scalars $(\xi^K, \tilde{\xi}_K)$ on the coordinate z of the circle we take

$$\partial_z \begin{pmatrix} \xi^K \\ \tilde{\xi}_K \end{pmatrix} = \begin{pmatrix} \mathcal{Q}^K{}_L & \mathcal{R}^{KL} \\ \mathcal{S}_{KL} & \mathcal{T}_K{}^L \end{pmatrix} \begin{pmatrix} \xi^L \\ \tilde{\xi}_L \end{pmatrix} , \tag{5.56}$$

and for the complex structure moduli z^r we choose in a similar fashion

$$\partial_z z^r = b^r - \frac{2}{3}\beta\, z^r + B^r{}_s z^s - \frac{1}{2} R^r{}_{st}{}^v a_v z^s z^t \equiv \mathcal{N}^r . \tag{5.57}$$

The finite version of these transformations can easily be written down for $(\xi^K, \tilde{\xi}_K)$. For z^r, one first expresses them as transformations for the sections Z^K, after which one can integrate. We choose $\partial_z a = \partial_z \mathcal{V} = 0$.

Note that, since we have chosen the dependence of the fields on the circle coordinate z such that they correspond to Killing vectors of the five-dimensional theory, the full action does not depend on z and so we can evaluate the terms at a particular reference point, say $z_0 = 0$.

For the five-dimensional metric, we make the following ansatz for the dimensional reduction

$$\tilde{g}_{\tilde{\mu}\tilde{\nu}} = \begin{pmatrix} R^{-1} g_{\mu\nu} + R^2 A^0_\mu A^0_\nu & -R^2 A^0_\mu \\ -R^2 A^0_\nu & R^2 \end{pmatrix} , \tag{5.58}$$

where $g_{\mu\nu}$ is the four-dimensional metric, R is the radius of the circle and where the four-vector A^0_μ will become the graviphoton. The factor R^{-1} is chosen such that we end up in Einstein frame. For the five-dimensional gauge fields appearing in the action (5.29), we choose

$$A^A_{(5)} = A^A_{(4)} + b^A\big(\mathrm{d}z - A^0\big) , \tag{5.59}$$

where we added subscripts to distinguish between five- and four-dimensional quantities. Using the above ansätze within the action (5.29), one can perform the dimensional reduction.

5.3.2 Reduction to the four-dimensional action

Computations

To perform the dimensional reduction of the five-dimensional action (5.29), we note that the inverse of the metric (5.58) reads

$$\tilde{g}^{\tilde{\mu}\tilde{\nu}} = \begin{pmatrix} R\, g^{\mu\nu} & R\, A^{0\,\mu} \\ R\, A^{0\,\nu} & R^{-2} + R\, A^0_\rho A^{0\,\rho} \end{pmatrix} , \tag{5.60}$$

where $A^{0\,\mu}$ is the graviphoton with indices raised by the inverse of the four-dimensional metric $g_{\mu\nu}$. The determinant of $\tilde{g}_{\tilde{\mu}\tilde{\nu}}$ is given by

$$\det g_{\tilde{\mu}\tilde{\nu}} = R^{-2} \det g_{\mu\nu} \ . \tag{5.61}$$

For the five-dimensional Ricci scalar, we then find

$$\int_{\mathbb{R}^{4,1}} \frac{1}{2}\, R_{(5)} \star_5 1 = \int_{\mathbb{R}^{3,1}} \left[\frac{1}{2}\, R_{(4)} \star_4 1 - \frac{3}{4} \mathrm{d}\log R \wedge \star_4 \mathrm{d}\log R - \frac{R^3}{4}\, \mathrm{d}A^0 \wedge \star_4 \mathrm{d}A^0 \right] . \tag{5.62}$$

Under the symmetries (5.31) discussed in section 5.2.2, due to equation (5.32), the volume $\mathcal{V}$ is independent of z and so we have chosen $\partial_z \mathcal{V} = 0$. Upon dimensional reduction, the corresponding term in the action keeps the same form, i.e.

$$\int_{\mathbb{R}^{4,1}} \left[-\frac{1}{4}\, \mathrm{d}\log \mathcal{V} \wedge \star_5 \mathrm{d}\log \mathcal{V} \right] = \int_{\mathbb{R}^{3,1}} \left[-\frac{1}{4}\, \mathrm{d}\log \mathcal{V} \wedge \star_4 \mathrm{d}\log \mathcal{V} \right] . \tag{5.63}$$

However, for the scalars ν^A there is a non-trivial dependence on the coordinate z of the circle, which we have specified in equation (5.53). This leads to

$$\begin{aligned} &\int_{\mathbb{R}^{4,1}} \left[\frac{1}{4} \mathcal{K}_{ABC}\, \nu^C \mathrm{d}\nu^A \wedge \star_5 \mathrm{d}\nu^B \right] \\ &= \int_{\mathbb{R}^{3,1}} \left[\frac{1}{4}\, \mathcal{K}_{ABC} \nu^C D\nu^A \wedge \star_4 D\nu^B + \frac{1}{4R^3}\, \mathcal{K}_{ABC} \nu^C \big(M^A{}_D \nu^D \big) \big(M^B{}_E \nu^E \big) \star_4 1 \right] , \end{aligned} \tag{5.64}$$

where we have defined

$$D\nu^A = \mathrm{d}\nu^A + A^0 M^A{}_B \nu^B \ . \tag{5.65}$$

The computation for the remaining five-dimensional scalar fields in the action (5.29) is completely analogous. On the other hand, the reduction of the five-dimensional vector fields is non-trivial. In particular, using (5.53) and (5.59), for the kinetic term one finds

$$\begin{aligned}
&\int_{\mathbb{R}^{4,1}} \left[\frac{1}{4}\Big(\mathcal{K}_{ABC}\nu^C - \frac{1}{4}\,\mathcal{K}_{ACD}\nu^C\nu^D\mathcal{K}_{BEF}\nu^E\nu^F\Big)\,\mathrm{d}A^A_{(5)} \wedge \star_5 \mathrm{d}A^B_{(5)} \right.\\
&= \int_{\mathbb{R}^{3,1}} \left[\frac{1}{4}\Big(\mathcal{K}_{ABC}\nu^C - \frac{1}{4}\,\mathcal{K}_{ACD}\nu^C\nu^D\mathcal{K}_{BEF}\nu^E\nu^F\Big)\times \right. \\
&\qquad\qquad \times \Big(R\,F^A_{(4)} \wedge \star_4 F^B_{(4)} + \frac{1}{R^2}\,Db^A \wedge \star_4 Db^B\Big) ,
\end{aligned} \tag{5.66}$$

with the definitions

$$F^A_{(4)} = \mathrm{d}A^A_{(4)} - M^A{}_B A^B_{(4)} \wedge A^0 , \qquad Db^A = \mathrm{d}b^A - M^A{}_B\big(A^B_{(4)} - b^B A^0\big) . \tag{5.67}$$

For the Chern-Simons term in the five-dimensional action (5.29), employing the constraint (5.32), we find in agreement with [64]

$$\begin{aligned}
&\int_{\mathbb{R}^{4,1}} \left[-\frac{1}{12}\,\mathcal{K}_{ABC}\,\mathrm{d}A^A_{(5)} \wedge \mathrm{d}A^B_{(5)} \wedge A^C_{(5)} \right] \\
&= \int_{\mathbb{R}^{3,1}} \left[-\frac{1}{6}\,\mathcal{K}_{ABC}\,F^A_{(4)} \wedge M^B{}_D\,A^D_{(4)} \wedge A^C_{(4)} - \frac{1}{4}\,\mathcal{K}_{ABC}b^C F^A_{(4)} \wedge F^B_{(4)} \right. \\
&\qquad \left. + \frac{1}{6}\,\mathcal{K}_{ABC}b^B b^C \mathrm{d}A^0 \wedge F^A_{(4)} - \frac{1}{12}\,\mathcal{K}_{ABC}b^A b^B b^C \mathrm{d}A^0 \wedge \mathrm{d}A^0 \right] .
\end{aligned} \tag{5.68}$$

Standard form of $\mathcal{N} = 2$ gauged supergravity

Let us now bring the above results into the standard form of $\mathcal{N} = 2$ gauged supergravity in four dimensions, which was given in section 2.4.1. For ease of notation we will drop all subscripts indicating four-dimensional quantities since this will be clear from the context.

- The Einstein-Hilbert term shown in equation (5.62) is already in the standard form.
- Concerning the scalars ν^A and b^A, we first define fields ϕ^A in the following way

  $$\phi^A = R\,\nu^A , \qquad R^3 = \frac{1}{6}\,\mathcal{K}_{ABC}\phi^A\phi^B\phi^C , \tag{5.69}$$

  where we have included the constraint (5.30) in terms of the ϕ^A. Collecting then all kinetic terms involving ϕ^A and b^A from above, we can express them as

  $$\int_{\mathbb{R}^{3,1}} \Big[-g_{AB}\,Dt^A \wedge \star_4 D\bar{t}^B\Big] , \tag{5.70}$$

  where we employed the definitions (5.77) as well as (5.78).

- For the four-dimensional vector fields A^A and A^0, we first recall the definitions (5.83) and (5.84) for the combined field strengths and structure constants, respectively. Next, we note that the period matrix derived from (5.81) reads

$$\begin{aligned} \mathrm{Im}\,\mathcal{N}_{AB} &= -4\,R^3\,g_{AB}\ , & \mathrm{Re}\,\mathcal{N}_{AB} &= -\mathcal{K}_{ABC}b^C\ , \\ \mathrm{Im}\,\mathcal{N}_{A0} &= +4\,R^3\,g_{AB}b^B\ , & \mathrm{Re}\,\mathcal{N}_{A0} &= +\tfrac{1}{2}\,\mathcal{K}_{ABC}b^Bb^C\ , \\ \mathrm{Im}\,\mathcal{N}_{00} &= -R^3\left(1+4g_{AB}b^Ab^B\right)\ , & \mathrm{Re}\,\mathcal{N}_{00} &= -\tfrac{1}{3}\,\mathcal{K}_{ABC}b^Ab^Bb^C\ . \end{aligned} \tag{5.71}$$

With $\Lambda, \Sigma = 0, \ldots, h^{1,1}$, the kinetic and topological terms for the vector fields are then expressed as

$$\int_{\mathbb{R}^{3,1}} \left[+\frac{1}{4}\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\,F^\Lambda \wedge \star_4 F^\Sigma + \frac{1}{4}\,\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}\,F^\Lambda \wedge F^\Sigma \right] . \tag{5.72}$$

- In equation (5.68), there is one term not contained in (5.72) which can be brought into the following form

$$\int_{\mathbb{R}^{3,1}} \left[-\frac{1}{6}\,A^A M_A{}^B \wedge A^C \wedge \mathrm{d}A^D \mathcal{K}_{BCD} \right] . \tag{5.73}$$

- For the hypermultiplets, we first note that the reduction from five to four dimensions is very similar to the one presented in (5.64). Defining then

$$\mathcal{V} = \mathrm{e}^{-2\phi}\ , \tag{5.74}$$

one arrives at the kinetic terms given in (5.85).

- Let us finally comment on the scalar potential. As one can see for instance from (5.64), the non-trivial dependence of the scalar fields on the circle coordinate z will lead to a scalar potential in four dimensions. Collecting these terms also for the remaining scalar fields, one arrives at the potential given in (5.3.2).

In particular, the four-dimensional action takes the form

$$\begin{aligned} S_4 = \int_{\mathbb{R}^{3,1}} \Bigg[\quad & \frac{1}{2}\,R_{(4)} \star_4 1 + \frac{1}{4}\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\,F^\Lambda \wedge \star_4 F^\Sigma + \frac{1}{4}\,\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}\,F^\Lambda \wedge F^\Sigma \\ & - g_{AB}\,Dt^A \wedge \star_4 D\bar{t}^B - \frac{1}{6}\,A^A M_A{}^B \wedge A^C \wedge \mathrm{d}A^D \mathcal{K}_{BCD} \\ & - h_{uv} Dq^u \wedge \star_4 Dq^v - V \quad \Bigg]\ , \end{aligned} \tag{5.75}$$

where $\Lambda, \Sigma = 0, \ldots, h^{1,1}$ while $A, B, \ldots = 1, \ldots, h^{1,1}$, and where we have omitted all labels indicating four-dimensional quantities. We have furthermore defined

$$\phi^A = R\nu^A \,, \qquad \mathcal{V} = \mathrm{e}^{-2\phi} \,, \tag{5.76}$$

as well as the complexified Kähler moduli and their derivatives

$$t^A = b^A + i\,\phi^A \,, \qquad Dt^A = \mathrm{d}t^A - M^A{}_B\big(A^B - t^B A^0\big) \,, \tag{5.77}$$

where b^A appeared in (5.59). The Kähler metric g_{AB} is written as

$$g_{AB} = -\frac{1}{4R^3}\left(\mathcal{K}_{AB} - \frac{\mathcal{K}_A\mathcal{K}_B}{4R^3}\right) \,, \tag{5.78}$$

where we have employed the following notation

$$\mathcal{K}_A = \mathcal{K}_{ABC}\phi^B\phi^C \,, \qquad \mathcal{K}_{AB} = \mathcal{K}_{ABC}\phi^C \,, \tag{5.79}$$

with $\mathcal{K}_{ABC}$ the triple intersection numbers defined in (5.8). Using these as well as (5.76) in the constraint (5.30), we also find

$$R^3 = \frac{1}{6}\,\mathcal{K}_{ABC}\phi^A\phi^B\phi^C \,. \tag{5.80}$$

The metric (5.78) is a special Kähler metric and can be derived from the prepotential of section 2.3.1

$$F = -\frac{1}{3!}\,\mathcal{K}_{ABC}\,\frac{X^A X^B X^C}{X^0} \,, \qquad A, B, C = 1, \ldots, h^{1,1} \,, \tag{5.81}$$

where we employ coordinates $\{X^0, X^A\}$ with $X^A = X^0\,t^A$. The corresponding Kähler potential reads

$$\mathcal{K}^{\mathrm{vec}} \equiv -\log\Big[\,i\,\overline{X}^\Lambda F_\Lambda - i\,X^\Sigma\overline{F}_\Sigma\,\Big] = -\log\big[\,8R^3\,\big] \,, \tag{5.82}$$

where due to the symmetries of the theory we can set $X^0 = 1$. The expressions for the period matrix $\mathcal{N}_{\Lambda\Sigma}$ are given in (5.71), and the field strengths appearing in (5.75) are written as

$$F^\Lambda = \mathrm{d}A^\Lambda + \frac{1}{2}\,f^\Lambda{}_{\Sigma\Gamma}A^\Sigma \wedge A^\Gamma \,, \qquad \Lambda, \Sigma, \Gamma = 0, \ldots, h^{1,1} \,. \tag{5.83}$$

The structure constants are [64, 118]

$$f^0{}_{AB} = 0 \,, \qquad f^C{}_{AB} = 0 \,, \qquad f^B{}_{A0} = -M^B{}_A \,, \tag{5.84}$$

and they define the gauge group which we elaborate on in the next subsection.

We mention here that gauge invariance of the action (5.75) requires the presence of Chern-Simons-like terms, which are inherited from the five-dimensional Chern-Simons term. These arise when the matrix $\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}$ transforms nontrivially under the action of the gauge group, in such a way that it needs to be compensated by an additional term in the action, the last term on the second line in (5.75). The existence of such terms in gauged supergravity was found in [38], and in the present context it was discussed in [64]. Some further applications of these terms in the study of $\mathcal{N} = 2$ supersymmetric vacua can be found in section 3.3.1.

Turning to the hypermultiplet sector, we find that it is described by

$$\begin{aligned} h_{uv} D_\mu q^u D^\mu q^v &= G_{r\bar{s}}\, D_\mu z^r D^\mu \overline{z}^{\bar{s}} + \partial_\mu \phi\, \partial^\mu \phi \\ &\quad + \frac{\mathrm{e}^{4\phi}}{4}\Big(\partial_\mu a + \xi^K D_\mu \tilde{\xi}_K - \tilde{\xi}_K D_\mu \xi^K\Big)\Big(\partial^\mu a + \xi^L D^\mu \tilde{\xi}_L - \tilde{\xi}_L D^\mu \xi^L\Big) \\ &\quad - \frac{\mathrm{e}^{2\phi}}{2}\big(\mathrm{Im}\,\mathcal{M}\big)^{-1\,KL}\Big(D_\mu \tilde{\xi}_K - \mathcal{M}_{KP} D_\mu \xi^P\Big)\Big(D^\mu \tilde{\xi}_L - \overline{\mathcal{M}}_{LQ} D^\mu \xi^Q\Big)\,, \end{aligned} \tag{5.85}$$

where $\mu = 0, \ldots, 3$ and $G_{r\bar{s}}$ has been introduced in (5.38). The covariant derivatives appearing here are

$$D_\mu z^r = \partial_\mu z^r - \mathcal{N}^r A^0_\mu\,, \qquad D_\mu \begin{pmatrix} \xi \\ \tilde{\xi} \end{pmatrix} = \partial_\mu \begin{pmatrix} \xi \\ \tilde{\xi} \end{pmatrix} - N \begin{pmatrix} \xi \\ \tilde{\xi} \end{pmatrix} A^0_\mu\,, \tag{5.86}$$

where $\mathcal{N}^r$ had been defined in (5.57), where appropriate indices for $(\xi, \tilde{\xi})$ are understood and where the matrix N is given by

$$N = \begin{pmatrix} \mathcal{Q}^K{}_L & \mathcal{R}^{KL} \\ \mathcal{S}_{KL} & \mathcal{T}_K{}^L \end{pmatrix}\,. \tag{5.87}$$

Finally, the scalar potential can be expressed in the following way

$$\boxed{\begin{aligned} V = \quad & \frac{1}{R^3}\,\mathcal{N}^r \overline{\mathcal{N}}^s\, G_{r\bar{s}} + \frac{\mathrm{e}^{4\phi}}{4R^3}\left[\begin{pmatrix} \xi \\ \tilde{\xi} \end{pmatrix}^T N^T \begin{pmatrix} \tilde{\xi} \\ -\xi \end{pmatrix}\right]^2 \\ & - \frac{\mathrm{e}^{2\phi}}{2R^3}\begin{pmatrix} \xi \\ \tilde{\xi} \end{pmatrix}^T N^T \begin{pmatrix} \mathcal{M} \\ -\mathbb{1} \end{pmatrix} \big(\mathrm{Im}\,\mathcal{M}\big)^{-1} \begin{pmatrix} \overline{\mathcal{M}} \\ -\mathbb{1} \end{pmatrix}^T N \begin{pmatrix} \xi \\ \tilde{\xi} \end{pmatrix} \\ & - \frac{1}{4R^6}\,\big(M^A{}_C \phi^C\big)\big(M^B{}_D \phi^D\big)\,\mathcal{K}_{AB}\,, \end{aligned}}$$

where matrix multiplication with correct contraction of indices is again understood. We will study the properties of this potential in section 6.

5.3.3 Gauged $\mathcal{N} = 2$ supergravity formulation

The ungauged part of the Lagrangian (5.75) is already written in the usual form of four-dimensional $\mathcal{N} = 2$ supergravity. The only changes we have to explain are the modifications due to the gauging, in particular the covariant derivatives for the scalars, and the scalar potential.

The covariant derivatives are given by

$$D_\mu q^u = \partial_\mu q^u + \tilde{k}^u_\Lambda A^\Lambda_\mu \ , \qquad\qquad D_\mu t^A = \partial_\mu t^A + k^A_\Lambda A^\Lambda_\mu \ , \tag{5.88}$$

where the quantities $\tilde{k}^u_\Lambda$ and k^A_Λ are Killing vectors on the quaternionic and special Kähler spaces, respectively. For the scalars t^A in the vector multiplets, from (5.77) we read off that

$$k^A_0 = M^A{}_B t^B \ , \qquad\qquad k^A_B = -M^A{}_B \ ,$$

which means that on the special Kähler space defined by (5.81), the isometries we are gauging are given by

$$\delta t^A = -M^A{}_B a^B + M^A{}_B t^B a^0 \ , \tag{5.89}$$

for some arbitrary parameters a^0 and a^A. That these are indeed isometries follows from the analysis of the special Kähler subsector of the hypermultiplets given in (5.48), which is completely analogous. Here, the symmetries (5.89) correspond to the first and third term in (5.48), namely a shift in t^A and a linear transformation with a matrix satisfying (5.32). The gauge group is thus a subgroup of the duality group of isometries on the special Kähler manifold. This duality group contains the one from the five-dimensional theory, but in four dimensions it gets extended to a larger group [136]. The structure constants of the gauge group are given by (5.84), and define a solvable Lie algebra which is the semi-direct product of two Abelian subalgebras of dimension one (graviphoton) and $h^{1,1}$ (the other vector potentials) [64, 118].

The isometry group for the hypermultiplets can easily be read off from (5.86). It is a U(1) group, though realized non-linearly on the scalars. The gauge group acts on it only via the graviphoton.

The explicit form of the scalar potential is given in (5.3.2), and can be written in the standard form of $\mathcal{N} = 2$ supergravity [21]

$$V = 2\, \mathrm{e}^{\mathcal{K}^{\text{vec}}} \Big(4\, h_{uv} \tilde{k}^u_\Lambda \tilde{k}^v_\Sigma + g_{AB}\, k^A_\Lambda\, \overline{k}^B_\Sigma \Big)\, \overline{X}^\Lambda X^\Sigma \ , \tag{5.90}$$

[21] The overall factor 2 compared to the potential of section 2.4.1 is due to the different normalization in (5.75). When rescaling the four-dimensional metric in (5.75) as $g \to \frac{1}{2} g$, one arrives at the form of section 2.4.1.

where $\mathcal{K}^{\rm vec}$ and g_{AB} were defined in (5.82) and (5.78). In the general expression (2.72) for the $\mathcal{N}=2$ scalar potential, there is an additional term proportional to the quaternionic moment maps (see e.g. [28, 56])

$$V^P = 2\left(g^{A\overline{B}} f_A^\Lambda f_{\overline{B}}^\Sigma - 3L^\Lambda \overline{L}^\Sigma\right) P_\Lambda^x P_\Sigma^x \ . \tag{5.91}$$

These moment maps in turn are proportional to a covariant derivative on $\tilde{k}_\Lambda^u$. However, as can be seen from (5.86), the hypermultiplets are only gauged with the graviphoton A_μ^0. Therefore $\tilde{k}_\Lambda^u = 0$ for $\Lambda \neq 0$ and their covariant derivative also vanishes, so $P_\Lambda^x = 0$ for $\Lambda \neq 0$. The only term in (5.91) that can contribute is the term with $\Lambda = 0$. We then utilize that the vector geometry is specified by (5.81), from which one calculates $g^{A\overline{B}} f_A^0 f_{\overline{B}}^0 - 3L^0\overline{L}^0 = 0$. Combining these properties, one finds that $V^P = 0$. This analogue of the $\mathcal{N}=1$ no-scale property reduces the full scalar potential to (5.90).

To see that (5.90) reproduces our scalar potential, we use $k_\Lambda^A \overline{X}^\Lambda = 2iM^A{}_B X^0 \phi^B$, and as (5.32) implies $\mathcal{K}_A M^A{}_B \phi^B = 0$, with the help of (5.82) we find

$$2\, \mathrm{e}^{\mathcal{K}} g_{AB}\, k_\Lambda^A k_\Sigma^{\overline{B}}\, \overline{X}^\Lambda X^\Sigma = -\frac{1}{4R^6}\, \mathcal{K}_{AB} M^A{}_C \phi^C M^B{}_D \phi^D \ . \tag{5.92}$$

Employing the expressions for the covariant derivatives of the hyperscalars above, it is then straight-forward to check that (5.90) reproduces (5.3.2).

5.4 M-theory on twisted seven-manifolds

The Scherk-Schwarz reduction described above, yielding the gauged supergravity Lagrangian (5.75), can also be obtained from a compactification of eleven-dimensional supergravity on a seven-manifold. This point of view had also been taken in [64] for the vector multiplets. We will briefly review and extend this procedure in the present section to also include the hypermultiplet sector.

The seven-dimensional space we are going to compactify on, denoted by $\mathcal{Y}$ in the following, is chosen as a fibration of a Calabi-Yau three-fold $\mathcal{X}$ over a circle S^1.

$$\begin{array}{ccc} \mathcal{X} & \rightarrow & \mathcal{Y} \\ & & \downarrow \\ & & S^1 \end{array} \tag{5.93}$$

The coordinates of $\mathcal{X}$ will be denoted by y and the coordinate z of the circle is again normalized such that $z \sim z+1$. At a particular reference point $z_0 = 0$, we choose a basis of harmonic two- and three-forms of the corresponding Calabi-Yau three-fold as in section 5.2. We then must indicate how this data changes when moving around the circle.

In words, this can be explained as follows: instead of specifying the z-dependence in the coefficient functions (i.e. the five-dimensional fields) as we do in the Scherk-Schwarz reduction, we can move the same z-dependent terms from the fields into the basis of two- and three-forms of $\mathcal{X}$. This produces a non-trivial seven-dimensional manifold of the type (5.93), which by construction is equivalent to the Scherk-Schwarz reduction. We now explain this in some more detail.

Cohomology

Let us begin our discussion with the cohomology of the compactification space $\mathcal{Y}$. Analogous to the harmonic $(1,1)$-forms on $\mathcal{X}$ we introduce

$$\hat{\omega}_A(y,z)\ , \qquad\qquad A = 1,\ldots,h^{1,1}(\mathcal{X})\ . \tag{5.94}$$

The dependence of $\hat{\omega}_A$ on the coordinate z of the circle is taken as

$$\hat{\omega}_A(y,z) = \Big[\exp(zM^T)\Big]_A^{\ \ B}\,\omega_B\ , \tag{5.95}$$

where the exponential of the matrix $(M^T)_A{}^B$ is understood as a matrix product and ω_B is a basis of harmonic $(1,1)$-forms on the Calabi-Yau three-fold at a particular reference point $z_0 = 0$. The matrix $M^B{}_A$ is not arbitrary but, as explained in [64], has to satisfy the constraint shown in (5.32). Infinitesimally, the relation (5.95) can be written as

$$\mathrm{d}\hat{\omega}_A = (M^T)_A{}^B\hat{\omega}_B \wedge \mathrm{d}z\ , \qquad\qquad \hat{\omega}_B(y,0) = \omega_B\ , \tag{5.96}$$

so we see that in general the forms $\hat{\omega}_A$ are not closed. Their non-closure will be the origin of the gaugings in the resulting four-dimensional action. The triple intersection numbers for the Calabi-Yau three-fold in the present context are given by

$$\hat{\mathcal{K}}_{ABC} \equiv \int_{\mathcal{Y}} \hat{\omega}_A \wedge \hat{\omega}_B \wedge \hat{\omega}_C \wedge \mathrm{d}z = \int_{\mathcal{X}} \omega_A \wedge \omega_B \wedge \omega_C = \mathcal{K}_{ABC}\ , \tag{5.97}$$

where the second equality follows by using (5.32).

Analogous to the second co-homology, for the third co-homology group we introduce

$$\big\{\hat{\alpha}_K(y,z), \hat{\beta}^L(y,z)\big\}\ , \qquad\qquad K,L = 0,\ldots,h^{2,1}(\mathcal{X})\ . \tag{5.98}$$

Their dependence on the coordinate z of the circle is chosen as

$$\begin{pmatrix} \hat{\alpha}(y,z) \\ -\hat{\beta}(y,z) \end{pmatrix} = \Big[\exp(zN^T)\Big] \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}\ , \tag{5.99}$$

where the matrix N was defined in (5.87) and proper contraction of indices is understood. Furthermore, $\{\alpha_K, \beta^K\}$ denotes the basis of harmonic three-forms on the Calabi-Yau manifold at a particular reference point $z_0 = 0$, and the minus sign has been chosen to match the results from the previous section. Infinitesimally, we can express (5.99) as

$$\mathrm{d}\begin{pmatrix} \hat\alpha \\ -\hat\beta \end{pmatrix} = -N^T \begin{pmatrix} \hat\alpha \\ -\hat\beta \end{pmatrix} \wedge \mathrm{d}z \;, \qquad \begin{pmatrix} \hat\alpha(y,0) \\ -\hat\beta(y,0) \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} , \tag{5.100}$$

where proper contraction of indices is again understood. Finally, using (5.10) and (5.99), one can show that

$$\int_{\mathcal{X}} \hat\alpha_K \wedge \hat\beta^L = \delta_K{}^L \;, \qquad \int_{\mathcal{X}} \hat\alpha_K \wedge \hat\alpha_L = 0 \;, \qquad \int_{\mathcal{X}} \hat\beta^K \wedge \hat\beta^L = 0 \;. \tag{5.101}$$

Dimensional reduction

For the dimensional reduction of the M-theory action (5.6) on the seven-manifold $\mathcal{Y}$ we make the following ansatz for the space-time metric

$$\mathrm{ds}^2_{11} = \mathrm{e}^{\frac{4}{3}\phi} R^{-1} g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{\frac{4}{3}\phi} R^2 \big(\mathrm{d}z - A^0\big)^2 + G_{mn} \mathrm{d}y^m \mathrm{d}y^n \;, \tag{5.102}$$

where R is the radius of the circle satisfying (5.80), A^0 denotes the graviphoton one-form and G_{mn} is the metric of the Calabi-Yau threefold, whose fluctuations depend on v^A and z^r. For the three-form potential $\hat C_3$ we consider an ansatz similar to [64] but are more specific about the sector corresponding to the hypermultiplets. In particular, we consider

$$\begin{aligned} \hat C_3 &= c_3 + B \wedge (dz - A^0) + (A^A - b^A A^0) \wedge \hat\omega_A + b^A \hat\omega_A \wedge dz + C_3 \;, \\ C_3 &= \sqrt{2}\, \xi^K \hat\alpha_K - \sqrt{2}\, \tilde\xi_K \hat\beta^K \;, \end{aligned} \tag{5.103}$$

where c_3 is a four-dimensional three-form, B denotes a four-dimensional two-form, A^A are one-forms and b^A as well as $(\xi^K, \tilde\xi_K)$ are scalars in four dimensions. For the corresponding field strength $\hat F_4 = \mathrm{d}\hat C_3$, employing (5.96) as well as (5.100), one finds

$$\begin{aligned} \hat F_4 = \quad & \mathrm{d}c_3 + \mathrm{d}B \wedge \big(\mathrm{d}z - A^0\big) - B \wedge F^0 + F^a \wedge \hat\omega_A - b^A F^0 \wedge \hat\omega_A \\ & + Db^A \wedge \hat\omega_A \wedge \big(\mathrm{d}z - A^0\big) + \sqrt{2} \left[\mathrm{d}\begin{pmatrix} \xi \\ \tilde\xi \end{pmatrix}^T - \begin{pmatrix} \xi \\ \tilde\xi \end{pmatrix}^T N^T \mathrm{d}z \right] \wedge \begin{pmatrix} \hat\alpha \\ -\hat\beta \end{pmatrix} , \end{aligned} \tag{5.104}$$

where F^0 and F^A are defined in (5.83). Using the above ansätze in the eleven-dimensional action (5.6), one can perform the dimensional reduction. However, to make contact with (5.75), we have to dualize B to a scalar a and c_3 to a constant e_0, chosen to be zero. A non-zero choice for e_0 would correspond to a non-trivial z-dependence for the five-dimensional field a in the Scherk-Schwarz reduction of section 5.3, which we did not consider. Taking into account these remarks, we then recover the four-dimensional action (5.75), as we have checked explicitly.

5.5 Truncation to $\mathcal{N} = 1$ supersymmetry

We now perform a truncation of the theory studied in section 5.3.2 from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry. To motivate this truncation, we note that M-theory compactifications on seven-manifolds of the form $\mathcal{X} \times S^1$ can be related to orientifold compactifications of type IIA string theory [137]. In particular, consider M-theory on

$$\frac{\mathcal{X} \times S^1}{(\overline{\sigma}, -1)} , \tag{5.105}$$

where $\overline{\sigma}$ is an anti-holomorphic involution acting on the Calabi-Yau three-fold $\mathcal{X}$ and where (-1) acts on the circle coordinate as $z \to -z$. Upon dimensionally reducing on S^1, the resulting theory is type IIA string theory on

$$\frac{\mathcal{X}}{(-1)^{F_L}\Omega\,\overline{\sigma}} , \tag{5.106}$$

where F_L is the left-moving space-time fermion number and Ω is the parity operator on the string world-sheet. The quotient notation here means that one keeps the string states which are invariant under the action of $(-1)^{F_L}\Omega\overline{\sigma}$. Motivated by this observation, in the present work we will impose a truncation similar to (5.105).

We also observe that $\overline{\sigma}$ being anti-holomorphic means that $\overline{\sigma}^*\Omega \sim \overline{\Omega}$, where Ω is the holomorphic three-form of the Calabi-Yau manifold and $\overline{\sigma}^*$ denotes the action of $\overline{\sigma}$ induced on the cohomology. Utilizing the relation

$$\Omega \wedge \overline{\Omega} \sim J \wedge J \wedge J , \tag{5.107}$$

and applying $\overline{\sigma}^*$ to both sides, we infer that the Kähler form J has to be odd under the anti-holomorphic involution $\overline{\sigma}^*$.

5.5.1 Defining the truncation

Cohomology

To define our truncation, we first consider an involution $\overline{\sigma}$ acting on a Calabi-Yau three-fold $\mathcal{X}$. The action $\overline{\sigma}^*$ induced on the cohomology groups of $\mathcal{X}$ splits them into even and odd sub-spaces. In particular, the basis of harmonic $(1,1)$-forms introduced in (5.7) can be separated as

$$\begin{aligned} \overline{\sigma}^*\omega_\alpha &= +\omega_\alpha \ , \qquad & \alpha &= 1,\ldots,h^{1,1}_+ \ , \\ \overline{\sigma}^*\omega_a &= -\omega_a \ , \qquad & a &= 1,\ldots,h^{1,1}_- \ , \end{aligned} \tag{5.108}$$

where $h^{1,1}_+ + h^{1,1}_- = h^{1,1}$. Since the Kähler form is odd under $\overline{\sigma}^*$, also the volume form on $\mathcal{X}$ is odd. Thus, some triple intersection numbers have to vanish which leads to

$$\mathcal{K}_{\alpha\beta\gamma} = \mathcal{K}_{\alpha bc} = 0 \ , \qquad \mathcal{K}_{\alpha b} = 0 \ , \qquad \mathcal{K}_\alpha = 0 \ . \tag{5.109}$$

For the basis of the third cohomology group of $\mathcal{X}$ introduced in (5.9), we similarly observe

$$\begin{aligned} \overline{\sigma}^*\alpha_k &= +\alpha_k \ , \qquad & \overline{\sigma}^*\beta^k &= -\beta^k \ , \\ \overline{\sigma}^*\alpha_\lambda &= -\alpha_\lambda \ , \qquad & \overline{\sigma}^*\beta^\lambda &= +\beta^\lambda \ , \end{aligned} \tag{5.110}$$

where the indices k and λ jointly range from 0 to $h^{2,1}$. For the period matrix $\mathcal{M}$ introduced in equations (5.24), from (5.110) we then infer that

$$\mathrm{Re}\,\mathcal{M}_{\kappa\lambda} = 0 \ , \qquad \mathrm{Re}\,\mathcal{M}_{kl} = 0 \ , \qquad \mathrm{Im}\,\mathcal{M}_{k\lambda} = \mathrm{Im}\,\mathcal{M}_{\lambda k} = 0 \ . \tag{5.111}$$

Truncation of vector multiplets

Motivated by our discussion at the beginning of this section about ordinary M-theory compactifications, we will truncate our $\mathcal{N}=2$ supersymmetric theory by

$$\overline{\Sigma} = \big(\overline{\sigma}, -1\big) \ , \tag{5.112}$$

where $\overline{\sigma}$ is the anti-holomorphic involution considered above and (-1) acts on the circle coordinate as $z \to -z$. As noted below (5.107), the Kähler form J is odd under $\overline{\sigma}^*$, which we extend to every point on the circle:

$$\overline{\Sigma}^* J = -J \ . \tag{5.113}$$

In terms of the expansion $J = v^A(z)\,\omega_A$,[22] we find that equation (5.113), evaluated at $z=0$, yields $v^\alpha(0) = 0$ and therefore, using (5.28) and (5.76), we find

$$\phi^\alpha(0) = 0 \ . \tag{5.114}$$

[22]To keep our notation short, we suppress the dependence of the fields on x^μ but only indicate the dependence on the circle coordinate z.

For general values of z, we employ (5.54) and (5.28) to express J as

$$J(z) = v^a(0) \left[\mathrm{e}^{zM^T} \right]_a^{\ \ B} \omega_B \ . \tag{5.115}$$

Inserting this expansion into (5.113) leads to the constraint that $M^a{}_b = 0$. Concerning the vector fields A^A, we require that the M-theory three-form $\hat{C}_3$, given in (5.14), satisfies

$$\overline{\Sigma}^* \hat{C}_3 = +\hat{C}_3 \ . \tag{5.116}$$

In particular, the term involving the five-dimensional vector fields $A^A_{(5)}$ has to be even under $\overline{\Sigma}^*$. Performing a similar analysis as for the Kähler form at $z = 0$, and using equation (5.59), we obtain

$$A^a_{(4)}(0) = 0 \ , \qquad\qquad b^\alpha(0) = 0 \ . \tag{5.117}$$

Furthermore, requiring $A^A_{(5)} \wedge \omega_A$ to be even under $\overline{\Sigma}^*$ for all values of z and employing (5.53) implies that $M^\alpha{}_\beta = 0$. We thus arrive at

$$M^A{}_B = \begin{pmatrix} 0 & M^\alpha{}_b \\ M^a{}_\beta & 0 \end{pmatrix} \ . \tag{5.118}$$

Finally, recalling the five-dimensional metric (5.58) and requiring it to be invariant under the action (5.113), we see that the graviphoton A^0 is projected out, that is

$$A^0 = 0 \ . \tag{5.119}$$

Truncation of hypermultiplets

To define the truncation of the hypermultiplets, let us consider the action of the anti-holomorphic involution on the holomorphic three-form Ω. Similarly as in [138], we write

$$\overline{\sigma}^* \Omega = \mathrm{e}^{2i\Theta}\, \overline{\Omega} \ , \tag{5.120}$$

where Θ is a constant phase. As for the Kähler form, we extend (5.120) to $\overline{\Sigma}$ in the following way

$$\overline{\Sigma}^* \Omega = \mathrm{e}^{2i\Theta}\, \overline{\Omega} \ . \tag{5.121}$$

Employing then the expansion of Ω given in (5.11), at $z = 0$ the relation (5.121) implies that $\mathrm{Im}(e^{-i\Theta} Z^k(0)) = 0$ and similar relations for Z^λ, G_k and G_λ. However, for later convenience, let us introduce the compensator C, which is defined in terms of the four-dimensional dilaton ϕ and the Kähler potential (5.36) for the complex structure moduli

$$C \equiv \mathrm{e}^{-\phi}\mathrm{e}^{\mathcal{K}^{\mathrm{cs}}/2}\mathrm{e}^{-i\Theta} \ . \tag{5.122}$$

Noting that ϕ as well as $\mathcal{K}^{\mathrm{cs}}$ are invariant under $\overline{\Sigma}$, equation (5.121) can be brought into the form $\overline{\Sigma}^*(C\Omega) = \overline{C\Omega}$, whose implications at $z = 0$ read

$$\begin{aligned} \mathrm{Im}\big(CZ^k(0)\big) &= 0 \ , \qquad & \mathrm{Re}\big(CG_k(0)\big) &= 0 \ , \\ \mathrm{Re}\big(CZ^\lambda(0)\big) &= 0 \ , \qquad & \mathrm{Im}\big(CG_\lambda(0)\big) &= 0 \ . \end{aligned} \tag{5.123}$$

As carefully discussed in [138], the equations on the left in (5.123) project out $h^{2,1}$ real scalars, corresponding to half of the complex structure deformations. The set of equations on the right should not be interpreted as further truncations, but as constraints on the triple intersection numbers d_{rst} in (5.34).

Next, requiring again C_3 in the M-theory three-form $\hat{C}_3$ of (5.14) to be invariant under $\overline{\Sigma}^*$ leads to

$$\xi^\lambda(0) = 0 \ , \qquad \tilde{\xi}_k(0) = 0 \ . \tag{5.124}$$

To study the five-dimensional three-form $\tilde{c}_3$ in (5.14), we write

$$\tilde{c}_3 = \mathcal{C}_3 + \mathcal{C}_2 \wedge dz \ , \tag{5.125}$$

where $\mathcal{C}_3$ and $\mathcal{C}_2$ respectively are three- and two-forms in four dimensions. Since $\tilde{c}_3$ has to be even under $\overline{\Sigma}^*$, we see that $\mathcal{C}_2$ is projected out. Furthermore, $\mathcal{C}_3$ in four dimensions is dual to a constant e_0, which in the analysis of section 5.3 and 5.4 we have chosen to be zero. Therefore, the contribution of $\tilde{c}_3$ in the truncated theory vanishes, that is

$$a = 0 \ . \tag{5.126}$$

Combining then all these constraints, we see that $2h^{2,1}$ out of the $4(h^{2,1}+1)$ original hyperscalars survive the truncation. We will later show that these remaining scalars form chiral multiplets and that their target space is Kähler.

Finally, in the above analysis we studied (5.116) and (5.121) at $z = 0$. To satisfy these constraints for all values of z, additional restrictions on the matrices $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ introduced in (5.40) arise. In particular, employing (5.40) as well as (5.56), in a similar fashion as in (5.115) one obtains

$$\begin{aligned} \mathcal{Q}^k{}_l &= 0 \ , \qquad & \mathcal{Q}^\lambda{}_\rho &= 0 \ , \qquad & \mathcal{R}^{k\lambda} &= \mathcal{R}^{\lambda k} = 0 \ , \\ \mathcal{T}_k{}^l &= 0 \ , \qquad & \mathcal{T}_\lambda{}^\rho &= 0 \ , \qquad & \mathcal{S}_{k\lambda} &= \mathcal{S}_{\lambda k} = 0 \ . \end{aligned} \tag{5.127}$$

5.5.2 Performing the truncation

After having specified the truncation of the fields appearing in the $\mathcal{N} = 2$ theory (at the point $z = 0$), we can now apply these results to (5.75). Note that this action was obtained by evaluating all five-dimensional fields at a particular reference point $z_0 = 0$. Employing the results from section 5.5.1, we then find

$$\begin{aligned} S_4^{\text{trunc.}} = \int_{\mathbb{R}^{3,1}} \Big[& \frac{1}{2} R_{(4)} \star_4 1 + \frac{1}{4} \operatorname{Im} \mathcal{N}_{\alpha\beta} \, \mathrm{d}A^\alpha \wedge \star_4 \mathrm{d}A^\beta + \frac{1}{4} \operatorname{Re} \mathcal{N}_{\alpha\beta} \, \mathrm{d}A^\alpha \wedge \mathrm{d}A^\beta \\ & - g_{ab} \, Dt^a \wedge \star_4 D\bar{t}^b - \frac{1}{6} A^\alpha M_\alpha{}^b \wedge A^\gamma \wedge \mathrm{d}A^\delta \, \mathcal{K}_{b\gamma\delta} \\ & - G_{I\overline{J}} \, \mathrm{d}M^I \wedge \star_4 \mathrm{d}\overline{M}^J - V^{\text{trunc.}} \Big] \, . \end{aligned} \tag{5.128}$$

Kinetic terms

In the present case the covariant derivative acting on the complexified Kähler moduli t^a takes the form

$$Dt^a = \mathrm{d}t^a - M^a{}_\beta A^\beta \, . \tag{5.129}$$

Furthermore, using the explicit formulas for the period matrix $\mathcal{N}$ given in (5.71) as well as (5.109), the gauge kinetic function for the vector fields is found as

$$f_{\alpha\beta} = -i \overline{\mathcal{N}}_{\alpha\beta} = i \, \mathcal{K}_{\alpha\beta c} t^c \, , \tag{5.130}$$

which is holomorphic in the chiral fields, as required by $\mathcal{N} = 1$ supersymmetry.

The next step is the reduction of the hypermultiplets. Since the graviphoton A^0 is projected out, the hyperscalars become uncharged. For the truncation of the hypermultiplets from $\mathcal{N} = 2$ to $\mathcal{N} = 1$, we can thus refer to the existing literature. In particular, employing the results of appendix C in [138], the kinetic terms for the hypermultiplet scalars are given by

$$- \int_{\mathbb{R}^{3,1}} G_{I\overline{J}} \, \mathrm{d}M^I \wedge \star_4 \mathrm{d}M^{\overline{J}} \, , \tag{5.131}$$

where $M^I = \{N^k, T_\lambda\}$ collectively denotes the chiral fields

$$N^k = \frac{1}{2} \xi^k + i \operatorname{Re}(C Z^k) \, , \qquad T_\lambda = i \tilde{\xi}_\lambda - 2 \operatorname{Re}(C G_\lambda) \, . \tag{5.132}$$

The metric $G_{I\overline{J}} = \partial_{M^I} \partial_{\overline{M}^J} \mathcal{K}^{\mathrm{Q}}$ in (5.131) is Kähler and the corresponding Kähler potential $\mathcal{K}^{\mathrm{Q}}$ is given by [138]

$$\mathcal{K}^{\mathrm{Q}} = -2 \log \left[2 \int_{\mathcal{X}} \operatorname{Re}\big(C\Omega\big) \wedge \star_6 \operatorname{Re}\big(C\Omega\big) \right] \, . \tag{5.133}$$

Potential

We now turn to the truncation of the scalar potential (5.3.2). For the scalars ϕ^A we employ (5.114) and (5.118) to find

$$V^{(1)} = -\frac{1}{4R^6}\left(M^{\alpha}{}_{c}\,\phi^c\right)\left(M^{\beta}{}_{d}\,\phi^d\right)\mathcal{K}_{\alpha\beta}\ . \tag{5.134}$$

For the truncation of the terms involving ξ and $\tilde{\xi}$ we use (5.124), (5.127) and (5.111). These merely imply that we have to restrict the index ranges of ξ and $\tilde{\xi}$ in (5.3.2). For later convenience, we express this result as

$$V^{(2)} = -\frac{\mathrm{e}^{2\phi}}{2R^3}\begin{pmatrix}\xi\\ \tilde{\xi}\end{pmatrix}^T N^T\,\Pi\,N\begin{pmatrix}\xi\\ \tilde{\xi}\end{pmatrix} + \frac{\mathrm{e}^{4\phi}}{4R^3}\left[\begin{pmatrix}\xi\\ \tilde{\xi}\end{pmatrix}^T N^T\Delta\begin{pmatrix}\xi\\ \tilde{\xi}\end{pmatrix}\right]^2\ , \tag{5.135}$$

where we have defined

$$\Pi = \begin{pmatrix}\mathcal{M}\\ -\mathbb{1}\end{pmatrix}\left(\mathrm{Im}\,\mathcal{M}\right)^{-1}\begin{pmatrix}\overline{\mathcal{M}}\\ -\mathbb{1}\end{pmatrix}^T\ , \qquad \Delta = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\ . \tag{5.136}$$

As mentioned, these formulas are understood with the restrictions (5.124), (5.127) and (5.111) applied.

To make the truncation of the potential for the complex structure moduli z^r more feasible, we first define

$$\mathcal{G}_{LK} = 2\,(\mathrm{Im}G)_{LK} - 2\,\frac{(\mathrm{Im}G)_{LN}\overline{Z}^N\,(\mathrm{Im}G)_{KM}Z^M}{Z^N\,(\mathrm{Im}G)_{NM}\overline{Z}^M}\ , \tag{5.137}$$

with $G_{LK} = \partial_{Z^L}G_K$ and where (Z^L, G_L) are the holomorphic sections introduced in equation (5.11). Recalling then (5.57) as well as that $z^r = \frac{Z^r}{Z^0}$, we can write

$$\begin{aligned}V^{(3)} &= \frac{1}{R^3}\,\mathcal{N}^r\overline{\mathcal{N}}^s\,G_{r\bar{s}}\\ &= \frac{\mathrm{e}^{\mathcal{K}^{\mathrm{cs}}}}{R^3|C|^2}\left(-CZ^K\mathcal{T}_K{}^L + CG_K\mathcal{R}^{KL}\right)\mathcal{G}_{LM}\left(\mathcal{Q}^M{}_N\overline{CZ^N} + \mathcal{R}^{MN}\overline{CG_N}\right),\end{aligned} \tag{5.138}$$

where the restrictions (5.123) and (5.127) are understood. Note that to arrive at the second line in (5.138) we utilized $\mathcal{G}_{LK}\overline{Z}^K = 0$, and that the compensator C was introduced in equation (5.122).

5.5.3 Superpotential and D-terms

We will now bring the potentials (5.134), (5.135) and (5.138) into the standard form of $\mathcal{N} = 1$ supergravity, which was discussed in section 2.4.2. The scalar potential is given by [23]

$$V = 2\,\mathrm{e}^{\mathcal{K}}\Big(G^{\hat{I}\bar{\hat{J}}}D_{\hat{I}}W D_{\bar{\hat{J}}}\overline{W} - 3|W|^2\Big) + \big(\mathrm{Re}f\big)^{-1|\alpha\beta}D_\alpha D_\beta = V_F + V_D\ , \tag{5.139}$$

where we use $\hat{M}^{\hat{I}} = \{N^k, T_\lambda, t^a\}$ to label all chiral fields in the theory. Here, the Kähler covariant derivatives reads $D_{\hat{I}}W = \partial_{\hat{I}}W + (\partial_{\hat{I}}\mathcal{K})W$, $\mathrm{Re}f_{\alpha\beta}$ is the real part of the gauge kinetic function (5.130) and D_α are the moment maps associated with the gauging of the chiral multiplets. The Kähler potential $\mathcal{K}$ in (5.139) is the sum of (5.82) subject to the truncation (5.114), and $\mathcal{K}^Q$ given in (5.133),

$$\mathcal{K} = \mathcal{K}^{\mathrm{vec}} + \mathcal{K}^Q\ . \tag{5.140}$$

D-term potential

The D-term potential arises as some of the chiral fields are gauged. In our case, as can be inferred from (5.129), only the chiral fields t^a arising from the projection of the $\mathcal{N}=2$ vector multiplets are gauged. We will therefore show that their potential term (5.134) is given by the D-term potential.

To find an expression for D_α, we can use the truncation of the original moment maps P_K on the special Kähler space, given in [118]. We then obtain

$$D_\alpha = i\,(M^T)_\alpha{}^a\,\partial_{t^a}\mathcal{K}^{\mathrm{vec}} = -\frac{1}{4R^3}(M^T)_\alpha{}^a\,\mathcal{K}_a\ . \tag{5.141}$$

Noting that the Killing vectors after the truncation are given by $k^a_\alpha = M^a{}_\alpha$, we see that the D_α's obey

$$k^a_\alpha = -ig^{a\bar{b}}\partial_{\bar{b}}D_\alpha\ , \tag{5.142}$$

as $\partial_{\bar{b}}\partial_a\mathcal{K}^{\mathrm{vec}} = g_{a\bar{b}}$, and they are therefore moment maps for the Killing vectors k^a_α.

Notice furthermore that the gauge group has now become abelian, $G = U(1)^{h^{1,1}_+}$, since the Killing vectors are constant and hence commute. However, as we will analyze in section 5.6, the gauge group can be broken further due to a Higgsing of the gauge fields.

Contracting then equation (5.32) with $\phi^b\phi^c$ and restricting the index A to α, we find $\mathcal{K}_a M^a{}_\alpha = -2\mathcal{K}_{\alpha\beta}M^\beta{}_a\phi^a$, which allows us to bring (5.141) into the form

$$\boxed{D_\alpha = \frac{1}{2R^3}\,\mathcal{K}_{\alpha\beta}M^\beta{}_a\phi^a\ .}$$

[23] Again, there is an overall factor 2 with respect to the standard literature; see footnote 21.

Employing finally the expression (5.130) for the real part of the gauge kinetic function, that is $\mathrm{Re} f_{\alpha\beta} = -\mathcal{K}_{\alpha\beta}$, we arrive at

$$V_D = \left(\mathrm{Re} f\right)^{-1|\alpha\beta} D_\alpha D_\beta = -\frac{1}{4R^6}\left(M^\alpha{}_c\,\phi^c\right)\left(M^\beta{}_d\,\phi^d\right)\mathcal{K}_{\alpha\beta} = V^{(1)}\ . \tag{5.143}$$

So indeed, as expected, in the truncated theory the potential term for the fields ϕ^a is a D-term potential.

F-term potential

Next, we turn to the F-term potential. As the chiral fields (N^k, T_λ) are ungauged, the contribution to their scalar potential terms has to come from the superpotential W. We will now show that indeed their potential, $V^{(2)} + V^{(3)}$, is described by

$$W = \frac{1}{2}\,\mathcal{U}^T \Delta N \,\mathcal{U}\ .$$

We remind the reader that Δ was defined in (5.136), the twisting matrix N had been introduced in (5.87) and the restrictions (5.127) are imposed. Furthermore, we have combined the chiral fields N^k and T_λ into the vector

$$\mathcal{U} = \begin{pmatrix} 2iN^k \\ T_\lambda \end{pmatrix} = \begin{pmatrix} i\,\xi^k - 2\mathrm{Re}(CZ^k) \\ i\,\tilde{\xi}_\lambda - 2\mathrm{Re}(CG_\lambda) \end{pmatrix} = i\,\mathcal{U}_I + \mathcal{U}_R\ . \tag{5.144}$$

To show that this superpotential (5.5.3) reproduces the scalar F-term potential, given by $V^{(2)} + V^{(3)}$, we first notice that

$$\partial_{t^a} W = 0\ , \qquad\qquad \partial_{t^a}\mathcal{K}\, G^{a\bar{b}}\,\partial_{\bar{t}^{\bar{b}}}\mathcal{K} = 3\ , \tag{5.145}$$

which reduces V_F in (5.139) to

$$V_F = 2\,\mathrm{e}^{\mathcal{K}}\Big(\,G^{I\overline{J}} D_I W D_{\overline{J}}\overline{W}\,\Big)\,, \tag{5.146}$$

with I labeling (T_λ, N^k). Next, we recall from [138] the expressions for the inverse Kähler metric $G^{I\overline{J}}$ which are given by

$$\begin{aligned}
G^{T_\kappa \overline{T}_\lambda} &= -2\,\mathrm{e}^{-2\phi}\ \Big[\mathrm{Im}\mathcal{M} + (\mathrm{Re}\mathcal{M})(\mathrm{Im}\mathcal{M})^{-1}(\mathrm{Re}\mathcal{M})\Big]_{\kappa\lambda}\ ,\\
G^{T_\lambda \overline{N}^k} &= -i\,\mathrm{e}^{-2\phi}\ \Big[(\mathrm{Re}\mathcal{M})(\mathrm{Im}\mathcal{M})^{-1}\Big]_\lambda^{\ \ k}\ ,\\
G^{N^k \overline{N}^l} &= -\frac{1}{2}\,\mathrm{e}^{-2\phi}\ \Big[(\mathrm{Im}\mathcal{M})^{-1}\Big]^{kl}\ ,
\end{aligned} \tag{5.147}$$

where in the present case (5.111) implies that some entries of $\mathrm{Re}\mathcal{M}$ and $\mathrm{Im}\mathcal{M}$ are vanishing. Furthermore, we have the contractions

$$G^{N^k\overline{M}^{\bar{I}}}\partial_{\overline{M}^{\bar{I}}}\mathcal{K} = -(N^k - \overline{N}^k)\,, \qquad\qquad G^{T_\lambda\overline{M}^{\bar{I}}}\partial_{\overline{M}^{\bar{I}}}\mathcal{K} = -(T_\lambda + \overline{T}_\lambda)\,. \tag{5.148}$$

Employing the above expressions as well as (5.82) and (5.133), one can bring equation (5.146) into the following form

$$\begin{aligned} V_F = \frac{1}{R^3}\,\frac{1}{\mathcal{U}_R^T\,\Pi\,\mathcal{U}_R}\Bigg[\quad & \mathcal{U}_R^T N^T \Delta^T \left(\Pi^{-1} - \frac{\mathcal{U}_R\,\mathcal{U}_R^T}{\mathcal{U}_R^T\,\Pi\,\mathcal{U}_R}\right)\Delta\,N\,\mathcal{U}_R \\ & + \mathcal{U}_I^T N^T \Delta^T\,\Pi^{-1}\,\Delta\,N\,\mathcal{U}_I + \frac{\left(\mathcal{U}_I^T\,\Delta\,N\,\mathcal{U}_I\right)^2}{\mathcal{U}_R^T\,\Pi\,\mathcal{U}_R} \qquad \Bigg]\,, \end{aligned} \tag{5.149}$$

where $\mathcal{U}_R$ and $\mathcal{U}_I$ had been defined in (5.144) and the matrices Π as well as Δ had been introduced in equation (5.136). To proceed, we compute

$$\mathcal{U}_R^T\,\Pi\,\mathcal{U}_R = -4\,\big|C\big|^2\,Z^K(\mathrm{Im}\mathcal{M})_{KL}\,\overline{Z}^L = 2\,\mathrm{e}^{-2\phi}\,, \tag{5.150}$$

and, by carefully taking (5.123) into account, one finds

$$\Pi^{-1} - \alpha\,\frac{\mathcal{U}_R\,\mathcal{U}_R^T}{\mathcal{U}_R^T\,\Pi\,\mathcal{U}_R} = \begin{pmatrix}\mathbb{1}\\ \mathcal{M}\end{pmatrix}\left(-(\mathrm{Im}\mathcal{M})^{-1} + \alpha\,\frac{Z\,\overline{Z}^T}{Z^T(\mathrm{Im}\mathcal{M})\overline{Z}}\right)\begin{pmatrix}\mathbb{1}\\ \mathcal{M}\end{pmatrix}^T\,, \tag{5.151}$$

where our case of interest is $\alpha = 0$ and $\alpha = 1$. With these relations, from the terms involving $\mathcal{U}_I$ one can now reproduce the potential $V^{(2)}$ for the fields ξ and $\tilde{\xi}$. For the remaining terms, we note that the period matrix $\mathcal{M}$ can be expressed using the matrix $G_{LK} = \partial_{Z^L}G_K$ as follows

$$\mathcal{M}_{KL} = \overline{G}_{KL} + 2i\,\frac{(\mathrm{Im}G)_{KM}Z^M\,Z^N(\mathrm{Im}G)_{NL}}{Z^T(\mathrm{Im}G)Z}\,. \tag{5.152}$$

Employing then the relation (5.43), one can bring the terms involving $\mathcal{U}_R$ into the form (5.138).

In summary, we have outlined how the superpotential (5.5.3) indeed reproduces the scalar potential $V^{(2)} + V^{(3)}$.

5.5.4 Connection to manifolds with G_2 structure

In this subsection, we indicate a connection of the truncated theory studied above to compactifications of M-theory on seven-manifolds with G_2 structure. A manifold has G_2 structure if its structure group is contained in G_2, and if they feature a globally defined, G_2-invariant, real and nowhere-vanishing three-form Φ. In our present setting, we define the three-form Φ as

$$\Phi = \sqrt{2} R\, \mathcal{V}^{-\frac{1}{3}}\, J \wedge \mathrm{d}z + 4\, \mathrm{Re}(C\Omega) \ , \tag{5.153}$$

although we will not show that this actually defines a G_2-structure. Using (5.153), one can express the Kähler potential (5.155) and the superpotential (5.157) in the following way

$$\mathcal{K} = -3 \ln \left(\frac{1}{7} \int_{\mathcal{Y}} \Phi \wedge \star_7 \Phi \right) , \qquad W = \frac{1}{8} \int_{\mathcal{Y}} \big(\sqrt{2}\, C_3 + i\Phi\big) \wedge \mathrm{d}_7 \big(\sqrt{2}\, C_3 + i\Phi\big) \ . \tag{5.154}$$

These formulas agree with the those in the literature of M-theory on manifolds with G_2-structure [138–142].

To verify that the expressions in (5.154) indeed reproduce the Kähler potential and superpotential of our truncated theory, we first note that the sum of (5.82) and (5.133) can be brought into the form

$$\mathcal{K} = -\log\left[\, 8R^3 \,\right] - 2 \log \left[2\mathcal{V}^{\frac{1}{3}} R^{-1} \int_{\mathcal{Y}} \mathrm{Re}\big(C\Omega\big) \wedge \star_7 \mathrm{Re}\big(C\Omega\big) \right] \ . \tag{5.155}$$

Note that in the second term the integral is over the seven-manifold $\mathcal{Y}$ and its prefactor arises from the zz-component of the metric (5.58) by taking into account the Weyl rescaling mentioned above equation (5.27). From (5.122) and (5.74), utilizing $\star_6 \mathrm{Re}\Omega = \mathrm{Im}\Omega$, one also finds the relation

$$2\mathcal{V}^{\frac{1}{3}} R^{-1} \int_{\mathcal{Y}} \mathrm{Re}\big(C\Omega\big) \wedge \star_7 \mathrm{Re}\big(C\Omega\big) = 2 \int_{\mathcal{X}} \mathrm{Re}\big(C\Omega\big) \wedge \star_6 \mathrm{Re}\big(C\Omega\big) = \mathrm{e}^{-2\phi} = \mathcal{V} \ . \tag{5.156}$$

With the help of these, one indeed reproduces (5.155) from the Kähler potential in (5.154). For the superpotential, we note that, employing (5.40) as well as (5.56), we can express (5.5.3) in the following way

$$W = \frac{1}{4} \int_{\mathcal{Y}} \Omega_c \wedge \mathrm{d}_7 \Omega_c \ , \qquad \Omega_c = C_3 + i\, \sqrt{8}\, \mathrm{Re}(C\Omega) \ , \tag{5.157}$$

where $\mathcal{Y}$ is the seven-dimensional space given by (5.93) and C_3, subject to the truncation (5.124), was defined in (5.103). One then shows that the superpotential in (5.154) reproduces (5.157).

We finally remark that in the literature on M-theory compactifications on manifolds with G_2 structure, one usually does not find D-terms. Studying this question would be an interesting extension of our work.

5.6 Vacuum structure

The $\mathcal{N}=2$ theory

Let us now briefly analyze the vacuum structure of the $\mathcal{N}=2$ theory derived in section 5.3.2. In particular, to determine the minima of the potential (5.3.2) we first compute

$$\phi^A \frac{\partial}{\partial \phi^A} V = -3V \ , \tag{5.158}$$

which means that the potential is a homogeneous function of degree three in the fields ϕ^A. Thus, a necessary condition for a minimum is that the potential V vanishes. Since the potential (5.3.2) is a sum of semi-positive terms, each of those has to vanish independently. Thus, the non-degenerate solutions are

$$\begin{aligned} 0 &\stackrel{!}{=} \mathcal{N}^r = b^r - \frac{2}{3}\beta\, z^r + B^r{}_s z^s - \frac{1}{2} R^r{}_{st}{}^v a_v z^s z^t \ , \\ 0 &\stackrel{!}{=} N \begin{pmatrix} \xi \\ \tilde{\xi} \end{pmatrix} \\ 0 &\stackrel{!}{=} M^A{}_B\, \phi^B \ . \end{aligned} \tag{5.159}$$

Notice that, from the last two equations, the vacua are counted by the number of zero eigenvalues of the twisting matrices N and M. The eigenvectors of these matrices define a finite dimensional subspace, defining the flat directions of the scalar potential. In the orthogonal directions, the moduli are stabilized.

Of course, there are also degenerate solutions which can lead to a vanishing potential. These include configurations such as $(\xi^k, \tilde{\xi}_\lambda)^T = 0$, $\phi^A = 0$, $\phi \to -\infty$, $R \to \infty$, or where the matrices $G_{r\bar{s}}$, $\mathcal{M}$ and $\mathcal{K}_{AB}$ have zero eigenvalues.

Furthermore, since some of the scalar fields of the theory are gauged, a mass term for the gauge fields A^Λ can be generated. More concretely, one finds terms of the form

$$\int_{\mathbb{R}^{3,1}} \Big[M_{\Lambda\Sigma}\, A^\Lambda \wedge \star_4 A^\Sigma \Big] \ , \tag{5.160}$$

where $\Lambda = 0, \ldots, h^{1,1}$ and the mass matrix $M_{\Lambda\Sigma}$ has components

$$\begin{aligned} M_{AB} &= -\big(M^T g\, M\big)_{AB}\Big|_{\text{min.}} \ , \\ M_{0A} &= +\big(b^T M^T g\, M\big)_A\Big|_{\text{min.}} \ , \\ M_{00} &= -b^T M^T g\, M b\Big|_{\text{min.}} \ . \end{aligned} \tag{5.161}$$

Here $g = g_{AB}$ denotes the Kähler metric (5.78), $b^A = \mathrm{Re}\,(t^A)$ and matrix products are understood. Note that M_{00} contains an additional term proportional to the scalar potential, which however vanishes in the minimum.

The $\mathcal{N} = 1$ theory

To study the vacua of the truncated theory, we first recall the D- and F-term potential given in (5.143) and (5.146)

$$V = V_D + V_F = -\frac{1}{4R^6}\left(M^\alpha{}_c\,\phi^c\right)\left(M^\beta{}_d\,\phi^d\right)\mathcal{K}_{\alpha\beta} + 2\mathrm{e}^{\mathcal{K}}\Big(G^{I\overline{J}}D_I W D_{\overline{J}}\overline{W}\Big)\ . \qquad (5.162)$$

Similarly as for the $\mathcal{N} = 2$ case, a necessary condition for minima in the fields ϕ^a is

$$0 \overset{!}{=} \phi^a\,\frac{\partial}{\partial\phi^a}\,V = -3V\ , \qquad (5.163)$$

which, since (5.162) is a sum of semi-positive definite terms, implies that $V_D = V_F = 0$. The non-degenerate solution to $V_D = 0$ is given by $D_\alpha = 0$ which implies

$$M^\beta{}_c\phi^c = 0\ , \qquad (5.164)$$

whereas the non-degenerate solution to $V_F = 0$ leads to $F_I = 0$. One configuration satisfying this constraint reads

$$N\,\mathcal{U} = 0\ , \qquad (5.165)$$

where $\mathcal{U}$ is the vector of chiral coordinates of (5.144). Other solutions containing for instance $(\xi^k, \tilde{\xi}_\lambda) = 0$ are also possible.

For the mass terms of the vector fields we recall that the graviphoton A^0 as well as the fields A^a are projected out. We are thus left with

$$\int_{\mathbb{R}^{3,1}} M_{\alpha\beta}\,A^\alpha \wedge \star_4 A^\beta\ , \qquad (5.166)$$

where the mass matrix $M_{\alpha\beta}$ is found, employing (5.118), to be

$$M_{\alpha\beta} = -\big(M^T g\,M\big)_{\alpha\beta}\Big|_{\text{min.}}\ , \qquad (5.167)$$

with the metric g_{ab} taking indices $a, b = 1, \ldots, h^{1,1}_-$.

Chapter 6

Volume stabilization with NS5 branes

6.1 Introduction

When performing a compactification, one of the important quantities is the volume of the internal dimensions. The scale set by the volume has important consequences for low-energy physics, such as supersymmetry breaking and inflation, see e.g. [21, 143–146]. However, in ordinary compactifications, the volume is usually not stabilized. It is then the inclusion of quantum corrections that might provide a mechanism to stabilize the moduli.

Quantum corrections

A simple example of the effect of quantum corrections can be illustrated with the universal hypermultiplet, which was shortly discussed in section 2.3.2. This space describes the dilaton r and three moduli χ, φ, σ, and can be obtained by a compactification on a Calabi-Yau (CY) three-fold $\mathcal{X}$ which has $h^{2,1}(\mathcal{X}) = 0$. The metric, as presented in section 2.3.2, is only the tree-level result. There is a one-loop correction [80, 81], which corrects the metric to

$$\mathrm{d}s^2 = \frac{1}{r^2}\left(\frac{r+2c}{r+c}\,\mathrm{d}r^2 + (r+2c)\,(\mathrm{d}\chi^2 + \mathrm{d}\varphi^2) + \frac{r+c}{r+2c}\big(\mathrm{d}\sigma + \chi\mathrm{d}\varphi\big)^2\right). \qquad (6.1)$$

The constant c is given by $c = -\frac{h^{1,1}}{6\pi}$. With a typical value of $h^{1,1} \simeq 100$ we find $|c| \simeq 5$. This quantum correction is therefore important when $r \simeq 5$, and would only be negligible when $r \gg 5$. The latter condition is equivalent to $\mathcal{V}g_s^{-2} \gg 5$, where g_s is the ten-dimensional string coupling constant and $\mathcal{V}$ is the volume of the CY. For large volume and small string coupling constant this is clearly satisfied, but when one moves to small coupling and/or small volume, these corrections can be important.

The one-loop correction above turns out to be the only perturbative correction to the UHM metric; all higher loop corrections can be absorbed into field redefinitions. There are, however, non-perturbative corrections, which depend on the inverse of the coupling constant. In type IIA and type IIB, such corrections come (among others) from the wrapping of Euclidean D-branes and NS5-branes over cycles of the internal space. D-branes can be seen as objects where strings can end on, but this is not the case for NS5-branes, which makes them more difficult to study. They were first discovered as supergravity backgrounds [147, 148]. Being six-dimensional objects, we can (after a Wick rotation to an Euclidean brane) wrap them around a CY three-fold. These possibilities give corrections to the four-dimensional effective action [149].

Volume stabilization

For type IIB flux compactifications on CY manifolds (and their orientifolds), the volume is stabilized by non-perturbative effects, such as stringy D3-brane instantons whose Euclidean worldvolume wraps a four-cycle in the CY. Such an instanton stabilizes the volume of the corresponding four-cycle and therefore the associated Kähler modulus. The relation between the four-cycle and two-cycle volumes is known in principle, but is complicated in practice since it requires inverting a set of coupled quadratic equations involving the triple intersection numbers, as we review in the next section. Therefore, although stabilization of all Kähler moduli indeed stabilizes the entire volume of the CY, it requires a case by case analysis to fix it at large values[24], see for instance [145] for a recent analysis.

The fact that a wrapped Euclidean p-brane can stabilize the volume of a $p+1$-cycle naturally raises the question of whether the overall CY-volume can be stabilized directly by wrapping a Euclidean fivebrane over the entire CY. From this point of view, one could expect NS5-brane instantons to play an important role in relation to volume stabilization [25]. In case such instantons contribute to the low-energy scalar potential, they will do so with exponentially suppressed terms of the form

$$\exp\left[-S_{\mathrm{NS5}}\right] = \exp\left[-\frac{\mathcal{V}}{g_s^2}\right] \,, \tag{6.2}$$

[24]The situation might seem to look better in type IIA theories, since there one can stabilize the Kähler moduli directly at the classical level by switching on fluxes [150]. However, the Kähler moduli are fixed again by solving a set of quadratic equations involving the triple intersection numbers, see equation (4.36) in [150]. So to find large values for the total volume, one ends up with similar difficulties as in type IIB.

[25]In $\mathcal{N}=2$ compactifications of type IIB, one also expects D5-brane instantons to contribute. However, after orientifold projection with O3/O7 planes, the D5 brane is not BPS and therefore harder to analyze. Furthermore, we look for a mechanism that also applies to IIA and heterotic string theories.

where $S_{\rm NS5}$ is the (real part of the) one-instanton action, $\mathcal{V}$ is the volume of the CY in dimensionless units, and g_s is the ten-dimensional string coupling constant. This can be compared to the contribution of a D3-brane instanton, of the form

$$\exp\left[-S_{\rm D3}\right] = \exp\left[-\frac{\mathrm{vol}(\gamma_4)}{g_s}\right] . \tag{6.3}$$

Here, vol(γ_4) is the volume of the four-cycle γ_4.

At weak string coupling constant, one expects the D3-brane instantons to dominate over the NS5-brane instantons, such that one can safely ignore the latter. The two exponents only are of the same order of magnitude when

$$\frac{\mathrm{vol}(\gamma_4)}{\mathcal{V}} = \frac{1}{g_s} . \tag{6.4}$$

For small four-cycles and weak string coupling, $g_s \leq 1$, this is never satisfied. However, there are plenty of CY manifolds which have four-cycles with larger volume than the total volume, as we review in the next section, so some care is needed to make this argument, especially when $g_s \sim 1$. Similar considerations hold for compactifications of type IIA strings on CY threefolds, in which membrane instantons arise by wrapping Euclidean D2-branes over three-cycles.

There is another reason to be careful in ignoring the fivebrane instantons. Assuming that both instantons contribute to the scalar potential in the effective action and vol $\gamma_4 < \mathcal{V}$, the exponents above can still be multiplied by prefactors to make them of the same order, especially at intermediate string coupling $g_s \sim 1$. We will show this more explicitly in the next section, for values $\mathcal{V} \approx 100$ (which are the typical values of the original KKLT approach). The existing vacua of the LVS scenarios at $\mathcal{V} \approx 10^{13}$ are not affected by NS5-brane instantons. However, at smaller volumes $\mathcal{V} \sim 100$, additional vacua can arise with interesting properties.

The purpose of this chapter is to analyze the effects of NS5-brane instantons in relation to the stabilization of the volume modulus. In particular, we show that under certain conditions, the contributions from NS5-brane instantons yield uplifting terms in the scalar potential that can lead to meta–stable de Sitter vacua. Most of the work on moduli stabilization has focused on $\mathcal{N} = 1$ supersymmetry in four dimensions, as they give rise to semi-realistic string vacua. In such models, like e.g. type IIB strings on Calabi-Yau orientifolds, moduli can be stabilized by combining the effects of fluxes and quantum corrections coming from perturbative corrections to the Kähler potential and D3-brane instanton corrections to the superpotential. However, the Kähler potential is subject to higher loop corrections in α' and g_s which are not known explicitly. For a recent discussion on this, see [145, 151, 152]. As we will show, the non-perturbative corrections of the form (6.2) also contribute to the Kähler potential, and are generically subleading with respect to the first perturbative corrections, but could compete with next–to–leading perturbative corrections. For this reason, our investigations are more meaningful in $\mathcal{N} = 2$ models, since in that case higher order corrections are absent due to the constraints from $\mathcal{N} = 2$ supersymmetry. This fact also motivated the authors of [61] to study $\mathcal{N} = 2$ moduli potentials in type IIA flux compactifications. These toy models can serve as good approximations for the more realistic $\mathcal{N} = 1$ string vacua. Moreover, for $\mathcal{N} = 2$ theories in type IIA, there are some explicit results known about the contribution of NS5-brane instantons [153–156] to the effective action for the hypermultiplets.

Plan of this chapter

The plan of this chapter is as follows. In section 2, we present the generic form of an NS5–brane instanton correction to the scalar potential. We study this in the setting of $\mathcal{N} = 1$ supergravity in four dimensions, and investigate the relation with the KKLT and LVS scenarios. In section 3, we discuss IIA strings compactified on a (rigid) CY, for which there is some explicit knowledge on NS5-brane instantons. We investigate the stability of the volume, and find the possibility that NS5-brane instantons can produce de Sitter vacua. We then truncate this model preserving local $\mathcal{N} = 1$ supersymmetry, and determine the Kähler and superpotential.

6.2 Volume stabilization

In this section, we review certain aspects of the KKLT scenario and discuss some of the subtleties that can arise in stabilizing the volume at large values in IIB orientifold compactifications. We then include terms that mimic the contributions from NS5-brane instantons to the Kähler potential, and re-analyze the stabilization of the volume modulus.

We consider an orientifold of type IIB string theory on a CY 3-fold with O3/O7 planes. The cohomology groups $H^{(p,q)}$ are split under the orientifold mapping into odd and even forms, and hence their dimensions split as $h^{p,q} = h^{p,q}_{+} + h^{p,q}_{-}$. We follow the notation of [157], although we change a few names and numerical factors.

Let us first list the various chiral fields. The field τ contains the axion and dilaton, and is defined by $\tau = l + i\mathrm{e}^{-\phi_{10}}$. The fields T_i are defined by

$$T_i = \tau_i + ih_i - 2\zeta_i, \quad i, j = 1, \ldots, h^{1,1}_{+}, \tag{6.5}$$

where

$$\zeta_j = -\frac{i}{2(\tau - \bar{\tau})}\mathcal{K}_{jab}G^a(G - \bar{G})^b, \quad G^a = c^a - \tau b^a, \quad a, b = 1, \ldots, h^{1,1}_{-}. \tag{6.6}$$

The τ_i capture the sizes of the even four–cycles under the orientifold projection and h_i are real fields that arise by expanding the C_4 gauge field over these four–cycles. The fields b^a, c^a are the expansions of the B_2 and C_2 forms respectively over the $h^{1,1}_-$ cycles. Notice that the definition of ζ_i contains intersection numbers of even and odd two–cycles.

The four–cycles τ_i are related to the two–cycles t^i by the triple intersection numbers $\mathcal{K}_{ijk}$ as

$$\tau_i = \mathcal{K}_{ijk} t^j t^k \,. \tag{6.7}$$

The total volume $\mathcal{V}$ is only implicitly known as a function of the $\mathcal{N} = 1$ chiral coordinates through the relation

$$\mathcal{V} = \frac{1}{6} \mathcal{K}_{ijk} t^i t^j t^k \,. \tag{6.8}$$

To write the volume in terms of the chiral fields we first use the definitions (6.5), (6.6) to find

$$\tau_i = \frac{1}{2}(T_i + \bar{T}_i) - \frac{i}{2}\, \mathcal{K}_{iab} b^a b^b (\tau - \bar{\tau}) \,, \tag{6.9}$$

or in terms of the chiral fields

$$\tau_i = \frac{1}{2}(T_i + \bar{T}_i) - \frac{i}{2} \frac{1}{(\tau - \bar{\tau})}\, \mathcal{K}_{iab} (G - \bar{G})^a (G - \bar{G})^b. \tag{6.10}$$

One then solves the quadratic equations in equation (6.7) to obtain functions $t^i(\tau_j)$, and one obtains the volume $\mathcal{V}$ depending on the chiral fields via $\{\tau - \bar{\tau}, T_i + \bar{T}_i, (G - \bar{G})^a\}$.

The type IIB Kähler potential is given by [157, 158]

$$\begin{aligned} \mathcal{K} &= \mathcal{K}^{\rm cs}(U, \bar{U}) + \mathcal{K}^{\rm k}(\tau, T, G) \,, \\ \mathcal{K}^{\rm k} &= -\ln\left[-i(\tau - \bar{\tau})\right] - 2\ln\left[\mathcal{V}(\tau, T, G) + \xi\, {\rm Im}(\tau)^{3/2}\right] . \end{aligned} \tag{6.11}$$

The Kähler potential $\mathcal{K}^{\rm k}$ in (6.11) is the tree-level expression, together with the leading perturbative α' correction proportional to the parameter $\xi = -\frac{\chi(CY)\zeta(3)}{2(2\pi)^3}$, containing the Euler number $\chi(CY)$ of the internal Calabi-Yau M (we use conventions where $l_s = 2\pi\sqrt{\alpha'}$). The complex structure deformations U are described by $\mathcal{K}^{\rm cs}$, whose precise form is not important. Higher string loop corrections could give a dilaton–dependence to $\mathcal{K}^{\rm cs}(U, \bar{U})$, but this is beyond the approximation we are working in.

The scalar potential for a Kähler potential $\mathcal{K}$ and superpotential W is given by

$$V = {\rm e}^{\mathcal{K}} \Big(\mathcal{K}^{\alpha\bar{\beta}} D_\alpha W D_{\bar{\beta}} \overline{W} - 3|W|^2 \Big), \tag{6.12}$$

where the indices $\alpha, \bar{\beta}$ run over all chiral fields, with $\mathcal{K}^{\alpha\bar{\beta}}$ the inverse Kähler metric.

6.2.1 Fluxes and D3–brane instantons

The tree–level superpotential is given by

$$W = W_0(\tau, U) \; = \int_M \Omega \wedge G_3 \,, \tag{6.13}$$

where G_3 is the complex combination of the background three–form fluxes, given by $G_3 = F_3 - \tau H_3$.

We now assume that the complex structure moduli U and the axio-dilaton τ are stabilized at a higher energy scale at a SUSY minimum, by demanding $D_\tau W = D_U W = 0$. To stabilize the Kähler moduli we add the non-perturbative instanton corrections of wrapping Euclidean D3–branes over four–cycles. The superpotential is given by [19]

$$W = W_0 + \sum_i A_i \mathrm{e}^{-a_i T_i} \,, \tag{6.14}$$

where A_i and a_i are treated as field–independent parameters. In the literature one often considers the case where the G^a fields are absent. Using the expressions (6.11) and (6.14), one finds the following scalar potential [158, 159]

$$V = \mathrm{e}^{\mathcal{K}} \left[\mathcal{K}^{j\bar{k}} \left(a_j A_j a_k \bar{A}_k \mathrm{e}^{-a_j T_j - a_k \overline{T_k}} - \left(a_j A_j \mathrm{e}^{-a_j T_j} \overline{W} \mathcal{K}_{\bar{k}} + \text{c.c.} \right) + \mathcal{K}_j \mathcal{K}_{\bar{k}} |W|^2 \right) - 3|W|^2 \right] , \tag{6.15}$$

where $\mathcal{K}^{i\bar{\jmath}}$ are the components of the inverse metric $\mathcal{K}^{\alpha\bar{\beta}}$ in the directions of the Kähler moduli. In the absence of the G^a fields, there is no–scale structure at tree–level, leading to $\mathcal{K}^{i\bar{\jmath}} \mathcal{K}_i \mathcal{K}_{\bar{\jmath}} = 3$. This no–scale structure is broken when α'–corrections are included, and one then finds (see [158], or for some further details of the calculation, see appendix E)

$$V = \mathrm{e}^{\mathcal{K}} \Bigg[\mathcal{K}^{j\bar{k}} \left(a_j A_j a_k \bar{A}_k \mathrm{e}^{-a_j T_j - a_k \overline{T_k}} - \left(a_j A_j \mathrm{e}^{-a_j T_j} \overline{W} \mathcal{K}_{\bar{k}} + \text{c.c.} \right) \right) + 3\xi \frac{\xi^2 + 7\xi\mathcal{V} + \mathcal{V}^2}{(\mathcal{V} - \xi)(2\mathcal{V} + \xi)^2} |W|^2 \Bigg] . \tag{6.16}$$

The various studies (see for example [160]) of these potentials indicate a large volume AdS vacuum, which can be realized in explicit models. In the $\mathbb{P}^4_{[1,1,1,6,9]}$ model, for example, which yields two Kähler moduli, the volume is expressed in terms of the 4–cycle volumes τ_s, τ_b as

$$\mathcal{V} = \frac{1}{9\sqrt{2}} \left(\tau_b^{3/2} - \tau_s^{3/2} \right) . \tag{6.17}$$

This already gives an example where a four–cycle volume can be bigger than the total volume, as mentioned below (6.4)[26].

To remain in the regime of a geometrical compactification, one needs $\tau_b > \tau_s$. To obtain a large volume, one needs to arrange $\tau_b \gg \tau_s$. This is then solved self–consistently: one assumes a cycle to be small and the other large, approximates the potential in this regime and then searches for sets of vacua. In such a two–Kähler model this is doable, but a more general model will have many different Kähler moduli. To express the volume in terms of the four-cycles, one first has to solve the system of many coupled quadratic equations (6.7) and use the explicit form of the triple intersection numbers. Even for simple models with a few Kähler moduli, this can lead to equations which are not solvable analytically. Numerical methods can be used, but have their own limitations. One then has to find a limit on the four–cycles that leads to a large volume $\mathcal{V}$. This will be very difficult without any analytical control. Overall, it seems desirable to have a different mechanism that stabilizes the volume at once, without the need to stabilize the individual cycles that build up the total volume. A prime candidate for such a mechanism is the NS5-brane instanton, to which we turn now.

6.2.2 Adding NS5-brane instantons

We now add a correction due to the wrapping of a Euclidean NS5–brane over the entire CY. Such an instanton configuration contributes to correlators proportional to $\exp(-\mathcal{V}/g_s^2)$, where g_s is the 10–dimensional string coupling constant. The volume can not be expressed as a holomorphic function of the $\mathcal{N} = 1$ chiral fields. We therefore expect that the NS5–brane does not correct the superpotential, but instead it will correct the Kähler potential,

$$\mathcal{K}_{\text{NS5}} = B\,\mathcal{V}^n \exp(-\mathcal{V}/g_s^2) \; = B\,\mathcal{V}^n \exp\left(\frac{1}{4}\mathcal{V}(\tau-\bar{\tau})^2\right). \tag{6.18}$$

The factor of $\mathcal{V}^n$ represents the leading power of the instanton measure and the one-loop determinant of the fluctuations around the instanton solution. There is a proportionality factor B that could – in principle – depend on the moduli G^a and the dilaton τ. We expect no dependence on the complex structure moduli U since the NS5–brane cannot probe the individual 3–cycles. Furthermore, before the orientifold projection, the NS5–branes correct the moduli space of Kähler deformations, and not the complex structure deformations. The prefactor $\mathcal{V}$ is absent in an instanton corrected superpotential: because the superpotential is a holomorphic function of the chiral fields T_i, any non–trivial function $A_i(T)$ in (6.14) breaks the shift symmetry on the imaginary

[26] Take for example the values $\tau_s \sim 4.6$ and $\tau_b \sim 120$ which give a total volume $\mathcal{V} \sim 100$.

parts h_i completely and is therefore forbidden. Since instantons are expected to break the shift symmetries to a discrete subgroup only, we can use superpotentials of the form $\exp(-a_i T_i)$. The Kähler potential does not need to be holomorphic, and we can therefore only use the real parts of the chiral fields T_i, which combine into powers of the volume $\mathcal{V}$.

A further argument in favor of (6.18) comes from the parent $\mathcal{N} = 2$ theory. NS5-branes correct the hypermultiplet moduli space even in the absence of fluxes [149]. In IIB compactifications, the hypermultiplet moduli are counted by the Kähler moduli, and the kinetic terms of these scalars receive corrections from NS5-brane instantons. After the orientifold projection, one expects these corrections to survive, even after fluxes are turned on. Our claim is that, to leading order, they enter the Kähler potential in the way described in (6.18). We elaborate further on this in the next section for type IIA compactifications, where we can determine the one–instanton NS5–brane contribution explicitly in some special cases [153].

We have to ask in which regime this approach makes sense. We want to remain in the one–instanton regime, and not consider multiple instanton contributions since nothing is known about them. Because an NS5–brane instanton scales as $\exp(-\mathcal{V}/g_s^2)$, this requires that

$$\frac{g_s^2}{\mathcal{V}} < 1 \,. \tag{6.19}$$

However, since the NS5-brane is the magnetic dual of the fundamental string, one also has to consider perturbative effects, which we expect to organize into an expansion in powers of $g_s^2/\mathcal{V}$. Besides that, there are also higher order α' corrections which are left out. Only the first perturbative correction is included in our analysis in (6.11). In $\mathcal{N} = 2$ theories these higher string-loop corrections are absent [80, 161], which makes the discussion of the NS5-brane instanton more reliable. We discuss such models in the next section. We take a pragmatic approach here and isolate the NS5-brane instanton correction from all other corrections in the Kähler potential $\mathcal{K}_0$. Hence we write

$$\mathcal{K} = \mathcal{K}_0(\tau, T, U, G) + \mathcal{K}_{\mathrm{NS5}}(\tau, T, G) \,, \tag{6.20}$$

and expand to leading order in $\mathcal{K}_{\mathrm{NS5}}$. To obtain the expression for the inverse Kähler metric $\mathcal{K}^{\alpha\bar{\beta}}$ (where $\alpha = \{\tau, T, G, U\}$ lists all the chiral fields) in the direction of the Kähler moduli, we use

$$\mathcal{K}^{i\bar{\jmath}} = \mathcal{K}_0^{i\bar{\jmath}} - (\mathcal{K}^{\mathrm{NS5}})^{i\bar{\jmath}}, \quad \mathcal{K}_{\mathrm{NS5}}^{i\bar{\jmath}} \equiv \mathcal{K}_0^{i\bar{\alpha}} \mathcal{K}_{\mathrm{NS5}\,\bar{\alpha}\beta} \mathcal{K}_0^{\beta\bar{\jmath}} \,, \tag{6.21}$$

and now $\mathcal{K}^{i\bar{\jmath}}\mathcal{K}_{\bar{\jmath}k} = \delta^i_k + \mathcal{O}\big(\mathcal{K}_{\mathrm{NS5}}^2\big)$.

In principle, there could be a dependence in $\mathcal{K}^{i\bar{\jmath}}_{\rm NS5}$ on the fields b^a, which enter through the definition of the chiral fields G^a in (6.5) and (6.6). As explained at the end of appendix E, the exact dependence on b^a is subleading in $\mathcal{K}^{i\bar{\jmath}}_{\rm NS5}$.

To obtain the scalar potential, we work out equation (6.12), setting $D_\tau W = D_U W = 0$. For simplicity, we now set G^a to zero. It would be interesting to consider the effects of non–zero G^a, but this is beyond the scope of this article. Using this in (6.12) leads to

$$V = V_0 + V_0 \mathcal{K}_{\rm NS5} - {\rm e}^{\mathcal{K}_0} \mathcal{K}^{i\bar{\jmath}}_{\rm NS5} \, |D^{(0)}_i W|^2 + {\rm e}^{\mathcal{K}_0} \mathcal{K}^{i\bar{\jmath}}_0 \Big(\partial_i \mathcal{K}_{\rm NS5} \, W D^{(0)}_{\bar{\jmath}} \overline{W} + \text{c.c.} \Big) . \qquad (6.22)$$

The potential in absence of $\mathcal{K}^{\rm NS5}$ is denoted V_0, $D^{(0)}_i W = \partial_i W + (\partial_i \mathcal{K}_0) W$ and $\mathcal{K}^{i\bar{\jmath}}_0$ is the inverse of $\mathcal{K}_{0,\alpha\bar{\beta}}$ in the directions of the Kähler moduli. Recall that all chiral fields are labeled by $\alpha, \bar{\beta}$, and the Kähler moduli are a subsector thereof labeled by $i, \bar{\jmath}$. Formula (6.22) in fact holds for any perturbation of the Kähler potential, labeled by $\mathcal{K}_{\rm NS5}$.

From equations (6.21)–(6.22) we can see that we have to compute $\mathcal{K}_{{\rm NS5}\bar{\alpha}\beta}$, so we have to take derivatives with respect to all chiral fields. This is to be contrasted to the situation in which one modifies the superpotential: the derivatives with respect to τ and U are contained in $D_\tau W$ and $D_U W$, which are set to zero.

Suppose for simplicity that W does not depend on the Kähler moduli (such as in equation (6.13)), then $D^{(0)}_i W_0 = \mathcal{K}^{(0)}_i W_0$ and hence

$$V = V_0 + V_0 \mathcal{K}_{\rm NS5} - {\rm e}^{\mathcal{K}_0} \mathcal{K}^{i\bar{\jmath}}_{\rm NS5} \, |\partial_i \mathcal{K}_0|^2 |W_0|^2 + {\rm e}^{\mathcal{K}_0} \mathcal{K}^{i\bar{\jmath}}_0 \Big(\partial_i \mathcal{K}_{\rm NS5} \, \partial_{\bar{\jmath}} \mathcal{K}_0 + \text{c.c.} \Big) |W_0|^2 \, . \quad (6.23)$$

Using expression (6.11) for $\mathcal{K}_0$, we find that $\mathcal{K}^{i\bar{\jmath}}_0$ is proportional to $t^i t^j$ and $\mathcal{K}^{ij}$ (the inverse of $\mathcal{K}_{ij} = \mathcal{K}_{ijk} t^k$), and both $\partial_i \mathcal{K}_0$ and $\partial_i \mathcal{K}_{\rm NS5}$ are proportional to $\mathcal{K}_i = \mathcal{K}_{ij} t^j$. Upon contracting indices, this will combine nicely into powers of $\mathcal{V}$. The precise calculation can be found in appendix E. Ignoring the D3–brane instantons, and other subleading corrections to the Kähler potential, the leading correction to V_0 is found to be

$$V_{\rm NS5} = -\frac{9}{8} B |W_0|^2 g_s^{-3} \mathcal{V}^n \exp(-\mathcal{V}/g_s^2) \equiv \hat{B} \mathcal{V}^n \exp(-\mathcal{V}/g_s^2) \, . \qquad (6.24)$$

In the last expression, we have absorbed all the constants into a new prefactor $\hat{B}$.

6.2.3 Analysis of the model

We have motivated the scalar potential

$$V = V_0(\mathcal{V}) + \hat{B} \mathcal{V}^n {\rm e}^{-\mathcal{V}/g_s^2} \, , \qquad (6.25)$$

where $\hat{B}, n$ are constants.

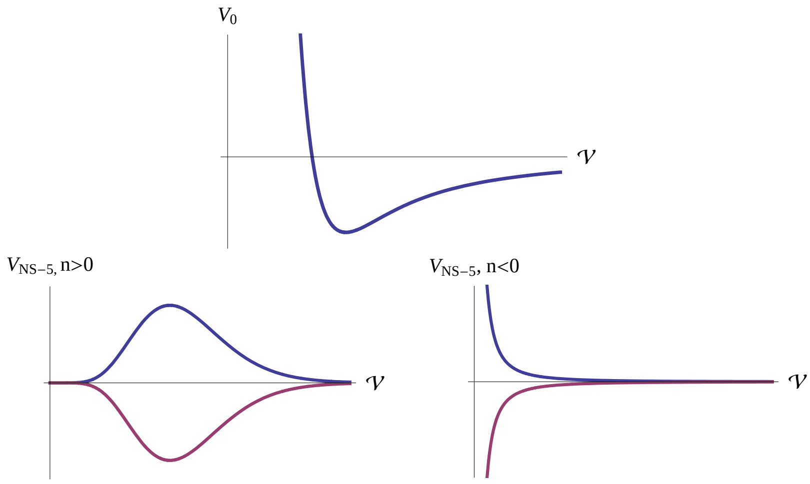


Figure 6.1: General shape of the three terms in the potential. The first is the AdS minimum of V_{pert}. The second and third are the NS5–brane contribution $\hat{B}\mathcal{V}^n \mathrm{e}^{-\mathcal{V}/g_s^2}$, with n positive and negative, respectively. The possible signs of $\hat{B}$ are given by the upper line ($\hat{B}$ positive) and the lower line ($\hat{B}$ negative).

A general situation achieved in moduli stabilization is indicated on the left plot below, which displays an AdS minimum. We will investigate the physics of the NS5–brane contribution, depending on the signs of the parameters $\hat{B}$ and n. The results are summarized in the table below. The most interesting case is when n is positive and $\hat{B}$ is positive (the upper line in the middle graph). The exact location of the bump depends on the value of n. Its value is at this point undetermined, but we will argue in the next section that

$$n = -3 - \frac{\chi}{12\pi} \,. \tag{6.26}$$

This is the value obtained in the $\mathcal{N} = 2$ theory, and we assume that its order of magnitude remains the same in the orientifolded $\mathcal{N} = 1$ theory.

For $n > 0$, the contribution from the five–brane is positive and will therefore certainly increase the value of the potential at the minimum. Depending on the strength $\hat{B}$ and the location of the contribution it can produce new vacua or uplift the existing minimum to a de Sitter vacuum. With n positive and $\hat{B}$ negative, the five–brane yields a negative energy contribution to the potential function. Depending on the parameters, it will introduce new vacua or lower the potential energy at the location of the existing vacuum. The situation is also interesting if the existing scenarios do not stabilize the volume and we obtain a run–away potential which behaves as $V_0 \simeq \mathcal{V}^m$, with m negative. A five–brane contribution with positive n can then provide the volume stabilization. The possible signs of n and $\hat{B}$ and their results are summarized in the table below.

n	$\hat{B}$	Result
+	+	Increased vacuum energy (uplift), with de Sitter vacua
+	–	Decreased vacuum energy, increased number of minima
–	+	Increased vacuum energy (uplift)
–	–	Decreased vacuum energy

We have outlined the model and the qualitative behavior, and will now investigate the numerics. The general scenario is described by

$$V = V_0(\mathcal{V}) + V_{\rm NS5}, \quad V_{\rm NS5} = \hat{B}\mathcal{V}^n {\rm e}^{-\mathcal{V}/g_s^2}. \tag{6.27}$$

We have not really specified V_0 here; one can choose a favorite scenario in which the Kähler moduli are stabilized, and then investigate the influence of the NS5–brane contribution. The expression for $V_{\rm NS5}$ will, in general, be more complicated: if we take a non–trivial W (e.g. including D3–brane instantons as in (6.14)), there will be mixing terms consisting of Kähler perturbations and superpotential perturbation in terms like $|D_i W|^2$. These terms are typically suppressed, or at most of the same order as in (6.27). In both cases, the numerical analysis below remains the same.

The scalar potential (6.27) contains an overall term $\exp(\mathcal{K}^{\rm cs}(U,\bar{U}))$, whose exact value depends on the details of the complex structure moduli stabilization. We set this factor to unity to compare with other models [27]. The KKLT scenario stabilizes the volume at $\mathcal{V} = \mathcal{O}(100)$ at a minimum of $V_0 = -2 \cdot 10^{-15}$. The LVS on $\mathbb{P}^4_{[1,1,1,6,9]}$ stabilizes the volume at $\mathcal{V} = \mathcal{O}(10^{12})$ with a value of $V_0 = -6 \cdot 10^{-37}$ [145].

Using $B = -10, W_0 = 1$ (implying $\hat{B} = 90/8g_s^{-3}$) and an Euler number of $\chi = -300$, we find (using (6.26)) the values

$V_{\rm NS5}$	$\mathcal{V}=10$	$\mathcal{V}=13$	$\mathcal{V}=20$	$\mathcal{V}=60$	$\mathcal{V}=130$
$g_s = 0.5$	10^{-10}	10^{-15}	10^{-26}	10^{-93}	10^{-213}
$g_s = 1$	10^{2}	10^{0}	10^{-1}	10^{-16}	10^{-44}

(6.28)

For very weakly coupled strings, $g_s < 0.5$, the effect of the NS5–brane instanton is negligible. For $g_s = 0.5$ one sees from the table that a volume of order $\mathcal{V} \simeq 15$ yields corrections to the potential that cannot be ignored in a KKLT scenario. For higher values of the string coupling constant, $g_s = 1$, the corrections to the potential at $\mathcal{V} \simeq 60$, are of the same order as the value of the KKLT–potential at its minimum. In that case, NS5–brane instantons cannot be ignored and can change the KKLT AdS vacuum to become dS, although fine tuning is required.

[27] In $V_{\rm NS5}$ it was absorbed in the factor $\hat{B}$.

Quantum corrections (both in α' and g_s) become very important at those scales. The exact form of those corrections is not known, but we can make some rough estimates. The first correction in α' scales as $\hat{\xi}^3/\mathcal{V}^4$, where $\hat{\xi} = \xi g_s^{-3/2}$, and higher corrections are expected to be further suppressed by factors $\hat{\xi}/\mathcal{V}$. For the values $g_s = 1$, $\mathcal{V} = 50$, $\chi \sim -300$ we find $\hat{\xi} \sim 0.7$ and

$$V_{\mathrm{NS5}} \sim \frac{\xi^5}{\mathcal{V}^6} \,. \tag{6.29}$$

This can be of the same order as next–to–subleading corrections in α'.

6.2.4 Fivebranes and orientifold projections

We have shown the influence of a NS5–brane instanton on the scalar potential. There is, however, a subtlety in the microscopic string theory that needs to be addressed. The NS5–brane instanton arises from the wrapping of the 10–dimensional NS5–brane soliton solution. In 10 dimensions, the NS5–brane is the magnetic source of the NS–NS B_2 field. Such a wrapping naturally arises when we compactify six internal dimensions. In such a compactification, the 4–dimensional part of the B_2 field is dualized to an axion σ, and the NS5–brane instanton yields exponential corrections of the form

$$\mathrm{e}^{-\frac{\mathcal{V}}{g_s^2}|Q| + iQ\sigma} \,, \tag{6.30}$$

where Q is the instanton charge. As usual, the instanton action contains an imaginary part, that distinguishes between instantons ($Q > 0$) and anti–instantons ($Q < 0$). Microscopically, this distinction arises when one has to specify the orientation of the wrapping relative to the orientation of the CY. However, the field σ gets projected out in an orientifold; see e.g. [138]. In the ten–dimensional picture, this corresponds to saying that the space–time part of B_2 gets projected out, but the NS5–brane couples magnetically to this field. These considerations seem to lead, on the one hand, to the conclusion that an NS5–brane instanton cannot exist, at least not in the traditional sense.

On the other hand, one could argue that at the level of the effective action, all even combinations in σ survive the projection. Examples of such terms are $\exp(-\mathcal{V}/g_s^2)$ and $\exp(-\mathcal{V}/g_s^2)\cos\sigma$. The last term can be interpreted as an instanton ($Q = 1$) – anti–instanton pair ($Q = -1$). Such a pair would annihilate, unless some other mechanism stabilizes the pair. They cannot be separated in the internal manifold as e.g. for D3–branes, because they wrap the entire CY. This suggests that they cannot preserve $\mathcal{N} = 1$ SUSY after the orientifold projection, but in the next section, we show that they still can be written in an $\mathcal{N} = 1$ supergravity action. So we are led to the conclusion that instantons do remain present after taking the orientifold projection.

The situation can be described with the following diagram:

$$\begin{array}{lcl} \text{II String theory}/CY & \longrightarrow & \mathcal{N}=2, \text{effective supergravity description} \\ \Big\downarrow \ \text{orientifold} & & \Big\downarrow \ \text{orientifold} \\ \text{II String theory}/CY_{\text{or}} & \longrightarrow & \mathcal{N}=1, \text{effective supergravity description} \end{array} \tag{6.31}$$

Starting from the full–fledged string theory in the top left, one can obtain an effective $\mathcal{N} = 2$ supergravity description, containing effects from NS5–brane instantons. We then orientifold by simply putting $\sigma = 0$ (and other fields) in this action and obtain an $\mathcal{N} = 1$ theory. This procedure shows that there is a contribution from NS5–branes consistent with $\mathcal{N} = 1$ SUSY. However, one could argue that the correct way to proceed is to incorporate the orientifold projection in string theory, and then calculate the low–energy effects of a NS5–brane. It would be interesting to compare these two approaches; we leave this question open for further investigation. The situation is better understood before the orientifold projection, when we still have $\mathcal{N} = 2$. We will now turn to this setting.

6.3 The $\mathcal{N} = 2$ scenario

In this section we will describe our results in the more stringent language of $\mathcal{N} = 2$ supergravity. In this setting we have good control over the possible quantum corrections. Furthermore, there are no subtleties with the orientifold projection of the NS5–brane, as discussed at the end of the previous section.

Although the previous section dealt with IIB string compactifications, we will change in this section to type IIA models. The reason is of technical origin, as the dimension of the hypermultiplet moduli space in IIA is given by $4(h_{1,2}+1)$ (as opposed to $4(h_{1,1}+1)$ for IIB), and $h_{1,2}$ can be set to zero for rigid CY's. This yields a four–dimensional moduli–space, which simplifies the analysis. Moreover, NS5–brane instantons were analyzed in these models in [153, 154], and we will make use of these results. We expect that the results for IIA carry over to IIB.

6.3.1 Gauged and ungauged $\mathcal{N} = 2$ supergravity

In ungauged $\mathcal{N} = 2$ supergravity, the moduli space has the local product structure

$$\mathcal{M}^K \times \mathcal{M}^Q \, . \tag{6.32}$$

For type IIA strings, the special Kähler manifold $\mathcal{M}^K$ has dimension equal to $2h^{1,1}$ and is spanned by the scalars in the vector multiplets, corresponding to the deformation of the Kähler form. The quaternionic-Kähler space $\mathcal{M}^Q$ is spanned by the scalars in the hypermultiplets and is $4(h^{1,2}+1)$ dimensional. The manifold $\mathcal{M}^K$ is described in terms of a prepotential $F(X^I)$, where $I = 0, \ldots, h^{1,1}$. In supergravity, this can be any holomorphic function of the X^I variables of degree two. The prepotentials obtained from IIA string theory have the specific form

$$F(X) = \frac{1}{3!}\frac{\mathcal{K}_{ijk}X^iX^jX^k}{X^0} + \frac{i}{2}\zeta(3)\chi(CY)X^0X^0 - i\sum_{k_a} n_{k_a}\mathrm{Li}_3(\mathrm{e}^{2\pi i k_a X^a/X^0}), \tag{6.33}$$

where the first term is a tree–level contribution, the second is a perturbative one-loop correction and the last terms are the non-perturbative worldsheet instanton contributions. Note that there is only one perturbative correction, so the perturbative regime is under complete control. The geometry of $\mathcal{M}^Q$ is known at tree–level and at one–loop. It is argued in [161] that higher loop corrections can be absorbed into field redefinitions, and if so, the entire perturbative corrected geometry is known [80, 81, 161]. If we restrict ourselves to a rigid CY manifold, which has $h^{2,1} = 0$ by definition, there is only one hypermultiplet, which is called the universal hypermultiplet (UHM).

Gauged supergravities arise when isometries on the moduli space are gauged. They give rise to scalar potentials that are consistent with $\mathcal{N} = 2$ supersymmetry. Microscopically, gauged supergravities arise when fluxes (in the RR and NS–NS sector) are turned on. For the purpose of this chapter, it suffices to look at abelian isometry groups.

The scalar potential is determined by the geometrical data of the moduli space, such as the choice of killing vectors k_I and their corresponding moment maps $\vec{\mu}_I$. For further details on the gauging, we refer to appendix E.1 and references therein. The result for the scalar potential is

$$V = -4\left[2G_{\alpha\bar\beta}k_I^\alpha k_J^\beta + 3\vec{\mu}_I\cdot\vec{\mu}_J\right]\frac{X^I\bar X^J}{N_{MN}X^M\bar X^N} - 4N_{MN}X^M\bar X^N\mathcal{M}_{IJ}N^{IK}N^{JL}\vec{\mu}_K\cdot\vec{\mu}_L\,. \tag{6.34}$$

In this formula, $G_{\alpha\bar\beta}$ is the metric on the hypermultiplet space. The gauged isometries are represented by the Killing vectors k_I^α and their moment maps $\vec{\mu}_I$. The matrices N_{MN} and $\mathcal{M}_{IJ}$ are defined by

$$\begin{aligned} N_{IJ} &= -iF_{IJ} + i\bar F_{IJ}, \\ \mathcal{M}_{IJ} &= \frac{1}{[N_{MN}X^M\bar X^N]^2}\left[N_{IJ}N_{KL} - N_{IK}N_{JL}\right]\bar X^K X^L\,, \end{aligned} \tag{6.35}$$

where $F_I = \partial_I F$ etc. In our conventions, both $G_{\alpha\bar\beta}$ and $\mathcal{M}_{IJ}$ are negative definite, so the first term is a positive contribution. The second is negative, whereas the last one is positive. There is an additional term in non-abelian gaugings which can be omitted for our analysis.

6.3.2 Including NS5–brane corrections

We will now make those expressions explicit for the UHM. The perturbatively corrected metric $G_{\alpha\beta}$ on the UHM space is given by [80]

$$\mathrm{ds}^2_{\mathrm{UHM}} = \frac{r+2c}{r^2(r+c)}\mathrm{d}r^2 + \frac{r+2c}{r^2}(\mathrm{d}\chi^2 + \mathrm{d}\varphi^2) + \frac{r+c}{r^2(r+2c)}(\mathrm{d}\sigma + \chi\mathrm{d}\varphi)^2 \,. \tag{6.36}$$

The four bosonic fields are an axion σ from the dualization of the NSNS two-form B_2, two RR scalars χ, φ and the four–dimensional dilaton g_4

$$r \equiv \mathrm{e}^{\phi_4} = \frac{1}{g_4^2} = \frac{\mathcal{V}}{g_s^2} \,. \tag{6.37}$$

The relation between the four–dimensional string coupling constant g_4 and the ten–dimensional string coupling constant g_s is important, as it will introduce factors of the volume into our future expressions.

The constant c encodes the one–loop correction and is proportional to the Euler number

$$c = -\frac{\chi(CY)}{12\pi} = -\frac{h^{1,1}}{6\pi} \,, \tag{6.38}$$

where the second equality holds on a rigid CY. The contributions from a single NS5–brane instanton to the UHM metric have been derived in [153, 154]. To leading order in the semi-classical approximation, the metric reads

$$\begin{aligned}\mathrm{ds}^2_{\mathrm{UHM}} = \frac{r+2c}{r^2(r+c)}\mathrm{d}r^2 + \frac{r+2c}{r^2}(1-Y)\mathrm{d}\chi^2 + \frac{r+2c}{r^2}(1+Y)\mathrm{d}\varphi^2 \\ + \frac{2}{r}\widetilde{Y}\mathrm{d}\chi\mathrm{d}\varphi + \frac{r+c}{r^2(r+2c)}(\mathrm{d}\sigma + \chi\mathrm{d}\varphi)^2.\end{aligned} \tag{6.39}$$

The quantities Y and $\widetilde{Y}$ are defined as[28]

$$\begin{aligned} Y &= 4C(2\chi^2-1)r^{-1-c}\cos(\sigma)\mathrm{e}^{-r-\frac{1}{2}\chi^2-c}, \\ \widetilde{Y} &= 4C(2\chi^2-1)r^{-1-c}\sin(\sigma)\mathrm{e}^{-r-\frac{1}{2}\chi^2-c}. \end{aligned} \tag{6.40}$$

The factor C is a numerical constant which could not be determined. This solution has a shift symmetry associated with φ which we can gauge, using the graviphoton as

[28] Compared to [154] we have taken $\chi_0 = 0$. Its dependence can easily be restored.

gauge field. The field φ is obtained by expanding the 10–dimensional RR field $\widehat{C}_3$ over one of the $2(h^{2,1}+1) = 2$ cycles in H^3; gauging the isometry associated with φ has a microscopic interpretation of adding NS flux over this cycle [162]. Moreover, the shift symmetry is not broken by NS5-brane instantons, so there is no obstruction in gauging this isometry by fluxes in the presence of instantons [163, 164].

The gauging of this isometry leads to a scalar potential of the type given in (6.34), with a Killing vector $k = \partial_\varphi$. Upon inserting the prepotential (6.33) without the world-sheet instantons, one finds that the moment maps drop out of the equation for the potential (6.34), and only the norm of the Killing vector remains. The only dependence on the vector multiplet moduli comes from the factor $(N_{MN}X^M X^{\bar{N}})^{-1}$. The details of this calculation can be found in appendix E.1.

Without the worldsheet instanton corrections, we then find the scalar potential

$$\begin{aligned} V &= \frac{2}{4\mathcal{V}+e}\left[-2G_{\alpha\bar{\beta}}k^\alpha k^\beta\right] \\ &= \frac{4}{4\mathcal{V}+e}\left(\frac{4(r+2c)^2+4(r+c)\chi^2}{r^2(r+2c)} + 16\,C\mathrm{e}^{-c-r-\chi^2/2}r^{-2-c}(2\chi^2-1)\cos(\sigma)\right)\,, \end{aligned} \tag{6.41}$$

where $e = \frac{1}{2}\zeta(3)\chi(CY)$ and C is the undetermined overall constant. Reinstating all volume factor dependencies using (6.37), this has the schematic form

$$V = V_0 + \widetilde{C}\mathcal{V}^{-3-c}\mathrm{e}^{-\mathcal{V}/g_s^2}\,, \tag{6.42}$$

where $\widetilde{C} = 16Cg_s^{4+2c}\mathrm{e}^{-c-\chi^2/2}(2\chi^2-1)\cos(\sigma)$. We also neglect the correction due to e in the 2nd term, because it is subleading.

We see how a NS5–brane contribution can be included into $\mathcal{N}=2$ type IIA supergravity. It would be interesting to repeat this exercise including the worldsheet instantons.

6.3.3 Truncation to $\mathcal{N}=1$

To clarify the relation to the previous section, we will now perform a truncation of this theory. We make an orientifold inspired truncation to $\mathcal{N}=1$ at the level of the effective action (see figure (6.31)). A similar truncation has been done in [159]. We follow the orientifold rules from [138]. Because we merely truncate the theory, there should still be a local product structure as in (6.32), but now the product is between two Kähler manifolds. Furthermore, for simplicity we restrict ourself to the cubic prepotential and therefore put $e=0$.

The universal hypermultiplet loses half of its fields under truncation to become a chiral $\mathcal{N}=1$ multiplet. We keep the four–dimensional dilaton r and project out the axion σ. From the RR scalars χ, φ we can choose which we keep. We gauged the isometry on φ, which corresponds to a NS–flux on the cycle of φ. The relevant part of the expansion of $\widehat{H}_3$ and $\widehat{C}_3$ is given by

$$\begin{aligned}\widehat{C}_3 &= \chi\alpha + \varphi\beta\,,\\ \widehat{H}_3 &= p\,\alpha + q\,\beta\,.\end{aligned} \tag{6.43}$$

The field $\widehat{C}_3$ is expanded over a basis of the third cohomology group H^3, given by three–forms α, β, which give the four–dimensional fields χ, φ. The flux of $\widehat{B}_2$ is likewise expanded, with flux parameters p, q. Under an orientifold, the RR form $\widehat{C}_3$ and the NS-NS flux $\widehat{H}_3 = \mathrm{d}\widehat{B}_2$ are even and odd respectively. We gauge the isometry associated with φ, so we want to keep the flux parameter q. This implies that β should be an odd cycle. In the expansion of the even form $\widehat{C}_3$ we only keep even forms, and hence φ gets projected out.

The metric then truncates to

$$\mathrm{d}s^2_{\mathrm{UHM}} = \frac{r+2c}{r^2(r+c)}\mathrm{d}r^2 + \frac{r+2c}{r^2}(1-Y)\mathrm{d}\chi^2\,, \tag{6.44}$$

where we have put $\sigma = 0$ in Y. The perturbatively corrected scalar potential is, with $e = 0$

$$V_0 = \frac{4}{\mathcal{V}}\frac{(r+2c)^2 + (r+c)\chi^2}{r^2(r+2c)}\,, \tag{6.45}$$

and the NS5–brane instantons yields equation (6.41) with $\sigma = 0$

$$V_{\mathrm{NS5}} = \frac{16}{\mathcal{V}}C\mathrm{e}^{-c-r-\chi^2/2}r^{-2-c}(2\chi^2-1)\,, \tag{6.46}$$

where C is independent of vector multiplet scalars.

We want to express these quantities in terms of a Kähler and superpotential. The Kähler potential $\mathcal{K}$ is a sum of the Kähler potential $\mathcal{K}^{\mathrm{k}}$ for the truncated Kähler moduli and a potential $\mathcal{K}^{\mathrm{Q}}$ for the truncated universal hypermultiplet. In the Kähler sector, we have the Kähler potential [138]

$$\mathcal{K}^{\mathrm{k}} = -\ln(\mathcal{V})\,, \tag{6.47}$$

which follows from the choice of the cubic prepotential we made in the $\mathcal{N} = 2$ calculation, earlier in this section. The important property of this Kähler potential is its no–scale structure $(\mathcal{K}^{\mathrm{k}})^{i\bar{\jmath}}\mathcal{K}^{\mathrm{k}}_i\mathcal{K}^{\mathrm{k}}_{\bar{\jmath}} = 3$.

For the scalar potential we use expression (6.12)

$$V = \mathrm{e}^{\mathcal{K}}\Big(\mathcal{K}^{\alpha\bar{\beta}}D_\alpha W D_{\bar{\beta}}\overline{W} - 3|W|^2\Big) = \frac{1}{\mathcal{V}}\,\mathrm{e}^{\mathcal{K}^{\mathrm{Q}}}\big((\mathcal{K}^{\mathrm{Q}})^{z\bar{z}}D_z W\overline{D_z W}\big)\,, \tag{6.48}$$

where the no–scale structure in directions orthogonal to the truncated universal hypermultiplet has been used.

The perturbative part of the metric (6.44) and the potential (6.45) are now exactly reproduced by the Kähler potential and superpotential [165]

$$\begin{aligned} \mathcal{K}^{\mathrm{Q}} &= -2\ln\left[(z+\bar{z})^2 - 16c\right], \\ W &= 16z\,. \end{aligned} \tag{6.49}$$

These are formulated in terms of the chiral field z defined by

$$z = 2\sqrt{r+c} + i\chi\,. \tag{6.50}$$

We use conventions for which $\mathrm{d}s^2 = 4\mathcal{K}_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}$, as in [165].

We now also want to describe the NS5–brane instanton contribution. This scales as

$$\exp\left[-r - \frac{\chi^2}{2} - c\right] = \exp\left[\frac{1}{16}(z^2 - 6z\bar{z} + \bar{z}^2)\right], \tag{6.51}$$

which is not holomorphic in z. Therefore, we cannot correct the superpotential with such a term. The correction will take place in the Kähler potential and in the definition of the $\mathcal{N}=1$ chiral field. Both the metric (6.44) and the potential (6.41) are reproduced up to leading order by the chiral field and the Kähler potential

$$\begin{aligned} z &= 2\sqrt{r+c} + i\chi + 2C\mathrm{e}^{-r-\frac{1}{2}\chi^2-c} r^{-2-c}\left(\sqrt{r}(1-2\chi^2) + i\chi(2\chi^2-5)\right), \qquad (6.52) \\ \mathcal{K}^{\mathrm{Q}} &= -2\ln\left[(z+\bar{z})^2 - 16c\right] + C\exp\left[\frac{1}{16}(z^2 - 6z\bar{z} + \bar{z}^2)\right] 4^{9+2c}\frac{(z+\bar{z})^{-4-2c}(1+\frac{1}{2}(z-\bar{z})^2)}{(z-3\bar{z})(\bar{z}-3z)}, \\ W &= 16z\,. \end{aligned}$$

Because our four–dimensional dilaton r contains a factor of the volume $\mathcal{V}$, the leading term in the Kähler potential is equal to

$$\mathcal{K}^{\mathrm{Q}} = \mathcal{K}_0 + B\mathcal{V}^n \mathrm{e}^{-\mathcal{V}/g_s^2}, \tag{6.53}$$

where we have defined

$$B = 64C\mathrm{e}^{-c-\chi^2/2}(1-2\chi^2)g_s^{-2n}, \quad n = -3 - c\,. \tag{6.54}$$

The overall factor B can depend on other moduli. In this setting, the leading dependence on χ is explicit in (6.54). We expect (6.53) to hold also for non–rigid CY with $h_{1,2} \neq 0$. In that case the factor B presumably depends on the other hypermultiplet scalars. This confirms our proposal of (6.18) and (6.26), where we apply it to type IIB string theory.

Chapter 7

Conclusions

In this thesis we have studied several aspects of compactifications and black holes in four-dimensional supergravity.

The first chapter dealt with fully supersymmetric vacua. We have found simple, algebraic conditions which need to be satisfied for an $\mathcal{N} = 2$ supersymmetric vacuum. These conditions are illustrated with several examples, which show that even complex, interacting theories can have fully supersymmetric vacua. This work joins the the ongoing research on the structure of $\mathcal{N} = 1$ vacua of $\mathcal{N} = 2$ supergravity to provide more insight in the vacuum structure of these theories. This research will give a better understanding of the possible supersymmetric compactifications of higher-dimensional supergravity theories.

A next topic were black holes, which play an important role in supergravity. Such configurations are usually found in ungauged supergravities without hypermultiplets. The second chapter studied black hole configurations in gauged supergravity with hypermultiplets. Using spontaneous symmetry breaking, we have shown how to embed known, ungauged black hole solutions into the gauged theory. We have furthermore searched for new solutions, but only found solutions where the scalars become ghost-like. As we have made some assumptions in this computation, it could be possible that a modification of these assumptions can lead to proper new solutions. Such configurations are interesting for e.g. the AdS/CFT correspondence, where black holes can describe the thermal behavior of the dual field theory.

Over the last decade, many techniques to construct supersymmetric solutions have been devised, but it is not so clear how these methods are related. It would be a major step forward for the field if we could come to a systematic way of classifying all supersymmetric solutions.

The next area of research is part of the program of moduli stabilization. The appearance of massless scalars in e.g. Calabi-Yau compactifications is in direct conflict with experiments and we need to resolve this issue. This should be possible using flux compactifications, which have therefore seen a lot of research.

We studied compactifications of M-theory and showed that compactification to five dimensions, followed by a Scherk-Schwarz reduction, leads to a gauged $\mathcal{N} = 2$ theory. We illustrated how this can also be obtained by a direct compactification on a seven-dimensional space, constructed as a twisted Calabi-Yau threefold over a circle. Inspired by orientifold compactifications, this theory can be truncated to an $\mathcal{N} = 1$ theory, and we derived the Kähler and superpotential for this theory. Interestingly, there are also D-terms in these models. Although these models do not directly stabilize all moduli, inclusion of quantum corrections might make them phenomenologically more relevant.

As a final project in the area of moduli stabilization, we studied the stabilization of the volume of the internal space. The possibility of (Euclidean) NS5-branes that wrap the entire internal space gives corrections to the effective four-dimensional action, which depend on the volume of the internal space. We motivate how these instantons lead to a correction of the $\mathcal{N} = 1$ Kähler potential, and then study when these effects are important. It is found that there are regimes in parameter space where they can be important, but other (unknown) effects are expected to be important as well, so it remains difficult to study their effects. The research in this direction should therefore either focus on models with an (exponentially large) volume, where many of these effects are expected to be negligible, or explicitly compute these corrections, which is a very challenging problem.

Appendix A

Notation, conventions and spacetimes

A.1 Notation and conventions

We mainly follow the notation and conventions from [28]. In particular, our spacetime has a $\{+,-,-,-\}$ signature. Self-dual and anti-self-dual tensors are defined as

$$F^{\pm}_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} \pm \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\right) , \tag{A.1}$$

where $\epsilon_{0123} = 1$.

Our gamma matrices satisfy

$$\begin{aligned} \{\gamma_a, \gamma_b\} &= 2\eta_{ab} \ , \\ [\gamma_a, \gamma_b] &\equiv 2\gamma_{ab} \ , \\ \gamma_5 &\equiv -i\gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ . \end{aligned} \tag{A.2}$$

In addition, they can be chosen such that

$$\gamma_0^\dagger = \gamma_0, \quad \gamma_0\gamma_i^\dagger\gamma_0 = \gamma_i, \quad \gamma_5^\dagger = \gamma_5, \quad \gamma_\mu^* = -\gamma_\mu \ , \tag{A.3}$$

and an explicit example of such a basis is the Majorana basis, given by

$$\begin{aligned} &\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \\ &\gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} , \end{aligned} \tag{A.4}$$

where the σ^i are the Pauli matrices.

The action is defined by $S = \int \sqrt{|g|}\mathcal{L}$. We start with the (ungauged) Lagrangian, whose Einstein-Hilbert and scalar derivative terms read

$$\mathcal{L} = \frac{1}{2}R + g_{i\bar{\jmath}}\partial_\mu z^i \partial^\mu z^{\bar{\jmath}} + h_{uv}\partial_\mu q^u \partial^\mu q^v \ . \tag{A.5}$$

We set the Newton constant $\kappa^2 = 1$. As we use a $\{+,-,-,-\}$ metric signature, we have to choose $g_{i\bar{\jmath}}$ and h_{uv} positive definite to get positive kinetic terms for the scalars.

We compute the Riemann curvature as follows[29]

$$\begin{aligned} R^{\rho}{}_{\sigma\mu\nu} &= \epsilon \left[\partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \right] , \\ R_{\mu\nu} &= R^{\rho}{}_{\mu\rho\nu} , \quad R = g^{\mu\nu} R_{\mu\nu} , \end{aligned} \tag{A.6}$$

where $\epsilon = 1$ for Riemann spaces (the quaternionic and special Kähler target spaces) and $\epsilon = -1$ for Lorentzian spaces (space-time). The overall minus sign in the latter case is needed to give AdS spaces a negative scalar curvature. This gives a sphere in Euclidean space (with signature $\{+,+,+,+\}$) a positive scalar curvature.

The spin connection enters in the covariant derivative

$$\begin{aligned} D_\mu &= \partial_\mu - \frac{1}{4}\omega^{ab}_\mu \gamma_{ab} , \\ \omega^{ab}_\mu &= \frac{1}{2} e_{\mu c} \left(\Omega^{cab} - \Omega^{abc} - \Omega^{bca} \right) , \\ \Omega^{cab} &= \left(e^{\mu a} e^{\nu b} - e^{\mu b} e^{\nu a} \right) \partial_\mu e^{c}{}_{\nu} . \end{aligned} \tag{A.7}$$

The Lagrangian (A.5) is only supersymmetric if the Riemann curvature of the hypermultiplet moduli space satisfies $R(h_{uv}) = -8n(n+2)$, where n is the number of hypermultiplets, so the dimension of the quaternionic manifold is $4n$ (in applications to the universal hypermultiplet, we have $n = 1$ and hence $R = -24$).

Our conventions for the sigma matrices follow [28]; in particular they are symmetric and satisfy $\left(\sigma^{xAB}\right)^* = -\sigma^{x}{}_{AB}$, and we have the relation

$$\sigma^x_{AB}\sigma^{yBC} = -\delta^C_A \delta^{xy} + i\epsilon_{AB}\epsilon^{xyz}\sigma^{zBC} . \tag{A.8}$$

Indices on bosonic quantities are raised and lowered as

$$\epsilon_{AB} V^B = V_A , \qquad \epsilon^{AB} V_B = -V^A . \tag{A.9}$$

As mentioned in the main text, all fermions with upper $SU(2)_R$ index have negative chirality and all fermions with lower index have positive chirality. We set γ_5 to be purely imaginary and then complex conjugation interchanges chirality.

A.2 Metrics and field strengths

- $AdS_2 \times S^2$

  The line element, in local coordinates $\{t, x, \theta, \phi\}$, is

$$\mathrm{d}s^2 = q_0^2 \left(\mathrm{d}t^2 - \sin^2(t)\mathrm{d}x^2 - \mathrm{d}\theta^2 - \sin^2(\theta)\mathrm{d}\phi^2 \right) , \tag{A.10}$$

[29] Note that this definition, when applied to the Riemann curvature of the quaternionic manifold, differs with a factor of 2 compared with [28, 55]. As a consequence, there one has $R(h_{uv}) = -4n(n+2)$.

where q_0 is a real, overall constant which determines the size of both AdS_2 and S^2. From (3.29) we find the only non–vanishing components

$$\begin{aligned} T^+_{tx} &= \frac{1}{2} q_0 \sin(t) \mathrm{e}^{i\alpha} \ , \\ T^+_{\theta\phi} &= -\frac{i}{2} q_0 \sin(\theta) \mathrm{e}^{i\alpha} \ . \end{aligned} \tag{A.11}$$

- The pp-wave
  The line element of a four–dimensional Cahen-Wallach space [76], in local coordinates $\{x^-, x^+, x^1, x^2\}$, is given by

$$\mathrm{ds}^2 = -2\mathrm{d}x^+\mathrm{d}x^- - A_{ij} x^i x^j (\mathrm{d}x^-)^2 - (\mathrm{d}x^i)^2 \ , \tag{A.12}$$

  where A_{ij} is a symmetric matrix. Conformal flatness requires $A_{11} = A_{22}$ and $A_{12} = 0$. We denote $A_{11} = -\mu^2$ as A_{11} should be negative. This space is known as the pp-wave. From (3.29) we find the only non–vanishing components

$$\begin{aligned} T^+_{x^- x^1} &= \frac{\mu}{2} \mathrm{e}^{i\alpha} \ , \\ T^+_{x^- x^2} &= -i\frac{\mu}{2} \mathrm{e}^{i\alpha} \ . \end{aligned} \tag{A.13}$$

Appendix B

Integrability conditions

B.1 Commutators of supersymmetry tranformations

A killing spinor ε_A satisfies

$$\delta_\varepsilon \psi_{\mu A} = \nabla_\mu \varepsilon_A + T^-_{\mu\nu}\gamma^\nu \epsilon_{AB}\varepsilon^B + igS_{AB}\gamma_\mu \varepsilon^B = 0\,, \tag{B.1}$$

whence the commutator is

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]\varepsilon_A =& - \epsilon_{AB} D_\mu T^-_{\nu\rho}\gamma^\rho \epsilon^B + \frac{g}{2}\sigma^x_{AB}\nabla_\mu P^x \gamma_\nu \varepsilon^B - (\mu\nu) \\
&+ T^-_{\nu\rho}\gamma^\rho T^+_{\mu\sigma}\gamma^\sigma \varepsilon_A - (\mu\nu) \\
&- \frac{g}{2} T^-_{\nu\rho}\gamma^\rho\gamma_\mu P^x_\Lambda \bar{L}^\Lambda \sigma^x{}_A{}^C \varepsilon_C + \frac{g}{2} T^+_{\mu\rho}\gamma_\nu\gamma^\rho P^x_\Lambda L^\Lambda \sigma^x{}_A{}^C\varepsilon_C - (\mu\nu) \\
&+ \frac{g^2}{2}\left(\delta_A{}^C P^x \overline{P^x} - i\epsilon^{xyz}\sigma^x{}_A{}^C P^y \overline{P^z}\right)\gamma_{\mu\nu}\varepsilon_C\,.
\end{aligned} \tag{B.2}$$

From the defintion (3.17) we obtain

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]\varepsilon_A =& -\frac{1}{4}R_{\mu\nu}{}^{ab}\gamma_{ab}\varepsilon_A - g_{i\bar{\jmath}}\nabla_{[\mu} z^i \nabla_{\nu]} z^{\bar{\jmath}}\varepsilon_A - igF^\Lambda_{\mu\nu}P_\Lambda \\
&+ 2i\Omega_{uvA}{}^B \nabla_{[\mu} q^u \nabla_{\nu]} q^v \varepsilon_B + ig\sigma^x{}_A{}^B F^\Lambda_{\mu\nu} P^x_\Lambda \varepsilon_B\,.
\end{aligned} \tag{B.3}$$

B.2 Fully BPS vacua

In the fully BPS case, all terms with a covariant derivative in (B.2) and (B.3) vanish. We furthermore see that (B.3) does not contain a term proportional to ϵ_{AB}, so $D_\mu T^-_{\nu\rho} = 0$.

Some algebra now yields the necessary and sufficient conditions to match the terms proportional to $\sigma^x{}_A{}^B$:

$$\begin{aligned}
T^-_{\mu\nu}\overline{P^x} &= 0\,, \\
\epsilon^{xyz}P^y\overline{P^z} &= 0\,,
\end{aligned} \tag{B.4}$$

which give the first conditions of section 2.3. The other conditions are obtained by comparing the parts proportional to $\mathbf{1}_A{}^B$.

B.3 Half BPS vacua

We use (4.51) to eliminate ε^A in terms of ε_A and for convenience define $b \equiv -ie^{i\alpha}$. The remaining equation should hold for any choice of ε_A. We can then use the independence of the gamma matrices and the SU(2) matrices $\epsilon_{AB}, \sigma^x_{AB}$ to find the conditions

1. Terms proportional to ϵ_{AB}, no gamma.
$$bD_\mu T^-_{\nu 0} - (\mu\nu) = -g_{i\bar{\jmath}}\partial_{[\mu} z^i \partial_{\nu]} z^{\bar{\jmath}} \; . \tag{B.5}$$

2. Terms proportional to ϵ_{AB}, two gamma
$$bD_\mu T^-_{\nu\rho}\gamma^{\rho 0} + T^-_{\nu\rho}T^+_{\mu\sigma}\gamma^{\rho\sigma} - (\mu\nu) + \frac{g^2}{2}P^x\overline{P^x}\gamma_{\mu\nu} = -\frac{1}{4}R_{\mu\nu}{}^{ab}\gamma_{ab} \; . \tag{B.6}$$

3. Terms proportional to σ^x_{AB}, no gamma
$$\begin{aligned} &\frac{g}{2}b\nabla_\mu P^x g_{\nu 0} - (\mu\nu) + gT^-_{\mu\nu}\overline{P^x} + gT^+_{\mu\nu}P^x \\ &\qquad = g\left(L^\Lambda T^+_{\mu\nu} - \bar{L}^\Lambda T^-_{\mu\nu} - 2if^\Lambda_{\bar{\imath}}G^{i+}_{\mu\nu} + 2if^\Lambda_i G^{i-}_{\mu\nu}\right)P^x_\Lambda \; , \end{aligned} \tag{B.7}$$
where we used that $-\Omega^x_{uv}\nabla_{[\mu}q^u\nabla_{\nu]}q^v = 0$, which follows from (4.56). Using $f^\Lambda_i P^x_\Lambda = 0$ from (4.53) we therefore find
$$\frac{g}{2}b\nabla_\mu P^x g_{\nu 0} - (\mu\nu) = -2gT^-_{\mu\nu}P^x_\Lambda \bar{L}^\Lambda \; . \tag{B.8}$$
We now take components $\mu = \theta$ and use $\nabla_\theta P^x = 0$ and $g_{\theta 0} = 0$. We then find $T^-_{\theta\nu}P^x = 0$, whence $P^x = 0$ or $T^-_{\theta\nu} = 0$. In the latter case also $T^-_{\mu\nu} = 0$, because of the anti-self-duality property, and then $T_{\mu\nu} = 0$. We conclude
$$T^-_{\mu\nu}P^x_\Lambda L^\Lambda = 0 \; . \tag{B.9}$$

4. Terms proportional to σ^x_{AB}, two gamma. Using (B.9) we find
$$\epsilon^{xyz}P^y\overline{P^z}\gamma_{\mu\nu} = 0 \; . \tag{B.10}$$

To summarize: we found two cases, one with $T^-_{\mu\nu} = 0$, the other with $P^x = 0$. We now list the remaining conditions for each case.

B.3.1 Case A: $F = 0$

The remaining conditions are

$$\begin{aligned} \frac{g^2}{2} P^x \overline{P^x} \gamma_{\mu\nu} &= -\frac{1}{4} R_{\mu\nu}{}^{ab} \gamma_{ab} \ , \\ g_{i\bar{\jmath}} \partial_{[\mu} z^i \partial_{\nu]} z^{\bar{\jmath}} &= 0 \ , \\ \epsilon^{xyz} P^y \overline{P^z} &= 0 \ . \end{aligned} \tag{B.11}$$

The first condition implies that the spacetime is maximally symmetric, with constant curvature $\propto P^x \overline{P^x}$, and is then solved.

B.3.2 Case B: $P^x = 0$

The remaining conditions are

$$\begin{aligned} b D_\mu T^-_{\nu 0} - (\mu\nu) &= -g_{i\bar{\jmath}} \partial_{[\mu} z^i \partial_{\nu]} z^{\bar{\jmath}} \ , \\ b D_\mu T^-_{\nu\rho} \gamma^{\rho 0} + T^-_{\nu\rho} T^+_{\mu\sigma} \gamma^{\rho\sigma} - (\mu\nu) &= -\frac{1}{4} R_{\mu\nu}{}^{ab} \gamma_{ab} \ . \end{aligned} \tag{B.12}$$

From the second condition we find the Riemann tensor

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= R^-_{\mu\nu\rho\sigma} + R^+_{\mu\nu\rho\sigma} \ , \\ R^-_{\mu\nu\rho\sigma} &= - b D_\mu T^-_{\nu\rho} e^0_\sigma + T^-_{\nu\rho} T^+_{\mu\sigma} - (\mu\nu) \\ &\quad - b D_\nu T^-_{\mu\sigma} e^0_\rho + T^-_{\mu\sigma} T^+_{\nu\rho} - (\mu\nu) \\ &\quad + b i \epsilon_{\rho\sigma}{}^{\lambda\kappa} D_\mu T^-_{\nu\lambda} e^0_\kappa + i \epsilon_{\rho\sigma}{}^{\lambda\kappa} T^-_{\nu\lambda} T^+_{\mu\kappa} - (\mu\nu) \ . \end{aligned} \tag{B.13}$$

Appendix C

Isometries of special Kähler manifolds

In this appendix, we present some further relevant formulae that are used in the main body of the paper. First, we have defined the moment maps on the special Kähler manifold as follows. Given an isometry, with a symplectic embedding (2.46), we can define the functions

$$P_\Lambda \equiv i(k_\Lambda^i \partial_i \mathcal{K} + r_\Lambda) \ . \qquad \text{(C.1)}$$

Since the Kähler potential satisfies (2.48), it is easy to show that P_Λ is real. From this definition, it is easy to verify that

$$k_\Lambda^i = -ig^{i\bar{\jmath}}\partial_{\bar{\jmath}} P_\Lambda \ . \qquad \text{(C.2)}$$

Hence the P_Λ can be called moment maps, but they are *not* subject to arbitrary additive constants. Using (2.49) and (2.56), it is now easy to prove the relation

$$k_\Lambda^i g_{i\bar{\jmath}} k_\Sigma^{\bar{\jmath}} - k_\Sigma^i g_{i\bar{\jmath}} k_\Lambda^{\bar{\jmath}} = i f_{\Lambda\Sigma}{}^\Pi P_\Pi \ , \qquad \text{(C.3)}$$

also called the equivariance condition.

We can obtain formulas for the moment maps in terms of the holomorphic sections. For this, one needs the identities

$$k_\Lambda^i \partial_i X^\Sigma = -f_{\Lambda\Pi}{}^\Sigma X^\Pi + r_\Lambda X^\Sigma \ , \qquad k_\Lambda^i \partial_i F_\Sigma = c_{\Lambda,\Sigma\Pi} X^\Pi + f_{\Lambda\Sigma}{}^\Pi F_\Pi + r_\Lambda F_\Sigma \ , \qquad \text{(C.4)}$$

which follow from the gauge transformations of the sections, see (2.46). Using the chain rule in (2.56), it is now easy to derive

$$P_\Lambda = \mathrm{e}^{\mathcal{K}} \Big[f_{\Lambda\Pi}{}^\Sigma (X^\Pi \bar{F}_\Sigma + F_\Sigma \bar{X}^\Pi) + c_{\Lambda,\Pi\Sigma} X^\Pi \bar{X}^\Sigma \Big] \ , \qquad \text{(C.5)}$$

and similarly

$$k_\Lambda^i = -ig^{i\bar{\jmath}} \Big[f_{\Lambda\Pi}{}^\Sigma (f_{\bar{\jmath}}^\Pi M_\Sigma + h_{\Sigma|\bar{\jmath}} L^\Pi) + c_{\Lambda,\Sigma\Pi} \bar{f}_{\bar{\jmath}}^\Pi L^\Sigma \Big] \ , \qquad \text{(C.6)}$$

where we introduced $M_\Lambda \equiv \mathrm{e}^{\mathcal{K}/2} F_\Lambda$ and $h_{\Lambda|i} \equiv \mathrm{e}^{\mathcal{K}/2}(\partial_i + \mathcal{K}_i) F_\Lambda$. The Killing vectors (C.6) are not manifestly holomorphic. This needs not be the case because otherwise we would have constructed isometries for arbitrary special Kähler manifolds, since holomorphic vector fields obtained from a (real) moment map solve the Killing equation.

We now show that $P_\Lambda L^\Lambda = 0$, following the discussion in [73]. We start from the consistency conditions on the symplectic embedding of the gauge transformations, equations (C.4). We eliminate r_Λ using (2.56), and rewrite them as

$$-f_{\Lambda\Pi}{}^{\Sigma} L^\Pi = k^i_\Lambda f^\Sigma_i + i P_\Lambda L^\Sigma \, , \tag{C.7}$$

$$f_{\Lambda\Gamma}{}^{\Sigma} M_\Sigma + c_{\Lambda,\Gamma\Sigma} L^\Sigma = k^i_\Lambda h_{\Gamma|i} + i P_\Lambda M_\Gamma \, , \tag{C.8}$$

with $h_{\Gamma|i} = \mathrm{e}^{K/2} D_i F_\Gamma$. Multiplication of the first equation with M_Σ and the second with L^Γ and subtracting leads to

$$2 f_{\Lambda\Gamma}{}^{\Sigma} L^\Gamma M_\Sigma + c_{\Lambda,\Gamma\Sigma} L^\Gamma L^\Sigma = 0 \, , \tag{C.9}$$

where we have used the identity $f^\Sigma_i M_\Sigma - h_{\Gamma|i} L^\Gamma = 0$. Contracting equation (C.5) with L^Λ and using (C.9) and (2.63) one finds

$$P_\Lambda L^\Lambda = 0 \, , \tag{C.10}$$

as announced below equation (3.16). Contrating the first equation of (C.7) with L^Λ gives $L^\Lambda k^i_\Lambda f^\Sigma_i = 0$. It follows from contracting with $\mathrm{Im}\, \mathcal{N}_{\Gamma\Sigma} f^\Sigma_{\bar{\jmath}}$ that

$$L^\Lambda k^i_\Lambda = 0 \, . \tag{C.11}$$

Here we have used the special geometry identities on the period matrix (2.27).

Appendix D

The universal hypermultiplet

The metric for the universal hypermultiplet is known to be

$$\mathrm{d}s^2 = \frac{1}{r^2}\left(\mathrm{d}r^2 + r\left(\mathrm{d}\chi^2 + \mathrm{d}\varphi^2\right) + \left(\mathrm{d}\sigma + \chi\mathrm{d}\varphi\right)^2\right) \ . \tag{D.1}$$

It describes the coset space $SU(2,1)/U(2)$ and therefore there are eight Killing vectors spanning the isometry group $SU(2,1)$. In the coordinates of (D.1), they can be written as

$$\begin{aligned}
k_{a=1} &= \partial_\sigma \ , \\
k_{a=2} &= \partial_\chi - \varphi\partial_\sigma \ , \\
k_{a=3} &= \partial_\varphi , \\
k_{a=4} &= -\varphi\partial_\chi + \chi\partial_\varphi + \frac{1}{2}(\varphi^2 - \chi^2)\partial_\sigma \ , \\
k_{a=5} &= 2r\partial_r + \chi\partial_\chi + \varphi\partial_\varphi + 2\sigma\partial_\sigma \ , \\
k_{a=6} &= 2r\varphi\partial_r + (-2\sigma + \varphi\chi)\partial_\chi + \frac{1}{2}(-3r + \varphi^2 - 3\chi^2)\partial_\varphi + (2\sigma\varphi + 2r\chi + \chi^3)\partial_\sigma \ , \\
k_{a=7} &= 2r\chi\partial_r + \frac{1}{2}(-4r - 3\varphi^2 + \chi^2)\partial_\chi + (2\sigma + 3\varphi\chi)\partial_\varphi + \frac{\varphi}{2}(\varphi^2 - 3\chi^2)\partial_\sigma \ , \\
k_{a=8} &= r(2\sigma + \varphi\chi)\partial_r + \frac{1}{4}(-4r\varphi - \varphi^3 + 4\sigma\chi + \varphi\chi^2)\partial_\chi + \frac{1}{4}(4r\chi + 4\sigma\varphi + 3\varphi^2\chi + \chi^3)\partial_\varphi \\
&\quad + \frac{1}{16}(-16r^2 + 16\sigma^2 + \varphi^4 - 2(8r + 3\varphi^2)\chi^2 - 3\chi^4)\partial_\sigma \ .
\end{aligned} \tag{D.2}$$

The moment maps P^x are computed from

$$P^x = \Omega^x_{uv} D^u k^v \ . \tag{D.3}$$

The quaternionic two-forms Ω^x satisfy $\Omega^x\Omega^y = -\frac{1}{4}\delta^{xy} + \frac{1}{2}\epsilon^{xyz}\Omega^z$, and can be written as

$$\begin{aligned}
\Omega^1 &= \frac{1}{2r^{3/2}}\left(\mathrm{d}r\wedge\mathrm{d}\chi + \mathrm{d}\varphi\wedge\mathrm{d}\sigma\right) \ , \\
\Omega^2 &= \frac{1}{2r^{3/2}}\left(-\mathrm{d}r\wedge\mathrm{d}\varphi + \mathrm{d}\chi\wedge\mathrm{d}\sigma - \chi\mathrm{d}\varphi\wedge\mathrm{d}\chi\right) \ , \\
\Omega^3 &= \frac{1}{2r^2}\left(\mathrm{d}r\wedge\mathrm{d}\sigma + \chi\mathrm{d}r\wedge\mathrm{d}\varphi - r\mathrm{d}\varphi\wedge\mathrm{d}\chi\right) \ .
\end{aligned} \tag{D.4}$$

We then find the moment maps

$$\begin{aligned}
P_{a=1} &= \left\{0, 0, -\frac{1}{2r}\right\}, \\
P_{a=2} &= \left\{-\frac{1}{\sqrt{r}}, 0, \frac{\varphi}{2r}\right\}, \\
P_{a=3} &= \left\{0, \frac{1}{\sqrt{r}}, -\frac{\chi}{2r}\right\}, \\
P_{a=4} &= \left\{\frac{\varphi}{\sqrt{r}}, \frac{\chi}{\sqrt{r}}, 1 - \frac{\chi^2+\varphi^2}{4r}\right\}, \\
P_{a=5} &= \left\{-\frac{\chi}{\sqrt{r}}, \frac{\varphi}{\sqrt{r}}, -\frac{\sigma + \frac{1}{2}\varphi\chi}{r}\right\}, \\
P_{a=6} &= \left\{\frac{2\sigma - \varphi\chi}{\sqrt{r}}, \frac{4r+\varphi^2-3\chi^2}{2\sqrt{r}}, \frac{-4\sigma\varphi - (12r+\varphi^2)\chi + \chi^3}{4r}\right\}, \\
P_{a=7} &= \left\{-\frac{4r-3\varphi^2+\chi^2}{2\sqrt{r}}, \frac{2\sigma+3\varphi\chi}{\sqrt{r}}, -\frac{-12r\varphi+\varphi^3+4\sigma\chi+3\varphi\chi^2}{4r}\right\}, \\
P_{a=8} &= \Bigg\{\frac{-4r\varphi+\varphi^3-4\sigma\chi-\varphi\chi^2}{4\sqrt{r}}, \frac{4\sigma\varphi-4r\chi+3\varphi^2\chi+\chi^3}{4\sqrt{r}}, \\
&\qquad -\frac{16r^2+16\sigma^2+\varphi^4+16\sigma\chi\varphi+6\varphi^2\chi^2+\chi^4-24r(\varphi^2+\chi^2)}{32r}\Bigg\}.
\end{aligned} \tag{D.5}$$

These formulae are needed for some of the examples that we consider in the main text of this paper.

Appendix E

Calculations with NS5-branes

In this appendix we give some details of the calculations which have been used in chapter 6.

We want to calculate the derivatives and inverses for the Kähler potentials (6.11)

$$\mathcal{K}_0 = -\ln(-i(\tau - \bar{\tau})) - 2\ln\left(\mathcal{V} + \frac{\hat{\xi}}{2}\right), \tag{E.1}$$

$$\mathcal{K}_{\rm NS5} = B\mathcal{V}^n \exp\big(-\mathcal{V}\tau_2^2\big), \tag{E.2}$$

where we use

$$\begin{aligned} \tau &= l + i\mathrm{e}^{\phi}, \\ \tau_2 &= \mathrm{Im}\,\tau\,, \\ 6\mathcal{V} &= \mathcal{K}_{ijk}t^i t^j t^k = \mathcal{K}_{ij}t^i t^j = \mathcal{K}_i t^i\,, \\ \hat{\xi} &= \xi(\mathrm{Im}\,\tau)^{3/2}. \end{aligned} \tag{E.3}$$

We introduce $\mathcal{K}^{ij}$ as the inverse of $\mathcal{K}_{ij}$, and denote $\mathcal{A} := \mathcal{V} + \hat{\xi}/2$. We first consider the case where $G^a = 0$. From $T_i = \tau_i + ib_i$ (equation (6.5) for $G^a = 0$) we find

$$\frac{\partial t^j}{\partial T_i} = \frac{1}{4}\mathcal{K}^{ij}, \quad \frac{\partial \mathcal{V}}{\partial T_i} = \frac{1}{8}t^i. \tag{E.4}$$

We can then calculate

$$\begin{aligned}
\mathcal{K}^0_{T_i} &= -\frac{1}{4}\frac{t^i}{\mathcal{A}}, \quad \mathcal{K}^0_{\tau} = i\tau_2^{-1}\left(\frac{3\hat{\xi}}{4\mathcal{A}} + \frac{1}{2}\right), \\
\mathcal{K}^0_{T_i\bar{T}_j} &= \frac{G^{ij}}{\mathcal{A}^2}, \quad G^{ij} = -\frac{1}{16}\mathcal{A}\mathcal{K}^{ij} + \frac{1}{32}t^i t^j, \\
\mathcal{K}^0_{i\bar{\tau}} &= \frac{3i}{32\mathcal{A}^2}\tau_2^{-1}\hat{\xi}t^i, \\
\mathcal{K}^0_{\tau\bar{\tau}} &= \frac{1}{16}\tau_2^{-2}\mathcal{A}^{-2}(4\mathcal{V}^2 + \mathcal{V}\hat{\xi} + 4\hat{\xi}^2).
\end{aligned} \tag{E.5}$$

This can be inverted to give

$$\begin{aligned}
\mathcal{K}_0^{\tau\bar{\tau}} &= \frac{4\mathcal{V} - \hat{\xi}}{\mathcal{V} - \hat{\xi}}\tau_2^2, \\
\mathcal{K}_0^{i\bar{\tau}} &= -\frac{3i\hat{\xi}}{\mathcal{V} - \hat{\xi}}\tau_2\mathcal{K}_i, \\
\mathcal{K}_0^{i\bar{j}} &= -8(2\mathcal{V} + \hat{\xi})\mathcal{K}_{ij} + \frac{4\mathcal{V} - \hat{\xi}}{\mathcal{V} - \hat{\xi}}\mathcal{K}_i\mathcal{K}_j.
\end{aligned} \tag{E.6}$$

We then find a familiar result, which leads directly to (6.16):

$$\mathcal{K}_0^{\alpha\bar{\beta}}\mathcal{K}^0_{\alpha}\mathcal{K}^0_{\bar{\beta}} = 3 + \frac{3\hat{\xi}(\mathcal{V}^2 + 7\mathcal{V}\hat{\xi} + \hat{\xi}^2)}{(\mathcal{V} - \hat{\xi})(2\mathcal{V} + \hat{\xi})^2}. \tag{E.7}$$

For the NS5–brane contribution we obtain (we write $\mathcal{K}_5 = \mathcal{K}^5 = \mathcal{K}_{\text{NS5}}$)

$$\begin{aligned}
\mathcal{K}^5_{\tau\bar{\tau}} &= -\frac{1}{4}\mathcal{K}_5\mathcal{V}((\tau - \bar{\tau})^2\mathcal{V} + 2), \\
\mathcal{K}^5_{i\bar{\tau}} &= \frac{i}{8}B\exp(-\mathcal{V}\tau_2^2)\mathcal{V}^n(-\mathcal{V}\tau_2 + n + 1)\tau_2 t^i, \\
\mathcal{K}^5_{i\bar{j}} &= \frac{1}{64}Bt^i t^j\mathcal{V}^{n-2}\left(\mathcal{V}^2\tau_2^4 - \mathcal{V}2n\tau_2^2 + (n-1)n\right)\exp(-\mathcal{V}\tau_2^2).
\end{aligned} \tag{E.8}$$

We are interested in the leading term in the potential, so we want to investigate the powers of the volume. If we denote volume powers with $[\cdot]$, then

$$[\mathcal{V}] = 1, \quad [\mathcal{K}_{ijk}] = 0, \quad [t^i] = \frac{1}{3}, \quad [\mathcal{K}_{ij}] = \frac{1}{3}. \tag{E.9}$$

All the one–instanton terms are multiplied by $\exp(-\mathcal{V}\tau_2^2)$. To determine the leading term, we have to find the highest power of the volume $\mathcal{V}$ in the polynomial which appears in front of this exponent. Therefore, we do not include the factor $\exp(-\mathcal{V}\tau_2^2)$ in the counting, or equivalently we put $[\exp(-\mathcal{V}\tau_2^2)] = 0$.

The various terms have the following leading volume dependencies:

$$
\begin{aligned}
&[\mathcal{K}_0^{i\bar{\jmath}}] = 4/3 && [\mathcal{K}^5_{i\bar{\jmath}}] = n + 2/3 && [\mathcal{K}_0^{i\bar{\jmath}}\mathcal{K}^5_{\bar{\jmath}k}\mathcal{K}_0^{\bar{k}l}] = n + 10/3 \\
&[\mathcal{K}_0^{i\bar{\tau}}] = -1/3 && [\mathcal{K}^5_{i\bar{\tau}}] = n + 4/3 && [\mathcal{K}_0^{i\bar{\tau}}\mathcal{K}^5_{\bar{\tau}k}\mathcal{K}_0^{\bar{k}l}] = n + 7/3 \\
&[\mathcal{K}_0^{\tau\bar{\tau}}] = 0 && [\mathcal{K}^5_{\tau\bar{\tau}}] = n + 2 && [\mathcal{K}_0^{i\bar{\tau}}\mathcal{K}^5_{\bar{\tau}\tau}\mathcal{K}_0^{\bar{\tau}l}] = n + 4/3\,.
\end{aligned}
\tag{E.10}
$$

The leading contribution is given by (we use $\simeq$ here to denote equality up to subleading terms)

$$
\mathcal{K}_5^{i\bar{\jmath}} = \mathcal{K}_0^{i\bar{\alpha}}\mathcal{K}^5_{\bar{\alpha}\beta}\mathcal{K}_0^{\beta\bar{\jmath}} \simeq \mathcal{K}_0^{i\bar{l}}\mathcal{K}^5_{\bar{l}k}\mathcal{K}_0^{k\bar{\jmath}} \simeq B\mathcal{V}^{n+2}g_s^{-4}\exp(-\mathcal{V}/g_s^2)\mathcal{K}_i\mathcal{K}_j\,. \tag{E.11}
$$

We remind that $\mathcal{K}_5^{i\bar{\jmath}}$ is the inverse of $\mathcal{K}_{5,\alpha\bar{\beta}}$ in the directions of the Kähler moduli. In the potential we find then $[\mathcal{K}_5^{i\bar{\jmath}}|\partial_i\mathcal{K}_0|^2] = (n+10/3) - 2/3 - 2/3 = n+2$. The other term is $[\mathcal{K}_0^{ij}\partial_i\mathcal{K}^5\partial_{\bar{\jmath}}\mathcal{K}_0] = 4/3 + (n+1/3) - 2/3 = n+1$, which is subleading with respect to the terms above. Then the leading contribution to the scalar potential is given by

$$
\begin{aligned}
V &\simeq -\mathrm{e}^{\mathcal{K}_0}\mathcal{K}_{\mathrm{NS5}}^{i\bar{\jmath}}\,|\partial_i\mathcal{K}_0|^2|W_0|^2 \\
&= -\frac{9}{8}B\mathcal{V}^n g_s^{-3}|W_0|^2\exp(-\mathcal{V}/g_s^2)\,.
\end{aligned}
\tag{E.12}
$$

Let us now consider the effects of non-zero G^a, to clarify the statements made after equation (6.21). From the definition (6.5)

$$
T_i = \tau_i + ib_i + \frac{i}{\tau-\bar{\tau}}\,\mathcal{K}_{iab}G^a(G-\bar{G})^c\,, \tag{E.13}
$$

we find that

$$
\frac{\partial t^j}{\partial T_i} = \frac{1}{4}\mathcal{K}^{ij}\,,\quad \frac{\partial t^i}{\partial G^a} = \frac{1}{4}\mathcal{K}^{ij}\mathcal{K}_{jab}b^b\,,\quad \frac{\partial t^i}{\partial\tau} = \frac{i}{2}\mathcal{K}_{iab}b^ab^b\,, \tag{E.14}
$$

and hence

$$
\frac{\partial\mathcal{V}}{\partial T_i} = \frac{1}{8}t^i\,,\quad \frac{\partial\mathcal{V}}{\partial G^a} = \frac{1}{2}t^j\mathcal{K}_{jac}b^c = \frac{1}{2}\mathcal{K}_{ac}b^c\,,\quad \frac{\partial\mathcal{V}}{\partial\tau} = \frac{i}{4}t^j\mathcal{K}_{jab}b^ab^b = \frac{i}{4}\mathcal{K}_{ab}b^ab^b\,. \tag{E.15}
$$

In the last two expressions the factor t^j is bound with the factor $\mathcal{K}_{jac}$ and cannot combine with a $\mathcal{K}_j$ to form a power of the volume.

The expression for (6.21) also contains inverse metrics. If we use the expressions for the tree–level Kähler metric in [157], we can explicitly determine the volume dependence, and we find

$$
\begin{aligned}
\mathcal{K}_0^{ia}\mathcal{K}^5_{ab}\mathcal{K}_0^{bj}t^it^j &\sim \mathcal{V}^{n+2}\,, \\
\mathcal{K}_0^{ik}\mathcal{K}^5_{kl}\mathcal{K}_0^{lj}t^it^j &\sim \mathcal{V}^{n+4}\,, \\
\mathcal{K}_0^{ia}\mathcal{K}^5_{ak}\mathcal{K}_0^{kj}t^it^j &\sim \mathcal{V}^{n+3}\,,
\end{aligned}
\tag{E.16}
$$

and the leading term does not contain the fields G^α. We do not know if this property holds when we include quantum corrections to the Kähler potential, but as quantum corrections are expected to be subleading in the volume, we expect this to be the case.

E.1 Potentials in $\mathcal{N} = 2$

In this appendix we give some more details of the calculation in the $N = 2$ setting. This appendix derives a general form of the scalar potential. The next appendix specializes this to the UHM and the Przanowski metric.

We use the formalism from [56] with the vector prepotential

$$F = \frac{1}{3!}\frac{\mathcal{K}_{ijk}X^iX^jX^k}{X^0} + \frac{i}{2}\zeta(3)\chi(CY)X^0X^0\,. \tag{E.17}$$

From the prepotential we define

$$\begin{aligned} N_{IJ} &= -iF_{IJ} + i\bar{F}_{IJ} = 2\mathrm{Im}\,F_{IJ}, \\ \mathcal{M}_{IJ} &= \frac{1}{[N_{MN}X^M\bar{X}^N]^2}\left[N_{IJ}N_{KL} - N_{IK}N_{JL}\right]\bar{X}^KX^L\,, \end{aligned} \tag{E.18}$$

and then the scalar potential is given by

$$\begin{aligned} V =& -4g^2\left[2G_{\alpha\bar{\beta}}k_I^\alpha k_J^\beta + 3\vec{\mu}_I\cdot\vec{\mu}_J\right]\frac{X^I\bar{X}^J}{N_{MN}X^M\bar{X}^N} \\ & - g^2N_{MN}X^M\bar{X}^N\mathcal{M}_{IJ}\left[4N^{IK}N^{JL}\vec{\mu}_L\cdot\vec{\mu}_L - \frac{f^I_{KL}X^K\bar{X}^L}{N_{PQ}X^P\bar{X}^Q}\frac{f^J_{MN}X^M\bar{X}^N}{N_{PQ}X^P\bar{X}^Q}\right], \end{aligned} \tag{E.19}$$

where g is an overall factor to make the terms which are a result of the gauging more explicit in the Lagrangian; we put $g = 1$ from now on. In general, each vector field in the vector multiplets can be used to gauge one of the $h^{1,1}+1$ different killing vectors k^I. In our setting, there is only one isometry k, which we gauge by the graviphoton. The index I therefore only attains the value 0. If we use $\vec{\mu}_I = \delta_I^0\vec{\mu}$ we obtain

$$V = -\frac{4}{N_{MN}X^M\bar{X}^N}\Big(\left[2G_{\alpha\bar{\beta}}k^\alpha k^\beta + 3\vec{\mu}^2\right]X^0\bar{X}^0 + (N_{KL}N^{00} - \delta_K^0\delta_L^0)\bar{X}^KX^L\vec{\mu}^2\Big),$$

and the term depending on f^I_{KL} is zero for abelian gaugings. We now use the prepotential

$$F = \frac{1}{3!}\mathcal{K}_{ijk}\frac{X^iX^jX^k}{X^0} + \frac{1}{2}eX^0X^0\,, \tag{E.20}$$

where $e = \frac{i}{2}\zeta(3)\chi(CY)$ is purely imaginary. Using $X^i/X^0 = z^i = b^i + it^i$, we find

$$\begin{aligned} F_{00} &= \frac{1}{3}\mathcal{K}_{ijk}z^i z^j z^k + e & \operatorname{Im} F_{00} &= \mathcal{K}_{ijk}b^i b^j t^k - \frac{1}{3}\mathcal{K}_{ijk}t^i t^j t^k \\ F_{0i} &= -\frac{1}{2}\mathcal{K}_{ijk}z^j z^k & \operatorname{Im} F_{0i} &= -\mathcal{K}_{ijk}b^j t^k \\ F_{ij} &= \mathcal{K}_{ijk}z^k & \operatorname{Im} F_{ij} &= \mathcal{K}_{ijk}t^k\,. \end{aligned} \tag{E.21}$$

Using the abbreviations $\mathcal{K}_{ij} = \mathcal{K}_{ijk}t^k, \mathcal{K}_i = \mathcal{K}_{ij}t^j, 6\mathcal{V} = \mathcal{K}_{ijk}t^i t^j t^k$ we find

$$\begin{aligned} N_{00} &= 2\mathcal{K}_{ij}b^i b^j - 4\mathcal{V}' & N^{00} &= \frac{-1}{4\mathcal{V}'} \\ N_{0i} &= -2\mathcal{K}_{ij}b^j & N^{0i} &= \frac{-b^i}{4\mathcal{V}'} \\ N_{ij} &= 2\mathcal{K}_{ij} & N^{ij} &= \frac{-b^i b^j}{4\mathcal{V}'} + \frac{1}{2}d^{ij} \\ \mathrm{e}^{-K} &\equiv N_{IJ}X^I \bar{X}^J = (8\mathcal{V} + 2e)X^0\bar{X}^0\,, \end{aligned} \tag{E.22}$$

where we have written $\mathcal{V}' := \mathcal{V} - \frac{1}{2}e$. For the scalar potential we then finally find equation (6.41)

$$\begin{aligned} V &= -\frac{4}{8\mathcal{V} + 2e}\left(\left[2G_{\alpha\bar{\beta}}k^\alpha k^\beta + 3\vec{\mu}^2\right] + (8\mathcal{V}'\frac{-1}{4\mathcal{V}'} - 1)\vec{\mu}^2\right), \\ &= \frac{2}{4\mathcal{V} + e}\left[-2G_{\alpha\bar{\beta}}k^\alpha k^\beta\right]. \end{aligned} \tag{E.23}$$

The scalar potential is positive definite.

Appendix F

The Przanowski metric

In this appendix we repeat some of the results of [153, 166], which are used to determine the NS5–brane one–instanton corrected $N = 2$ moduli space in section 3.2.

In [166], it has been shown that a four-dimensional quaternionic-Kähler manifold M can be described in terms of a partial differential equation for a single, real function. Locally, the metric takes the form

$$\begin{aligned} g &= g_{\alpha\bar{\beta}}(dz^{\alpha} \otimes dz^{\bar{\beta}} + dz^{\bar{\beta}} \otimes dz^{\alpha}) \\ &= g_{1\bar{1}} dz^1 dz^{\bar{1}} + g_{1\bar{2}} dz^1 dz^{\bar{2}} + g_{2\bar{1}} dz^2 dz^{\bar{1}} + g_{2\bar{2}} dz^2 dz^{\bar{2}} + c.c.\,, \end{aligned} \tag{F.1}$$

where indices $\alpha, \beta, \bar{\alpha}, \bar{\beta} = 1, 2$, and we have used the usual convention of complex conjugation $z^{\bar{\alpha}} := \overline{z^{\alpha}}$. The Hermicity of this metric is encoded in the requirement $\overline{g_{\alpha\bar{\beta}}} = g_{\beta\bar{\alpha}}$. The elements $g_{\alpha\bar{\beta}}$ are now defined in terms of a real function $h = h(z^{\alpha}, z^{\bar{\alpha}})$ via

$$g_{\alpha\beta} = 2\left(h_{\alpha\bar{\beta}} + 2\delta^2_{\alpha}\delta^2_{\bar{\beta}}\mathrm{e}^h\right), \tag{F.2}$$

where the subscript α on h_{α} indicates differentiation of the function with respect to z^{α}. We have changed the sign of our defining function h with respect to the original function u used by Przanowski, as it offers a slightly more convenient form to work with.

The differential equation which determines the function h is the non–linear partial differential equation

$$h_{1\bar{1}}h_{2\bar{2}} - h_{1\bar{2}}h_{\bar{1}2} + (2h_{1\bar{1}} - h_1 h_{\bar{1}})\mathrm{e}^h = 0\,. \tag{F.3}$$

F.1 Solutions to the master equation

The equation (F.3) is a difficult partial differential equation. There have been various approaches in the literature which found exact and approximate solutions to the master equation. By imposing additional symmetries on the manifold M, one can simplify the master equation. Imposing one isometry reduces this equation to the Toda equation [166]. Upon imposing two commuting isometries one obtains the Calderbank-Pedersen metrics [167].

In [153], solutions to the master equation where obtained which corresponded to NS5–brane instantons. The relation between the complex coordinates and the real coordinates is given by

$$z^1 = \frac{1}{2}(u + i\sigma)\,, \quad z^2 = \frac{1}{2}(\chi + i\varphi)\,, \quad u \equiv r - \frac{1}{2}\chi^2 + c\log(r + c)\,. \tag{F.4}$$

The leading term of the one–instanton contribution is captured by

$$\begin{aligned} h &= h_0 + \Lambda, \quad h_0 = \log(r + c) - 2\log r\,, \\ \Lambda &= Cr^{-2-c}\cos(\sigma)\exp\left[-r + \frac{1}{2}\chi^2\right]. \end{aligned} \tag{F.5}$$

From the metric we only need the length of the Killing vector $k = \partial_\varphi$, which can be found from (F.1), (F.2) and (F.5) and is given by

$$-G_{\alpha\beta}k^\alpha k^\beta = \frac{4((r+2c)^2 + (r+c)\chi^2)}{r^2(r+2c)} + 16Cr^{-2-c}(2\chi^2 - 1)\exp(-c - r - \chi^2/2)\,.$$

Inserting this into (E.23) yields (6.41).

F.2 Moment maps

Although the moment maps are not present in the scalar potential, we include their calculation for completeness. We follow the conventions on quaternionic-Kähler geometry from [168].

We want to find vielbeins a, b for the metric (F.1) such that

$$a \otimes \bar{a} + b \otimes \bar{b} + c.c. = \mathrm{d}s^2\,. \tag{F.6}$$

Using the Ansatz $a = \alpha \mathrm{d}z^1 + \beta \mathrm{d}z^2, b = \gamma \mathrm{d}z^1 + \delta \mathrm{d}z^2$ we find

$$\begin{aligned} a &= \sqrt{2h_{1\bar{1}}}\,\mathrm{d}z^1 + \sqrt{2}\frac{h_{\bar{1}2}}{\sqrt{h_{1\bar{1}}}}\,\mathrm{d}z^2\,, \\ b &= \sqrt{2}\mathrm{e}^{h/2}\sqrt{\frac{h_1 h_{\bar{1}}}{h_{1\bar{1}}}}\,\mathrm{d}z^2\,. \end{aligned} \tag{F.7}$$

From those, we determine the $SU(2)$ connection one–forms

$$\begin{aligned}\omega^1 &= i\frac{\mathrm{e}^{h/2}}{\sqrt{h_1 h_{\bar 1}}}(h_{\bar 1}\mathrm{d}z^2 - h_1 \mathrm{d}z^{\bar 2})\,,\\ \omega^2 &= -\frac{\mathrm{e}^{h/2}}{\sqrt{h_1 h_{\bar 1}}}(h_{\bar 1}\mathrm{d}z^2 + h_1 \mathrm{d}z^{\bar 2})\,,\\ \omega^3 &= -\frac{i}{2}\left(h_1 - \frac{h_{1\bar 1}}{h_{\bar 1}} + \frac{h_{11}}{h_1}\right)\mathrm{d}z^1\\ &\quad -\frac{i}{2}\left(h_2 - \frac{h_{\bar 1 2}}{h_{\bar 1}} + \frac{h_{12}}{h_1}\right)\mathrm{d}z^2 + c.c.\,.\end{aligned} \tag{F.8}$$

As a non-trivial check, we can use the tree-level UHM metric, and these one-forms agree with the those obtained in [168]. Notice that the situation drastically simplifies when there is an additional killing vector in the direction $i(\partial_1 - \partial_{\bar 1})$, because then $h_1 = h_{\bar 1}$.

We now gauge the isometry associated with φ. In the complex coordinates, this is the vector

$$k = \frac{1}{2} i(\partial_2 - \partial_{\bar 2}), \tag{F.9}$$

where the normalization is such that $k = \partial_\varphi$. Calculations of the moment maps is now straight-forward and after some algebra we find

$$\vec{\mu} = \begin{pmatrix} \frac{\mathrm{e}^{h/2}}{\sqrt{h_1 h_{\bar 1}}}(h_1 + h_{\bar 1}) \\ -i\frac{\mathrm{e}^{h/2}}{\sqrt{h_1 h_{\bar 1}}}(h_1 - h_{\bar 1}) \\ -h_2 \end{pmatrix}, \tag{F.10}$$

which are real ($h_2 = h_{\bar 2}$).

The square of the moment maps therefore reads

$$\vec{\mu}^2 = (4\mathrm{e}^h + h_2^2) = 4\mathrm{e}^h + (\partial_\chi h)^2\,, \tag{F.11}$$

where we have used $\partial_\varphi h = 0$. This last expression is valid in the coordinates $(u, \sigma, \chi, \varphi)$. Changing to the coordinates $(r, \sigma, \chi, \varphi)$ amounts to changing the derivatives according to

$$\partial_\chi \to \partial_\chi + \chi\frac{r + c}{r + 2c}\partial_r\,. \tag{F.12}$$

Nederlandse samenvatting

De theoretische natuurkunde probeert de wereld om ons heen te beschrijven en te verklaren. Dit doet ze door theorieën op te stellen. In een theorie worden een aantal aannames gemaakt, op basis waarvan bestaande verschijnselen verklaard en nieuwe fenomenen voorspeld worden. Als dit goed werkt, wordt de theorie geaccepteerd. Als blijkt dat dit niet goed werkt, wordt de theorie verworpen. Hoewel in de loop der jaren al vele theorieën naar de prullenbak zijn verwezen, hebben we nu een redelijk goed begrip van de natuurkunde om ons heen.

Klassieke mechanica

Klassieke mechanica is de natuurkundige theorie van Newton. Veel mensen van mijn generatie die niks meer met natuurkunde te maken hebben, herinneren zich nog steeds de formule '$F = m \times a$' van de middelbare school. Deze formule zegt dat een kracht, ter grootte van F, een object met massa m een versnelling a zal geven. Vervolgens zijn er afzonderlijke formules waarmee de grootte van een kracht berekend kan worden, bijvoorbeeld voor zwaartekracht, elektrische kracht en wrijvingskracht.

De klassieke mechanica is een intuïtieve theorie, en werkt goed in alledaagse situaties[30]. Onder alledaagse situaties verstaan we: snelheden klein ten opzichte van de lichtsnelheid, objecten groter dan atomen en zwakke zwaartekracht.

Speciale en algemene relativiteitstheorie

Klassieke mechanica biedt echter geen volledige beschrijving van onze wereld. Zo blijkt de snelheid van licht altijd hetzelfde te zijn: als je in een trein zit en een lichtsignaal met de trein meestuurt, gaat deze net zo snel als wanneer je 'm tegen de richting van de trein in stuurt. Dit is in sterk contrast met de situatie als je op een trein staat en een balletje weggooit: iemand die op het perron staat zal zien dat het balletje harder gaat als

[30] Deze twee uitspraken zijn waarschijnlijk equivalent.

je 'm naar voren gooit, en langzamer als hij naar achteren gaat. Einstein kon dit verklaren met behulp van zijn speciale relativiteitstheorie. In deze theorie worden de drie ruimtelijke dimensies (lengte, breedte en hoogte) samengevoegd met de tijdsdimensie tot vier ruimte-tijd dimensies.

Na het ontwikkelen van de speciale relativiteitstheorie bedacht Einstein hoe hij hier vervolgens de zwaartekracht kon beschrijven. In 1915 publiceerde hij zijn algemene relativiteitstheorie. De vier-dimensionale ruimte-tijd is hierin gekromd door de aanwezigheid van materie, en de kromming van de ruimte-tijd bepaalt vervolgens hoe de materie beweegt. De theorie voorspelde een aantal nieuwe dingen, zoals het bestaan van zwarte gaten, waarover later meer.

Andere voorspellingen van algemene relativiteitstheorie zijn goed getest: de baan van Mercurius wijkt een beetje af van de klassieke mechanica, maar wordt nu correct beschreven. Ook het afbuigen van licht door zwaartekracht en energieverlies door zwaartekrachtsgolven zijn nauwkeurig gemeten en in overeenstemming met de theorie. De tijdsmetingen die GPS-satelieten gebruiken krijgen een kleine correctie ten gevolge van relativistische effecten; zonder deze correcties zou GPS minder nauwkeurig zijn.

Quantummechanica

Een andere ontwikkeling van de 20e eeuw was de quantummechanica, die de wereld op atoomschaal beschrijft. De theorie heeft zijn oorsprong in het werk van Max Planck uit 1900, waar hij straling beschreef door aan te nemen dat de energie die vrijkomt bij straling niet continu is, maar in kleine 'pakketjes' verdeeld was, de energiequanta. Einstein nam deze quantisatie serieus en beschreef hiermee hoe ook licht gequantiseerd is, in lichtdeeltjes die fotonen heten. In het dagelijks leven komen gigantische hoeveelheden fotonen voor, waardoor het ons niet opvalt dat het gequantiseerd is in individuele fotonen. Later werd de quantummechancia door Niels Bohr gebruikt om een model voor atomen te maken. De wereld van de quantummechancia is een bizarre wereld: deeltjes hebben geen vaste locatie, maar hebben alleen maar een kans om ergens te zijn op het moment dat er gemeten wordt. Het blijkt echter dat zodra men quantummechanica toepast op alledaagse objecten, dit soort correcties compleet te verwaarlozen is, waardoor we het quantumgedrag niet opmerken.

Quantumveldentheorie en het standaardmodel

De volgende stap was een volledige quantum-beschrijving van het elektro-magnetische veld. Al snel bleek dat een dergelijke theorie ook de speciale relativiteitstheorie moest bevatten. Het onderzoek leidde uiteindelijk tot de quantumveldentheorie. Het standaardmodel is de quantumveldentheorie die bijna alle materie en krachten beschrijft.

De theorie is uitvoerig getest, vooral in deeltjesversnellers, zoals bij CERN in Genève. De enige kracht die ontbreekt in dit model is de zwaartekracht.

Open vragen

Hoewel de algemene relativiteitstheorie en het standaardmodel voortreffelijk werken, zijn er nog steeds een aantal grote open vragen. Een direct probleem is dat het niet mogelijk is om zwaartekracht consistent te schrijven als een quantumveldentheorie. Hierdoor hebben we geen volledige beschrijving wanneer beide theorieën belangrijk zijn, zoals het bestuderen van zwarte gaten of de oerknal.

Andere open vragen komen uit de kosmologie. Uit metingen blijkt dat de materie van het standaardmodel maar 4% van alle materie in het helaal is. We denken dat de overige 96% uit donkere materie en donkere energie bestaat, welke beiden niet door de algemene relativiteitstheorie en het standaardmodel beschreven worden. Donkere materie zijn deeltjes die niet in het standaardmodel zitten. Donkere energie kan beschreven worden door de kosmologische constante; dit is een parameter in de vergelijkingen van Einstein. De waarde van deze constante is van groot belang voor de ontwikkeling van het heelal. Huidige metingen wijzen op een kleine, positieve waarde voor deze constante, maar een combinatie van algemene relativiteitstheorie en quantumveldentheorie geeft echter een heel grote waarde.

Om deze open vragen te beantwoorden, hebben we een theorie nodig die zwaartekracht en quantummechancia verenigt - een theorie van quantumzwaartekracht.

Snaartheorie, supersymmetrie en superzwaartekracht

Een van de meest veelbelovende theorieën voor quantumzwaartekracht is snaartheorie. Deze theorie, ontstaan rond 1970, gaat er van uit dat de fundamentele bouwstenen geen puntdeeltjes zijn, zoals in het standaardmodel, maar snaren. De deeltjes die we zien, zoals quarks, elektronen en gravitonen, worden dan beschreven door trillingen van deze snaren. Een recent verschenen Nederlandstalig, populair wetenschappelijk boek is ‘Snaartheorie’ van Marcel Vonk [169], dat een uitgebreide introductie tot snaartheorie geeft.

De fundamentele deeltjes in een quantumveldentheorie kunnen worden verdeeld in twee types: bosonen en fermionen. Twee gelijke fermionen kunnen niet bij elkaar zitten, terwijl bosonen dat juist wel kunnen. In het standaardmodel zijn de quarks en elektronen fermionen, en het Higgs deeltje, het foton en andere krachten-dragende deeltjes bosonen. Supersymmetrie is een symmetrie die de twee types relateert. In een supersymmetrische theorie heeft elk boson een fermionische partner, en omgekeerd; deze worden elkaars de superpartners genoemd. Hoewel het standaardmodel bosonen en fermionen bevat, zijn deze niet elkaars superpartners, maar moeten er extra deeltjes worden toegevoegd om het standaardmodel supersymmetrisch te maken. De snaartheorieën die het meest bestudeerd worden zijn ook supersymmetrisch.

Supersymmetrie heeft vele aantrekkelijke eigenschappen. De superpartners van het standaardmodel zijn nog niet ontdekt, maar ze zijn wel een kandidaat voor donkere materie; de supersymmetrie zou dan hun stabiliteit kunnen verklaren. Op dit moment worden er experimenten uitgevoerd in de Large Hadron Collider van CERN, die hopen om supersymmetrie te ontdekken.

Veel van de effecten van snaartheorie spelen zich af bij heel hoge energieën. Omdat het technisch voorlopig onmogelijk is om experimenten bij zulke hoge energieën uit te voeren, is het nuttig om naar een beschrijving te gaan waar alleen de effecten op lagere energie in voorkomen. Een dergelijke beschrijving is een quantumveldentheorie met zwaartekracht, die ook supersymmetrisch is, en daarom superzwaartekracht wordt genoemd.

Compactificatie

Hoewel snaartheorie zijn successen heeft als theorie van quantumzwaartekracht, leidt het ook tot nieuwe open vragen. Een direct probleem is dat de supersymmetrische snaartheorieën uitgaan van tien ruimte-tijd dimensies, in plaats van de vier die wij gewend zijn. Dit klinkt moeilijker dan het is. In ons dagelijks leven gebruiken we vier coördinaten om een gebeurtenis aan te duiden: drie voor de plek en eentje voor de tijd. In een tien-dimensionale wereld moeten er nog zes plaats-coördinaten bij om een volledige beschrijving van de gebeurtenis te geven. Als deze zes extra dimensies echter heel erg klein zijn (bijvoorbeeld in de orde van de Planck lengte, 10^{-35} m), dan maakt het niet veel uit wat de locatie in deze zes dimensies is, en de wereld lijkt dan weer vier-dimensionaal. Als we aannemen dat zes dimensies erg klein zijn, kunnen we de snaartheorie beschrijven met een effectieve theorie in vier dimensies. Dit proces heet compactificatie.

In dit proefschrift bekijk ik enkele open vragen rondom compactificatie. Zo is er een groot aantal verschillende ruimtes voor de interne zes dimensies, en verschillende keuzes leiden tot verschillende modellen. Zelfs voor een specifieke keuze kunnen er kleine verschillen in de interne ruimte zijn als we deze vanuit twee verschillende plekken in de vier-dimensionale ruimte bekijken. Deze verschillen komen uiteindelijk in de vier-dimensionale theorie terug als massaloze deeltjes, die moduli genoemd worden. Zulke moduli zijn echter niet waargenomen in ons universum. We moeten dus een mechanisme bedenken dat deze moduli een massa geeft. Als deze massa groot genoeg is, kan dit verklaren waarom de deeltjes tot nu toe niet waargenomen zijn.

Zwarte gaten

Een ander onderwerp van dit proefschrift zijn zwarte gaten. Een zwart gat ontstaat als er heel veel materie in een heel klein volume komt: de zwaartekracht (of eigenlijk: de kromming van de ruimte-tijd) wordt dan zo sterk dat er niks meer aan kan ontsnappen, zelfs licht niet. De meeste astronomen zijn er van overtuigd dat er in het midden van ons melkwegstelsel een zwart gat zit.

Stephan Hawking bestudeerde de theorie van quantummechanica in de omgeving van een zwart gat en ontdekte dat er, door quantum-effecten, wél straling uit het zwarte gat kan ontsnappen. Deze ontdekking leidde later tot grote discussie onder theoretisch fysici over de vraag of de informatie van deeltjes die in het zwarte gat vallen, verloren gaat. Men is er nu van overtuigd dat de informatie bewaard blijft. Zulke discussies laten zien dat zwarte gaten belangrijk zijn voor de studie van quantumzwaartekracht, omdat het gebieden zijn waar algemene relativiteitstheorie en quantummechanica elkaar tegenkomen.

Conclusie

In dit proefschrift heb ik een aantal vragen rondom compactificaties en zwarte gaten bestudeerd. Allereerst hebben we de volledig supersymmetrische configuraties beschreven in bepaalde theorieën van vier-dimensionale superzwaartekracht. Vervolgens gingen we op zoek naar nieuwe types zwarte gaten in deze theorieën. Hierna hebben we laten zien hoe deze vier-dimensionale theorieën verkregen kunnen worden uit de elf-dimensionale superzwaartekrachtstheorie. Een van de moduli is het volume van de interne zes dimensies. We hebben bestudeerd hoe deze modulus een massa kan krijgen.

Hoewel er nog vele interessante open vragen blijven in de snaartheorie, hoop ik dat dit proefschrift de antwoorden hierop dichterbij gebracht heeft.

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Hugo Looijestijn,
23 augustus 2010.

Curriculum Vitae

Ik ben geboren op 8 januari 1983 in Delft, waar ik de VWO-opleiding aan het Stanislas College van 1995 tot 2001 heb gevolgd. Hierna ben ik natuur- en wiskunde gaan studeren aan de Universiteit Utrecht. In 2006 heb ik beide doctoraaldiploma's cum laude behaald. Vanaf oktober 2006 heb ik mijn promotieonderzoek gedaan onder begeleiding van dr. Stefan Vandoren. De resultaten van mijn onderzoek zijn het onderwerp van dit proefschrift.

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