Interacting quantum field theories and topological defects

Antonino Flachi and Vincenzo Vitagliano

Department of Physics & Research and Education Center for Natural Sciences, Keio University, 4-1-1 Hiyoshi, Kanagawa 223-8521, Japan

We present here the results of a recent analysis of interacting quantum field theories on a curved manifold with a topological defect constructed by geometrically deforming a lattice. We discuss an explicit example where curvature and boundary conditions compete in altering the way the quantum vacuum is destabilized. We show that the competing action of the locally induced curvature and of boundary conditions generated by the non-trivial topology allows configurations where symmetries can be spontaneously broken close to the defect core. Inspired by this effect, we propose a novel mechanism to induce a superconducting phase by triggering particle condensation along cosmic strings.

As the energy scales of a physical system become higher, non-perturbative effects get into the game and the system is said to have entered an strongly coupled regime. Remarkable examples of strongly coupled systems include QCD, high temperature superconductors and the very primordial plasma filling the universe a few instants after the big bang. Interactions are also important in describing the propagation of conducting electrons in graphene.

Precise study of strong coupled dynamics is only possible at the cost of demanding numerical simulations. On the other hand, general phenomenological guidelines for the study of strongly coupled systems can be precisely draft exploiting mathematical considerations on the underlying symmetries. The physics of symmetry breaking, in fact, notably relies on exact mathematical statements which are intimately non-perturbative, as in the case of the Goldstone theorem.

A seminal paper by Coleman and Weinberg[1] showed how symmetries can be spontaneously broken due to quantum fluctuations. This mechanism of *dynamical* symmetry breaking, originally developed in the context of scalar field self-interactions, can be naturally extended to fermions. In a field theory with massless fermions, the interactions between particles preserve helicity: the polarised left- and right-handed sectors evolve separately. However, in the celebrated Nambu–Jona-Lasinio model (λ is the coupling constant and N the total number of fermion dof,

$$S_{\rm NJL} = \int d^4x\sqrt{g}\left\{\bar{\psi}i\gamma^\mu\nabla_\mu\psi + \frac{\lambda}{2N}(\bar{\psi}\psi)^2\right\},$$

and in its progeny of models for strong interactions, such symmetry is spontaneously broken by a non-vanishing order parameter: when the composite operator $\phi\sim\bar{\psi}\psi$ acquires a non-zero vacuum expectation value, then a dynamical effective mass for the fermions, $M_{\rm eff}\sim\langle\bar{\psi}\psi\rangle$, is generated. This mechanism is responsible for most of the observed mass of hadrons.

Transitions between symmetry broken and symmetry restored phases are generally triggered by changes in the external conditions. In particular a nonzero temperature leads to temperature-dependent mass generation and to restoration of broken symmetries once a certain critical value of the temperature is reached. Other factors modifying the phase diagram of self-interacting theories span from finite density effects to the action of a non-vanishing chemical potential or of an external gauge field. Here we will instead concentrate on geometrical effects, and how these challenge the vacuum stability of a theory with four fermions interactions.

Many are the configurations in which geometry affects the symmetry breaking of strongly interacting systems: in flat spacetime with $R^3 \times S^1$ topology and periodic boundary conditions, for example, the consequences of the non-trivial topology are very similar to those of nonzero temperature [2]; on the other hand, in curved spacetime the effects of external gravitational fields resemble those of an effective extra mass [3]. The combination of these external factors, and in particular topology, nonzero temperature, and curvature, acting on self-interacting theories is likely to have been of considerable importance in the early stages of the evolution of the universe. During those eras a spontaneous breaking of an internal symmetry group results in the production of topological defects – the well-known Kibble–Zurek mechanism [4,5].

Suppose the dynamical symmetry breaking for some particle model to be at the origin of the formation of a static straight cosmic string [6,7] lying along the z-axis, namely an infinitely long thin tube of false vacuum generated in the sudden temperature-driven transition from a phase to another (here, the transverse size of the cosmic string is neglected while 'sudden transition' means a transition with a rate that is fast if compared with the size of the system). Away from the defect, the spacetime associated with the gravitating string is accurately described by the vacuum Einstein equations. It turns out that at large distance from the string, the geometry is locally flat, $ds^2_{\rm con} = dt^2 - dz^2 - dr^2 - r^2 d\theta^2$, but with an important *caveat*: it is not globally Euclidean, since the angular coordinate does not run on the entire 2π circle; instead, $0 \leq \theta < 2\pi - \Delta$, with $\Delta > 0$ ($\Delta < 0$) being the deficit (excess) angle: surfaces at constant t and z are cones, not planes.

This is a remembrance of defects insertion in crystal lattices: starting from a (locally) flat lattice, the subtraction (or addition) of atoms is equivalent to the extraction (or insertion) of sections of the lattice with a given angle. The procedure results in an ice-cream-shaped lattice (or a saddle) which is locally flat everywhere apart for the apex. Note yet that the spacetime surgery does not come for free: the price to pay is the implementation of some non-trivial new boundary conditions along the cut, which will be a reminder of the deficit (excess) angle at the apex.

To investigate the role of the background geometry in modifying the vacuum structure, we will consider a $(2 + 1)$–dimensional honeycomb lattice, whose hexagonal structure is obtained by the superposition of two triangular sublattices: the breakdown of this discrete symmetry is behind the phenomenon of condensation we will describe here[8]. The dynamics of the delocalized electrons on such a lattice is often described in terms of a generalisation of the Hubbard model, that in the continuum limit is mapped onto a field theory with nine different couplings[9]; however, considering the limit for a large number of fermion flavours N and after bosonization, the Hubbard model for the honeycomb lattice has the following form

$$\mathcal{L}_{\rm GN} = \bar{\psi}_\sigma \imath \partial\!\!\!/ \psi_\sigma + \sigma \bar{\psi}_\sigma \phi \psi_\sigma + \frac{\phi^2}{2\lambda} \,, \tag{1}$$

where $\sigma = \pm$ is a spin index on which one sums. The final goal is to study the behaviour of the order parameter ϕ when moving toward the apex of the cone.

Using a regularization of the space generated by the topological 'stringy' defect with a family of smoothed versions of the conical solution (in Euclidean time), $ds^2 = d\tau^2 + f_\epsilon(r) dr^2 + r^2 d\theta^2$, where $f_\epsilon(r)$ is a regularising function: in the limit $\epsilon \to 0$ one might recover the singular cone. The Schrödinger–Lichnerowicz–Weitzenböck formula allows us to write the effective action as

$$\Gamma\left[\phi\right] = -\int d^3x \sqrt{g} \frac{\phi^2}{2\lambda} + \frac{1}{2} \sum_{p=\pm} \log\det\left(\Box + \frac{R}{4} + \phi_p^2\right) , \quad \phi_\pm^2 = \phi^2 \pm \sqrt{g^{rr}} \phi' \tag{2}$$

where the metric here employed is the smoothed one, ds^2, rather than $ds^2_{\rm con}$. As previously mentioned, the change in the topology has a prominent role in altering the boundary conditions on the glued side of the lattice; the response to this change is captured by the employment of a modified covariant derivative, $D_\mu = \nabla_\mu + \imath \mathcal{A}_\mu$, acting on spinors and encompassing an effective non-dynamical gauge field, $\mathcal{A}_\mu$, that depends on the deficit angle[10].

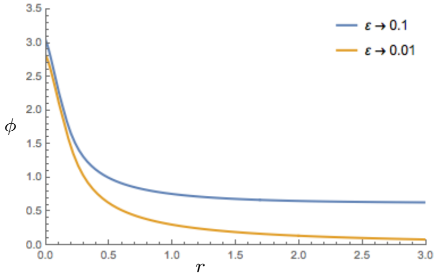

Fig. 1. Profile of the condensate ϕ as a function of the distance from the apex, r. Different colours refer to different values of the ϵ parameter regularising the cone.

Expressing the functional determinant of the (squared) Dirac operator in curved space in terms of its heat kernel expansion[11], and using zeta function regularization, it is possible to calculate the effective action (2) and find out the effective equations of motion of the order parameter ϕ, whose solutions can be reconstructed numerically (See figure and see Ref. 12 for some examples). An alternative way to proceed could make use of the world-line formalism adapted to the present case (see Ref. 13 for a relevant calculation). Although the presence of curvature (which acts, as for the chiral gap effect[14], as an effective mass term) is supposed to enhance a phase of symmetry restoration, the presence of the extra effective gauge field, another reminder of the geometry of the system, catalyse the formation of a bubble of condensed particles close the apex.

For relativistic cosmic strings, the possibility of particle condensation in the region surrounding the string core due only to configurational elements is a interesting way to induce a superconducting phase around the defect. However, it is worth mentioning that in a more realistic setup, phase transitions in the early universe did not seed the formation of a single straight string, but of a network of cosmic strings, which renders eventually even more striking and intriguing the connection with a crystal lattice, where a distribution of defects is more natural to occur. There is finally another interesting aspect: in order to simplify the discussion we have here considered the spontaneous symmetry breaking for the Gross–Neveu model catalysed by an external string defect, namely originated by the breakdown of some symmetry of a different field. Different would be the situation in which the responsible for the defect formation is the very same field, in which case one might take into account possible back-reaction effects.

References

1. S. R. Coleman, E. J. Weinberg, "*Radiative Corrections as the Origin of Spontaneous Symmetry Breaking*," Phys. Rev. D **7**, 1888 (1973).
2. L. Dolan, R. Jackiw, "*Symmetry Behavior at Finite Temperature*," Phys. Rev. D **9**, 3320 (1974).
3. A. Flachi, T. Tanaka, "*Chiral Phase Transitions around Black Holes*," Phys. Rev. D **84**, 061503 (2011); A. Flachi, "*Deconfinement transition and Black Holes*," Phys. Rev. D **88**, no. 4, 041501 (2013); A. Flachi, "*Black holes as QCD laboratories*," Int. J. Mod. Phys. D **24**, 1542017 (2015).
4. T. W. B. Kibble, "*Topology of Cosmic Domains and Strings*," J. Phys. A **9**, 1387 (1976).
5. W. H. Zurek, "*Cosmological Experiments in Superfluid Helium?*," Nature **317**, 505 (1985).
6. A. Vilenkin, E. P. S. Shellard, "*Cosmic strings and other topological defects*", Cambridge University Press, (2000).
7. R. H. Brandenberger, T. Huang, K. Yang and X. Zhang, "*Cosmic strings in theories with dynamical symmetry breaking*," Phys. Rev. D **55**, 4511 (1997).
8. E. V. Castro, A. Flachi, P. Ribeiro and V. Vitagliano, "*Symmetry Breaking and Lattice Kirigami*," Phys. Rev. Lett. **121**, no. 22, 221601 (2018).
9. I. F. Herbut, "*Interactions and phase transitions on graphene's honeycomb lattice*," Phys. Rev. Lett. **97**, 146401 (2006).
10. Y. A. Sitenko, N. D. Vlasii, "*Electronic properties of graphene with a topological defect*," Nucl. Phys. B **787**, 241 (2007).

11. L. Parker, D. Toms, "*Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*", Cambridge University Press, (2009).
12. A. Flachi and T. Tanaka, "*Chiral Modulations in Curved Space I: Formalism*," JHEP **1102**, 026 (2011); A. Flachi, "*Chiral Modulations in Curved Space II: Conifold Geometries*," JHEP **1201**, 023 (2012); A. Flachi, "*Dual Fermion Condensates in Curved Space*," Phys. Rev. D **88**, no. 8, 085011 (2013)
13. F. Bastianelli, O. Corradini, P. A. G. Pisani and C. Schubert, "*Scalar heat kernel with boundary in the worldline formalism*," JHEP **0810** (2008) 095; O. Corradini and C. Schubert, "*Spinning Particles in Quantum Mechanics and Quantum Field Theory*," arXiv:1512.08694 [hep-th].
14. A. Flachi, K. Fukushima, "*Chiral Mass-Gap in Curved Space*," Phys. Rev. Lett. **113**, 091102 (2014); A. Flachi, K. Fukushima and V. Vitagliano, "*Geometrically induced magnetic catalysis and critical dimensions*," Phys. Rev. Lett. **114**, no. 18, 181601 (2015).