I. REPRESENTATION THEORY OF FINITE AND INFINITE DIMENSIONAL GROUPS

Horospheres and Twistors. *

S.G.Gindikin
A.N.Belozersky Laboratory of Molecular Biology
and Bioorganic Chemistry,Building A
Moscow State University,
Moscow 119899, USSR.

The aim of the present lecture is to expound a new way of deriving the Plancherel formula for complex semisimple Lie groups, a way based enterely on the ideas of integral geometry. That approach can be immediately extended to all Pseudo-Riemannian symmetric spaces for which the problem of integral geometry can be solved.

The Plancherel formula for complex semisimple Lie groups was first obtained in 1950-1951 by Gelfand-Naimark and Harish-Chandra[1],[2]. Some years later a new explicit proof of it, which used a regularisation procedure for parameter dependent distributions, has been obtained in [3]. In 1959 Gelfand and Graev [4] have formulated a problem of integral geometry equivalent to the Plancherel formula: how can one reconstruct a function on the group from its integrals over horospheres, i.e. shifts of the maximal unipotent subgroup.

A natural plan was to include that problem into a more general class of problems of integral geometry replacing horospheres by other submanifolds and to develop general inversion procedures for such integral transformations.

In 1967 Gelfand, Graev and Shapiro [5] have found the general structure of the inversion formulas (the κ form) in the particular case when the integration was carried out over some family of p-dimensional planes in $\mathbf{C}^n$. That result made it possible [6] to obtain the Plancherel formula for the group SL(1,**C**) since its horospheres can be interpreted as $l(l-1)/2$ dimensional planes in $\mathbf{C}^{l^2-1}$. For other groups one has to be able to solve problems of integral geometry involving integration over *curved* submanifolds. Some fairly complete results in that direction have been obtained in the case of 1-dimensional curves [7],[8]. In the present lecture we give some results for curved submanifolds of higher dimensions sufficient for the derivation of the Plancherel formula.

Derivation of the Plancherel formula with the use of integral geometry is of a special interest since it clarifies geometrical structures responsible for the existence of the explicit Plancherel formula. There is no doubt that the existence of a group action is too heavy a price for that. We show that it is in fact connected with the existence of a very simple differential geometric structure on the space of horospheres. That structure can be naturally interpreted as a possible generalisation of structures appearing in the Penrose twistor theory [9]. This explains the title of this lecture. We have tried to formulate it in the most general way in order to stress our belief that the structures in question are very significant also outside the scope of integral geometry.

*Talk given at the XVI. International Colloquium on Group Theoretical Methods in Physics, Varna, Bulgaria, June 15-21, 1987

The two main goals of our investigation can be summarized in the following way:

(i) *obtain a sufficiently general formula for the problems of integral geometry that can be applied, in particular, to the case of semisimple complex Lie groups;*

(ii) *specify a geometric structure on the space of horospheres responsible for the possibility to apply that formula.*

Accordingly, the lecture splits into three main parts: derivation of a general inversion formula for the non-linear problems of integral geometry; verification of the applicability of that formulation to the case of complex semisimple Lie groups, and, finally, the description of the geometrical structure on the space of horospheres.

1 An introduction into integral geometry

1.1 The Gelfand-Graev-Shapiro operator in the planar problem of integral geometry [5].

Consider the manifold $H = H_{n,p}$ of p-dimensional planes in $\mathbf{C}^n_z$ with a fixed parametrisation. Denote by $\pi(\alpha,\beta)$, $\alpha = (\alpha^1,\cdots,\alpha^p)$, $\alpha^j,\beta \in \mathbf{C}^n$ the plane defined by the formula

$$z = \alpha t + \beta = \sum_{j=1}^{p} \alpha^j t_j + \beta,\ t = (t_1,\cdots,t_p) \in \mathbf{C}^p. \tag{1}$$

Here (α,β) are coordinates in H.

To each function $f(z) \in C_0^\infty(\mathbf{C}^n)$ we associate its integrals over $\pi(\alpha,\beta)$:

$$\hat{f}(\alpha,\beta) = \int_{\mathbf{C}^p} f(\alpha t+\beta)\,\mathrm{d}t \wedge \overline{\mathrm{d}t},\ \mathrm{d}t = \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_p. \tag{2}$$

Denote by H_z the submanifold consisting of planes $\pi(\alpha,\beta)$ going through the point z. Consider the operators κ_j acting on forms and increasing the degree of each form by 1:

$$\kappa_j = \frac{\partial}{\partial_\beta}\mathrm{d}\alpha^j = \sum_i \frac{\partial}{\partial_{\beta_i}}\mathrm{d}\alpha^j_i,\ \overline{\kappa_j} = \frac{\partial}{\partial\overline{\beta}}\overline{\mathrm{d}\alpha^j}. \tag{3}$$

Let

$$\kappa = \kappa_1 \wedge \cdots \wedge \kappa_p,\ \overline{\kappa} = \wedge\overline{\kappa_j}.$$

Proposition 1 *[5] The (p,p) form $(\kappa\wedge\overline{\kappa})\hat{f}\,|_{H_z}$ is closed for each $z \in \mathbf{C}^n$, and if γ is a 2p-dimensional (over $\mathbf{R}$) cycle in H_z then*

$$\int_\gamma (\kappa\wedge\overline{\kappa})\hat{f} = c(\gamma) f(z) \tag{4}$$

where $c(\gamma)$ does not depend on f.

The only statement for which a proof is really needed is that the form $(\kappa \wedge \overline{\kappa})\hat{f}\,|_{H_z}$ is closed and that is verified by a direct computation.

Note that if for a function φ on H the forms $(\kappa \wedge \overline{\kappa})\varphi\,|_{H_z}$ are closed for all z then there exists a function f such that $\varphi = \hat{f}$.

Let us now discuss some implications of formula (4) for integral geometry. Let K be a submanifold in H of (complex) dimension n and suppose that our task is to reconstruct $f(z)$ from $\hat{f}\,|_K$. Let $K_z = K \bigcap H$. One can assume that $\dim_{\mathbf{C}}(K_z) = p$ for allmost all z. If those K_z are cycles such that $c(K_z) \neq 0$ then one can reconstruct the function $f(z)$ using formula (4). However, that procedure involves a delicate point: information given only by $\hat{f}$ on K may turn out to be insufficient for the computation of $(\kappa \wedge \overline{\kappa})\hat{f}\,|_{H_z}$ (tangent derivations might be insufficient). It is only for special types of submanifolds K that the inversion problem can be solved in that way (such submanifolds are called *admissible*). However the space of horospheres in SL(1,C) turned out to be admissible and that made it possible to obtain in [6] the Plancherel formula for that group.

It is important to be able to compute the coefficient $c(\gamma)$. That can be accomplished using test functions but it may be advisible to take into account its geometrical meaning: $c(\gamma)/(2\pi)^{2p}$ is the number of planes $\pi \in \gamma$ contained in a general hyperplane.

1.2 Closed continuation of κ to curved manifolds

A crucial point in the preceding considerations involved the closeness of the form $(\kappa \wedge \overline{\kappa})\hat{f}$. We now wish to continue it, while preserving that property, to the manifold (infinite dimensional) of all p-dimensional submanifolds. Considering a local situation let Π denote the set of *all* smooth parametrized submanifolds in a neighborhood $U \subset \mathbf{C}^n_z$ of the origin:

$$z = \varphi(t),\ \varphi = (\varphi_1, \cdots, \varphi_n),\ z \in \mathbf{C}^n,\ t \in \mathbf{C}^p. \tag{5}$$

For each $f(z) \in \mathbf{C}^\infty_0(U)$ let

$$\hat{f}(\varphi) = \int f(\varphi(t))\, \mathrm{d}t \wedge \overline{\mathrm{d}t}. \tag{6}$$

Denote by Π_z the set of all φ for which $\varphi(0) = z$. One has $H \subset \Pi$: since $\varphi(t) = \alpha t + \beta$ for each $\pi(\alpha, \beta)$. To simplify the notation we conduct our computations for $z = 0$.

The elements of the tangent space $T_\varphi \Pi$ will be identified with the variations $\delta\varphi$. Consider a canonical decomposition of $T_\varphi \Pi_0$ of the form

$$T_\varphi \Pi_0 = T^{(1)} \oplus \cdots \oplus T^{(p)} \tag{7}$$

and denote by $\delta^{(j)}\varphi$ the component of $\delta\varphi$ in $T^{(j)}$.

The following conditions have to be satisfied:

$$(i)\quad \delta^{(j)}\varphi(t) \text{ depends only on } t_1, \cdots, t_j;$$
$$(ii)\quad \delta(j)\varphi\,|_{t_j=0} \equiv 0$$

Those conditions imply that

$$\delta^{(j)}\varphi(t) = \delta\varphi(t_1, \cdots, t_j, 0, \cdots, 0) - \delta\varphi(t_1, \cdots, t_{j-1}, 0, \cdots, 0),$$

and, consequently, descomposition (7) is defined uniquely.

Define the operator κ_j on functionals $F(\varphi)$ in Π taking them into 1-forms in Π_0;

$$\kappa_j F(\varphi;\delta\varphi) = \delta F(\varphi;\delta^{(j)}\varphi/t_j) \tag{8}$$

and continue it on forms. In other words one has to take the value of the variation δF of the functional F at the point φ on the variation $\delta^{(j)}\varphi/t_j$. By (ii) it is a regular variation, although, of course, it is not tangent to Π_0. In particular, for the functional $\hat{f}(\varphi)$ one has

$$\kappa_j\hat{f} = \int <\text{grad } f(\varphi(t)), \delta^j\varphi(t)> /t_j \ \ \mathrm{d}t \wedge \overline{\mathrm{d}t}.$$

Let $\kappa = \bigwedge \kappa_j$ and consider a (p,p)-form :

$$(\kappa\wedge\overline{\kappa})f = \int \sum_{I,J} \frac{\partial^{2p} f(\varphi(t))}{\partial z_{i_1}\cdots\partial z_{i_p}\partial\overline{z}_{j_1}\cdots\partial\overline{z}_{j_p}} \delta^{(1)}\varphi_{i_1}\wedge\cdots\delta^{(p)}\varphi_{i_p}\wedge\overline{\delta^{(1)}\varphi_{j_1}}\cdots\wedge\overline{\delta^{(p)}\varphi_{j_p}} \ \frac{\mathrm{d}t\wedge\overline{\mathrm{d}t}}{\Pi t_{i_k}\Pi\overline{t_{j_l}}}.$$

The following two main properties of the form $(\kappa\wedge\overline{\kappa})\hat{f}$ are verified directly:

(A) it is closed on Π_0

(B) it coincides with the form from section 1.1, on the submanifolds of planes $H_0 \subset \Pi_0$ (see (3)).

The form $(\kappa\wedge\overline{\kappa})F\,|_{\Pi_z}$ is defined in a similar way. These forms are closed if and only if there exist a function f such that $F = \hat{f}$.

Theorem 1 *For each 2p-dimensional (over* $\mathbf{R}$*) cycle* $\gamma \in \Pi_z$ *one has*

$$\int (\kappa\wedge\overline{\kappa})\hat{f} = c(\gamma)f(z). \tag{9}$$

This follows immediately from (A),(B) and Proposition 1.1.

For the computation of $c(\gamma)$ it is useful to take into account that $c(\gamma) = c(\tilde{\gamma})$ where $\tilde{\gamma}$ is a cycle of tangent planes to the submanifolds $\varphi \in \gamma$ at the point z (the cycles γ and $\tilde{\gamma}$ are homological).

Now all the statements of the preceding section about links with integral geometry can be formulated for the more general situation considered here: one can solve the problem of integral geometry for the manifolds $K \subset \Pi$, $\dim_{\mathbf{C}} K = n$ using formula (9) provided the forms $(\kappa\wedge\overline{\kappa})F\,|_{K_z}$ are defined by $F\,|_K$ for almost all z, K_z are cycles and $c(K_z) \neq 0$. We shall presently see that this condition is satisfied by spaces of horospheres in complex semisimple Lie groups.

2 Solution of the integral geometry problem for complex semisimple Lie groups

2.1 Notation

Let G be such a group, dim G = n; denote by $\mathcal{G}$ its Lie algebra, and by H and $\mathcal{H}$ its Cartan subgroup and Cartan subalgebra respectively, dim $\mathcal{H} = \ell$. Let $\{\alpha\}$ be a system of positive roots numbered in such a way that

$$\textit{if} \ \ \alpha_i + \alpha_j = \alpha_k \ \ \textit{then } k > i \textit{ and } k > j. \tag{10}$$

The existence of such a numbering (which is not unique) is easily proved [10]. Denote by $e_{\pm j}$, $j > 0$, the root vectors corresponding to $\pm\alpha_j$, respectively, and let

$$f_i = [e_{-i}, e_i],\ [f_i, e_j] = < f_i, f_j > e_j,$$

where $<,>$ denotes the Cartan scalar product. For the basis in $\mathcal{G}$ we take all $\{e_{\pm j}\}$ and f_j for simple roots. Consider the exponential coordinate system in G (we are treating a local situation in a neighborhood of unity $e \in G$).

Let $\mathcal{Z}$ be a subalgebra generated by the vectors e_j, $j > 0, \dim \mathcal{Z} = (n-1)/2 \stackrel{\rm def}{=} p$. Denote by Z the corresponding maximal unipotent group: $Z = \exp(\sum_{j>0} t_j e_j)$ and by Z_- the subgroup corresponding to negative roots. Each horosphere in G is of the form $Z(g_1, g_2) = g_1 Z g_2$. One obtains almost all horosphere taking $g_1 = \zeta_1 h$, $g_2 = \zeta_2$ for some $\zeta_1, \zeta_2 \in Z_-, h \in H$. Accordingly $\{\zeta_1, \zeta_2, h\}$ define a coordinate system on the dense chart on the space of horospheres Ξ, $\dim \Xi = n$. Almost all horospheres going through e are of the form $Z(\zeta^{-1}, \zeta)$, $\zeta \in Z_-$, $\dim \Xi_e = \dim Z = p$ where Ξ_e denotes the set of all horospheres going through e.

For each function $f \in C_0(G)$ consider its integrals over horospheres

$$\hat{f}(\zeta_1, \zeta_2, h) = j(h) \int f(\zeta_1 h \exp\,(\sum_{j>0} t_j e_j)\zeta_2)\, \mathrm{d}t \wedge \overline{\mathrm{d}t}, \tag{11}$$

where

$$j(\exp\, f) = \exp\,(\frac{1}{2} < f, \sum_{i>0} f_i >),\ f \in \mathcal{H}.$$

2.2 Restriction of the form $(\kappa \wedge \overline{\kappa})\hat{f}$ to the space of horospheres

The homogeneity considerations imply that it is sufficient to carry out our computations only for the unity element $e \in G$, i.e. to compute $(\kappa \wedge \overline{\kappa})\hat{f}\,|_{\Xi_e}$. In view of the parametrisation chosen on Ξ_e it is a form on Z_-. The homogeneity considerations again imply that it is sufficient to carry out the computations only for $\zeta = e$ on Z_-.

Thus, one has to study a variation δZ of an horosphere Z in Ξ_e. Consider the exponential coordinate system on Z_-: $\zeta = \exp\,(\sum_{i>0} s_i e_{-i})$. Then

$$\delta Z(t) = \mathrm{d}_s \ln[\zeta^{-1} \exp\,(\sum t_j e_j)\zeta],$$

where the differential with respect to s is taken for $s = 0$. Evidently, $\mathrm{d}Z$ takes values in the Lie algebra $\mathcal{G}$. Using the formula $\exp(-\varepsilon Y)\exp(X)\exp(\varepsilon Y) = \exp(X + \varepsilon[X, Y] + o(\varepsilon))$, one has

$$\delta Z = [\sum_{j>0} t_j e_j, \sum_{l>0} \mathrm{d}s_l e_{-l}];$$

which for the canonical decomposition components yields

$$\delta^{(j)} Z = t_j [e_j, \sum_{l>0} \mathrm{d}s_l e_{-l}]; \tag{12}$$

Thus $\delta^{(j)} Z / t_j$ is a regular variation, and in our case it does not depend on t. Those variations are not tangent to the manifold of horospheres. However, as we shall presently see, the form $(\kappa \wedge \overline{\kappa})\hat{f}$ can be computed. Let us study the variation $\delta^{(j)} Z / t_j$ in more detail:

$$\delta^{(j)} Z / t_j = -\mathrm{d}s_j f_j + \delta_1^{(j)} Z + \delta_2^{(j)} Z,$$

where

(i) $\delta_1^{(j)}Z$ is expressed in terms of $\mathrm{d}s_k$, $k > j$;

(ii) $\delta_2^{(j)}Z$ takes values in the Lie algebra $\mathcal{Z}$ of the group Z and can be expressed as a linear combination of e_m, $m < j$.

We obtain the first summand by taking $l = j$ in (12). In $\delta_1^{(j)}Z$ we group together the summands with $l > j$ and in $\delta_2^{(j)}Z$ the summands with $l < j$. One has to verify (ii).

Consider$[e_j, e_{-j}]$. One has to check that if $\alpha_j - \alpha_l$ is a root then it is a positive root α_m. Indeed, let $\alpha_j - \alpha_l = -\alpha_k$. Then condition (10) implies that $j > k$ contradicting the above inequality. If $\alpha_j - \alpha_l = \alpha_m$ then using (10) once again one has $m < j$.

Let us now compute $\kappa\hat{f}\mid_{\Xi_e}$ for $\zeta = e$. Recall that in order to compute κ_j one has to evaluate the variation on $\delta^{(j)}Z/t_j$. Note first of all that the terms $\delta_2^{(j)}$ vanish as, according to (ii), they correspond to shifts along the horosphere Z itself, and those shifts preserve the volume element $\mathrm{d}t$ and, consequently, the integral $\hat{f}$.

By induction in j starting with $j = p$ and working in the direction of lower values we now prove that $\delta_1^{(j)}Z$ also vanish. For $j = p = (n-1)/2$ one has: $\delta_1^{(j)}Z = 0$. Now we verify by induction that $\bigwedge_{j>k}\kappa_j f$ is of the form $c\bigwedge_{j>k}\mathrm{d}s_j$. Indeed, after addition of κ_k, the forms $\delta_1^{(j)}Z$ vanish because, in view of (i), they are linear combinations of $\mathrm{d}s_m$, $m > j$. Therefore $\bigwedge_{j\geq k}\kappa_j f$ is of the form $c\bigwedge_{j>k}\mathrm{d}s_j$. Thus, for each j only the value of the variation on $-\mathrm{d}s_j f_j$ is essential, and that is a tangent variation to Ξ corresponding to shifts by the elements of the Cartan subgroup H. Denoting by D_j the derivation in the direction f_j the result may be formulated in the following way:

$$\kappa\hat{f}(e,e,e) = (-1)^p\Pi_{j>0}D_j f(e,e,h)\mid_{h=e}\wedge\mathrm{d}s_i.$$

One has to take into account that

$$\hat{f}(e,e,\exp(\varepsilon f)) = \int f(\exp(\varepsilon f + \sum t_j e_j))\mathrm{d}t\wedge\overline{\mathrm{d}t} + o(\varepsilon).$$

It is to ensure conformity with that relation that the normalisation factor $j(h)$ has been introduced into definition (11). The operator $\overline{\kappa}$ is of a similar form and, by homogeneity, the formulas are valid for other points $\zeta\in Z$.

Thus

$$(\kappa\wedge\overline{\kappa})\hat{f}(\zeta^{-1},\zeta,e) = \Pi_{j>0}D_j\overline{D_j}f(\zeta^{-1},\zeta,h)\mid_{h=e}\mathrm{d}s_j\wedge\mathrm{d}s_j.$$

It remains to consider the integral $\int_{\Xi_e}(\kappa\wedge\overline{\kappa})\hat{f}$ and to compute $c(\Xi_e)$.

For the cycle Ξ_e the cycle of tangent planes $\widetilde{\Xi_e}$ consists of those subspaces in $\mathcal{G}$ that are of the form $\mathrm{T}_g\mathcal{Z}$, $g\in G$, where T_g is the adjoint representation of G in $\mathcal{G}$. It is sufficient to take $g\in U$, where U is the maximal compact subgroup, and taking $g\in Z_-$ one gets almost all of the planes. Let π_u be a hyperplane in $\mathcal{G}$ of the form $< u, x > = 0$, $u\in\mathcal{G}$. It is sufficient to consider the case $u\in\mathcal{H}$ so that general hyperplanes corresponds to regular elements of $\mathcal{H}$. In that case π_u contains $\mid W\mid$ planes from $\widetilde{\Xi_e}$ where W is the order of the Weyl group of algebra corresponding to different orderings of roots on $\mathcal{H}$. That can be easily deduced, e.g., from the Bruhat decomposition. Therefore $c(\Xi_e) = (2\pi)^{n-1}\mid W\mid$ and one has :

$$f(e) = \frac{1}{(2\pi)^{n-1}\mid W\mid}\int_{Z_-}\Pi_{j>0}D_j\overline{D_j}\hat{f}(\zeta^{-1},\zeta,h)\mid_{h=e}\bigwedge_{j>0}(\mathrm{d}s_j\wedge\mathrm{d}s_j). \qquad (13)$$

3 Infinitesimal structure on the space of horospheres Ξ

Finally we present without proof the geometrical structure on the horosphere manifold which is sufficient for the existence of the inversion formula based on the form $(\kappa \wedge \overline{\kappa})\hat{f}$.

Let us now study the structure of incidence relations between horospheres of the type of projective duality. Consider the horosphere manifold Ξ, $\dim \Xi = n$. To the group elements $g \in G$ there correspond on Ξ submanifolds of horospheres Ξ_g going through g, $\dim \Xi_g = p = (n-1)/2$. Let $\xi \in \Xi$ (fixing an horosphere $Z(\xi)$ on G) and consider submanifolds $\Xi_g \ni \xi$ (*i.e.* $g \in Z(\xi)$) and tangent planes $\mathcal{O}_g \in T_\xi\Xi$ to Ξ_g at the point ξ. Thus there arises a p-parameter family of p-planes in the tangent space $T_\xi\Xi$.

That configuration of planes turns out to have a remarkable property which bears the main responsibility for the existence of the inversion formula. Let us call the set of p-planes that lie in a fixed (p+1)-plane and contain a fixed (p-1)-plane λ a p-bunch (σ, λ) and the (p-1)-plane σ the axis of the p-bunch (σ, λ). Then the family of planes σ_g splits into a (p-1)-parameter family of p-bunches; the axes of those bunches $\{\sigma^{(p-1)}\}$, in their turn, split into (p-2)-parameter family of (p-1)-bunches etc. At the last step one obtains a one-parameter family of straight lines (the axes of 2-bunches of the previous step) that lie in the same 2-plane and go through the origin. That stratification corresponds to the enumeration roots (10) and is, accordingly, non-unique. Each step involves a finite number of possibilities to choose a descomposition into bunches. Axes of bunches may be related to degenerate horospheres.

If for a family Ξ of p-dimensional submanifolds on the n-dimensional manifold G the above inductive decomposition into bunches of tangent planes holds for each point $\xi \in \Xi$, then we shall say that Ξ satisfies the **(H)**-*condition* , or *horospheric condition*. For the problem of integral geometry to be solvable one more condition has to be satisfied.

We shall say that Ξ satisfies the infinitesimal Desargues condition if in a neighborhood of each point there exists a diffeomorphism "flattening" Ξ_g up to the 3-d order. That condition can be expressed analytically.

On each family Ξ of submanifolds $Z_\xi \subset G$ satisfying both the **(H)**-condition and the infinitesimal Desargues condition the form $(\kappa \wedge \overline{\kappa})\hat{f}$ induces an inversion formula.

The **(H)**-condition can be reformulated in the "G-representation": there is a canonical rational-triangular structure on submanifolds Z_ξ (in particular, on horospheres), viz., there is a mapping of Z_ξ on $\mathbf{CP}^1$, then a mapping of the inverse image of points again on $\mathbf{CP}^1$, etc. It is a geometrical expression of the root structure and undoubtedly deserves a detailed study. In particular, it is interesting to consider how that approach is related to the integration theory of those non-linear equations that can be represented as compatibility conditions for systems of linear equations with several spectral parameters.

References

[1] Gelfand I.M., Naimark M.A., Unitary Representations of classical Groups, Trudy Matem. Instituta AN SSSR, **36** (1950)(in Russian).

[2] Harish-Chandra. Plancherel Formula for Complex Semisimple Lie Groups, Proc. Nat. Acad. Sci. USA, **37**, 12 (1951), 813.

[3] Gelfand I.M., Graev M.I., An Analogue of the Plancherel Formula for Classical Groups, Trudy Moskovskogo Matem. Ob-va,**4** (1955), 375-404 (in Russian)

[4] Gelfand I.M., Graev M.I., Geometry of Homogeneous Spaces, Representations of groups in Homogeneous Spaces and related Problems of Integral Geometry, Trudy Moskovskogo Matem. Ob-va, **8** (1959), 321-390 (in Russian)

[5] Gelfand I.M., Graev M.I., Shapiro Z.Ya Integral Geometry on k-planes, Functional Analysis and Its Applications, **1**, 1 (1967), 15-31, (in Russian)

[6] Gelfand I.M., Graev M.I., Complexes of k-Planes in $\mathbf{C}^n$ and the Plancherel Formula for GL(n,**C**), Doklady AN SSSR, **3** (1968), 522-525 (in Russian)

[7] Gelfand I.M., Gindikin S.G., Shapiro Z.Ya.,Local Problem of Integral Geometry in the Space of Curves, Functional Analysis and Its Applications, **13**, 2 (1979), 11-31 (in Russian)

[8] Gindikin S.G.,Reductions of Manifolds of Rational Curves and Related Problems of the Theory of Differential Equations, Functional Analysis and its Applications, **18**,4 (1984), 14-39 (in Russian).

[9] Penrose R., Non-linear Gravitons and Curved Twistor Theory, Gen. Rel. Grav., **7** (1976), 31-52

[10] Gindikin S.G., Karpelevich F.I., Plancherel Measure for Riemannian Symmetric Spaces of Non-positive Curvature, Doklady AN SSSR, **145**, 2 (1962), 252-255 (in Russian).