

# Quantum Cosmology of Tachyons in String Theory

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### Abstract.

The Hamiltonian formulation of the effective bosonic action of the  $SDp$ -brane in Type II superstring theory is given. This effective description involves the Tachyon driven matter coupled to bosonic ten dimensional Type II supergravity. An exact solution to the corresponding Wheeler-deWitt equation in the late-time limit of the rolling tachyon is found. Finally some comments about the incorporation of electromagnetic fluxes are given.

## 1. Introduction

Recently an effective model of a coupled system of ten dimensional Type II supergravity and tachyon driven matter have been proposed [1, 2, 3]. This model is a good candidate for the effective action of a space-like  $Dp$ -brane, called  $SDp$ -brane, which describes (time dependent) decaying processes in string theory. In this model our universe is contained in the bulk and the brane is considered as a topological defect that is embedded also in the bulk. If all the Dirichlet boundary conditions on the brane are spacelike, then usual  $Dp$ -branes arise. However if one of these Dirichlet conditions is timelike, then the tachyon rolls down the potential and a time-dependent process occurs [4]. The full picture of passing from one minimum to the other is called  $SDp$ -brane solution. As soon as the tachyon field rolls down from the top of the tachyon

potential  $V(T)$  towards one of its minima, it starts to excite open and closed string modes in such a way that the energy of the unstable  $Dp$ -brane is radiated away into the bulk. When the tachyon arrives to its minimum, the radiation is in the form of only closed strings because open strings cannot exist in the bulk.

On the other hand, Sen proposed a field theory describing the dynamics of the rolling tachyon [5, 6, 7]. In this context, he found that the tachyon field can be interpreted as the *time* in quantum cosmology [8]. This was done by coupling the “tachyon matter” to a gravitational field and then performing its canonical quantization. From it, a Wheeler-deWitt equation turns out, which can be regarded as a time-dependent Schrödinger equation for this gravity-tachyon matter system. The coupling of the tachyon to gravity has been studied in connection with classical cosmological evolution [9, 10, 11]. In particular its role related to inflation has been discussed, see [12] and references therein.

In the present note we survey the *minisuperspace* approach [13] of the  $SDp$ -brane decay corresponding to the model [1, 2, 3]. Although minisuperspace contains a further reduced sector of the degrees of freedom of the full string theory, at the level of the effective theory it becomes a suitable approach to have under control some features of the classical theory, as well as the lowest excited states of the genuine quantum dynamics of the  $SDp$ -branes [11]. Thus, it is expected that the canonical quantization description presented here corresponds to the *s*-wave approximation of the low energy quantum theory of the effective action of Refs. [1, 2, 3]. Furthermore it could give some insight into the solution of the Wheeler-deWitt equation for finite values of the tachyon  $T$ . The quantum properties of the considered field theory seem to be an interesting problem by itself, as already pointed out by Sen in Ref. [8], where he considers a quantum cosmology model coupled to the tachyon matter. The effective model [1, 2, 3] we are going to consider can be understood as a cosmological model with dilaton and RR fields, driven by the tachyon matter. We show that the proposal by Sen, concerning the interpretation of the tachyon as time, in the late ‘time’ decoupling limit, is valid for the model under consideration [13]. In this case we find an exact wave function, finite and continuous everywhere for the corresponding Schrödinger equation. It was shown also in Ref [13] that in the presence of a uniform electric field, the interpretation of the tachyon as time seems to be spoiled. In the late-time limit, the tachyon does not decouple from the electric field.

This paper is organized as follows: in section 2 we briefly discuss the model under consideration. In section 3 we find the Hamiltonian constraint for the system. Section 4 is devoted to the study of quantum solutions with the rolling tachyon approximation in the decoupling limit. Our final remarks are presented in section 5.

## 2. The Model

The action proposed in [1, 2] for this theory is given by:

$$S = S_{bulk} + S_{brane}, \quad (1)$$

where

$$S_{bulk} = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} \left( R - \frac{1}{2}(\partial\phi)^2 - \frac{e^{a\phi}}{2(p+2)!} F_{p+2}^2 \right), \quad (2)$$

$$S_{brane} = \frac{\Lambda}{16\pi G_{10}} \int d^{p+2}x_{\parallel} \hat{\varrho}_{\perp} \left( -V(T)e^{-\phi}\sqrt{-\mathcal{A}} \right) + \frac{\Lambda}{16\pi G_{10}} \int \hat{\varrho}_{\perp} \mathcal{F}(T) dT \wedge C_{p+1}, \quad (3)$$

where  $G_{10}$  is the Newton’s constant in the ten-dimensional theory,  $\Lambda$  is the brane coupling,  $a \equiv (3-p)/2$  is the dilaton coupling,  $\mathcal{A} = \det \mathcal{A}_{\alpha\beta}$ ,  $\mathcal{A}_{\alpha\beta} = g_{\alpha\beta}e^{\phi/2} + \partial_{\alpha}T\partial_{\beta}T$  is the tachyon metric,  $\mathcal{F}(T)$  is the factor of coupling between the tachyon and the RR fields  $C_{p+1}$ , and  $V(T)$  is the tachyon potential.  $\hat{\varrho}_{\perp}$  is the “density of branes”, which does not depend on the parallel coordinates of the brane  $x_{\parallel}$ . Greek indices  $\alpha, \beta = 0, 1, \dots, p+1$ , label the time and

parallel coordinates (denoted by  $\parallel$ ) to the  $SDp$ -brane. Latin indices  $i, j = 1, \dots, 8-p$  label the perpendicular coordinates of the brane (denoted by  $\perp$ ), and capital letters  $A, B, \dots$ , etc. stand for space-time coordinates of the bulk.

Following Refs. [1, 2], the simplest model that we can study is assuming the homogeneous (but non-isotropic) FRW metric. Making the space decomposition into maximal symmetric direct product  $ISO(p+1) \times SO(8-p, 1)$  we have the metric

$$ds^2 = -N^2(t)dt^2 + a_{\parallel}^2(t)dx_{\parallel}^2 + a_{\perp}^2(t)dx_{\perp}^2, \quad (4)$$

where  $a_{\parallel}(t)$  and  $a_{\perp}(t)$  are the parallel and perpendicular scaling factors of the brane and  $N(t)$  is the lapse function. In this homogeneous model the space parallel (perpendicular) to the brane is characterized by the curvature constant  $k_{\parallel}$  ( $k_{\perp}$ ). For simplicity we will take  $k_{\parallel}$  and  $k_{\perp}$  to be zero.

We introduce the following coordinates  $\beta_1, \beta_2$  defined as,

$$\beta_1 = \frac{1}{9} \left[ (p+1)\beta_{\parallel} + (8-p)\beta_{\perp} \right], \quad \beta_2 = \beta_{\parallel} - \beta_{\perp}, \quad (5)$$

where  $\beta_{\parallel} = \ln a_{\parallel}$  and  $\beta_{\perp} = \ln a_{\perp}$ . Also the space volume is given by  $V_S = \frac{1}{16\pi G_{10}} \int d^{p+1}x_{\parallel} d^{8-p}x_{\perp}$ , so  $S = \int d^{10}x \mathcal{L}$ , with  $\mathcal{L} = V_S \int dt L$ .

In  $\beta_1$  and  $\beta_2$  coordinates and with the ansatz (4) we have the Lagrangian

$$L = -\frac{e^{9\beta_1}}{N} \left[ 72\dot{\beta}_1^2 - \frac{(p+1)(8-p)}{9}\dot{\beta}_2^2 - \frac{1}{2}\dot{\phi}^2 - \frac{e^{a\phi}}{2(p+2)!}\dot{C}^2 \right] - \lambda e^{9\beta_1 - a\phi/2} V(T) \sqrt{N^2 e^{\phi/2} - \dot{T}^2} \\ + \lambda e^{(8-p)[\beta_1 - \frac{1}{9}(p+1)\beta_2]} \mathcal{F}(T) \dot{T} C. \quad (6)$$

We can formulate an equivalent theory if we introduce a Lagrange multiplier  $\Omega$  [14] into the Lagrangian (6) as follows,

$$L = -\frac{e^{9\beta_1}}{N} \left[ 72\dot{\beta}_1^2 - \frac{(p+1)(8-p)}{9}\dot{\beta}_2^2 - \frac{1}{2}\dot{\phi}^2 - \frac{e^{a\phi}}{2(p+2)!}\dot{C}^2 \right] - \frac{1}{2}\Omega^{-1} \left( N^2 e^{\phi/2} - \dot{T}^2 \right) \\ - \frac{1}{2}\lambda^2 e^{18\beta_1 - a\phi} V^2(T) \Omega + \lambda e^{(8-p)[\beta_1 - \frac{1}{9}(p+1)\beta_2]} \mathcal{F}(T) \dot{T} C, \quad (7)$$

where  $\lambda = \Lambda\rho_0$ . As usual, the variation of this action with respect to  $\Omega$ ,  $\frac{\partial L}{\partial \Omega} = 0$ , and substituting  $\Omega$  from it into Lagrangian (7) the Lagrangian (6) follows.

### 3. The Hamiltonian Analysis

As a consequence of reparametrization invariance of the theory we expect the associated hamiltonian constraint, which will be used in the next section to give the corresponding Wheeler-deWitt equation. The canonical momenta obtained from the Lagrangian (7) are given by,  $P_1 = \frac{\partial L}{\partial \dot{\beta}_1} = -\frac{144}{N} e^{9\beta_1} \dot{\beta}_1$ ,  $P_2 = \frac{\partial L}{\partial \dot{\beta}_2} = \frac{2}{9} \frac{(p+1)(8-p)}{N} e^{9\beta_1} \dot{\beta}_2$ ,  $P_{\phi} = \frac{\partial L}{\partial \dot{\phi}} = \frac{e^{9\beta_1}}{N} \dot{\phi}$ ,  $P_C = \frac{\partial L}{\partial \dot{C}} = \frac{e^{9\beta_1 + a\phi}}{N(p+2)!} \dot{C}$ , and  $P_T = \frac{\partial L}{\partial \dot{T}} = \Omega^{-1} \dot{T} + \lambda e^{(8-p)[\beta_1 - \frac{1}{9}(p+1)\beta_2]} \mathcal{F}(T) C$ .

The first class constraints are  $P_{\Omega} = P_N = 0$ . When we implement these constraints, the Hamiltonian, after eliminating  $\Omega$ , is given by

$$H = \dot{\beta}_1 P_1 + \dot{\beta}_2 P_2 + \dot{\phi} P_{\phi} + \dot{C} P_C + \dot{T} P_T - L \\ H_0 = -\frac{1}{144} e^{-9\beta_1} P_1^2 + \frac{9e^{-9\beta_1}}{2(p+1)(8-p)} P_2^2 + e^{-9\beta_1} P_{\phi}^2 + (p+2)! e^{-(9\beta_1 + a\phi)} P_C^2$$

$$+2e^{\phi/4} \left\{ \lambda^2 V^2(T) e^{18\beta_1 - a\phi} + [P_T - \lambda e^{(8-p)[\beta_1 - \frac{1}{9}(p+1)\beta_2}] \mathcal{F}(T) C]^2 \right\}^{1/2} = 0, \quad (8)$$

is the first class Hamiltonian constraint associated with the invariance under time reparametrizations.

It is worth to notice that when this constraint is applied at the quantum level, the resulting Wheeler-deWitt equation does not provide a time evolution of the system, and the corresponding wave function is not normalizable. The spectrum of the whole system (bulk-brane) has zero energy. This is known as the “time problem” [15]. However, as we will see in the next section, in the late-time limit when the tachyon potential and the RR fields are negligible, we can find a “time” dependent Wheeler-deWitt equation.

#### 4. Tachyon Driven Quantum Cosmology

In this section we will overview a way of finding an exact solution of the quantum cosmology of the cosmological model reviewed in Sec. 2. These results were discussed in the Ref. [13].

Exact expressions for the potential  $V(T)$  and the coupling factor  $\mathcal{F}(T)$  are not known. However, their asymptotic universal form  $V(T) = e^{-\alpha T/2}$  and  $\mathcal{F}(T) = \text{sign}(T)e^{-\alpha T/2}$  as  $T \rightarrow \infty$ , are known from string theory [5, 6, 7, 8]. Here  $\alpha = 1$  for bosonic string theory, and  $\alpha = \sqrt{2}$  for superstring theory. We only assume that  $V(T)$  has a maximum at  $T = 0$  and a minimum at  $T \rightarrow \infty$ , where  $V(T) = 0$ . Also, we see that in this limit the tachyon decouples also from the RR fields as  $\mathcal{F}(T) \rightarrow 0$ . Because we want to interpret tachyon as time, we restrict to the late-time non-interacting sector where tachyon decouples from the other fields. The “strong” (near the top of the potential) coupling sector, does not allow to interpret tachyon as time directly, and therefore the Wheeler-deWitt equation does not lead in a simple way to a solvable wave function.

##### 4.1. Late Time Limit

In the sector where  $V(T) \approx 0$ , the canonical Hamiltonian (8) takes the form

$$H_0 = -\frac{1}{144} e^{-9\beta_1} P_1^2 + \frac{9}{2} \frac{e^{-9\beta_1}}{(p+1)(8-p)} P_2^2 + e^{-9\beta_1} P_\phi^2 + (p+2)! e^{-(9\beta_1 + a\phi)} P_C^2 + 2e^{\phi/4} P_T = 0. \quad (9)$$

The resulting equation is the Wheeler-deWitt equation which in the minisuperspace approach is given by

$$\hat{H}_0 \Psi = 0, \quad (10)$$

where  $\hat{H}_0$  is given by (9), with  $P_1 = -i\frac{\partial}{\partial\beta_1}$ ,  $P_2 = -i\frac{\partial}{\partial\beta_2}$ ,  $P_C = -i\frac{\partial}{\partial C}$  and  $P_T = -i\frac{\partial}{\partial T}$ . As it is known, the quantum analysis of the Wheeler-deWitt equation has the problem of the factor ordering. In order to solve this problem there are some proposals in the literature [16], however none of them seems to be satisfactory. This problem is beyond the scope of this paper and here we only assume a particular factor ordering. Assuming that the dilaton field is given by its vacuum expectation value in the late-time, i.e.  $g_s = e^{\langle\phi\rangle}$ , where  $g_s$  is the string coupling constant, then  $P_\phi = 0$ , and we have

$$e^{-9\beta_1} \left[ C_1 \frac{\partial^2 \Psi}{\partial \beta_1^2} - C_2 \frac{\partial^2 \Psi}{\partial \beta_2^2} - C_3 \frac{\partial^2 \Psi}{\partial C^2} \right] = i \frac{\partial \Psi}{\partial T}, \quad (11)$$

where  $C_1 = \frac{1}{288} g_s^{-1/4}$ ,  $C_2 = \frac{9}{4(p+1)(8-p)} g_s^{-1/4}$ ,  $C_3 = ((p+2)! g_s^{-(a+1/4)})/2$ . Now, we see that the Wheeler-deWitt equation (10) leads to a Schrödinger-like equation. Thus in this limit, the tachyon is an usual scalar field which provides a useful parametrization of time, because the tachyon momentum enters linearly in (9). This matter accompanies gravitation ( $S_{bulk}$ ) in a natural and consistent manner. So it seems that at least some of the criticisms and problems related to a “matter clock” can be in this case avoided.

The general solution of (11) is straightforward. Assuming separation of variables for  $\Psi$  is of the form:  $\Psi = \psi_{\beta_1}(\beta_1)\psi_{\beta_2}(\beta_2)\psi_T(T)\psi_C(C)$  is given by

$$\Psi^\pm = \mathcal{N} e^{i\mu T} e^{\pm i\sqrt{\frac{\xi}{c_3}}C} e^{\pm i\sqrt{\frac{\sigma}{c_2}}\beta_2} K_{i\nu}\left(\frac{8}{3}\sqrt{\mu}e^{\frac{9}{2}\beta_1}\right), \quad (12)$$

where  $\mu$ ,  $\nu$ ,  $\xi$  and  $\sigma$  are separation variables and  $\mathcal{N}$  is a normalization constant. This is a plane wave, that represents a free particle, with respect to the variables  $C$  and  $\beta_2$  and with  $T$  playing the role of time. Also notice that the constant  $\mu \equiv E$  enters as an energy when tachyon is interpreted as time. This energy would correspond to the lowest level of the closed string spectrum. In terms of the radii  $a_{\parallel}$  and  $a_{\perp}$  we have,

$$\Psi^\pm = \mathcal{N} e^{i\mu T} e^{\pm i g_s^{a/2+1/8} \sqrt{2\xi/(p+2)!} C} \left(\frac{a_{\parallel}}{a_{\perp}}\right)^{\pm \frac{2i}{3} g_s^{1/8} \sqrt{(p+1)(8-p)}\sigma} K_{i\nu}\left(\frac{8}{3}\sqrt{\mu a_{\parallel}^{p+1} a_{\perp}^{8-p}}\right). \quad (13)$$

This wave function represents the state of the system in the asymptotic limit  $T \rightarrow \infty$ . In this sector the brane has completely decayed, therefore, this wave function can be associated with the stable final state of the system where only closed string spectrum exists.

We can compute the expectation value of  $a_{\parallel}$ , for a certain constant value of  $a_{\perp}$

$$\langle a_{\parallel} \rangle = \mathcal{N}' \int_0^\infty \Psi^* a_{\parallel} \Psi da_{\parallel} = \mathcal{N}' \int_0^\infty \left[ K_{i\nu}\left(\frac{8}{3}\sqrt{\mu a_{\parallel}^{p+1} a_{\perp}^{8-p}}\right) \right]^2 a_{\parallel} da_{\parallel}, \quad (14)$$

where  $\Psi$  is a linear combination of  $\Psi^\pm$  and  $\mathcal{N}'$  is an overall normalization constant. Here  $\Psi^*$  denotes complex conjugate of  $\Psi$ . From (14) we get

$$\langle a_{\parallel} \rangle = \mathcal{N}' \sqrt{\pi} \frac{\Gamma\left(\frac{2}{p+1}\right) \Gamma\left(\frac{2}{p+1} + i\nu\right) \Gamma\left(\frac{2}{p+1} - i\nu\right)}{(3-p)\Gamma\left(\frac{3-p}{2(p+1)}\right)} \left(\frac{9}{64\mu\langle a_{\perp} \rangle^{8-p}}\right)^{\frac{2}{p+1}}. \quad (15)$$

This relation can also be written as a sort of uncertainty relation between the two radii  $\langle a_{\parallel} \rangle \sim \langle a_{\perp} \rangle^{-2\frac{8-p}{p+1}}$ , where the proportionality factor is, for  $\nu = 1$ , of the order of  $10^{-2} \mathcal{N}'$  and decreases exponentially as  $\nu$  increases. Note that the denominator in (15) does not diverge at  $p = 3$  due to the properties of the Gamma function, in fact  $(3-p)\Gamma\left(\frac{3-p}{2(p+1)}\right) = 2(p+1)\Gamma\left(\frac{5+p}{2(p+1)}\right)$ .

The wave function determines the probability density  $|\Psi|^2$  whose continuum of maxima give, in the  $a_{\perp} - a_{\parallel}$  plane, a path in minisuperspace. This relation shows an inverse relation between the two radii given by Eq. (15).

## 5. Final Remarks

In this work, we have provided an exact solution to the canonical quantization of the effective tachyon model [1, 2, 3]. For this effective action, a Wheeler-deWitt equation (for a particular factor ordering) in minisuperspace approach has been obtained from the Hamiltonian analysis. Following Ref. [14], the square root in the tachyonic matter action (6) was eliminated by the introduction of a Lagrange multiplier  $\Omega$ . From the resulting action the Hamiltonian (8) has been computed and the decoupling late-time limit ( $T \rightarrow \infty$ ) has been done. Even though we have considered the canonical quantization of the effective action with a maximally symmetric metric (4), the quantization of this field theory and in particular of the model under consideration is interesting in its own right [8]. Although the minisuperspace approach reduces the number of degrees of freedom of the full string theory, it still contains information of  $s$ -wave approximation of the genuine tachyonic quantum dynamics [11]. Moreover it may give some insight to describe

beyond the classical limit, by using the methods of quantum cosmology, the dynamical degrees of freedom for the decay region where  $V(T)$  and  $\mathcal{F}(T)$  are different from zero.

Further we show that the proposal by Sen, concerning the interpretation of the tachyon as time, in the late-time decoupling limit, is still valid for this model. In this limit we find an exact wave function for the corresponding Schrödinger equation. The decoupled limit certainly behaves different than the quantum cosmology associated to the closed string sector without an initial decaying non-BPS configuration. Thus, the quantum cosmological description with tachyons and without them is different, even in this decoupling limit. The associated probability density is a finite and continuous function of the radii  $a_{\parallel}$  and  $a_{\perp}$ , it gives (a non-singular) continuum of maxima along a definite trajectory, in such a way that if the mean value of one of the radii increases, the mean value of the other one decreases.

Finally, if quantum corrections of the string theory have to be taken into account and if the open-closed duality holds (see remarks of review, [11]), it would be very interesting to explore if solutions of the Schrödinger equation for intermediate values of time (representing open-closed states), correspond to a description (at higher level) of the physics of the quantum string theory associated to SD $p$ -branes.

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