Mixed three-point functions of conserved currents in three-dimensional superconformal field theory

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(Received 12 March 2021; accepted 5 April 2021; published 28 April 2021)

We consider mixed three-point correlation functions of the supercurrent and flavor current in three-dimensional $1 \leq \mathcal{N} \leq 4$ superconformal field theories. Our method is based on the decomposition of the relevant tensors into irreducible components to guarantee that all possible tensor structures are systematically taken into account. We show that only parity-even structures appear in the correlation functions. In addition to the previous results obtained in [E. I. Buchbinder *et al.*J. High Energy Phys. 06 (2015) 138], it follows that supersymmetry forbids parity-odd structures in three-point functions involving the supercurrent and flavor current multiplets.

DOI: 10.1103/PhysRevD.103.086023

I. INTRODUCTION

It is a well-known property of conformal field theories that the functional form of two- and three-point functions of conserved currents such as the energy-momentum tensor and vector current are fixed up to finitely many parameters. In [1,2] a systematic formalism was developed to construct two- and three-point functions of primary operators in diverse dimensions. The method was based on properly imposing the relevant symmetries arising from scale transformations and permutations of points as well as the conservation laws for the conserved currents (see also Refs. [3–10] for earlier work). More recently it was shown in [11] that a peculiar feature of three-dimensional (and perhaps, in general, odd-dimensional) conformal field theories is the appearance of parity-violating contributions in three-point functions of conserved currents. These structures were overlooked in the original study by Osborn and Petkou [1] (also [2]) and have since been shown to arise in Chern-Simons theories interacting with parity-violating matter. Parity-violating (or parity-odd) structures were studied in [12–20]. Recently they were also studied in light-cone gauge [21] and in momentum space [22].[1]

In contrast with the nonsupersymmetric case studied in [1,2], supersymmetry imposes additional restrictions on the structure of three-point functions of conserved currents. In supersymmetric field theories the energy-momentum tensor is replaced with the supercurrent multiplet [32], which contains the energy-momentum tensor, the supersymmetry current and additional components such as the R-symmetry current. Similarly, a conserved vector current becomes a component of the flavor current supermultiplet. The general formalism to construct the two- and three-point functions of primary operators in three-dimensional superconformal field theories was developed in [33–36].[2] Within this formalism it was shown in [34] that the three-point function of the supercurrent (and, hence, of the energy-momentum tensor) in three-dimensional $\mathcal{N} = 1$ superconformal theory is comprised of only one tensor structure. It was also shown that the three-point function of the non-Abelian flavor current (and, hence, the three-point function of conserved vector currents) also contains only one tensor structure. In both cases the tensor structures are parity even.

The aim of this paper is to apply the approach of [34] to the case of mixed correlators involving the supercurrent and flavor current multiplets in theories with $1 \leq \mathcal{N} \leq 4$ superconformal symmetry. Our method is based on a systematic decomposition of the relevant tensors into irreducible components, which guarantees that all possible linearly independent structures are consistently taken into account. We demonstrate that these correlation functions contain only parity-even structures; hence, in combination with the results of [34] we conclude that supersymmetry forbids

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[1] Parity-even correlation functions in momentum space were discussed in [23–31].

[2] A similar formalism in four dimensions was developed in [37–39] and in six dimensions in [40].

parity-odd structures in the three-point functions of conserved low-spin currents such as the energy-momentum tensor, supersymmetry current and conserved vector current. In [41] Maldacena and Zhiboedov showed under quite general assumptions that if a three-dimensional conformal field theory possesses a conserved higher spin current, then it is free. Since a free theory results in only parity-even contributions to correlation functions, we arrive at the conclusion that if the assumptions of [41] are fulfilled, one cannot obtain parity-odd structures in three-point functions of all conserved currents in supersymmetric conformal field theories.

The paper is organized as follows. In Sec. II we review the construction of the two-point and three-point building blocks which appear in correlation functions of primary superfields. We also review the general form of two- and three-point correlation functions of primary operators. In Sec. III we introduce a systematic approach to solve for correlation functions of conserved currents. We illustrate our method by reconsidering the flavor current three-point function which was previously computed in [34]. In Sec. IV we study three-point functions of mixed correlators involving both the supercurrent and the flavor current multiplet. We show that the three-point function involving one supercurrent and two flavor current multiplets is fixed by the $\mathcal{N}=1$ superconformal symmetry up to an overall coefficient. We also show that the three-point function involving two supercurrents and one flavor current vanishes. In Sec. V we present a systematic discussion regarding the absence of parity-violating structures in our results. In Sec. VI we generalize our method to superconformal theories with $\mathcal{N}=2$ supersymmetry. We show that both mixed correlators are fixed up to an overall coefficient. In Sec. VII we extend our analysis to the case of $\mathcal{N}=3$ and $\mathcal{N}=4$ superconformal symmetry. In the Appendix we summarize our three-dimensional notation and conventions.

The nonvanishing of the three-point function of two supercurrents and one flavor current in $\mathcal{N}=2$ theories is quite a surprise given that a similar three-point function vanishes in the $\mathcal{N}=1$ case. Naively it appears to be a contradiction, as any theory with $\mathcal{N}=2$ supersymmetry is also a theory with $\mathcal{N}=1$ supersymmetry. From an intuitive standpoint, the number of independent tensor structures cannot grow as one increases the number of supersymmetries. Nevertheless, we explain that our results in the $\mathcal{N}=1$ and $\mathcal{N}=2$ cases are fully consistent.

II. SUPERCONFORMAL BUILDING BLOCKS

The formalism to construct correlation functions of primary operators for conformal field theories in general dimensions was first elucidated in [1] using an efficient group theoretic formalism. In four dimensions the method was then extended to the case of $\mathcal{N}=1$ supersymmetry in [37,38,42] and was later generalized to higher $\mathcal{N}$ in [39]. Here we review the pertinent details of the three-dimensional formalism [33,34] necessary to construct correlation functions of the 3D supercurrent and flavor current multiplets.

A. Superconformal transformations and primary superfields

Let us begin by reviewing infinitesimal superconformal transformations and the transformation laws of primary superfields. This section closely follows the notation of [43–45]. Consider 3D $\mathcal{N}$-extended Minkowski superspace $\mathbb{M}^{3|2\mathcal{N}}$, parameterized by coordinates $z^A=(x^a,\theta^\alpha_I)$, where $a=0,1,2$, and $\alpha=1,2$ are Lorentz and spinor indices, while $I=1,\ldots,\mathcal{N}$ is the R-symmetry index. The 3D $\mathcal{N}$-extended superconformal group cannot act by smooth transformations on $\mathbb{M}^{3|2\mathcal{N}}$; in general, only infinitesimal superconformal transformations are well defined. Such a transformation

$$\begin{aligned}\delta z^A=\xi z^A \Leftrightarrow \delta x^a &= \xi^a(z)+\mathrm{i}(\gamma^a)_{\alpha\beta}\xi^\alpha_I(z)\theta^\beta_I,\\ \delta\theta^\alpha_I &= \xi^\alpha_I(z)\end{aligned}\tag{2.1}$$

is associated with the real first-order differential operator

$$\xi=\xi^A(z)\partial_A=\xi^a(z)\partial_a+\xi^\alpha_I(z)D^I_\alpha,\tag{2.2}$$

which satisfies the master equation $[\xi,D^I_\alpha]\propto D^J_\beta$. From the master equation we find

$$\xi^\alpha_I=\frac{\mathrm{i}}{6}D_{\beta I}\xi^{\alpha\beta},\tag{2.3}$$

which implies the conformal Killing equation

$$\partial_a\xi_b+\partial_b\xi_a=\frac{2}{3}\eta_{ab}\partial_c\xi^c.\tag{2.4}$$

The solutions to the master equation are called the conformal Killing supervector fields of Minkowski superspace [44,46]. They span a Lie algebra isomorphic to the superconformal algebra $\mathfrak{osp}(\mathcal{N}|2;\mathbb{R})$. The components of the operator ξ were calculated explicitly in [33] and are found to be

$$\begin{aligned}\xi^{\alpha\beta} &= a^{\alpha\beta}-\lambda^\alpha{}_\gamma x^{\gamma\beta}-x^{\alpha\gamma}\lambda_\gamma{}^\beta+\sigma x^{\alpha\beta}+4\mathrm{i}\epsilon^{(\alpha}_I\theta^{\beta)}_I+2\mathrm{i}\Lambda_{IJ}\theta^\alpha_J\theta^\beta_I\\ &\quad+x^{\alpha\gamma}x^{\beta\delta}b_{\gamma\delta}+\mathrm{i}b^{(\alpha}_\delta x^{\beta)\delta}\theta^2-\frac{1}{4}b^{\alpha\beta}\theta^2\theta^2\\ &\quad-4\mathrm{i}\eta_{\gamma I}x^{\gamma(\alpha}\theta^{\beta)}_I+2\eta^{(\alpha}_I\theta^{\beta)}_I\theta^2,\end{aligned}\tag{2.5a}$$

$$\begin{aligned}\xi^\alpha_I &= \epsilon^\alpha_I-\lambda^\alpha{}_\beta\theta^\beta_I+\frac{1}{2}\sigma\theta^\alpha_I+\Lambda_{IJ}\theta^\alpha_J+b_{\beta\gamma}\boldsymbol{x}^{\beta\alpha}\theta^\gamma_I\\ &\quad+\eta_{\beta J}(2\mathrm{i}\theta^\beta_I\theta^\alpha_J-\delta_{IJ}\boldsymbol{x}^{\beta\alpha}),\end{aligned}\tag{2.5b}$$

$$a_{\alpha\beta} = a_{\beta\alpha}, \qquad \lambda_{\alpha\beta} = \lambda_{\beta\alpha}, \qquad \lambda^\alpha{}_\alpha = 0,$$
$$b_{\alpha\beta} = b_{\beta\alpha}, \qquad \Lambda_{IJ} = -\Lambda_{JI}. \tag{2.6}$$

The bosonic parameters $a_{\alpha\beta}, \lambda_{\alpha\beta}, \sigma, b_{\alpha\beta}$, and Λ_{IJ} correspond to infinitesimal translations, Lorentz transformations, scale transformations, special conformal transformations and R-symmetry transformations, respectively, while the fermionic parameters ϵ_I^α and η_I^α correspond to Q-supersymmetry and S-supersymmetry transformations. Furthermore, the identities

$$D^I_{[\alpha}\xi^J_{\beta]} \propto \varepsilon_{\alpha\beta}, \qquad D^I_{(\alpha}\xi^J_{\beta)} \propto \delta^{IJ}, \qquad D^{(I}_{[\alpha}\xi^{J)}_{\beta]} \propto \delta^{IJ}\varepsilon_{\alpha\beta} \tag{2.7}$$

imply that

$$[\xi, D^I_\alpha] = -(D^I_\alpha \xi^\beta_J)D^J_\beta = \lambda_\alpha{}^\beta(z)D^I_\beta + \Lambda^{IJ}(z)D^J_\alpha - \frac{1}{2}\sigma(z)D^I_\alpha, \tag{2.8}$$

$$\lambda_{\alpha\beta}(z) = -\frac{1}{\mathcal{N}}D^I_{(\alpha}\xi^I_{\beta)}, \qquad \Lambda^{IJ}(z) = -2D^{[I}_\alpha\xi^{J]\alpha},$$
$$\sigma(z) = \frac{1}{\mathcal{N}}D^I_\alpha\xi^\alpha_I. \tag{2.9}$$

The local parameters $\lambda^{\alpha\beta}(z)$, $\Lambda_{IJ}(z)$, and $\sigma(z)$ are interpreted as being associated with combined special-conformal or Lorentz, R-symmetry and scale transformations, respectively, and appear in the transformation laws for primary tensor superfields. For later use let us also introduce the z-dependent S-supersymmetry parameter

$$\eta_{I\alpha}(z) = -\frac{\mathrm{i}}{2}D_{I\alpha}\sigma(z). \tag{2.10}$$

Explicit calculations of the local parameters give [33]

$$\lambda^{\alpha\beta}(z) = \lambda^{\alpha\beta} - x^{\gamma(\alpha}b^{\beta)}_\gamma - \frac{\mathrm{i}}{2}b^{\alpha\beta}\theta^2 + 2\mathrm{i}\eta_I^{(\alpha}\theta_I^{\beta)}, \tag{2.11a}$$

$$\Lambda_{IJ}(z) = \Lambda_{IJ} + 4\mathrm{i}\eta^\alpha_{[I}\theta_{J]\alpha} + 2\mathrm{i}b_{\alpha\beta}\theta^\alpha_I\theta^\beta_J, \tag{2.11b}$$

$$\sigma(z) = \sigma + b_{\alpha\beta}x^{\alpha\beta} + 2\mathrm{i}\theta^\alpha_I\eta_{\alpha I}, \tag{2.11c}$$

$$\eta_{\alpha I}(z) = \eta_{\alpha I} - b_{\alpha\beta}\theta^\beta_I. \tag{2.11d}$$

Now consider a generic tensor superfield $\Phi^{\mathcal{I}}_{\mathcal{A}}(z)$ transforming in a representation T of the Lorentz group with respect to the index $\mathcal{A}$, and in the representation D of the R-symmetry group $\mathsf{O}(\mathcal{N})$ with respect to the index $\mathcal{I}$.[3] Such a superfield is called primary with dimension q if its superconformal transformation law is

[3]We assume the representations T and D are irreducible.

$$\delta\Phi^{\mathcal{I}}_{\mathcal{A}} = -\xi\Phi^{\mathcal{I}}_{\mathcal{A}} - q\sigma(z)\Phi^{\mathcal{I}}_{\mathcal{A}} + \lambda^{\alpha\beta}(z)(M_{\alpha\beta})_{\mathcal{A}}{}^{\mathcal{B}}\Phi^{\mathcal{I}}_{\mathcal{B}}$$
$$+ \Lambda^{IJ}(z)(R_{IJ})^{\mathcal{I}}{}_{\mathcal{J}}\Phi^{\mathcal{J}}_{\mathcal{A}}, \tag{2.12}$$

where ξ is the superconformal Killing vector, $\sigma(z)$, $\lambda^{\alpha\beta}(z)$, and $\Lambda_{IJ}(z)$ are the z-dependent parameters associated with ξ, and the matrices $M_{\alpha\beta}$ and R_{IJ} are the Lorentz and $\mathsf{O}(\mathcal{N})$ generators, respectively.

B. Two-point functions

Given two superspace points z_1 and z_2, we can define the two-point functions

$$\boldsymbol{x}^{\alpha\beta}_{12} = (x_1 - x_2)^{\alpha\beta} + 2\mathrm{i}\theta^{(\alpha}_{1I}\theta^{\beta)}_{2I} - \mathrm{i}\theta^\alpha_{12I}\theta^\beta_{12I},$$
$$\theta^{\alpha I}_{12} = \theta^{\alpha I}_1 - \theta^{\alpha I}_2, \tag{2.13}$$

which transform under the superconformal group as follows:

$$\tilde{\delta}\boldsymbol{x}^{\alpha\beta}_{12} = \left(\frac{1}{2}\delta^\alpha{}_\gamma\sigma(z_1) - \lambda^\alpha{}_\gamma(z_1)\right)\boldsymbol{x}^{\gamma\beta}_{12}$$
$$+ \boldsymbol{x}^{\alpha\gamma}_{12}\left(\frac{1}{2}\delta_\gamma{}^\beta\sigma(z_2) - \lambda_\gamma{}^\beta(z_2)\right), \tag{2.14a}$$

$$\tilde{\delta}\theta^\alpha_{12I} = \left(\frac{1}{2}\delta^\alpha{}_\beta\sigma(z_1) - \lambda^\alpha{}_\beta(z_1)\right)\theta^\beta_{12I}$$
$$- \boldsymbol{x}^{\alpha\beta}_{12}\eta_{\beta I}(z_2) + \Lambda_{IJ}(z_2)\theta^\alpha_{12J}. \tag{2.14b}$$

Here the total variation $\tilde{\delta}$ is defined by its action on an n-point function $\Phi(z_1, \ldots, z_n)$ as

$$\tilde{\delta}\Phi(z_1, \ldots, z_n) = \sum_{i=1}^n \xi_{z_i}\Phi(z_1, \ldots, z_n). \tag{2.15}$$

It should be noted that (2.14b) contains an inhomogeneous piece in its transformation law; hence, it will not appear as a building block in two- or three-point functions. Due to the useful property $\boldsymbol{x}^{\alpha\beta}_{21} = -\boldsymbol{x}^{\beta\alpha}_{12}$, the two-point function (2.13) can be split into symmetric and antisymmetric parts as follows:

$$\boldsymbol{x}^{\alpha\beta}_{12} = x^{\alpha\beta}_{12} + \frac{\mathrm{i}}{2}\varepsilon^{\alpha\beta}\theta^2_{12}, \qquad \theta^2_{12} = \theta^\alpha_{12I}\theta_{12\alpha I}. \tag{2.16}$$

The symmetric component

$$x^{\alpha\beta}_{12} = (x_1 - x_2)^{\alpha\beta} + 2\mathrm{i}\theta^{(\alpha}_{1I}\theta^{\beta)}_{2I} \tag{2.17}$$

is recognized as the bosonic part of the standard two-point superspace interval. Next let us introduce the two-point objects

$$\boldsymbol{x}_{12}^2 = -\frac{1}{2}\boldsymbol{x}_{12}^{\alpha\beta}\boldsymbol{x}_{12\alpha\beta}, \tag{2.18a}$$

$$\hat{\boldsymbol{x}}_{12}^{\alpha\beta} = \frac{\boldsymbol{x}_{12}^{\alpha\beta}}{\sqrt{\boldsymbol{x}_{12}^2}}, \qquad \hat{\boldsymbol{x}}_{12\alpha}{}^{\gamma}\hat{\boldsymbol{x}}_{12\gamma}{}^{\beta} = \delta_\alpha{}^\beta. \tag{2.18b}$$

Hence, we find

$$(\boldsymbol{x}_{12}^{-1})^{\alpha\beta} = -\frac{\boldsymbol{x}_{12}^{\beta\alpha}}{\boldsymbol{x}_{12}^2}. \tag{2.19}$$

Under superconformal transformations, (2.18a) transforms with local scale parameters, while (2.18b) transforms with local Lorentz parameters:

$$\tilde{\delta}\boldsymbol{x}_{12}^2 = (\sigma(z_1)+\sigma(z_2))\boldsymbol{x}_{12}^2, \tag{2.20a}$$

$$\tilde{\delta}\hat{\boldsymbol{x}}_{12}^{\alpha\beta} = -\lambda^\alpha{}_\gamma(z_1)\hat{\boldsymbol{x}}_{12}^{\gamma\beta} - \hat{\boldsymbol{x}}_{12}^{\alpha\gamma}\lambda_\gamma{}^\beta(z_2). \tag{2.20b}$$

Thus, both objects are essential in the construction of correlation functions of primary superfields. We also have the useful differential identities

$$D^I_{(1)\gamma}\boldsymbol{x}_{12}^{\alpha\beta} = -2\mathrm{i}\theta_{12}^{I\beta}\delta^\alpha_\gamma, \qquad D^I_{(1)\alpha}\boldsymbol{x}_{12}^{\alpha\beta} = -4\mathrm{i}\theta_{12}^{I\beta}, \tag{2.21}$$

where $D^I_{(i)\alpha}$ is the standard covariant spinor derivative (A16) acting on the superspace point z_i. Finally, for completeness, the SO($\mathcal{N}$) structure of primary superfields in correlation functions is addressed by the $\mathcal{N}\times\mathcal{N}$ matrix

$$u_{12}^{IJ} = \delta^{IJ} + 2\mathrm{i}\theta_{12}^{I\alpha}(\boldsymbol{x}_{12}^{-1})_{\alpha\beta}\theta_{12}^{J\beta}, \tag{2.22}$$

which is orthogonal and unimodular,

$$u_{12}^{IK}u_{12}^{KJ} = \delta^{IJ}, \qquad \det u_{12} = 1. \tag{2.23}$$

The infinitesimal variation of this matrix is

$$\tilde{\delta}u_{12}^{IJ} = \Lambda^{IK}(z_1)u_{12}^{KJ} - u_{12}^{IK}\Lambda^{KJ}(z_2). \tag{2.24}$$

Hence, (2.22) is expected to appear in the construction of correlation functions of primary superfields with SO($\mathcal{N}$) indices.

The two-point correlation function of a primary superfield $\Phi^{\mathcal{I}}_{\mathcal{A}}$ and its conjugate $\bar{\Phi}^{\mathcal{B}}_{\mathcal{J}}$ is fixed by the superconformal symmetry as follows:

$$\langle\Phi^{\mathcal{I}}_{\mathcal{A}}(z_1)\bar{\Phi}^{\mathcal{B}}_{\mathcal{J}}(z_2)\rangle = c\frac{T_{\mathcal{A}}{}^{\mathcal{B}}(\hat{\boldsymbol{x}}_{12})D^{\mathcal{I}}{}_{\mathcal{J}}(u_{12})}{(\boldsymbol{x}_{12}^2)^q}, \tag{2.25}$$

where c is a constant coefficient. The denominator of the two-point function is determined by the conformal dimension of $\Phi^{\mathcal{I}}_{\mathcal{A}}$, which guarantees that the correlation function transforms with the appropriate weight under scale transformations.

C. Three-point functions

Given three superspace points z_i, $i = 1, 2, 3$, one can define the three-point building blocks $\mathcal{Z}_i = (\boldsymbol{x}_i, \Theta_i)$ as follows:

$$\boldsymbol{X}_{1\alpha\beta} = -(\boldsymbol{x}_{21}^{-1})_{\alpha\gamma}\boldsymbol{x}_{23}^{\gamma\delta}(\boldsymbol{x}_{13}^{-1})_{\delta\beta}, \qquad \Theta^I_{1\alpha} = (\boldsymbol{x}_{21}^{-1})_{\alpha\beta}\theta_{12}^{I\beta} - (\boldsymbol{x}_{31}^{-1})_{\alpha\beta}\theta_{13}^{I\beta}, \tag{2.26a}$$

$$\boldsymbol{X}_{2\alpha\beta} = -(\boldsymbol{x}_{32}^{-1})_{\alpha\gamma}\boldsymbol{x}_{31}^{\gamma\delta}(\boldsymbol{x}_{21}^{-1})_{\delta\beta}, \qquad \Theta^I_{2\alpha} = (\boldsymbol{x}_{32}^{-1})_{\alpha\beta}\theta_{23}^{I\beta} - (\boldsymbol{x}_{12}^{-1})_{\alpha\beta}\theta_{21}^{I\beta}, \tag{2.26b}$$

$$\boldsymbol{X}_{3\alpha\beta} = -(\boldsymbol{x}_{13}^{-1})_{\alpha\gamma}\boldsymbol{x}_{12}^{\gamma\delta}(\boldsymbol{x}_{32}^{-1})_{\delta\beta}, \qquad \Theta^I_{3\alpha} = (\boldsymbol{x}_{13}^{-1})_{\alpha\beta}\theta_{31}^{I\beta} - (\boldsymbol{x}_{23}^{-1})_{\alpha\beta}\theta_{32}^{I\beta}. \tag{2.26c}$$

These objects, along with their corresponding transformation laws, may be obtained from one another by cyclic permutation of superspace points. The building blocks transform covariantly under the action of the superconformal group:

$$\tilde{\delta}\boldsymbol{X}_{1\alpha\beta} = \lambda_\alpha{}^\gamma(z_1)\boldsymbol{X}_{1\gamma\beta} + \boldsymbol{X}_{1\alpha\gamma}\lambda^\gamma{}_\beta(z_1) - \sigma(z_1)\boldsymbol{X}_{1\alpha\beta}, \tag{2.27a}$$

$$\tilde{\delta}\Theta^I_{1\alpha} = \left(\lambda_\alpha{}^\beta(z_1) - \frac{1}{2}\delta_\alpha{}^\beta\sigma(z_1)\right)\Theta^I_{1\beta} + \Lambda^{IJ}(z_1)\Theta^J_{1\alpha}. \tag{2.27b}$$

Therefore (2.26a), (2.26b) and (2.26c) will appear as building blocks in three-point correlation functions. It should be noted that under scale transformations of superspace, $z^A = (x^a, \theta^\alpha) \mapsto z'^A = (\lambda^{-2}x^a, \lambda^{-1}\theta^\alpha)$, the three-point building blocks transform as $\mathcal{Z} = (\boldsymbol{X}, \Theta) \mapsto \mathcal{Z}' = (\lambda^2\boldsymbol{X}, \lambda\Theta)$. Next we define

$$X_1^2 = -\frac{1}{2}X_1^{\alpha\beta}X_{1\alpha\beta} = \frac{\boldsymbol{x}_{23}^2}{\boldsymbol{x}_{13}^2\boldsymbol{x}_{12}^2}, \qquad \Theta_1^2 = \Theta_1^{I\alpha}\Theta^I_{1\alpha}, \tag{2.28}$$

which, due to (2.27a) and (2.27b), have the transformation laws

$$\tilde{\delta} \boldsymbol{X}_1^2 = -2\sigma(z_1)\boldsymbol{X}_1^2, \qquad \tilde{\delta}\Theta_1^2 = -\sigma(z_1)\Theta_1^2. \tag{2.29}$$

We also define the inverse of $\boldsymbol{X}_1$,

$$(\boldsymbol{X}_1^{-1})^{\alpha\beta} = -\frac{\boldsymbol{X}_1^{\beta\alpha}}{\boldsymbol{X}_1^2}, \tag{2.30}$$

and introduce useful identities involving $\boldsymbol{X}_i$ and Θ_i at different superspace points, e.g.,

$$\boldsymbol{x}_{13}^{\alpha\alpha'}\boldsymbol{X}_{3\alpha'\beta'}\boldsymbol{x}_{31}^{\beta'\beta} = -(\boldsymbol{X}_1^{-1})^{\beta\alpha}, \tag{2.31a}$$

$$\Theta^I_{1\gamma}\boldsymbol{x}_{13}^{\gamma\delta}\boldsymbol{X}_{3\delta\beta} = u_{13}^{IJ}\Theta^J_{3\beta}. \tag{2.31b}$$

As a consequence of (2.29), we can identify the three-point superconformal invariant

$$\frac{\Theta_1^2}{\sqrt{\boldsymbol{X}_1^2}} \Rightarrow \tilde{\delta}\left(\frac{\Theta_1^2}{\sqrt{\boldsymbol{X}_1^2}}\right) = 0. \tag{2.32}$$

Hence, the superconformal symmetry fixes the functional form of three-point correlation functions up to this combination. Indeed, using (2.31a) and (2.31b) one can show that the superconformal invariant is also invariant under permutation of superspace points, i.e.,

$$\frac{\Theta_1^2}{\sqrt{\boldsymbol{X}_1^2}} = \frac{\Theta_2^2}{\sqrt{\boldsymbol{X}_2^2}} = \frac{\Theta_3^2}{\sqrt{\boldsymbol{X}_3^2}}. \tag{2.33}$$

The three-point objects (2.26a), (2.26b) and (2.26c) have many properties similar to those of the two-point building blocks. After decomposing $\boldsymbol{X}_1$ into symmetric and antisymmetric parts similar to (2.16) we have

$$\boldsymbol{X}_{1\alpha\beta} = X_{1\alpha\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}\Theta_1^2, \qquad X_{1\alpha\beta} = X_{1\beta\alpha}, \tag{2.34}$$

where the symmetric spinor $X_{1\alpha\beta}$ can be equivalently represented by the three-vector $X_{1m} = -\frac{1}{2}(\gamma_m)^{\alpha\beta}X_{1\alpha\beta}$. It is now convenient to introduce analogs of the covariant spinor derivative and supercharge operators involving the three-point objects,

$$\mathcal{D}^I_{(1)\alpha} = \frac{\partial}{\partial\Theta^\alpha_{1I}} + \mathrm{i}(\gamma^m)_{\alpha\beta}\Theta_1^{I\beta}\frac{\partial}{\partial X_1^m},$$
$$\mathcal{Q}^I_{(1)\alpha} = \mathrm{i}\frac{\partial}{\partial\Theta^\alpha_{1I}} + (\gamma^m)_{\alpha\beta}\Theta_1^{I\beta}\frac{\partial}{\partial X_1^m}, \tag{2.35}$$

which obey the standard commutation relations

$$\{\mathcal{D}^I_{(i)\alpha}, \mathcal{D}^J_{(i)\beta}\} = \{\mathcal{Q}^I_{(i)\alpha}, \mathcal{Q}^J_{(i)\beta}\} = 2\mathrm{i}\delta^{IJ}(\gamma^m)_{\alpha\beta}\frac{\partial}{\partial X_i^m}. \tag{2.36}$$

Some useful identities involving (2.35) are

$$\mathcal{D}^I_{(1)\gamma}\boldsymbol{X}_{1\alpha\beta} = -2\mathrm{i}\varepsilon_{\gamma\beta}\Theta^I_{1\alpha}, \qquad \mathcal{Q}^I_{(1)\gamma}\boldsymbol{X}_{1\alpha\beta} = -2\varepsilon_{\gamma\alpha}\Theta^I_{1\beta}. \tag{2.37}$$

We must also account for the fact that various primary superfields obey certain differential equations. Using (2.21) we arrive at the following:

$$D^I_{(1)\gamma}\boldsymbol{X}_{3\alpha\beta} = 2\mathrm{i}(\boldsymbol{x}_{13}^{-1})_{\alpha\gamma}u_{13}^{IJ}\Theta^J_{3\beta}, \quad D^I_{(1)\alpha}\Theta^J_{3\beta} = -(\boldsymbol{x}_{13}^{-1})_{\beta\alpha}u_{13}^{IJ}, \tag{2.38a}$$

$$D^I_{(2)\gamma}\boldsymbol{X}_{3\alpha\beta} = 2\mathrm{i}(\boldsymbol{x}_{23}^{-1})_{\beta\gamma}u_{23}^{IJ}\Theta^J_{3\beta}, \qquad D^I_{(2)\alpha}\Theta^J_{3\beta} = (\boldsymbol{x}_{23}^{-1})_{\beta\alpha}u_{23}^{IJ}. \tag{2.38b}$$

Now given a function $f(\boldsymbol{X}_3, \Theta_3)$, there are the following differential identities which arise as a consequence of (2.37), (2.38a) and (2.38b):

$$D^I_{(1)\gamma}f(\boldsymbol{X}_3, \Theta_3) = (\boldsymbol{x}_{13}^{-1})_{\alpha\gamma}u_{13}^{IJ}\mathcal{D}^{J\alpha}_{(3)}f(\boldsymbol{X}_3, \Theta_3), \tag{2.39a}$$

$$D^I_{(2)\gamma}f(\boldsymbol{X}_3, \Theta_3) = \mathrm{i}(\boldsymbol{x}_{23}^{-1})_{\alpha\gamma}u_{23}^{IJ}\mathcal{Q}^{J\alpha}_{(3)}f(\boldsymbol{X}_3, \Theta_3). \tag{2.39b}$$

These will prove to be essential for imposing differential constraints on correlation functions, e.g., those arising from conservation equations in the case of correlators involving the supercurrent and flavor current multiplets.

Finally, for completeness, let us introduce the three-point objects which take care of the R-symmetry structure of correlation functions. We define

$$U_1^{IJ} = u_{12}^{IK}u_{23}^{KL}u_{31}^{LJ} = \delta^{IJ} + 2\mathrm{i}\Theta^I_{1\alpha}(\boldsymbol{X}_1^{-1})^{\alpha\beta}\Theta^J_{1\beta}, \tag{2.40}$$

which transforms as an $\mathrm{O}(\mathcal{N})$ tensor at z_1,

$$\tilde{\delta}U_1^{IJ} = \Lambda^{IK}(z_1)U_1^{KJ} - U_1^{IK}\Lambda^{KJ}(z_1), \tag{2.41}$$

and is orthogonal and unimodular by construction. The others are obtained by cyclic permutation of superspace points and are related by the useful identities

$$U_2^{IJ} = u_{21}^{IK}U_1^{KL}u_{12}^{LJ}, \qquad U_3^{IJ} = u_{31}^{IK}U_1^{KL}u_{13}^{LJ}. \tag{2.42}$$

As concerns three-point correlation functions, let Φ, Ψ, and Π be primary superfields with conformal dimensions q_1, q_2 and q_3, respectively. The three-point function may be constructed using the general expression

$$\langle\Phi^{\mathcal{I}_1}_{\mathcal{A}_1}(z_1)\Psi^{\mathcal{I}_2}_{\mathcal{A}_2}(z_2)\Pi^{\mathcal{I}_3}_{\mathcal{A}_3}(z_3)\rangle = \frac{T^{(1)}{}_{\mathcal{A}_1}{}^{\mathcal{B}_1}(\hat{\boldsymbol{x}}_{13})T^{(2)}{}_{\mathcal{A}_2}{}^{\mathcal{B}_2}(\hat{\boldsymbol{x}}_{23})D^{(1)\mathcal{I}_1}{}_{\mathcal{J}_1}(u_{13})D^{(2)\mathcal{I}_2}{}_{\mathcal{J}_2}(u_{23})}{(\boldsymbol{x}^2_{13})^{q_1}(\boldsymbol{x}^2_{23})^{q_2}}\mathcal{H}^{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}_{\mathcal{B}_1\mathcal{B}_2\mathcal{A}_3}(\boldsymbol{X}_3,\Theta_3,U_3), \tag{2.43}$$

where the tensor $\mathcal{H}^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}$ is highly constrained by the superconformal symmetry as follows.

(i) Under scale transformations of superspace the correlation function transforms as

$$\langle\Phi^{\mathcal{I}_1}_{\mathcal{A}_1}(z_1')\Psi^{\mathcal{I}_2}_{\mathcal{A}_2}(z_2')\Pi^{\mathcal{I}_3}_{\mathcal{A}_3}(z_3')\rangle = (\lambda^2)^{q_1+q_2+q_3}\langle\Phi^{\mathcal{I}_1}_{\mathcal{A}_1}(z_1)\Psi^{\mathcal{I}_2}_{\mathcal{A}_2}(z_2)\Pi^{\mathcal{I}_3}_{\mathcal{A}_3}(z_3)\rangle, \tag{2.44}$$

which implies that $\mathcal{H}$ obeys the scaling property

$$\mathcal{H}^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(\lambda^2\boldsymbol{X},\lambda\Theta,U) = (\lambda^2)^{q_3-q_2-q_1}\mathcal{H}^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(\boldsymbol{X},\Theta,U), \quad \forall\ \lambda\in\mathbb{R}\backslash\{0\}. \tag{2.45}$$

This guarantees that the correlation function transforms correctly under conformal transformations.

(ii) If any of the fields Φ, Ψ, and Π obey differential equations, such as conservation laws in the case of conserved current multiplets, then the tensor $\mathcal{H}$ is also constrained by differential equations. Such constraints may be derived with the aid of identities (2.39a) and (2.39b).

(iii) If any (or all) of the superfields Φ, Ψ, and Π coincide, the correlation function possesses symmetries under permutations of superspace points, e.g.,

$$\langle\Phi^{\mathcal{I}_1}_{\mathcal{A}_1}(z_1)\Phi^{\mathcal{I}_2}_{\mathcal{A}_2}(z_2)\Pi^{\mathcal{I}_3}_{\mathcal{A}_3}(z_3)\rangle = (-1)^{\epsilon(\Phi)}\langle\Phi^{\mathcal{I}_2}_{\mathcal{A}_2}(z_2)\Phi^{\mathcal{I}_1}_{\mathcal{A}_1}(z_1)\Pi^{\mathcal{I}_3}_{\mathcal{A}_3}(z_3)\rangle, \tag{2.46}$$

where $\epsilon(\Phi)$ is the Grassmann parity of Φ. As a consequence, the tensor $\mathcal{H}$ obeys constraints which will be referred to as "point-switch identities." To analyze these constraints, we note that under permutations of any two superspace points, the three-point building blocks transform as

$$\boldsymbol{X}_{3\alpha\beta}\overset{1\leftrightarrow 2}{\to}-\boldsymbol{X}_{3\beta\alpha}, \qquad \Theta^I_{3\alpha}\overset{1\leftrightarrow 2}{\to}-\Theta^I_{3\alpha}, \tag{2.47a}$$

$$\boldsymbol{X}_{3\alpha\beta}\overset{2\leftrightarrow 3}{\to}-\boldsymbol{X}_{2\beta\alpha}, \qquad \Theta^I_{3\alpha}\overset{2\leftrightarrow 3}{\to}-\Theta^I_{2\alpha}, \tag{2.47b}$$

$$\boldsymbol{X}_{3\alpha\beta}\overset{1\leftrightarrow 3}{\to}-\boldsymbol{X}_{1\beta\alpha}, \qquad \Theta^I_{3\alpha}\overset{1\leftrightarrow 3}{\to}-\Theta^I_{1\alpha}. \tag{2.47c}$$

The constraints above fix the functional form of $\mathcal{H}$ (and therefore the correlation function) up to finitely many parameters. Hence the procedure described above reduces the problem of computing three-point correlation functions to deriving the tensor $\mathcal{H}$ subject to the above constraints. In the next sections, we will apply this formalism to compute three-point correlation functions involving the supercurrent and flavor current multiplets.

III. CORRELATION FUNCTIONS OF CONSERVED CURRENTS IN $\mathcal{N}=1$ SUPERCONFORMAL FIELD THEORY

A. Supercurrent and flavor current multiplets

The 3D, $\mathcal{N}=1$ conformal supercurrent is a primary, dimension $5/2$ totally symmetric spin tensor $J_{\alpha\beta\gamma}$, which contains the three-dimensional energy-momentum tensor along with the supersymmetry current [46–48]. It obeys the conservation equation

$$D^\alpha J_{\alpha\beta\gamma} = 0 \tag{3.1}$$

and has the following superconformal transformation law:

$$\delta J_{\alpha\beta\gamma} = -\xi J_{\alpha\beta\gamma} - \frac{5}{2}\sigma(z)J_{\alpha\beta\gamma} + 3\lambda(z)_{(\alpha}^{\delta}J_{\beta\gamma)\delta}. \tag{3.2}$$

The $\mathcal{N}=1$ supercurrent may be derived from, for example, supergravity prepotential approaches [46] or the superfield Noether procedure [49,50].

The general formalism in Sec. II allows the two-point function to be determined up to a single real coefficient:

$$\langle J_{\alpha\beta\gamma}(z_1)J^{\alpha'\beta'\gamma'}(z_2)\rangle = \mathrm{i}b_{\mathcal{N}=1}\frac{\boldsymbol{x}_{12(\alpha}{}^{\alpha'}\boldsymbol{x}_{12\beta}{}^{\beta'}\boldsymbol{x}_{12\gamma)}{}^{\gamma'}}{(\boldsymbol{x}^2_{12})^4}. \tag{3.3}$$

It is then a simple exercise to show that the two-point function has the right symmetry properties under permutation of superspace points

$$\langle J_{\alpha\beta\gamma}(z_1)J_{\alpha'\beta'\gamma'}(z_2)\rangle = -\langle J_{\alpha'\beta'\gamma'}(z_2)J_{\alpha\beta\gamma}(z_1)\rangle \tag{3.4}$$

and also satisfies

$$D^\alpha_{(1)}\langle J_{\alpha\beta\gamma}(z_1)J_{\alpha'\beta'\gamma'}(z_2)\rangle = 0. \tag{3.5}$$

Next let us consider the 3D $\mathcal{N}=1$ flavor current, which is represented by a primary, dimension $3/2$ spinor superfield L_α obeying the conservation equation[4]

$$D^\alpha L_\alpha = 0. \tag{3.6}$$

It transforms covariantly under the superconformal group as

$$\delta L_\alpha = -\xi L_\alpha - \frac{3}{2}\sigma(z) L_\alpha + \lambda(z)_\alpha{}^\beta L_\beta. \tag{3.7}$$

We can also consider the case when there are several flavor current multiplets (represented by the flavor index $\bar{a}$) corresponding to a simple flavor group. According to general formalism in Sec. II, the two-point function for $\mathcal{N}=1$ flavor current multiplets is fixed up to a single real coefficient $a_{\mathcal{N}=1}$:

$$\langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2)\rangle = \mathrm{i} a_{\mathcal{N}=1} \frac{\delta^{\bar{a}\bar{b}} \boldsymbol{x}_{12\alpha\beta}}{(\boldsymbol{x}^2_{12})^2}. \tag{3.8}$$

It is easy to see that the two-point function obeys the correct symmetry properties under permutation of superspace points, $\langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2)\rangle = -\langle L^{\bar{b}}_\beta(z_2) L^{\bar{a}}_\alpha(z_1)\rangle$. One can also check that it satisfies the conservation equation (3.6):

$$D^\alpha_{(1)}\langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2)\rangle = 0. \tag{3.9}$$

Three-point correlation functions of the flavor current and particularly the supercurrent are considerably more complicated and were derived in [34,35]. However, correlators of *combinations* of these fields (mixed correlators) were not studied previously and will be analysed in Sec. IV.

B. Correlation functions of conserved current multiplets

The possible three-point correlation functions that may be constructed from the conserved $\mathcal{N}=1$ supercurrent and flavor current multiplets are

$$\langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2) L^{\bar{c}}_\gamma(z_3)\rangle, \qquad \langle J_{\mathcal{A}}(z_1) J_{\mathcal{B}}(z_2) J_{\mathcal{C}}(z_3)\rangle, \tag{3.10}$$

$$\langle L^{\bar{a}}_\alpha(z_1) J_{\mathcal{A}}(z_2) L^{\bar{b}}_\beta(z_3)\rangle, \qquad \langle J_{\mathcal{A}}(z_1) J_{\mathcal{B}}(z_2) L^{\bar{a}}_\alpha(z_3)\rangle, \tag{3.11}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ each denote a totally symmetric combination of three spinor indices. The correlators $\langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2) L^{\bar{c}}_\gamma(z_3)\rangle$ and $\langle J_{\mathcal{A}}(z_1) J_{\mathcal{B}}(z_2) J_{\mathcal{C}}(z_3)\rangle$ were studied in [34]. Before we compute the mixed correlators, let us demonstrate our method on the three-point function $\langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2) L^{\bar{c}}_\gamma(z_3)\rangle$, which is comparatively straightforward.

The general form of the flavor current three-point function is[5]

$$\langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2) L^{\bar{c}}_\gamma(z_3)\rangle = f^{\bar{a}\bar{b}\bar{c}} \frac{\boldsymbol{x}_{13\alpha}{}^{\alpha'} \boldsymbol{x}_{23\beta}{}^{\beta'}}{(\boldsymbol{x}^2_{13})^2 (\boldsymbol{x}^2_{23})^2} \mathcal{H}_{\alpha'\beta'\gamma}(\boldsymbol{X}_3, \Theta_3). \tag{3.12}$$

The correlation function is required to satisfy the following properties.

(i) *Scaling constraint.*—Under scale transformations the correlation function must transform as

$$\langle L^{\bar{a}}_\alpha(z'_1) L^{\bar{b}}_\beta(z'_2) L^{\bar{c}}_\gamma(z'_3)\rangle = (\lambda^2)^{9/2} \langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2) L^{\bar{c}}_\gamma(z_3)\rangle, \tag{3.13}$$

which gives rise to the homogeneity constraint on $\mathcal{H}$:

$$\mathcal{H}_{\alpha\beta\gamma}(\lambda^2 \boldsymbol{X}, \lambda\Theta) = (\lambda^2)^{-3/2} \mathcal{H}_{\alpha\beta\gamma}(\boldsymbol{X}, \Theta). \tag{3.14}$$

(ii) *Differential constraints.*—The conservation equation for the flavor current results in

$$D^\alpha_{(1)}\langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2) L^{\bar{c}}_\gamma(z_3)\rangle = 0. \tag{3.15}$$

Using identities (2.39a) and (2.39b), we obtain a differential constraint on $\mathcal{H}$:

$$\mathcal{D}^\alpha \mathcal{H}_{\alpha\beta\gamma}(\boldsymbol{X}, \Theta) = 0. \tag{3.16}$$

We need not consider the conservation law at z_2 as we can use an algebraic constraint instead.

(iii) *Point permutation symmetry.*—The symmetry under permutation of points (z_1 and z_2) results in the following constraint on the correlation function:

$$\langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2) L^{\bar{c}}_\gamma(z_3)\rangle = -\langle L^{\bar{b}}_\beta(z_2) L^{\bar{a}}_\alpha(z_1) L^{\bar{c}}_\gamma(z_3)\rangle, \tag{3.17}$$

which constrains the tensor $\mathcal{H}$ so that

$$\mathcal{H}_{\alpha\beta\gamma}(\boldsymbol{X}, \Theta) = \mathcal{H}_{\beta\alpha\gamma}(-\boldsymbol{X}^{\mathrm{T}}, -\Theta). \tag{3.18}$$

On the other hand, the symmetry under permutation of points z_1 and z_3 results in

[4] The tensor structure and the conservation law of the $1 \le \mathcal{N} \le 4$ flavor currents follow from the structure of unconstrained prepotentials for $1 \le \mathcal{N} \le 4$ vector multiplets [51–57].

[5] Here we consider only the contribution proportional to the totally antisymmetric structure constants $f^{\bar{a}\bar{b}\bar{c}}$. Similarly, one can consider the contribution totally symmetric in flavor indices. However, this contribution vanishes [34] so it is omitted here.

$$\langle L^{\bar{a}}_{\alpha}(z_1)L^{\bar{b}}_{\beta}(z_2)L^{\bar{c}}_{\gamma}(z_3)\rangle = -\langle L^{\bar{c}}_{\gamma}(z_3)L^{\bar{b}}_{\beta}(z_2)L^{\bar{a}}_{\alpha}(z_1)\rangle, \tag{3.19}$$

which gives rise to the point-switch identity

$$\mathcal{H}_{\alpha\beta\gamma}(\boldsymbol{X}_3,\Theta_3) = \frac{\boldsymbol{x}_{13\gamma}^{\gamma'}(\boldsymbol{x}_{13}^{-1})_{\alpha}{}^{\alpha'}\boldsymbol{x}_{13}^{\beta'\sigma}\boldsymbol{X}_{3\sigma\beta}}{\boldsymbol{X}_3^4\boldsymbol{x}_{13}^4} \times \mathcal{H}_{\gamma'\beta'\alpha'}(-\boldsymbol{X}_1^{\mathrm{T}},-\Theta_1). \tag{3.20}$$

To solve this problem systematically let us decompose the tensor $\mathcal{H}$ into irreducible components:

$$\mathcal{H}_{\alpha\beta\gamma}(\boldsymbol{X},\Theta) = \sum_i c_i \mathcal{H}_{i\alpha\beta\gamma}(\boldsymbol{X},\Theta). \tag{3.21}$$

It is also more convenient to work with X_m instead of $\boldsymbol{X}_{\alpha\beta}$. We have

$$\mathcal{H}_{1\alpha\beta\gamma} = \varepsilon_{\alpha\beta}\Theta_\gamma A(X), \tag{3.22a}$$

$$\mathcal{H}_{2\alpha\beta\gamma} = \varepsilon_{\alpha\beta}(\gamma^a)_\gamma{}^\delta\Theta_\delta B_a(X), \tag{3.22b}$$

$$\mathcal{H}_{3\alpha\beta\gamma} = (\gamma^a)_{\alpha\beta}\Theta_\gamma C_a(X), \tag{3.22c}$$

$$\mathcal{H}_{4\alpha\beta\gamma} = (\gamma^a)_{\alpha\beta}(\gamma^b)_\gamma{}^\delta\Theta_\delta D_{ab}(X). \tag{3.22d}$$

Here we have used the fact that every matrix antisymmetric in α,β is proportional to $\varepsilon_{\alpha\beta}$, every matrix symmetric in α,β is proportional to a gamma matrix, and that since $\mathcal{H}$ is Grassmann odd it follows that $\mathcal{H}$ is linear in Θ due to $\Theta_\alpha\Theta_\beta\Theta_\gamma = 0$. Due to the scaling property (3.14) it follows that the functions A, B, C, and D have dimension -2. From Eq. (3.18) it also follows that

$$A(X) = A(-X), \qquad B_a(X) = B_a(-X), \tag{3.23a}$$

$$C_a(X) = -C_a(-X), \qquad D_{ab}(X) = -D_{ab}(-X). \tag{3.23b}$$

It is easy to see that the conservation equation (3.16) splits into the two independent equations:

$$\partial^\alpha\mathcal{H}_{\alpha\beta\gamma} = 0, \tag{3.24a}$$

$$(\gamma^t)^{\alpha\tau}\Theta_\tau\partial_t\mathcal{H}_{\alpha\beta\gamma} = 0. \tag{3.24b}$$

Imposing (3.24a) results in the algebraic equations

$$A(X) = -D^a{}_a(X), \qquad C_a(X) = B_a(X) + \epsilon_a{}^{mn}D_{mn}(X), \tag{3.25}$$

while, on the other hand, from (3.24b) we obtain

$$\partial^a\{B_a(X) + C_a(X) - \epsilon_a{}^{mn}D_{mn}(X)\} = 0, \tag{3.26a}$$

$$\partial_t A(X) + \epsilon_t{}^{ma}\partial_m B_a(X) - \epsilon_t{}^{ma}\partial_m C_a(X) - \partial^m D_{mt}(X) + \partial_t D^a{}_a(X) - \partial^m D_{tm}(X) = 0. \tag{3.26b}$$

Using Eqs. (3.25), (3.26a), and (3.26b) we obtain that B_a and D_{ab} satisfy

$$\partial^a B_a(X) = 0, \qquad \partial^a D_{ab}(X) = 0. \tag{3.27}$$

Thus, the problem is reduced to finding transverse tensors B_a and D_{ab} of dimension -2 satisfying (3.23b). The tensors A and C are then found using Eq. (3.25). It is not difficult to show that the solution to this problem is given by

$$A(X) = 0, \qquad B_a(X) = 0, \tag{3.28a}$$

$$C_a(X) = \frac{X_a}{X^3}, \qquad D_{ab}(X) = \epsilon_{abc}\frac{X^c}{X^3}, \tag{3.28b}$$

with $c_3 = -2c_4$. Hence this correlation function is fixed up to a single real coefficient which we denote $d_{\mathcal{N}=1}$. Converting back to spinor notation we find[6]

$$\mathcal{H}_{\alpha\beta\gamma}(\boldsymbol{X},\Theta) = \frac{\mathrm{i}d_{\mathcal{N}=1}}{\boldsymbol{X}^3}\{\boldsymbol{X}_{\alpha\beta}\Theta_\gamma - \varepsilon_{\alpha\gamma}\boldsymbol{X}_\beta{}^\delta\Theta_\delta - \varepsilon_{\beta\gamma}\boldsymbol{X}_\alpha{}^\delta\Theta_\delta\}. \tag{3.29}$$

One may also check that this solution satisfies the point-switch identity (3.20). This agrees with the result in [34], which was computed in a different way. Our method has the advantage that it systematically takes care of all possible irreducible components of $\mathcal{H}$ and, hence, is more useful when $\mathcal{H}$ is a tensor of high rank.

IV. MIXED CORRELATORS IN $\mathcal{N}=1$ SUPERCONFORMAL FIELD THEORY

A. The correlation function $\langle LJL\rangle$

Let us first consider the correlation function $\langle L^{\bar{a}}_{\alpha}(z_1) \times J_{\gamma_1\gamma_2\gamma_3}(z_2)L^{\bar{b}}_{\beta}(z_3)\rangle$. Using the general expression (2.43), it has the form

$$\langle L^{\bar{a}}_{\alpha}(z_1)J_{\gamma_1\gamma_2\gamma_3}(z_2)L^{\bar{b}}_{\beta}(z_3)\rangle = \frac{\delta^{\bar{a}\bar{b}}\hat{\boldsymbol{x}}_{13\alpha}{}^{\alpha'}\hat{\boldsymbol{x}}_{23(\gamma_1}{}^{\gamma_1'}\hat{\boldsymbol{x}}_{23\gamma_2}{}^{\gamma_2'}\hat{\boldsymbol{x}}_{23\gamma_3)}{}^{\gamma_3'}}{(\boldsymbol{x}_{13}^2)^{3/2}(\boldsymbol{x}_{23}^2)^{5/2}} \times \mathcal{H}_{\alpha'\beta,\gamma_1'\gamma_2'\gamma_3'}(\boldsymbol{X}_3,\Theta_3), \tag{4.1}$$

where $\mathcal{H}$ is totally symmetric in three of its indices, $\mathcal{H}_{\alpha\beta,\gamma_1\gamma_2\gamma_3} = \mathcal{H}_{\alpha\beta,(\gamma_1\gamma_2\gamma_3)}$. The correlation function is also required to satisfy the following.

(i) *Scaling constraint.*—Under scale transformations the correlation function transforms as

[6] Note that since $\Theta_\alpha\Theta_\beta\Theta_\gamma = 0$ we can replace $\boldsymbol{X}$ with X in (3.29).

$$\langle L^{\bar{a}}_{\alpha}(z_1')J_{\gamma_1\gamma_2\gamma_3}(z_2')L^{\bar{b}}_{\beta}(z_3')\rangle = (\lambda^2)^{11/2}\langle L^{\bar{a}}_{\alpha}(z_1)J_{\gamma_1\gamma_2\gamma_3}(z_2)L^{\bar{b}}_{\beta}(z_3)\rangle, \quad (4.2)$$

which implies that we have the following homogeneity constraint on $\mathcal{H}$:

$$\mathcal{H}_{\alpha\beta,\gamma_1\gamma_2\gamma_3}(\lambda^2 \boldsymbol{X}, \lambda\Theta) = (\lambda^2)^{-5/2}\mathcal{H}_{\alpha\beta,\gamma_1\gamma_2\gamma_3}(\boldsymbol{X},\Theta). \quad (4.3)$$

(ii) *Differential constraints.*—The differential constraints on the flavor current and supercurrent result in the following constraints on the correlation function:

$$D^{\alpha}_{(1)}\langle L^{\bar{a}}_{\alpha}(z_1)J_{\gamma_1\gamma_2\gamma_3}(z_2)L^{\bar{b}}_{\beta}(z_3)\rangle = 0, \quad (4.4a)$$

$$D^{\gamma_1}_{(2)}\langle L^{\bar{a}}_{\alpha}(z_1)J_{\gamma_1\gamma_2\gamma_3}(z_2)L^{\bar{b}}_{\beta}(z_3)\rangle = 0. \quad (4.4b)$$

Using identities (2.39a) and (2.39b), these result in the following differential constraints on $\mathcal{H}$:

$$\mathcal{D}^{\alpha}\mathcal{H}_{\alpha\beta,\gamma_1\gamma_2\gamma_3}(\boldsymbol{X},\Theta) = 0, \quad (4.5a)$$

$$\mathcal{Q}^{\gamma_1}\mathcal{H}_{\alpha\beta,\gamma_1\gamma_2\gamma_3}(\boldsymbol{X},\Theta) = 0. \quad (4.5b)$$

(iii) *Point permutation symmetry.*—The symmetry under permutation of points (z_1 and z_3) results in the following constraint on the correlation function:

$$\langle L^{\bar{a}}_{\alpha}(z_1)J_{\gamma_1\gamma_2\gamma_3}(z_2)L^{\bar{b}}_{\beta}(z_3)\rangle = -\langle L^{\bar{b}}_{\beta}(z_3)J_{\gamma_1\gamma_2\gamma_3}(z_2)L^{\bar{a}}_{\alpha}(z_1)\rangle, \quad (4.6)$$

which results in the point-switch identity

$$\mathcal{H}_{\alpha\beta,\gamma_1\gamma_2\gamma_3}(\boldsymbol{X}_3,\Theta_3) = -\frac{\boldsymbol{x}_{13}{}^{\beta'}{}_{\beta}(\boldsymbol{x}_{13}^{-1})_{\alpha}{}^{\alpha'}\boldsymbol{x}_{13}^{\gamma_1'\delta_1}\boldsymbol{X}_{3\delta_1\gamma_1}\boldsymbol{x}_{13}^{\gamma_2'\delta_2}\boldsymbol{X}_{3\delta_2\gamma_2}\boldsymbol{x}_{13}^{\gamma_3'\delta_3}\boldsymbol{X}_{3\delta_3\gamma_3}}{\boldsymbol{X}_3^8\boldsymbol{x}_{13}^8} \times \mathcal{H}_{\beta'\alpha',\gamma_1'\gamma_2'\gamma_3'}(-\boldsymbol{X}_1^{\mathrm{T}},-\Theta_1). \quad (4.7)$$

Thus we need to solve for the tensor $\mathcal{H}$ subject to the constraints (4.3), (4.5a), (4.5b) and (4.7). To start with we combine two of the three γ indices into a vector index and impose a γ-trace constraint to remove the component antisymmetric in γ_1 and γ_2:

$$\mathcal{H}_{\alpha\beta,\gamma_1\gamma_2\gamma_3} = (\gamma^m)_{\gamma_2\gamma_3}\mathcal{H}_{\alpha\beta,\gamma_1 m}, \quad (\gamma^m)^{\tau\gamma}\mathcal{H}_{\alpha\beta,\gamma m} = 0. \quad (4.8)$$

Since our correlator is Grassmann odd the function $\mathcal{H}_{\alpha\beta,\gamma m}$ must be linear in Θ. Just like the flavor current three-point function, linearity in Θ implies that the differential constraints (4.5a) and (4.5b) are, respectively, equivalent to

$$\partial^{\alpha}\mathcal{H}_{\alpha\beta,\gamma m} = 0, \qquad (\gamma^t)^{\alpha\tau}\Theta_{\tau}\partial_t\mathcal{H}_{\alpha\beta,\gamma m} = 0, \quad (4.9a)$$

$$\partial^{\gamma}\mathcal{H}_{\alpha\beta,\gamma m} = 0, \qquad (\gamma^t)^{\gamma\tau}\Theta_{\tau}\partial_t\mathcal{H}_{\alpha\beta,\gamma m} = 0. \quad (4.9b)$$

Now let us decompose $\mathcal{H}$ into irreducible components:

$$\mathcal{H}_{\alpha\beta,\gamma m} = \sum_i c_i\mathcal{H}_{i\alpha\beta,\gamma m}, \quad (4.10)$$

where

$$\mathcal{H}_{1\alpha\beta,\gamma m} = \varepsilon_{\alpha\beta}\Theta_{\gamma}A_m(X), \quad (4.11a)$$

$$\mathcal{H}_{2\alpha\beta,\gamma m} = \varepsilon_{\alpha\beta}(\gamma^a)_{\gamma}{}^{\delta}\Theta_{\delta}B_{ma}(X), \quad (4.11b)$$

$$\mathcal{H}_{3\alpha\beta,\gamma m} = (\gamma^a)_{\alpha\beta}\Theta_{\gamma}C_{ma}(X), \quad (4.11c)$$

$$\mathcal{H}_{4\alpha\beta,\gamma m} = (\gamma^a)_{\alpha\beta}(\gamma^b)_{\gamma}{}^{\delta}\Theta_{\delta}D_{mab}(X). \quad (4.11d)$$

It follows from Eq. (4.3) that the dimension of A, B, C, and D is -3. We now impose the differential constraints (4.9a) and (4.9b), along with the gamma-trace constraint (4.8). After imposing (4.9a) and (4.9b) the terms $O(\Theta^0)$ imply

$$A_m(X) = 0, \qquad C_{mn}(X) = 0, \quad (4.12a)$$

$$B_{ma}(X) = -\epsilon_{nra}D_{mnr}(X), \quad \eta^{na}D_{mna}(X) = 0, \quad (4.12b)$$

while the terms $O(\Theta^2)$ give the differential constraints

$$\partial^t B_{mt}(X) = 0, \quad (4.13a)$$

$$\partial^t D_{mnt}(X) = 0, \quad (4.13b)$$

$$\partial^t\{B_{mt}(X) + \epsilon_t{}^{an}D_{mna}(X)\} = 0, \quad (4.13c)$$

$$\partial^t\{D_{mnt}(X) + D_{mtn}(X) - \eta_{tn}D_m{}^a{}_a(X) + \epsilon_{nt}{}^aB_{ma}(X)\} = 0. \quad (4.13d)$$

Imposing the gamma-trace condition (4.8) results in

$$\eta^{ma}B_{ma}(X) = 0, \qquad \epsilon^{qma}B_{ma}(X) = 0, \quad (4.14a)$$

$$\eta^{ma}D_{mna}(X) = 0, \qquad \epsilon^{qma}D_{mna}(X) = 0. \quad (4.14b)$$

One may show that the differential and algebraic constraints above are mutually consistent and reduce to

$$\partial^t B_{mt}(X) = 0, \qquad \partial^t D_{mnt}(X) = 0, \quad (4.15a)$$

$$\eta^{na}D_{mna}(X) = 0, \qquad \eta^{ma}D_{mna}(X) = 0, \quad (4.15b)$$

$$B_{ma}(X) = -\epsilon_{nra}D_{mnr}(X), \quad (4.15c)$$

where B_{ma} is symmetric and traceless and D_{mna} is symmetric in the first and last index. After some calculation one can show that general solutions consistent with the scaling property (4.3) and the above constraints are

$$B_{ma}(X) = \frac{\eta_{ma}}{X^3} - \frac{3X_m X_a}{X^5}, \tag{4.16}$$

$$D_{mna}(X) = \epsilon_{ndm}\frac{X^d X_a}{X^5} + \epsilon_{nda}\frac{X^d X_m}{X^5}, \tag{4.17}$$

with $c_2 = c_4$. Hence, the three-point correlation function is determined up to a single free parameter which we denote $c_{\mathcal{N}=1}$. Our solution is then

$$\langle L^{\bar a}_\alpha(z_1) J_{\gamma_1\gamma_2\gamma_3}(z_2) L^{\bar b}_\beta(z_3)\rangle = \frac{\delta^{\bar a\bar b}\boldsymbol{x}_{13\alpha}{}^{\alpha'}\boldsymbol{x}_{23(\gamma_1}{}^{\gamma_1'}\boldsymbol{x}_{23\gamma_2}{}^{\gamma_2'}\boldsymbol{x}_{23\gamma_3)}{}^{\gamma_3'}}{(\boldsymbol{x}^2_{13})^2(\boldsymbol{x}^2_{23})^4}\mathcal{H}_{\alpha'\beta,\gamma_1'\gamma_2'\gamma_3'}(\boldsymbol{X}_3,\Theta_3), \tag{4.18}$$

where

$$\mathcal{H}_{\alpha\beta,\gamma_1\gamma_2\gamma_3}(\boldsymbol{X},\Theta) = (\gamma^m)_{\gamma_2\gamma_3}\mathcal{H}_{\alpha\beta,\gamma_1 m}(\boldsymbol{X},\Theta), \tag{4.19}$$

$$\mathcal{H}_{\alpha\beta,\gamma m}(\boldsymbol{X},\Theta) = \mathrm{i}c_{\mathcal{N}=1}(\gamma^a)_\gamma{}^\delta\Theta_\delta\{\varepsilon_{\alpha\beta}B_{ma}(X) + (\gamma^n)_{\alpha\beta}D_{mna}(X)\}, \tag{4.20}$$

with B and D given in Eqs. (4.16) and (4.17). In spinor notation, this is equivalent to

$$\begin{aligned}\mathcal{H}_{\alpha\beta,\gamma_1\gamma_2\gamma_3}(\boldsymbol{X},\Theta) = \mathrm{i}c_{\mathcal{N}=1}\bigg\{&\frac{\varepsilon_{\alpha\beta}}{\boldsymbol{X}^3}(\varepsilon_{\gamma_1\gamma_2}\Theta_{\gamma_3} + \varepsilon_{\gamma_1\gamma_3}\Theta_{\gamma_2}) + \frac{1}{\boldsymbol{X}^5}(\varepsilon_{\gamma_2\alpha}\boldsymbol{X}_{\beta\gamma_3}\boldsymbol{X}_{\gamma_1}{}^\delta\Theta_\delta \\ &+ \varepsilon_{\gamma_2\beta}\boldsymbol{X}_{\alpha\gamma_3}\boldsymbol{X}_{\gamma_1}{}^\delta\Theta_\delta + \varepsilon_{\gamma_1\alpha}\boldsymbol{X}_{\gamma_2\gamma_3}\boldsymbol{X}_\beta{}^\delta\Theta_\delta + \varepsilon_{\gamma_1\beta}\boldsymbol{X}_{\gamma_2\gamma_3}\boldsymbol{X}_\alpha{}^\delta\Theta_\delta \\ &- \varepsilon_{\gamma_2\gamma_3}\boldsymbol{X}_{\alpha\beta}\boldsymbol{X}_{\gamma_1}{}^\delta\Theta_\delta - \boldsymbol{X}_{\gamma_2\gamma_3}\boldsymbol{X}_{\alpha\beta}\Theta_{\gamma_1} - 3\varepsilon_{\alpha\beta}\boldsymbol{X}_{\gamma_2\gamma_3}\boldsymbol{X}_{\gamma_1}{}^\delta\Theta_\delta)\bigg\}.\end{aligned} \tag{4.21}$$

Finally, one must check that this solution also satisfies the point-switch identity. With the aid of identities (2.31a) and (2.31b), it is a relatively straightforward exercise to show that the point-switch identity (4.7) is indeed satisfied.

B. The correlation function $\langle JJL\rangle$

Let us now discuss the remaining mixed correlation function

$$\langle J_{\beta_1\beta_2\beta_3}(z_1) J_{\gamma_1\gamma_2\gamma_3}(z_2) L_\alpha(z_3)\rangle. \tag{4.22}$$

Here the correlator can exist only if the flavor group contains $U(1)$ factors, so we will assume that the flavor group is just $U(1)$. At the component level this correlation function contains $\langle T_{ab}(x_1)T_{mn}(x_2)L_c(x_3)\rangle$, which was shown to vanish in any conformal field theory after imposing all differential constraints and symmetries [11]. As we will show, the same occurs in the supersymmetric theory. However, we will see that (4.22) vanishes without needing to impose the conservation equation for $L_\alpha(z_3)$. The general expression for this correlation function is

$$\begin{aligned}\langle J_{\beta_1\beta_2\beta_3}(z_1) J_{\gamma_1\gamma_2\gamma_3}(z_2) L_\alpha(z_3)\rangle = &\frac{\hat{\boldsymbol{x}}_{13(\beta_1}{}^{\beta_1'}\hat{\boldsymbol{x}}_{13\beta_2}{}^{\beta_2'}\hat{\boldsymbol{x}}_{13\beta_3)}{}^{\beta_3'}\hat{\boldsymbol{x}}_{23(\gamma_1}{}^{\gamma_1'}\hat{\boldsymbol{x}}_{23\gamma_2}{}^{\gamma_2'}\hat{\boldsymbol{x}}_{23\gamma_3)}{}^{\gamma_3'}}{(\boldsymbol{x}^2_{13})^{5/2}(\boldsymbol{x}^2_{23})^{5/2}} \\ &\times\mathcal{H}_{\beta_1'\beta_2'\beta_3'\gamma_1'\gamma_2'\gamma_3'\alpha}(\boldsymbol{X}_3,\Theta_3),\end{aligned} \tag{4.23}$$

where $\mathcal{H}$ has the symmetry property $\mathcal{H}_{\beta_1\beta_2\beta_3\gamma_1\gamma_2\gamma_3\alpha} = \mathcal{H}_{(\beta_1\beta_2\beta_3)(\gamma_1\gamma_2\gamma_3)\alpha}$. The correlation function is required to satisfy the following.

(i) *Scaling constraint.*—Under scale transformations it transforms as

$$\langle J_{\beta_1\beta_2\beta_3}(z_1') J_{\gamma_1\gamma_2\gamma_3}(z_2') L_\alpha(z_3')\rangle = (\lambda^2)^{13/2}\langle J_{\beta_1\beta_2\beta_3}(z_1) J_{\gamma_1\gamma_2\gamma_3}(z_2) L_\alpha(z_3)\rangle, \tag{4.24}$$

which results in the constraint

$$\mathcal{H}_{\beta_1\beta_2\beta_3\gamma_1\gamma_2\gamma_3\alpha}(\lambda^2\boldsymbol{X},\lambda\Theta) = (\lambda^2)^{-7/2}\mathcal{H}_{\beta_1\beta_2\beta_3\gamma_1\gamma_2\gamma_3\alpha}(\boldsymbol{X},\Theta). \tag{4.25}$$

(ii) *Differential constraint.*—The conservation law on the supercurrent implies

$$D_{(1)}^{\beta_1}\langle J_{\beta_1\beta_2\beta_3}(z_1)J_{\gamma_1\gamma_2\gamma_3}(z_2)L_\alpha(z_3)\rangle = 0, \qquad (4.26)$$

which results in a differential constraint on $\mathcal{H}$:

$$\mathcal{D}^{\beta_1}\mathcal{H}_{\beta_1\beta_2\beta_3\gamma_1\gamma_2\gamma_3\alpha}(\boldsymbol{X},\Theta) = 0. \qquad (4.27)$$

(iii) *Point permutation symmetry.*—The symmetry under permutation of points z_1 and z_2 implies the following constraint on the correlation function:

$$\langle J_{\beta_1\beta_2\beta_3}(z_1)J_{\gamma_1\gamma_2\gamma_3}(z_2)L_\alpha(z_3)\rangle = -\langle J_{\gamma_1\gamma_2\gamma_3}(z_2)J_{\beta_1\beta_2\beta_3}(z_1)L_\alpha(z_3)\rangle, \qquad (4.28)$$

which results in the identity

$$\mathcal{H}_{\beta_1\beta_2\beta_3\gamma_1\gamma_2\gamma_3\alpha}(\boldsymbol{X},\Theta) = -\mathcal{H}_{\gamma_1\gamma_2\gamma_3\beta_1\beta_2\beta_3\alpha}(-\boldsymbol{X}^{\mathrm{T}},-\Theta). \qquad (4.29)$$

Thus, we need to solve for the tensor $\mathcal{H}$ subject to the constraints (4.25), (4.27) and (4.29). Note that we also must impose one more differential constraint:

$$D_{(3)}^{\alpha}\langle J_{\beta_1\beta_2\beta_3}(z_1)J_{\gamma_1\gamma_2\gamma_3}(z_2)L_\alpha(z_3)\rangle = 0, \qquad (4.30)$$

which is quite nontrivial in this formalism. Fortunately, constraints (4.25), (4.27) and (4.29) are sufficient to show that correlator (4.22) vanishes; hence, we will not need to consider (4.30).

To start, we combine two of the three β, γ indices into a vector index and impose γ-trace constraints to remove antisymmetric components:

$$\mathcal{H}_{\beta_1\beta_2\beta_3\gamma_1\gamma_2\gamma_3\alpha}(\boldsymbol{X},\Theta) = (\gamma^a)_{\beta_2\beta_3}(\gamma^b)_{\gamma_2\gamma_3}\mathcal{H}_{\beta_1 a,\gamma_1 b,\alpha}(\boldsymbol{X},\Theta), \qquad (4.31)$$

$$(\gamma^a)^{\tau\beta}\mathcal{H}_{\beta a,\gamma b,\alpha}(\boldsymbol{X},\Theta) = 0, \qquad (\gamma^b)^{\tau\gamma}\mathcal{H}_{\beta a,\gamma b,\alpha}(\boldsymbol{X},\Theta) = 0. \qquad (4.32)$$

Now let us split $\mathcal{H}$ into symmetric and antisymmetric parts in the first and second pair of indices:

$$\mathcal{H}_{\beta a,\gamma b,\alpha} = \mathcal{H}_{(\beta a,\gamma b),\alpha} + \mathcal{H}_{[\beta a,\gamma b],\alpha}. \qquad (4.33)$$

Due to the symmetry properties, (4.29) implies that $\mathcal{H}_{(\sigma a,\gamma b),\alpha}$ is an even function of X,[7] while $\mathcal{H}_{[\beta a,\gamma b],\alpha}$ is odd. Therefore they do not mix in the conservation law (4.27) and may be considered independently. In irreducible components, $\mathcal{H}_{(\beta a,\gamma b),\alpha}$ has the decomposition

$$\mathcal{H}_{(\beta a,\gamma b),\alpha} = \sum_i \mathcal{H}_{i(\beta a,\gamma b),\alpha}, \qquad (4.34)$$

where

$$\mathcal{H}_{1(\beta a,\gamma b),\alpha} = \varepsilon_{\beta\gamma}\Theta_\alpha A_{[ab]}(X), \qquad (4.35a)$$

$$\mathcal{H}_{2(\beta a,\gamma b),\alpha} = \varepsilon_{\beta\gamma}(\gamma^m)_\alpha{}^\delta\Theta_\delta B_{m[ab]}(X), \qquad (4.35b)$$

$$\mathcal{H}_{3(\beta a,\gamma b),\alpha} = (\gamma^m)_{\beta\gamma}\Theta_\alpha C_{m(ab)}(X), \qquad (4.35c)$$

$$\mathcal{H}_{4(\beta a,\gamma b),\alpha} = (\gamma^m)_{\beta\gamma}(\gamma^n)_\alpha{}^\delta\Theta_\delta D_{mn(ab)}(X). \qquad (4.35d)$$

Here we have made explicit the algebraic symmetry properties of A, B, C, and D, which by virtue of (4.29) are all even functions of X. Now due to linearity in Θ, the differential constraint (4.27) is equivalent to the pair of equations

$$\partial^\beta\mathcal{H}_{\beta a,\gamma b,\alpha} = 0, \qquad (\gamma^t)^{\beta\tau}\Theta_\tau\partial_t\mathcal{H}_{\beta a,\gamma b,\alpha} = 0. \qquad (4.36)$$

After imposing (4.36), the terms $O(\Theta^0)$ imply

$$A_{m[ab]}(X) = 0, \qquad B_{m[ab]}(X) = 0, \qquad (4.37a)$$

$$C_{m(ab)}(X) + \epsilon_m{}^{rs}D_{rs(ab)}(X) = 0, \qquad (4.37b)$$

$$\eta^{mn}D_{mn(ab)}(X) = 0, \qquad \eta^{ma}D_{mn(ab)}(X) = 0, \qquad (4.37c)$$

so $\mathcal{H}_{1(\beta a,\gamma b),\alpha} = \mathcal{H}_{2(\beta a,\gamma b),\alpha} = 0$. The terms $O(\Theta^2)$ then result in the differential constraints

$$\partial^m\{-C_{m(ab)}(X) + \epsilon_m{}^{rs}D_{rs(ab)}(X)\} = 0, \qquad (4.38a)$$

$$\epsilon_c{}^{tm}\partial_t C_{m(ab)}(X) - \partial^m D_{mc(ab)}(X) - \partial^m D_{cm(ab)}(X) = 0. \qquad (4.38b)$$

Imposing the gamma-trace condition (4.32) results in

$$\eta^{ma}C_{m(ab)}(X) = 0, \qquad \epsilon_c{}^{ma}C_{m(ab)}(X) = 0, \qquad (4.39a)$$

$$\eta^{ma}D_{mn(ab)}(X) = 0, \qquad \epsilon_c{}^{ma}D_{mn(ab)}(X) = 0. \qquad (4.39b)$$

Altogether (4.37b), (4.38a) and (4.39a) imply that C is a totally symmetric, traceless, transverse and even function of X. Let us try to construct such a tensor by analyzing its irreducible components. To determine which irreducible components are permitted, let us trade each vector index for a pair of spinor indices. Since C is completely symmetric and traceless, it is equivalent to $C_{(\alpha_1\ldots\alpha_6)}$. In addition since C is even in $X_{\alpha\beta}$ only irreducible structures (that is, totally symmetric tensors) of rank 4 and 0 in $X_{\alpha\beta}$ can contribute to

[7]As in the previous case, our correlator is Grassmann odd which means we can replace $\boldsymbol{X}$ with X.

the solution. Going back to vector indices, let us denote these components of C as $C_{1(mn)}(X)$ and $C_2(X)$.

Since it is not possible to construct a rank-three tensor $C_{(mnk)}$ out of $C_{1(mn)}(X)$ and $C_2(X)$, the tensor C_{mnk} vanishes. Hence, $\mathcal{H}_{3(\beta a,\gamma b),\alpha} = 0$.

Given this information, the remaining set of equations implies that D is now a totally symmetric, traceless and transverse tensor that is even in X. Following a similar argument, the symmetries imply that it has irreducible components $D_{1(mnab)}(X)$, $D_{2(mn)}(X)$ and $D_3(X)$. We are now equipped with enough information to construct an explicit solution for D. Using the symmetries and the scaling property (4.25) we have the most general ansatz

$$D_{(mnab)}(X) = \frac{d_1}{X^4}[\eta_{ma}\eta_{nb} + \eta_{mb}\eta_{na} + \eta_{mn}\eta_{ab}] + \frac{d_2}{X^6}[\eta_{mn}X_aX_b + \eta_{ma}X_nX_b + \eta_{mb}X_aX_n + \eta_{na}X_mX_b + \eta_{nb}X_mX_a + \eta_{ab}X_mX_n] + \frac{d_3}{X^8}X_mX_nX_aX_b. \tag{4.40}$$

Requiring that D be traceless and transverse fixes all the d_i to 0. Hence, $D = 0$, and $\mathcal{H}_{(\beta a,\gamma b),\alpha}$ vanishes.

In a similar way we consider $\mathcal{H}_{[\beta a,\gamma b],\alpha}$ for which we have the following decomposition:

$$\mathcal{H}_{[\beta a,\gamma b],\alpha} = \sum_i \mathcal{H}_{i[\beta a,\gamma b],\alpha}, \tag{4.41}$$

where

$$\mathcal{H}_{1[\beta a,\gamma b],\alpha} = \varepsilon_{\beta\gamma}\Theta_\alpha A_{(ab)}(X), \tag{4.42a}$$

$$\mathcal{H}_{2[\beta a,\gamma b],\alpha} = \varepsilon_{\beta\gamma}(\gamma^m)_\alpha{}^\delta\Theta_\delta B_{m(ab)}(X), \tag{4.42b}$$

$$\mathcal{H}_{3[\beta a,\gamma b],\alpha} = (\gamma^m)_{\beta\gamma}\Theta_\alpha C_{m[ab]}(X), \tag{4.42c}$$

$$\mathcal{H}_{4[\beta a,\gamma b],\alpha} = (\gamma^m)_{\beta\gamma}(\gamma^n)_\alpha{}^\delta\Theta_\delta D_{mn[ab]}(X). \tag{4.42d}$$

In this case, A, B, C, and D are now odd functions in X. Imposing the conservation equations and vanishing of the γ trace we obtain the following set of constraints:

$$A_{(ab)}(X) = 0, \qquad B_{m(ab)}(X) = 0, \tag{4.43a}$$

$$D^m_{m[ab]}(X) = 0, \quad C_{m[ab]}(X) + \epsilon_m{}^{rs}D_{rs[ab]}(X) = 0, \tag{4.43b}$$

$$C^m_{[mb]}(X) = 0, \qquad D^m_{n[mb]}(X) = 0, \tag{4.43c}$$

$$\epsilon^{cma}C_{m[ab]}(X) = 0, \tag{4.43d}$$

$$\epsilon^{cma}D_{mn[ab]}(X) = 0. \tag{4.43e}$$

We see that the functions A and B vanish. To show that $C_{m[ab]}$ vanishes we consider Eq. (4.43d) and use the fact that in three dimensions an antisymmetric tensor is equivalent to a vector:

$$C_{m[ab]}(X) = \epsilon_{ab}{}^q\tilde{C}_{mq}(X). \tag{4.44}$$

Hence from (4.43d) it follows that

$$\tilde{C}_{ab}(X) - \eta_{ab}\tilde{C}^d_d(X) = 0. \tag{4.45}$$

Contracting with η^{ab} we find that $\tilde{C}^d_d = 0$, and hence $\tilde{C}_{ab} = 0$. It also implies that $C_{m[ab]} = 0$. In a similar way using Eq. (4.43e) one can show that $D_{mn[ab]} = 0$. This means that $\mathcal{H}_{[\beta a,\gamma b],\alpha} = 0$. Hence the three-point function of two supercurrents and one flavor current (4.22) vanishes.

V. COMMENTS ON THE ABSENCE OF PARITY VIOLATING STRUCTURES

In [11] it was shown that correlation functions of conserved current in three-dimensional conformal field theories can have parity-violating structures. Specifically, it was defined as follows. Given a conserved current

$$J_{\alpha_1\alpha_2\ldots\alpha_{2s-1}\alpha_{2s}}(x) = (\gamma^{m_1})_{\alpha_1\alpha_2}\cdots(\gamma^{m_s})_{\alpha_{2s-1}\alpha_{2s}}J_{m_1\ldots m_s}(x), \tag{5.1}$$

we can construct

$$J_s(x,\lambda) = J_{\alpha_1\alpha_2\ldots\alpha_{2s-1}\alpha_{2s}}(x)\lambda^{\alpha_1}\ldots\lambda^{\alpha_{2s}}, \tag{5.2}$$

where λ^α are auxiliary commuting spinors. The action of parity is then $x \to -x$, $\lambda \to i\lambda$. In theories with a parity symmetry, $J_{\mu_1\ldots\mu_s}(x)$ acquires a sign $(-1)^s$ under parity and $J_s(x,\lambda)$ is invariant. However, as was shown in [11] correlation functions admit contributions which are odd under parity. In particular, it was shown that a parity-odd contribution to the mixed correlator of the energy-momentum tensor T_{mn} and two flavor currents $L^{\bar{a}}_k$ can arise. Translating their result into our notation it can be written as follows:

$$\langle T_{mn}(x_1)L^{\bar{a}}_k(x_2)L^{\bar{b}}_p(x_3)\rangle_{\text{odd}} = \frac{\delta^{\bar{a}\bar{b}}}{x_{13}^3x_{12}^3x_{23}}\mathcal{I}_{mn,m'n'}(x_{13})I_{kk'}(x_{23})t_{m'n'k'p}(X_3), \tag{5.3}$$

where

$$t_{mnkp}(X) = \epsilon_{npq}\frac{X^qX_mX_k}{X^3} + \epsilon_{nkq}\frac{X^qX_mX_p}{X^3}. \tag{5.4}$$

Here X_i are three-point building blocks introduced by Osborn and Petkou in [1], while the object $I_{mn}(x)$ is the inversion tensor, and $\mathcal{I}_{mn,m'n'}(x)$ is an inversion tensor which extracts the symmetric traceless component. They are defined as follows:

$$I_{mn}(x) = \eta_{mn} - 2\frac{x_m x_n}{x^2}, \tag{5.5}$$

$$\mathcal{I}_{mn,m'n'}(x) = \frac{1}{2}\{I_{mm'}(x)I_{nn'}(x) + I_{mn'}(x)I_{nm'}(x)\} - \frac{1}{3}\eta_{mm'}\eta_{nn'}. \tag{5.6}$$

An important and specific feature of all parity-violating terms is appearance of the ϵ tensor.

In $\mathcal{N}=1$ supersymmetric theories the supercurrent $J_{\alpha\beta\gamma}$ and the flavor current multiplet $L^{\bar a}_\alpha$ contain the following conserved currents:

$$T_{\alpha\beta\gamma\delta} = D_{(\delta}J_{\alpha\beta\gamma)}|, \qquad T_{\alpha\beta\gamma\delta} = (\gamma^m)_{(\alpha\beta}(\gamma^n)_{\gamma\delta)}T_{mn}, \qquad \partial^m T_{mn} = 0, \qquad \eta^{mn}T_{mn} = 0, \tag{5.7a}$$

$$Q_{\alpha\beta\gamma} = J_{\alpha\beta\gamma}|, \qquad Q_{\alpha\beta\gamma} = (\gamma^m)_{\alpha\beta}Q_{m\gamma}, \qquad \partial^m Q_{m\alpha} = 0, \qquad (\gamma^m)^{\alpha\beta}Q_{m\alpha} = 0, \tag{5.7b}$$

$$V^{\bar a}_{\alpha\beta} = D_{(\alpha}L^{\bar a}_{\beta)}|, \qquad V^{\bar a}_{\alpha\beta} = (\gamma^m)_{\alpha\beta}V^{\bar a}_m, \qquad \partial^m V^{\bar a}_m = 0, \tag{5.7c}$$

where T_{mn} is the energy-momentum tensor, $Q_{m\gamma}$ is the supersymmetry current and $V^{\bar a}_m$ is a vector current. Hence, the mixed correlators studied in the previous section give rise to the following correlators in terms of components:

$$\langle T_{mn}(x_1)T_{pq}(x_2)V_k(x_3)\rangle, \qquad \langle Q_{m\alpha}(x_1)Q_{n\beta}(x_2)V_k(x_3)\rangle, \qquad \langle V^{\bar a}_k(x_1)T_{mn}(x_2)V^{\bar b}_p(x_3)\rangle. \tag{5.8}$$

The first two correlators vanish because the entire superspace correlator (4.22) vanishes. The last one is, in general, nonzero and fixed up to one overall coefficient. It can be computed using Eqs. (4.18) and (4.21) using the superspace reduction procedure

$$\langle V^{\bar a}_k(x_1)T_{mn}(x_2)V^{\bar b}_p(x_3)\rangle = \frac{1}{16}(\gamma_k)^{\alpha_1\alpha_2}(\gamma_m)^{(\beta_1\beta_2}(\gamma_n)^{\beta_3\beta_4)}(\gamma_p)^{\gamma_1\gamma_2}D_{(1)\alpha_1}D_{(2)\beta_1}D_{(3)\gamma_1}\langle L^{\bar a}_{\alpha_2}(z_1)J_{\beta_2\beta_3\beta_4}(z_2)L^{\bar b}_{\gamma_2}(z_3)\rangle|. \tag{5.9}$$

Here the bar projection denotes setting the fermionic coordinates θ_α to zero. We will not perform the reduction explicitly; instead, we will indirectly determine whether (5.9) is even or odd under parity. For this it is sufficient to study whether or not the ϵ tensor appears upon reduction. Since

$$\epsilon_{mnp} = \frac{1}{2}\mathrm{tr}(\gamma_m\gamma_n\gamma_p), \tag{5.10}$$

it is enough to count the number of gamma matrices: If the number of γ matrices appearing in the superspace reduction is even, the ϵ tensor cannot arise and the contribution is parity even; if the number of γ matrices is odd, the contribution is parity odd. Let us perform the counting. Since in (5.9) we act with just three covariant derivatives before setting all $\theta_i = 0$ (where $i = 1, 2, 3$ is the index labeling the three points) only term linear and cubic in θ_i will contribute. Let us concentrate on the terms linear on θ_i. Since the function $\mathcal{H}$ in (4.21) is already linear in θ_i we can set $\theta_i = 0$ in $\boldsymbol{x}_{ij}$ and $\boldsymbol{X}$. This makes $\boldsymbol{x}_{ij}$ and $\boldsymbol{X}$ symmetric and proportional to a gamma matrix. Now we have four gamma matrices in (5.9), four gamma matrices coming from $\boldsymbol{x}_{ij}$ in Eq. (4.18), zero or two gamma matrices coming from $\mathcal{H}$ in (4.21) and also one more gamma matrix contained in Θ_3; see Eq. (2.26c). Overall we have odd number of gamma matrices at this point. However, superspace covariant derivatives also contain gamma matrices; see Eq. (A16). Since we are considering terms linear in θ_i and setting $\theta_i = 0$ upon differentiating it is easy to realize that in the three derivatives $D_{(1)\alpha_1}D_{(2)\beta_1}D_{(3)\gamma_1}$ we must take one derivative with respect to x_i and two derivatives with respect to θ_i. This gives one more gamma matrix, making the total number even. Terms cubic in θ can be considered in a similar way. They also yield an even number of gamma matrices. Hence, the entire contribution (5.9) is parity even.

In a similar way we can count the number of gamma matrices in the superspace reduction of (3.12) and (3.29):

$$\langle V^{\bar a}_m(x_1)V^{\bar b}_n(x_2)V^{\bar c}_k(x_3)\rangle = -\frac{1}{8}(\gamma_m)^{\alpha_1\alpha_2}(\gamma_n)^{\beta_1\beta_2}(\gamma_k)^{\gamma_1\gamma_2}D_{(1)\alpha_1}D_{(2)\beta_1}D_{(3)\gamma_1} \times \langle L^{\bar a}_{\alpha_2}(z_1)L^{\bar a}_{\beta_2}(z_2)L^{\bar b}_{\gamma_2}(z_3)\rangle|. \tag{5.11}$$

An analysis similar to the above shows that this contribution is also parity even. Finally, one can also consider the superspace reduction of the three-point function of the supercurrent

$$\langle T_{mn}(x_1)T_{k\ell}(x_2)T_{pq}(x_3)\rangle = \frac{1}{64}(\gamma_m)^{(\alpha_1\alpha_2}(\gamma_n)^{\alpha_3\alpha_4)}(\gamma_k)^{(\beta_1\beta_2}(\gamma_\ell)^{\beta_3\beta_4)}(\gamma_p)^{(\gamma_1\gamma_2}(\gamma_q)^{\gamma_3\gamma_4)} \times D_{(1)\alpha_1}D_{(2)\beta_1}D_{(3)\gamma_1}\langle J_{\alpha_2\alpha_3\alpha_4}(z_1)J_{\beta_2\beta_3\beta_4}(z_2)J_{\gamma_2\gamma_3\gamma_4}(z_3)\rangle| \tag{5.12}$$

and

$$\langle T_{mn}(x_1)Q_{k\beta}(x_2)Q_{p\gamma}(x_3)\rangle = \frac{1}{16}(\gamma_m)^{(\alpha_1\alpha_2}(\gamma_n)^{\alpha_3\alpha_4)}(\gamma_k)^{\beta_1\beta_2}(\gamma_p)^{\gamma_1\gamma_2}D_{(1)\alpha_1}\langle J_{\alpha_2\alpha_3\alpha_4}(z_1)J_{\beta\beta_1\beta_2}(z_2)J_{\gamma\gamma_1\gamma_2}(z_3)\rangle|. \tag{5.13}$$

The three-point function of the supercurrent was found in [34]. We will not repeat it here since the expression for it is quite long. However, a similar analysis shows that the contributions (5.12) and (5.13) are parity even.[8]

This means that no parity-violating structures can arise in three-point functions of T_{mn}, $Q_{m\alpha}$ and $V_m^{\bar{a}}$ in superconformal field theories. Maldacena and Zhiboedov proved in [41] that if a three-dimensional conformal field theory possesses a higher spin conserved current, then it is essentially a free theory. Since a free theory has only parity-even contributions to the three-point functions of conserved currents, the correlators involving one or more higher spin-conserved currents admit only parity-even structures. This leads us to conclude that $\mathcal{N}=1$ supersymmetry forbids parity-violating structures in all three-point functions of conserved currents unless the assumptions of the Maldacena-Zhiboedov theorem are violated. The strongest assumption of the theorem is that the theory under consideration contains unique conserved current of spin two which is the energy-momentum tensor. Some properties of theories possessing more than one conserved current with spin two were discussed in [41]. In supersymmetric theory the energy-momentum tensor is a component of the supercurrent. One can also consider a different supermultiplet containing a conserved spin two current, namely

$$J_{(\alpha_1\alpha_2\alpha_3\alpha_4)}, \qquad D^{\alpha_1}J_{(\alpha_1\alpha_2\alpha_3\alpha_4)} = 0. \tag{5.14}$$

The lowest component of $J_{(\alpha_1\alpha_2\alpha_3\alpha_4)}$ is a conserved spin two current which is not the energy-momentum tensor. Note that $J_{(\alpha_1\alpha_2\alpha_3\alpha_4)}$ also contains a conserved higher-spin current. It will be interesting to perform a systematic study of three-point functions of $J_{(\alpha_1\alpha_2\alpha_3\alpha_4)}$ to see if they allow any parity-violating structures.

VI. MIXED CORRELATORS IN $\mathcal{N}=2$ SUPERCONFORMAL FIELD THEORY

Now we will generalize our method to mixed three-point functions in superconformal field theory with $\mathcal{N}=2$ supersymmetry. A specific feature of three-dimensional $\mathcal{N}=2$ superconformal field theories is contact terms in correlation functions of the conserved currents [58,59]. In this paper, we study correlation functions at noncoincident points where the contact terms do not contribute.

A. Supercurrent and flavor current multiplets

The 3D, $\mathcal{N}=2$ supercurrent was studied in [54,60–62]. It is a primary, dimension 2 symmetric spin tensor $J_{\alpha\beta}$, which obeys the conservation equation

$$D^{I\alpha}J_{\alpha\beta} = 0, \tag{6.1}$$

and has the following superconformal transformation law:

$$\delta J_{\alpha\beta} = -\xi J_{\alpha\beta} - 2\sigma(z)J_{\alpha\beta} + 2\lambda(z)^{\gamma}_{(\alpha}J_{\beta)\gamma}. \tag{6.2}$$

The general formalism in Sec. II allows the two-point function to be determined up to a single real coefficient

$$\langle J_{\alpha\beta}(z_1)J^{\alpha'\beta'}(z_2)\rangle = b_{\mathcal{N}=2}\frac{\boldsymbol{x}_{12(\alpha}{}^{\alpha'}\boldsymbol{x}_{12\beta)}{}^{\beta'}}{(\boldsymbol{x}_{12}^2)^3}. \tag{6.3}$$

It is then a simple exercise to show that the two-point function has the right symmetry properties under permutation of superspace points:

$$\langle J_{\alpha\beta}(z_1)J_{\alpha'\beta'}(z_2)\rangle = \langle J_{\alpha'\beta'}(z_2)J_{\alpha\beta}(z_1)\rangle \tag{6.4}$$

and also satisfies the conservation equation

$$D^{I\alpha}_{(1)}\langle J_{\alpha\beta}(z_1)J_{\alpha'\beta'}(z_2)\rangle = 0, \quad z_1 \neq z_2. \tag{6.5}$$

Similarly, the 3D $\mathcal{N}=2$ flavor current is a primary, dimension 1 scalar superfield L, which obeys the conservation equation

$$\left(D^{\alpha(I}D^{J)}_{\alpha} - \frac{1}{2}\delta^{IJ}D^{\alpha K}D^{K}_{\alpha}\right)L = 0, \tag{6.6}$$

and transforms under the superconformal group as

$$\delta L = -\xi L - \sigma(z)L. \tag{6.7}$$

As in the $\mathcal{N}=1$ case, we assume the $\mathcal{N}=2$ superconformal field theory in question has a set of flavor

[8] In general, if a superspace three-point function is fixed up to an overall coefficient, it is expected to be parity even because this contribution is expected to exist in a free theory of a real scalar superfield.

currents $L^{\bar{a}}$ associated with a simple flavor group. Due to the absence of spinor or R-symmetry indices, the $\mathcal{N}=2$ flavor current two-point function is fixed up to a single real coefficient $a_{\mathcal{N}=2}$ as follows:

$$\langle L^{\bar{a}}(z_1)L^{\bar{b}}(z_2)\rangle = a_{\mathcal{N}=2}\frac{\delta^{\bar{a}\bar{b}}}{\boldsymbol{x}_{12}^2}. \tag{6.8}$$

The two-point function obeys the correct symmetry properties under permutation of superspace points, $\langle L^{\bar{a}}(z_1)L^{\bar{b}}(z_2)\rangle = \langle L^{\bar{b}}(z_2)L^{\bar{a}}(z_1)\rangle$, and also satisfies the conservation equation

$$\left(D_{(1)}^{\alpha(I}D_{(1)\alpha}^{J)} - \frac{1}{2}\delta^{IJ}D_{(1)}^{\alpha K}D_{(1)\alpha}^{K}\right)\langle L^{\bar{a}}(z_1)L^{\bar{b}}(z_2)\rangle = 0, \quad z_1 \neq z_2. \tag{6.9}$$

In the next section we will compute the mixed correlation functions associated with the $\mathcal{N}=2$ supercurrent and flavor current multiplets. There are two possibilities to consider; they are

$$\langle L^{\bar{a}}(z_1)J_{\alpha\beta}(z_2)L^{\bar{b}}(z_3)\rangle, \langle J_{\alpha\beta}(z_1)J_{\gamma\delta}(z_2)L(z_3)\rangle. \tag{6.10}$$

Note that in second case we are considering a $U(1)$ flavor current.

B. The correlation function $\langle LJL\rangle$

First let us consider the $\langle LJL\rangle$ case first. Using the general ansatz, we have

$$\langle L^{\bar{a}}(z_1)J_{\alpha\beta}(z_2)L^{\bar{b}}(z_3)\rangle = \frac{\delta^{\bar{a}\bar{b}}\boldsymbol{x}_{23\alpha}{}^{\alpha'}\boldsymbol{x}_{23\beta}{}^{\beta'}}{\boldsymbol{x}_{13}^2(\boldsymbol{x}_{23}^2)^3}\mathcal{H}_{\alpha'\beta'}(\boldsymbol{X}_3,\Theta_3), \tag{6.11}$$

where $\mathcal{H}_{\alpha\beta} = \mathcal{H}_{(\alpha\beta)}$. The correlation function is also required to satisfy the following.

(i) *Scaling constraint.*—Under scale transformations the correlation function transforms as

$$\begin{aligned}&\langle L^{\bar{a}}(z_1')J_{\alpha\beta}(z_2')L^{\bar{b}}(z_3')\rangle\\ &\quad = (\lambda^2)^4\langle L^{\bar{a}}(z_1)J_{\alpha\beta}(z_2)L^{\bar{b}}(z_3)\rangle,\end{aligned} \tag{6.12}$$

from which we find the homogeneity constraint

$$\mathcal{H}_{\alpha\beta}(\lambda^2\boldsymbol{X},\lambda\Theta) = (\lambda^2)^{-2}\mathcal{H}_{\alpha\beta}(\boldsymbol{X},\Theta). \tag{6.13}$$

(ii) *Differential constraints.*—The differential constraints on the flavor current and supercurrent result in the following constraints on the correlation function:

$$\begin{aligned}&\left(D_{(1)}^{\sigma(I}D_{(1)\sigma}^{J)} - \frac{1}{2}\delta^{IJ}D_{(1)}^{\sigma K}D_{(1)\sigma}^{K}\right)\\ &\quad\times\langle L^{\bar{a}}(z_1)J_{\alpha\beta}(z_2)L^{\bar{b}}(z_3)\rangle = 0,\end{aligned} \tag{6.14a}$$

$$D_{(2)}^{I\alpha}\langle L^{\bar{a}}(z_1)J_{\alpha\beta}(z_2)L^{\bar{b}}(z_3)\rangle = 0. \tag{6.14b}$$

These result in the following differential constraints on $\mathcal{H}$:

$$\left(\mathcal{D}^{\sigma(I}\mathcal{D}_{\sigma}^{J)} - \frac{1}{2}\delta^{IJ}\mathcal{D}^{\sigma K}\mathcal{D}_{\sigma}^{K}\right)\mathcal{H}_{\alpha\beta}(\boldsymbol{X},\Theta) = 0, \tag{6.15a}$$

$$\mathcal{Q}^{I\alpha}\mathcal{H}_{\alpha\beta}(\boldsymbol{X},\Theta) = 0. \tag{6.15b}$$

(iii) *Point permutation symmetry.*—The symmetry under permutation of points (z_1 and z_3) results in the following constraint on the correlation function:

$$\langle L^{\bar{a}}(z_1)J_{\alpha\beta}(z_2)L^{\bar{b}}(z_3)\rangle = \langle L^{\bar{b}}(z_3)J_{\alpha\beta}(z_2)L^{\bar{a}}(z_1)\rangle, \tag{6.16}$$

which results in the point-switch identity

$$\mathcal{H}_{\alpha\beta}(\boldsymbol{X}_3,\Theta_3) = \frac{\boldsymbol{x}_{13}^{\sigma\sigma'}\boldsymbol{X}_{3\sigma'\alpha}\boldsymbol{x}_{13}^{\rho\rho'}\boldsymbol{X}_{3\rho'\beta}}{\boldsymbol{X}_3^6\boldsymbol{x}_{13}^6}\mathcal{H}_{\sigma\rho}(-\boldsymbol{X}_1^{\mathrm{T}},-\Theta_1). \tag{6.17}$$

The symmetry properties of $\mathcal{H}$ allow us to trade the spinor indices for a vector index

$$\mathcal{H}_{\alpha\beta}(\boldsymbol{X},\Theta) = (\gamma^m)_{\alpha\beta}\mathcal{H}_m(\boldsymbol{X},\Theta). \tag{6.18}$$

The most general expansion for $\mathcal{H}_m(\boldsymbol{X},\Theta)$ is then

$$\begin{aligned}\mathcal{H}_m(\boldsymbol{X},\Theta) &= A_m(X) - \frac{\mathrm{i}}{2}\Theta^2B_m(X) + (\Theta\Theta)^nC_{mn}(X)\\ &\quad + \frac{1}{8}\Theta^4D_m(X),\end{aligned} \tag{6.19}$$

where we have defined

$$(\Theta\Theta)_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}(\Theta\Theta)_{\alpha\beta}, \qquad (\Theta\Theta)_{\alpha\beta} = \Theta_\alpha^I\Theta_\beta^J\varepsilon_{IJ} \tag{6.20}$$

and accounted for the $\mathcal{N}=2$ identity

$$\Theta^2\Theta_\alpha^I\Theta_\beta^J\varepsilon_{IJ} = 0. \tag{6.21}$$

The prefactors in front of B and D have been chosen for convenience, and as in the $\mathcal{N}=1$ case it is more convenient to work with X^m instead of $\boldsymbol{X}^{\alpha\beta}$. Imposing (6.15b) results in the differential constraints

$$\partial^m A_m(X) = 0, \tag{6.22a}$$

$$\partial^m B_m(X) = 0, \tag{6.22b}$$

$$\epsilon^{mnt}\partial_n C_{mt}(X) = 0, \tag{6.22c}$$

$$B_q(X) + \epsilon_{qmn}\partial^n A^m(X) = 0, \tag{6.22d}$$

$$D_q(X) - \epsilon_{qmn}\partial^n B^m(X) = 0, \tag{6.22e}$$

$$\partial^m\{C_{mt}(X) + C_{tm}(X) - \eta_{mt}C^a{}_a(X)\} = 0 \tag{6.22f}$$

and the algebraic constraints

$$C^a{}_a(X) = 0, \tag{6.23a}$$

$$\epsilon^{qmt}C_{mt}(X) = 0, \tag{6.23b}$$

which imply that C is symmetric and traceless. Furthermore the scaling condition (6.13) allows us to construct the solutions

$$A_m(X) = a\frac{X_m}{X^3}, \tag{6.24a}$$

$$B_m(X) = b\frac{X_m}{X^4}, \tag{6.24b}$$

$$C_{mn}(X) = c\left(\frac{\eta_{mn}}{X^3} - \frac{3X_mX_n}{X^5}\right), \tag{6.24c}$$

$$D_m(X) = d\frac{X_m}{X^5}. \tag{6.24d}$$

Together (6.22) imply $B_m(X) = D_m(X) = 0$, while a and c remain as two free parameters. Hence the solution for $\mathcal{H}$ becomes

$$\mathcal{H}_{\alpha\beta}(\boldsymbol{X},\Theta) = \tilde{c}_{\mathcal{N}=2}\frac{X_{\alpha\beta}}{X^3} + \mathrm{i}c_{\mathcal{N}=2}\left\{\frac{\Theta^I_\alpha\Theta^J_\beta\varepsilon_{IJ}}{X^3} + \frac{3}{2}\frac{X_{\alpha\beta}X^{\gamma\delta}\Theta^I_\gamma\Theta^J_\delta\varepsilon_{IJ}}{X^5}\right\}. \tag{6.25}$$

After some lengthy calculation it turns out that only the second structure satisfies the conservation equation (6.15a). Hence there is only one linearly independent structure in the correlation function that is compatible with the differential constraints. Therefore we find that the final solution is

$$\langle L^{\bar{a}}(z_1)J_{\alpha\beta}(z_2)L^{\bar{b}}(z_3)\rangle = \frac{\delta^{\bar{a}\bar{b}}\boldsymbol{x}_{23\alpha}{}^{\alpha'}\boldsymbol{x}_{23\beta}{}^{\beta'}}{\boldsymbol{x}_{13}^2(\boldsymbol{x}_{23}^2)^3}\mathcal{H}_{\alpha'\beta'}(\boldsymbol{X}_3,\Theta_3), \tag{6.26}$$

with

$$\mathcal{H}_{\alpha\beta}(\boldsymbol{X},\Theta) = \mathrm{i}c_{\mathcal{N}=2}\left\{\frac{\Theta^I_\alpha\Theta^J_\beta\varepsilon_{IJ}}{\boldsymbol{X}^3} + \frac{3}{2}\frac{\boldsymbol{X}_{\alpha\beta}\boldsymbol{X}^{\gamma\delta}\Theta^I_\gamma\Theta^J_\delta\varepsilon_{IJ}}{\boldsymbol{X}^5}\right\}. \tag{6.27}$$

In deriving this result, we Taylor expanded the denominator in (6.25) using $X^2 = \boldsymbol{X}^2 - \frac{1}{4}\Theta^4$, which follows from (2.28) and (2.34), and then used the $\mathcal{N}=2$ identity (6.21). It may also be shown that this structure satisfies the point-switch identity (6.17).

The supercurrent $J_{\alpha\beta}$ leads to the following $\mathcal{N}=1$ supermultiplets (here the bar projection denotes setting $\theta^{I=\mathbf{2}}$ to zero and $D^\alpha = D^{\alpha,I=\mathbf{1}}$)[9]:

$$S_{\alpha\beta} = J_{\alpha\beta}|, \qquad D^\alpha S_{\alpha\beta} = 0, \tag{6.28a}$$

$$J_{\alpha\beta\gamma} = \mathrm{i}D^{\mathbf{2}}_{(\alpha}J_{\beta\gamma)}, \qquad D^\alpha J_{\alpha\beta\gamma} = 0. \tag{6.28b}$$

In these equations $J_{\alpha\beta\gamma}$ is the $\mathcal{N}=1$ supercurrent and $S_{\alpha\beta}$ is the additional $\mathcal{N}=1$ supermultiplet containing the second supersymmetry current and the R-symmetry current. Similarly, the $\mathcal{N}=2$ flavor current leads to

$$S = L^{\bar{a}}|, \tag{6.29a}$$

$$L^{\bar{a}}_\alpha = \mathrm{i}D^{\mathbf{2}}_\alpha L^{\bar{a}}, \qquad D^\alpha L^{\bar{a}}_\alpha = 0, \tag{6.29b}$$

where $L^{\bar{a}}_\alpha$ is the $\mathcal{N}=1$ flavor current and S is unconstrained. Hence, the $\mathcal{N}=2$ three-point function $\langle L^{\bar{a}}(z_1)J_{\alpha\beta}(z_2)L^{\bar{b}}(z_3)\rangle$ contains three-point functions of the following conserved component currents: the energy-momentum tensor, conserved vector currents, the supersymmetry currents and the R-symmetry current. All these three-point functions can be found by superspace reduction and are fixed by the $\mathcal{N}=2$ superconformal symmetry up to one overall coefficient (or vanish). A simple gamma-matrix-counting procedure similar to the one discussed in the previous section shows that all these correlators are parity even.

C. The correlation function $\langle JJL\rangle$

For this example, the general ansatz gives

$$\langle J_{\alpha\beta}(z_1)J_{\gamma\delta}(z_2)L(z_3)\rangle = \frac{\boldsymbol{x}_{13\alpha}{}^{\alpha'}\boldsymbol{x}_{13\beta}{}^{\beta'}\boldsymbol{x}_{23\gamma}{}^{\gamma'}\boldsymbol{x}_{23\delta}{}^{\delta'}}{(\boldsymbol{x}_{13}^2)^3(\boldsymbol{x}_{23}^2)^3}\mathcal{H}_{\alpha'\beta'\gamma'\delta'}(\boldsymbol{X}_3,\Theta_3), \tag{6.30}$$

[9]From here we will use bold R-symmetry indices to distinguish them from other types of indices.

where $\mathcal{H}_{\alpha\beta\gamma\delta} = \mathcal{H}_{(\alpha\beta)(\gamma\delta)}$. The correlation function is required to satisfy the following.

(i) *Scaling constraint.*—Under scale transformations the correlation function transforms as

$$\langle J_{\alpha\beta}(z_1')J_{\gamma\delta}(z_2')L(z_3')\rangle = (\lambda^2)^5\langle J_{\alpha\beta}(z_1)J_{\gamma\delta}(z_2)L(z_3)\rangle, \tag{6.31}$$

from which we find the homogeneity constraint

$$\mathcal{H}_{\alpha\beta\gamma\delta}(\lambda^2\boldsymbol{X}, \lambda\Theta) = (\lambda^2)^{-3}\mathcal{H}_{\alpha\beta\gamma\delta}(\boldsymbol{X}, \Theta). \tag{6.32}$$

(ii) *Differential constraints.*—The differential constraints on the flavor current and supercurrent result in the following constraints on the correlation function:

$$D^{I\alpha}_{(1)}\langle J_{\alpha\beta}(z_1)J_{\gamma\delta}(z_2)L(z_3)\rangle = 0. \tag{6.33a}$$

$$\left(D^{\sigma(I}_{(3)}D^{J)}_{(3)\sigma} - \frac{1}{2}\delta^{IJ}D^{\sigma K}_{(3)}D^{K}_{(3)\sigma}\right) \times \langle J_{\alpha\beta}(z_1)J_{\gamma\delta}(z_2)L(z_3)\rangle = 0. \tag{6.33b}$$

The first equation results in the following differential constraints on $\mathcal{H}$:

$$\mathcal{D}^{I\alpha}\mathcal{H}_{\alpha\beta\gamma\delta}(\boldsymbol{X}, \Theta) = 0. \tag{6.34}$$

The second constraint (6.33b) is more difficult to handle in this formalism; however, we will demonstrate how to deal with it later.

(iii) *Point permutation symmetry.*—The symmetry under permutation of points z_1 and z_2 results in the following constraint on the correlation function:

$$\langle J_{\alpha\beta}(z_1)J_{\gamma\delta}(z_2)L(z_3)\rangle = \langle J_{\gamma\delta}(z_2)J_{\alpha\beta}(z_1)L(z_3)\rangle, \tag{6.35}$$

which results in the point-switch identity

$$\mathcal{H}_{\alpha\beta\gamma\delta}(\boldsymbol{X}, \Theta) = \mathcal{H}_{\gamma\delta\alpha\beta}(-\boldsymbol{X}^{\mathrm{T}}, -\Theta). \tag{6.36}$$

Now due to the symmetry properties of $\mathcal{H}$, we may trade pairs of symmetric spinor indices for vector indices:

$$\mathcal{H}_{(\alpha\beta)(\gamma\delta)}(\boldsymbol{X}, \Theta) = (\gamma^m)_{\alpha\beta}(\gamma^n)_{\gamma\delta}\mathcal{H}_{mn}(\boldsymbol{X}, \Theta). \tag{6.37}$$

Now if we split $\mathcal{H}_{mn}$ into symmetric and antisymmetric parts

$$\begin{aligned}\mathcal{H}_{mn}(\boldsymbol{X}, \Theta) &= \mathcal{H}_{(mn)}(\boldsymbol{X}, \Theta) + \mathcal{H}_{[mn]}(\boldsymbol{X}, \Theta)\\ &= \mathcal{H}_{(mn)}(\boldsymbol{X}, \Theta) + \epsilon_{mnt}\mathcal{H}^t(\boldsymbol{X}, \Theta),\end{aligned} \tag{6.38}$$

then the point-switch identity implies

$$\begin{aligned}\mathcal{H}_{(mn)}(\boldsymbol{X}, \Theta) &= \mathcal{H}_{(mn)}(-\boldsymbol{X}^{\mathrm{T}}, -\Theta),\\ \mathcal{H}_t(\boldsymbol{X}, \Theta) &= -\mathcal{H}_t(-\boldsymbol{X}^{\mathrm{T}}, -\Theta).\end{aligned} \tag{6.39}$$

General expansions consistent with the index structure and symmetries are

$$\mathcal{H}_{(mn)}(\boldsymbol{X}, \Theta) = A_{(mn)}(X) + \Theta^2 B_{(mn)}(X) + (\Theta\Theta)^s C_{(mn)s}(X) + \Theta^4 D_{(mn)}(X), \tag{6.40a}$$

$$\mathcal{H}_t(\boldsymbol{X}, \Theta) = A_t(X) + \Theta^2 B_t(X) + (\Theta\Theta)^s C_{ts}(X) + \Theta^4 D_t(X). \tag{6.40b}$$

All the tensors comprising $\mathcal{H}_{(mn)}$ are even functions of X, while those in the expansion for $\mathcal{H}_t$ are odd functions of X. Furthermore, due to symmetry arguments the tensors $\mathcal{H}_{(mn)}$ and $\mathcal{H}_t$ do not mix in the conservation law (6.34); hence, they may be considered independently. First let us analyze $\mathcal{H}_{(mn)}$; imposing (6.15a) results in the differential constraints

$$\partial^m A_{(mn)}(X) = 0, \tag{6.41a}$$

$$\partial^m B_{(mn)}(X) = 0, \tag{6.41b}$$

$$\epsilon^{mrs}\partial_r C_{(mn)s}(X) = 0, \tag{6.41c}$$

$$2B_{(qn)}(X) + \mathrm{i}\epsilon_q{}^{mt}\partial_t A_{(mn)}(X) = 0, \tag{6.41d}$$

$$4D_{(qn)}(X) + \mathrm{i}\epsilon_q{}^{mt}\partial_t B_{(mn)}(X) = 0, \tag{6.41e}$$

$$\partial^m\{C_{(mn)s}(X) + C_{(sn)m}(X) - \eta_{ms}C^a{}_{na}(X)\} = 0 \tag{6.41f}$$

and the algebraic constraints $N = 2$ JJL—algebraic constraints 1

$$C^m{}_{nm}(X) = 0, \tag{6.42a}$$

$$\epsilon^{rms}C_{(mn)s}(X) = 0. \tag{6.42b}$$

The scaling condition (6.32) along with (6.42) implies that C is totally symmetric, traceless and even in X. Following the argument presented in Sec. IV B we find that no such tensor exists; hence, $C = 0$. Furthermore, evenness in X allows us to identify solutions for the remaining tensors:

$$A_{(mn)}(X) = a_1\frac{\eta_{mn}}{X^3} + a_2\frac{X_m X_n}{X^5}, \tag{6.43a}$$

$$B_{(mn)}(X) = b_1\frac{\eta_{mn}}{X^4} + b_2\frac{X_m X_n}{X^6}, \tag{6.43b}$$

$$D_{(mn)}(X) = d_1\frac{\eta_{mn}}{X^5} + d_2\frac{X_m X_n}{X^7}. \tag{6.43c}$$

Imposing (6.41a) and (6.41b) results in $a_2 = -3a_1$ and $b_2 = -2b_1$; however, for this choice of coefficients (6.41d) implies $B = 0$, while the tensor A survives. It is then easy to see that (6.41e) implies $D = 0$. Therefore the only solution is

$$A_{(mn)}(X) = a\left(\frac{\eta_{mn}}{X^3} - \frac{3X_m X_n}{X^5}\right). \tag{6.44}$$

Now let us direct our attention to $\mathcal{H}_t$; imposing (6.34) results in the set of equations

$$\epsilon^{mnt}\partial_m A_t(X) = 0, \tag{6.45a}$$

$$\epsilon^{mnt}\partial_m B_t(X) = 0, \tag{6.45b}$$

$$\partial^m\{C_{mn}(X) - \eta_{mn}C^s{}_s(X)\} = 0, \tag{6.45c}$$

$$2\epsilon_{qt}{}^s B_s(X) - \mathrm{i}\partial_t A_q(X) + \mathrm{i}\eta_{qt}\partial^s A_s(X) = 0, \tag{6.45d}$$

$$4\epsilon_{qt}{}^s D_s(X) - \mathrm{i}\partial_t B_q(X) + \mathrm{i}\eta_{qt}\partial^s B_s(X) = 0 \tag{6.45e}$$

and the algebraic constraints

$$\epsilon_n{}^{ma} C_{ma}(X) = 0, \tag{6.46a}$$

$$C_{mn}(X) - \eta_{mn}C^s{}_s(X) = 0. \tag{6.46b}$$

The algebraic constraints (6.46) imply that $C = 0$. Now since A, B and D are odd in X we can construct the solutions

$$A_t(X) = a\frac{X_t}{X^4}, \tag{6.47a}$$

$$B_t(X) = b\frac{X_t}{X^5}, \tag{6.47b}$$

$$D_t(X) = d\frac{X_t}{X^6}. \tag{6.47c}$$

However it is not too difficult to show that imposing (6.45d) and (6.45e) requires that A, B and D must all vanish. Hence $\mathcal{H}_t(\boldsymbol{X}, \Theta) = 0$.

So far we have found a single solution consistent with the supercurrent conservation equation and the point-switch identity:

$$\mathcal{H}_{mn}(\boldsymbol{X}, \Theta) = a\left(\frac{\eta_{mn}}{X^3} - \frac{3X_m X_n}{X^5}\right), \tag{6.48}$$

$$\begin{aligned}\mathcal{H}_{\alpha\beta\gamma\delta}(\boldsymbol{X}, \Theta) &= (\gamma^m)_{\alpha\beta}(\gamma^n)_{\gamma\delta}\mathcal{H}_{mn}(\boldsymbol{X}, \Theta) \\ &= d_{\mathcal{N}=2}\left(\frac{\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma}}{X^3} + \frac{3X_{\alpha\beta}X_{\gamma\delta}}{X^5}\right).\end{aligned} \tag{6.49}$$

Therefore the correlation function is

$$\langle J_{\alpha\beta}(z_1) J_{\gamma\delta}(z_2) L(z_3)\rangle = \frac{\boldsymbol{x}_{13\alpha}{}^{\alpha'}\boldsymbol{x}_{13\beta}{}^{\beta'}\boldsymbol{x}_{23\gamma}{}^{\gamma'}\boldsymbol{x}_{23\delta}{}^{\delta'}}{(\boldsymbol{x}_{13}^2)^3(\boldsymbol{x}_{23}^2)^3}\mathcal{H}_{\alpha'\beta'\gamma'\delta'}(\boldsymbol{X}_3, \Theta_3), \tag{6.50}$$

where, after writing our solution in terms of the variable $\boldsymbol{X}$,

$$\begin{aligned}\mathcal{H}_{\alpha\beta\gamma\delta}(\boldsymbol{X}, \Theta) = d_{\mathcal{N}=2}\bigg\{&\frac{\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}}{\boldsymbol{X}^3} + \frac{\varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma}}{\boldsymbol{X}^3} + \frac{3}{8}\frac{\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}\Theta^4}{\boldsymbol{X}^5} + \frac{3}{8}\frac{\varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma}\Theta^4}{\boldsymbol{X}^5} \\ &+ \frac{3\boldsymbol{X}_{\alpha\beta}\boldsymbol{X}_{\gamma\delta}}{\boldsymbol{X}^5} + \frac{3\mathrm{i}}{2}\frac{\varepsilon_{\alpha\beta}\boldsymbol{X}_{\gamma\delta}\Theta^2}{\boldsymbol{X}^5} + \frac{3\mathrm{i}}{2}\frac{\varepsilon_{\gamma\delta}\boldsymbol{X}_{\alpha\beta}\Theta^2}{\boldsymbol{X}^5} \\ &- \frac{3}{4}\frac{\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\Theta^4}{\boldsymbol{X}^5} + \frac{15}{8}\frac{\boldsymbol{X}_{\alpha\beta}\boldsymbol{X}_{\gamma\delta}\Theta^4}{\boldsymbol{X}^5}\bigg\}.\end{aligned} \tag{6.51}$$

However it remains to check whether this solution satisfies the flavor current conservation equation. As mentioned earlier it is difficult to check conservation laws on the third superspace point in this formalism as there are no identities that allow differential operators acting on the z_3 dependence to pass through the prefactor of (2.43). To deal with this we will rewrite our solution in terms of the three-point building block $\boldsymbol{X}_1$ using identities (2.31a) and (2.33). This ultimately has the effect

$$\langle J_{\alpha\beta}(z_1) J_{\gamma\delta}(z_2) L(z_3)\rangle \to \langle L(z_3) J_{\gamma\delta}(z_2) J_{\alpha\beta}(z_1)\rangle. \tag{6.52}$$

Written in terms of the variable $\boldsymbol{X}_1$, the correlation function is found to be

$$\langle L(z_3) J_{\gamma\delta}(z_2) J_{\alpha\beta}(z_1)\rangle = \frac{\boldsymbol{x}_{21\gamma}{}^{\gamma'}\boldsymbol{x}_{21\delta}{}^{\delta'}}{\boldsymbol{x}_{31}^2(\boldsymbol{x}_{21}^2)^3}\mathcal{H}_{\gamma'\delta'\alpha\beta}(\boldsymbol{X}_1, \Theta_1), \tag{6.53}$$

where

$$\mathcal{H}_{\gamma\delta\alpha\beta}(\boldsymbol{X},\Theta) = d_{\mathcal{N}=2}\Big\{\frac{\boldsymbol{X}_{\gamma\alpha}\boldsymbol{X}_{\delta\beta}}{\boldsymbol{X}^3} + \frac{\boldsymbol{X}_{\gamma\beta}\boldsymbol{X}_{\delta\alpha}}{\boldsymbol{X}^3} + \frac{3}{8}\frac{\boldsymbol{X}_{\gamma\alpha}\boldsymbol{X}_{\delta\beta}\Theta^4}{\boldsymbol{X}^5} + \frac{3}{8}\frac{\boldsymbol{X}_{\gamma\beta}\boldsymbol{X}_{\delta\alpha}\Theta^4}{\boldsymbol{X}^5} - \frac{3\boldsymbol{X}_{\alpha\beta}\boldsymbol{X}_{\gamma\delta}}{\boldsymbol{X}^3} - \frac{3\mathrm{i}}{2}\frac{\varepsilon_{\alpha\beta}\boldsymbol{X}_{\gamma\delta}\Theta^2}{\boldsymbol{X}^3} - \frac{3\mathrm{i}}{2}\frac{\varepsilon_{\gamma\delta}\boldsymbol{X}_{\alpha\beta}\Theta^2}{\boldsymbol{X}^3} + \frac{3}{4}\frac{\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\Theta^4}{\boldsymbol{X}^3} - \frac{15}{8}\frac{\boldsymbol{X}_{\alpha\beta}\boldsymbol{X}_{\gamma\delta}\Theta^4}{\boldsymbol{X}^5}\Big\}. \tag{6.54}$$

We are now able to check the conservation equation (6.33b), which after using identities equivalent to (2.39a) becomes the constraint

$$\left(\mathcal{D}^{\sigma(I}\mathcal{D}^{J)}_{\sigma} - \frac{1}{2}\delta^{IJ}\mathcal{D}^{\sigma K}\mathcal{D}^{K}_{\sigma}\right)\mathcal{H}_{\gamma\delta\alpha\beta}(\boldsymbol{X},\Theta) = 0. \tag{6.55}$$

After a very lengthy calculation one can show that the solution above satisfies this conservation equation; hence, this correlation function is nontrivial and is determined up to a single parameter.

This is a peculiar result, as it was shown in Sec. IV B that the correlation function $\langle JJL\rangle$ vanishes for $\mathcal{N}=1$. At first glance this appears to be a contradiction since any theory with $\mathcal{N}=2$ supersymmetry is also $\mathcal{N}=1$ supersymmetric. However, as was discussed in the previous subsection, the $\mathcal{N}=2$ current supermultiplets $J_{\alpha\beta}$ and L contain not only the $\mathcal{N}=1$ supercurrent and flavor currents, but also the unconstrained scalar superfield S and the supermultiplet of currents $S_{\alpha\beta}$. Hence, nonvanishing of the $\mathcal{N}=2$ three-point function (6.49) and (6.50) implies nonvanishing of some of the three-point functions involving these additional $\mathcal{N}=1$ currents. For example, from Eqs. (6.49) and (6.50) it follows that the following $\mathcal{N}=1$ correlator is, in general, nonzero:

$$\langle S_{\alpha_1\alpha_2}(z_1)J_{\beta_1\beta_2\beta_3}(z_2)L_{\gamma}(z_3)\rangle = -D^{\mathbf{2}}_{(2)(\beta_1}D^{\mathbf{2}}_{(3)\gamma}\langle J_{\alpha_1\alpha_2}(z_1)J_{\beta_2\beta_3)}(z_2)L(z_3)\rangle|, \tag{6.56}$$

where the bar projection means setting $\theta^{\mathbf{2}}_i$ to zero. In components this correlator contains (among others) $\langle R_m(x_1)T_{pq}(x_2)V_s(x_3)\rangle$, where R_m is the $U(1)$ R-symmetry current which exists in theories with $\mathcal{N}=2$ supersymmetry. In theories with $\mathcal{N}=1$ supersymmetry such a correlator does not exist because there is no R-symmetry current.[10] On the other hand, the $\mathcal{N}=2\to\mathcal{N}=1$ superspace reduction

$$\langle J_{\alpha_1\alpha_2}(z_1)J_{\beta_1\beta_2}(z_2)L(z_3)\rangle \to \langle J_{\alpha_1\alpha_2\alpha_3}(z_1)J_{\beta_1\beta_2\beta_3}(z_2)L_{\gamma}(z_3)\rangle \tag{6.57}$$

must give zero to be consistent with the result of the previous subsection. Let us check that this is indeed the case. To perform the reduction we compute

$$-\mathrm{i}D^{\mathbf{2}}_{(1)(\alpha_1}D^{\mathbf{2}}_{(2)(\beta_1}D^{\mathbf{2}}_{(3)\gamma}\langle J_{\alpha_2\alpha_3)}(z_1)J_{\beta_2\beta_3)}(z_2)L(z_3)\rangle|. \tag{6.58}$$

That is, we must act with three covariant derivatives with respect to $\theta^{\mathbf{2}}_i$ and then set all $\theta^{\mathbf{2}}_i$ to zero. From the explicit form of the correlator $\langle J_{\alpha_1\alpha_2}(z_1)J_{\beta_1\beta_2}(z_2)L(z_3)\rangle$ in Eqs. (6.49) and (6.50) it follows that it depends on $\theta^{\mathbf{2}}_i\theta^{\mathbf{2}}_j$. Since it is Grassmann even it contains only even powers of $\theta^{\mathbf{2}}_i$. Therefore, acting on $\langle J_{\alpha_1\alpha_2}(z_1)J_{\beta_1\beta_2}(z_2)L(z_3)\rangle$ with three derivatives as in (6.58) will give a result either linear or higher order in $\theta^{\mathbf{2}}_i$, so it vanishes when we set $\theta^{\mathbf{2}}_i=0$. This shows that despite being nonzero our result (6.49) and (6.50) is consistent with vanishing of the similar correlator in the $\mathcal{N}=1$ case.

VII. MIXED CORRELATORS IN $\mathcal{N}=3$, 4 SUPERCONFORMAL FIELD THEORY

In this section we will generalize our method for $\mathcal{N}=3$ and $\mathcal{N}=4$ superconformal theories. An essential difference with the previous cases is that the flavor current now carries R-symmetry indices which must be taken into account in the irreducible decompositions. We will start with reviewing the properties of the $\mathcal{N}=3$ and $\mathcal{N}=4$ supercurrent [63,64] and flavor current multiplets and then apply our formalism to compute the mixed correlation functions involving these multiplets.

A. Supercurrent and flavor current multiplets

1. $\mathcal{N}=3$ theories

The 3D, $\mathcal{N}=3$ supercurrent is a primary, dimension 3/2 spinor superfield J_{α}, which satisfies the conservation equation

$$D^{I\alpha}J_{\alpha} = 0, \tag{7.1}$$

and has the following superconformal transformation law:

$$\delta J_{\alpha} = -\xi J_{\alpha} - \frac{3}{2}\sigma(z)J_{\alpha} + \lambda(z)_{\alpha}{}^{\beta}J_{\beta}. \tag{7.2}$$

[10]Note that all component three-point functions contained in (6.49) and (6.50) are parity even.

The two-point function is again determined up to a single real coefficient

$$\langle J_\alpha(z_1) J_\beta(z_2)\rangle = b_{\mathcal{N}=3} \frac{\boldsymbol{x}_{12\alpha\beta}}{(\boldsymbol{x}_{12}^2)^2}. \tag{7.3}$$

It has the right symmetry properties under permutation of superspace points

$$\langle J_\alpha(z_1) J_\beta(z_2)\rangle = -\langle J_\beta(z_2) J_\alpha(z_1)\rangle \tag{7.4}$$

and also satisfies the conservation equation

$$D^{I\alpha}_{(1)} \langle J_\alpha(z_1) J_\beta(z_2)\rangle = 0, \quad z_1 \neq z_2. \tag{7.5}$$

The $\mathcal{N}=3$ flavor current is a primary, dimension 1 isovector L^I, which obeys the conservation equation

$$D^{(I}_\alpha L^{J)} - \frac{1}{3}\delta^{IJ} D^K_\alpha L^K = 0, \tag{7.6}$$

and transforms under the superconformal group as

$$\delta L^I = -\xi L^I - \sigma(z) L^I + \Lambda^{IJ}(z) L^J. \tag{7.7}$$

The $\mathcal{N}=3$ flavor current two-point function is fixed up to a single real coefficient $a_{\mathcal{N}=3}$:

$$\langle L^{I\bar a}(z_1) L^{J\bar b}(z_2)\rangle = a_{\mathcal{N}=3} \frac{\delta^{\bar a\bar b} u^{IJ}_{12}}{\boldsymbol{x}^2_{12}}, \tag{7.8}$$

where we have introduced the flavor group index $\bar a$. The two-point function obeys the correct symmetry properties under permutation of superspace points, $\langle L^{I\bar a}(z_1) L^{J\bar b}(z_2)\rangle = \langle L^{J\bar b}(z_2) L^{I\bar a}_\alpha(z_1)\rangle$, and also satisfies the conservation equation

$$D^{(I}_{(1)\alpha} \langle L^{J)\bar a}(z_1) L^{K\bar b}(z_2)\rangle - \frac{1}{3}\delta^{IJ} D^L_{(1)\alpha} \langle L^{L\bar a}(z_1) L^{K\bar b}(z_2)\rangle = 0, \quad z_1 \neq z_2. \tag{7.9}$$

2. $\mathcal{N}=4$ theories

The $\mathcal{N}=4$ supercurrent is a primary, dimension 1 scalar superfield J, which satisfies the conservation equation

$$\left(D^{I\alpha} D^K_\alpha - \frac{1}{4}\delta^{IK} D^{L\alpha} D^L_\alpha\right) J = 0, \tag{7.10}$$

and has the following superconformal transformation law:

$$\delta J_\alpha = -\xi J_\alpha - \sigma(z) J_\alpha. \tag{7.11}$$

The dimension of the supercurrent is fixed by the conservation equation (7.10). The two-point function is determined up to a single real coefficient

$$\langle J(z_1) J(z_2)\rangle = b_{\mathcal{N}=4} \frac{1}{\boldsymbol{x}^2_{12}}. \tag{7.12}$$

Under permutation of superspace points, we have

$$\langle J(z_1) J(z_2)\rangle = \langle J(z_2) J(z_1)\rangle. \tag{7.13}$$

The two-point function also satisfies the conservation equation

$$\left(D^{I\alpha}_{(1)} D^K_{(1)\alpha} - \frac{1}{4}\delta^{IK} D^{L\alpha}_{(1)} D^L_{(1)\alpha}\right)\langle J(z_1) J(z_2)\rangle = 0, \quad z_1 \neq z_2. \tag{7.14}$$

In the $\mathcal{N}=4$ case there exists two inequivalent flavor current multiplets, described by $\mathsf{SO}(4)$ bivectors L^{IJ}_+, L^{IJ}_-, which are primary with dimension 1 and satisfy

$$L^{IJ}_\pm = -L^{JI}_\pm, \qquad \frac{1}{2}\epsilon^{IJKL} L^{KL}_\pm = \pm L^{IJ}_\pm. \tag{7.15}$$

where $\bar a$ is the index for the flavor group. The flavor current multiplets are subject to the conservation equation

$$D^I_\alpha L^{JK}_\pm = D^{[I}_\alpha L^{JK]}_\pm - \frac{2}{3} D^L_\alpha L^{L[J}_\pm \delta^{K]I} \tag{7.16}$$

and transform under the superconformal group as

$$\delta L^{IJ}_\pm = -\xi L^{IJ}_\pm - \sigma(z) L^{IJ}_\pm + \Lambda^{K[I}(z) L^{J]K}_\pm. \tag{7.17}$$

Since the flavor current multiplets $L^{IJ}_\pm$ are inequivalent, they may be studied independently when deriving correlation functions.

B. Mixed correlation functions in $\mathcal{N}=3$ theories

There are two mixed correlation functions in $\mathcal{N}=3$ theories; they are

$$\langle L^I(z_1) J_\alpha(z_2) L^J(z_3)\rangle, \qquad \langle J_\alpha(z_1) J_\beta(z_2) L^I(z_3)\rangle. \tag{7.18}$$

1. The correlation function $\langle LJL\rangle$

Using the general ansatz, we have

$$\langle L^{I\bar a}(z_1) J_\alpha(z_2) L^{J\bar b}(z_3)\rangle = \frac{\delta^{\bar a\bar b} u^{II'}_{13} \boldsymbol{x}_{23\alpha}{}^{\alpha'}}{\boldsymbol{x}^2_{13}(\boldsymbol{x}^2_{23})^2} \mathcal{H}^{I'J}_{\alpha'}(\boldsymbol{X}_3, \Theta_3). \tag{7.19}$$

The correlation function is required to satisfy the following.

(i) *Scaling constraint.*—Under scale transformations the correlation function must transform as

$$\langle L^{I\bar{a}}(z_1')J_\alpha(z_2')L^{J\bar{b}}(z_3')\rangle = (\lambda^2)^{7/2}\langle L^{I\bar{a}}(z_1)J_\alpha(z_2)L^{J\bar{b}}(z_3)\rangle, \quad (7.20)$$

which gives rise to the homogeneity constraint on $\mathcal{H}$:

$$\mathcal{H}^{IJ}_\alpha(\lambda^2\boldsymbol{X},\lambda\Theta) = (\lambda^2)^{-3/2}\mathcal{H}^{IJ}_\alpha(\boldsymbol{X},\Theta). \quad (7.21)$$

(ii) *Differential constraints.*—The differential constraints on the flavor current and supercurrent result in the following constraints on the correlation function:

$$D^{(I}_{(1)\alpha}\langle L^{J)\bar{a}}(z_1)J_\beta(z_2)L^{K\bar{b}}(z_3)\rangle - \frac{1}{3}\delta^{IJ}D^{L}_{(1)\alpha}\langle L^{L\bar{a}}(z_1)J_\beta(z_2)L^{K\bar{b}}(z_3)\rangle = 0, \quad (7.22a)$$

$$D^{I\alpha}_{(2)}\langle L^{J\bar{a}}(z_1)J_\alpha(z_2)L^{K\bar{b}}(z_3)\rangle = 0. \quad (7.22b)$$

These equations result in the following differential constraints on $\mathcal{H}$:

$$\mathcal{D}^{(I}_\alpha\mathcal{H}^{J)K}_\beta(\boldsymbol{X},\Theta) - \frac{1}{3}\delta^{IJ}\mathcal{D}^{L}_\alpha\mathcal{H}^{LK}_\beta(\boldsymbol{X},\Theta) = 0, \quad (7.23a)$$

$$\mathcal{Q}^{I\alpha}\mathcal{H}^{JK}_\alpha(\boldsymbol{X},\Theta) = 0. \quad (7.23b)$$

(iii) *Point permutation symmetry.*—The symmetry under permutation of points (z_1 and z_3) imposes the following constraint on the correlation function:

$$\langle L^{I\bar{a}}(z_1)J_\alpha(z_2)L^{J\bar{b}}(z_3)\rangle = \langle L^{J\bar{b}}(z_3)J_\alpha(z_2)L^{I\bar{a}}(z_1)\rangle, \quad (7.24)$$

which results in the point-switch identity

$$\mathcal{H}^{IJ}_\alpha(\boldsymbol{X}_3,\Theta_3) = -\frac{(u_{13}^{-1})^{II'}u_{13}^{JJ'}\boldsymbol{x}_{13}^{\alpha'\sigma}\boldsymbol{X}_{3\sigma\sigma}}{\boldsymbol{X}_3^4\boldsymbol{x}_{13}^4} \times \mathcal{H}^{J'I'}_{\alpha'}(-\boldsymbol{X}_1^{\mathrm{T}},-\Theta_1). \quad (7.25)$$

Now let us find the general solution for $\mathcal{H}$ consistent with the above constraints. To do this systematically, we note that since $\mathcal{H}$ is Grassmann odd we must find all the linearly independent structures that are odd in Θ that can be constructed out of the $\mathcal{N}=3$ building blocks. A general expansion for $\mathcal{H}^{IJ}_\alpha$ is

$$\mathcal{H}^{IJ}_\alpha(\boldsymbol{X},\Theta) = \mathcal{H}^{IJ}_{(1)\alpha}(\boldsymbol{X},\Theta) + \mathcal{H}^{IJ}_{(3)\alpha}(\boldsymbol{X},\Theta) + \mathcal{H}^{IJ}_{(5)\alpha}(\boldsymbol{X},\Theta) \quad (7.26)$$

$$= \mathcal{H}_{(1)\alpha\beta}(X)A^{IJK}\Theta^{K\beta} + \mathcal{H}_{(3)\alpha\beta\gamma\delta}(X)B^{IJKLM}\Theta^{K\beta}\Theta^{L\gamma}\Theta^{M\delta} + \mathcal{H}_{(5)\alpha\beta\gamma\delta\mu\nu}(X)C^{IJKLMNP}\Theta^{K\beta}\Theta^{L\gamma}\Theta^{M\delta}\Theta^{N\mu}\Theta^{P\nu}, \quad (7.27)$$

where A, B, and C are tensors formed out of the $\mathcal{N}=3$ invariant tensors δ^{IJ} and ϵ^{IJK}. At $O(\Theta^1)$ the only choice we can make for A is $A^{IJK}=\epsilon^{IJK}$, from which we find the linearly independent structures

$$\mathcal{H}^{IJ}_{(1)\alpha}(\boldsymbol{X},\Theta) = a_1\epsilon^{IJK}\frac{\Theta^K_\alpha}{X^2} + a_2\epsilon^{IJK}\frac{X_{\alpha\beta}\Theta^{K\beta}}{X^3}. \quad (7.28)$$

The conservation equation (7.23b) implies that the terms $O(\Theta^1)$ are odd in $X_{\alpha\beta}$, while the terms $O(\Theta^3)$ must be even in $X_{\alpha\beta}$. At $O(\Theta^3)$ we have the following choices for B:

$$B_1^{IJKLM} = \delta^{IJ}\epsilon^{KLM}, \qquad B_2^{IJKLM} = \epsilon^{IJK}\delta^{LM}, \quad (7.29)$$

$$B_3^{IJKLM} = \delta^{IK}\epsilon^{JLM} + \delta^{JK}\epsilon^{ILM}, \quad (7.30)$$

from which we find the linearly independent structures

$$\mathcal{H}^{IJ}_{(3)\alpha}(\boldsymbol{X},\Theta) = b_1\epsilon^{IJK}\frac{\Theta^K_\alpha\Theta^2}{X^3} + b_2\delta^{IJ}\epsilon^{KPQ}\Theta^{K\delta}\Theta^{P\beta}\Theta^{K\gamma}\frac{X_{\alpha(\delta}X_{\beta\gamma)}}{X^5} + b_3(\epsilon^{IKP}\Theta^{J\delta} + \epsilon^{JKP}\Theta^{I\delta})\Theta^{K\beta}\Theta^{P\gamma}\frac{X_{\alpha\delta}X_{\beta\gamma}}{X^5}. \quad (7.31)$$

If we follow the same procedure at $O(\Theta^5)$, we find the structures

$$\mathcal{H}^{IJ}_{(5)\alpha}(\boldsymbol{X},\Theta) = c_1\epsilon^{IJK}\frac{\Theta^K_\alpha\Theta^4}{X^5} + c_2(\epsilon^{IKP}\Theta^J_\alpha + \epsilon^{JKP}\Theta^I_\alpha)\Theta^{K\beta}\Theta^{P\gamma}\frac{X_{\beta\gamma}\Theta^2}{X^5}. \quad (7.32)$$

In determining the linearly independent terms we make use of the $\mathcal{N}=3$ identity

$$\epsilon^{IJK}\Theta^{I\alpha}\Theta^{J\beta}\Theta^{K\gamma}\Theta^2 = 0, \quad (7.33)$$

in addition to

$$\Theta^{I\alpha}\Theta^J_\alpha\Theta^{K\beta}\Theta^L_\gamma\epsilon^{JKL} = 2\Theta^2\Theta^{J\beta}\Theta^K_\gamma\epsilon^{IJK}, \quad (7.34a)$$

$$\Theta^{I\alpha}\Theta^{J\beta}\Theta^{K\gamma}\Theta^{L\delta}\epsilon^{JKL} = -\frac{1}{2}\varepsilon^{\alpha\beta}\Theta^2\Theta^{J\gamma}\Theta^{K\delta}\epsilon^{IJK} - \frac{1}{2}\varepsilon^{\alpha\gamma}\Theta^2\Theta^{J\beta}\Theta^{K\delta}\epsilon^{IJK} \quad (7.34b)$$

$$-\frac{1}{2}\varepsilon^{\alpha\delta}\Theta^2\Theta^{J\beta}\Theta^{K\gamma}\epsilon^{IJK},$$

$$\Theta^{(P\alpha}\epsilon^{I)MN}\Theta^{M\mu}\Theta^{N\nu} = -\frac{1}{3}\varepsilon^{\alpha\mu}\Theta^{(P\beta}\Theta^M_\beta\Theta^{N\nu}\epsilon^{I)MN} - \frac{1}{3}\varepsilon^{\alpha\nu}\Theta^{(P\beta}\Theta^M_\beta\Theta^{N\mu}\epsilon^{I)MN}, \quad (7.34c)$$

which arise as differential consequences of (7.33). Applying the conservation law (7.23b) results in

$$a_1 = b_1 = b_3 = c_1 = c_2 = 0, \quad (7.35)$$

which leaves us with only two structures. Next we must impose the flavor current conservation equation (7.23a). After a lengthy calculation we find $b_2 = \mathrm{i}a_2$. Hence the solution is

$$\mathcal{H}^{IJ}_\alpha(\boldsymbol{X},\Theta) = c_{\mathcal{N}=3}\left\{\epsilon^{IJK}\frac{X_{\alpha\beta}\Theta^{K\beta}}{X^3} + \mathrm{i}\delta^{IJ}\epsilon^{KPQ}\Theta^{K\delta}\Theta^{P\sigma}\Theta^{K\gamma}\frac{X_{\alpha(\delta}X_{\sigma\gamma)}}{X^5}\right\} \quad (7.36)$$

$$= c_{\mathcal{N}=3}\left\{\epsilon^{IJK}\left(\frac{\boldsymbol{X}_{\alpha\beta}\Theta^{K\beta}}{\boldsymbol{X}^3} + \frac{\mathrm{i}}{2}\frac{\Theta^K_\alpha\Theta^2}{\boldsymbol{X}^3} + \frac{3}{8}\frac{\boldsymbol{X}_{\alpha\beta}\Theta^{K\beta}\Theta^4}{\boldsymbol{X}^5}\right) + \mathrm{i}\delta^{IJ}\epsilon^{KPQ}\Theta^{K\delta}\Theta^{P\sigma}\Theta^{K\gamma}\frac{\boldsymbol{X}_{\alpha(\delta}\boldsymbol{X}_{\sigma\gamma)}}{\boldsymbol{X}^5}\right\}. \quad (7.37)$$

After some additional calculation it can be shown that this solution also satisfies the point-switch identity (7.25), which completes our study of the $\mathcal{N}=3$ correlation function.

2. The correlation function $\langle JJL\rangle$

Using the general ansatz, we have

$$\langle J_\alpha(z_1)J_\beta(z_2)L^I(z_3)\rangle = \frac{\boldsymbol{x}_{23\alpha}{}^{\alpha'}\boldsymbol{x}_{23\beta}{}^{\beta'}}{(\boldsymbol{x}^2_{13})^2(\boldsymbol{x}^2_{23})^2}\mathcal{H}^I_{\alpha'\beta'}(\boldsymbol{X}_3,\Theta_3). \quad (7.38)$$

Note that in this case the flavor current is $U(1)$. The correlation function is required to satisfy the following.

(i) *Scaling constraint.*—Under scale transformations the correlation function must transform as

$$\langle J_\alpha(z_1')J_\beta(z_2')L^I(z_3')\rangle = (\lambda^2)^4\langle J_\alpha(z_1)J_\beta(z_2)L^I(z_3)\rangle, \quad (7.39)$$

which gives rise to the homogeneity constraint on $\mathcal{H}$:

$$\mathcal{H}^I_{\alpha\beta}(\lambda^2\boldsymbol{X},\lambda\Theta) = (\lambda^2)^{-2}\mathcal{H}^I_{\alpha\beta}(\boldsymbol{X},\Theta). \quad (7.40)$$

(ii) *Differential constraints.*—The differential constraints on the flavor current and supercurrent result in the following constraints on the correlation function:

$$D^{(I}_{(3)\gamma}\langle J_\alpha(z_1)J_\beta(z_2)L^{J)}(z_3)\rangle - \frac{1}{3}\delta^{IJ}D^K_{(3)\gamma}\langle J_\alpha(z_1)J_\beta(z_2)L^K(z_3)\rangle = 0, \quad (7.41a)$$

$$D^{I\alpha}_{(1)}\langle J_\alpha(z_1)J_\beta(z_2)L^J(z_3)\rangle = 0. \quad (7.41b)$$

Since (7.41a) involves a covariant derivative acting on the third point, it is more difficult to impose. However it turns out that the second equation is sufficient to show that this correlation function vanishes. From (7.41b) we obtain

$$\mathcal{D}^{I\alpha}\mathcal{H}^J_{\alpha\beta}(\boldsymbol{X},\Theta) = 0. \quad (7.42)$$

(iii) *Point permutation symmetry.*—The symmetry under permutation of points (z_1 and z_2) imposes the following constraint on the correlation function:

$$\langle J_\alpha(z_1)J_\beta(z_2)L^I(z_3)\rangle = -\langle J_\beta(z_2)J_\alpha(z_1)L^I(z_3)\rangle, \quad (7.43)$$

which results in the point-switch identity

$$\mathcal{H}^I_{\alpha\beta}(\boldsymbol{X},\Theta) = -\mathcal{H}^I_{\beta\alpha}(-\boldsymbol{X}^{\mathrm{T}},-\Theta). \quad (7.44)$$

To proceed we start by decomposing $\mathcal{H}$ into symmetric and antisymmetric parts as follows:

$$\mathcal{H}^I_{\alpha\beta}(\boldsymbol{X},\Theta) = \mathcal{H}^I_{(\alpha\beta)}(\boldsymbol{X},\Theta) + \mathcal{H}^I_{[\alpha\beta]}(\boldsymbol{X},\Theta) = \mathcal{H}^I_{(\alpha\beta)}(\boldsymbol{X},\Theta) + \varepsilon_{\alpha\beta}\mathcal{H}^I(\boldsymbol{X},\Theta). \quad (7.45)$$

The symmetry under permutation of points (7.44) implies

$$\mathcal{H}^I_{(\alpha\beta)}(\boldsymbol{X},\Theta) = -\mathcal{H}^I_{(\alpha\beta)}(-\boldsymbol{X}^{\mathrm{T}},-\Theta),$$
$$\mathcal{H}^I(\boldsymbol{X},\Theta) = \mathcal{H}^I(-\boldsymbol{X}^{\mathrm{T}},-\Theta), \quad (7.46)$$

therefore $\mathcal{H}^I_{(\alpha\beta)}$ is an odd function, while $\mathcal{H}^I$ is an even function. They will not mix in the conservation law (7.42); hence, we may consider each of them independently.

Starting with $\mathcal{H}^I$, we note that since it is Grassmann even it must be an even function of Θ; hence, it admits the expansion

$$\mathcal{H}^I(\boldsymbol{X},\Theta) = \mathcal{H}_{(2)\alpha\beta}(X)A^{IJK}\Theta^{J\alpha}\Theta^{K\beta} + \mathcal{H}_{(4)\alpha\beta\gamma\delta}(X)B^{IJKLM}\Theta^{J\alpha}\Theta^{K\beta}\Theta^{L\gamma}\Theta^{M\delta} + \mathcal{H}_{(6)\alpha\beta\gamma\delta\sigma\mu}(X)C^{IJKLMNP}\Theta^{J\alpha}\Theta^{K\beta}\Theta^{L\gamma}\Theta^{M\delta}\Theta^{N\sigma}\Theta^{P\mu}. \quad (7.47)$$

Here we have replaced the variable $\boldsymbol{X}$ with X and introduced the arbitrary tensors A, B, and C, which are constructed out of the invariant tensors for the $\mathcal{N}=3$ R-symmetry group. The $\mathcal{H}_{(i)}$ are all even functions of X. At $O(\Theta^2)$ the only choice is $A^{IJK}=\epsilon^{IJK}$, so we have the contribution

$$\mathcal{H}_{(2)(\alpha\beta)}(X)\epsilon^{IJK}\Theta^{J\alpha}\Theta^{K\beta}. \tag{7.48}$$

However, it is not too hard to see that we cannot construct an even, symmetric function of X with the required index structure. Hence $\mathcal{H}_{(2)(\alpha\beta)}(X)=0$. Now at $O(\Theta^4)$ we have the choices

$$B_1^{IJKLM}=\delta^{IJ}\epsilon^{KLM}, \qquad B_2^{IJKLM}=\epsilon^{IJK}\delta^{LM}. \tag{7.49}$$

The choice B_1 results in the contribution

$$\mathcal{H}_{(4)\alpha(\beta\gamma\delta)}(X)\Theta^{I\alpha}\Theta^{K\beta}\Theta^{L\gamma}\Theta^{M\delta}\epsilon^{KLM}. \tag{7.50}$$

After applying the $\mathcal{N}=3$ identity (7.34a), this is equivalent to the contribution

$$\mathcal{F}_{(\gamma\delta)}(X)\epsilon^{IJK}\Theta^2\Theta^{J\gamma}\Theta^{K\delta}, \tag{7.51}$$

where $\mathcal{F}$ is a symmetric and even function of X. We cannot construct such a function; hence, $\mathcal{F}_{(\gamma\delta)}(X)=0$. Indeed if we follow the same procedure for B_2, we arrive at the same conclusion. Concerning contributions $O(\Theta^6)$, no terms are permitted due to the $\mathcal{N}=3$ identity (7.34a). Hence we find $\mathcal{H}^I(\boldsymbol{X},\Theta)=0$ as there are no contributions that are consistent with the algebraic symmetries.

Let us now follow the same procedure for the symmetric contribution $\mathcal{H}^I_{(\alpha\beta)}$. Since it is Grassmann even it must be an even function of Θ; hence, it must be odd in X. The general expansion for this contribution reads

$$\begin{aligned}\mathcal{H}^I_{(\alpha\beta)}(\boldsymbol{X},\Theta) &= \mathcal{H}_{(2)(\alpha\beta)\mu\nu}(X)A^{IJK}\Theta^{J\mu}\Theta^{K\nu} \\ &\quad + \mathcal{H}_{(4)(\alpha\beta)\mu\nu\gamma\delta}(X)B^{IJKLM}\Theta^{J\mu}\Theta^{K\nu}\Theta^{L\gamma}\Theta^{M\delta},\end{aligned} \tag{7.52}$$

where the $\mathcal{H}_{(i)}$ are odd functions of X. Here we have neglected the contribution $O(\Theta^6)$ as it will vanish due to $\mathcal{N}=3$ identities. Following the same procedure outlined above we find that to $O(\Theta^2)$ we have the contribution

$$\mathcal{H}_{(2)(\alpha\beta)(\mu\nu)}(X)\epsilon^{IJK}\Theta^{J\mu}\Theta^{K\nu}. \tag{7.53}$$

Since the tensor $\mathcal{H}_{(2)}$ is symmetric in each pair of spinor indices, we may trade them for vector ones:

$$\mathcal{H}_{(2)(\alpha\beta)(\mu\nu)}(X)=(\gamma^a)_{\alpha\beta}(\gamma^b)_{\mu\nu}\mathcal{H}_{(2)ab}(X). \tag{7.54}$$

The general expansion for $\mathcal{H}_{(2)ab}$ with the scaling condition (7.40) is

$$\mathcal{H}_{(2)ab}(X)=\frac{h_1}{X^3}\eta_{ab}+\frac{h_2}{X^4}\epsilon_{abc}X^c+\frac{h_3}{X^5}X_aX_b, \tag{7.55}$$

however, only the second term is odd in X, which results in the contribution

$$\mathcal{H}^I_{(\alpha\beta)}(\boldsymbol{X},\Theta)\propto\frac{1}{X^4}\epsilon^{IJK}\Theta^{J\gamma}\Theta^K_{(\alpha}X_{\beta)\gamma}. \tag{7.56}$$

Concerning the terms $O(\Theta^4)$ we follow the same procedure outlined above; for each choice of B in (7.49) we obtain the contribution

$$\mathcal{H}^I_{(\alpha\beta)}(\boldsymbol{X},\Theta)\propto\frac{1}{X^5}\epsilon^{IJK}\Theta^2\Theta^{J\gamma}\Theta^K_{(\alpha}X_{\beta)\gamma}, \tag{7.57}$$

where we have made use of (7.34a). Hence our solution for $\mathcal{H}^I_{(\alpha\beta)}$ is of the form

$$\mathcal{H}^I_{(\alpha\beta)}(\boldsymbol{X},\Theta)=\frac{a_1}{X^4}\epsilon^{IJK}\Theta^{J\gamma}\Theta^K_{(\alpha}X_{\beta)\gamma}+\frac{a_2}{X^5}\epsilon^{IJK}\Theta^2\Theta^{J\gamma}\Theta^K_{(\alpha}X_{\beta)\gamma}. \tag{7.58}$$

It remains to impose the conservation equation (7.42). After a short calculation we find $a_1=a_2=0$; hence, this correlation function vanishes.

C. Mixed correlation functions in $\mathcal{N}=4$ theories

For $\mathcal{N}=4$ superconformal symmetry there are two possible mixed correlation functions (for concreteness we will consider only L_+^{IJ}); they are

$$\langle L_+^{IK}(z_1)J(z_2)L_+^{JL}(z_3)\rangle, \qquad \langle J(z_1)J(z_2)L_+^{IJ}(z_3)\rangle, \tag{7.59}$$

where in the second case we require a $U(1)$ flavor group. The first correlator $\langle LJL\rangle$ was previously studied in [35], so we will not analyze it here.

The general ansatz for the correlation function $\langle JJL\rangle$ is

$$\langle J(z_1)J(z_2)L_+^{IJ}(z_3)\rangle=\frac{1}{\boldsymbol{x}_{13}^2\boldsymbol{x}_{23}^2}\mathcal{H}^{IJ}(\boldsymbol{X}_3,\Theta_3). \tag{7.60}$$

As we will soon find out, the algebraic symmetries on the tensor $\mathcal{H}$ are sufficient to show that this correlation function vanishes. The relevant constraints are the following.

(i) *Scaling constraint.*—The correlation function must transform as

$$\langle J(z_1')J(z_2')L_+^{IJ}(z_3')\rangle=(\lambda^2)^3\langle J(z_1)J(z_2)L_+^{IJ}(z_3)\rangle, \tag{7.61}$$

from which we find the homogeneity constraint on $\mathcal{H}$:

$$\mathcal{H}^{IJ}(\lambda^2 \boldsymbol{X}, \lambda\Theta) = (\lambda^2)^{-1}\mathcal{H}^{IJ}(\boldsymbol{X}, \Theta). \tag{7.62}$$

(ii) *Algebraic constraints.*—The symmetry under permutation of points (z_1 and z_2) constrains the correlation function as follows:

$$\langle J(z_1)J(z_2)L_+^{IJ}(z_3)\rangle = \langle J(z_2)J(z_1)L_+^{IJ}(z_3)\rangle, \tag{7.63}$$

which is equivalent to

$$\mathcal{H}^{IJ}(\boldsymbol{X}, \Theta) = \mathcal{H}^{IJ}(-\boldsymbol{X}^{\mathrm{T}}, -\Theta). \tag{7.64}$$

In addition, we also have constraints arising from antisymmetry and self-duality of the flavor current, which give rise to

$$\mathcal{H}^{IJ}(\boldsymbol{X}, \Theta) = -\mathcal{H}^{JI}(\boldsymbol{X}, \Theta), \qquad \mathcal{H}^{IJ}(\boldsymbol{X}, \Theta) = \frac{1}{2}\epsilon^{IJKL}\mathcal{H}^{KL}(\boldsymbol{X}, \Theta). \tag{7.65}$$

The constraint (7.64) implies that $\mathcal{H}^{IJ}$ is an even function, while (7.65) implies that $\mathcal{H}^{IJ}$ must be antisymmetric in the R-symmetry indices. Furthermore since $\mathcal{H}$ is Grassmann even it must be an even function of Θ, which implies it must also be even in X. It is not too difficult to see that it is impossible to construct any structures consistent with these requirements out of the available building blocks; hence, this correlation function must vanish.

ACKNOWLEDGMENTS

The authors thank Jessica Hutomo, Sergei Kuzenko and Michael Ponds for valuable discussions. The work of E. I. B. is supported in part by the Australian Research Council, Project No. DP200101944. The work of B. S. is supported by the Bruce and Betty Green Postgraduate Research Scholarship under the Australian Government Research Training Program.

APPENDIX: 3D CONVENTIONS AND NOTATION

For the Minkowski metric we use the "mostly plus" convention: $\eta_{mn} = \mathrm{diag}(-1, 1, 1)$. Spinor indices are then raised and lowered with the $\mathrm{SL}(2, \mathbb{R})$ invariant antisymmetric ε tensor

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta} = \delta_\alpha{}^\beta, \tag{A1}$$

$$\phi_\alpha = \varepsilon_{\alpha\beta}\phi^\beta, \qquad \phi^\alpha = \varepsilon^{\alpha\beta}\phi_\beta. \tag{A2}$$

The γ matrices are chosen to be real and are expressed in terms of the Pauli matrices σ as follows:

$$(\gamma_0)_\alpha{}^\beta = -\mathrm{i}\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad (\gamma_1)_\alpha{}^\beta = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A3a}$$

$$(\gamma_2)_\alpha{}^\beta = -\sigma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{A3b}$$

$$(\gamma_m)_{\alpha\beta} = \varepsilon_{\beta\delta}(\gamma_m)_\alpha{}^\delta, \qquad (\gamma_m)^{\alpha\beta} = \varepsilon^{\alpha\delta}(\gamma_m)_\delta{}^\beta. \tag{A4}$$

The γ matrices are traceless and symmetric:

$$(\gamma_m)^\alpha{}_\alpha = 0, \qquad (\gamma_m)_{\alpha\beta} = (\gamma_m)_{\beta\alpha}, \tag{A5}$$

and also satisfy the Clifford algebra

$$\gamma_m\gamma_n + \gamma_n\gamma_m = 2\eta_{mn}. \tag{A6}$$

Products of γ matrices are then

$$(\gamma_m)_\alpha{}^\rho(\gamma_n)_\rho{}^\beta = \eta_{mn}\delta_\alpha{}^\beta + \epsilon_{mnp}(\gamma^p)_\alpha{}^\beta, \tag{A7a}$$

$$(\gamma_m)_\alpha{}^\rho(\gamma_n)_\rho{}^\sigma(\gamma_p)_\sigma{}^\beta = \eta_{mn}(\gamma_p)_\alpha{}^\beta - \eta_{mp}(\gamma_n)_\alpha{}^\beta + \eta_{np}(\gamma_m)_\alpha{}^\beta + \epsilon_{mnp}\delta_\alpha{}^\beta, \tag{A7b}$$

where we have introduced the 3D Levi-Civita tensor ϵ, with $\epsilon^{012} = -\epsilon_{012} = 1$. It satisfies the following identities:

$$\epsilon_{mnp}\epsilon_{m'n'p'} = -\eta_{mm'}(\eta_{nn'}\eta_{pp'} - \eta_{np'}\eta_{pn'}) - (n' \leftrightarrow m') - (m' \leftrightarrow p'), \tag{A8a}$$

$$\epsilon_{mnp}\epsilon^m{}_{n'p'} = -\eta_{nn'}\eta_{pp'} + \eta_{np'}\eta_{pn'}, \tag{A8b}$$

$$\epsilon_{mnp}\epsilon^{mn}{}_{p'} = -2\eta_{pp'}, \tag{A8c}$$

$$\epsilon_{mnp}\epsilon^{mnp} = -6. \tag{A8d}$$

We also have the orthogonality and completeness relations for the γ matrices:

$$(\gamma^m)_{\alpha\beta}(\gamma_m)^{\rho\sigma} = -\delta_\alpha{}^\rho\delta_\beta{}^\sigma - \delta_\alpha{}^\sigma\delta_\beta{}^\rho, \qquad (\gamma_m)_{\alpha\beta}(\gamma_n)^{\alpha\beta} = -2\eta_{mn}. \tag{A9}$$

Finally, the γ matrices are used to swap from vector to spinor indices. For example, given some three-vector x_m, it may equivalently be expressed in terms of a symmetric second-rank spinor $x_{\alpha\beta}$ as follows:

$$x^{\alpha\beta} = (\gamma^m)^{\alpha\beta}x_m, \qquad x_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}x_{\alpha\beta}, \tag{A10}$$

$$\det(x_{\alpha\beta}) = \frac{1}{2}x^{\alpha\beta}x_{\alpha\beta} = -x^m x_m = -x^2. \tag{A11}$$

The same conventions are also adopted for the spacetime partial derivatives ∂_m:

$$\partial^{\alpha\beta} = \partial^m (\gamma_m)^{\alpha\beta}, \qquad \partial_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}\partial_{\alpha\beta}, \tag{A12}$$

$$\partial_m x^n = \delta_m^n, \qquad \partial_{\alpha\beta} x^{\rho\sigma} = -\delta_\alpha{}^\rho \delta_\beta{}^\sigma - \delta_\alpha{}^\sigma \delta_\beta{}^\rho, \tag{A13}$$

$$\xi^m \partial_m = -\frac{1}{2}\xi^{\alpha\beta}\partial_{\alpha\beta}. \tag{A14}$$

We also define the supersymmetry generators Q_α^I:

$$Q_\alpha^I = \mathrm{i}\frac{\partial}{\partial\theta_I^\alpha} + (\gamma^m)_{\alpha\beta}\theta^{I\beta}\frac{\partial}{\partial x^m} \tag{A15}$$

and the covariant spinor derivatives:

$$D_\alpha^I = \frac{\partial}{\partial\theta_I^\alpha} + \mathrm{i}(\gamma^m)_{\alpha\beta}\theta^{I\beta}\frac{\partial}{\partial x^m}, \tag{A16}$$

which anticommute with the supersymmetry generators, $\{Q_\alpha^I, D_\beta^J\} = 0$, and obey the standard anticommutation relations

$$\{D_\alpha^I, D_\beta^J\} = 2\mathrm{i}\delta^{IJ}(\gamma^m)_{\alpha\beta}\partial_m. \tag{A17}$$

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