Zimmermann's subtraction scheme and the perturbative solution to R.G. evolution equations

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Abstract

In the framework of Euclidean field theory we show that an infrared safe slightly modified version of Zimmermann's subtraction scheme generates the perturbative solutions to the Wilson-Polchinski renormalization group equations.

1 Introduction

On the occasion of Wolfhart Zimmermann's 80th birthday I think that a short look at the present status of Quantum Field Theory is certainly timely. I would like in particular to give an example of the persisting fundamental role of many Zimmermann's contributions in the development of Quantum Field Theory.

No doubt quantum field theory is one of the major achievements of twenty's century physics [1]. Even if no interacting four dimensional model has yet been solved, an axiomatic framework leading to a well defined scattering theory is now clearly defined and different constructive approaches have been set up for a class of models. Lehmann-Symanzik-Zimmermann construction of scattering amplitudes has been and remains a basic step in the construction of a complete theory. Among the constructive methods the most important are loop ordered perturbative renormalization [2] and Wilson's renormalization group (R.G.) [3]. I think that a short comparison of the use of these methods in the framework of perturbation theory is timely.

Loop ordered perturbative renormalization is the natural development of QED and has produced exceptionally successful phenomenological analyses in the framework of the Standard Model of Electro-Weak and Strong Interactions. Forgetting the problems related to infra-red divergences the construction of scattering amplitudes and operator matrix elements is based on the Feynman expansion with suitable subtraction prescriptions of the ultra-violet divergences. A systematic solution to the ultra-violet problem first described by Bogoliubov and Pasiuk, has found, after Hepp corrections [4], a clear and handy form in Zimmermann's scheme subsequently extended in collaboration with Lowenstein to the massless case [2], and by Breitenlhoner and Maison to dimensional regularization [5]. The availabilty of this approach has led to many achievements such as a rigorous renormalized construction of gauge theories, systematic construction of renormalized operators, a clear and rigorous study of short distance physics.

Wilson's renormalization group was introduced as an alternative approach to Quantum Field Theory based on a systematic analysis of the scale transformation properties of Green functions. The natural framework is Euclidean field theory which can be related to a corresponding Minkowskian theory on the basis of Osterwalder-Schrader axioms [1]. The main goal consists in the construction of the Feynman-Kac functional integral. The most relevant application is the construction of gauge theories regularized on a lattice. The main purpose was and still is a non-perturbative construction of QCD and, in particular, the proof of confinement. On a lattice a scale transformation corresponds to the repeated replacement of the local fields with their averages over lattice cells. One studies the behavior of Feynman-Kac integral under these repeated substitutions. In the case of a theory built over a continuous manifold the analysis of scale transformation on the Feynman-Kac functional measure leads to a differential evolution equation for the measure.

In principle these evolution equations apply to the exact functional measure and do not rely on any Feynman graph expansion, however, until now, direct application of Wilson's approach to the construction of field theories beyond perturbation theory have been limited to special, however important, classes of models among which the most successful have been those involving only fermionic, and hence nilpotent, field variables. The construction of the Gross-Neveu model is the best known example[6]. In the general case one has to deal with an infinite sequence of equations that, in the case of bosonic variables, have no natural truncation. In some situation it is possible to justify the assumption of a measure remaining local after scale transformations [7]; this opens a further way toward non-perturbative results. However quite often the infinite sequence of evolution equations is truncated in a completely arbitrary way, often mimicking results that traditionally were obtained from naively simplified and truncated versions of the Schwinger-Dyson equations. The exact renormalized version of the Schwinger-Dyson equations has been studied in the early sixties by Symanzik and by Wu; the case of a scalar theory in four dimensions has been discussed by Johnson [8]. The analogy of this technique with Wilson's method should be better understood.

The application of the evolution equations to the construction of renormalized perturbation theory described by Polchinski in his thesis attracted new attention on Wilson's construction[9][10]. The essential reason for this interest lies in the major simplicity of the approach which is not directly based on a diagrammatic expansion. That is: the perturbative expansion of the functional measure leads to a series of terms each of which corresponds to a set of diagrams. Thus, even in a perturbative approach, the evolution equations deal with sets of diagrams, instead of dealing with single diagrams as the subtraction method does. Furthermore the differential nature of the evolution equations overcomes the problem of overlapping divergences. This, as shown by Hepp [4], is the most difficult part of the Bogoliubov's renormalization project. In the renormalization group approach the overlapping divergences are disentangled by the cut-off derivative appearing in the evolution equation. This is just a pedagogical advantage, since one does not need anymore to have recourse to forests, however one should not underestimate a pedagogical advantage in a moment in which field theory is loosing part of the original interest being often presented as a special limit of a more general string "theory". On the other hand one should not consider Wilson-Polchinski method as an alternative computational method of renormalized amplitudes. Indeed the purpose of the short note is to prove that the pertubative solution to the evolution equations leads to a Zimmermann subtracted Euclidean field theory.

Taking into account the limits of this note we shall try to give a general idea of the reasons for this equivalence avoiding the formal aspects of a rigorous proof [11].

2 The Renormalization Group evolution equations

With the aim described in the introduction we shall limit our discussion to the most simple situation considering an Euclidean scalar field theory in 4 dimensions.

Wilson's functional measure corresponds to an *Effective Interaction* which, when expanded into Feynman diagrams, is identified with the functional generator of connected amputated amplitudes built with the bare interaction and a doubly cut-off propagator, that is, with a propagator carrying an ultra-violet cut-off Λ_0 and an infra-red one Λ.

Wilson's equations describe the evolution of the measure with respect to Λ. The crucial part of the analysis consists in the proof that the Effective Interaction has a regular $\Lambda_0 \to \infty$, fixed Λ, limit. The final goal should be the study of the infra-red limit, i.e. $\Lambda \to 0$, which leads back to the renormalized (Schwinger) functions. However, fixing our mind on the ultra-violet problem, we limit our discussion to a pre-infra-red situation in which the infra-red cut-off Λ

does not vanish. In this situation, if we restrict our discussion to perturbation theory, the role of the mass turns out to be of limited interest. On the other hand, inserting a mass into the propagator in perturbation theory, the $\Lambda \to 0$ limit becomes trivial. [1] Thus we do not pay particular attention to the $\Lambda \to 0$ limit and hence to the difference between Wilson's Effective Interaction and the generator of connected Green functions. This difference becomes relevant whenever there are infrared problems that we do not want to face.

Therefore we introduce the ultra-violet-infra-red cut-off Fourier transformed propagator:

$$\tilde{\hat{S}}(p) = \frac{e^{-\frac{p^2}{\Lambda_0^2}} - e^{-\frac{p^2}{\Lambda^2}}}{p^2} \tag{1}$$

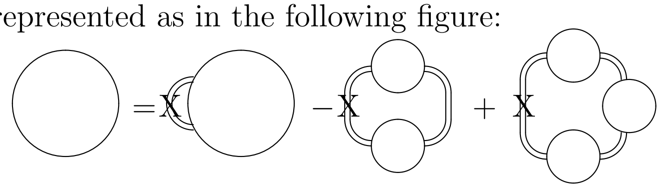
and we define:

$$\Lambda^2 \frac{\partial}{\partial \Lambda^2} \tilde{\hat{S}}(p) \equiv \dot{\tilde{\hat{S}}}(p) = -\frac{e^{-\frac{p^2}{\Lambda^2}}}{\Lambda^2} \ . \tag{2}$$

Even if the best known version of the renormalization group evolution equation describes the Λ dependence of the Effective Interaction, for renormalization purposes it is convenient to consider the evolution equation of the Legendre transform of the Effective Interaction which is identified with the functional generator of the one-particle irreducible (1-P.I.) diagrams built with the bare interaction and the above propagator [12][11]. We call this new functional *1-P.I. Effective Action* and we label it with V_{Λ,Λ_0}.

The evolution equation of the 1-P.I. Effective Action can be easily deduced noticing that the Λ-derivative of each term of its expansion in Feynman diagrams only acts on propagators. If one selects and cuts a line into an one-particle irreducible diagram, what remains is an amputated connected diagram consisting of a chain of 1-P.I. parts linked by single lines. Therefore the evolution equation can be represented as in the following figure:

$\Lambda\partial_\Lambda V_{\Lambda,\Lambda_0} \equiv \Lambda\partial_\Lambda$



$$+ \cdots \equiv R_{\Lambda,\Lambda_0} \tag{3}$$

where double lines correspond to the propagator $\hat{S}$ and the crossed double one to $\dot{\hat{S}}$ while circles correspond to the 1-P.I. parts generated by V_{Λ,Λ_0}. The same equation in functional form appears as:

$$\Lambda^2 \frac{\partial}{\partial \Lambda^2} V_{\Lambda,\Lambda_0}[\phi] \equiv \dot{V}_{\Lambda,\Lambda_0}[\phi] = \frac{1}{2} Tr \left(\dot{\hat{S}} \frac{\delta^2 V_{\Lambda,\Lambda_0}}{\delta \phi^2} \sum_{n=0}^{\infty} (- * \hat{S} * \frac{\delta^2 V_{\Lambda,\Lambda_0}}{\delta \phi^2})^n \right)$$
$$\equiv \frac{1}{2} R_{\Lambda,\Lambda_0}[\phi] \ . \tag{4}$$

[1]If however one tries to have a look beyond perturbation theory one immediately encounters well known naturalness problems concerning the masses of scalars.

In the right-hand side of this equation $\dot{\hat{S}}$, $\hat{S}$ and $\delta^2 V_{\Lambda,\Lambda_0}/\delta\phi^2$ are multiplied as matrices and the traces of products are taken.

We translate this equation into a system of ordinary differential equations expanding:

$$V_{\Lambda,\Lambda_0}[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} (dp_i \tilde{\phi}(p_i)) \delta(\sum_{j=1}^{n} p_j) V_n(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_0)$$

and introducing an analogous expansion for $R_{\Lambda,\Lambda_0}[\phi]$. Notice that the coefficients $R_n(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_0)$ of the field expansion of $R_{\Lambda,\Lambda_0}[\phi]$ are sums of series of terms corresponding to increasing numbers of 1-P.I. parts. Indeed this is apparent from Fig.(3). However, if we consider loop expanded quantities, the contribution of loop order ν to $R_n(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_0)$ appears as a finite sum of terms built with the contribution of lower loop order of the coefficients $V_{n'}$ with $n' \leq n+2$. Thus, if V_2 vanishes at zero loop order, one never encounters infinite series[2].

Next step consists in translating this infinite system of differential equations into a corresponding system of integral equations accounting for the initial conditions of the evolution equation. In order to do this we need consistent bounds on the coefficients V_n and R_n. Using Eq.(4) it is not difficult to show [11] that, if up to loop order ν and uniformly in Λ_0, one has

$$\sup_p |\partial_p^k \; V_n(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_0)| \leq \Lambda^{4-n-k} P_{n,k,\nu}\left(\log\left(\Lambda\right)\right)$$

a completely analogous bound holds true for $\sup_p |\partial_p^k \; R_n(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_0)|$. Then the system of integral equations:

$$\begin{aligned}
V_2(0,0,\Lambda,\Lambda_0) &= \mu^2 + \int_{\Lambda_R}^{\Lambda} \frac{d\lambda}{\lambda} R_2(0,0,\lambda,\Lambda_0) \\
\partial_{p^2} V_2(p,-p,\Lambda,\Lambda_0)|_{p=0} &= \zeta^2 + \int_{\Lambda_R}^{\Lambda} \frac{d\lambda}{\lambda} \partial_{p^2} R_2(p,-p,\lambda,\Lambda_0)|_{p=0} \\
V_4(0,\cdot\cdot,0,\Lambda,\Lambda_0) &= g + \int_{\Lambda_R}^{\Lambda} \frac{d\lambda}{\lambda} R_4(0,\cdot\cdot,0,\lambda,\Lambda_0) \; ,
\end{aligned}$$

and, for $n+k > 4$,

$$\partial_p^k \; V_n(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_0) = \int_{\Lambda_0}^{\Lambda} \frac{d\lambda}{\lambda} \; \partial_p^k \; R_n(p_1, \cdot\cdot, p_n, \lambda, \Lambda_0)$$

solves the evolution equations generating higher loop order terms in V_n satisfying analogous bounds. Now these bounds turn our to involve polynomials in $\log(\Lambda/\Lambda_R)$.

Furthermore both V_n and R_n have regular $\Lambda_0 \to \infty$ limits.

In this way one proves that the evolution equations produce a formally loop expanded 1-P.I. Wilson's Effective Action $V_R[\phi, \Lambda, \Lambda_R]$ which is defined as $\lim_{\Lambda_0 \to \infty} V_{\Lambda,\Lambda_0}[\phi]$ and whose field expansion coefficients satisfy the system of integral equations:

$$\begin{aligned}
V_{R,2}(0,0,\Lambda,\Lambda_R) &= \mu^2 + \int_{\Lambda_R}^{\Lambda} \frac{d\lambda}{\lambda} R_{R,2}(0,0,\lambda,\Lambda_R) \\
\partial_{p^2} V_{R,2}(p,-p,\Lambda,\Lambda_R)|_{p=0} &= \zeta^2 + \int_{\Lambda_R}^{\Lambda} \frac{d\lambda}{\lambda} \partial_{p^2} R_{R,2}(p,-p,\lambda,\Lambda_R)|_{p=0} \\
V_{R,4}(0,\cdot\cdot,0,\Lambda,\Lambda_R) &= g + \int_{\Lambda_R}^{\Lambda} \frac{d\lambda}{\lambda} R_{R,4}(0,\cdot\cdot,0,\lambda,\Lambda_R) \; , \qquad (5)
\end{aligned}$$

[2] A mass term at zero loop order should be inserted into the propagator Eq. (1).

and, for $n + k > 4$,

$$\partial_p^k \, V_{R,n}(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_R) = \int_\infty^\Lambda \frac{d\lambda}{\lambda} \, \partial_p^k \, R_{R,n}(p_1, \cdot\cdot, p_n, \lambda, \Lambda_R) \tag{6}$$

where $R_{R,n}(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_R) = \lim_{\Lambda_0 \to \infty} R_n(p_1, \cdot\cdot, p_n, \Lambda, \Lambda_0)$. It is apparent that the renormalized 1-P.I. Effective Action satisfies a differential evolution equation which is straightforwardly obtained from Fig.(3) and Eq.(4) replacing V_{Λ,Λ_0} with V_R and the propagator in Eq.(1) with: $(1 - \exp(p^2/\Lambda^2))/p^2$.

Now we can specify the purpose of this note as follows: we want to show that the contribution of every single diagram to the solutions to the integral equations (5) and (6) and hence to the renormalized version of Fig.(3) and Eq.(4) corresponds to a suitably subtracted version of the Feynman amplitude associated with the diagram.

Notice that our set of integral equations ((5) and (6)) can be extended to the 1-P.I. Effective Action in the presence of local composite operators. Formally to every operator one couples an independent external field, whose dimension is obviously related to that of the operator. The evolution equations for the coefficients of the field-external-field expansion of the 1-P.I. Effective Action can be translated into integral equations accounting for initial conditions strictly analogous to Eq.s(5) and (6). It turns out [11] that the resulting renormalized composite operators directly correspond the Zimmermann's $N_\delta[P(\phi)]$ renowned operators.

3 Comparison with Zimmermann's subtraction approach

Here we come to the main goal of this note showing that in the $\Lambda_0 \to \infty$ limit an alternative construction of the iterative, loop expanded, solutions to the R.G. integral equations is given by an Euclidean variant of Zimmermann's (Lowenstein-Zimmermann) subtraction method. It is worth noticing that in many important instances the evolution equations are constrained by invariance conditions for the measure. The most frequently met are the Slavnov-Taylor-Ward identities. These conditions constrain the choice of the initial parameters. There are situation in which the constraints have no solution and hence one finds *anomalies*, the typical case is that of naive scale invariance. The analysis of invariance conditions is a crucial step of renormalization theory, we do not discuss it here since it is shown in the existing literature that this analysis follows the same lines in Wilson-Polchinski and subtraction approaches [11][10].

In our simplified example the unsubtracted, and hence possibly divergent, Feynman integral corresponding to the diagram Γ contributing to the Schwinger function $S_n^{(m)}$ with an even number, n, of external legs and m loops, has the form:

$$S_\Gamma(p) = \int \frac{d^{4m}k}{(2\pi)^{4m}} I_\Gamma(p, k) \ ,$$

where $k \equiv k_1, ..., k_m$ is a basis of internal momenta of the diagram and $p \equiv p_1, ..., p_{n-1}$ a basis of external momenta. $I_\Gamma(p, k)$ is built with the propagator:

$$\tilde{\hat{S}}(p) = \frac{1 - e^{-\frac{p^2}{\Lambda^2}}}{p^2} \tag{7}$$

and vertices

$$(\mu^2\phi^2 + \zeta^2(\partial\phi)^2)/2 \quad , \quad g\phi^4/4! \tag{8}$$

The subtraction procedure consists in replacing $I_\Gamma(p,k)$ with the renowned *forest formula*:

$$R_\Gamma(p,k) \equiv \mathcal{S}_\Gamma \sum_{F\in\mathcal{F}_\Gamma} \prod_{\gamma\in F} (-t^d_\gamma \mathcal{S}_\gamma) I_\Gamma(p.k) \ . \tag{9}$$

where:

- $\mathcal{F}_\Gamma$ is the set of all forests of Γ
- $\mathcal{S}_\gamma$ defines the *momentum routing* in the sub-diagram γ
- t^d_γ takes the $\hat{p}^{(\gamma)}$ Taylor expansion of $I_\gamma(p,k)$ up to degree d_γ, the superficial divergence of γ,
- t^d_γ **replaces** Λ **with** Λ_R **in the propagators**

Notice the analogy with Lowenstein-Zimmermann's [2] infrared subtraction scheme where an auxiliary parameter s is introduced, analogous to our Λ, and the ultra-violet subtraction is made at $s = 0$, in our example at $\Lambda = \Lambda_R$. We do not perform any infra-red subtraction that we should apply if we were interested in the $\Lambda \to 0$ mass-less limit.

Let us call $\mathcal{V}_\Lambda[\phi]$ the functional generator of the subtracted 1-P.I. Feynman amplitudes. The coefficient function $\mathcal{V}_n(p,\Lambda)$ of its field expansion appears as loop ordered formal series whose term of order ν is the sum of all the n-legs, ν-loops, subtracted 1-P.I. diagrams. We have to show that these coefficient functions satisfy the system of integral evolution equations (5) and (6).

The basic point that we have to show is that the Λ-derivative commutes with the subtraction operator as a consequence of the Λ-independence of the subtraction point. We start our analysis studying the Λ-derivative of a subtracted graph.

In order to do this let us take the Λ-derivative of a generic subtracted Feynman integral corresponding to a 1-P.I. diagram and hence contributing to $\mathcal{V}_\Lambda$. Due to the absolute convergence of the momentum integral we are allowed to commute this derivative with the internal momentum integration and hence we come to the k-momentum integral of $\partial_\Lambda R_\Gamma(p,k)$. As already done we notice that an un-subtracted Feynman integrand depends on Λ only through the propagators $\hat{S}$ and that the sub-diagram subtraction terms generated by the Taylor operators $t^{d_\gamma}_\gamma$ are Λ-independent since they are computed at $\Lambda = \Lambda_R$. Thus, in order to compute the Λ-derivative, we have single out in Eq.(9) the contributions of the propagators of un-subtracted sub-diagrams.

For a generic 1-P.I. diagram Γ we define:

$$R_\Gamma(p,k) = (1 - t^{d_\Gamma}_\Gamma)\hat{R}_\Gamma(p,k) \tag{10}$$

where

$$\hat{R}_\Gamma(p,k) = \mathcal{S}_\Gamma \sum_{F\in\mathcal{F}'_\Gamma} \prod_{\gamma\in F} (-t^d_\gamma \mathcal{S}_\gamma) I_\Gamma(p.k) \tag{11}$$

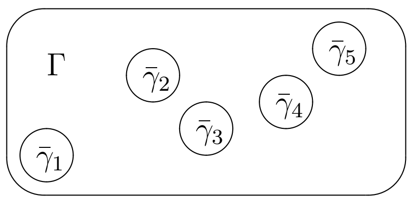
and $\mathcal{F}'_\Gamma$ is the set of forests non containing Γ as an element. In other words, computing $\hat{R}_\Gamma(p,k)$ we exclude the subtraction of the whole diagram. The reason for this definition lies in the equation:

$$\partial_\Lambda R_\Gamma(p,k) = \partial_\Lambda \hat{R}_\Gamma(p,k) \ , \tag{12}$$

which means that, computing the Λ-derivative, one restricts the sum over the forests in Eq.(9) to $\mathcal{F}'_\Gamma$.

Now some more diagrammatic analysis is needed. For every forest F in $\mathcal{F}'_\Gamma$, we say that $\bar{\gamma} \in F$ is a maximal element of F if it is not contained into other elements of F. Then we call $\bar{F}$, *maximal sub-forest* of F, the set of maximal elements of F. Finally we label by $\bar{\mathcal{F}}'_\Gamma$ the set of maximal sub-forests in $\mathcal{F}'_\Gamma$. Notice that $\bar{\mathcal{F}}'_\Gamma$ coincides with the set of forests made of mutually disjoint sub diagrams of Γ.

A generic maximal sub-forest can be graphically represented as in the following figure:



It is clear that any forest F in $\mathcal{F}'_\Gamma$ is equal to the union of forests $F_{\bar{\gamma}}$ contained in $\bar{\gamma}$ and including it as an element, for every $\bar{\gamma}$, element of the maximal sub-forest $\bar{F}$, that is:

$$F \equiv \cup_{\bar{\gamma}\in\bar{F}} F_{\bar{\gamma}}|_{\bar{\gamma}\in F_{\bar{\gamma}}} \ . \tag{13}$$

Therefore we can write Eq.(11) in the form:

$$\hat{R}_\Gamma(p,k) = \mathcal{S}_\Gamma \sum_{\bar{F}\in\bar{\mathcal{F}}'_\Gamma} \prod_{\bar{\gamma}\in\bar{F}} \sum_{F_{\bar{\gamma}}\in\mathcal{F}_{\bar{\gamma}},\bar{\gamma}\in F_{\bar{\gamma}}} \prod_{\gamma\in F_{\bar{\gamma}}} (-t^d_\gamma \mathcal{S}_\gamma) I_\Gamma(p.k) \ . \tag{14}$$

Given a maximal sub-forest $\bar{F}$ of Γ we define the *reduced diagram* $\Gamma/(\bigoplus_{\bar{\gamma}\in\bar{F}}\bar{\gamma})$ which is built with the lines and vertices of Γ not belonging to any element of $\bar{F}$ and of a further set of vertices corresponding to the elements $\bar{\gamma}$ of $\bar{F}$ shrunk to point vertices. The reduced diagram $\Gamma/(\bigoplus_{\bar{\gamma}\in\bar{F}}\bar{\gamma})$ is relevant to our discussion since the corresponding integrand identifies the part of $I_\Gamma(p.k)$ which is not concerned by the subtraction operation corresponding to the forest F. Indeed one can write:

$$\hat{R}_\Gamma(p,k) = \mathcal{S}_\Gamma \sum_{\bar{F}\in\bar{\mathcal{F}}'_\Gamma} \left[\prod_{\bar{\gamma}\in\bar{F}} ((-t^{d_{\bar{\gamma}}}_{\bar{\gamma}} \mathcal{S}_{\bar{\gamma}})\hat{R}_{\bar{\gamma}}(p,k))\right] I_{\Gamma/(\bigoplus_{\bar{\gamma}\in\bar{F}}\bar{\gamma})}(p,k) \tag{15}$$

This expression is identical to that associated with a diagram coinciding with the reduced diagram in which the vertices corresponding to the elements $\bar{\gamma}$ in $\bar{F}$ carry factors equal to $(-t^{d_{\bar{\gamma}}}_{\bar{\gamma}} \mathcal{S}_{\bar{\gamma}})\hat{R}_{\bar{\gamma}}(p,k)$. These factors, i.e. the brackets above, are, of course, Λ-independent. Therefore, inserting Eq.(15) into Eq.(12) one has:

$$\Lambda^2\partial_{\Lambda^2} R_\Gamma(p,k) \qquad = \mathcal{S}_\Gamma \sum_{\bar{F}\in\bar{\mathcal{F}}'_\Gamma} \left[\prod_{\bar{\gamma}\in\bar{F}} ((-t^{d_{\bar{\gamma}}}_{\bar{\gamma}} \mathcal{S}_{\bar{\gamma}})\hat{R}_{\bar{\gamma}}(p,k))\right] \Lambda^2\partial_{\Lambda^2} I_{\Gamma/(\bigoplus_{\bar{\gamma}\in\bar{F}}\bar{\gamma})}(p,k)$$

$$
= \mathcal{S}_\Gamma \sum_{\bar F \in \bar{\mathcal{F}}'_\Gamma} \left[\prod_{\bar\gamma \in \bar F} ((-t^{d_{\bar\gamma}}_{\bar\gamma} \mathcal{S}_{\bar\gamma}) \hat R_{\bar\gamma}(p,k)) \right]
$$

$$
\sum_{l \in L(\Gamma/(\prod_{\bar\gamma\in\bar F}\bar\gamma))} \dot{\hat S}(\hat p_l + \hat k_l) I_{\Gamma/(l \bigoplus_{\bar\gamma\in\bar F}\bar\gamma)}(p,k)
$$

where $\Gamma/(l \bigoplus_{\bar\gamma\in\bar F}\bar\gamma)$ means the reduced diagram $\Gamma/(\bigoplus_{\bar\gamma\in\bar F}\bar\gamma)$ deprived of the line l and we have used the fact that the Λ-dependence comes from the propagators.

Now we interchange the summation over the forests with that over the lines of Γ upon which the Λ-derivative acts. This is possible since every line l contributes to the above sum in correspondence with the forests F in $\mathcal{F}'_\Gamma$ whose elements do not contain it. If we extend the idea of forest to diagrams, such as Γ/l which are connected but not necessarily 1-P.I., the set of forests we are speaking of is $\mathcal{F}_{\Gamma/l}$ which, of course, is contained in $\mathcal{F}'_\Gamma$. Thus we get:

$$
\Lambda^2 \partial_{\Lambda^2} R_\Gamma(p,k) = \mathcal{S}_\Gamma \sum_{l\in L(\Gamma)} \dot{\hat S}(\hat p_l + \hat k_l) \sum_{F\in\mathcal{F}_{\Gamma/l}} \prod_{\gamma\in F} (-t^d_\gamma \mathcal{S}_\gamma) I_{\Gamma/l}(p,k) \ , \tag{16}
$$

Let us now consider the possibility of Γ/l not being 1-P.I.. The diagram Γ/l is however connected and it decomposes according to its skeleton structure into lines linking 1-P.I. parts. In the present situation, in which the diagram is obtained from a 1-P.I. diagram cutting the line l, Γ/l is either 1-P.I. or consists in a chain 1-P.I. sub-diagrams linked by lines. Therefore $I_{\Gamma/l}(p,k)$ factorizes into a product of line and 1-P.I. factors, one of the end points of the line l being attached to the first 1-P.I. sub-diagram of the chain, the other one to the last. Labelling these sub-diagrams by α_i , $i = 0,..,n(\Gamma,l)$, where $n(\Gamma,l)$ is a non-negative integer, we can write:

$$
I_{\Gamma/l}(p,k) = I_{\alpha_0}(p,k) \prod_{i=1}^{n(\Gamma,l)} \hat S(\hat p_i + \hat k_i)) I_{\alpha_i}(p,k) \ . \tag{17}
$$

If Γ/l is 1-P.I., the product above reduces to one.

Now a forest F in Γ/l appears as the union of, possibly trivial, forests in the above mentioned chain of 1-P.I. sub-diagrams, therefore the sum over the forests in $\mathcal{F}_{\Gamma/l}$ decomposes into the product of the sums over the forests in each sub-diagram α_i and hence we have:

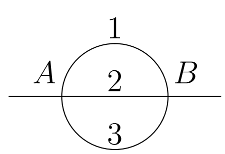
$$
\begin{aligned}
\Lambda^2 \partial_{\Lambda^2} R_\Gamma(p,k) \quad &= \mathcal{S}_\Gamma \sum_{l\in L(\Gamma)} \dot{\hat S}(\hat p_l + \hat k_l) \sum_{F\in\mathcal{F}_{\Gamma/l}} \prod_{\gamma\in F} (-t^d_\gamma \mathcal{S}_\gamma) I_{\Gamma/l}(p,k) \\
&= \mathcal{S}_\Gamma \sum_{l\in L(\Gamma)} \dot{\hat S}(\hat p_l + \hat k_l) \left[\mathcal{S}_{\alpha_0} \sum_{F\in\mathcal{F}_{\alpha_0}} \prod_{\gamma\in F} (-t^{d_\gamma}_\gamma \mathcal{S}_\gamma) I_{\alpha_0}(p,k) \right] \\
&\prod_{i=1}^{n(\Gamma,l)} \left[\hat S(\hat p_i + \hat k_i)) \left[\mathcal{S}_{\alpha_i} \sum_{F'\in\mathcal{F}_{\alpha_i}} \prod_{\gamma'\in F'} (-t^{d_{\gamma'}}_{\gamma'} \mathcal{S}_{\gamma'}) I_{\alpha_i}(p,k) \right] \right] \\
&= \mathcal{S}_\Gamma \sum_{l\in L(\Gamma)} \dot{\hat S}(\hat p_l + \hat k_l) R_{\alpha_0}(p,k) \prod_{i=1}^{n(\Gamma,l)} \left[\hat S(\hat p_i + \hat k_i)) R_{\alpha_i}(p,k) \right] \ .
\end{aligned} \tag{18}
$$

Now we consider how the Λ-derivative of a diagram must be subtracted in order to have absolute convergent internal momentum integrals. The basic remark is that the Λ-derivative only acts on lines giving (Eq.(2)) $-\exp((\hat{p}_l + \hat{k}_l)^2/\Lambda^2)/\Lambda^2$. Therefore we see that $\dot{\hat{S}}$ introduces a cut-off in the corresponding line momentum ($\hat{k}_l$) and the needed subtraction formula must be limited to the forests in $\mathcal{F}_{\Gamma/l}$. Hence one gets back Eq.(16) and the commutativity of subtraction and Λ-derivative is proven.

Summing over all diagrams one also sums over all the possible values of $n(\Gamma, l)$ and it clearly appears that the structure of the rightmost term of Eq.(18) coincides with that of the right-hand side of the evolution equation of the Effective Action $V_R[\phi]$ and, of course, with that of its coefficient functions. Indeed one finds a sum over the chains of n 1-P.I. amplitudes linked by propagators $\hat{S}$ and closed by $\dot{\hat{S}}$.

It remains to verify the correct counting of diagrams. In other words until now we have shown that, computing the Λ-derivative of every 1-P.I. subtracted diagram, one gets a combination of subtracted diagrams with the structure appearing in Fig.(3). What remains is a purely combinatorial problem, that is to verify that computing the Λ-derivative of $\mathcal{V}_\Lambda$, that is summing all diagrams together, one gets an expression in which all the expected diagrams appear with the expected combinatorial factor. This is just the consequence of the discussed commutativity of subtraction and Λ-derivative. Indeed the fact that before subtraction all the expected diagrams appear with the right factors is proven by a straightforward application of the functional method. At the formal level, disregarding divergences, the functional generator of Feynman diagrams Z is perfectly well defined, the generator of connected diagrams is $\ln Z$ and that of 1-P.I. diagrams is the Legendre transform of $\ln Z$. Now it is easy to show [11] that Eq.(4) is satisfied by the formal graph expansion under the hypothesis that the derivative only acts on lines. This guarantees the correct counting of diagrams and completes our proof.

In order to give a significant example let us consider the three line, two leg, diagram shown in Fig.(19):



(19)

This diagram seems to violate what just claimed, indeed it contains three indistinguishable internal lines, and hence its Λ-derivative gives three identical contributions in which $\dot{\hat{S}}$ is linked to a single diagram with two identical lines. On the contrary, in a diagrammatic expansion of Fig.(3) and Eq.(4) this diagram should appear only once. This is however a wrong argument since it forgets the combinatorial factors of the diagrams. A diagram with N sets of n_i, $i = 1, ..., N$, indistinguishable lines carries a combinatorial factor equal to $1/(\prod_{i=1}^N n_i!)$ that is $1/6$ in the example. Combining the three identical contributions from the three lines together we get the resulting contribution to the evolution equation with weight $1/2$ which is exactly the combinatorial factor of the corresponding diagram with two identical lines.

In conclusion we have shown that, applying a slightly modified subtraction method to the Feynman diagrams built with the propagator $\hat{S}$ given in Eq.(7), and possibly with its spinor, or gauge field variants, yields to a diagrammatic construction of $\mathcal{V}_\Lambda[\phi]$ solving the R.G. evolution equation (4).

However we want also to show that the field expansion coefficients of $\mathcal{V}_\Lambda[\phi]$ satisfy Eqs.(5) and (6) with the initial conditions at $\Lambda = \Lambda_R$ appearing in Eq.(5), and furthermore that the limit $\Lambda \to \infty$ of $\partial_p^k \, \mathcal{V}_{\Lambda,n}$ for $n + k > 4$ vanishes.

It is apparent that $\mathcal{V}_{\Lambda,2}$ and $\mathcal{V}_{\Lambda,4}$ satisfy Eqs.(5). Indeed $\mathcal{V}_{\Lambda,2}$ is the sum of two leg proper diagrams which, with the exception of the trivial diagrams generated by the first two vertices in Eq.(8), are superficially divergent and hence subtracted to zero at $p = 0$ and $\Lambda = \Lambda_R$ with their first derivative in p^2. Furthermore $\mathcal{V}_{\Lambda,4}$ is the sum of four leg proper diagrams which, with the exception of the trivial diagram generated by the third vertex in Eq.(8), are superficially divergent and hence subtracted to zero at $p = 0$ and $\Lambda = \Lambda_R$.

Concerning the derivatives of the coefficients $\partial_p^k \, \mathcal{V}_{\Lambda,n}$ for $n + k > 4$, they only receive contributions from superficially convergent diagrams which are easily seen to vanish in the $\Lambda \to \infty$ limit using the inequality:

$$(1 - \exp(-p^2/\Lambda^2))/p^2 \leq 2/(p^2 + \Lambda^2), \tag{20}$$

and pure scale arguments.

Therefore we conclude that the construction of the Effective Action $\mathcal{V}_\Lambda[\phi]$ by the above defined subtraction method leads to a solution of Wilson-Polchinski evolution equation satisfying the boundary conditions characterizing Wilson's construction, thus it leads to the same functional:

$$\mathcal{V}_\Lambda[\phi] \equiv V_R[\phi, \Lambda, \Lambda_R] \ .$$

4 Conclusions

In conclusion, comparing the Wilson-Polchinski renormalization group and the BPHZ subtraction approach one sees that in both cases one is dealing with an infinity of quantities related by an infinity of equations and hence the chosen ordering is a crucial step of the construction procedure.

The subtraction approach deals with one diagram at a time and the physical amplitudes appear as formal expansions into subtracted diagrams which must be ordered in some way. The loop ordering is the typical choice.

The R.G. integral equations (5) and (6) for the coefficient functions of the field expanded 1-P.I. effective action are not strictly related to diagrams, hence a wider class of recursive construction is *in principle* open. However the right-hand sides of the evolution equations appear as the sum of series which are infinite due to the presence of chains of two-point insertions which in principle can be summed. This is particularly critical in the scalar field case due to the quadratic divergence of the mass terms.

The set of evolution equations for the coefficient functions is infinite and open, in the sense that it does not contain any closed finite sub-set, that is, any finite sub-set of equations involving a finite number of coefficient functions. Indeed the evolution equation of the coefficient $V_{R,n}$ involves $V_{R,n+2}$. Thus, in order to build a solution, one must truncate in some way the sequence of the $V_{R,n}$ evolution equations.

We have limited our study to the loop ordered perturbative expansion in which the sequence of evolution equations appears closed at any order. This has allowed us to study the details of the resulting amplitudes proving that their expansion into diagrams coincides with that generated by the subtraction method with a suitable, however natural, choice of the subtraction prescriptions.

With the aim of simplifying our presentation we have also limited our discussion to the simplest scalar model disregarding invariance properties and possible infra-red singularities, thus, in a sense, remaining far apart from the physical applications.

Our hope is that the present discussion could further clarify the relations among different construction techniques of Quantum Field Theory confirming the central role of Zimmermann's work.

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