# Currents of the Quantum Boussinesq Hierarchy

*By*

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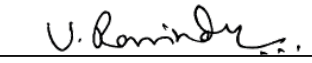

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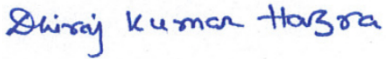

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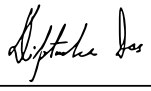

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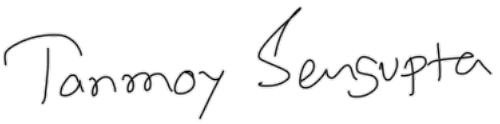

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## List of Publications arising from the thesis[1]

### Journal

1. **Sujay K. Ashok, Sanhita Parihar, Tanmoy Sengupta, Adarsh Sudhakar, Roberto Tateo, Thermal Correlators and Currents of the $\mathcal{W}_3$ Algebra, JHEP 01 (2025) [arXiv:2410.11748 [hep-th]].**

2. **Sujay K. Ashok, Sanhita Parihar, Tanmoy Sengupta, Adarsh Sudhakar, Roberto Tateo, Integrable Structure of Higher Spin CFT and the ODE/IM Correspondence, JHEP07(2024)179, [arXiv:2405.12636 [hep-th]].**

## List of other Publications, Not included in the thesis

### Journal

1. **Ratul Mahanta and Tanmoy Sengupta, Modular linear differential equations for four-point sphere conformal blocks, JHEP 02 (2023) 158 [arXiv:2211.05158 [hep-th]].**

### Preprint

1. **Max Downing, Faisal Karimi, Tanmoy Sengupta, Adarsh Sudhakar and G´erard M.T. Watts , Modular Properties of $\mathcal{W}_3$ Generalised Gibbs Ensembles, [arXiv:2508.16258 [hep-th]].**

[1] As it is standard in the High Energy Physics Theory (hep-th) community the names of the authors on any paper appear in their alphabetical order.

## Presentations

**Talks Presented :**

1. **"Integrable Structure of Quantum Boussinesq Hierarchy" invited seminar at TIFR, Mumbai, India on January 2025.**

2. **"Integrable Structure of Quantum Boussinesq Hierarchy" invited seminar at IIT Bombay, Mumbai, India on January 2025.**

3. **"Integrable Structure of Quantum Boussinesq Hierarchy" National Strings Meeting, 2024 at IIT Ropar, India on December 2024.**

4. **ODE/IM and Quantum Boussinesq Hierarchy," Chennai Strings Meeting, 2024, IMSc, Chennai, India.**

5. **"Modular linear differential equations for four-point sphere conformal blocks", Indian Strings Meeting, 2023, IIT Bombay, Mumbai, India.**

6. **"Some results on KdV and Boussinesq Hierarchy," Chennai Strings Meeting, 2023, IMSc, Chennai, India.**

7. **"Modular linear differential equations for four-point sphere conformal blocks", invited seminar at IIT Kanpur, Kanpur, India on June 2023.**

8. **"Modular linear differential equations for four-point sphere conformal blocks", Chennai Strings Meeting, 2023, IMSc, Chennai, India.**

**Poster Presentation:**

1. **"Integrable Structure of Quantum Boussinesq Hierarchy", Strings, NYUED, Abu Dhabi, UAE on Jan 2025.**

2. **"Integrable Structure of Higher Spin CFT and the ODE/IM Correspondence", Integrability in Gauge and String Theory, ICTP-SAIFR, São Paulo, Brazil on June 2024.**

3. **"Modular linear differential equations for four-point sphere conformal blocks", Indian Strings Meeting, 2023, IIT Bombay, Mumbai, India on December 2023.**

**To Maa and Baba**

*Education was a luxury we couldn't afford. Thank you for the immense sacrifices you made to put me and Dada through college. I owe everything to you both.*

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It seems a lifetime ago that I first dared to dream this dream, as I embarked on my bachelor's degree at Scottish Church College in 2015. This thesis is the culmination of that dream, but it is a journey I would never have completed alone. There were moments I stumbled, days I wanted to surrender, and challenges that seemed insurmountable. Yet, I was never truly alone. I was carried by the unwavering support of incredible individuals who offered their strength, wisdom, and light when my own grew dim. Without them, this achievement, and the person I have become, would not exist.

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I have been extraordinarily fortunate to have been guided by amazing teachers throughout my journey, each of whom left an indelible mark.

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# Summary

A conformal field theory (CFT) possessing the Virasoro algebra as its local symmetry algebra is endowed with an infinite set of mutually commuting local integrals of motion, commonly referred to as quantum Korteweg-de Vries (KdV) charges. This system defines the quantum KdV problem, where in the limit of infinite central charge, these quantum charges reduce to their classical counterparts in the KdV hierarchy. Similarly, a CFT with an extended $\mathcal{W}_3$ symmetry algebra is expected to admit an infinite tower of conserved local integrals of motion, known as quantum Boussinesq charges. While explicit expressions for the first four such charges are known, this thesis presents a systematic procedure for constructing the higher charges.

Our construction relies on two key methodological components:

1. We first compute the eigenvalues of these charges on the highest-weight state using the **ODE/IM correspondence**

2. We then formulate the most general ansatz for the current densities and compute the thermal correlators of each composite operator appearing in the ansatz. These correlators are evaluated via the **Zhu recursion relations**, where for an operator $O$, the thermal expectation value is defined as:

$$\langle O \rangle = \mathrm{Tr}_{\mathcal{V}}\, O\, q^{L_0 - \frac{c}{24}}, \quad q = e^{-\beta}, \tag{1}$$

with $\beta$ denoting the inverse temperature and the trace is taken over the higher spin

module generated by the action of $L_{-n}$ and $W_{-m}$ on a highest weight state.

By matching the low-temperature limit ($q \to 0$) of these thermal correlators with the computed charge eigenvalues, we determine the undetermined coefficients in the current ansatz. To extend this construction to higher-order currents, we require additional spectral data, which we obtain by:

- Computing the eigenvalues of the higher charges in the first and second excited level states.
- Evaluating thermal two-point functions of currents with composite operators.

Through a systematic comparison of the $q$-expansion coefficients (up to $\mathcal{O}(q^2)$) with our spectral calculations, we determine all remaining undetermined constants in the current densities. This approach enables us to explicitly construct three previously unknown current densities of the quantum Boussinesq hierarchy: $J_8$, $J_9$, and $J_{11}$.

Our main results can be summarized as follows:

- We perform a systematic study of the integrable structure of $\mathcal{W}_3$-symmetric CFTs through their quantum Boussinesq charges.
- We compute the eigenvalues of these charges on the low-lying states of the higher-spin module.
- We compute the thermal one point functions of the currents up to weight eleven.
- We establish a systematic procedure to construct higher conserved currents by combining the spectral data with thermal correlators.
- We explicitly derive three new current densities, all of which correctly reduce to their classical Boussinesq counterparts in the large central charge limit.

# Chapter 1

# Introduction

The investigation of Integrable quantum field theories (IQFTs) in two dimensions has proved to be very fruitful in modern physics. Unlike generic QFTs, which require perturbative or numerical methods to derive physical quantities of interest, integrable 2D theories can be solved non-perturbatively using techniques such as the Bethe ansatz, bootstrap methods, and the ODE/IM correspondence. This class of models serves as a unique framework to obtain exact solutions that enhance our understanding of strongly coupled systems while bridging connections between high-energy physics, statistical mechanics, and mathematical physics. The study of IQFTs continues to be an active area of research, with implications for both fundamental theoretical developments and practical applications in physical systems.

Physically, IQFTs can be understood as deformations of a two dimensional CFT [1] by relevant operators, which trigger a renormalization group (RG) flow while preserving integrability—yielding a massive theory with infinitely many conserved currents. Since the mid eighties 2D CFTs have been extensively studied due to their broad applications in string theory, critical phenomena, and mathematical Physics [2–4]. The conformal symmetry in two dimensions is generated by the energy-momentum tensor $T(u)$, whose operator product expansion with itself encodes the Virasoro algebra. Within the univer-

sal enveloping algebra of the Virasoro algebra, which is generated by $T(u)$ along with the normal ordered composite operators built out of $T(u)$ and its derivatives, an infinite-dimensional abelian sub-algebra can be identified, which is constructed from local integrals of motion (IMs) [5–7]. The integrability of such systems is related to the existence of the Yang-Baxter equation, which guarantees that an infinite tower of mutually commuting local integrals of motion are present and leads to factorization the S-matrix. Thus the spectral problem for these conserved quantities can be solved exactly through Bethe ansatz methods. In a series of seminal works during the mid-nineties, the diagonalization of these IMs was systematically investigated [8–11]. These charges are termed quantum KdV charges, because the quantum IMs reduce to their classical counterparts in the Korteweg–de Vries (KdV) hierarchy in the limit of infinite central charge. Here we list the first few quantum KdV charges:

$$\begin{aligned}
\mathbf{Q}_1 &= \int_0^1 T(u)\, du \ , \\
\mathbf{Q}_3 &= \int_0^1 (TT)(u) du \ , \\
\mathbf{Q}_5 &= \int_0^1 [(T(TT))(u) + \frac{c+2}{12}(2\pi)^2 (T'T')(u)] du \ ,
\end{aligned} \tag{1.1}$$

where $u$ is a co-ordinate on a cylinder with a circumference $2\pi$ and the parentheses indicate conformal normal ordering[1]. Here $T(u)$ is the energy-momentum tensor and $T'(u)$ denotes the derivative of the energy-momentum tensor with respect to $u$. In the large central charge limit $c \to -\infty$, we can substitute

$$T(u) \longrightarrow -\frac{c}{6} U(u), \quad \text{where} \quad U(u + 2\pi) = U(u) \tag{1.3}$$

[1]The normal ordered product of two fields $A_1(u_1)A_2(u_1)$ is defined using the two-point thermal correlator [12]:

$$\langle (A_1 A_2)(u_1) \rangle = \frac{1}{2\pi i} \oint_{u_1} \frac{du_2}{u_2 - u_1} \langle A_1(u_1) A_2(u_2) \rangle \ . \tag{1.2}$$

and scale the quantum charges as

$$\mathbf{Q}_{2n-1} \to \left(-\frac{c}{6}\right)^n I^{cl}_{2n-1} \,. \tag{1.4}$$

Then (1.1) will reduce to

$$\begin{aligned} I^{cl}_1 &= \int du\, U(u)\,, \\ I^{cl}_3 &= \int du\, U^2(u)\,, \\ I^{cl}_5 &= \int du\, \left(U^3(u) - \frac{1}{2}(2\pi U'(u))^2\right)\,. \end{aligned} \tag{1.5}$$

These are the conserved charges of the classical KdV hierarchy that follows from $U(u)$ satisfying the KdV equation with periodic boundary conditions.

In this thesis, we investigate the higher-spin generalization of the quantum integrability problem by studying the $\mathcal{W}_3$ algebra, an extension of the Virasoro algebra generated by the spin-2 energy-momentum tensor $T(u)$, along with a spin-3 current $W(u)$ [13]. Analogous to the Virasoro case, the universal enveloping algebra of $\mathcal{W}_3$ algebra is conjectured to admit an infinite-dimensional abelian sub-algebra [7, 14]. However, explicit construction of this sub-algebra presents significant challenges due to the non-linear nature of the $\mathcal{W}_3$ algebra. The first few non-zero IMs are known due to the work of [15], which are given by:

$$\mathbf{I}_n = \int_0^1 du\, J_{n+1},\ \forall\, n \in \mathbb{Z}^+\,. \tag{1.6}$$

where $J_{n+1}$ is the current for a given charge $I_n$, which are given by

$$\begin{aligned} J_2(u) &= T(u)\,, \\ J_3(u) &= W(u)\,, \\ J_5(u) &= (TW)(u)\,, \\ J_6(5) &= (T(TT))(u) + 9(WW)(u) + \frac{c-10}{32}(2\pi)^2(T'T')(u)\,. \end{aligned} \tag{1.7}$$

In literature, this problem is known as quantum Boussinesq hierarchy and IMs are quantum Boussinesq charges as in the classical limit $c \to -\infty$, these IMs reduces to its classical counterpart which is the classical Boussinesq charges[2]. The higher order currents are not known and the goal of this thesis is to give a systematic procedure to construct the currents.

In integrable systems, the existence of infinitely many commuting conserved charges allows for the definition of a Generalized Gibbs Ensemble (GGE). When a system possesses additional conserved quantities beyond the Hamiltomian, its equilibrium state must be described by a grand canonical ensemble with a density matrix of the form

$$\rho = e^{-\beta H + \sum_i \mu_i I_i} \tag{1.8}$$

where $\mu_i$ are fugacities conjugate to the conserved charges $I_i$. The study of GGEs is of significant importance due to multiple compelling theoretical and practical considerations. The GGE is also of interest within a holographic framework. For instance, higher spin theories in $AdS$ with gauge fields $s > 2$ (in addition to the graviton) are dual to CFTs with extended symmetry algebra. A particularly relevant case is the $AdS_3/CFT_2$ correspondence; in the Chern-Simons formulation, the higher spin gravity theory with $\mathrm{SL}(3,\mathbb{R})$ gauge group is dual to a CFT with an extended $\mathcal{W}_3$ algebra. For such cases the CFT partition function is given by

$$Z_{CFT} = Tr(q^{L_0 - \frac{c}{24}} y^{W_0}) \tag{1.9}$$

where $q = e^{2\pi i \tau}$, $y = e^{2\pi i \alpha}$ and $W_0$ is the zero mode of the spin three field which is also the second Boussinesq charge. $\tau$ is the complex structure of the torus which is related to the inverse temperature and $\alpha$ is the chemical potential. The exact expression of the partition function is not known and it remains an outstanding problem. Another important question is to understand the modular properties of the partition function in (1.9). Apart from

[2]We will discuss the classical Boussinesq charges in detail in the next chapter.

mathematical interest, it will also help us to understand the high temperature behaviour of the partition function. As first step towards understanding the modular transformation properties of the generalized partition function (where fugacities are turned on for multiple Boussinesq charges) is the calculation of thermal correlators of quantum Boussinesq charges, a topic that is studied in detail in this thesis.

The GGE has been studied recently in presence of one KdV charge [16–19] for simpler minimal models. The GGE for in presence of one KdV charge[3] can be written as

$$Z_{CFT} = Tr(q^{L_0-\frac{c}{24}} e^{\alpha \mathbf{Q}_{2n-1}}). \tag{1.10}$$

For small chemical potential $\alpha$, the partition can be written as a asymptotic series in $\alpha$. In this scenario, all the thermodynamically relevant quantities are the thermal correlator of the KdV charge present in the partition function which are known in terms of Ramanujam-Serre derivative acting on the character [12]. Then one can take the modular transformation of the thermal correlators and for simpler minimal models one can re sum it back. It turns out that the modular transformed partition function contains the information of all other KdV charges and quasi-primaries.

A similar strategy can be taken to understand the CFT partition with an extended $\mathcal{W}$ algebra. In order to do so, one needs to construct the higher charges of the quantum Boussinesq charges.

In this thesis, we significantly extend the program initiated in [15] (also [24]) to explicitly construct the infinite-dimensional abelian sub-algebra within the universal enveloping algebra of the $\mathcal{W}_3$ algebra. In order to so, we study a CFT defined on a cylinder of circumference 1 with an extended $\mathcal{W}_3$ algebra.

While in principle one could construct the commuting charges by writing a general ansatz

[3] GGE for the KdV hierarchy has been extensively studied in the large central charge $c$ limit [20–22], as these theories are holographically dual to KdV-charged black holes in $AdS_3$ [23], and while an exact expression for the GGE partition function remains unknown, some progress has been made for simple minimal models [16–19]

and imposing commutativity with all lower-order charges to fix the undetermined constants, this direct approach becomes prohibitively difficult in practice due to the nonlinear structure of the $\mathcal{W}_3$ algebra. To circumvent these challenges, we adopt an alternative strategy focusing on the construction of currents that generate these charges. In this thesis, we will give a systematic method to construct the quantum Boussinesq hierarchy. We will achieve this by using two ingredients

- The eigenvalues of the quantum Boussinesq charge acting on the highest weight states of a conformal field theory (CFT) will be derived using the **ODE/IM correspondence**, establishing a connection between integrable models and ordinary differential equations.

- A general ansatz for the desired current is formulated as a linear combination of normal-ordered composite operators constructed from the energy-momentum tensor $T(u)$, spin three field $W(u)$ and their derivatives. For a CFT defined on a torus, the thermal correlator of each composite operator is computed using the Zhu recursion relations [25, 26]. The thermal correlator of an operator $\mathcal{O}$ is defined as

$$\langle \mathcal{O} \rangle = Tr_V\left(\mathcal{O}\, q^{L_0 - \frac{c}{24}}\right), \quad q = e^{-\beta}, \tag{1.11}$$

  where $\beta$ is related to the inverse temperature.

- In the low-temperature limit, the total thermal correlator of the ansatz must coincide with the eigenvalue of the charge obtained via the ODE/IM correspondence. By enforcing this matching condition, the undetermined coefficients in the current ansatz can be systematically fixed.

For higher-order charges, matching the thermal one-point function of the ansatz with the eigenvalue of the charge alone is insufficient to uniquely determine the current. This arises because certain linear combinations of a subset of composite operators in the ansatz vanish

identically, leaving only a subset of undetermined constants fixed. To resolve the higher-order currents, additional data are required. We address this issue through the following systematic procedure:

- We first compute the eigenvalues of the charges on the first and second excited states using the ODE/IM correspondence.
- Next, we evaluate the thermal two-point function of the ansatz in the presence of a composite operator $\mathcal{Q}$, employing Zhu's recursion relations. The choice of $\mathcal{Q}$ is non-unique; however, it must be selected such that the thermal two-point function of the vanishing operator combination remains non-zero when inserted alongside $\mathcal{Q}$.
- Using the $\mathcal{W}_3$ algebra, we explicitly compute the action of the composite operator (constructed from the charge and $\mathcal{Q}$) on the first and second excited states.
- Finally, we enforce consistency by matching the $q$-expansion of the thermal two-point function with the results obtained from steps (1) and (3) at $\mathcal{O}(q)$ and $\mathcal{O}(q^2)$. This allows us to systematically fix the higher-order currents up to $J_{11}$.

This thesis is based on the work done in [27, 28]. Below we summarize our main results.

- We have put forward a systematic method to construct the currents of quantum Boussinesq hierarchy.
- We have computed the eigenvalues of the first eleven quantum Boussinesq charges on the highest weight state, and the first and second excited states using the ODE/IM correspondence.
- We have computed the thermal one-point functions of the aforementioned charges.
- Furthermore, we have constructed three new currents of the quantum Boussinesq hierarchy. In the classical limit, these currents are in agreement with their classical counterparts.

**Plan of the Thesis:**

- In chapter 2, we review the necessary background for this thesis, including the $\mathcal{W}_3$ algebra,the classical Boussinesq hierarchy, ODE/IM correspondence and the Zhu recursion relations.
- In chapter 3, we use the ODE/IM correspondence to compute the eigenvalues of the charges on highest, first and second excited states.
- Chapter 4 is devoted to the construction the currents of the Boussinesq hierarchy.
- In chapter 5, we summarize our results and present our conclusions.

# Chapter 2

# Background

This chapter establishes the theoretical framework necessary for the development of this thesis. Our discussion begins with an analysis of the Boussinesq hierarchy in both its classical and quantum formulations. Building on this foundation, we then explore the ODE/IM correspondence and employ this framework to derive explicit expressions for the eigenvalues of KdV charges acting on highest weight states. The chapter concludes with a systematic presentation of thermal correlator computations using Zhu's recursion relations, providing essential tools for later chapters.

## 2.1 The Boussinesq Equation: Classical Analysis

The historical foundation of this class of equations is attributed to the French mathematician Joseph Valentin Boussinesq. In his seminal publication [29], Boussinesq first derived the nonlinear partial differential equation:

$$\partial_t^2 U = -\frac{1}{3}\partial_x^2(\partial_x^2 U - 2U^2)\,. \tag{2.1}$$

Boussinesq's objective was to formulate a refined model for the propagation of long, bidirectional waves in a shallow water channel, improving upon the existing hydrodynamical theories of the era. A critical distinction from the contemporaneous Korteweg-de Vries (KdV) equation is the second-order temporal derivative, which allows the Boussinesq equation to describe waves propagating in both directions. This feature renders it a more fundamental, though mathematically more complex, model for wave phenomena in a conservative system.

For nearly a century, the Boussinesq equation was studied as a nonlinear PDE of physical interest. The paradigm shift came with the discovery of the soliton and the concept of integrability in the 1960s. V. E. Zakharov in 1973 [30] demonstrated that the Boussinesq equation could be solved using the Inverse Scattering Transform, formally establishing it as a completely integrable system. A fundamental insight originating from the work of Lax is that any integrable equation inherently belongs to an infinite family of compatible equations, known as a hierarchy. The construction of this hierarchy is rigorously formalized through the algebra of pseudodifferential operators. Within this framework, the Boussinesq equation is identified as the second flow; the first flow corresponds to spatial translation, while subsequent flows represent higher-order nonlinear partial differential equations. All members share a common infinite set of conserved quantities and symmetries.

A pivotal advancement was the discovery of the system's bi-Hamiltonian structure, whereby the entire hierarchy admits two compatible Hamiltonian formulations. This property provides a recursive mechanism that generates the infinite sequence of conserved quantities.

The Boussinesq hierarchy serves as a paradigmatic example of a Drinfel'd-Sokolov hierarchy [31] associated with the affine Lie algebra $\hat{\mathfrak{sl}}_3$. Its algebraic structure is governed by the classical $\mathcal{W}_3$-algebra [13], and its bi-Hamiltonian structure is well-documented in [32, 33]. The modern interpretation utilizing pseudodifferential operators was largely

developed by Gelfand and Dickey [34].

We review some well-known results related to the classical Boussinesq equation in this section. We begin with the observation that the Boussinesq equation is essentially the consistency condition between the two coupled equations:

$$\partial_t U = 2\partial_x V \,, \tag{2.2}$$

$$\partial_t V = -\frac{1}{6}\partial_x^3 U + \frac{2}{3} U \partial_x U \,. \tag{2.3}$$

Furthermore, this problem is integrable, as it can be recast in the form of a Lax equation:

$$\partial_t L = [L, A] \,, \tag{2.4}$$

with the relevant Lax pair given by [35]:

$$L = \partial_x^3 - U\partial_x - \frac{1}{2}\partial_x U + V \,, \tag{2.5}$$

$$A = \partial_x^2 - \frac{2}{3} U \,. \tag{2.6}$$

### 2.1.1 Conserved Charges

The scalar Lax equation is useful to us in the following way: the WKB coefficients from the solution to its eigenvalue problem directly give us the conserved densities. We show this in detail, following the ideas in [30, 35]. Consider the scalar Lax equation, and its associated eigenvalue problem:

$$\partial_t L = [A, L] \,, \tag{2.7}$$

$$L\psi = \lambda\psi \,. \tag{2.8}$$

We now differentiate the eigenvalue equation with respect to $t$ and use the Lax equation, keeping in mind its iso-spectral property. Further, assuming that the span of eigenvectors with eigenvalue $\lambda$ is one dimensional, we arrive at

$$\begin{aligned}(L-\lambda)(\partial_t\psi - A\psi) &= 0\ ,\\ \text{or}\quad (\partial_t\psi - A\psi) &= g(t)\psi\ .\end{aligned} \tag{2.9}$$

Substituting the WKB ansatz $\psi(x,t) = \exp\left(\frac{1}{\epsilon}\int_0^x dx'\ P(x',t,\epsilon)\right)$ into the $x$-derivative of (2.9) gives us:

$$\partial_t P(x,t,\epsilon) - \epsilon\ \partial_x A = 0\ . \tag{2.10}$$

From this, it follows that the spatial integrals of $P(x,t,\epsilon)$ are conserved quantities. Therefore, we must compute the WKB solutions of the diagonalised Lax operator to find the conserved charges. To this end, we substitute into the eigenvalue problem of (2.5) the following ansatz:

$$\psi(x,t) = \exp(i\frac{x}{\epsilon} + i\int_0^x \chi(x,t,\epsilon))\ . \tag{2.11}$$

For small $\epsilon$ we have the asymptotic expansion:

$$\chi(x,t,\epsilon) = \sum_{n=1}^{\infty}\chi_n(x,t)\ \epsilon^n\ . \tag{2.12}$$

Substituting this into the eigenvalue equation $(L-\lambda)\psi = 0$, with $i\lambda = \frac{1}{\epsilon^3}$, we obtain the first two terms in the WKB expansion:

$$\chi_1 = -\frac{1}{3}U\ , \qquad \chi_2 = -\frac{i}{6}U_x - \frac{i}{3}V\ . \tag{2.13}$$

For $n > 2$, we obtain a recurrence relation for the $\chi_n$:

$$\chi_{n+1} = i(\chi_n)_x - \sum_{k=1}^{n-1}\chi_k\chi_{n-k} + \frac{1}{3}(\chi_{n-1})_{xx} + i\sum_{k=1}^{n-2}\chi_k(\chi_{n-k-1})_x$$

$$-\frac{1}{3}\sum_{p+m+l=n-1}\chi_p\chi_m\chi_l-\frac{U}{3}\chi_{n-1}\,. \tag{2.14}$$

By $(\chi_n)_x$ we mean the $x$-derivatives of $\chi_n$. The conserved charges are given by

$$I_n^{\text{class}}(x,t)=\int dx\, J_{n+1}(x,t)\,, \tag{2.15}$$

where the $J_{n+1}(x,t)$ are proportional to $\chi_n(x,t)$. We list the first few non-vanishing charges:

$$I_1^{\text{class}}=\int dx\, U\,, \tag{2.16a}$$

$$I_2^{\text{class}}=\int dx\, V\,, \tag{2.16b}$$

$$I_4^{\text{class}}=\int dx\, UV\,, \tag{2.16c}$$

$$I_5^{\text{class}}=\int dx\,(U^3+9V^2+\frac{3}{4}U_x^2)\,, \tag{2.16d}$$

$$I_7^{\text{class}}=\int dx\,(U^4+18V^2U+9V_x^2+\frac{3}{4}U_{xx}^2+\frac{9}{2}UU_x^2)\,, \tag{2.16e}$$

$$I_8^{\text{class}}=\int dx\,(U^3V+3V^3-3UVU''-\frac{9}{4}VU'^2+\frac{3}{5}U''V'')\,, \tag{2.16f}$$

$$\begin{aligned}I_{10}^{\text{class}}=\int dx\,(&U^4V+6UV^3+9VV'^2-6U^2VU''-\frac{15}{2}UVU'^2+\frac{21}{4}VU''^2\\&+\frac{15}{2}VU'U^{(3)}+3UVU^{(4)}+\frac{3}{7}V^{(3)}U^{(3)})\,.\end{aligned} \tag{2.16g}$$

We observe that all charges $I_{3n}=0$, as the corresponding densities are total derivatives. We have normalized the conserved current densities such that

$$J_{2n}=U^n+\dots\,,\quad\text{and}\quad J_{2n+1}=U^{n-1}\,V+\dots \tag{2.17}$$

One of our aims in this work is to find the conserved currents of the quantum Boussinesq hierarchy. One check on our eventual results will be to compute the classical limit of those currents and match with these classical currents.

## 2.2 The $\mathcal{W}_3$ algebra and its relation to classical Boussinesq hierarchy

In this section, we examine the quantum Boussinesq hierarchy and its relation to the $\mathcal{W}_3$ algebra. To establish the necessary framework, we first review the structure of the $\mathcal{W}_3$ algebra. Consider an energy-momentum tensor $T(u)$ and spin three field $W(u)$, with Fourier expansions given by

$$T(u) = -\frac{c}{24} + \sum_{n=-\infty}^{\infty} L_n e^{-2\pi i n u}, \qquad W(u) = \sum_{n=-\infty}^{\infty} W_n e^{-2\pi i n u} , \tag{2.18}$$

where $c$ denotes the central charge. As demonstrated in [13], the modes $L_n$ and $W_n$ satisfy the $\mathcal{W}_3$ algebra, defined by the commutation relations:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{m+n,0} , \tag{2.19}$$

$$[L_n, W_m] = (2n-m)W_{n+m} , \tag{2.20}$$

$$\begin{aligned}[W_n, W_m] =& \frac{(n-m)}{3}\Lambda_{n+m} + \frac{n-m}{3b^2}\left(\frac{1}{15}(n+m+3)(n+m+2) - \frac{1}{6}(n+2)(m+2)\right)L_{n+m} \\ &+ \frac{c}{1080b^2} n(n^2-4)(n^2-1)\delta_{n+m,0} ,\end{aligned} \tag{2.21}$$

$c$ is the central charge and $b^2 = \frac{16}{5c+22}$. The $\Lambda_n$ are the modes of the composite normal ordered operator $(TT)(u)$ built out of the stress tensor:

$$\Lambda_n = \sum_{k=-\infty}^{\infty} :L_k L_{n-k}: + \frac{1}{5} x_n L_n , \tag{2.22}$$

with $x_{2l} = 1 - l^2$ and $x_{2l+1} = 2 - l - l^2$. Here, the normal ordering symbol $::$ indicates as usual that we put the operators with larger index $n$ to the right. These relations were originally derived using operator product expansions and the imposition of the Jacobi identity. For an alternative derivation, see [36]. To connect this quantum algebra to its

classical counterpart, we introduce the following scaling transformation:

$$T(u) \to \frac{c}{24}U(u), \quad W(u) \to i\left(-\frac{c}{24}\right)^{\frac{3}{2}} V(u), \quad [\ ,\ ] \to \frac{48\pi}{ic}\{\ ,\ \} \tag{2.23}$$

Taking the limit $c \to -\infty$ limit, the $\mathcal{W}_3$ algebra reduces to the classical Poisson algebra:

$$\begin{aligned}
\{U(u), U(v)\} &= -(U(u) + U(v))\delta'(u-v) + 2\delta'''(u-v), \\
\{U(u), V(v)\} &= -(V(u) + 2V(v))\delta'(u-v), \\
\{V(u), V(v)\} &= -\left[\frac{1}{4}(U''(u) + U''(v)) + \frac{1}{3}(U^2(u) + U^2(v))\right]\delta'(u-v) \\
&\quad + \frac{5}{12}(U(u) + U(v))\delta'''(u-v) - \frac{1}{6}\delta^{(5)}(u-v).
\end{aligned} \tag{2.24}$$

Given a Lax equation of the form (2.4), the fundamental Poisson brackets between the fields $U(u)$ and $V(u)$ can be derived using the techniques outlined in [37]. Remarkably, these brackets coincide precisely with (2.24), confirming that a conformal field theory with an extended $\mathcal{W}_3$ symmetry algebra corresponds to the classical Boussinesq hierarchy in the classical limit. In the $c \to -\infty$ limit, the classical charges and the quantum charges are related by

$$\mathbf{I}_k \longrightarrow i^{-k-1}\left(-\frac{c}{24}\right)^{\frac{k+1}{2}} I_k^{\text{class}}\ . \tag{2.25}$$

## 2.3 ODE/IM correspondence: A review

In this section, we will briefly review the ODE/IM correspondence and as an exercise, we will compute the eigen-values of the KdV charges on the highest weight state of a CFT. This section is essentially a review of [38], for more details, readers may refer to the original article.

A remarkable connection exists between Ordinary Differential Equations (ODE) and Integrable Models (IM) [38–40]. This section will be dedicated to examining these con-

nections, commonly referred to as the ODE/IM correspondence. The spectrum of finite dimensional integrable models are typically obtained by solving the Bethe ansatz equations. Subsequently, the Bethe equations were reformulated by Baxter in terms of functions possessing specific analytic properties, known as Baxter $T$ and $Q$ functions. For an integrable model, the spectrum of the conserved charges can be derived from the asymptotic expansion of the logarithm of Baxter $T$ function. The ODE/IM correspondence establishes a connection by identifying Baxter $T$ and $Q$ functions with the Stokes data of a given ordinary differential equation (ODE). Specifically, the $T$ and $Q$ functions can be expressed in terms of the Wronskians of subdominant solutions in different Stokes sectors [41]. Therefore, the WKB periods of the ODE should encode the spectrum of the quantum charges.

In this section, we will review the ODE/IM correspondence for a second order ODE [41, 42]. We begin with a Schrödinger type ODE:

$$(-\partial_x^2 + V(x))\psi(x) = 0 \tag{2.26}$$

where the potential is given by[1]

$$V(x) = x^{2M} + \frac{l(l+1)}{x^2} - E \tag{2.27}$$

where $x \in C$. $C$ is a cover of the complex plane $\tilde{\mathbb{C}} = \mathbb{C} \setminus \{0\}$. In the region $Re[x] \to \infty$, we can write a asymptotically growing and decreasing solution $\psi_\pm(x)$ of the equation (2.26)

$$\begin{aligned} \psi_+(x) &\sim c_+ V(x)^{-\frac{1}{4}} exp\Big( \int^x dx' \sqrt{V(x')} \Big), \quad Re[x] \to \infty \,, \\ \psi_-(x) &\sim c_- V(x)^{-\frac{1}{4}} exp\Big( - \int^x dx' \sqrt{V(x')} \Big), \quad Re[x] \to \infty. \end{aligned} \tag{2.28}$$

Here $c_\pm$ are normalization constants. The asymptotic solution (2.28) not only holds in the

[1] For the specified potential,ODE/IM correspondence demonstrates that the spectral data of the ordinary differential equation encode the vacuum eigenvalues of the conserved integrals of motion for the KdV hierarchy.

real line but also in a general direction in the complex plane. We define a general direction in the complex plane as $x = \rho e^{i\nu}$ where $\rho$ and $\nu$ are real numbers. Upon substituting in (2.28), we get

$$\psi_{\pm}(x) \sim V^{-\frac{1}{4}} exp\Big( \pm \frac{1}{M+1} e^{i\nu(1+M)\rho^{(1+M)}} \Big) \tag{2.29}$$

Depending upon the values of $\nu$, these solutions will either grow or decay exponentially. We will call the exponentially growing solution as dominant whereas we will call the exponentially decaying solution subdominant. However, for certain values of $\nu$, both solutions are oscillatory in nature. These special solutions are given by

$$Re[e^{i\nu(1+M)}] = 0, \tag{2.30}$$

In the complex plane, the lines which satisfy the above equation are called *Anti-Stokes line*. Whereas a *Stokes line* is a line where both solutions exhibit their fastest growth or decay. The Stokes lines are characterized by

$$Im[e^{i\nu(1+M)}] = 0. \tag{2.31}$$

The wedges between two different Anti-Stokes lines are called Stokes Sector and we can label them as

$$\mathcal{S}_k = \left\{ x \in \mathbb{C} : \left| arg(x) - \frac{2\pi k}{2M+2} \right| < \frac{\pi}{2M+2} \right\} \tag{2.32}$$

The region of validity of the asymptotic solutions (2.29) is the union wedges

$$\mathcal{S}_{WKB} = \mathcal{S}_{-1} \cup \bar{\mathcal{S}}_0 \cup \mathcal{S}_1 \tag{2.33}$$

where $\bar{\mathcal{S}}_0$ is the closure of $\mathcal{S}_0$ (see figure (2.1)).

Finding solutions outside $\mathcal{S}_{WKB}$ in $|x| \to \infty$ is non-trivial as this is related to the *Stokes Phenomenon*. In the event of crossing the Stokes line, the previously negligible subdominant term becomes significant. Here the principal quantities of interest are the Stokes

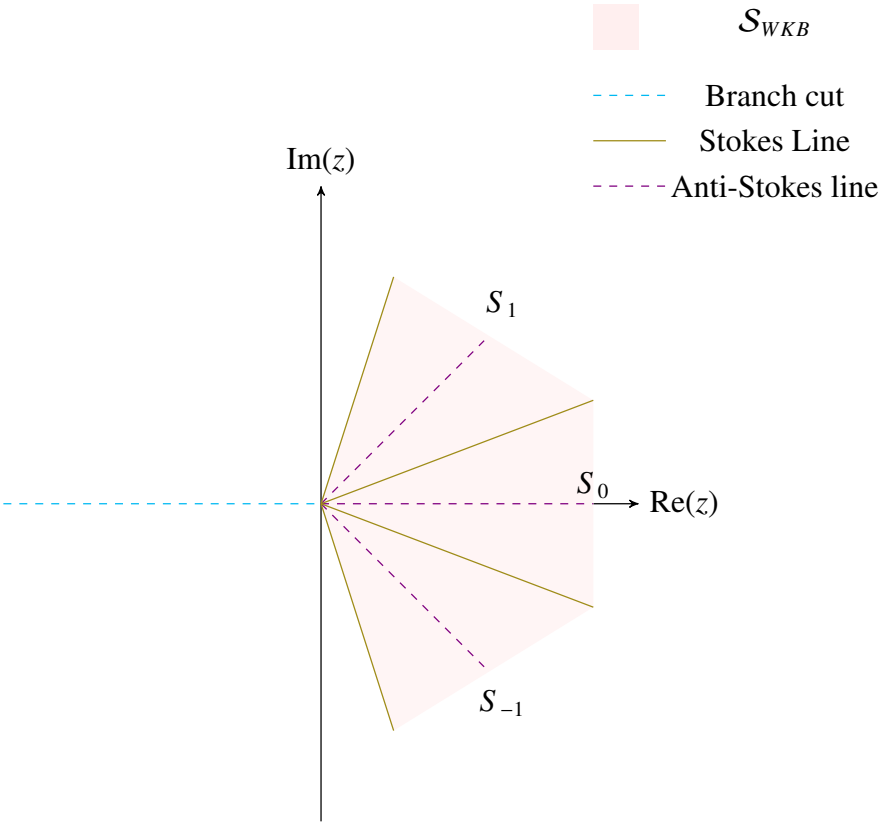


Figure 2.1: Stokes lines, anti-Stokes lines , and the convention of the branch cut when $2M \geq \mathbb{Z}_{\geq 0}$.

multipliers which contains the information about sudden discontinuities of the solution while crossing the stokes line. So far we have only talked about large $x$ asymptotics. For $M > -1$, the differential equation (2.26) has a regular singularity in the region $x \sim 0$. In this region as $x \to 0$, one can write a general solution of (2.26) as a linear combination of $x^{l+1}$ and $x^{-l}$. Defining one of the solutions as

$$\psi(x, E, l) \sim x^{l+1} + O(x^{l+3}) \tag{2.34}$$

one can write the other solution in terms of the previous solution by using the fact that (2.26) is invariant under the analytic continuation $l \to -l-1$ as $\psi(x, E, -l-1)$. These two solutions are linearly independent as the Wronskian doesn't vanish[2].

[2]Subtleties arises at some isolated points where this two solution doesn't remain linearly independent. For simplicity, we will not discuss those in this review but readers may look [41] for further details.

For our ODE in equation (2.26), we want to find a solution which vanishes when $|x| \to \infty$ and behaves as $x^{l+1}$ near origin. In order to find such solution, we first define a solution $y(x, E, l)$ of (2.26) which is an entire function of $E$ and has the following asymptotic behavior

$$y \sim \frac{1}{\sqrt{2i}} x^{-\frac{M}{2}} exp\Big( - \frac{x^{M+1}}{M+1}\Big) \text{ for } |x| \to \infty \text{ with } |Arg(x)| < \frac{2\pi}{2M+2} \tag{2.35}$$

Next, we start the solution $y(x, E, l)$ which is subdominant in the Stokes sector $\mathcal{S}_0$ and generate a set of functions [43] which are solutions of(2.26)

$$y_k(x, E, l) = \omega^{\frac{k}{2}} y(\omega^{-k} x, \omega^{2k}, l), \quad \omega = e^{\frac{2\pi i}{2M+2}}, \quad k \in \mathbb{Z}. \tag{2.36}$$

The asymptotic behavior of $y_{\pm 1}$ are given by

$$y_{\pm 1}(x, E, l) \sim \pm \sqrt{i} \frac{x^{-\frac{M}{2}}}{\sqrt{2}} exp\Big(\frac{x^{M+1}}{M+1}\Big) \tag{2.37}$$

Since the Wronskian $W[y, y_{\pm 1}] = \pm 1$, $\{y, y_{\pm 1}\}$ serve as the basis vectors of the two-dimensional solution space of (2.26). In principle, one can write $y_{-1}$ in terms of $y$ and $y_1$ or alternatively

$$\alpha\, y(x, E, l) = y_{-1}(x, E, l) + \beta y(x, E, l). \tag{2.38}$$

We can fix $\beta = 1$ in order to match the exponentially decreasing behavior of $y$. Here $\alpha$ is the Stokes multiplier $C(E, l)$ which can be written as

$$\alpha = C(E, l) = W[y_{-1}, y_1]. \tag{2.39}$$

In the region $x \to 0$, we define rotated solutions of (2.26) similar to (2.36)

$$\psi_k(x, E, l) = \omega^{\frac{k}{2}} \psi_k(\omega^{-k} x, \omega^{2k} E, l) \text{ for } k \in \mathbb{Z}. \tag{2.40}$$

Using the behavior of $\psi$ near the origin, we can write

$$\psi_k(x, E, l) = \omega^{(l+\frac{1}{2})k}\psi_k(x, E, l). \tag{2.41}$$

Using the above definition, we can compute

$$W[y_k, \psi](E, l) = \omega^{(l+\frac{1}{2})k}W[y, \psi](\omega^{2k}E, l). \tag{2.42}$$

In order to write the projected Stokes relation, we take Wrosnkian on both sides of (2.38) with $\psi$ and get

$$C(E, l)D(E, l) = \omega^{(l+\frac{1}{2})}D(\omega^{-2}E, l) + \omega^{(l+\frac{1}{2})}D(\omega^{2}E, l), \tag{2.43}$$

where

$$D(E, l) = W[y, \psi](E, l). \tag{2.44}$$

Upon identifying $T = C$ and $Q = D$, the Stokes relation becomes

$$T(E, l)Q(E, l) = \omega^{(l+\frac{1}{2})}Q(\omega^{-2}E, l) + \omega^{(l+\frac{1}{2})}Q(\omega^{2}E, l), \tag{2.45}$$

which exactly matches the Baxter $TQ$ relation for quantum KdV theory described by BLZ in [42].

### 2.3.1 Eigenvalues of the KdV charges:

We begin by observing that, through a straightforward change of variables [44] the equation (2.26) can be transformed into the form

$$\Big( -\epsilon^2\partial_w^2 + Z(w)\Big)y(w) = 0, \tag{2.46}$$

where

$$Z(w) = \frac{1}{4\hat{l}^2} w^{\frac{1}{l}-2}(w^{\frac{M}{l}} - 1), \;\; \hat{l} = l + \frac{1}{2}, \;\; \epsilon = E^{\frac{(M+1)}{2M}}. \tag{2.47}$$

Next, we substitute an ansatz of the form

$$y(w) = exp\Big(\frac{1}{\epsilon}\sum_{n=0}^{\infty} \epsilon^n S_n(w)\Big) \tag{2.48}$$

and solve the equation order by order in $\epsilon$. We get a recursion relation of the form

$$S_0^{(1)} = -\sqrt{Z(w)}, \;\; 2S_0^{(1)}S_n^{(1)} + \sum_{j=1}^{n-1} S_j^{(1)} S_{n-j}^{(1)} + S_{n-1}^{(2)}, \tag{2.49}$$

where $S_i^{(j)}$ represents $j$ number of derivatives on $S_i$ with respect to $w$. The WKB approximation fails in the vicinity of turning points, necessitating the derivation of connection formulas to smoothly match the solutions across distinct regions where $Z \neq 0$. For an entire function $Z(w)$ which possesses only a pair of well-separated simple zeros on the real axis, a simple quantization formula can be written using [45]

$$\frac{1}{i}\oint dw\Big(\sum_{n=0}^{\infty} \epsilon^{2n-1} S_{2n}^{(1)}(w)\Big) = (2k+1)\pi. \tag{2.50}$$

Note that we have discarded the terms of the form $S_{2n+1}^{(1)}$, as they are total derivatives. In the present analysis, we focus on the radial connection problem, wherein the integration contour initially lies along the real axis segment between two turning points $w \in (0, 1)$. However, to do this integration, we need a consistent regularisation prescription as $S_{2n}^{(1)}$ diverges at the end points of the line segment. So instead of integrating over the line segment, we integrate over the Pochhammer contour $\gamma_P$. There for one can write the period integrals as

$$\hat{I}_{2n-1}(M, \hat{l}) = \frac{1}{2\Big(1 - e^{i\frac{\pi(1-2n)}{M}}\Big)} \oint_{\gamma_p} dz \bar{S}_{2n}(z), \tag{2.51}$$

where $z = w^{\frac{M}{l}}$ and

$$\bar{S}_{2n}(z) = \frac{2}{i}\frac{\hat{l}}{M} S_{2n}^{(1)}\Big(z^{\frac{\hat{l}}{M}}\Big) z^{\frac{\hat{l}}{M}-1}. \tag{2.52}$$

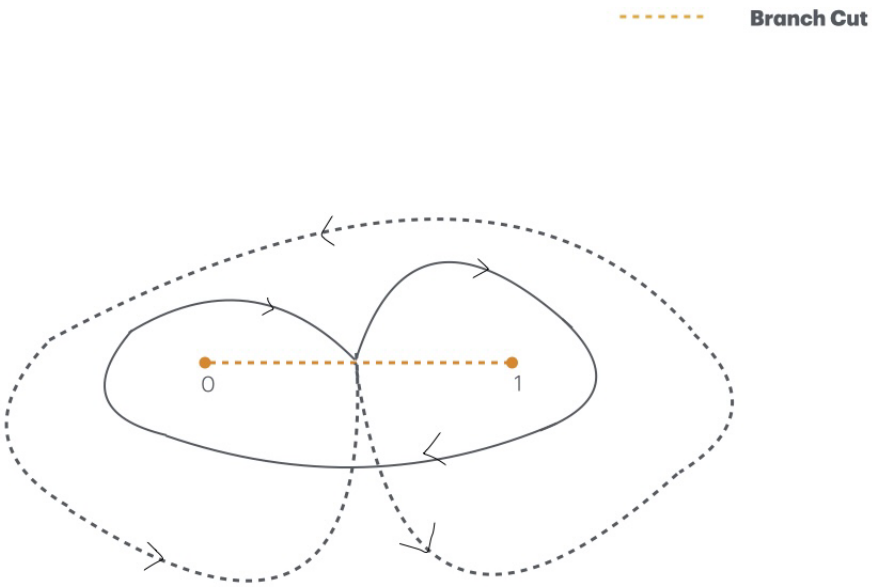


Figure 2.2: Pochhammer contour $\gamma_P$

The extra factor introduced in the integral (2.51) is an appropriate normalization factor. Now the integral is finite for generic values of parameters and can be written as a finite sum of Euler Beta functions. Performing the integration explicitly, we get

$$\hat{I}_{2n-1}(M,\hat{l}) = (-1)^n \frac{\sqrt{\pi}\,\Gamma\left(\frac{1-(2n-1)}{2M}\right)}{\Gamma\left(\frac{3}{2}-n-\frac{(2n-1)}{2M}\right)} \frac{(4M+4)^n}{(2n-1)\,n!} I_{2n-1}(M,\hat{l}), \tag{2.53}$$

where $I_{-1} = 1$. The coefficients $I_{2n-1}(M,\hat{l})$ are the eigenvalues of the KdV charges on the highest weight state if we use the map between the differential equation parameter and the CFT parameters:

$$c = 1 - \frac{6M^2}{(M+1)}, \quad h = \frac{(2l+1)^2 - 4M^2}{16(M+1)} \tag{2.54}$$

where $c$ is the central charge and $h$ is the conformal dimension of the highest weight state. Using this analysis, one can find the eigenvalue of the KdV charges on the highest weight state. Here we list the first few eigenvalues $Q_i$ of the KdV charge $\mathbf{Q}_i$ on the higehst weight

state: [41, 42]

$$\begin{aligned} Q_1 &= \left(\Delta - \frac{c}{24}\right) \\ Q_3 &= \left(\Delta^2 - \frac{c+2}{12}\Delta + \frac{c(5c+22)}{2880}\right) \\ &\vdots \end{aligned} \tag{2.55}$$

## 2.4 Zhu recursion relation: A review

In this section, we develop the technology needed to compute the thermal correlators of the conserved charges in the higher spin conformal field theory. This amounts to first computing the thermal correlators of the Boussinesq currents $J_n$. As seen from (1.7), the currents are linear combinations of normal ordered operators, each of which is made up of the energy-momentum tensor $T(u)$ and the spin 3 field $W(u)$ and their derivatives. At this point, it is also important to mention that we do not have explicit expressions for the conserved currents $J_n$ for $n > 5$. As we shall see, the thermal one-point functions of the composite operators will provide us with a route to derive the conserved current densities.

The problem of computing the thermal correlators of conserved charges in the quantum KdV case (which only involved the energy-momentum tensor and derivatives) was solved in [12], and a key element that greatly aided the analysis was that of modular covariance. It was shown that each such correlator could be written as a modular covariant differential operator acting on the Virasoro character. With the higher spin conformal field theory, however, this luxury is not afforded us. Deriving the modular transformation properties of the generalized character $\mathrm{Tr}\left(q^{L_0-\frac{c}{24}}\, y^{W_0}\right)$ is still an outstanding problem (see [46, 47] for some useful remarks in this direction) and we shall instead work with a reduced (higher spin) Virasoro module, setting $y$ to 1. The modular transformation properties of the character, including insertions of the zero modes of $W_0$ for small values of $n$ have been discussed in detail in [48, 49] and we shall make extensive use of these results in our analysis. In the higher spin CFT, we shall find that the thermal correlators are

quasi-modular differential operators acting on the torus character in the reduced Virasoro module.

Recall that our spatial direction is compact, with period one, $u \sim u + 1$. In addition, we compactify the Euclidean time direction as well, with $\tau \sim \tau + \beta$, so that we have a higher spin conformal field theory at a temperature $T = 1/\beta$. Thus, we study the higher spin conformal field theory on a Riemann surface with the topology of a torus. As explained in [12, 20, 50], the correlators of the conserved charges are best understood as the leading terms in a series expansion for the partition function of a generalized Gibbs ensemble, in which chemical potentials are turned on for each of the conserved Boussinseq charges $\mathbf{I}_k$. Following [48, 49], we shall consider the trace over a restricted module and define the reduced character as

$$\chi_{V,\text{red}} = \text{Tr}_V \left( q^{L_0 - \frac{c}{24}} \right) , \tag{2.56}$$

with $q = e^{-\beta}$, and where $V$ is a highest-weight representation of the higher spin CFT. We shall omit the subscripts in what follows and simply denote the character as $\chi$. The trace is over the restricted Verma module, which is obtained by acting with the creation operators of the $\mathcal{W}_3$ algebra, namely $L_{-n}$ and $W_{-n}$ with $n > 0$ on a specific primary $|\Delta_2, \Delta_3\rangle$. We write a generic state in the restricted Verma module as

$$\mathcal{P} = \prod_{n>0} (L_{-n}^{a_n} W_{-n}^{b_n} |\Delta_2, \Delta_3\rangle = \dots (L_{-n}^{a_n} W_{-n}^{b_n})(L_{-n+1}^{a_{n+1}} W_{-n+1}^{b_{n+1}}) \dots |\Delta_2, \Delta_3\rangle \text{ where } a_n, b_n \geq 0 . \tag{2.57}$$

In this restricted Verma module, for the non-degenerate case, assuming no null vectors for generic values of $c$, $\Delta_2$ and $\Delta_3$, the character $\chi$ is given by:

$$\chi(\tau) = \text{Tr}_V(q^{L_0 - \frac{c}{24}}) = q^{\Delta_2 - \frac{c}{24}} \prod_n \frac{1}{(1-q^n)^2} = \frac{q^{\Delta_2 - \frac{c}{24}}}{(\phi(q))^2} . \tag{2.58}$$

In contrast to the Virasoro characters, where the $q$ expansion coefficients give the partition of $n$, in this case, the $q$ expansion coefficients give the square of the partition of $n$ because we have two sets of generators.

The thermal correlator of any composite operator $\mathcal{O}$ in a highest-weight module $V$ of the higher spin conformal field theory is then defined to be

$$\langle \mathcal{O} \rangle = \mathrm{Tr}_V\left(\mathcal{O} q^{L_0 - \frac{c}{24}}\right). \tag{2.59}$$

We begin with the calculation of the thermal one-point functions of the low-weight currents $J_n(u)$ given in (1.7). Calculating these traces in the restricted Verma module is challenging due to the non-linear nature of the $\mathcal{W}_3$ algebra. So we use a recursive method due to Zhu to compute the thermal correlators involving the $T(u)$ and $W(u)$ fields. We review the basic idea here and refer the reader to Appendix C for details of the calculation using Zhu recursion relation. We will follow the notations and conventions of [25, 47], where the $n$-point functions are denoted

$$F((a^1, z_1), ..., (a^n, z_n); \tau) := z_1^{h_1} \dots z_n^{h_n} \mathrm{Tr}\left(V(a^1, z_1) \dots V(a^n, z_n) q^{L_0 - \frac{c}{24}}\right), \tag{2.60}$$

where $V(a^i, z_i)$ represents the $i$-th vertex operator, with $a^i$ being the field and $z_i := e^{2\pi i u_i}$ can be thought of as its location on the plane. For our purposes, the $a^i$ will be either $T$ or $W$. The central idea of the recursion is to expand the first vertex operator $V(a^1, z_1)$ from the $n$ point function in terms of its modes and push the non-zero modes of $V$ around the trace. As a result, the full $n$ point function can be described as the sum of an $n-1$ point function with one zero mode insertion (coming from the first vertex operator) and terms involving $n-1$-point functions, in which nonzero modes of $V(a^1, z_1)$ act on one of the fields appearing in the $n-1$ point functions.

Thus, the thermal correlator of the individual terms that appear in the Boussinesq currents can eventually be written as a sum of products of thermal correlators of zero modes of the energy-momentum tensor $\langle (L_0 - \frac{c}{24})^n \rangle$ and the spin-3 current $\langle W_0^m \rangle$ on the torus. While the former is easily evaluated, the latter is much more difficult to compute, with explicit results for $m = 1, 2$ given in [48, 49].

Finally, it is important to note that all the composite operators appearing in the currents are conformally normal ordered. For a composite operator made up of two local fields $A_1(u_1)$ and $A_2(u_2)$, the normal ordered product $(A_1A_2)(u_1)$ is defined using the two-point thermal correlator [12]:

$$\langle (A_1A_2)(u_1)\rangle = \frac{1}{2\pi i}\oint_{u_1}\frac{du_2}{u_2-u_1}\langle A_1(u_1)A_2(u_2)\rangle\ . \tag{2.61}$$

For a $n$-point correlator involving $n$ such fields, we perform the normal ordering successively from right to left, starting with the normal ordering of $A_n(u_n)$ and $A_{n-1}(u_{n-1})$, and so on, until we end with a composite operator defined at $u_1$. We shall now illustrate these points by explicitly evaluating the thermal one-point functions of the low-weight Boussinesq currents.

### 2.4.1 Vertex Operators and Square Modes

The vertex operator associated to a CFT state $a$ is a local operator whose action on the CFT vacuum $\Omega$ is given by:

$$V(a,z)\Omega = e^{L_{-1}z}a\ . \tag{2.62}$$

For a system of local vertex operators, the action on the vacuum given above determines the vertex operator completely [51] (see also [52]). A formal series expansion of the operator associated with a state $a$ in terms of the coordinate on the plane $z$ defines the plane modes $a_n$ for it:

$$V(a,z) = \sum_{n\in\mathbb{Z}}\frac{a_n}{z^{n+h_a}}\ , \tag{2.63}$$

where $h_a$ is the conformal dimension of the field $V(a,z)$. Since we compute the correlator of fields on the torus, it would be natural to use vertex operators and mode expansions in coordinates that have the right periodicity properties built in. Therefore, we transform vertex operators to the cylinder through a conformal map $z = e^{2\pi i u}$, $u$ being the coordinate on a cylinder of circumference 1. The conformal map from the plane to the cylinder has

the following effect on a primary operator $V(a,z)$ :

$$V(a,z) \to V(e^{2\pi i u L_0} a, e^{2\pi i u}) \,. \tag{2.64}$$

We now introduce a new type of vertex operator $V[a,u]$ on the cylinder through the following transformation:

$$V[a,u] = V(e^{2\pi i u L_0} a, e^{2\pi i u} - 1) \,. \tag{2.65}$$

This type of vertex operator naturally appears in the computation of the operator product expansions of vertex operators on a cylinder [53, 54]. They play an important role in the Zhu recursion relations, which relate n-point correlators on the torus to lower-point correlators. A formal series expansion of $V[a,u]$ in the cylinder coordinates defines the square modes for us:

$$V[a,u] = \sum_{n\in\mathbb{Z}} \frac{a[n]}{u^{n+h_a}} \,. \tag{2.66}$$

Explicitly, the square bracket modes can be written in terms of plane modes as:

$$a[n] = \frac{1}{(2\pi i)^{n+1}} \sum_{j\geq n+1-h_a} c(h_a, j+h_a-1, n) a_j \,, \tag{2.67}$$

where the expansion coefficients are given by

$$(\log(1+z))^s (1+z)^{h-1} = \sum_{j\geq s} c(h,j,s) z^j \,. \tag{2.68}$$

This formula will be used for instance to compute the square modes of $W(z)$, whose vertex operator representation is given by $V(W_{-3}\Omega, z)$. Here we tabulate just the first few of these square modes :

$$\begin{aligned}
W[0] &= (2\pi i)^{-1}\Big(W_{-2} + 2W_{-1} + W_0\Big) \,, \\
W[1] &= (2\pi i)^{-2}\Big(W_{-1} + \frac{3}{2}W_0 + \frac{1}{3}W_1 - \frac{1}{12}W_2 + \frac{1}{30}W_3 - \frac{1}{60}W_4 + \dots\Big) \,, \\
W[2] &= (2\pi i)^{-3}\Big(W_0 + W_1 - \frac{1}{12}W_2 + \frac{1}{90}W_4 + \dots\Big) \,,
\end{aligned}$$

$$W[3] = (2\pi i)^{-4}\Big(W_1 + \frac{1}{2}W_2 - \frac{1}{4}W_3 + \frac{1}{8}W_4 + \dots\Big) \ . \tag{2.69}$$
$$W[4] = (2\pi i)^{-5}\Big(W_2 - \frac{1}{6}W_4 + \dots\Big)$$
$$W[5] = (2\pi i)^{-6}\Big(W_3 - \frac{1}{2}W_4 + \dots\Big)$$
$$W[6] = (2\pi i)^{-7}\Big(W_4 + \dots\Big)$$

In contrast to primary fields, general vertex operators transform non-trivially under conformal transformations [55]. For instance, under the aforementioned map from the plane to the cylinder, the energy-momentum tensor transforms as:

$$V(L_{-2}\Omega, z) \to (2\pi i)^2 z^2 V(L_{-2}\Omega, z) - (2\pi i)^2 V\left(\frac{c}{24}\Omega, z\right) \ . \tag{2.70}$$

This motivates the introduction of the state $\tilde{\omega}$ and its modes defined through

$$\tilde{\omega} := \left(L_{-2} - \frac{c}{24}\right)\Omega \ , \quad \text{with} \quad V(\tilde{\omega}, z) = \sum_n \tilde{\omega}_n z^{-n-1} = \sum_n L_n z^{-n-2} - \frac{c}{24} \ . \tag{2.71}$$

The conformal dimension of the quasi-primary $\tilde{\omega}$ is 2, and the square modes of the stress tensor on the torus are defined through the formal series expansion in $u$ of the following vertex operator:

$$V(e^{2\pi i u L_0}\tilde{\omega}, e^{2\pi i u} - 1) \ . \tag{2.72}$$

Using the power series expansion discussed previously, the square modes take the form:

$$L_{[-n]} := (2\pi i)^2 \tilde{\omega}[n+1] = (2\pi i)^{-n} \sum_{j \geq n+1} c(2, j, n+1) L_{j-1} - (2\pi i)^2 \frac{c}{24}\delta_{n,-2} \ . \tag{2.73}$$

Here we have defined the modes $L_{[-n]}$ that satisfy the Virasoro algebra. Evaluating these square modes for the first few values of $n$, we get:

$$L_{[-1]} = (2\pi i)^2 \tilde{\omega}[0] = (2\pi i)(L_{-1} + L_0) \ ,$$

$$
\begin{aligned}
L_{[0]} &= (2\pi i)^2 \tilde{\omega}[1] = (L_0 + \frac{1}{2}L_1 - \frac{1}{6}L_2 + \frac{1}{12}L_3 - \frac{1}{20}L_4 + \ldots) \,, \\
L_{[1]} &= (2\pi i)^2 \tilde{\omega}[2] = \frac{1}{2\pi i}(L_1 - \frac{1}{12}L_3 + \frac{1}{12}L_4 + \ldots) \,, \\
L_{[2]} &= (2\pi i)^2 \tilde{\omega}[3] = \frac{1}{(2\pi i)^2}(L_2 - \frac{1}{2}L_3 + \frac{1}{4}L_4 + \ldots) \,, \\
L_{[3]} &= (2\pi i)^2 \tilde{\omega}[4] = \frac{1}{(2\pi i)^3}(L_3 - L_4 + \ldots), \\
L_{[4]} &= (2\pi i)^2 \tilde{\omega}[5] = \frac{1}{(2\pi i)^4}(L_4 - \ldots).
\end{aligned}
\tag{2.74}
$$

### 2.4.2 Zhu Recursion Relation

We will follow the notations and conventions of [26,47]. The $n$-point correlation functions are denoted

$$
F((a^1, z_1), ..., (a^n, z_n); \tau) := \mathrm{Tr}\,[V(e^{2\pi i u_1 L_0}\tilde{a}_1, e^{2\pi i u_1})...V(e^{2\pi i u_n L_0}\tilde{a}_n, e^{2\pi i u_n})q^{L_0 - \frac{c}{24}}] \,. \tag{2.75}
$$

Here, the $z_i := e^{2\pi i u_i}$ are coordinates on the plane. For later purposes, it is more convenient to write the recursion for the $n$-point correlation function, with $l$ insertions of the zero mode of another field $b$. As shown in [47], this satisfies the (modified) Zhu recursion relation:

$$
\begin{aligned}
&F(b_0^l; (a^1, z_1), ..., (a^n, z_n); \tau) = F(b_0^l a_0^1; (a^2, z_2), ..., (a^n, z_n); \tau) + \\
&\sum_{i=0}^{l}\sum_{j=2}^{n}\sum_{m=0}^{\infty}\binom{l}{i} g^i_{m+1}(z_{j1}, q) F(b_0^{l-i}; (a^2, z_2), ..., (d^i[m]a^j, z_j), ..., (a^n, z_n); \tau) \,,
\end{aligned}
\tag{2.76}
$$

where $z_{ij} = \frac{z_i}{z_j}$ and the functions $g^i_k$ are defined in terms of the Weierstrass functions[3] $\mathcal{P}_k(x, q)$ as

$$
g^i_k(x, q) = \frac{(2\pi i)^k}{(k-1)!}\sum_{n\neq 0} n^{k-i-1}x^n\partial^i\frac{1}{(1-q^n)} = (2\pi i)^{2i}\frac{(k-i-1)!}{(k-1)!}\partial_i\mathcal{P}_{k-i}(x, q) \,. \tag{2.77}
$$

[3] More details on this can be found in Appendix A of [12].

In the recursion, the mode denoted $d^i[m]$ is defined to be

$$d^i[m] = (-1)^i((b[0])^i a^1)[m] \ . \tag{2.78}$$

To find the square modes of composite objects of the form $(b[n])a$ , we use the identity (see equation (4.2.4) of [26])

$$(b[n]a)[m] = \sum_i \binom{n}{i} \Big( (-1)^i b[n-i]a[m+i] - (-1)^{n+i} a[n+m-i]b[i] \Big) \ . \tag{2.79}$$

**Thermal Expectation Values of Zero Modes**

As we show in many examples, the Zhu recursion eventually reduces a higher point correlation function to the thermal expectation values of zero modes of the various operators. So it is useful to tabulate these for the higher spin conformal field theory. The character in the higher spin module is defined to be

$$\chi(\tau) = \mathrm{Tr}\left(q^{L_0 - \frac{c}{24}}\right) = \frac{q^{\Delta_2 - \frac{c}{24}}}{\prod_{n=1}^{\infty}(1-q^n)^2} \ . \tag{2.80}$$

The zero mode of the energy-momentum tensor on the torus is given by

$$\tilde{\omega}_{(0)} := \int du \ V(e^{2\pi i u L_0}\tilde{\omega}, e^{2\pi i u}) = L_0 - \frac{c}{24} \ . \tag{2.81}$$

Thus, the insertion of the zero mode is equivalent to a derivation:

$$\mathrm{Tr}\left(\tilde{\omega}^n_{(0)} \, q^{L_0 - \frac{c}{24}}\right) = (q\frac{d}{dq})^n \chi(\tau) = \partial^n \chi(\tau) \ , \tag{2.82}$$

where we have defined $\partial = q\frac{d}{dq}$.

The insertion of the $W_0$ operator is more complicated and these have been evaluated for $n = 1, 2$ exactly, while for $n > 2$, the first few terms in the power series expansion have

been calculated in [48,49]. We list the results that are of relevance to our calculations:

$$\text{Tr}\, W_0 q^{L_0-\frac{c}{24}} = \Delta_3 \chi(\tau) = q^{\Delta_2-\frac{c}{24}}\Big(\Delta_3 + 2\Delta_3 q + 5\Delta_3 q^2 + O(q^3)\Big)\,, \tag{2.83}$$

$$\begin{aligned}\text{Tr}\, W_0^2 q^{L_0-\frac{c}{24}} &= \Big(\Delta_3^2 - \frac{1}{9}(\Delta_2 - \frac{c-2}{24})E_2'(\tau) + \frac{1}{27}E_2''(\tau) + \frac{1}{6}\frac{c+30}{1440}E_4'(\tau)\Big)\chi(\tau) \\ &= q^{\Delta_2-\frac{c}{24}}\Big(\Delta_3^2 + (2\Delta_3^2 + \frac{1}{16}(32\Delta_2 - c + 2))q + O(q^2)\Big)\,,\end{aligned} \tag{2.84}$$

$$\text{Tr}\, W_0^3 q^{L_0-\frac{c}{24}} = q^{\Delta_2-\frac{c}{24}}\Big(\Delta_3^3 + (2\Delta_3^3 + \frac{\Delta_3}{4}(32\Delta_2 - c + 2))q + O(q^2)\Big)\,. \tag{2.85}$$

This thesis has enabled the explicit construction of the quantum currents for the Boussinesq hierarchy up to the eleventh current, $J_{11}$. Extending this construction to higher-order currents necessitates the evaluation of the traces $\text{Tr}\left(W_0^n q^{L_0-c/24}\right)$ for $n \geq 4$. In a later paper [56], we have computed those expressions.

## 2.5 The Affine Toda Theory

In the previous section we showed that the classical conserved charges are encoded in the spatial integrals of the WKB solution of the scalar Lax equation. Following the recent work [57], we shall show how the form of the scalar Lax operator $L$ in (2.5) arises in the context of the modified affine Toda equation. In that reference, the WKB analysis for the linear problem associated to the modified affine Toda system was used to obtain the classical charges associated to the integrable system. We shall review this and then show that a suitable limit of the Lax operator reduces it to the higher order differential operator that has previously appeared in the context of the ODE/IM correspondence [58,59]. Since our goal in the next section will be the WKB analysis of this higher-order ODE in order to derive the quantum conserved charges, we believe this to be a useful detour to establish a path between the classical analysis of this section and the subsequent quantum analysis.

We begin with the modified affine Toda equation associated to an affine Lie algebra of

rank $r$:

$$\partial_{\bar{z}}\partial_z\phi(z,\bar{z}) - \sum_{i>0}^{r} \alpha_i \exp(\alpha_i \cdot \phi) - p(z)\bar{p}(\bar{z})\alpha_0 \exp(\alpha_0 \cdot \phi) = 0\,. \tag{2.86}$$

Here, $\phi$ is a Lie algebra valued field, and $\alpha_i$ are the simple roots of the affine Lie algebra $\mathfrak{g}$. This equation can also be put in the form of a compatibility condition $[\bar{\mathcal{L}}, \mathcal{L}] = 0$ of two operators $\mathcal{L}$ and $\bar{\mathcal{L}}$:

$$\mathcal{L} = \partial_z + \sum_{i=1}^{r} \partial_z\phi_i H_i + \lambda\left(\sum_{i=1}^{r} E_{\alpha_i} + p(z)E_{\alpha_0}\right)\,, \tag{2.87}$$

$$\bar{\mathcal{L}} = \partial_{\bar{z}} + \lambda^{-1} e^{-\sum_{i=1}^{r}\phi_i H_i}\left(\sum_{i=1}^{r} E_{-\alpha_i} + \bar{p}(\bar{z})E_{\alpha_0}\right) e^{\sum_{i=1}^{r}\phi_i H_i}\,, \tag{2.88}$$

where $H_i$, $E_{\alpha_i}$ and $E_{-\alpha_i}$ are the generators of $\mathfrak{g}$ in the Chevalley basis, and $\phi_i$ are defined through (with $\alpha_i^\vee$ being the co-roots):

$$\phi = \sum_i \alpha_i^\vee\ \phi_i(z,\bar{z})\,. \tag{2.89}$$

We now restrict ourselves to the affine algebra $A_2^{(1)}$ and write out the equation $\mathcal{L}\psi = 0$ explicitly, choosing a particular representation for the generators (we refer the reader to [57] for details):

$$\left(\partial_z + \begin{pmatrix} \partial_z\phi_1 & \lambda & 0 \\ 0 & \partial_z\phi_2 - \partial_z\phi_1 & \lambda \\ \lambda p(z) & 0 & -\partial_z\phi_2 \end{pmatrix}\right)\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = 0\,. \tag{2.90}$$

Eliminating $\psi_2$ and $\psi_3$ using the above equation ($\lambda \neq 0$), we obtain a higher order ordinary differential equation for $\psi_1$:

$$(-\lambda)^{-3}(\partial_z - \partial_z\phi_2)(\partial_z + \partial_z\phi_2 - \partial_z\phi_1)(\partial_z + \partial_z\phi_1)\psi_1 = p(z)\psi_1\,. \tag{2.91}$$

The left hand side of (2.91) is in the form of a generalized Miura transform [31], where an operator of the form $\partial_z^3 + u_1\partial_z + u_0$ is expressed in factorized form. We can read off the

$u_i(z)$ to be:

$$u_1(z) = \partial_z^2\phi_1 - (\partial_z\phi_1)^2 + \frac{1}{2}\partial_z\phi_1\partial_z\phi_2 + (\phi_1 \leftrightarrow \phi_2)\,, \tag{2.92}$$

$$u_0(z) = \partial_z^3\phi_1 + \partial_z\phi_1\left(\partial_z\phi_1\partial_z\phi_2 - (\partial_z\phi_2)^2 - 2\partial_z^2\phi_1 + \partial_z^2\phi_2\right)\,. \tag{2.93}$$

Since the scalar Lax operator of the Boussinesq equation is an operator that fits this form, we may equate the coefficient functions with that of (2.5) to find the map:

$$U(z) = -\left(\partial_z^2\phi_1 - (\partial_z\phi_1)^2 + \frac{1}{2}\partial_z\phi_1\partial_z\phi_2 + (\phi_1 \leftrightarrow \phi_2)\right)\,, \tag{2.94}$$

$$V(z) = \frac{1}{2}\left(\partial_z^3\phi_1 + 2\partial_z\phi_1\left(\partial_z\phi_1\partial_z\phi_2 - \partial_z^2\phi_1\right) - \partial_z\phi_2\partial_z^2\phi_1\right) - (\phi_2 \leftrightarrow \phi_1)\,. \tag{2.95}$$

**The null conformal limit**

In order to study the integrable structure of the higher spin conformal field theory, we perform a (conformal) double scaling limit, as in [60], on the scalar Lax operator associated to the affine Toda system. This essentially corresponds to taking the massless limit of the higher rank integrable system [61,62]. We begin by fixing the asymptotic behavior of the $\phi_i(z)$ as $z, \bar{z} \to 0$. This is obtained by setting

$$p(z) = s^{3M} - z^{3M}\,, \quad \text{and} \quad \phi_i(z,\bar{z}) = \ell_i log\,(z\bar{z}) + O(1)\,. \tag{2.96}$$

In addition, we take a conformal double-scaling limit [60]

$$\bar{z} \to 0, \quad z \sim s \to 0, \quad \text{and} \quad \lambda \to \infty, \tag{2.97}$$

keeping fixed the quantities

$$x = \lambda^{\frac{1}{1+M}} z\,,\; E = s^{3M}\lambda^{\frac{3M}{1+M}}\,. \tag{2.98}$$

As a result of taking these limits, the scalar Lax operator reduces to the familiar differential equation associated to the higher spin conformal field theory[4] [15, 57, 63]:

$$\prod_{j=1}^{3}\left(\frac{d}{dx}+\frac{\ell_j}{x}-\frac{\ell_{j-1}}{x}\right)\psi=(x^{3M}-E)\,\psi\,. \tag{2.99}$$

Let us summarize what we have reviewed in this section. We started with the linear problem associated to the integrable model that is the modified affine Toda system. What we have shown is that the scalar Lax operator of this integrable system, in the null conformal limit, reduces to the differential operator familiar from the ODE/IM correspondence for the higher spin conformal field theory. We now turn to the calculation of the spectrum of the quantum conserved charges in a higher spin module of the conformal field theory via the ODE/IM correspondence.

[4]Here we set $\ell_0=\ell_3=0$, but in this way of writing the operator, the generalization to the higher rank affine Lie algebras is evident [58, 59].

# Chapter 3

# Eigenvalues from the ODE/IM Correspondence

In this chapter, we will use ODE/IM correspondence to compute the eigenvalues of the quantum Boussinesq charges on the highest weight, first and second excited states. The reader can find a review of ODE/IM correspondence in 2.3.

We have seen in the classical analysis that the WKB solution of the linear problem associated with the Lax operator encodes the classical conserved charges of the Boussinesq equation. In the quantum case, the action of the charges on a state can be read off from the asymptotic expansion of the logarithm of Baxter's $T$ functions. Recall that in the classical case [15]:

$$\log\{\mathbf{T}(\lambda)\} = c_0\,\lambda - \sum_{n>0} c_n\, I_n^{\text{class}}\,\lambda^{-n}\,. \tag{3.1}$$

In the context of conformal field theories, these are generalizations of the $T$ functions, where functions appearing in the Lax operator are promoted to fields and products of fields are regularised through the conformal normal ordering [8]. As demonstrated in Section 2.3, the ODE/IM correspondence enables the systematic extraction of the eigenvalues associated with the integrals of motion.

In this chapter, we now extend the analysis to the third order ODE

$$\left(\frac{d}{dx}-\frac{\ell_1}{x}\right)\left(\frac{d}{dx}+\frac{\ell_1-\ell_2}{x}\right)\left(\frac{d}{dx}+\frac{\ell_2}{x}\right)\psi(x)=-(x^{3M}-E)\psi(x)\,, \tag{3.2}$$

and perform ODE/IM correspondence to find the eigenvalue of the conserved charges.

It is important to note that the Stokes data of the associated ordinary differential equation yields the $TQ$-relation for the quantum Boussinesq hierarchy. This connection, established in [15], is the direct analogue of the relations for the KdV hierarchy given in equations (2.43) and (2.45).

## 3.1 Computation of the eigenvalue on the highest weight state

### 3.1.1 The Langer modification and the Modified ODE

The first task is to transform this equation into a form in which we can use the WKB ansatz, with the role of $\hbar$ being played by $\epsilon = f(E)$, a specific function of the energy $E$. We follow the general logic of [44] (see section 6 of that reference, which in turn follows [64]).

1. We begin by expanding out the differential equation:

$$\frac{d^3\psi}{dx^3}-\frac{1}{x^2}(\ell_2^2+\ell_1^2+\ell_1+\ell_2-\ell_1\ell_2)\frac{d\psi}{dx}+\frac{1}{x^3}\ell_2(\ell_1+2)(1+\ell_2-\ell_1)\psi=-(x^{3M}-E)\psi\,. \tag{3.3}$$

2. We change coordinates to $x=e^z$, in combination with the redefinition

$$y=e^z\psi\,, \tag{3.4}$$

leading to the equation

$$\frac{d^3y}{dz^3}-(1+\ell_2^2+\ell_1^2+\ell_1+\ell_2-\ell_1\ell_2)\frac{dy}{dz}+(1+\ell_1)(1+\ell_2)(\ell_2-\ell_1)y=-e^{3z}(e^{3Mz}-E)y\,. \quad (3.5)$$

3. We now scale $z \to \gamma z$ and see how the differential operator acting on $y$ is mapped by this scaling.

4. We then do the inverse of the map in point 1. above, and obtain the equation

$$\frac{\left(\gamma^3\left(x^{3\gamma}\left(x^{3\gamma M}-E\right)\right)+(\gamma+\gamma\ell_2-1)(\gamma\ell_2-\gamma\ell_1+1)(\gamma+\gamma\ell_1+1)\right)}{x^3}y(x) - \frac{\left(\gamma^2\left(\ell_1^2-(\ell_2-1)\ell_1+\ell_2^2+\ell_2+1\right)-1\right)}{x^2}y'(x)+y^{(3)}(x)=0\,. \quad (3.6)$$

We choose $\gamma$ such that the coefficient of the $\frac{1}{x^3}$ term is set to zero. This is a cubic equation in $\gamma$, that has three solutions:

$$\gamma^{(1)}=-\frac{1}{\ell_1+1}\,, \quad \text{or} \quad \gamma^{(2)}=\frac{1}{\ell_2+1}\,, \quad \text{or} \quad \gamma^{(3)}=\frac{1}{\ell_1-\ell_2}\,. \quad (3.7)$$

We note that there is a complete symmetry between the 3 roots of the su(3) Lie algebra. To make this manifest, we introduce the parameters:

$$r_1=\ell_1+1\,, \qquad r_2=-\ell_2-1\,, \qquad r_3=\ell_2-\ell_1\,. \quad (3.8)$$

The $r_i$ satisfy the relation $\sum r_i = 0$, and the three solutions for the $\gamma$ that simplify the ODE is given by $\gamma^{(i)}=-\frac{1}{r_i}$.

5. Without loss of generality, we choose the last of these solutions and obtain

$$y^{(3)}(x)+\frac{r_1r_2}{r_3^2}\frac{y'(x)}{x^2}-\frac{1}{r_3^3}\,x^{-\frac{3(r_3+M+1)}{r_3}}\left(1-Ex^{\frac{3M}{r_3}}\right)y(x)=0\,. \quad (3.9)$$

6. Lastly we redefine the coordinate variable and the parameter

$$x = E^{-\frac{r_3}{3M}} t \,, \qquad E = \epsilon^{-\frac{3M}{M+1}} \,, \tag{3.10}$$

and obtain the final form of the ordinary differential equation whose WKB solution will encode the eigenvalues of the quantum Boussinesq charges in a highest weight module:

$$\epsilon^3 \left( y^{(3)}(t) + \frac{r_1 r_2}{r_3^2} \frac{y'(t)}{t^2} \right) - \frac{1}{r_3^3} t^{-\frac{3}{r_3}(r_3+M+1)} (1 - t^{\frac{3M}{r_3}})\, y(t) = 0 \,. \tag{3.11}$$

Lastly, we note in passing that we would have obtained the same third order ODE in the $t$-variables if we had begun with the original ODE in (2.99), but with a different map between the $r_i$ and $\ell_i$ variables. It follows that one can perform the WKB analysis on either the ODE in (2.99) or its adjoint ODE in order to derive the eigenvalues of the local conserved charges in the highest weight state.

### 3.1.2 WKB Analysis

We now compute the WKB solutions to the following class of third-order equations (see [65] for the WKB analysis of a similar third-order ODE but without the single derivative term):

$$\epsilon^3 \left( \partial_t^3 y + \frac{r_1 r_2}{r_3^2 t^2} \partial_t y \right) + p(t) y = 0 \,. \tag{3.12}$$

For the case at hand, we have

$$p(t) = -\frac{1}{r_3^3} t^{-\frac{3}{r_3}(r_3+M+1)} (1 - t^{\frac{3M}{r_3}}) \,. \tag{3.13}$$

To find the WKB solution, we plug in the usual exponential ansatz

$$y(t) = \exp\left[ \int^t dt \, \sum_{i \geq 0} \epsilon^{i-1} a_i(t) \right] \tag{3.14}$$

into the differential equation. Collecting the terms at each order in $\epsilon$, we find, at the first two orders, the equations:

$$a_0^3 + p(t) = 0 \,, \qquad a_1 + \frac{a_0'}{a_0} = 0 \,. \tag{3.15}$$

From an analysis of the remaining set of equations we find that the $a_n$, for $n > 1$, satisfy the recursion relation:

$$a_{n-2}'' + 3\sum_{i=0}^{n-1} a_i a_{n-i-1}' + \sum_{i_1+i_2+i_3=n} a_{i_1} a_{i_2} a_{i_3} + \frac{r_1 r_2}{r_3^2 t^2} a_{n-2} = 0 \,, \tag{3.16}$$

with initial conditions

$$a_0 = -p(t)^{1/3} \,, \qquad a_1 = -\frac{a_0'}{a_0} \,. \tag{3.17}$$

### 3.1.3 Vacuum Eigenvalues from WKB coefficients

#### Period Integrals

Once the $a_n$ are computed explicitly in terms of $p(t)$, the conserved charges of the Boussinesq hierarchy can be calculated from the period integrals of the $a_n$. The period integrals in the $t$-plane are equivalent to twice the line integral from $[0, 1]$ in the $t$-plane. Much of this analysis is similar to the analogous one carried out for the quantum KdV case [38]. We first define

$$\widehat{I_n} = \int_0^1 dt\, a_{n+1}(t) \,. \tag{3.18}$$

We first make a change of variable $t = z^{\frac{r_3}{3M}}$ , and rewrite the integral as

$$\widehat{I_n} = \int_0^1 dz\, S_{n+1}(z) = \frac{r_3}{3M} \int_0^1 dz\, z^{\frac{r_3}{3M}-1}\, a_{n+1}(z^{\frac{r_3}{3M}}) \,. \tag{3.19}$$

This defines the function $S_n(z)$. We can now write a recursion relation for $S_n(z)$, with $n > 1$ that follows directly from the recursion satisfied by the $a_n(t)$. The boundary condition for the recursion is given by

$$S_0(z) = \frac{1}{3M}\, z^{-1-\frac{M+1}{3M}}\, (1-z)^{\frac{1}{3}}\,, \tag{3.20}$$

with $S_1(z)$ given by a total derivative[1].

We now convert the definite integral in (3.19) to an integral over the Pochhammer contour $\Gamma_P$, resulting in

$$\widehat{I}_n = \frac{1}{(1-m_n^{(0)})(1-m_n^{(1)})} \int_{\Gamma_P} dz\, S_{n+1}(z)\,, \tag{3.21}$$

where $m_n^{(0)}$ and $m_n^{(1)}$ are the monodromy of $S_{n+1}(z)$ around $z = 0$ and $z = 1$. These monodromies in turn can be calculated directly from the recursion relation satisfied by the $S_n(z)$. A key input for this calculation is the monodromy of $S_0(z)$ about $z = 0$ and $z = 1$, which can be read off from (3.20). In addition we also impose trivial monodromy for $S_1(z)$ around $z = 0$ and $z = 1$, on account of it being proportional to the total derivative of $S_0(z)$. We omit the details of this calculation and present the result (for $n \geq 1$, and for $n \neq 0 \bmod 3$):

$$\widehat{I}_n = \frac{1}{(1-e^{-\frac{2\pi i n}{3}})(1-e^{2\pi i \frac{n(1+M)}{3M}})} \int_{\Gamma_P} dz\, S_{n+1}(z)\,. \tag{3.22}$$

The change of variables and the recursion relation ensures that the integrand for every $n$ has a linear combination of terms involving only powers of $z$ and $(1-z)$. Thus the integral over the Pochhammer contour turns out to be linear combinations of the Euler beta-function, on account of the integral:

$$\int_{\Gamma_P} dz\, z^{a-1}(1-z)^{b-1} = (1-e^{2\pi i a})(1-e^{2\pi i b})\, \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,. \tag{3.23}$$

[1]Note that for $\widehat{I}_{-1}$, we obtain a constant on performing the integral of $S_0(z)$, while we have $\widehat{I}_0 = 0$, since the integrand $S_{-1}(z)$ is a total derivative. So the integrals only lead to non-trivial conserved charges for $n \geq 1$.

**Eigenvalues of Conserved Charges**

The main point of this analysis is that the eigenvalues of the conserved charges in the highest weight states, which we denote by $I_n$, are directly proportional to the contour integrals $\widehat{I_n}$ in (3.22). Our normalization (see Appendix B.1 for more details regarding normalization) is such that the quantum charges have the appropriate classical limit (when $c \to \infty$).

From the form of the classical conserved currents, it is not difficult to infer that the first two eigenvalues must correspond respectively to the eigenvalues of the Cartan generators $L_0 - \frac{c}{24}$ and $W_0$ in the highest weight modules labeled by $(\Delta_2, \Delta_3)$. These in turn should be proportional to the integrals $\widehat{I_1}$ and $\widehat{I_2}$ respectively. Given that the central charge should be purely $M$-dependent, this motivates the following map between the parameters appearing in the ODE and those in the higher spin conformal field theory[2]:

$$c = 2 - 24\,\frac{M^2}{M+1}\,, \quad \Delta_2 = \frac{\left(r_1^2 + r_1 r_2 + r_2^2\right)}{9(M+1)} - \frac{M^2}{M+1}\,, \quad \Delta_3 = -\frac{r_1 r_2 (r_1 + r_2)}{27(1+M)^{\frac{3}{2}}}\,. \tag{3.26}$$

Using the inverse map, one can write out the eigenvalues of the higher conserved charges in terms of the conformal field theory data. We find that the charges $I_{3n} = 0$, which agrees with the classical analysis. The first few non-vanishing eigenvalues are listed below[3]:

$$I_1 = \Delta_2 - \frac{c}{24}\,, \tag{3.27a}$$

[2] A comparison with the results in [15] confirms this proposal. In particular, one can find the map to the parameters appearing in that reference:

$$g = \frac{1}{1+M}\,, \quad \text{in terms of which} \quad c = 50 - 24(g + g^{-1})\,, \tag{3.24}$$

$$r_1 = \frac{\sqrt{3}}{g}(\sqrt{3}p_1 + p_2)\,, \quad r_2 = \frac{\sqrt{3}}{g}(-\sqrt{3}p_1 + p_2)\,, \tag{3.25}$$

where $p_1$ and $p_2$ are the momenta associated with the two bosons that correspond to a free field realization of the higher spin conformal field theory in terms of two bosons with background charges.

[3] These eigenvalues have also been obtained by computing the higher integrals of motion in the affine Gaudin models [66, 67].

$$I_2 = \Delta_3 \,, \tag{3.27b}$$

$$I_4 = \Delta_3 \left( \Delta_2 - \frac{c+6}{24} \right) , \tag{3.27c}$$

$$I_5 = \Delta_2^3 + 9\Delta_3^2 - \frac{1}{8}(c+8)\Delta_2^2 + \frac{1}{192}(c+2)(c+15)\Delta_2 - \frac{c(c+23)(7c+30)}{96768} \,. \tag{3.27d}$$

The full list of eigenvalues in the highest weight state of the higher spin module obtained using the ODE/IM correspondence is given in Appendix B.1.

## 3.2 Excited State Eigenvalues from ODE/IM

In this section we first review the results of [68] in which they have proposed the ODE corresponding to higher excited states of the quantum Boussinesq model. The third order ODE that is considered in [68] is given by

$$\phi^{(3)}(z) - W_1(z)\phi'(z) + W_2(z)\phi(z) = 0 \,, \tag{3.28}$$

where the coefficient functions are given by

$$W_1 = \frac{\bar{r}_1}{z^2} + \sum_{j=1}^{N} \left( \frac{a_{11}^{(j)}}{z\left(z-w_j\right)} + \frac{3}{\left(z-w_j\right)^2} \right) , \tag{3.29}$$

$$W_2 = \frac{\bar{r}_2}{z^3} + \frac{1}{z^2} + \lambda z^k + \sum_{j=1}^{N} \left( \frac{a_{22}^{(j)}}{z^2\left(z-w_j\right)} + \frac{a_{21}^{(j)}}{z\left(z-w_j\right)^2} + \frac{3}{\left(z-w_j\right)^3} \right) . \tag{3.30}$$

Here $N$ is the level of excited state; for $N = 0$, the ODE should therefore be mapped to the one we studied in Section 3.1. We will show this explicitly in the following subsection. For now, we simply state that the parameters $(\bar{r}_1, \bar{r}_2)$ will be mapped to $(\ell_1, \ell_2)$ variables, while $k$ and $\lambda$ will be expressed as functions of $M$ and $E$ (see equation (3.40) for the map of parameters).

Starting from the ODE (3.28), we can see There are $4N$ further parameters $(a_{11}^{(j)}, a_{21}^{(j)}, a_{22}^{(j)}, w_j)$, with $j = 1, \ldots N$, that appear in the ODE. These are determined by imposing that there

is trivial monodromy around each $w_j$ [68–70]. This leads to algebraic constraints between the $4N$ parameters. $2N$ of these are immediately determined in terms of the rest as follows:

$$a_{11}^{(j)} = k\,, \qquad a_{22}^{(j)} = -\frac{k^2}{3} + a_{21}^{(j)} + \frac{2}{3}k a_{21}^{(j)}\,. \tag{3.31}$$

Renaming $a_{21}^{(j)} = a_j$, the remaining $2L$ parameters $(a_j, w_j)$ satisfy the non-trivial constraints [68]:

$$\begin{aligned} a_\ell^2 - k a_\ell + k^2 + 3k - 3\bar{r}^1 &= \sum_{\substack{j=1\\ j\neq\ell}}^{L}\left(\frac{9w_\ell^2}{(w_\ell - w_j)^2} + \frac{3kw_\ell}{w_\ell - w_j}\right), \\ \mathcal{A}a_\ell + \mathcal{B} - 9(k+2)w_\ell &= \sum_{\substack{j=1\\ j\neq\ell}}^{L}\left(\frac{18(-k + a_\ell + a_j)w_\ell^3}{(w_\ell - w_j)^3} + \frac{(3w_\ell^2\left(k\,(3a_\ell - 4k - 3) + a_j(5k+6)\right)}{(w_\ell - w_j)^2}\right. \\ &\qquad \left. - \frac{(2k+3)w_\ell\left(k\,(-a_\ell + 2k + 3) - 3a_j(k+2)\right)}{w_\ell - w_j}\right). \end{aligned} \tag{3.32}$$

Here the $\mathcal{A}$ and $\mathcal{B}$ are given in terms of the CFT data by

$$\mathcal{A} = (18 + 2k^3 + 14k^2 - 2k\bar{r}_1 + 30k - 6\bar{r}_1)\,, \tag{3.33}$$

$$\mathcal{B} = (k+3)\left((k+9)\bar{r}_1 - k(k+1)(k+3) - 9\bar{r}_2\right)\,. \tag{3.34}$$

The non-trivial result of [68, 70] is that there are exactly as many solutions to these algebraic constraints as the number of states at the $N$-th excited level in the higher spin module.

We shall set $N = 1$ in the rest of this section. For this case there are four undetermined coefficients $(a_{11}, a_{21}, a_{22}, w_1)$ in the excited state ODE, and these are fixed by imposing the constraint that the monodromy about $w_1$ is trivial[4]. These will lead to exactly four

[4]From the Bethe ansatz and $TQ$ approach to the integrable structure of the conformal field theory, this condition guarantees that the $Q$ function corresponding to the excited state (whose eigenvalue we compute using the ODE) has the same asymptotics as the vacuum solution [69]. See [71] for a proof of this claim for the quantum KdV models associated with simply laced Lie algebras.

algebraic constraints that completely fix these parameters. The imposition of the flatness conditions is carried out in detail in [68, 70], and we only present the results[5]. The coefficients $a_{11}, a_{22}$ and the location of the pole $w_1$ are completely determined in terms of $a_{21}$, as follows:

$$a_{11} = k\,, \tag{3.35}$$

$$a_{22} = \frac{2k+3}{3}a_{21} - \frac{k^2}{3}\,, \tag{3.36}$$

$$w_1 = \frac{1}{9(k+2)}\Big(2a_{21}(3+k)((k+1)(k+3) - \bar{r}_1) - (3+k)(k(k+1)(k+3) - (k+9)r_1 + 9r_2)\Big)\,. \tag{3.37}$$

The coefficient $a_{21}$ is in turn determined by the quadratic equation:

$$(a_{21})^2 - ka_{21} + k(3+k) - 3\bar{r}_1 = 0\,. \tag{3.38}$$

Putting all this together and relabeling $a_{21} = a_1$, we have the ODE corresponding to the first excited states of the higher spin module:

$$\left(\frac{\bar{r}_2}{z^3} + \frac{1}{z^2} + \lambda z^k + \left(\frac{(2k+3)a_1 - k^2}{3z^2\,(z-w_1)} + \frac{a_1}{z\,(z-w_1)^{\,2}} + \frac{3}{(z-w_1)^{\,3}}\right)\right)\phi(z) - \left(\frac{\bar{r}_1}{z^2} + \left(\frac{k}{z\,(z-w_1)} + \frac{3}{(z-w_1)^{\,2}}\right)\right)\phi'(z) + \phi^{(3)}(z) = 0\,. \tag{3.39}$$

### 3.2.1 WKB Analysis

As a first step towards performing the WKB analysis for the excited state ODE, we make a variable change to map the differential equation in (3.39) as a perturbation of the vacuum ODE in (2.99). We make the following change of parameters and variables:

$$k = -\frac{3M+2}{M+1}\,, \quad \lambda = (-1)^{\frac{-1}{(M+1)}}(3(M+1))^{-\frac{3M}{M+1}}E,$$

[5] We have corrected some minor typographical errors in the results of [68].

$$\begin{aligned}
\bar{r}_1 &= -1 + \frac{\ell_1(\ell_1+1)+\ell_2(\ell_2+1)+1-\ell_1\ell_2}{9(M+1)^2}\,, \\
\bar{r}_2 &= \frac{(\ell_1-3M-2)(\ell_2+3M+4)(\ell_1-l_2+3+3M)}{27(1+M)^3}\,, \\
z &= -\frac{x^{3(M+1)}}{27(1+M)^3}\,, \qquad \phi(z) = x^{3M+2}\,\psi(x)\,.
\end{aligned} \tag{3.40}$$

It is then straightforward to check that the excited state ODE takes the form

$$\begin{aligned}
&\frac{d^3\psi}{dx^3} - \frac{1}{x^2}(\ell_2^2+\ell_1^2+\ell_1+\ell_2-\ell_1\ell_2)\frac{d\psi}{dx} + \frac{1}{x^3}\ell_1(\ell_2+2)(1+\ell_1-\ell_2)\psi + (E-x^{3M})\psi \\
&\quad - \frac{9(M+1)x^{3M+1}\left(x^{3M+3}-27(M+1)^3(3M+2)w_1\right)}{\left(x^{3M+3}+27(M+1)^3w_1\right)^2}\psi'(x) \\
&\quad + \frac{9a_1(M+1)^2x^{3M}\left(2x^{3M+3}-27(M+1)^3(3M+1)w_1\right)}{\left(x^{3M+3}+27(M+1)^3w_1\right)^2}\psi(x) \\
&\quad + \frac{27(M+1)^2x^{6M+3}\left(x^{3M+3}-27(M+1)^3(3M+2)w_1\right)}{\left(x^{3M+3}+27(M+1)^3w_1\right)^3}\psi(x) = 0\,.
\end{aligned} \tag{3.41}$$

The terms in the first line are exactly those that appear in the vacuum ODE in (2.99). From the discussion in Section 3.1 it follows that one has to take the adjoint of this differential equation and perform a WKB analysis to extract the eigenvalues of the conserved charges in the excited states. As in the vacuum case, we find that the results obtained for the conserved charges are the same whether we perform the WKB analysis for (3.41) or its adjoint. So we continue to work with the differential equation in (3.41).

The additional terms in the differential equation are the simplest ($L = 1$) higher spin generalization of the monster potentials proposed in [69] to compute the eigenvalues of the quantum KdV charges in the excited states. The data of the conformal field theory can be expressed in terms of the $\ell_i$, as explained in the previous sections. From here onwards, the path to deriving the excited state eigenvalues is precisely what has been described previously. But first, we rewrite the constraints satisfied by $a_1$ and $w_1$ in terms of the CFT parameters. The constraint on $a_1$ reads:

$$a_1^2 + \frac{3M+2}{M+1}a_1 + \frac{3M+1}{(M+1)^2} - \frac{3\Delta_2}{M+1} = 0\,. \tag{3.42}$$

The quadratic equation has two solutions, and for each solution of the parameter $a_1$, the parameter $w_1$ is given by the equation

$$w_1 = -\frac{\Delta_3}{M(M+1)^{3/2}} + \frac{\Delta_2}{9M(M+1)}\left(2a_1 + \frac{3M+2}{M+1}\right) . \tag{3.43}$$

Now, at each order in the WKB expansion, the period integral of the wavefunction at that order encodes the eigenvalues of the conserved charge in the excited states. The period integrals are obtained in terms of the parameters $(a_1, w_1)$. Substituting the two independent solutions for these, we obtain the two eigenvalues in terms of the CFT parameters. These correspond to the eigenvalues of the eigenstates (of the conserved charges) that are particular linear combinations of the states $\{L_{-1}|\Delta_2,\Delta_3\rangle, W_{-1}|\Delta_2,\Delta_3\rangle\}$. As we shall see, while the individual eigenvalues are complicated functions of the CFT data, the sum of the eigenvalues are polynomials in $(c,\Delta_2,\Delta_3)$ and will precisely match the $O(q)$ coefficient of the thermal one-point function of the conserved charge.

The map from the ODE written in the $x$-coordinates to the $t$-coordinates most suited to the WKB analysis is the same as the one that worked for the vacuum ODE:

$$x = E^{-\frac{r_3}{3M}}\, t\ , \qquad \epsilon = E^{-\frac{M+1}{3M}}\ . \tag{3.44}$$

The final version of the ODE takes the following form:

$$\begin{aligned}
&\epsilon^3\Big(y^{(3)}(t) + \frac{r_1 r_2}{r_3^2}\frac{y'(t)}{t^2}\Big) + \frac{1}{r_3^3}t^{-\frac{3}{r_3}(r_3+M+1)}(1-t^{\frac{3M}{r_3}})\,y(t) \\
&\quad - \frac{9(M+1)\epsilon^3}{r_3^2 t^2}\Bigg\{\frac{1-27(M+1)^3(3M+2)w_1\epsilon^3 t^{\frac{3(M+1)}{r_3}}}{\left(1+27(M+1)^3 w_1\epsilon^3 t^{\frac{3(M+1)}{r_3}}\right)^2}\,y'(t) \\
&\quad + \frac{a_1(M+1)\left(2-27(M+1)^3(3M+1)w_1\epsilon^3 t^{\frac{3(M+1)}{r_3}}\right)}{r_3 t\left(1+27(M+1)^3 w_1\epsilon^3 t^{\frac{3(M+1)}{r_3}}\right)^2}y(t) \\
&\quad + \frac{27(M+1)^3 w_1\epsilon^3(3M(r_3-3M-4)+r_3-5)t^{\frac{3(M+1)}{r_3}} + 3M - r_3 + 2}{r_3 t\left(1+27(M+1)^3 w_1\epsilon^3 t^{\frac{3(M+1)}{r_3}}\right)^3}y(t)
\end{aligned}$$

$$+\frac{729(M+1)^6(3M+2)(r_3+1)w_1^2\epsilon^6 t^{\frac{6(M+1)}{r_3}}}{r_3 t\left(1+27(M+1)^3 w_1\epsilon^3 t^{\frac{3(M+1)}{r_3}}\right)^3}y(t)\Bigg\}=0\,. \tag{3.45}$$

The WKB analysis proceeds exactly as before. We plug in the exponential ansatz:

$$y(t)=\exp\Big[\int^t dt\,\sum_{i\geq 0}\epsilon^{i-1}a_i(t)\Big] \tag{3.46}$$

into the differential equation. We collect the terms at each order in $\epsilon$, and solve for the $a_n(t)$ recursively. The period integrals of the $a_n(t)$ encode the eigenvalues of the conserved charges in the $L=1$ excited states, up to some constant coefficients:

$$\widehat{I}_{n-1}^{L=1}=\int_0^1 dt\,a_n(t)\,. \tag{3.47}$$

### 3.2.2 Eigenvalues and Eigenstates at Level One

In order to find the eigenvalue of the charges in level one, we perform the integral (3.47). The eigenvalues at level one are related to the integral (3.47) up to a normalisation, which is gievn by:

$$\widehat{I}_n^{L=1}=c_n^{L=1}\,e^{-\frac{n\pi i}{3}}(1+M)^{\frac{n+1}{2}}\frac{\Gamma(\frac{n}{3M}+\frac{n}{3})\Gamma\left(-\frac{n}{3}\right)}{\Gamma(\frac{n}{3M})}\,I_n^{L=1}\,, \tag{3.48}$$

where $c^{L=1}$ are the relative numerical factor. The first non-trivial eigenvalue is given by

$$I_1^{(L=1)}=\Delta_2-\frac{c}{24}+1\,, \tag{3.49}$$

which is consistent with the fact that the conformal dimension of the excited state is one more than that of the primary in the module. The next charge is $\mathbf{I}_2\equiv W_0$, whose action on

the level one subspace we compute explicitly:

$$\begin{aligned}\mathbf{I}_2 L_{-1}|\Delta_2,\Delta_3\rangle &= \Delta_3 L_{-1}|\Delta_2,\Delta_3\rangle + 2W_{-1}|\Delta_2,\Delta_3\rangle\,,\\ \mathbf{I}_2 W_{-1}|\Delta_2,\Delta_3\rangle &= \frac{1}{15}\left(2-\frac{1}{b^2}+10\Delta_2\right)L_{-1}|\Delta_2,\Delta_3\rangle + \Delta_3 W_{-1}|\Delta_2,\Delta_3\rangle\,.\end{aligned} \tag{3.50}$$

Since the charges are mutually commuting, we can use this result to compute the level one eigenbasis for the Boussinesq charges:

$$\begin{aligned}|e_1\rangle &= -\frac{\sqrt{-c+32\Delta_2+2}}{4\sqrt{6}}L_{-1}|\Delta_2,\Delta_3\rangle + W_{-1}|\Delta_2,\Delta_3\rangle\,,\\ |e_2\rangle &= \frac{\sqrt{-c+32\Delta_2+2}}{4\sqrt{6}}L_{-1}|\Delta_2,\Delta_3\rangle + W_{-1}|\Delta_2,\Delta_3\rangle\,.\end{aligned} \tag{3.51}$$

We introduce the notation

$$\mathbf{I}_n|e_i\rangle = I^{(1)}_{n,i}|e_i\rangle\,, \tag{3.52}$$

where the index $i$ distinguishes the two eigenvectors at the first excited level. For $n = 2$, the two eigenvalues are easily calculated from (3.50) to be

$$I^{(1)}_{2,i} = \Delta_3 \pm \frac{\sqrt{-c+32\Delta_2+2}}{2\sqrt{6}}\,. \tag{3.53}$$

On the other hand, the excited state energies calculated from the ODE/IM correspondence give us:

$$I^{(1)}_{2,i} = \Delta_3 - \frac{2\sqrt{M+1}}{3}\left(a_{1,i}+\frac{3M+2}{2(M+1)}\right)\,. \tag{3.54}$$

Substituting the two values of $a_{1,i}$ obtained by solving (3.42), and recalling the relation (3.26) between the central charge and $M$

$$c = 2-\frac{24M^2}{M+1}\,, \tag{3.55}$$

we find a precise match with the results obtained by directly diagonalizing the operator $W_0$ on the level one subspace of the $\mathcal{W}_3$ module. Through this, we identify the common eigenvector of the Boussinesq charges (at level 1), whose corresponding eigenvalue is obtained by setting the parameter $a_1$ to each solution of (3.42). By summing over both eigenvalues, we obtain

$$\mathrm{Tr}_{L=1}\mathbf{I}_2 = 2\Delta_3 \,. \tag{3.56}$$

A similar analysis can be carried out for the higher charges, and we obtain the following expressions for the sum of the eigenvalues from the excited state ODE:

$$\mathrm{Tr}_{L=1}\mathbf{I}_4 = 2\Delta_3\left(\Delta_2 - \frac{c-90}{24}\right), \tag{3.57}$$

$$\mathrm{Tr}_{L=1}\mathbf{I}_5 = 2\Delta_2^3 + 18\Delta_3^2 + \frac{1}{4}(112-c)\Delta_2^2 + \frac{1}{96}\left(c^2-103c+2670\right)\Delta_2 + \frac{-7c^3+313c^2-40506c+87696}{48384}, \tag{3.58}$$

$$\mathrm{Tr}_{L=1}\mathbf{I}_7 = 2\Delta_2^4 + 36\Delta_3^2\Delta_2 - \frac{1}{6}(c-156)\left(2\Delta_2^3+9\Delta_3^2\right) + \frac{1}{240}\left(5c^2-713c+25794\right)\Delta_2^2 + \frac{\left(-10c^3+853c^2-93636c+534996\right)\Delta_2}{17280} + \frac{10c^4-373c^3+107916c^2-1475460c+2671488}{1658880}, \tag{3.59}$$

$$\mathrm{Tr}_{L=1}\mathbf{I}_8 = 2\Delta_3\Delta_2^3 + 6\Delta_3^3 + \frac{1}{4}(178-c)\Delta_3\Delta_2^2 + \frac{1}{480}\left(5c^2-812c+33540\right)\Delta_3\Delta_2 + \frac{\left(-35c^3+2378c^2-521028c+637560\right)\Delta_3}{241920}. \tag{3.60}$$

### 3.2.3 Eigenvalues and Eigenstates at Level two

In this section, we will discuss the construction of the eigenvalues of the charge on the second excited state.

For the first excited level, the procedure to solve the ODE (3.28) and extract the eigenvalues of the conserved charges has already been discussed in detail in 3.2.2. In this section we only highlight the distinct features of the ODE (3.28) for the second excited states.

The main change going from level one ODE to level two ODE is in the nature of the constraint equations for $\{a_1, w_1, a_2, w_2\}$ (3.32). As we can see from the equations, up to level one, the contribution from the right-hand side vanishes. However, for levels two and beyond, we cannot solve these equations analytically due to the non-trivial contributions from the right-hand side. Still, we can proceed numerically and solve these equations for particular choices of ODE parameters $\{k, \bar{r}_1, \bar{r}_2\}$ (or, equivalently, for a given central charge and conformal dimensions $(c, \Delta_2, \Delta_3)$).

Other than this non-triviality with the constraint equations, the WKB analysis for the level two case remains the same as the level 1 case, and the level $L$ eigenvalues can be found from the WKB periods:

$$\widehat{I}_n^{(L)} = \oint dz\, S_{n+1}^{(L)}(z)\,. \tag{3.61}$$

For $L = 2$, the $S_n^{(2)}$'s are related to the functions that appear at the $n$-th order in the WKB ansatz for the wave function that solves the third order ODE. There are multiple ways to calculate the period integrals (see the recent paper [72] for a recursive way to perform the period integrals in the vacuum case). We calculate these period integrals following the methods outlined for the vacuum and first excited level in the previous sections, and these are related to the eigenvalues of the conserved charge $\mathbf{I}_n$ in one of the $L$-th excited states:

$$\widehat{I}_n^{(L)} = c_n^{(L)}\, e^{-\frac{n\pi i}{3}} (1+M)^{\frac{n+1}{2}} \frac{\Gamma(\frac{n}{3M}+\frac{n}{3})\Gamma(-\frac{n}{3})}{\Gamma(\frac{n}{3M})}\, I_n^{(L)}\,. \tag{3.62}$$

A crucial point here is that the relative numerical factor between the period integrals and the eigenvalues turns out to be exactly the same[6] as it was for the vacuum case (up to a sign $c_n^{(2)} = (-1)^n c_n^{(0)}$).

We list the expressions of the first few excited state eigenvalues at the second level:

$$I_1^{(2)} = \Delta_2 - \frac{c}{24} + 2\,, \tag{3.63}$$

[6]Getting this numerical prefactor correct is crucial especially in light of the work done in Appendix D, where the eigenvalues are computed numerically for specific choices of $(c, \Delta_2, \Delta_3)$.

$$I_2^{(2)} = \Delta_3 - \frac{2(3M+2)}{3\sqrt{M+1}} - \frac{2}{3}\sqrt{M+1}(a_1+a_2)\,, \tag{3.64}$$

$$I_4^{(2)} = \frac{\Delta_3\left(3M^2+5M+5\right)}{3(M+1)} + \Delta_2\left(\Delta_3 - \frac{2(3M+2)}{3\sqrt{M+1}}\right) - \frac{2(3M+2)\left(3M^2+5M+5\right)}{9(M+1)^{3/2}}$$
$$+\left(-\frac{2\left(3M^2+5M+5\right)}{9\sqrt{M+1}} - \frac{2}{3}\Delta_2\sqrt{M+1}\right)(a_1+a_2) - 3M(M+1)^{3/2}(w_1+w_2)\,. \tag{3.65}$$

The $(a_i, w_i)$ that appear in the above expressions are solutions to the algebraic constraint equations. Although the eigenvalues of the higher charges can be similarly calculated, the expressions get increasingly complicated for higher charges, so we do not list them.

# Chapter 4

# Construction of the Currents

In this chapter, we present an explicit construction of the currents in the quantum Boussinesq hierarchy. We begin by outlining our foundational approach for determining the current densities. Applying this methodology, we demonstrate that one can systematically recover the previously established current densities reported in [15]. However, we find that this initial strategy proves insufficient for fixing the higher-order charges. To address this limitation, we subsequently develop an enhanced analytical framework. Within this advanced formulation, we explicitly derive three novel current densities for the quantum Boussinesq hierarchy.

## 4.1 Initial Strategy:

**1.** Our starting point will always be an ansatz[1] for the current $J_n$. For Example:

$$J_{2n} = \alpha_1 \underbrace{(T\ldots(T(TT)))}_{n \text{ numbers of } T} + \alpha_2 \underbrace{(T\ldots(T(WW)))}_{n-3 \text{ numbers of } T} + \alpha_3 \underbrace{(T\ldots(T(T'T')))}_{n-3 \text{ numbers of } T} + \alpha_4 \underbrace{(T\ldots(T(W'W')))}_{n-4 \text{ numbers of } T}$$
$$+ \ldots \text{all possible combinations up to total derivatives} \tag{4.1}$$

[1] The knowledge of the currents of the classical Boussinesq hierarchy dictates the composite operators appearing in the ansatz. In section (2.1.1), we have given a list of these currents.

The stress tensor $T(u)$ has weight 2, the $W(u)$ field has weight 3, while each derivative adds 1 to the conformal weight. One can check that each term in the ansatz for $J_9$ has weight 9. The parentheses indicate the conformal normal ordering of the operators (see Section 2.4 for details).

**2.** The next step involves computing the thermal one-point function of each of the composite operators appearing in the ansatz in a higher spin module. The thermal correlator is defined as

$$\langle \mathcal{O} \rangle = \text{Tr} \left( q^{L_0 - \frac{c}{24}} \mathcal{O} \right) , \tag{4.2}$$

where $q = e^{-\beta}$, with $\beta$ being the inverse temperature. The trace is taken in the higher spin module constructed over a highest weight state labeled by the eigenvalues of the commuting operators $(L_0, W_0)$. A generic state in the module will be of the form

$$\mathcal{P} = \prod_{n>0} L_{-n}^{a_n} W_{-n}^{b_n} |\Delta_2, \Delta_3\rangle , \tag{4.3}$$

where we shall associate to every state in the module a level $L$, defined to be

$$L = \sum_n n(a_n + b_n) . \tag{4.4}$$

We first calculate the thermal correlator of the fields appearing in the composite operator using the Zhu recursion relations, and then normal order them, as described in Appendix C. The thermal one-point functions of each of the composite operators appearing in $J_9$ have been listed in Appendix C.2.2. It is evident from the expressions that it is $u$-independent. Thus, computing the one-point function of the current $J_9(u)$ is equivalent to computing the one-point function of the conserved charge $\mathbf{I}_8$.

**3.** At the next step, we take the low-temperature limit of the thermal one-point functions, which essentially amounts to the $q \to 0$ limit. From the definition of the trace, it is clear

that one is effectively computing the eigenvalue of the conserved charge $\mathbf{I}_n$ in the highest weight state, at level $L = 0$. Now, this eigenvalue has already been computed in section 3 using the ODE/IM correspondence and subsequently, we have listed all the vacuum eigen-values in Appendix B. As we will see, for higher order currents there exits a set of combination of composite operators in the currents which are actually trace-free. This means that their one-point functions are identically zero (and not just the leading term as $q \to 0$). Due to this reason, our initial strategy will not be enough to construct the higher currents.

**4.** The next step is to take the classical limit and match with the classical charges. For this purpose we take the classical limit, which amounts to taking the large-$c$ limit. The precise way to do this has been outlined in [15]; we re-scale and redefine

$$T(u) \longrightarrow \left(\frac{c}{24}\right) U(u)\,, \qquad W(u) \longrightarrow i\left(-\frac{c}{24}\right)^{\frac{3}{2}} V(u)\,, \tag{4.5}$$

while simultaneously re-scaling the quantum Boussinesq charges in the following manner:

$$\mathbf{I}_k \longrightarrow i^{-k-1}\left(-\frac{c}{24}\right)^{\frac{k+1}{2}} I_k^{\text{class}}\,, \tag{4.6}$$

in the limit that $c \to -\infty$. The right-hand side comprises the classical conserved charges of the Boussinesq hierarchy, as enumerated in (2.1.1). For lower-order currents that can be determined via the preceding three-step procedure, agreement with these classical charges serves as a crucial consistency verification. In the case of higher-order currents—where our initial methodology proves inadequate—matching with the classical expressions enables the determination of a subset of the remaining undetermined coefficients.

## 4.2 Thermal One-point Functions

In this section, we compute the thermal one-point functions of the conserved charges of the quantum Boussinesq hierarchy. One check of these one-point functions is that, in the low-temperature limit, which corresponds to $q = e^{-\beta} \to 0$, the thermal one-point function of the charges $\mathbf{I}_k$ should coincide with the eigenvalues $I_k$ computed using the ODE/IM correspondence in the previous section. Of course, except for the first few, the explicit forms of the conserved currents are not known. In our analysis, we show how the evaluation of the thermal one-point functions of composite operators, along with the eigenvalues $I_k$ in the highest-weight states, provide important constraints on the form of the conserved current densities. We shall systematically work our way up the currents in order of increasing weight.

**$\mathbf{I}_1$:** We start with the thermal one-point function $J_2(u) = T(u)$. The one-point function of a single vertex operator reduces to the evaluation of its zero mode on the torus. We therefore have

$$\langle T(u) \rangle = F((T)_0; \tau) \,, \tag{4.7}$$

where $(O)_0$ denotes the zero mode of the operator $O$. Since the zero mode of the energy-momentum tensor on the cylinder is $L_0 - \frac{c}{24}$, we obtain

$$\langle T(u) \rangle = \mathrm{Tr}_V\Big(\Big(L_0 - \frac{c}{24}\Big) q^{L_0 - \frac{c}{24}}\Big) = q \frac{\partial}{\partial q} \chi(\tau) \,. \tag{4.8}$$

It is manifestly independent of the $u$ coordinate; thus, doing the $u$-integral is trivial, and we obtain

$$\langle \mathbf{I}_1 \rangle = \int_0^1 du \, \langle T(u) \rangle = q \frac{\partial}{\partial q} \chi(\tau) \,. \tag{4.9}$$

In the zero temperature limit, we obtain

$$\langle \mathbf{I}_1 \rangle_{q\to 0} = \Delta_2 - \frac{c}{24} \,, \tag{4.10}$$

which matches $I_1$, as expected.

**$\mathbf{I}_2$:** Next we compute the $\langle J_3(u)\rangle = \langle W(u)\rangle$, for which we get

$$\langle J_3(u)\rangle = \langle W_0\rangle = \mathrm{Tr}_V\Big(W_0 q^{L_0-\frac{c}{24}}\Big) = \Delta_3\chi(\tau) \,, \tag{4.11}$$

where we have used the result given in [48]. The integral over $u$ is trivially done, and we obtain

$$\langle \mathbf{I}_2\rangle = \Delta_3\chi(\tau) \,. \tag{4.12}$$

The low-temperature limit is again trivially taken, and we obtain $I_2 = \Delta_3$.

**$\mathbf{I}_4$:** For $\mathbf{I}_4$, we obtain

$$\langle \mathbf{I}_4\rangle = \int_0^1 du\, \langle (TW)\rangle(u_1) = \Delta_3\Big(\partial\chi(\tau) - \frac{1}{4}E_2(\tau)\chi(\tau)\Big) \,, \tag{4.13}$$

where we have defined $\partial = q\frac{\partial}{\partial q}$, and $E_2(\tau)$ is the level two Eisenstein series. We refer the reader to Appendix C.1 for a derivation of the one-point function. In the low temperature limit, we have $E_2(\tau)\to 1$ and we therefore obtain

$$\langle \mathbf{I}_4\rangle_{q\to 0} = \Delta_3(\Delta_2 - \frac{c}{24}) - \frac{\Delta_3}{4} = \Delta_3\left(\Delta_2 - \frac{(c+6)}{24}\right) \,, \tag{4.14}$$

which perfectly matches $I_4$ in (3.27c).

### 4.2.1 Currents and Thermal Expectation Values

So far we have seen that the thermal one-point functions behave as expected in the low-temperature limit and reproduce the eigenvalues obtained in the previous section. We now illustrate the use of the thermal one-point functions of composite operators in the determination of the conserved currents in the Boussinesq hierarchy.

**$\mathbf{I}_5$:** At weight 6, the expression for the current is already given in [15]. We shall rederive this result by using the results for the thermal correlators. We begin with the most general ansatz consistent with the classical current at weight-6:

$$J_6(u) = \alpha_1 \,(T(TT))(u) + \alpha_2 \,(T'T')(u) + \alpha_3 \,(WW)(u)\,. \tag{4.15}$$

The thermal one-point functions of each of the composite operators have been derived in the appendix. In fact, all we need at this point is the low-temperature limit, and we have

$$\langle (T(TT))\rangle_{q\to 0} = \Delta_2^3 - \frac{1}{8}(c+4)\Delta_2^2 + \frac{1}{960}(c+2)(5c+32)\Delta_2 - \frac{c\left(35c^2+462c+1504\right)}{483840}\,, \tag{4.16}$$

$$(2\pi)^2\langle (T'T')\rangle_{q\to 0} = \frac{\Delta_2}{60} - \frac{31c}{30240}\,, \tag{4.17}$$

$$\langle (WW)(z)\rangle_{q\to 0} = \Delta_3^2 - \frac{1}{18}\Delta_2^2 + \Big(\frac{17c}{3456} + \frac{91}{8640}\Big)\Delta_2 - \frac{191c^2}{1741824} - \frac{2101c}{4354560}\,. \tag{4.18}$$

These are independent of $u$, as expected. Demanding that the low-temperature limit of the thermal one-point function of the current in (4.15) should reproduce the eigenvalue $I_5$ in (3.27d) turns out to uniquely fix the coefficients $\alpha_i$ and we find that

$$J_6(u) = (T(TT))(u) + \frac{c-10}{32}(2\pi)^2(T'T')(u) + 9(WW)(u)\,, \tag{4.19}$$

which exactly matches[2] the result for the current in [15], and which we quoted in (1.7). Having completely fixed the coefficients appearing in our ansatz, we present the result for the thermal one-point function of the conserved charge:

$$\langle \mathbf{I}_5 \rangle = \partial^3\chi(\tau) - E_2(\tau)\partial^2\chi(\tau) + \Big(\frac{c+6}{192}E_4(\tau) + \frac{1}{8}E_2^2(\tau)\Big)\partial\chi(\tau) - \frac{c(84E_2(\tau)E_4(\tau) + (66+5c)E_6(\tau))}{241920}\chi(\tau) + 9\langle W_0^2\rangle \,. \quad (4.20)$$

Here we have chosen to write the thermal one-point function in terms of the expectation values of powers of the zero mode of the spin-3 current, which have been listed in the Appendix (see equation (2.84)).

**$\mathbf{I}_7$:** The current $J_8$, whose integral gives the conserved charge $\mathbf{I}_7$, can be written in terms of the following combinations of normal ordered composite operators:

$$J_8 = \alpha_1(T(T(TT))) + \alpha_2(T(T'T')) + \alpha_3(T''T'') + \alpha_4(TWW) + \alpha_5(W'W') \,. \quad (4.21)$$

Using the low-temperature limit of the one-point functions, and matching to the eigenvalue $I_7$ listed in the Appendix B.1, we find that the current is fixed up to a single constant:

$$J_8 = (T(T(TT))) + 18(T(WW)) + \frac{3}{16}(5c+46)(2\pi)^2(W'W') - \frac{1}{7680}(5c^2-28c+1124)(2\pi)^4(T''T'') + (2\pi)^2\,\alpha\left((T(T'T')) - 3(W'W') + (2\pi)^2\frac{c-34}{96}(T''T'')\right) \,. \quad (4.22)$$

So the low temperature limit alone does not fix the form of the current. Substituting the thermal one-point functions for each of the composites (see Appendix (C.1.1)), one can

[2]Note that in our conventions, the circumference of the cylinder is 1.

check that a much stronger statement is true:

$$\left\langle (T(T'T')) - 3(W'W') + (2\pi)^2 \frac{c-34}{96}(T''T'') \right\rangle = 0 \,. \tag{4.23}$$

In other words, the thermal one-point function of the operator multiplying the undetermined coefficient $\alpha$ vanishes. Thus, the coefficient $\alpha$ cannot be fixed, even if we compute the trace of the eigenvalues in the higher excited states. However, a happy consequence of this fact is that the thermal one-point function of the charge can be computed without knowing $\alpha$. Thus, we obtain

$$\begin{aligned}\langle \mathbf{I}_7 \rangle =& \partial^4\chi(\tau) - 2E_2(\tau)\partial^3\chi(\tau) + \Big(\frac{5}{6}E_2^2(\tau) + \frac{1}{480}(7c+194)E_4(\tau)\Big)\partial^2\chi(\tau) \\ &- \frac{1}{34560}\Big(2400E_2^3(\tau) + 108(26+c)E_2(\tau)E_4(\tau) + (1092 + c(252+5c))E_6(\tau)\Big)\partial\chi(\tau) \\ &+ \frac{c}{1105920}\Big(64E_2^2(\tau)E_4(\tau) + (580 + c(60+c))E_4^2(\tau) + 256E_2(\tau)E_6(\tau)\Big)\chi(\tau) \\ &+ 18\Big(q\frac{\partial}{\partial q} - \frac{1}{2}E_2(\tau)\Big)\langle W_0^2\rangle \,.\end{aligned}$$

In Section 4.5, we shall return to this point of fixing the undetermined parameter $\alpha$. For now, we go on and list the thermal one-point functions of the higher Boussinesq charges.

**$\mathbf{I}_8$:** By similar considerations, we find that the weight nine current $J_9$ can be determined up to a single constant:

$$\begin{aligned}J_9 =& (T(T(TW))) + 3(W(WW)) \\ &+ \frac{1}{32}(7c+2)(2\pi)^2(T'(T'W)) - \frac{1}{480}(c+2)(2c+55)(2\pi)^4(T''W'') \\ &+ (2\pi)^2\,\gamma\left((T(WT'')) + \frac{5}{2}(T'(T'W)) - \frac{1}{24}(c+25)(2\pi)^2(T''W'')\right) \,.\end{aligned} \tag{4.24}$$

We refer to Appendix (C.1.1) for the results of individual thermal correlators of composite fields present in $J_9$. From these results it is easy to check that the operator that multiplies $\gamma$ not only has a vanishing low-temperature limit, but its thermal one-point function also

vanishes. Thus, while the current is not determined uniquely by these considerations, it is possible to calculate the thermal one-point function of the conserved charge:

$$\begin{aligned}\langle \mathbf{I}_8 \rangle =& 3\langle W_0^3 \rangle + \partial^3\langle W_0\rangle - \frac{7}{4}E_2(\tau)\partial^2\langle W_0\rangle + \frac{1}{240}(150E_2^2(\tau) + (75+2c)E_4(\tau))\partial\langle W_0\rangle \\ &- \frac{1}{60480}(2730E_2^3(\tau) + 21(285+7c)E_2(\tau)E_4(\tau) + (2310 + c(123+2c))E_6(\tau))\langle W_0\rangle\,.\end{aligned}$$

**$\mathbf{I}_{10}$:** The vacuum eigenvalues are much less constraining on the ansatz for $\mathbf{I}_{10}$ and we find that the current $J_{11}$ is only fixed up to four undetermined constants:

$$\begin{aligned}J_{11} =& (T(T(T(TW)))) + 6(W(W(WT))) + (2\pi)^2\frac{3(11c+10)}{16}(W'(W'W)) \\ &+ (2\pi)^4\frac{(19c^2+304c+1492)}{1536}(T'''(WT')) + (2\pi)^6\frac{(480c^3+12733c^2+52424c-93444)}{1935360}(T'''W''') \\ &+ (2\pi)^4\delta_1\left[(T'''(WT')) + (T''(WT''))\right] \\ &+ (2\pi)^4\delta_2\left((T(WT'''')) + \frac{5}{2}(T'''(WT')) + (2\pi)^2\frac{1}{120}(5c-21)(T'''W''')\right) \\ &+ (2\pi)^2\delta_3\Big((T(T(T''W))) + \frac{21}{2}(W'(W'W)) + \frac{(2\pi)^2}{192}(19c-318)(T'''(WT')) \\ &\qquad + (2\pi)^4\frac{20c^2-11c+2318}{11520}(T'''W''')\Big) \\ &+ (2\pi)^2\delta_4\Big(T'(T'(TW))) - 3(W'(W'W)) - \frac{1}{96}(2\pi)^2(c-198)(T'''(WT')) \\ &\qquad + (2\pi)^4\frac{101c-598}{5760}(T'''W''')\Big)\,. \qquad (4.25)\end{aligned}$$

Following the pattern found for the previous two currents, we find that the four combinations multiplying the $\delta_i$ are trace-free. Thus, the thermal one-point function of the conserved charge can be calculated without ambiguity and we find:

$$\begin{aligned}\langle \mathbf{I}_{10}\rangle =& \partial^4\langle (W_0)\rangle - 3E_2(\tau)\partial^3\langle (W_0)\rangle \\ &+ 6\,\partial\langle (W_0)^3\rangle - \frac{9}{2}E_2(\tau)\langle (W_0)^3\rangle + \Big(\frac{7}{3}E_2^2(\tau) + \Big(\frac{7}{6}+\frac{c}{48}\Big)E_4(\tau)\Big)\partial^2\langle (W_0)\rangle \\ &- \Big(\frac{13}{24}E_2^3(\tau) + \Big(\frac{5}{6}+\frac{7c}{480}\Big)E_2(\tau)E_4(\tau) + \Big(\frac{87}{224}+\frac{193c}{10080}+\frac{5c^2}{24192}\Big)E_6(\tau)\Big)\partial\langle (W_0)\rangle \\ &+ \frac{1}{3870720}\Big(8\left(25c^2+2428c+53700\right)E_6(\tau)E_2(\tau) + 5\left(c^3+126c^2+4972c+72504\right)E_4^2(\tau)\end{aligned}$$

$$+ 448(17c + 945)E_4(\tau)E_2^2(\tau) + 107520E_2^4(\tau)\Big)\langle W_0\rangle \,. \quad (4.26)$$

### 4.2.2 The Classical Limit

In the large central charge limit, the quantum Boussinesq charges should go over to the classical conserved charges derived in Section 2.1.1. As we shall see, this imposes some constraints on the form of the undetermined coefficients $\alpha$, $\gamma$ and $\delta_i$. The way to take the classical limit has already been outlined in [15], and we review this now. We first rescale and redefine

$$T(u) \longrightarrow \left(\frac{c}{24}\right) U(u)\,, \qquad W(u) \longrightarrow i\left(-\frac{c}{24}\right)^{\frac{3}{2}} V(u)\,. \quad (4.27)$$

Then, the quantum conserved charges go over to the classical ones in the following manner:

$$\mathbf{I}_k \longrightarrow i^{-k-1}\left(-\frac{c}{24}\right)^{\frac{k+1}{2}} I_k^{\text{class}}\,, \quad (4.28)$$

in the limit that $c \to -\infty$. Doing this for the first three charges is a trivial exercise, as there is only a single term in both the classical and quantum currents. For $\mathbf{I}_5$, it has been checked already in [15] that the classical limit gives the expected classical current.

We now turn to the constraints that follow from taking the classical limit for the higher Boussinesq currents. We find a perfect agreement with the classical charges, provided we set

$$\begin{aligned} \alpha &= \frac{3c}{16} + \alpha^{(0)}\,, & \gamma &= -\frac{c}{8} + \gamma^{(0)} \\ \delta_1 &= \frac{7c^2}{768} + \delta_1^{(1)}c + \delta_1^{(0)}\,, & \delta_2 &= \frac{c^2}{192} + \delta_2^{(1)}c + \delta_2^{(0)}\,, \\ \delta_3 &= -\frac{c}{4} + \delta_3^{(0)}\,, & \delta_4 &= -\frac{5c}{16} + \delta_4^{(0)}\,. \end{aligned} \quad (4.29)$$

Thus the leading-in-$c$ behavior of the various constants is uniquely fixed by the classical limit[3]. We emphasize here that $\alpha^{(0)}$, $\gamma^{(0)}$ and $\delta_i^{(j)}$ are $c$-independent numbers. Requiring a

[3]The fact that $\delta_2$ and $\delta_3$ are both quadratic polynomials of the central charge follows by requiring that the operators multiplying these contribute in the classical limit. A linear polynomial in $c$ would lead to a vanishing contribution in each case, and there would be a mismatch with the classical limit.

consistent classical limit of the proposed conserved quantum currents is a non-trivial constraint since the constants $\alpha$, $\gamma$ and, $\delta_i$ feed into multiple terms that appear in the classical conserved charges. Thus, it is also a useful consistency check, both of the expressions for the currents and of our results for the thermal correlators.

### 4.2.3 Discussion

We conclude this section with a few remarks about the nature of our results. We have calculated the thermal one-point functions of all Boussinesq charges up to weight 10. We have expressed these as differential operators acting on the character (with insertions of the zero mode $W_0$), and with coefficients that are quasi-modular forms. As shown in [48, 49] for $n = 1, 2$, the thermal expectation values $\langle W_0^n \rangle$ can also be computed as quasi-modular differential operators acting on the character. Thus, the thermal one-point functions can be expressed as quasi-modular differential operators acting on the character of the higher spin module. This is expected from a study of the generalized Gibbs ensemble, in which one defines a partition function that includes chemical potentials for all the higher spin conserved currents [73]. The thermal correlators we have defined should be thought of as the lowest-order terms in a power series expansion (in the fugacities) of the generalized partition function.

Secondly, we emphasize the fact that we could calculate the thermal one-point functions even without fixing the form of the current densities exactly. The currents $J_8$, $J_9$ and $J_{11}$ have undetermined parameters, but these did not appear in the thermal one-point functions. This is on account of certain trace-free combinations appearing in the current densities.

As a first step towards the determination of the undetermined constants, we took the classical limit of large central charge and fixed the leading coefficients in the large-$c$ limit in equation (4.29). To completely fix the remaining parameters, we need more data from

the conformal field theory side. For this purpose, we now calculate the eigenvalues of the conserved currents in the first level of excited states in the higher spin module. This will prove to be useful in two ways: first of all, we should match the sum of the eigenvalues (of the conserved currents) in the first excited level with the terms that appear at sub-leading order in the $q$-expansion of the thermal one-point functions. This is indeed what we shall find in our analysis in Section 3.2. Secondly, as we shall show in Section 4.5, the excited state eigenvalues open up the possibility of fixing the undetermined parameters of the current densities via two-point functions involving the conserved currents.

## 4.3 Updated Strategy

**1.** To compute the undetermined constants, it is clear that we have to go beyond the one-point functions we have calculated so far. A natural idea is to consider the two-point function involving the conserved charge and another operator, which we denote by $\mathbf{A}$:

$$\langle \mathbf{I}_n \, \mathbf{A} \rangle = \mathrm{Tr} \left( q^{L_0 - \frac{c}{24}} \, \mathbf{I}_n \, \mathbf{A} \right) . \tag{4.30}$$

We shall choose $\mathbf{A}$ to be the zero mode of a composite operator built out of the stress tensor and/or the spin-3 current. For the undetermined coefficients to appear in the correlator (4.30), the operator $\mathbf{A}$ must be chosen so that its two-point correlator with the trace-free combinations appearing in the currents is non-vanishing[4]. The two-point function can be calculated by taking, in turn, each composite operator in $J_n$ and computing the two-point function with $\mathbf{A}$ by using the Zhu recursion relations. This is a modification of the calculations we performed for the one-point functions, and the results are presented in Appendix C.2.

[4]Such an operator is likely to exist due to the trace being non-degenerate (when considered as a bi-linear form), and the fact that there are infinitely many linearly independent local operators to choose from.

**2.** We now compute the two-point function in a completely independent manner by using the operator formalism. To facilitate this, we make a simple choice for $\mathbf{A}$ so that its action on states in the higher spin module is well understood. However, it is crucial that $\mathbf{A}$ be the zero mode of a composite operator. Being the zero mode ensures that $\mathbf{A}$ acts as a linear operator on the level-$L$ excited states of the higher spin module, just like the conserved charge itself. Thus, one can restrict attention to states at a given level and consider the contribution to the trace (4.30) from the level-$L$ excited states. Let $|\psi_i\rangle$ denote the usual basis elements of weight $\Delta_2 + L$ in the higher spin module, built out of the $L_{-m}$ and $W_{-n}$ operators. Then, one can straightforwardly compute

$$\mathbf{A}\,|\psi_i\rangle = \sum_j A_{ji}\,|\psi_j\rangle\ , \tag{4.31}$$

by using the $\mathcal{W}_3$ algebra. Now, at each level, the mutually commuting conserved charges $\mathbf{I}_n$ can be diagonalized, and we have corresponding eigen-states $|e_i\rangle$ such that

$$\mathbf{I}_n\,|e_i\rangle = I^{(L)}_{n,i}\,|e_i\rangle\ . \tag{4.32}$$

The basis elements $|\psi_i\rangle$ are obviously linear combinations of these eigen-states:

$$|\psi_j\rangle = \sum_k R_{kj}|e_k\rangle\ . \tag{4.33}$$

It is now a matter of simple algebra to check that the contribution to the trace in (4.30) at level $L$ is given by

$$\langle \mathbf{I}_n\,\mathbf{A}\rangle_{\mathcal{H}_L} = q^{\Delta_2+L}\,\mathrm{Tr}\left(A\,R\,I^{(L)}_n\,(R^{-1})\right)\ . \tag{4.34}$$

The matrices $A$ and $R$ can be obtained using the $\mathcal{W}_3$ algebra, as we shall show in the following section. Thus, the one remaining ingredient that cannot be directly calculated in the operator formalism is the set of eigenvalues of the Boussinesq charges at a given level.

**3.** The eigenvalues at a given level in the higher spin module are obtained via the ODE/IM correspondence [42, 68, 70, 74]. This has been discussed in detail in chapter 3.

**4.** With this final piece of the puzzle in place, we compare and match the result in (4.34) with the $O(q^{\Lambda_2+L})$ contribution to the trace computed using the Zhu recursion, as explained in point 5 above. By equating the two results, the idea is to unambiguously fix the undetermined coefficients in (4.29). A crucial point here is that the operator $\mathbf{A}$ must necessarily have a non-vanishing two-point function with the trace-free combinations appearing in the currents. This ensures that equating the trace computed in two different ways leads to simple linear equations for the coefficients appearing in the ansatz.

## 4.4 The Operator Formalism

In this section, we extend the analysis to the second excited level and outline the derivation of the matrices $A$ and $R$ required to compute the trace in (4.34). Since we make use of the $\mathcal{W}_3$ algebra for this purpose, we provide the commutation relations between the modes $L_n$ of the spin-2 current and the modes $W_n$ of the spin-3 current, as presented in [13]:

$$\begin{aligned}
[L_n, L_m] =&(n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{m+n,0}\,, \qquad [L_n, W_m] = (2n-m)W_{n+m}\,,\\
[W_n, W_m] =&\frac{(n-m)}{3}\Lambda_{n+m} + \frac{n-m}{3b^2}\left(\frac{1}{15}(n+m+3)(n+m+2) - \frac{1}{6}(n+2)(m+2)\right)L_{n+m}\\
&+ \frac{c}{1080b^2}n(n^2-4)(n^2-1)\delta_{n+m,0}\,.
\end{aligned}$$

Here $c$ is the central charge and $b^2 = \frac{16}{5c+22}$. The composite operator appearing on the right-hand side of the $W_n$ commutators is the normal ordered operator $\Lambda(u) = (TT)(u)$,

whose modes are given by

$$\Lambda_n = \sum_{k=-\infty}^{\infty} :L_k L_{n-k}: + \frac{1}{5} x_n L_n \ , \tag{4.35}$$

where $x_{2l} = 1 - l^2$ and $x_{2l+1} = 2 - l - l^2$. The normal ordering symbol : : indicates that we put the operators with larger index $n$ to the right.

### 4.4.1 Level 1 matrices

The basis vectors at level 1 are given by

$$|\psi\rangle \in \{L_{-1}|\Delta_2, \Delta_3\rangle, W_{-1}|\Delta_2, \Delta_3\rangle\} \ . \tag{4.36}$$

To construct the eigenstates of the mutually commuting charges at level 1, we begin by deriving the matrix representation of the operator $W_0$ in this subspace. Using $\mathcal{W}_3$ algebra, we find:

$$W_0 \begin{pmatrix} L_{-1}|\Delta_2, \Delta_3\rangle \\ W_{-1}|\Delta_2, \Delta_3\rangle \end{pmatrix} = \begin{pmatrix} \Delta_3 & 2 \\ \frac{1}{15}\left(2 - \frac{1}{b^2} + 10\Delta_2\right) & \Delta_3 \end{pmatrix} \begin{pmatrix} L_{-1}|\Delta_2, \Delta_3\rangle \\ W_{-1}|\Delta_2, \Delta_3\rangle \end{pmatrix} . \tag{4.37}$$

We diagonalise the above matrix to get the eigenvectors of all the charges $I_n$:

$$\begin{pmatrix} |e_1\rangle \\ |e_2\rangle \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{-c+32\Delta_2+2}}{4\sqrt{6}} & 1 \\ \frac{\sqrt{-c+32\Delta_2+2}}{4\sqrt{6}} & 1 \end{pmatrix} \begin{pmatrix} L_{-1}|\Delta_2, \Delta_3\rangle \\ W_{-1}|\Delta_2, \Delta_3\rangle \end{pmatrix} . \tag{4.38}$$

The inverse relation fixes the form of the R-matrix defined in (4.33):

$$R = \begin{pmatrix} -\frac{2\sqrt{6}}{\sqrt{-c+32\Delta_2+2}} & \frac{1}{2} \\ \frac{2\sqrt{6}}{\sqrt{-c+32\Delta_2+2}} & \frac{1}{2} \end{pmatrix} . \tag{4.39}$$

In our attempt to fix the form of the currents $J_n$, we will find it useful to consider two

different **A**-operators:

$$\mathbf{A}_3 = \int_0^1 du\,(TT)(u) = 2\sum_{n\geq 0} L_{-n}L_n + L_0^2 - \frac{c+2}{12}L_0 + \frac{c(5c+22)}{2880}\,, \tag{4.40}$$

$$\mathbf{A}_5 = (2\pi)^2 \int_0^1 du\,(T'T')(u) = 2\sum_{n>0} n^2 L_{-n}L_n + \frac{1}{60}L_0 - \frac{31c}{30240}\,. \tag{4.41}$$

The subscript indicates the weight of each operator. At level 1, it is straightforward to find their action on the basis states using the commutation relations of the $\mathcal{W}_3$ algebra. The matrix representations[5] of the operators $\mathbf{A}_3$ and $\mathbf{A}_5$ on level one subspace are given by:

$$(\mathbf{A}_3)_{ij} = \begin{pmatrix} \frac{5c^2-218c+2400}{2880} - \frac{1}{12}(c-70)\Delta_2 + \Delta_2^2 & 0 \\ 6\Delta_3 & \frac{5c^2-218c+2400}{2880} - \frac{1}{12}(c-22)\Delta_2 + \Delta_2^2 \end{pmatrix} \tag{4.42}$$

$$(\mathbf{A}_5)_{ij} = \begin{pmatrix} -\frac{31c}{30240} + \frac{241\Delta_2}{60} + \frac{1}{60} & 6\Delta_3 \\ 0 & \frac{1}{60}(\Delta_2+1) - \frac{31c}{30240} \end{pmatrix}. \tag{4.43}$$

Substituting these matrices, we find that the level-1 contribution to the traces from the operator formalism can be written compactly:

$$\langle \mathbf{I}_n\,\mathbf{A}_5\rangle_{\mathcal{H}_1} = q^{\Delta_2+1}\left\{\left(\frac{-31c+60984\Delta_2+504}{30240}\right)(I^{(1)}_{n,1}+I^{(1)}_{n,2}) + \frac{12\sqrt{6}\Delta_3}{\sqrt{-c+32\Delta_2+2}}(I^{(1)}_{n,2}-I^{(1)}_{n,1})\right\}. \tag{4.44}$$

The result for the trace with $\mathbf{A}_3$ inserted is:

$$\langle \mathbf{I}_n\,\mathbf{A}_3\rangle_{\mathcal{H}_1} = q^{\Delta_2+1}\left\{\left(\Delta_2^2 + \frac{\left(5c^2-218c+2400\right)}{2880} - \frac{1}{12}(c-46)\Delta_2\right)(I^{(1)}_{n,1}+I^{(1)}_{n,2})\right.$$

[5] For the computation of the matrix elements, we have used an implementation of $\mathcal{W}_3$ algebra in Mathematica. For this purpose, we have used the Mathematica notebook Virasoro by Matthew Headrick with minor modifications.

$$+\frac{12\sqrt{6}\Delta_3}{\sqrt{-c+32\Delta_2+2}}(I^{(1)}_{n,2}-I^{(1)}_{n,1})\Bigg\}\,. \tag{4.45}$$

What is left to compute are the eigenvalues $I^{(1)}_{n,i}$, which has been computed using the ODE/IM correspondence ine 3. With this in place, the trace with the **A**-operators insertion can be calculated explicitly from the operator formalism for any of the Boussinesq charges.

### 4.4.2 Level 2 matrices

The basis vectors at the $L = 2$ level are given by

$$\psi \in \left\{L_{-2}|\Delta_2,\Delta_3\rangle, L^2_{-1}|\Delta_2,\Delta_3\rangle, L_{-1}W_{-1}|\Delta_2,\Delta_3\rangle, W^2_{-1}|\Delta_2,\Delta_3\rangle, W_{-2}|\Delta_2,\Delta_3\rangle\right\}\,. \tag{4.46}$$

To find the eigenstates of the mutually commuting charges, we begin with the $5\times5$ matrix representation of the simplest non-trivial conserved charge $\mathbf{I}_2 = W_0$, given by

$$\begin{pmatrix}
\Delta_3 & 0 & \frac{1}{15}\left(-\frac{1}{b^2}+10\Delta_2+12\right) & 0 & \frac{2}{3}\\
0 & \Delta_3 & \frac{4\Delta_2}{3} & 2\Delta_3 & \frac{4\Delta_2}{3}\\
4 & 0 & \Delta_3 & \frac{2}{15}\left(-\frac{1}{b^2}+10\Delta_2+7\right) & 0\\
0 & 0 & 2 & \Delta_3 & 0\\
-2 & 4 & 0 & \frac{1}{15}\left(\frac{1}{b^2}-10\Delta_2-12\right) & \Delta_3
\end{pmatrix}\,. \tag{4.47}$$

This is easily derived by acting with $W_0$ on the basis of vectors using the basic commutation relations. The eigenvectors at $L = 2$ are obtained as column vectors of the matrix used to diagonalize this matrix. This will also automatically provide the form of the R-matrix, as the similarity transformation that diagonalizes this matrix. Since we have a general quintic equation to solve, unlike the $L = 1$ case, it is not possible to obtain a closed-form expression for the eigenvectors at the second excited level. However, for a

given numerical choice of $(c, \Delta_2, \Delta_3)$ this is easily done, and this is all we shall need to fix the form of the currents, as we shall explain in the subsequent analysis.

In the basis (4.46), one can similarly compute the matrix representation of $\mathbf{A}_3$:

$$\begin{pmatrix} \alpha_{11} & 6 & 6\Delta_3 & -\frac{2}{5b^2}+4\Delta_2+\frac{4}{5} & 0 \\ 12\Delta_2 & \alpha_{22} & 30\Delta_3 & \frac{2\Delta_2\left(10b^2\Delta_2+2b^2-1\right)}{3b^2} & 12\Delta_3 \\ 0 & 0 & \alpha_{33} & 12\Delta_3 & 8 \\ 0 & 0 & 0 & \alpha_{44} & 0 \\ 0 & 0 & -4\left(\Delta_2+3\right) & -12\Delta_3 & \alpha_{55} \end{pmatrix}, \tag{4.48}$$

where

$$\begin{aligned} \alpha_{11} &= \frac{5c^2-458c+22080}{2880}-\frac{1}{12}(c-142)\Delta_2+\Delta_2^2, \\ \alpha_{22} &= \frac{5c^2+2422c+10560}{2880}-\frac{1}{12}(c-142)\Delta_2+\Delta_2^2, \\ \alpha_{33} &= \frac{5c^2-458c+45120}{2880}-\frac{1}{12}(c-94)\Delta_2+\Delta_2^2, \\ \alpha_{44} &= \frac{5c^2-458c+10560}{2880}-\frac{1}{12}(c-46)\Delta_2+\Delta_2^2, \\ \alpha_{55} &= \frac{5c^2-458c-12480}{2880}-\frac{1}{12}(c-46)\Delta_2+\Delta_2^2. \end{aligned}$$

Similarly, $\mathbf{A}_5$ can be written as:

$$\begin{pmatrix} \beta_{11} & 6 & 6\Delta_3 & -\frac{2}{5b^2}+4\Delta_2+\frac{4}{5} & 0 \\ 48\Delta_2 & \beta_{22} & 120\Delta_3 & \frac{8\Delta_2\left(10b^2\Delta_2+2b^2-1\right)}{3b^2} & 48\Delta_3 \\ 0 & 0 & \beta_{33} & 12\Delta_3 & 8 \\ 0 & 0 & 0 & \beta_{44} & 0 \\ 0 & 0 & -4\left(\Delta_2+3\right) & -12\Delta_3 & \beta_{55} \end{pmatrix}, \tag{4.49}$$

where

$$\beta_{11} = -\frac{31c}{30240}+\frac{481\Delta_2}{60}+\frac{121}{30},$$

$$\beta_{22} = \frac{120929c}{30240} + \frac{1921\Delta_2}{60} + \frac{1}{30},$$
$$\beta_{33} = -\frac{31c}{30240} + \frac{241\Delta_2}{60} + \frac{361}{30},$$
$$\beta_{44} = -\frac{31c}{30240} + \frac{\Delta_2}{60} + \frac{1}{30},$$
$$\beta_{55} = -\frac{31c}{30240} + \frac{\Delta_2}{60} - \frac{239}{30}.$$

## 4.5 Fixing the Higher Boussinesq Currents

### 4.5.1 Fixing the current $J_8$

In this section, we shall show how the form of the quantum current $J_8$ can be fixed by using the excited state eigenvalues and two-point thermal correlators. We begin by recalling the form of the current $J_8$:

$$\begin{aligned} J_8 = (T(T(TT))) + 18(T(WW)) + \frac{3}{16}(5c+46)(2\pi)^2(W'W') \\ - \frac{1}{7680}(5c^2 - 28c + 1124)(2\pi)^4(T''T'') \\ + (2\pi)^2\,\alpha\left((T(T'T')) - 3(W'W') + (2\pi)^2\frac{c-34}{96}(T''T'')\right), \end{aligned} \quad (4.50)$$

where $\alpha$ is a central charge dependent constant which cannot be determined by the thermal one-point function $\langle \mathbf{I}_7 \rangle$. We know that this coefficient has to be non-zero since we have computed the $c$-dependent component of $\alpha$ in (4.2.2). The combination of fields in the last line of (4.50) does not contribute to the trace but does have non-trivial matrix elements on the level one subspace of the $\mathcal{W}_3$ module. Therefore, we may determine $\alpha$ by computing a trace weighted by an operator that does not act identically on the basis states of the level one subspace. The simplest such operator is

$$\mathbf{A}_3 \equiv ((TT)(u))_0 = L_0^2 - \frac{c+2}{12}L_0 + \frac{c(5c+22)}{2880} + 2\sum_{n\geq 0} L_{-n}L_n\,. \quad (4.51)$$

We now consider the trace involving the KdV charge and any of the Boussinesq charges at the first excited level:

$$\mathrm{Tr}_{\mathcal{H}^{(1)}}\Big(\mathbf{I}_n\mathbf{Q}_3\Big)\,. \tag{4.52}$$

We compute the trace in two distinct ways: directly using the operator formalism, and more indirectly, via calculating the thermal correlators involving the various terms in $J_9(u_1)$ and $(TT)(u_2)$, and by extracting the subleading coefficient in the $q$-expansion, after appropriate normal ordering. We work with the operator method first. At the first excited level, one can compute the action of $\mathbf{A}_3$ on the basis states:

$$\mathbf{A}_3 L_{-1}|\Delta_2,\Delta_3\rangle = \left(\frac{5c^2-218c+2400}{2880}-\frac{1}{12}(c-70)\Delta_2+\Delta_2^2\right)L_{-1}|\Delta_2,\Delta_3\rangle\,, \tag{4.53}$$

$$\begin{aligned}\mathbf{A}_3 W_{-1}|\Delta_2,\Delta_3\rangle &= 6\Delta_3 L_{-1}|\Delta_2,\Delta_3\rangle \\ &\quad+\left(\frac{5c^2-218c+2400}{2880}-\frac{1}{12}(c-22)\Delta_2+\Delta_2^2\right)W_{-1}|\Delta_2,\Delta_3\rangle\,.\end{aligned} \tag{4.54}$$

To calculate the action of the conserved charges on the basis, we recall the discussion from Section 3.2.2 and the derivation of the eigenstates $|e_i\rangle$ that simultaneously diagonalize all the Boussinesq charges in (4.38). It is straightforward to derive the following relations:

$$\mathbf{I}_n L_{-1}|\Delta_2,\Delta_3\rangle = \frac{(I^{(1)}_{n,1}+I^{(1)}_{n,2})}{2}L_{-1}|\Delta_2,\Delta_3\rangle+\frac{2\sqrt{6}\,(I^{(1)}_{n,2}-I^{(1)}_{n,1})}{\sqrt{-c+32\Delta_2+2}}W_{-1}|\Delta_2,\Delta_3\rangle\,, \tag{4.55}$$

$$\mathbf{I}_n W_{-1}|\Delta_2,\Delta_3\rangle = \frac{\sqrt{-c+32\Delta_2+2}}{8\sqrt{6}}\,(I^{(1)}_{n,2}-I^{(1)}_{n,1})L_{-1}|\Delta_2,\Delta_3\rangle+\frac{(I^{(1)}_{n,1}+I^{(1)}_{n,2})}{2}W_{-1}|\Delta_2,\Delta_3\rangle\,. \tag{4.56}$$

The trace of the product of these two charges can now be computed for all $n$ and we obtain

$$\begin{aligned}Tr_{\mathcal{H}^{(1)}}\Big(\mathbf{I}_n\mathbf{A}_3\Big) &= \left(\Delta_2^2+\frac{\Big(5c^2-218c+2400\Big)}{2880}-\frac{1}{12}(c-46)\Delta_2\right)(I^{(1)}_{n,1}+I^{(1)}_{n,2}) \\ &\quad+\frac{12\sqrt{6}\Delta_3}{\sqrt{-c+32\Delta_2+2}}(I^{(1)}_{n,2}-I^{(1)}_{n,1})\,.\end{aligned} \tag{4.57}$$

We note that this result does not require knowing the form of the current density $J_8$ but instead relies on our knowledge of the excited state level-one eigenvalues and eigenvectors of the Boussinesq charges from the ODE/IM correspondence. Thus, the expression in (4.57) can be thought of as the ODE/IM prediction for the level-one contribution to the trace of the product of charges in the first excited level.

We now independently calculate the two-point correlator, but now using the explicit form of the current density $J_8$ in (4.50). This is done by normal ordering appropriately the required $n$-point correlators of the stress tensor and the spin 3 field, which in turn are computed via the Zhu recursion formula. For details, we refer the reader to Appendix C.2. By comparing the result of this computation for the ansatz current given in (4.50) and (4.57), we find a perfect match with the ODE/IM result if we fix $\alpha$ to be

$$\alpha = \frac{1}{16}(3c - 14) \ . \tag{4.58}$$

We note that this is consistent with the linear-in-$c$ piece deduced from the classical limit of the current in section 4.2.2 (see the expression for $\alpha$ in (4.29)). The analysis in this section fixes $\alpha^{(0)} = -\frac{7}{8}$. In conclusion, we have completely determined the Boussinesq current $J_8$ (up to total derivatives), and it is given by:

$$\begin{aligned} J_8 = (T(T(TT))) + 18(T(WW)) + \frac{5c^2 - 276c + 628}{3840}(2\pi)^4 (T''T'') \\ + \frac{(3c - 14)}{16}(2\pi)^2 (T(T'T')) + \frac{3}{8}(2\pi)^2 (c + 30)(W'W') \ . \end{aligned} \tag{4.59}$$

### 4.5.2 Fixing the current $J_9$

Let us first discuss the case of the current $J_9$. We recall from (4.24) and (4.29) that the form of the current is given by:

$$J_9(u) = (T(T(TW)))(u) - \frac{3}{4}(2\pi)^2 (T(WT''))(u) + \frac{1}{32}(7c - 58)(2\pi)^2 (T'(T'W))(u)$$

$$+ 3(W(WW))(u) - \frac{1}{480}(2c^2 + 44c - 265)(2\pi)^4(T''W'')(u) + \left(-\frac{c}{8} + \gamma^{(0)}\right)\mathcal{F}(u)\,, \quad (4.60)$$

where we have denoted the trace-free combination appearing in (4.24) by $\mathcal{F}(u)$. The goal is to determine one $c$-independent coefficient $\gamma^{(0)}$ to completely fix the current.

We first compute the trace with the insertion of $\mathbf{A}_3$. However, we find that the two-point function

$$\left\langle \mathbf{A}_3\, \mathcal{F}(u) \right\rangle = 0\,. \quad (4.61)$$

Thus, the two-point function of $\mathbf{A}_3$ with $\mathbf{I}_8$ cannot be used to determine the coefficient $\gamma^{(0)}$. We next try with the insertion of $\mathbf{A}_5$; the results are given in Appendix C.2.2. We find that

$$\left\langle \mathbf{A}_5\, \mathcal{F}(u) \right\rangle = 216(c+2)\Delta_3\, q^{\Delta_2 - \frac{c}{24}}\,(q^2 + 58q^3 + \ldots)\,. \quad (4.62)$$

We see that there is a non-trivial contribution at $O(q^2)$. Thus, if we can compute the contribution to the trace in (4.34) from the second excited level $L = 2$ in the operator formalism, it would be possible to match with the result for the trace from the thermal two-point function and determine $\gamma^{(0)}$.

Since the expressions are rather cumbersome, we have relegated the details of the calculation to the appendices. The thermal two-point functions of $\mathbf{A}_5$ with each of the composite operators in the ansatz for $J_9$ are given in Appendix C.2.2, while the steps leading to the calculation of the eigenvalues of $\mathbf{I}_8$ at the second excited level have already been given in Section 3.2.3. The matching of the two sets of results is given in Appendix D. From the results summarized in Table D.1, we deduce that the coefficient $\gamma^{(0)}$ takes the simple value

$$\gamma^{(0)} = -\frac{3}{4}\,. \quad (4.63)$$

This fixes the form of the current density to be

$$J_9 = (T(T(TW))) + 3(W(WW)) - \frac{3}{4}(2\pi)^2(T(WT'')) + \frac{1}{32}(7c-58)(2\pi)^2(T'(T'W)) - \frac{1}{480}(2c^2+44c-265)(2\pi)^4(T''W'') \,. \quad (4.64)$$

### 4.5.3 Fixing the current $J_1 1$

We next turn to fix the undetermined coefficients appearing in the weight-11 current. We begin by recalling the form of the current that we obtained after using the low-temperature limit of the one-point function and the classical limit:

$$\begin{aligned} J_{11} = {} & (T(T(T(TW)))) + 6(W(W(WT))) + (2\pi)^2\frac{3(11c+10)}{16}(W(W'W')) \\ & + (2\pi)^4\frac{(19c^2+832c+1972)}{1536}(T''''(WT')) + (2\pi)^6\frac{(120c^3+6937c^2+24719c-15906)}{483840}(T''''W'''') \\ & + \left(\frac{7c^2}{768}+\delta_1^{(1)}c+\delta_1^{(0)}\right)\mathcal{G}_1 + \left(\frac{c^2}{192}+\delta_2^{(1)}c+\delta_2^{(0)}\right)\mathcal{G}_2 + \left(-\frac{c}{4}+\delta_3^{(0)}\right)\mathcal{G}_3 + \left(-\frac{5c}{16}+\delta_4^{(0)}\right)\mathcal{G}_4 \,. \end{aligned} \quad (4.65)$$

where

$$\begin{aligned} \mathcal{G}_1 = {} & (T''''(WT')) + (T''(WT'')) \\ \mathcal{G}_2 = {} & \left((T(WT'''')) + \frac{5}{2}(T''''(WT')) + (2\pi)^2\frac{1}{120}(5c-21)(T''''W'''')\right) \\ \mathcal{G}_3 = {} & \Big((T(T(T''W))) + \frac{21}{2}(W'(W'W)) + \frac{(2\pi)^2}{192}(19c-318)(T''''(WT')) \\ & + (2\pi)^4\frac{20c^2-11c+2318}{11520}(T''''W'''')\Big) \\ \mathcal{G}_4 = {} & \Big(T'(T'(TW))) - 3(W'(W'W)) - \frac{1}{96}(2\pi)^2(c-198)(T''''(WT')) \\ & + (2\pi)^4\frac{101c-598}{5760}(T''''W'''')\Big) \end{aligned}$$

We first compute the two-point functions of the $\mathcal{G}_i$ with the operators $\mathbf{A}_j$ to identify the level at which the eigenvalues of $\mathbf{I}_{10}$ must be calculated to determine the coefficients. The complete expressions for thermal correlators are rather cumbersome, and we present here only the leading terms in the $q$-expansion. We first list the two-point function of the $\mathcal{G}_i$ with $\mathbf{A}_5$:

$$\left\langle \mathbf{A}_5\, \mathcal{G}_i \right\rangle = \begin{cases} q^{\Delta_2-\frac{c}{24}+2}\Big(-96(2c+1)\Delta_3 - 288\Delta_2\Delta_3\Big) + \dots & i=1 \\ O(q^3) & i=2 \\ 20160\Delta_3 q^{\Delta_2-\frac{c}{24}+1}\Big((c-2)\Delta_2^2 - 32\Delta_2^3 + 216\Delta_3^2\Big) & i=3 \\ -5760 q^{\Delta_2-\frac{c}{24}+2}\Big(-51408\Delta_3^3 + 7616\Delta_3\Delta_2^3 + (1778c+5516)\Delta_3\Delta_2^2 \\ \quad + \Big(112c^2+4805c+4246\Big)\Delta_3\Delta_2 + \Big(60c^2+1715c+6202\Big)\Delta_3\Big) + \dots & \\ -2\Delta_3 q^{\Delta_2-\frac{c}{24}+1}\Big((c-2)\Delta_2^2 - 32\Delta_2^3 + 216\Delta_3^2\Big) & i=4 \\ \quad + q^{\Delta_2-\frac{c}{24}+2}\Big(1088\Delta_2^3\Delta_3 - 7344\Delta_3^3 + (254c+788)\Delta_2^2\Delta_3 \\ \qquad + \Big(16c^2+779c+1522\Big)\Delta_2\Delta_3 + \Big(6c^2+809c+1294\Big)\Delta_3\Big) + \dots & \end{cases}$$

We make a few observations about these results. Firstly, the two-point function of $\mathbf{A}_5$ with $\mathcal{G}_2$ is zero up to the second level. So, this thermal two-point function will not be able to fix $\delta_2^{(0)}$ and $\delta_2^{(1)}$. Secondly, the $O(q)$ contribution to the two-point function of $\mathbf{A}_5$ with $\mathcal{G}_3$ and $\mathcal{G}_4$ are equal up to a numerical coefficient. So, this should give a linear relation between $\delta_3^{(0)}$ and $\delta_4^{(0)}$. However, by setting different values of $(c, \Delta_2, \Delta_3)$ in the $O(q^2)$ contribution to the trace, one should be able to find linearly independent equations to fix four of the six undetermined coefficients by comparing with the results of the trace from the operator formalism. The specific numerical details of their derivation are provided in Appendix D.

Since we need more constraints to fix the constants, we move on to consider the two-point function of the $\mathcal{G}_i$ with $\mathbf{A}_3$. This is given by

$$\left\langle \mathbf{A}_3\, \mathcal{G}_i \right\rangle = \begin{cases} q^{\Delta_2-\frac{c}{24}+2}\Big(-48(c+2)\Delta_3\Big)+\dots & i=1 \\ \\ q^{\Delta_2-\frac{c}{24}+2}\Big(216(c+2)\Delta_3\Big)+\dots & i=2 \\ \\ 20160\Delta_3 q^{\Delta_2-\frac{c}{24}+1}\Big((c-2)\Delta_2^2-32\Delta_2^3+216\Delta_3^2\Big) & i=3 \\ -2880\times 14 q^{\Delta_2-\frac{c}{24}+2}\Big(320\Delta_2^3\Delta_3-2160\Delta_3^3+(62c+308)\Delta_2^2\Delta_3 & \\ \quad +\Big(4c^2+218c-92\Big)\Delta_2\Delta_3+\tfrac{1}{14}(-15c^2+1540c+6500)\Delta_3\Big)+\dots & \\ \\ -2\Delta_3 q^{\Delta_2-\frac{c}{24}+1}\Big((c-2)\Delta_2^2-32\Delta_2^3+216\Delta_3^2\Big) & i=4 \\ \quad +q^{\Delta_2-\frac{c}{24}+2}\Big(320\Delta_2^3\Delta_3-2160\Delta_3^3+(62c+308)\Delta_2^2\Delta_3 & \\ \quad\quad +\Big(4c^2+218c-92\Big)\Delta_2\Delta_3+\Big(\tfrac{3c^2}{2}+404c+1042\Big)\Delta_3\Big)+\dots & \end{cases}$$

We note that the two-point functions of $\mathbf{A}_3$ with $\mathcal{G}_1$ and $\mathcal{G}_2$ are proportional to each other by a purely numerical factor. In fact, one can check that this is true for all orders in the $q$-expansion. Interestingly, the $O(q)$ contribution to the two-point function of $\mathbf{A}_3$ with $\mathcal{G}_3$ and $\mathcal{G}_4$ are identical to those that appear in the two-point function with $\mathcal{A}_5$ (including the numerical factors). So these do not give any new constraints on the coefficients. However, the $O(q^2)$ contribution for $i=2$ does appear, and it turns out that the two-point function

at level 2 is sufficient to fix the remaining two coefficients. We find

$$\begin{aligned}
\delta_1^{(1)} &= -\frac{253}{192}, \quad \delta_1^{(0)} = -\frac{879}{64}\,, \\
\delta_2^{(1)} &= \frac{11}{96}\,, \quad \delta_2^{(0)} = \frac{43}{12}\,, \\
\delta_3^{(0)} &= -\frac{43}{8}\,, \quad \delta_4^{(0)} = -\frac{3}{2}\,.
\end{aligned} \tag{4.66}$$

The current $J_{11}$ therefore takes the form,

$$\begin{aligned}
J_{11} = {} & (T(T(T(TW)))) + 6(W(W(WT))) - \frac{5c+86}{16}(2\pi)^2(T'(T'(TW))) \\
& - \frac{(c+6)}{8}(2\pi)^2\Big(2(T(T(T''W))) - 3(W(W'W'))\Big) + \frac{\left(c^2+22c+688\right)}{192}(2\pi)^4(T(WT'''')) \\
& + \frac{\left(7c^2-1012c-10548\right)}{768}(2\pi)^4(T''(WT'')) + \frac{\left(5c^2-444c-4764\right)}{384}(2\pi)^4(T'''(WT')) \\
& + \frac{\left(30c^3+2513c^2+44188c-405612\right)}{967680}(2\pi)^6(T''''W(4)67)
\end{aligned}$$

This completes the determination of the higher currents $J_9$ and $J_{11}$ of the Boussinesq hierarchy.

# Chapter 5

# Summary and conclusion

The main result of this work is the calculation of the thermal one-point functions of all conserved charges up to weight ten, and the derivation of the higher conserved currents $J_8$,$J_9$ and $J_{11}$ in the quantum Boussinesq hierarchy. The thermal one-point functions are obtained by first calculating the thermal correlators using the Zhu recursion relations, followed by conformal normal ordering. Independently, we have also computed the eigenvalues of the conserved charges in the vacuum and first excited state of a higher spin module through the ODE/IM correspondence, and shown the mutual consistency of these two sets of results. All thermal one-point functions we have obtained are quasi-modular linear differential operators acting on the character of the higher spin module.

It is interesting to note that the information content coming from the ODE/IM correspondence is in some sense complementary to that coming from the thermal one-point functions. From the ODE/IM side, we computed the eigenvalues of the conserved charges in the vacuum, first and second excited levels of the higher spin module. Each of these calculations involved a WKB analysis of ordinary differential equations. As one moves on to higher excited states, the calculations get progressively more difficult, and explicit expressions in terms of the conformal field theory data, namely the central charge, conformal dimension and higher spin of the highest weight state, would be hard to obtain (since this

rewriting would involve solving algebraic equations of high degree). In contrast, the thermal one-point functions contain information about all the excited levels, but only about the sum of the eigenvalues at each level. These, in turn, have simple expressions that are polynomial in the conformal field theory data.

One curious feature of our analysis is the fact that there are trace-free combinations of normal ordered composite operators; that is, linear combinations of composite operators of a given conformal weight, whose one-point functions give a vanishing contribution to the trace. It would be important to better understand this feature of the one-point functions. As a consequence, the thermal one-point functions alone do not lead to a unique determination of the current densities. However, the excited state eigenvalues of the charges that we compute once again using the ODE/IM correspondence, along with the higher point thermal correlators come to the rescue in this case. By computing thermal two-point functions and extracting the subleading terms in the low-temperature limit, we were able to uniquely fix the current density $J_8$,$J_9$ and $J_{11}$, whose integral gives rise to the quantum integral of motions $\mathbf{I}_7$,$\mathbf{I}_8$ and $\mathbf{I}_{11}$ of the Boussinesq hierarchy. We believe that this method, which involves both the excited state eigenvalues and the thermal higher point functions, provides a systematic way to compute the higher conserved charges unambiguously.

It is worth pointing out that in earlier approaches to derive the commuting Boussinesq charges (see for instance [7]), the operator product expansion on the plane was used extensively. However, the lowest charges in that case turn out to be $(L_{-1}, W_{-2})$ in this case and naturally, all the higher currents listed in that reference differ from the ones we have derived, and in particular, these are not zero modes of composite operators. In the basis we have chosen however, the lowest two Cartan generators are $(L_0, W_0)$ and, as we have seen, all higher quantum Boussinesq charges are themselves zero modes of linear combinations of composite operators built out of $T$ and $W$. Thus, they act as linear operators at a given level of the higher spin module. This in turn allowed us to extract the excited state contribution to traces of the form $\langle \mathbf{I}_n (TT)_0 \rangle$ or $\langle \mathbf{I}_n (T'T')_0 \rangle$ by using oscillator methods and

the commutation relations of the $\mathcal{W}_3$ algebra. The calculations at the second excited level are done numerically, and we have provided the details in Appendix D. Having computed the trace by two completely independent means, a comparison leads to simple algebraic equations for the unknown coefficients appearing in the ansatz for the currents. Despite the numerical approach to the derivation of the equations, the result for these coefficients turns out to be simple rational numbers.

We note that we needed to use two sets of two-point functions to fix $J_{11}$ completely. We also had to compute the eigenvalues of the Boussinesq charges at the second excited level, which required some numerical interludes. The complications we have encountered in our analysis could be because we restricted our analysis to the insertion of operators built solely out of the stress tensor and its derivatives. It is likely that information regarding the undetermined constants could have been deduced more easily (perhaps using data from lower excited levels) if we inserted into the respective thermal correlators, operators involving the spin-3 current, such as $W_0$ or $(TW)_0$. It turns out, however, that taking these insertions into account necessitates the use of a generalized Zhu recursion (allowing for zero modes of multiple types of operators to appear in the recursion), which would require analysis of a different sort. In any case, it should be possible to extend our methods to compute any of the higher currents, provided we find simple enough (zero-mode) operators such that (i) they have a non-vanishing correlator with the trace-free combinations appearing in the ansatz and (ii) we can compute the eigenvalues of the conserved charges in the higher excited levels.

Finally, we note that the main check on our construction is the consistency with the spectrum of eigenvalues of the quantum Boussinesq charges in higher excited states, computed independently using the ODE/IM correspondence. It is implicit in our analysis that all the derived conserved charges are in involution since their eigenvalues match –at lower levels, partially by construction– the ones coming from the ODE side of the correspondence. Since the ODE/IM correspondence involves, on the IM side, exact Bethe Ansatz equations

and eigenvalues of commuting quantities such as quantum T and Q Baxter operators, there is no ambiguity apart from total derivatives.

It is important to note that our methods would only fix the form of the current densities. In order to obtain the operator that corresponds to the conserved charge, one would have to work out the zero modes of the composite operators appearing in the currents. This can be done by following the methods of [75]. Once the charge densities and conserved charges are determined, there are many interesting directions to explore. The calculation of higher point functions of the conserved charges is an important next step. These will be crucial to the study of the statistics of the conserved charges (see [50] for work in the quantum KdV case). In accordance with the generalized eigenstate thermal hypothesis (ETH) for conformal field theories [76, 77], in the high-temperature limit, correlators receive their dominant contributions from the states at higher levels. We expect that the higher point correlators of the conserved charges, akin to those of [50], should factor into a product of one-point functions in the large temperature limit.

One could also aim to obtain exact results for the generalized partition function in the large-$c$ limit for the higher spin conformal field theory [20, 21]. We have obtained the eigen functions and eigenvalues of the conserved charges at the first excited level. It would be a difficult but worthwhile exercise to attempt a similar calculation at the higher excited levels.

The eigenvalues obtained for $\mathcal{W}_n$ hierarchies in [72] can be employed within the analytical framework developed here to systematically construct the currents of those hierarchies.

Finally, we would like to recall that, in this work, we have expressed the thermal one-point functions of the conserved charges as quasi-modular differential operators acting on the character of the higher spin module, with insertions of powers of the zero mode $W_0$. These are known in closed form only for $n = 1, 2$ [48, 49].It should be noted that the calculations for higher values of $n$ were completed in a later paper [56] and thus constitute

an external result not contained within this thesis. The availability of this result, however, allows for the complete determination of the higher-order currents.

These are all questions we hope to address in the future.

# Appendix A

# Quasimodular Forms and Elliptic Functions

We denote the Eisenstein series as $E_{2k}(\tau)$. They are defined as [78]

$$E_{2k}(\tau) = 1 + \frac{2}{\zeta(1-2k)} \sum_{n\geq 1} \frac{n^{2k-1} q^n}{(1-q^n)^2} \quad \text{with} \quad n, k \in \mathbb{Z}^+ \,, \tag{A.1}$$

where $q = e^{2\pi i \tau}$. For the first few Eisenstein series, we present the power series expansions:

$$E_2(\tau) = 1 - 24q - 72q^2 - 96q^3 + \ldots \tag{A.2}$$

$$E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + \ldots \tag{A.3}$$

$$E_6(\tau) = 1 - 504q - 16632q^2 - 122976q^3 + \ldots \tag{A.4}$$

All $E_{2k}(\tau)$ for $k \geq 2$ transform as a weight $2k$ modular form under modular transformations, while $E_2$ is a quasi-modular form:

$$\begin{aligned} E_2\left(\frac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)^2 E_2(\tau) - \frac{6i}{\pi} c\,(c\tau+d) \ . \\ E_{2k}\left(\frac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)^{2k} E_{2k}(\tau) \ . \end{aligned} \tag{A.5}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. To compute the derivatives of Eisenstein series, we set $\partial = q\frac{q}{dq}$ and recall Ramanujan identities:

$$\begin{aligned} \partial E_2(\tau) &= \frac{1}{12}(E_2^2(\tau) - E_4(\tau)) \ , \\ \partial E_4(\tau) &= \frac{1}{3}\left(E_2(\tau)\, E_4(\tau) - E_6(\tau)\right) \ , \\ \partial E_6(\tau) &= \frac{1}{2}\left(E_2(\tau)\, E_6(\tau) - E_4^2(\tau)\right) \ . \end{aligned} \tag{A.6}$$

The Weierstrass functions are defined for $k \geq 1$, as [25]

$$\mathcal{P}_k(x,q) = \frac{(2\pi i)^k}{(k-1)!} \sum_{n\neq 0} \frac{n^{k-1}x^n}{(1-q^n)} \ . \tag{A.7}$$

These expansions converge for $|q| < |x| < 1$. We write $x = e^{2\pi i u}$, where $u$ is the coordinate on a cylinder with a period 1. The derivative with respect to $u$ is given by

$$\partial_u \mathcal{P}_k(e^{2\pi i u}, q) = k\mathcal{P}_{k+1}(e^{2\pi i u}, q) \ . \tag{A.8}$$

The relationship between $\mathcal{P}(e^{2\pi i u}, q)$ and Weierstrass functions $\rho_k(u,\tau)$ are given by [25]:

$$\begin{aligned} \mathcal{P}_1(e^{2\pi i u}, q) &= -\rho_1(u,\tau) + 2\zeta(2)E_2(\tau)u - i\pi \ , \\ \mathcal{P}_2(e^{2\pi i u}, q) &= \rho_2(u,\tau) + 2\zeta(2)E_2(\tau) \ , \\ \mathcal{P}_k(e^{2\pi i u}, q) &= (-1)^k \rho_k(u,\tau) \quad \text{for } k \geq 2 \ , \end{aligned} \tag{A.9}$$

where $q = e^{2\pi i\tau}$ and $E_{2k}(\tau)$ is the Eisenstein series. The Laurent expansion of $\rho_k(u,\tau)$ near small $u$ is given by

$$\rho_k(u,\tau) = \frac{1}{u^k} + (-1)^k \sum_{n=1}^{\infty} \binom{2n+1}{k-1} 2\zeta(2n+2) E_{2n+2}(\tau) u^{2n+2-k} \,, \tag{A.10}$$

with $\zeta(n)$ the Rienmann zeta function. Note that the above sum vanishes for $2n+2-k < 0$.

# Appendix B

# Eigenvalues of Quantum Boussinesq Charges

## B.1 Vacuum Eigenvalues of Quantum Boussinesq Charges

In this section, we list the eigenvalues of the quantum Boussinesq charges in the highest weight vector of the higher spin module labeled by $(\Delta_2, \Delta_3)$. The charges $I_{3n}$ vanish, and the rest can be related to the contour integrals in (3.18) by the formula

$$\widehat{I_n} = c_n \, e^{-\frac{n\pi i}{3}} (1+M)^{\frac{n+1}{2}} \frac{\Gamma(\frac{n}{3M} + \frac{n}{3})\Gamma\left(-\frac{n}{3}\right)}{\Gamma(\frac{n}{3M})} \, I_n \, . \tag{B.1}$$

For the first twelve charges, the $c_n$ are given by the list $\{1, 0, 3, 9, 0, 108\}$ for odd $n$, and the list $\{3, 9, 0, 135, 567, 0\}$ for even $n$. The eigenvalues are given by

$$I_1 = \Delta_2 - \frac{c}{24} \, , \tag{B.2a}$$

$$I_2 = \Delta_3 \, , \tag{B.2b}$$

$$I_4 = \Delta_2\Delta_3 - \frac{c+6}{24}\Delta_3 \, , \tag{B.2c}$$

$$I_5 = \mathcal{A}_2^3 + 9\mathcal{A}_3^2 - \left(\frac{c+8}{8}\right)\mathcal{A}_2^2 + \frac{1}{192}(c+2)(c+15)\mathcal{A}_2 - \frac{c(c+23)(7c+30)}{96768}\,, \tag{B.2d}$$

$$\begin{aligned} I_7 = \mathcal{A}_2^4 + 18\mathcal{A}_3^2\mathcal{A}_2 - \frac{3}{4}(c+12)\mathcal{A}_3^2 \\ - \frac{c+12}{6}\mathcal{A}_2^3 + \frac{1}{480}\left(5c^2+127c+594\right)\mathcal{A}_2^2 \\ - \frac{(c+2)\left(10c^2+387c+3150\right)}{34560}\mathcal{A}_2 + \frac{c(5c+186)\left(2c^2+43c+150\right)}{3317760}\,, \end{aligned} \tag{B.2e}$$

$$\begin{aligned} I_8 = \mathcal{A}_2^3\mathcal{A}_3 + 3\mathcal{A}_3^3 - \frac{(c+14)}{8}\mathcal{A}_2^2\mathcal{A}_3 + \frac{(c(5c+148)+900)}{960}\mathcal{A}_2\mathcal{A}_3 \\ - \frac{(c+30)(c(35c+604)+2940)}{483840}\mathcal{A}_3\,, \end{aligned} \tag{B.2f}$$

$$\begin{aligned} I_{10} = \mathcal{A}_2^4\mathcal{A}_3 + 6\mathcal{A}_2\mathcal{A}_3^3 - \frac{1}{6}(c+18)\mathcal{A}_2^3\mathcal{A}_3 - \frac{1}{4}(c+18)\mathcal{A}_3^3 \\ + \frac{1}{96}(c+14)(c+24)\mathcal{A}_3\mathcal{A}_2^2 - \frac{(c+30)\left(7c^2+215c+1422\right)}{24192}\mathcal{A}_2\mathcal{A}_3 \\ + \frac{(c+30)\left(7c^3+401c^2+5844c+26460\right)}{2322432}\mathcal{A}_3\,, \end{aligned} \tag{B.2g}$$

$$\begin{aligned} I_{11} = \mathcal{A}_2^6 + \frac{135}{2}\mathcal{A}_3^4 + 45\mathcal{A}_3^2\mathcal{A}_2^3 - \frac{1}{8}(c+20)\left(2\mathcal{A}_2^5 + 45\mathcal{A}_3^2\mathcal{A}_2^2\right) \\ + \frac{\mathcal{A}_2}{192}\left(\mathcal{A}_2^3 + 9\mathcal{A}_3^2\right)\left(5c^2+211c+2154\right) \\ - \frac{35c^3+2353c^2+50514c+383400}{10752}\mathcal{A}_3^2 - \frac{\left(140c^3+9445c^2+202320c+1233036\right)}{96768}\mathcal{A}_2^3 \\ + \frac{350c^4+34061c^3+1154072c^2+14378340c+50425200}{7741440}\mathcal{A}_2^2 \\ - \frac{(c+2)\left(140c^4+18577c^3+865560c^2+15423300c+87318000\right)}{185794560}\mathcal{A}_2 \\ + \frac{c(7c+470)\left(1820c^4+216001c^3+6997896c^2+78367140c+227026800\right)}{2434651914240}\,. \end{aligned} \tag{B.2h}$$

The constant prefactors are chosen in such a way that, in terms of the conformal field theory data, the eigenvalues of the even-dimension charges are normalized to be written as $\mathcal{A}_2^n + \ldots$, while the odd-dimension charges are normalized to be written as $\mathcal{A}_2^{n-1}\mathcal{A}_3 + \ldots$. This ensures that the quantum Boussinesq charges go over to the charges of the classical

Boussinesq hierarchy in the large-$c$ limit.

# Appendix C

# List of all thermal correlators

## C.1 Thermal Correlators

In this section, we illustrate how the Zhu recursion works in practice by computing some two and three-point correlators involving the energy-momentum tensor and the spin-3 current. The thermal correlators involving just the energy-momentum tensors have been worked out in detail in [12], so after reviewing the simplest two-point correlator of the stress tensor, we shall mostly present the results for the new correlators involving the spin-3 field.

**$\langle \mathbf{T(u_1)T(u_2)} \rangle$:** We start with the two-point function, which is $\langle T(u_1)T(u_2)\rangle$. For this purpose we can put $l = 0$ and $a^1 = a^2 = \tilde{\omega}$ in (2.76) and we obtain

$$\begin{aligned} F((\tilde{\omega}, z_1), (\tilde{\omega}, z_2); \tau) &= F(\tilde{\omega}_{(0)}; (\tilde{\omega}, z_2); \tau) + \sum_{m=0}^{\infty} g^0_{m+1}(z_{21}) F((d^0[m]\tilde{\omega}, z_2); \tau) \\ &= F(\tilde{\omega}_{(0)}; (\tilde{\omega}, z_2); \tau) + \sum_{m=0}^{\infty} g^0_{m+1}(z_{21}) F((\tilde{\omega}[m]\tilde{\omega}, z_2); \tau) \\ &= F((\tilde{\omega}_{(0)})^2; \tau) + \sum_{m=0}^{\infty} g^0_{m+1}(z_{21}) F((\tilde{\omega}[m]\tilde{\omega})_0; \tau) \, . \end{aligned} \tag{C.1}$$

Using the fact that the zero mode of $\tilde{\omega}$ is $L_0 - \frac{c}{24}$, we find that $F((\tilde{\omega}_{(0)})^n;\tau) = \partial^n\chi(\tau)$ where $\partial = q\frac{\partial}{\partial q}$ and $\chi(\tau)$ is the reduced character defined in (2.58). Since $\tilde{\omega}[0] \propto L_{[-1]}$, the action of $\tilde{\omega}[0]$ on a state corresponds to the action of a derivative with respect to the cylinder coordinate; thus the zero mode for such a state is vanishing. Thus, the $m = 0$ term in (C.1) does not contribute to the correlator. To compute the contribution of the $m > 0$ terms in (C.1), we use the square modes listed in (2.74). By using all of the above arguments, we find that

$$\begin{aligned} F((\tilde{\omega},z_1),(\tilde{\omega},z_2);\tau) &= F((\tilde{\omega}_{(0)})^2;\tau) + \frac{2}{(2\pi i)^2}\mathcal{P}_2(z_{21})F((\tilde{\omega}_{(0)});\tau) + \frac{c}{2(2\pi i)^4}\mathcal{P}_4(z_{21})F(\tau) \\ &= \partial^2\chi(\tau) + \frac{2}{(2\pi i)^2}\mathcal{P}_2(z_{21})\partial\chi(\tau) + \frac{c}{2(2\pi i)^4}\mathcal{P}_4(z_{21})\chi(\tau)\,. \end{aligned} \tag{C.2}$$

Next, we perform the normal ordering using the definition (see (2.61)):

$$\langle(TT)(u_1)\rangle = \frac{1}{2\pi i}\oint_{u_1}\frac{du_2}{u_2-u_1}\langle T(u_1)T(u_2)\rangle\,. \tag{C.3}$$

In order to do so, we use expansions of the Weierstrass functions (see (A.9) and (A.10))

$$\begin{aligned} \mathcal{P}_2(e^{2\pi i(u_2-u_1)},q) =& \frac{1}{(u_2-u_1)^2} + 2\zeta(2)E_2(\tau) + 6\zeta(4)E_4(\tau)(u_2-u_1)^2 + \ldots \\ \mathcal{P}_4((e^{2\pi i(u_2-u_1)},q) =& \frac{1}{(u_2-u_1)^4} + 2\zeta(4)E_4(\tau) + 20\zeta(6)E_6(\tau)(u_2-u_1)^2 + \ldots \end{aligned} \tag{C.4}$$

Only the third term in each of the expansions contributes to the integral over $u_2$ in (C.3), and substituting the values $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$, we obtain

$$\langle(TT(u_1))\rangle = \partial^2\chi(\tau) - \frac{1}{6}E_2(\tau)\partial\chi(\tau) + \frac{c}{1440}E_4(\tau)\chi(\tau)\,. \tag{C.5}$$

**$\langle\mathbf{T(u_1)W(u_2)}\rangle$:** For this correlator we set $l = 0$, $a^1 = \tilde{\omega}$ and $a^2 = W$ in (2.76) and we obtain

$$F((\tilde{\omega},z_1),(W,z_2);\tau) = F((\tilde{\omega})_0(W)_0;\tau) + \sum_{m=0}^{\infty} g^0_{m+1}(z_{21})F((\tilde{\omega}[m]W)_0;\tau)\,. \tag{C.6}$$

Using the square modes of $\tilde{\omega}$ given in (2.74) we can show that $\tilde{\omega}[m]W = 0$ for $m \geq 2$. The trace of the insertion of the zero mode $W_0$ over the restricted Verma module was computed in [48]. We use that result to write the first term on the right-hand side of (C.6) as

$$F((\tilde{\omega})_0(W)_0;\tau) = \Delta_3 q\frac{\partial}{\partial q}\chi(\tau)\,. \tag{C.7}$$

Combining the aforementioned arguments gives

$$F((\tilde{\omega},z_1),(W,z_2);\tau) = \Delta_3\partial\chi(\tau) + \frac{3}{(2\pi i)^2}\mathcal{P}_2(z_{21})\Delta_3\chi(\tau)\,. \tag{C.8}$$

Normal ordering is done as before, and we obtain

$$\langle(TW(u_1))\rangle = \Delta_3\Big(\partial\chi(\tau) - \frac{1}{4}E_2(\tau)\chi(\tau)\Big)\,. \tag{C.9}$$

**$\langle\mathbf{T(u_1)T(u_2)T(u_3)}\rangle$:** We now compute a thermal correlator involving three energy-momentum tensors. We set $l = 0$ and $a^1 = a^2 = a^3 = \tilde{\omega}$ in (2.76) and we obtain

$$\begin{aligned}
F((\tilde{\omega},z_1),(\tilde{\omega},z_2),(\tilde{\omega},z_3);\tau) &= \partial^3\chi(\tau) + \frac{2}{(2\pi i)^2}\Big\{\mathcal{P}_2(z_{21}) + \mathcal{P}_2(z_{31}) + \mathcal{P}_2(z_{32})\Big\}\partial^2\chi(\tau)\\
&+ \frac{1}{(2\pi i)^4}\Big\{4\mathcal{P}_2(z_{32})\Big(\mathcal{P}_2(z_{31}) + \mathcal{P}_2(z_{21})\Big) + 2(2\pi i)^2\partial\mathcal{P}_2(z_{32})\\
&\quad + 4\mathcal{P}_1(z_{31})\mathcal{P}_3(z_{32}) - 4\mathcal{P}_1(z_{21})\mathcal{P}_3(z_{32})\\
&\quad + \frac{c}{2}\Big(\mathcal{P}_4(z_{21}) + \mathcal{P}_4(z_{31}) + \mathcal{P}_4(z_{32})\Big)\Big\}\partial\chi(\tau)\\
&+ \frac{c}{(2\pi i)^6}\Big\{\frac{1}{2}(2\pi i)^2\partial\mathcal{P}_4(z_{32}) + \mathcal{P}_2(z_{31})\mathcal{P}_4(z_{32}) + \mathcal{P}_2(z_{21})\mathcal{P}_4(z_{32})\\
&\quad + 2\mathcal{P}_1(z_{31})\mathcal{P}_5(z_{32}) - 2\mathcal{P}_1(z_{21})\mathcal{P}_5(z_{32})\Big\}\chi(\tau)\,.
\end{aligned} \tag{C.10}$$

The normal ordering for three-point function is defined as

$$\langle(T(TT))(u_1)\rangle = \frac{1}{(2\pi i)^2}\oint_{u_1}\frac{du_2}{u_2-u_1}\oint_{u_2}\frac{du_3}{u_3-u_2}\langle T(u_1)T(u_2)T(u_3)\rangle\,. \tag{C.11}$$

Note that here we have coupled Weierstrass functions in the integrand. To do the integration sequentially, we use the method discussed in [12], where we Taylor expand $\mathcal{P}(z_{ij})$ using (A.8) about $z_{i-1} - z_j = 0$ if $j \neq i-1$. Then we use the expansion of the Weierstrass function in terms Eisenstein series given in (A.9) and (A.10) to do the integral, where again only the simple pole will contribute.

After performing the normal ordering, we get

$$\begin{aligned}\langle (T(TT))(u_1)\rangle = \partial^3\chi(\tau) - \frac{1}{2}E_2(\tau)\partial^2\chi(\tau) \\ + \Big(\frac{1}{24}E_2^2(\tau) + \frac{1}{40}E_4(\tau) + \frac{c}{480}E_4(\tau)\Big)\partial\chi(\tau) - \frac{c}{3024}E_6(\tau)\chi(\tau)\,. \end{aligned} \tag{C.12}$$

We take the low-temperature limit $q \to 0$ and find

$$\begin{aligned}\lim_{q\to 0} q^{\frac{c}{24}-\Delta_2}\langle (T(TT))\rangle = \Delta_2^3 - \frac{1}{8}(c+4)\Delta_2^2 \\ + \frac{1}{960}(c+2)(5c+32)\Delta_2 - \frac{c\left(35c^2+462c+1504\right)}{483840}\,. \end{aligned} \tag{C.13}$$

**$\langle \mathbf{W}(\mathbf{u_1})\mathbf{W}(\mathbf{u_2})\rangle$:** For this case we set $l = 0$, $a^1 = W$ and $a^2 = W$ in (2.76) and we have

$$\begin{aligned} F((W,z_1),(W,z_2);\tau) &= F(W_0^2;\tau) + \sum_{m=0}^{\infty} g^0_{m+1}(z_{21})F((W[m]W)_0;\tau) \\ &= F(W_0^2;\tau) + g_2^0(z_{21})F((W[1]W)_0;\tau) \\ &\quad + \frac{2}{3b^2(2\pi i)^4}g_4^0(z_{21})F(\tilde{\omega}_{(0)};\tau) + \frac{c}{9b^2(2\pi i)^6}g_6^0(z_{21})\chi(\tau)\,. \end{aligned} \tag{C.14}$$

We use the square modes of $W$ given in (2.69) to compute the zero mode of $W[m]W$. The calculation of the zero mode of $W[1]W$ is straightforward when $m \neq 1$. When $m = 1$, we have

$$W[1]W_{-3}\Omega = (2\pi i)^{-2}\Big[\Big(\frac{1}{5b^2} - \frac{2}{5}\Big)L_{-4}\Omega + \frac{1}{2b^2}L_{-3}\Omega + \frac{2}{9b^2}L_{-2}\Omega + \frac{c}{270b^2}\Omega + \frac{2}{3}L_{-2}^2\Omega\Big]\,. \tag{C.15}$$

For all but the last term, the zero modes are easy to compute, as

$$\left(L_{-k}\Omega\right)_0 = (-1)^k(k-1)L_0 \ . \tag{C.16}$$

To compute the zero mode of $(L_{-2})^2\Omega$, we use the fact that it is identical to the zero mode of $(TT)(z)$ on the plane, which we computing using the formula for the $n$th mode of $(AB)(z)$ in terms of the modes of $A(z)$ snd $B(z)$ [4] :

$$(AB)_m = \sum_{n\leq -h_A} A_n B_{m-n} + \sum_{n>-h_A} B_{m-n}A_n \tag{C.17}$$

We get

$$\left(L_{-2}^2\,\Omega\right)_0 = (TT)_0 = L_0^2 + 2L_0 + 2\sum_{n\geq 0} L_{-n}L_n \ . \tag{C.18}$$

Collecting all the zero modes, and substituting into (C.14), we get

$$\begin{aligned} F((W,z_1),(W,z_2);\tau) = F(W_0^2;\tau) &+ \frac{1}{(2\pi i)^2}\mathcal{P}_2(z_{21})\Big(\frac{2}{3}\partial^2\chi(\tau) - \frac{1}{9}\partial\chi(\tau) + \frac{c}{2160}\chi(\tau)\Big) \\ &+ \frac{1}{(2\pi i)^2}\mathcal{P}_2(z_{21})\left\langle\frac{4}{3}\sum_{n\geq 0} L_{-n}L_n\right\rangle \\ &+ \frac{2}{3b^2(2\pi i)^4}\mathcal{P}_4(z_{21})\partial\chi(\tau) + \frac{c}{9b^2(2\pi i)^6}\mathcal{P}_6(z_{21})\chi(\tau) \ . \end{aligned} \tag{C.19}$$

Finally we apply the technique described in [12] to compute $\left\langle\sum_{n\geq 0} L_{-n}L_n\right\rangle$ by moving $L_{-n}$ through the trace, and obtain

$$\left\langle\sum_{n\geq 0} L_{-n}L_n\right\rangle = \frac{1-E_2(\tau)}{12}\partial\chi(\tau) + c\frac{E_4(\tau)-1}{2880}\chi(\tau) \ . \tag{C.20}$$

Substituting this into the correlator we have

$$\begin{aligned} F((W,z_1),(W,z_2);\tau) = F(W_0^2;\tau) &+ \frac{1}{(2\pi i)^2}\mathcal{P}_2(z_{21})\Big(\frac{2}{3}\partial^2\chi(\tau) - \frac{1}{9}\partial\chi(\tau) + \frac{c}{2160}\chi(\tau)\Big) \\ &+ \frac{1}{(2\pi i)^2}\mathcal{P}_2(z_{21})\Big(\frac{1-E_2(\tau)}{12}\partial\chi(\tau) + c\frac{E_4(\tau)-1}{2880}\chi(\tau)\Big) \\ &+ \frac{2}{3b^2(2\pi i)^4}\mathcal{P}_4(z_{21})\partial\chi(\tau) + \frac{c}{9b^2(2\pi i)^6}\mathcal{P}_6(z_{21})\chi(\tau) \ . \end{aligned} \tag{C.21}$$

We normal order the above correlator and we get

$$\langle(WW(u_1))\rangle = \langle(W_0)^2\rangle - \frac{1}{18}E_2(\tau)\partial^2\chi(\tau) + \frac{1}{17280}(160E_2^2 + (22+5c)E_4(\tau))\partial\chi(\tau) \\ - \frac{c}{4354560}(168E_2(\tau)E_4(\tau) + (22+5c)E_6(\tau))\chi(\tau)\,. \quad \text{(C.22)}$$

The thermal one-point function of $W_0^2$ and its $q$-expansion has been listed previously in (2.84). Substituting this into the normal ordered one-point function, and taking the low-temperature limit, we get

$$\lim_{q\to 0} q^{\frac{c}{24}-\Delta_2}\langle(WW)(u_1)\rangle = \Delta_3^2 - \frac{1}{18}\Delta_2^2 + \Big(\frac{17c}{3456} + \frac{91}{8640}\Big)\Delta_2 - \frac{191c^2}{1741824} - \frac{2101c}{4354560}\,. \quad \text{(C.23)}$$

$\langle \mathbf{T'(u_1)T'(u_2)}\rangle$ : To compute the thermal correlator involving the derivatives of $T$, we us the fact that $\tilde{\omega}[0]$ operating on $T$ is $T'$. We put $l = 0$ and $a^1 = a^2 = \tilde{\omega}[0]\tilde{\omega}$ in (2.76) and we obtain

$$F((\tilde{\omega}[0]\tilde{\omega}, z_1), (\tilde{\omega}[0]\tilde{\omega}, z_2); \tau) = -\frac{12}{(2\pi i)^6}\mathcal{P}_4(z_{21})\partial\chi(\tau) - \frac{10c}{(2\pi i)^8}\mathcal{P}_6(z_{21})\chi(\tau)\,. \quad \text{(C.24)}$$

Here we use the fact that the zero mode of $\tilde{\omega}[0]\tilde{\omega}$ is zero. We perform the normal ordering and get

$$(2\pi)^2\langle(T'T')(u_1)\rangle = \frac{E_4(\tau)}{60}\partial\chi(\tau) - \frac{cE_6(\tau)}{3024}\chi(\tau)\,. \quad \text{(C.25)}$$

In the low-temperature limit, we obtain

$$\lim_{q\to 0}\; q^{\frac{c}{24}-\Delta_2}\;(2\pi)^2\langle(T'T')(u_1)\rangle = \frac{\Delta_2}{60} - \frac{31c}{30240}\,. \quad \text{(C.26)}$$

### C.1.1 Thermal One-Point Function of Composite Operators

#### Weight Eight

Here we present the thermal one-point functions of composite objects which are relevant to compute the thermal one-point function of $I_7$.

$$
\begin{aligned}
\langle (T(T(TT))) \rangle =& \partial^4\chi(\tau) - E_2(\tau)\partial^3\chi(\tau) + \frac{1}{240}((c+32)E_4(\tau) + 60E_2^2(\tau))\partial^2\chi(\tau) \\
&- \frac{1}{4320}(3(c+32)E_4(\tau)E_2(\tau) + 10(c+6)E_6(\tau) + 60E_2^3(\tau))\partial\chi(\tau) \\
&+ \frac{1}{4838400}c\left((7c+1024)E_4^2(\tau) + 10(7c+32)E_8(\tau)\right)\chi(\tau) \,, \\
\langle (T(WW)) \rangle =& -\frac{1}{18}E_2(\tau)\partial^3\chi(\tau) + \frac{1}{17280}\Big((5c+214)E_4(\tau) + 560E_2^2(\tau)\Big)\partial^2\chi(\tau) \\
&- \frac{1}{4354560}\Big(42E_2(\tau)(320E_2^2(\tau) + 3(98+3c)E_4(\tau)) \\
&+ (150+c)(22+5c)E_6(\tau)\Big)\partial\chi(\tau) \\
&+ \frac{c}{174182400}\Big(560E_2^2(\tau)E_4(\tau) + 4(446+25c)E_4^2(\tau) + 2240E_2(\tau)E_6(\tau) \\
&+ 15(22+5c)E_8(\tau)\Big)\chi(\tau) + \Big(q\frac{\partial}{\partial q} - \frac{1}{2}E_2(\tau)\Big)\langle W_0^2 \rangle \,, \\
(2\pi)^2\langle (T(T'T')) \rangle =& \frac{1}{60}E_4(\tau)\partial^2\chi(\tau) - \frac{1}{15120}\Big(42E_2(\tau)E_4(\tau) + 5(14+c)E_6(\tau)\Big)\partial\chi(\tau) \\
&+ \frac{c}{604800}(100E_4^2(\tau) - 9E_8(\tau))\chi(\tau) \,, \\
(2\pi)^4\langle (W'W') \rangle =& \frac{1}{180}E_4(\tau)\partial^2\chi(\tau) - \frac{1}{181440}\Big(168E_2(\tau)E_4(\tau) + 5(22+5c)E_6(\tau)\Big)\partial\chi(\tau) \\
&+ \frac{c}{4147200}\Big(16E_4^2(\tau) + (22+5c)E_8(\tau)\Big) \,, \\
(2\pi)^4\langle (T''T'') \rangle =& -\frac{1}{126}E_6(\tau)\partial\chi(\tau) + \frac{c}{2880}E_8(\tau)\chi(\tau) \,.
\end{aligned}
$$

Using the above results, we can show

$$
\left\langle (T(T'T')) - 3(W'W') + (2\pi)^2\frac{c-34}{96}(T''T'') \right\rangle = 0 \,. \tag{C.27}
$$

**Weight Nine**

Next, we present the thermal one-point functions of composite objects which are relevant to compute the thermal one-point function of $I_8$.

$$\begin{aligned}
\langle(T(T(TW)))\rangle =&\partial^3\langle W_0\rangle - \frac{5}{4}E_2(\tau)\partial^2\langle W_0\rangle + \frac{1}{480}(200E_2^2(\tau) + (c+108)E_4(\tau))\partial\langle W_0\rangle \\
&-\frac{1}{120960}\left(4200E_2^3(\tau) + 63(108+c)E_2(\tau)E_4(\tau) + 130(30+c)E_6(\tau)\right)\langle W_0\rangle\,, \\
\langle(W(WW))\rangle =&\langle W_0^3\rangle - \frac{1}{6}E_2(\tau)\partial^2\langle W_0\rangle + \frac{1}{5760}(400E_2^2(\tau) + (166+5c)E_4(\tau))\partial\langle W_0\rangle \\
&-\frac{1}{1451520}(5040E_2^3(\tau) + 21(978+23c)E_2(\tau)E_4(\tau) \\
&+ (3930 + c(217+5c))E_6(\tau))\langle W_0\rangle\,, \\
(2\pi)^2\langle(T(WT''))\rangle =&-\frac{1}{24}E_4(\tau)\partial\langle W_0\rangle + \frac{1}{6048}(63E_2(\tau)E_4(\tau) + 2(30+c)E_6(\tau))\langle W_0\rangle\,, \\
(2\pi)^2\langle(T'(T'W))(u)\rangle =&\frac{E_4}{60}\partial\langle W_0\rangle - \left(\frac{(c+27)E_6}{3024} + \frac{E_2E_4}{240}\right)\langle W_0\rangle\,, \\
(2\pi)^4\langle(T''W'')\rangle =&-\frac{1}{84}E_6(\tau)\langle W_0\rangle\,.
\end{aligned}$$

Using the above results, we can show

$$\left\langle(T(WT''))(u) + \frac{5}{2}(T'(T'W))(u) - \frac{1}{24}(c+25)(2\pi)^2(T''W'')(u)\right\rangle = 0\,. \tag{C.28}$$

**Weight Eleven**

Next, we present the thermal one-point functions of composite objects which are relevant to compute the thermal one-point function of $I_{10}$.

$$\begin{aligned}
\langle(T(T(T(TW))))\rangle =&\partial^4\langle W_0\rangle - 2E_2(\tau)\partial^3\langle W_0\rangle + \Big(\frac{1}{240}(c+152)E_4(\tau) + \frac{5}{4}E_2^2(\tau)\Big)\partial^2\langle W_0\rangle \\
&-\Big(\frac{1}{360}(c+152)E_4(\tau)E_2(\tau) + \frac{1}{378}(2c+81)E_6(\tau) + \frac{5}{18}E_2^3(\tau)\Big)\partial\langle W_0\rangle \\
&+\frac{1}{4838400}\Big((7c^2+4882c+175356)E_4^2(\tau) + 2(35c^2+3439c+36978)E_8(\tau)
\end{aligned}$$

$$
\begin{aligned}
&+1680(c+152)E_4(\tau)E_2^2(\tau)+3200(2c+81)E_6(\tau)E_2(\tau)+84000E_2^4(\tau)\Big)\langle W_0\rangle\,,\\
\langle(W(W(WT)))\rangle =&\partial\langle W_0^3\rangle-\frac{3}{4}E_2(\tau)\langle W_0^3\rangle-\frac{1}{6}E_2(\tau)\partial^3\langle W_0\rangle+\frac{1}{5760}\Big((5c+502)E_4(\tau)\\
&+1040E_2^2(\tau)\Big)\partial^2\langle W_0\rangle-\frac{1}{1451520}\Big(\big(5c^2+1717c+42210\big)E_6(\tau)\\
&+1008(c+97)E_4(\tau)E_2(\tau)+63840E_2^3(\tau)\Big)\partial\langle W_0\rangle+\frac{1}{58060800}\times\\
&\Big(10(5c^2+1941c+55650)E_6(\tau)E_2(\tau)+(235c^2+16481c+371826)E_4(\tau)^2\\
&+(185c^2+5989c+51954)E_8(\tau)+280(23c+1926)E_4(\tau)E_2(\tau)^2\\
&+100800E_2(\tau)^4\Big)\langle W_0\rangle\,,\\
(2\pi)^6\langle(T'''W''')\rangle =&\frac{1}{80}E_8(\tau)\langle W_0\rangle\,,\\
(2\pi)^4\langle(T'''(WT'))\rangle =&\frac{1}{126}E_6(\tau)\partial\langle W_0\rangle-\frac{1}{20160}\Big(7(c+9)E_8(\tau)+40E_2(\tau)E_6(\tau)\Big)\langle W_0\rangle\,,\\
(2\pi)^2\langle(W'(W'W))\rangle =&\frac{1}{180}E_4(\tau)\partial^2\langle W_0\rangle-\Big(\frac{1}{270}E_2(\tau)E_4(\tau)+\frac{142+5c}{36288}E_6(\tau)\Big)\partial\langle W_0\rangle\\
&+\frac{1}{29030400}\Big(13440E_2(\tau)^2E_4(\tau)+7(4368+254c+5c^2)E_4(\tau)^2\\
&+200(142+5c)E_2(\tau)E_6(\tau)\Big)\langle W_0\rangle\,.
\end{aligned}
$$

The thermal one-point functions of the remaining four composite operators can be inferred from the four trace-free relations, which we list:

$$
\begin{aligned}
\mathcal{G}_1 =&(T'''(WT'))+(T''(WT'')),\\
\mathcal{G}_2 =&\left((T(WT''''))+\frac{5}{2}(T'''(WT'))+(2\pi)^2\frac{1}{120}(5c-21)(T'''W''')\right),\\
\mathcal{G}_3 =&\Big((T(T(T''W)))+\frac{21}{2}(W'(W'W))+\frac{(2\pi)^2}{192}(19c-318)(T'''(WT'))\\
&+(2\pi)^4\frac{20c^2-11c+2318}{11520}(T'''W''')\Big),\\
\mathcal{G}_4 =&\Big(T'(T'(TW)))-3(W'(W'W))-\frac{1}{96}(2\pi)^2(c-198)(T'''(WT'))\\
&+(2\pi)^4\frac{101c-598}{5760}(T'''W''')\Big),
\end{aligned}
$$

where

$$\langle \mathcal{G}_i \rangle = 0$$

## C.2 A Two Point Correlator

In this section, we show how to calculate the two point correlator

$$\left\langle (TT)(u_1) J_n(v_1) \right\rangle , \tag{C.29}$$

for $n = 8$ using the Zhu recursion relations. The procedure described here is fairly general and can be used to compute two-point functions of any two composite operators. To illustrate the ideas involved let us discuss the thermal two point function $\langle (TT)(u_1)(T(WW))(v_1)\rangle$. The other correlators involving the composite operators in $J_8$ can be computed similarly.

The first step is to use the Zhu recursion to compute

$$\left\langle T(u_1)T(u_2)T(v_1)W(v_2)W(v_3)\right\rangle , \tag{C.30}$$

and then perform normal ordering in a particular manner. Using the arguments used in (C.1) we have that

$$\begin{aligned}\langle (TT)(u_1)(T(WW))(v_1)\rangle = \frac{1}{(2\pi i)^3} \oint_{u_1} \frac{du_2}{u_2 - u_1} \oint_{v_1} \frac{dv_2}{v_2 - v_1} \\ \oint_{v_3} \frac{dv_3}{v_3 - v_2} \langle T(u_1)T(u_2)T(v_1)W(v_2)W(v_3)\rangle .\end{aligned} \tag{C.31}$$

This two-point function depends only on the difference in the positions of the insertions, and expanding it in these variables and discarding all but just the constant term gives us the zero mode. In this computation, we will encounter the product of two Weierstrass functions with the same argument, for which we are only interested in the constant piece.

The constant piece is given by

$$\mathcal{P}_{m_1}(x)\mathcal{P}_{m_2}(x)|_{x\to 0} = \frac{(2\pi i)^{m_1+m_2}(-1)^{m_2}}{(m_1-1)!(m_2-1)!}\zeta(3-m_1-m_2)\partial E_{m_1+m_2-2}\,. \tag{C.32}$$

We once again refer the reader to Appendix G of [12] for further details.

### C.2.1 Two-point functions involving weight eight composites

In this section, we provide the thermal two-point functions of composite objects, which are relevant to compute the thermal two-point function of $\langle \mathbf{A}_3 I_7\rangle$.

$$\begin{aligned}
(2\pi)^4\left\langle \mathbf{A}_3\int du\,(T''T'')(u)\right\rangle &= \frac{c}{958003200}\Big(3(79200+16877c)E_4(\tau)^3\\
&-800(484+105c)E_2(\tau)E_4(\tau)E_6(\tau)+800(187+42c)E_6(\tau)^2\Big)\chi(q)\\
&+\frac{\Big(3(752+273c)E_2(\tau)E_4(\tau)^2+1480E_2(\tau)^2E_6(\tau)-2(1868+421c)E_4(\tau)E_6(\tau)\Big)\partial\chi(q)}{362880}\\
&+\frac{\Big(3(480+7c)E_4(\tau)^2-1360E_2(\tau)E_6(\tau)\Big)\partial^2\chi(q)}{60480}-\frac{1}{126}E_6(\tau)\partial^3\chi(q)
\end{aligned}$$

$$\begin{aligned}
(2\pi)^2\left\langle \mathbf{A}_3\int du\,(W'W')(u)\right\rangle &= \frac{c}{1379524608000}\Big(-2661120E_2(\tau)^2E_4(\tau)^2\\
&+3(3432000+405326c+385c^2)E_4(\tau)^3\quad-1280(8657+1345c)E_2(\tau)E_4(\tau)E_6(\tau)\\
&+160(21538+3215c)E_6(\tau)^2\Big)\chi(q)+\frac{1}{522547200}\Big(362880E_2(\tau)^3E_4(\tau)\\
&+3(-36704+19590c+245c^2)E_2(\tau)E_4(\tau)^2+40(2006+325c)E_2(\tau)^2E_6(\tau)\\
&-2(166504+36562c+445c^2)E_4(\tau)E_6(\tau)\Big)\partial\chi(q)+\frac{1}{87091200}\Big(-443520E_2(\tau)^2E_4(\tau)\\
&+3(150720+4378c+35c^2)E_4(\tau)^2-400(-34+25c)E_2(\tau)E_6(\tau)\Big)\partial^2\chi(q)\\
&+\frac{1}{181440}\Big(336E_2(\tau)E_4(\tau)-(782+25c)E_6(\tau)\Big)\partial^3\chi(q)+\frac{1}{180}E_4(\tau)\partial^4\chi(q)\\
&+\frac{1}{20}\Big(E_2(\tau)E_4(\tau)-E_6(\tau)\Big)\langle W_0^2\rangle
\end{aligned}$$

$$
\begin{aligned}
(2\pi)^2\left\langle \mathbf{A}_3 \int du\,(T(T'T'))(u)\right\rangle &= \frac{c}{28740096000}\Big(9(374000+74467c)E_4(\tau)^3\\
&\quad -200(28072+5615c)E_2(\tau)E_4(\tau)E_6(\tau)+200(11242+2279c)E_6(\tau)^2\Big)\chi(q)\\
&+\frac{1}{21772800}\Big(35280E_2(\tau)^3E_4(\tau)+12(4048+2335c+70c^2)E_2(\tau)E_4(\tau)^2\\
&\quad +20(3118+305c)E_2(\tau)^2E_6(\tau)-(146216+34778c+845c^2)E_4(\tau)E_6(\tau)\Big)\partial\chi(q)\\
&+\frac{1}{907200}\Big(-16380E_2(\tau)^2E_4(\tau)+9(2720+183c)E_4(\tau)^2\\
&\quad -10(698+145c)E_2(\tau)E_6(\tau)\Big)\partial^2\chi(q)+\left(\frac{1}{36}E_2(\tau)E_4(\tau)-\frac{574+5c}{15120}E_6(\tau)\right)\partial^3\chi(q)\\
&+\frac{1}{60}E_4(\tau)\partial^4\chi(q)
\end{aligned}
$$

Using the above two-point functions, one can show

$$
\left\langle (\mathbf{A}_3)(T(T'T'))-3(\mathbf{A}_3)(W'W')+(2\pi)^2\frac{c-34}{96}(\mathbf{A}_3)(T''T'')\right\rangle \neq 0\,. \tag{C.33}
$$

We have additionally computed the two-point correlation functions involving the composite operators $(\mathbf{A}_3)(T(T(TT)))$ and $(\mathbf{A}_3)(T(WW))$, which play a crucial role in determining the structure of the current $J_8$. Due to their complexity, we omit their explicit expressions here.

### C.2.2 Two-point functions involving weight nine composites

In this section, we provide the thermal two-point functions of composite objects, which are relevant to compute the thermal two-point function of $\langle \mathbf{A}_5 I_8\rangle$.

$$
\begin{aligned}
(2\pi)^4\left\langle \mathbf{A}_5 \int du\,(T''W'')(u)\right\rangle &= \left(\frac{1}{30}E_2E_4^2-\frac{169E_4E_6}{5040}\right)\partial\langle W_0\rangle\\
&+\left(\frac{(210c+1031)E_4^3}{110880}-\frac{(350c+633)E_2E_6E_4}{110880}+\frac{(17695c+66276)E_6^2}{13970880}-\frac{E_2^2E_4^2}{120}\right)\langle W_0\rangle\,,
\end{aligned}
$$

$$(2\pi)^2\left\langle \mathbf{A}_5 \int du\,(T'(T'W))(u)\right\rangle = \left(\frac{407E_4^2}{25200} - \frac{E_2E_6}{63}\right)\partial^2\langle W_0\rangle$$
$$+\left(\frac{(560c+7447)E_4^2E_2}{302400} + \frac{(-1690c-32139)E_4E_6}{907200} + \frac{2}{189}E_6E_2^2\right)\partial\langle W_0\rangle$$
$$+\left(\frac{\left(-3500c^2-106705c-180486\right)E_4E_6E_2}{39916800} + \frac{\left(2100c^2+73780c+340341\right)E_4^3}{39916800}\right.$$
$$\left.+\frac{\left(17695c^2+649881c+2424492\right)E_6^2}{502951680} + \left(-\frac{c}{2160} - \frac{121}{16128}\right)E_4^2E_2^2 - \frac{1}{756}E_6E_2^3\right)\langle W_0\rangle\,,$$

$$(2\pi)^2\left\langle \mathbf{A}_5 \int du\,(T(WT''))(u)\right\rangle = \left(\frac{5E_2E_6}{126} - \frac{407E_4^2}{10080}\right)\partial^2\langle W_0\rangle$$
$$+\left(\frac{(-392c-3247)E_2E_4^2}{120960} + E_6\left(\frac{(1183c+19464)E_4}{362880} - \frac{5E_2^2}{189}\right)\right)\partial\langle W_0\rangle$$
$$+\left(\frac{\left(-840c^2-31774c-177051\right)E_4^3}{15966720} + \frac{\left(-17695c^2-671214c-3194856\right)E_6^2}{502951680}\right.$$
$$\left.+E_6\left(\frac{\left(1400c^2+40039c+64800\right)E_4E_2}{15966720} + \frac{5E_2^3}{1512}\right) + \left(\frac{7c}{8640} + \frac{325}{32256}\right)E_2^2E_4^2\right)\langle W_0\rangle\,,$$

$$\left\langle \mathbf{A}_5 \int du\,(T(T(TW)))(u)\right\rangle = \frac{E_4}{60}\partial^4\langle W_0\rangle - \left(\frac{(5c+3024)E_6}{15120} - \frac{43E_2E_4}{240}\right)\partial^3\langle W_0\rangle$$
$$+\left(\frac{(-85c-1044)E_6E_2}{12096} + \frac{(1507c+68906)E_4^2}{201600} - \frac{47}{192}E_4E_2^2\right)\partial^2\langle W_0\rangle$$
$$+\left(\frac{\left(280c^2+30007c+153316\right)E_4^2E_2}{2419200} + \frac{145E_4E_2^3}{1728}\right.$$
$$\left.+E_6\left(\frac{\left(-845c^2-126754c-2106252\right)E_4}{7257600} + \frac{(175c+5112)E_2^2}{36288}\right)\right)\partial\langle W_0\rangle$$
$$+\left(\frac{\left(-56c^2-7207c-85788\right)E_4^2E_2^2}{1935360} + \frac{\left(54570c^2+2088991c+17218668\right)E_4^3}{319334400}\right.$$
$$+E_6\left(\frac{\left(-80555c^2-1987936c-3875928\right)E_2E_4}{319334400} - \frac{5(53c+1836)E_2^3}{435456}\right)$$
$$\left.+\frac{\left(222979c^2+8161302c+61790904\right)E_6^2}{2011806720} - \frac{49E_4E_2^4}{6912}\right)\langle W_0\rangle\,,$$

$$\left\langle \mathbf{A}_5 \int du\,(W(WW))(u)\right\rangle = \frac{E_4}{60}\partial\langle W_0^3\rangle + \left(\frac{E_2E_4}{5} - \frac{(5c+3024)E_6}{15120}\right)\langle W_0^3\rangle$$
$$+ \frac{E_2E_4}{360}\partial^3\langle W_0\rangle + \left(\frac{(5c-1476)E_6E_2}{90720} + \frac{(35c+121722)E_4^2}{2419200} - \frac{7}{216}E_4E_2^2\right)\partial^2\langle W_0\rangle$$
$$+ \left(E_6\left(\frac{\left(-10c^2-53689c-1274918\right)E_4}{29030400} + \frac{(4176-5c)E_2^2}{217728}\right)\right.$$
$$\left. + \frac{(53179c+301162)E_4^2E_2}{29030400} + \frac{727E_4E_2^3}{51840}\right)\partial\langle W_0\rangle$$
$$+ \left(\frac{\left(46460c^2+2074265c+29875806\right)E_4^3}{3832012800} + \frac{(-168469c-2879910)E_4^2E_2^2}{348364800}\right.$$
$$+E_6\left(\frac{\left(-238735c^2-4379196c+10311732\right)E_4E_2}{11496038400} + \frac{(5c-14976)E_2^3}{4354560}\right)$$
$$\left. + \frac{\left(275c^3+2124955c^2+79379178c+898468200\right)E_6^2}{241416806400} - \frac{49E_4E_2^4}{69120}\right)\langle W_0\rangle\,.$$

From these explicit results, one can check that

$$\begin{aligned}\left\langle \mathbf{A}_5\left((T(WT'')) + \frac{5}{2}(T'(T'W)) - \frac{1}{24}(c+25)(2\pi)^2(T''W'')\right)(u)\right\rangle & \\ = \frac{c+2}{18480}(5E_4^3+7E_6^2-12E_2E_4E_6)\langle W_0\rangle & \\ = 216(c+2)\Delta_3\, q^{\Delta_2-\frac{c}{24}}\,(q^2+58q^3+\ldots)\,. &\end{aligned} \tag{C.34}$$

Thus, there will be a $\gamma$-dependent contribution at $O(q^2)$ to the two-point function of $\mathbf{A}_5$ and $\mathbf{I}_8$. By combining the thermal two-point functions of $\mathbf{A}_5$ and the composite operators appearing in the ansatz for $J_9$, the $O(q^2)$ contribution to the two-point function is given by:

$$\begin{aligned}\langle \mathbf{A}_5\,\mathbf{I}_8\rangle_{L=2} = \frac{529}{12}(\Delta_2^4\Delta_3+3\Delta_3^3\Delta_2) + \left(\frac{144337}{48} - \frac{4583c}{3024}\right)\Delta_2^3\Delta_3 + \left(\frac{24161c}{2016}+\frac{17329}{2}\right)\Delta_3^3 & \\ + \left(-\frac{3263c^2}{12096} + \frac{45792023c}{120960} + \frac{2285741}{64}\right)\Delta_2^2\Delta_3 & \\ + \left(\frac{10229c^3}{580608} + \frac{71822327c^2}{2903040} + \frac{52879403c}{30240} + \frac{152645845}{2304}\right)\Delta_2\Delta_3 & \\ + \Delta_3\Big(-\frac{24161c^4}{83607552} + \frac{87748349c^3}{1463132160} + \frac{3402082927c^2}{81285120} + \frac{4070183401}{5806080}c + \frac{24989101}{1152} &\end{aligned} \tag{C.35}$$

$$+ 216\gamma^{(0)}(c+2)\Big) .$$

This result will be used in Appendix D, and compared with the result for the two-point function calculated using the operator formalism.

To fix the form of $\mathbf{I}_{10}$, similar results for the one and two-point correlators have been computed for the individual composite operators appearing in the $J_{!1}$ ansatz. We do not present the analytical form of the results as they are quite cumbersome. However, we shall present numerical results for these in the next section for specific choices of $(c, \Delta_2, \Delta_3)$.

# Appendix D

# Fixing the Currents: Numerical Details

In this section we collect the details of the calculations that fix the higher Boussinesq currents.

## D.1 $J_9$

We begin with fixing the undetermined constant in $J_9$.

- First, we choose a random set of values for the central charge and conformal dimensions of the module under consideration. This can be seen in the first column of Table D.1.
- Secondly, we solve for the algebraic constraints in (3.32). We note that we have solved the equations numerically up to 20 digits of precision. We find exactly five solutions that correspond to the eigenvectors of the quantum Boussinesq charges at the second excited level. Substituting these values into the eigenvalue of $\mathbf{I}_8$ computed in the previous section gives the five eigenvalues at the second excited level. These are listed in the second column of Table D.1 for $\mathbf{I}_8$.
- The next step computes the trace in the operator formalism. To do so, we first com-

| $(c, \Delta_2, \Delta_3)$ | Eigenvalues at level 2: $I^{(2)}_{8,i}$ | $\langle \mathbf{A}_5 \mathbf{I}_8 \rangle_{L=2}$ (osc.) | $\langle \mathbf{A}_5 \mathbf{I}_8 \rangle_{L=2}$ (therm.) |
|---|---|---|---|
| $(-10, \frac{20}{9}, -\frac{10\sqrt{2}}{9})$ | -8245.7932795505<br>-663.54732556260<br>515.28751158324<br>43.77545199884<br>-483.08161460450 | -542700.22989155 | 2715.29003975634 $\gamma^{(0)}$<br>-540663.76236173 |

Table D.1: Numerical results that lead to the determination of $\gamma^{(0)}$. We first choose particular values of $(c, \Delta_2, \Delta_3)$. This allows us to solve for the $(a_i, w_i)$ numerically. We find five distinct solutions. Using these, we find the five distinct eigenvalues of $\mathbf{I}_8$ at the second excited level and the trace with $\mathbf{A}_5$ inserted. We compare this with the result for the trace calculated using the thermal two-point functions, and this allows us to fix $\gamma^{(0)}$ uniquely.

pute the five eigenvalues of $W_0 = \mathbf{I}_2$ using (3.64). We then diagonalize the matrix (4.47) to compare with this list and find the R-matrix that implements this diagonalization. Since the charges all mutually commute, this gives us the R-matrix that also diagonalizies $\mathbf{I}_8$. Substituting this into the formula for the trace (see (4.34)):

$$\langle \mathbf{I}_8 \, \mathbf{A} \rangle_{\mathcal{H}_2} = q^{\Delta_2+2} \, \mathrm{Tr} \left( A \, R \, I^{(L)}_n \, (R^{-1}) \right) , \tag{D.1}$$

we find the entries in the third column of Table D.1.

- To obtain the final column, we simply substitute the values of $(c, \Delta_2, \Delta_3)$ into the $O(q^2)$ contribution to the trace computed using the thermal correlators, and presented in (C.35).

- Equating the entries in the third and fourth column, we find $\gamma^{(0)}$ to be given by:

$$\gamma^{(0)} = -\frac{3}{4} . \tag{D.2}$$

We obtain the same value of $\gamma^{(0)}$ by choosing various other choices of $(c, \Delta_2, \Delta_3)$.

| $(c, \Delta_2, \Delta_3)$ | Eigenvalues at level 2: $I^{(2)}_{10,i}$ | $\langle \mathbf{A}_5 \mathbf{I}_{10} \rangle_{L=2}$ (osc.) | $\langle \mathbf{A}_5 \mathbf{I}_{10} \rangle_{L=2}$ (therm.) |
|---|---|---|---|
| $\left(-\frac{671}{2}, -\frac{1009}{72}, -\frac{1}{288}\right)$ | 3533.5725819+ $i$2.5725964257922×$10^7$<br>3533.5725819- $i$2.5725964257922×$10^7$<br>19.40887956686+ $i$794.15593683822<br>19.40887956686- $i$794.15593683822<br>64.743553579092 | -8.547506010×$10^6$ | -8.42712458599599174×$10^6$<br>-237.3472222222222222 $\delta_1^{(0)}$<br>+79629.9930555555556 $\delta_1^{(1)}$<br>-494757.394811820083 $\delta_4^{(0)}$<br>+141553.1584382978014 $\delta_3^{(0)}$ |
| $\left(-\frac{374}{5}, -\frac{116}{45}, -\frac{16}{45\sqrt{5}}\right)$ | 131.773179900298+ $i$2561.55911926717<br>131.773179900298- $i$2561.55911926717<br>-15.1220957985725<br>-3.00689419219823+ $i$2.42552596973312<br>-3.00689419219823- $i$2.42552596973312 i | -82937.229195105 | 158427.841348086225<br>-2386.41125022697556 $\delta_1^{(0)}$<br>+178503.561516977772 $\delta_1^{(1)}$<br>-116476.3098370625835 $\delta_4^{(0)}$<br>+39746.9342833526784 $\delta_3^{(0)}$ |
| $\left(-145, -\frac{217}{36}, -\frac{1}{72\sqrt{2}}\right)$ | 431.226110418+ $i$338276.69028687<br>431.226110418- $i$338276.69028687<br>-8.25576901577- $i$645.02009814402<br>-8.25576901577+ $i$645.02009814402<br>9.9242777690861 | -478157.823246 | -420167.815576069929<br>-289.520943185825292 $\delta_1^{(0)}$<br>+41980.5367619446673 $\delta_1^{(1)}$<br>-97403.79290878278594$\delta_4^{(0)}$<br>+28419.3456945738868 $\delta_3^{(0)}$ |
| $\left(-\frac{374}{5}, -\frac{101}{45}, -\frac{14}{45\sqrt{5}}\right)$ | 131.773179900298+ $i$2561.55911926717<br>131.773179900298- $i$2561.55911926717<br>-15.1220957985725<br>-3.00689419219823+ $i$2.42552596973312<br>-3.00689419219823- $i$2.42552596973312 | -54298.791240984 | 156415.623479858187<br>-2074.75306456300487 $\delta_1^{(0)}$<br>+155191.529229312764 $\delta_1^{(1)}$<br>-78260.5005359722806 $\delta_3^{(0)}$<br>-78260.5005359722806 $\delta_4^{(0)}$ |

Table D.2: Numerical results required to find constants involved in $\mathbf{I}_{10}$. As it can be seen explicitly the thermal correlator of $\mathbf{I}_{10}$ with $\mathbf{A_5}$ involves only four undetermined constants ($\delta_1^{(0)}$,$\delta_1^{(1)}$,$\delta_3^{(0)}$,$\delta_4^{(0)}$). These constants can be fixed from four numerical choices of $(c, \Delta_2, \Delta_3)$.

## D.2 $J_{11}$

We now repeat these steps for the case of $\mathbf{I}_{10}$. Recall that there are six undetermined constants in the expression for $J_{11}$ in (4.65). We first consider the two-point function of the charge with $\mathbf{A}_5$. At level 1, we find that no matter what numerical values for $(c, \Delta_2, \Delta_3)$ are chosen, we obtain a single equation, given by

$$8\delta_3^{(0)} - 28\delta_4^{(0)} + 1 = 0 \,. \tag{D.3}$$

Moving on to level 2, we repeat the steps outlined earlier; we obtain the results in Table D.2.

We note that when we equate the entries of the third and fourth columns for each of the rows, we find four linearly independent equations in four variables, which we solve to

find

$$\delta_1^{(1)} = -\frac{253}{192}, \quad \delta_1^{(0)} = -\frac{879}{64}, \\ \delta_3^{(0)} = -\frac{43}{8}, \quad \delta_4^{(0)} = -\frac{3}{2}. \tag{D.4}$$

We can immediately check that the values of $\delta_3^{(0)}$ and $\delta_4^{(0)}$ satisfy the equation in (D.3). Furthermore, we have done analogous calculations for other values of $(c, \Delta_2, \Delta_3)$ and found exactly the same results. This leaves two parameters to fix. To fix these, we compute the two-point function of $\mathbf{I}_{10}$ with $\mathbf{A}_3$, and the results are shown in Table D.3.

| $(c, \Delta_2, \Delta_3)$ | Eigenvalues at level 2: $I_{10}^{(2)}$ | $\langle \mathbf{A}_3 \mathbf{I}_{10} \rangle_{L=2}$ (osc.) | $\langle \mathbf{A}_3 \mathbf{I}_{10} \rangle_{L=2}$ (therm.) |
|---|---|---|---|
| $\left(-52, -\frac{31}{18}, -\frac{5}{36}\right)$ | -13.08889619- $i$216.69461435<br>-13.08889619+ $i$216.69461435<br>-45.72705707<br>-0.11270168- $i$1.21762082<br>-0.11270168+ $i$1.21762082 | 891.8628026497 | 28809.94074427 - 333.33333333 $\delta_1^{(0)}$<br>17333.33333333 $\delta_1^{(1)}$<br>+1500 $\delta_2^{(0)}$-78000 $\delta_2^{(1)}$<br>3497.40131077 $\delta_3^{(0)}$<br>-8470.07125438 $\delta_4^{(0)}$ |
| $\left(-145, -\frac{199}{36}, -\frac{7}{72\sqrt{2}}\right)$ | 2570.33616417+ $i$273006.46983069<br>2570.33616417- $i$273006.46983069<br>-37.18463907 -$i$262.65391607<br>-37.18463907+ $i$262.65391607<br>23.58335121 | -608361.8110576 | -471.87592531$\delta_1^{(0)}$+68422.00917021$\delta_1^{(1)}$<br>+2123.44166390$\delta_2^{(0)}$-495890.02280547<br>+43653.74702321$\delta_3^{(0)}$<br>-155678.35462377$\delta_4^{(0)}$<br>-307899.04126596$\delta_2^{(1)}$ |

Table D.3: Numerical results required to find $(\delta_2^{(0)}, \delta_2^{(1)})$. These constants can be fixed from the thermal correlator of $\mathbf{I}_{10}$ with $\mathbf{A_3}$ with two different numerical choices of $(c, \Delta_2, \Delta_3)$.

Equating the third and fourth columns for each row and substituting the values in (D.4), we obtain

$$\delta_2^{(1)} = \frac{11}{96}, \quad \delta_2^{(0)} = \frac{43}{12}. \tag{D.5}$$

We have checked that the same results for the coefficients are obtained for a variety of values chosen for $(c, \Delta_2, \Delta_3)$.

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