FERMILAB-PUB-23-0828-STUDENT

Analytical Meridional, Non-Paraxial Ray Tracing

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(Dated: December 25, 2023)

We present an analytical method for non-paraxial, meridional ray tracing through a system of basic geometric surfaces derivable from a generalized ellipse; i.e. the most general ray tracing method for a 2-dimensional system comprised of elliptical, circular, and linear surfaces. This work is meant as a non-small angle generalization of standard ray tracing for geometric optics.

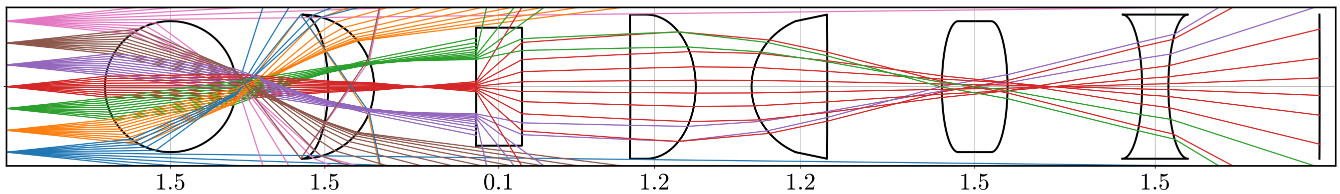

FIG. 1. Propagation of optical rays (color corresponds to initial position) from left to right through an arbitrary lens system (black) using the methods introduced in this paper. The system's medium has a refractive index of 1.0 with the refractive indices for each lens shown on the bottom. Rays are not propagated past the last plane or a point of total internal reflection. Lines connecting the left and right surfaces (e.g. top and bottom of the third, rectangular lens) are only visual.

INTRODUCTION

The study of light propagation in optical systems is crucial for various scientific and technological applications. The paraxial approximation is widely used and is easily accessible both online and through introductory optics textbooks owing to it providing an elegant ray propagation method in the form of matrices. However, the paraxial approximation fails to accurately describe light behavior at non-small angles (as shown in Figure 2) or with surfaces (lenses or mirrors) which are not approximately linear and perpendicular to the optical axis.

We acknowledge that previous work similar to ours exists in this area [1, 2], although our search did not yield any results which where directly and easily implementable to general optical systems. The intention of this work is not to present a novel idea, but rather to provide an accessible resource for generalized ray propagation not limited by the paraxial approximation.

To this end, we provide a simulation framework written in Python that leverages the analytical methods introduced in this paper for fast ($\mathcal{O}(N)$ where N is the number of surfaces) ray propagation through an arbitrary optical system as seen in Figure 1. Examples of its uses include analyzing sensitivity of the final-state ray to deviations in the rays initial state, implementation as the reward function for reinforcement learning-based lens design, and labeled data generation for supervised deep learning-based optical models.

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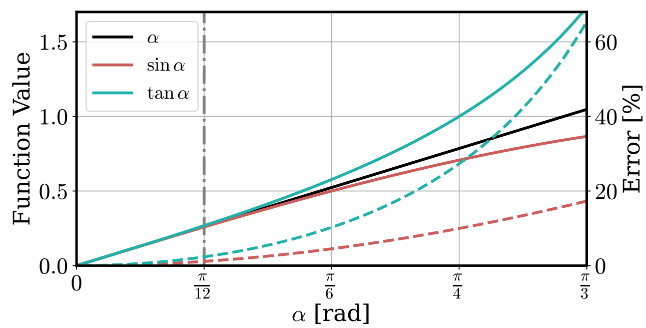

FIG. 2. Visualization of the small angle approximations $\sin\alpha \approx \alpha$ and $\tan\alpha \approx \alpha$ (solid lines) and their percent errors (dashed lines). $\pi/12$ $(= 15°)$ is commonly used as the maximum angle for which the small angle approximation holds.

LIMITATIONS OF MERIDIONAL RAY TRACING

This work uses meridional rays, which are confined to a plane coinciding with the optical axis. The utilization of a 2-dimensional simplification within a 3-dimensional system necessitates a careful examination of the optical systems to which these methods can be applied. Within the context of a 3-dimensional space ($\mathbb{R}^3$) where the optical axis aligns with the x-axis, the ray must lie in the meridional plane which we define as the xy-plane. Then, to ensure the ray stays in the same meridional plane throughout its propagation, we require $dx/dz = 0$ for all points on every surface within the xy-plane.

This manuscript has been authored by Fermi Research Alliance, LLC under Contract No. DE-AC02-07CH11359 with the U.S. Department of Energy, Office of Science, Office of High Energy Physics.

NOTATION

We define γ to be the ray state vector where $\gamma \equiv [x, y, \theta, 1]^T$ such that a ray γ is at position (x, y) and directed at angle θ with respect to the positive x-axis.

Unlike the paraxial propagation matrices, non-paraxial propagation matrices depend on $x, y, \theta \in \gamma$ such that x, y, and/or θ are in a propagation matrix. This mandates that γ is incrementally (in single steps) and chronologically (from initial-state to final-state) propagated through a system. This can be written as

$$\gamma' = \left(\mathcal{P}_n \left(\mathcal{P}_{n-1} \left(\ldots \left(\mathcal{P}_2 \left(\mathcal{P}_1 \gamma\right)\right) \ldots\right)\right)\right)$$

where $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$ is a set of propagation matrices and γ' is the final transmitted ray. To more easily notate this, we define a matrix propagation operator Ξ which is used to create directed processes, such that

$$\gamma' = \mathop{\Xi} \mathcal{P} \odot \gamma \equiv \left(\mathcal{P}_n \left(\mathcal{P}_{n-1} \left(\ldots \left(\mathcal{P}_2 \left(\mathcal{P}_1 \gamma\right)\right) \ldots\right)\right)\right).$$

RAY PROPAGATION MATRICES

In this section, we present matrices for translation ($\mathcal{T}$), refraction ($\mathcal{R}$), and reflection ($\mathcal{M}$) that enable the propagation of a ray through an arbitrary optical system composed of surfaces which pass the horizontal line test.

The most general form of a lens surface $\mathcal{L}$ in our optical system is an ellipse

$$1 = \left[\frac{(x' - x_l)\cos\phi + (y' - y_l)\sin\phi}{r_x}\right]^2 + \left[\frac{(x' - x_l)\sin\phi - (y' - y_l)\cos\phi}{r_y}\right]^2 \quad (1)$$

such that $(x', y') \in \mathcal{L}$ where r_x and r_y are the x- and y-radii of the ellipse respectively, (x_l, y_l) is the center position of the ellipse, and ϕ is the angular rotation of the ellipse about its center.

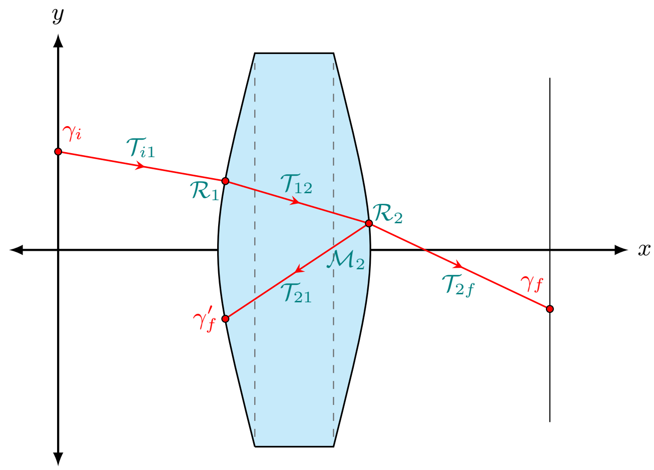

FIG. 3. Visualization of ray propagation in a one-lens optical system. We track the initial ray γ_i to two arbitrary final states γ_f and γ'_f where $\gamma_f = \Xi \{\mathcal{T}_{i1}, \mathcal{R}_1, \mathcal{T}_{12}, \mathcal{R}_2, \mathcal{T}_{2f}\} \odot \gamma_i$ and $\gamma'_f = \Xi \{\mathcal{T}_{i1}, \mathcal{R}_1, \mathcal{T}_{12}, \mathcal{M}_2, \mathcal{T}_{21}\} \odot \gamma_i$.

To distinguish between the two sides of the ellipse, we define $r_x < 0$ to indicate that the surface is the left side of the ellipse ($x' < x_l$), and $r_x > 0$ denotes that the surface is the right side of the ellipse ($x' > x_l$). Additionally, when $r_x \approx 0$, (1) becomes a linear surface such that $y' - y_l \approx (x' - x_l)\tan(\phi + \pi/2)$. Likewise, a circular surface can be found from (1) when $r_x = r_y = r$ such that $r^2 = (x' - x_l)^2 + (y' - y_l)^2$.

Translation Matrix

The translation matrix $\mathcal{T}$ translates a ray $\gamma = [x, y, \theta, 1]^T$ to the point $(x', y') \in \mathcal{L}$ such that the final state ray is $\gamma' = [x', y', \theta, 1]^T = \mathcal{T}\gamma$.

To find (x', y'), we start by writing γ's position as

$$y' = m(x' - x) - y, \quad (2)$$

where $m = \tan\theta$ and $x, y, \theta \in \gamma$. Then, (2) intersects the surface (1) at

$$x' = \frac{-c_2 + (r_x/|r_x|)\sqrt{c_2^2 - 4c_1c_3}}{2c_1}$$

where

$$c_1 = \left(m^2 r_y^2 + r_x^2\right)\sin^2\phi + \left(m^2 r_x^2 + r_y^2\right)\cos^2\phi + m(r_y - r_x)(r_x + r_y)\sin 2\phi,$$

$$c_2 = (r_x - r_y)(r_x + r_y)((m(x + x_l) - y + y_l)\cos 2\phi + (x_l - m(mx - y + y_l))\sin 2\phi) - \left(r_x^2 + r_y^2\right)(m(mx - y + y_l) + x_l),$$

$$c_3 = (r_y^2(mx - y + y_l)^2 + r_x^2 x_l^2)\sin^2\phi + (r_x^2(mx - y + y_l)^2 + r_y^2 x_l^2)\cos^2\phi + x_l(r_y - r_x)(r_x + r_y)(mx - y + y_l)\sin 2\phi - r_x^2 r_y^2.$$

Finally, we can simply write the translation matrix as

$$\mathcal{T} = \begin{bmatrix} 0 & 0 & 0 & x' \\ 0 & 0 & 0 & y' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Refraction Matrix

The refraction matrix $\mathcal{R}$ refracts a ray $\gamma_i' = [x', y', \theta, 1]^T$ through a surface $\mathcal{L}$ such that the final state ray is $\gamma_t' = [x', y', \theta', 1]^T = \mathcal{R}\gamma_i'$ where γ_t' is the transmitted ray and $(x', y') \in \mathcal{L}$ as seen in Figure 4. Using the law of refraction (Snell's law), $n_1 \sin\psi = n_2 \sin\psi'$, where n_1 and n_2 are the refractive indices in the initial and final mediums respectively, we find the refracted angle ψ' with respect to the normal angle η where γ is incident with the surface such that $\psi = \theta - \eta$.

To find η, we simply use the orthogonal line to the tangent of (1) such that

$$\tan\eta = -\frac{dx'}{dy'} = r_x^2 r_y^2 \Big[2(y_l - y')\left(r_x^2 \cos^2\phi + r_y^2 \sin^2\phi\right) - (r_x - r_y)(r_x + r_y)(x_l - x')\sin 2\phi \Big]^{-1}.$$

Then, $\theta' = \eta + \psi'$, such that the refraction matrix can be written as

$$\mathcal{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \theta' \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Reflection Matrix

The reflection matrix $\mathcal{M}$ reflects a ray off a surface $\mathcal{L}$ such that, for an incident ray $\gamma_i' = [x', y', \theta, 1]^T$ where $(x', y') \in \mathcal{L}$, $\gamma_t' = [x', y', \theta', 1] = \mathcal{M}\gamma_i'$ where γ_t' is the reflected ray. Using the law of reflection $\psi = -\psi'$, where $\psi = \theta - \eta$ and $\psi' = \theta_M' - \eta$, we can simply solve for $\theta' = 2\eta - \theta$. Then, we simply write the reflection matrix as

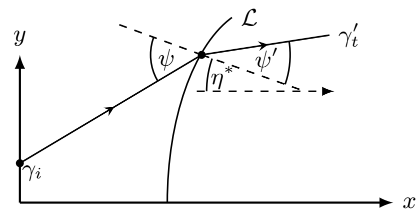

FIG. 4. Refraction diagram. The ray γ_i is propagated from right to left. γ_i is incident with the surface $\mathcal{L}$ at $(x', y') \in \gamma_i' = \mathcal{T}\gamma_i$. Then, γ_i' is refracted becoming $[x', y', \theta', 1]^T = \gamma_t' = \mathcal{R}\gamma_i'$. Alternatively, this system can be written as $\gamma_t' = \Xi\, \{\mathcal{T}, \mathcal{R}\} \odot \gamma_i$. The angles ψ and ψ' are relative to the angle η or $\eta^* = \pi - \eta$, which is normal to the surface $\mathcal{L}$ at (x', y'), and are used with the law of refraction to calculate the angle of refraction.

$$\mathcal{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \theta' \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

CONCLUSION

In this work, we have presented a matrix-based ray propagation method for transmission, refraction, and reflection of meridional, non-paraxial rays. While this is not a novel concept, this work is mean to serve as a direct and easily implementable 2-dimensional ray tracing method for non-paraxial rays. This method can be used for any ray which is confined to a 2-dimensional space with surfaces derivable from a general ellipse, i.e. elliptical, circular, and linear surfaces. The propagation matrices derived in this work and are also implemented in a Python simulation framework.

[1] A. Nussbaum, in *Education in Optics*, Vol. 1603, edited by G. B. Altshuler and B. J. Thompson, International Society for Optics and Photonics (SPIE, 1992) pp. 389 – 400.
[2] M. Born and E. Wolf, in *Principles of Optics* (Pergamon Press, 1980) p. 190–196, 6th ed.