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Jan Manschot

Stability and Duality in $\mathscr{N}=2$ Supergravity

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Abstract The BPS-spectrum is known to change when moduli cross a wall of marginal stability. This paper tests the compatibility of wall-crossing with *S*-duality and electric-magnetic duality for $\mathscr{N}=2$ supergravity. To this end, the BPS-spectrum of D4-D2-D0 branes is analyzed in the large volume limit of Calabi-Yau moduli space. Partition functions are presented, which capture the stability of BPS-states corresponding to two constituents with primitive charges and supported on very ample divisors in a compact Calabi-Yau. These functions are "mock modular invariant" and therefore confirm *S*-duality. Furthermore, wall-crossing preserves electric-magnetic duality, but is shown to break the "spectral flow" symmetry of the $\mathscr{N}=(4,0)$ CFT, which captures the degrees of freedom of a single constituent.

Contents

1 Introduction

The study of BPS-states in physics has been very fruitful. Their invariance under (part of the) supersymmetry transformations of a theory makes them insensitive to variations of certain parameters. This allows the calculation of some quantities in a different regime than the regime of interest. BPS-states have been specifically

Rutgers University, Piscataway, NJ 08854-8019, USA. manschot@physics.rutgers.edu

useful in testing various dualities, for example S-duality in $\mathscr{N}=4$ Yang-Mills theory (43) or in string theory (39). Another major application is the understanding of the spectrum of supersymmetric theories of gravity, leading to the microscopic account of black hole entropy for various supersymmetric black holes in string theory (32; 41).

This article considers the BPS-spectrum of $\mathscr{N}=2$ supergravity theories in 4 dimensions. $\mathscr{N}=2$ supersymmetry is the least amount of supersymmetry, which allows massive states to be BPS. It appears in string theory by compactifying the 10-dimensional space-time on a compact 6-dimensional Calabi-Yau manifold X. A large class of BPS-states are formed by wrapping D-branes around cycles of X, which might correspond to black hole states if the number of D-branes is sufficiently large. The Witten index Ω (degeneracy counted with $(-1)^F$) is insensitive to perturbations of the string coupling constant g_s, and plays therefore a central role in this paper. It allows to show for certain cases that the magnitude of the index agrees with black hole entropy: $\log\Omega \sim S_{\mathrm{BH}}$. The study of D-branes on X revealed many connections to objects in mathematics, like vector bundles, coherent sheaves and derived categories, which helps to understand their nature, see for a review Ref. (2). The index Ω corresponds from this perspective to the Euler number $\chi(\mathscr{M})$ of their moduli space $\mathscr{M}$ (43), or an analogous but better defined invariant like Donaldson-Thomas invariants (42).

An intriguing aspect of BPS-states is their behavior as a function of the moduli of the theory. The moduli parametrize the Calabi-Yau X and appear in supergravity as scalar fields. Under variations of the moduli, conservation laws allow BPS-states to become stable or unstable at codimension 1 subspaces (walls) of the moduli space. Such changes in the spectrum indeed occur, and were first observed in 4 dimensions by Seiberg and Witten (38). Denef (11) has given an illuminating picture of stability in supergravity as multi black hole solutions whose relative distances depend on the value of the moduli at infinity. At a wall, these distances might diverge or become positive and finite. The changes in the degeneracies $\Delta\Omega$ at a wall show the impact on the spectrum of these processes. Ref. (12) derives formulas for $\Delta\Omega$ for n-body semi-primitive decay using arguments from supergravity. The notion of stability for D-branes is closely related to the notion of stability in mathematics (16; 17). In this context, Kontsevich and Soibelman (30) derive a very general wall-crossing formula for (generalized) Donaldson-Thomas invariants. Gaiotto et al. (23) shows that this generic formula applied to the indices of 4-dimensional $\mathscr{N}=2$ quantum field theory, is implied by properties of the field theory.

Much evidence exists for the presence of an S-duality and electric-magnetic duality group in $\mathscr{N}=2$ supergravity (7; 44). S-duality is an $SL(2,\mathbb{Z})$ group which exchanges weak and strong coupling; electric-magnetic duality is the action of a symplectic group on the vector multiplets. These dualities impose strong constraints on the spectrum of the theory. The wall-crossing formulas are very generic on the other hand, and the walls form a very intricate web in the moduli space. It is therefore appropriate to ask: *are wall-crossing and duality compatible with each other?* This paper analyses this question, concentrating on D4-D2-D0 BPS-states or M-theory black holes, in the large volume limit of Calabi-Yau moduli space. The BPS-objects correspond in this limit to coherent sheaves on a Calabi-Yau 3-fold supported on an ample divisor. The analysis considers the walls, the primitive wall-crossing formula and (part of) the supergravity partition function $\mathcal{Z}_{\rm sugra}(\tau,C,t)$, which enumerates the indices as a function of D2- and D0-brane charges for fixed D4-brane charge. $\mathcal{Z}_{\rm sugra}(\tau,C,t)$ captures the changes in the spectrum by wall-crossing. *S*-duality predicts modularity for this function, which is tested in this paper.

The degrees of freedom of a single D4-D2-D0 black hole are related via M-theory to a 2-dimensional $\mathcal{N}=(4,0)$ superconformal field theory (SCFT) (32). One of the symmetries of the SCFT spectrum is the "spectral flow symmetry" (4; 21; 31), which are certain transformations of the charges, which do not change the value of the moduli at infinity. This imposes additional constraints on the spectrum to the ones imposed by the supergravity duality groups. A single constituent cannot decay any further, and conjectures by (1; 5) indicate that the SCFT description of the spectrum (for given charge) might only be valid for a specific value of the moduli. Therefore, interesting dependence of the SCFT spectrum as a function of the moduli at infinity is not expected. This suggests that a natural decomposition for the supergravity partition function with fixed magnetic charge P might be

$$\mathcal{Z}_{\rm sugra}(\tau,C,t)=\mathcal{Z}_{\rm CFT}(\tau,C,t)+\mathcal{Z}_{\rm wc}(\tau,C,t), \tag{1.1}$$

where $\mathcal{Z}_{\rm CFT}(\tau,C,t)$ is the well-studied SCFT elliptic genus (4; 21; 31; 33), and all wall-crossing in the moduli space is captured by $\mathcal{Z}_{\rm wc}(\tau,C,t)$. $\mathcal{Z}_{\rm CFT}(\tau,C,t)$ is known to transform as a modular form from arguments of CFT; the modular properties of $\mathcal{Z}_{\rm wc}(\tau,C,t)$ are however unknown.

This paper considers a small part of $\mathcal{Z}_{\rm wc}(\tau,C,t)$, namely $\sum_{\substack{P_1+P_2=P \\ \text{ample, primitive}}} \mathcal{Z}_{P_1\leftrightarrow P_2}(\tau,C,t)$, which enumerates the indices of composite BPS-configurations with two constituents, with ample and primitive magnetic charges P_1 and P_2. An important building block of these functions is the newly introduced "mock Siegel-Narain theta function". Mock modular forms do not transform exactly as modular forms, but can be made so by the addition of a relatively simple correction term (46), which is applied to mock Siegel-Narain theta functions in the appendix. Using its transformation properties, one can show that the corrected partition function transforms precisely as the SCFT elliptic genus, thereby confirming *S*-duality.

From the analysis follows also that electric-magnetic duality remains present in the theory, but the "spectral flow" symmetry of the SCFT is generically not present. This is not quite unexpected since this is not a symmetry of supergravity. Another indication that the spectral flow symmetry is not present appears in Ref. (1), which explains that the jump in the D4-D2-D0 index by wall-crossing can be larger than the index of a single BPS-object (this effect is known as the entropy enigma (12)).

A special property of $\mathcal{Z}_{P_1\leftrightarrow P_2}(\tau,C,t)$ is that it does not contribute to the index if the moduli are chosen at the corresponding attractor point. However, $\mathcal{Z}_{P_1\leftrightarrow P_2}(\tau,C,t)$ is generically not zero, and therefore $\mathcal{Z}_{\rm sugra}(\tau,C,t)$ is nowhere equal to $\mathcal{Z}_{\rm CFT}(\tau,C,t)$ generically. Section 4 explains how these observations are in agreement with conjectures of Refs. (1; 5) about the uplift of these BPS-configurations to five dimensions.

Although the compatibility with the dualities is expected, it is very interesting to see how it is realized. The stability condition and primitive wall-crossing formula combine in an almost miraculous way to the mock Siegel-Narain theta function, which gives insights in the way wall-crossing is captured by $\mathcal{N}=2$ BPS partition functions for compact Calabi-Yau 3-folds. An intriguing property of the corrected partition function is that it is continuous as a function of the Kähler moduli t, which is reminiscent of earlier discussions (23; 29).

The outline of this paper is as follows. Section 2 reviews briefly the relevant aspects of $\mathcal{N}=2$ supergravity. Section 3 describes the BPS-states of interest and the expected properties of their partition function. Section 4 is the heart of the paper; it describes the walls and the partition functions capturing wall-crossing. Section 5 finishes with discussions and suggestions for further research. The Appendix defines two mock Siegel-Narain theta functions and gives some of their properties.

2 BPS-States in $\mathcal{N}=2$ Supergravity

If IIA string theory is compactified on a compact Calabi-Yau 3-fold X, one obtains $\mathcal{N}=2$ supergravity as the low energy theory in the non-compact dimensions. The most essential part of the field content for this article is the b_2+1 vector multiplets, which each contain a $U(1)$ gauge field $F^A_{\mu\nu}$ and complex scalar X^A, $A=1,\dots,b_2+1$ (with b_2 the second Betti number of X). The gauge fields lead to a vector of conserved charges $\Gamma=(P^0,P^a,Q_a,Q_0)^{\mathrm{T}}$, $a=1\dots b_2$, which take value in the $(2b_2+2)$-dimensional lattice L. The magnetic charges are denoted by P^A and electric charges by Q_A. The charges arise in IIA string theory as wrapped D-branes on the even homology of X; the components of Γ represent 6-, 4-, 2- and 0-dimensional cycles. A symplectic pairing is defined on the charge lattice

$$\langle\Gamma_1,\Gamma_2\rangle=-P^0_1Q_{0,2}+P_1\cdot Q_2-P_2\cdot Q_1+P^0_2Q_{0,1}.$$

The symplectic inner product is thus

$$\mathbf{I}=\begin{pmatrix} & & & -1\\ & & \mathbf{1} & \\ & -\mathbf{1} & & \\ 1 & & & \end{pmatrix},$$

where $\mathbf{1}$ denotes a $b_2\times b_2$ unit matrix.

The scalars X^A parametrize the Kähler moduli space of the Calabi-Yau X: the complexified Kähler moduli are given by $t^a=B^a+iJ^a=X^a/X^0$. Here, B^a and J^a are periods of the B-field and the Kähler form respectively.[1] The B-field takes values in $H^2(X,\mathbb{R})$. The Kähler forms are restricted to the Kähler cone C_X, which is defined to be the space of 2-forms such that $\int_\gamma J>0$, $\int_P J^2>0$ and $\int_X J^3>0$ for any holomorphic curve γ and surface $P\in X$. An accurate Lagrangian description of supergravity requires that the volume of X is parametrically larger than the Planck length, thus $J^a\to\infty$. This article is mainly concerned with this parameter regime. Loop and instanton corrections can here be neglected, such that the prepotential simplifies to the cubic expression

$$F(t)=\frac{1}{6}d_{abc}t^at^bt^c,$$

[1] The moduli t^a will sometimes be viewed as 2-forms instead of scalars. Similarly, the charges Γ can also be viewed as homology cycles or their Poincaré dual forms.

where d_{abc} is the triple intersection number of 4-cycles in X.

The supergravity Lagrangian is invariant under the electric-magnetic duality group, which acts on the vector multiplets and more specifically on the electric-magnetic fields and moduli. This duality group is essentially a gauge redundancy, which appears by working on the universal covering space of the moduli space instead of the moduli space itself. The group is $Sp(2b_2+2,\mathbb{Z})$: the group of $(2b_2+2)\times(2b_2+2)$ matrices $\mathbf{K}$ which leave invariant $\mathbf{I}$ (44):[2]

$$\mathbf{K}^{\mathrm{T}}\mathbf{I}\mathbf{K}=\mathbf{I}.$$

The arguments that the group is $Sp(2b_2+2,\mathbb{Z})$ are valid in the large volume limit. The correct electric-magnetic duality group, which is valid for any value of J, is a subgroup of this and generated by the monodromies around singularities in the moduli space. These generators are generically hard to determine, except for the monodromies in the limit $J\to\infty$. They are simply the translations[3]

$$\mathbf{K}(k)=\begin{pmatrix} 1 & & & \\ k^a & \mathbf{1} & & \\ \frac{1}{2}d_{abc}k^bk^c & d_{abc}k^c & \mathbf{1} & \\ \frac{1}{6}d_{abc}k^ck^bk^c & \frac{1}{2}d_{abc}k^bk^c & k^a & 1 \end{pmatrix}, \quad k\in\mathbb{Z}^{b_2}. \tag{2.1}$$

In addition, an $SL(2,\mathbb{Z})$ duality group is present, which exchanges the weak and strong coupling regime. This group acts on the hypermultiplets, and is most manifest in the IIB description for large Kähler parameters (7).

If time is considered as Euclidean and compactified, another $SL(2,\mathbb{Z})$ duality group appears. This can be seen from the M-theory viewpoint, where the total geometry is $\mathbb{R}^3\times T^2\times X$, and T^2 is the product of the time and M-theory circle $S^1_{\mathrm{t}}\times S^1_{\mathrm{M}}$.

A Kaluza-Klein reduction to $\mathbb{R}^3$, leads to a 3-dimensional $\mathcal{N}=4$ supergravity theory. Since the physics in $\mathbb{R}^3$ is independent of large coordinate reparametrizations of T^2, it should exhibit an $SL(2,\mathbb{Z})$ duality group. The duality (complex structure) parameter is given by

$$\tau=C_1+ie^{-\Phi}=C_1+i\beta/g_{\mathrm{s}}, \tag{2.2}$$

where $C_1\in\mathbb{R}$ is the component of the RR-potential 1-form along the time direction. Note that this $SL(2,\mathbb{Z})$ exchanges S^1_{t} and S^1_{M}, which changes the physical interpretation of the states on both sides of the duality. D2-branes become for example worldsheet instantons and vice versa. The full BPS-spectrum should however be invariant under these $SL(2,\mathbb{Z})$ transformations. These transformations also transform the B- and C-fields into each other, which is easily seen from the M-theory perspective: the B- and C-field are reductions of the M-theory 3-form over different 2-cycles of the torus. The duality transformations are summarized by

$$\tau\to\frac{a\tau+b}{c\tau+d},\quad C\to aC+bB,\quad B\to cC+dB,\quad J\to|c\tau+d|J \tag{2.3}$$

[2] Note that we use here a different notation as in e.g. (44), which is more natural from the point of view of geometry.

[3] Note that the upper or lower indices might label either rows or columns in the matrix.

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. Note that this $SL(2,\mathbb{Z})$ is not the weak-strong duality of the 4-dimensional supergravity. But it is possible to relate this "M-theory" $SL(2,\mathbb{Z})$ to the S-duality $SL(2,\mathbb{Z})$ of IIB, by a T-duality along the time circle (12). This transforms C_1 into C_0 and (2.2) becomes the familiar IIB duality parameter. The physical D4-D2-D0 branes of IIA become D3-D1-D-1 instantons of IIB. Therefore, a test of the M-theory $SL(2,\mathbb{Z})$ is equivalent to testing S-duality, and in the rest of the paper the M-theory $SL(2,\mathbb{Z})$ is referred to as S-duality.

The $\mathcal{N}=2$ supersymmetry algebra contains a central element, the central charge $Z(\Gamma) \in \mathbb{C}$. The central charge of a BPS-state is a linear function of its charge Γ and a non-linear function of the Kähler or complex structure moduli of X. Only the complexified Kähler moduli t^a appear in $Z(\Gamma)$ for the relevant BPS-states in this article, thus $Z(\Gamma,t)$.

The mass M of supersymmetric states is determined by the supersymmetry algebra to be $M = |Z(\Gamma,t)|$. In a theory of gravity, a sufficiently massive BPS-state corresponds to a black hole state in the non-compact dimensions. The moduli depend generically on the spatial position $t(\vec{x})$ in a black hole solution. Their value at the horizon is determined in terms of the charge Γ by the attractor mechanism (19), whereas the value at infinity is imposed as boundary condition. The mass M is determined by the moduli at infinity. The following sections deal with the stability of BPS-states, which is determined by these values at infinity. Also the $SL(2,\mathbb{Z})$ duality group is acting on the complex structure parameter τ of T^2 at infinity.

The expression for the central charge as a function of the moduli is generically highly non-trivial. However in the limit $J \to \infty$ it simplifies to (2)

$$Z(\Gamma,t) = -\int_X e^{-t} \wedge \Gamma,$$

where the moduli t and the charge Γ are viewed as forms on X. Alternatively, one can write

$$Z(\Gamma,t) = \left(1, t^a, \frac{1}{2}d_{abc}t^b t^c, \frac{1}{6}d_{abc}t^a t^b t^c\right) \mathbf{I}\Gamma = \Pi^{\mathrm{T}}\mathbf{I}\Gamma,$$

where we defined the vector of the periods Π.

A very intriguing aspect of BPS-states is their stability. The simplest example is the case with two BPS-objects with primitive charges Γ_1 and Γ_2. Their total mass is larger than or equal to the mass of a single BPS-object with the same total charge: $|Z(\Gamma_1,t)| + |Z(\Gamma_2,t)| \geq |Z(\Gamma_1+\Gamma_2,t)|$. The equality is generically not saturated, but for special values of the moduli $t = t_{\mathrm{ms}}$, the central charges can align $Z(\Gamma_1,t_{\mathrm{ms}})/Z(\Gamma_2,t_{\mathrm{ms}}) \in \mathbb{R}^+$, and the equality holds. These values form a real codimension 1 subspace of the moduli space, appropriately called the "walls of marginal stability". They decompose the moduli space into chambers. BPS-states might decay or become stable, whenever the moduli cross a wall.

Denef (11) has shown how wall-crossing phenomena are manifested in supergravity. The equations of motions allow for BPS-solutions with multiple black holes. The ones of interest for the present discussion are solutions with only two black holes. The relative distance between the two centers is given by

$$|x_1 - x_2| = \sqrt{G_4}\,\frac{\langle \Gamma_1, \Gamma_2\rangle}{2}\,\frac{|Z(\Gamma_1,t)+Z(\Gamma_2,t)|}{\mathrm{Im}(Z(\Gamma_1,t)\bar{Z}(\Gamma_2,t))}\Bigg|_{\infty},$$

where $|_\infty$ means that the central charges are evaluated at asymptotic infinity in the black hole solution; G_4 is the 4-dimensional Newton constant.[4] In the limit $G_4 \to 0$, or equivalently $g_\mathrm{s} \to 0$, the distance between the centers also approaches 0. This is the regime, where a microscopic analysis is typically carried out, it is the D-brane regime as opposed to the black hole regime.

Since distances must be positive, the solution can only exist for

$$\langle \Gamma_1, \Gamma_2\rangle\, \mathrm{Im}(Z(\Gamma_1,t)\bar{Z}(\Gamma_2,t)) > 0. \tag{2.4}$$

Importantly, $|x_1 - x_2|$ depends on the moduli: if t approaches a wall of marginal stability

$$\mathrm{Im}(Z(\Gamma_1,t)\bar{Z}(\Gamma_2,t)) = 0, \tag{2.5}$$

$|x_1 - x_2| \to \infty$ and the 2-center solution decays. An implication of the mechanism for stability in supergravity is that single center black holes cannot decay into BPS-configurations with multiple constituents. If the moduli are chosen at the attractor point at infinity, and are thus constant throughout the black hole solution, 2-center solutions cannot exist. Moreover, the moduli flow in a 2-center solution from a stable chamber at infinity, to unstable chambers at the attractor points.

In the following, we will analyze wall-crossing between two chambers $\mathscr{C}_\mathrm{A}$ and $\mathscr{C}_\mathrm{B}$. To avoid ambiguities, one can choose Γ_1 and Γ_2 such that $\mathrm{Im}(Z(\Gamma_1,t)\bar{Z}(\Gamma_2,t)) < 0$ in $\mathscr{C}_\mathrm{B}$, which is equivalent to the convention in the mathematical literature, see for example (45). This means that a stable object with charge Γ satisfies

$$\frac{\mathrm{Im}(Z(\Gamma_2,t))}{\mathrm{Re}(Z(\Gamma_2,t))} < \frac{\mathrm{Im}(Z(\Gamma,t))}{\mathrm{Re}(Z(\Gamma,t))},$$

with $\langle \Gamma, \Gamma_2\rangle > 0$.

Coherent sheaves are expected to be the proper mathematical description of D-branes in the limit $J \to \infty$ (2). The charge Γ of the BPS-state is determined by the Chern character of the corresponding sheaf $\mathscr{E}$ and of the $\hat{A}$ genus of the Calabi-Yau (35)

$$\Gamma = \mathrm{ch}(i_!\mathscr{E})\,\sqrt{\hat{A}(TX)}, \tag{2.6}$$

where $i : P \hookrightarrow X$ is the inclusion map of the divisor into the Calabi-Yau.

Of central interest are the degeneracies of BPS-states with charge Γ. Most useful is actually the index

$$\Omega(\Gamma;t) = \frac{1}{2}\mathrm{Tr}_{\mathscr{H}(\Gamma;t)}\,(2J_3)^2(-1)^{2J_3}, \tag{2.7}$$

[4] G_4 is the 4-dimensional Newton constant, and is given in terms of IIA and M-theory parameters by $G_4 = g_\mathrm{s}^2\alpha'\frac{(\alpha')^3}{V_\mathrm{CY}}$ and $G_4 = \ell_\mathrm{P}^2\frac{\ell_\mathrm{P}}{2\pi R}\frac{\ell_\mathrm{P}^6}{V_\mathrm{CY}}$, respectively.

where J_3 is a generator of the rotation group Spin(3). $\Omega(\Gamma;t)$ is a protected quantity against variations of g_s. The degeneracies are only constant in chambers of the moduli space, but jump if a wall is crossed. This is easily understood from the mechanism for decay in supergravity: the constituents separate, leading to a factorization of the Hilbert spaces, and consequently a loss of the number of states. The change in the index is (12):

$$\begin{aligned}\Delta\Omega(\Gamma;t_{\mathrm{s}}\to t_{\mathrm{u}}) &= \Omega(\Gamma;t_{\mathrm{u}})-\Omega(\Gamma;t_{\mathrm{s}})\\ &= -(-1)^{\langle\Gamma_1,\Gamma_2\rangle-1}\,|\langle\Gamma_1,\Gamma_2\rangle|\,\Omega(\Gamma_1;t_{\mathrm{ms}})\,\Omega(\Gamma_2;t_{\mathrm{ms}}).\end{aligned}$$

Of course, in crossing a wall towards stability one gains states. Therefore the change of the index is in this case

$$\Delta\Omega(\Gamma;t_{\mathrm{u}}\to t_{\mathrm{s}})=(-1)^{\langle\Gamma_1,\Gamma_2\rangle-1}\,|\langle\Gamma_1,\Gamma_2\rangle|\,\Omega(\Gamma_1;t_{\mathrm{ms}})\,\Omega(\Gamma_2;t_{\mathrm{ms}}).$$

Wall-crossing occurs more generally between two chambers $\mathscr{C}_{\mathrm{A}}$ and $\mathscr{C}_{\mathrm{B}}$. If Γ_1 and Γ_2 are chosen such that $\mathrm{Im}(Z(\Gamma_1,t_{\mathrm{B}})\bar{Z}(\Gamma_2,t_{\mathrm{B}}))<0$ in $\mathscr{C}_{\mathrm{B}}$, the change of the index between the two chambers is

$$\Delta\Omega(\Gamma;t_{\mathrm{A}}\to t_{\mathrm{B}})=(-1)^{\langle\Gamma_1,\Gamma_2\rangle}\,\langle\Gamma_1,\Gamma_2\rangle\,\Omega(\Gamma_1;t_{\mathrm{ms}})\,\Omega(\Gamma_2;t_{\mathrm{ms}}). \tag{2.8}$$

This is consistent with jumps of the invariants in mathematics at walls of marginal stability. We can of course choose the points t_{A} and t_{B} more generally and allow them to lie in the same chamber. Then the change in the index is

$$\begin{aligned}\Delta\Omega(\Gamma;t_{\mathrm{A}}\to t_{\mathrm{B}}) &= (-1)^{\langle\Gamma_1,\Gamma_2\rangle}\,\langle\Gamma_1,\Gamma_2\rangle\,\Omega(\Gamma_1;t)\,\Omega(\Gamma_2;t)\\ &\quad\times\frac{1}{2}\,(\,\mathrm{sgn}(\mathrm{Im}(Z(\Gamma_1;t_{\mathrm{A}})\bar{Z}(\Gamma_2;t_{\mathrm{A}})))\\ &\quad-\mathrm{sgn}(\mathrm{Im}(Z(\Gamma_1;t_{\mathrm{B}})\bar{Z}(\Gamma_2;t_{\mathrm{B}})))\,),\end{aligned} \tag{2.9}$$

where $\mathrm{sgn}(z)$ is defined as $\mathrm{sgn}(z)=1$ for $z>0$, 0 for $z=0$, and -1 for $z<0$. Note that $\Delta\Omega(\Gamma;t_{\mathrm{A}}\to t_{\mathrm{B}})$ satisfies a cocycle relation $[\mathrm{AC}]=[\mathrm{AB}]+[\mathrm{BC}]$.

Compatibility of the earlier described dualities with wall-crossing is non-trivial. Consider here the compatibility of electric-magnetic duality. As a gauge redundancy, $Sp(2b_2+2,\mathbb{Z})$ (or the relevant subgroup) leaves invariant the central charge: $Z(\Gamma;t)=Z(\mathbf{K}\Gamma;\mathbf{K}t)$ ($\mathbf{K}t$ denotes the transformed vector of moduli), and the indices:

$$\Omega(\Gamma;t)=\Omega(\mathbf{K}\Gamma;\mathbf{K}t), \tag{2.10}$$

for every $\Gamma\in L$. Since the walls are determined by the central charges, and $\langle\Gamma_1,\Gamma_2\rangle=\langle\mathbf{K}\Gamma_1,\mathbf{K}\Gamma_2\rangle$, it is clear that wall-crossing does not obstruct the electric-magnetic duality group. Note that generically $\Omega(\Gamma;t)\neq\Omega(\mathbf{K}\Gamma;t)$, and that no symmetry exists in supergravity which relates these two indices. Section 3 comes back to this point.

The $SL(2,\mathbb{Z})$-duality group also implies non-trivial constraints for the degeneracies and their wall-crossing. The test of this duality is however much more involved and the subject of Sect. 4, after general aspects of D4-D2-D0 BPS-states and their partition functions are explained in the next section.

3 D4-D2-D0 BPS-States

This section specializes the general considerations of the previous section to the set of states with charge $\Gamma=(0,P,Q,Q_0)$, and discusses the supergravity partition functions for this class of charges. These BPS-states correspond to D4-branes wrapping a divisor in X, with homology class $P\in H_4(X,\mathbb{Z})$. This class of BPS-states is well-described in the literature, see for example (4; 21; 32; 36), therefore the review here will only include the most essential parts for the discussion.

The divisor is also denoted by P and taken to be very ample, which means among others that it has non-zero positive components in all 4-dimensional homology classes. The intersection form on P leads to a quadratic form $D_{ab}=d_{abc}P^c$ for magnetic charges $k\in H_4(X,\mathbb{Z})$; the signature of D_{ab} is $(1,b_2-1)$. The lattice is denoted by Λ. The electric charge Q takes its value in $\Lambda^*+P/2$ (20; 35). The conjugacy class of $Q-P/2$ in Λ^*/Λ is denoted by μ. If necessary, the dependence of D_{ab} on P will be made explicit, like $P\cdot J^2$, otherwise simply J^2 is used.

The real and imaginary part of the central charge $Z((P,Q,Q_0),t)$ of these states are

$$\begin{aligned}\mathrm{Re}(Z(\Gamma,t)) &= \frac{1}{2}P\cdot(J^2-B^2)+Q\cdot B-Q_0,\\ \mathrm{Im}(Z(\Gamma,t)) &= (Q-BP)\cdot J.\end{aligned}$$

The mass $|Z(\Gamma;t)|$ of BPS-states in the regime $P\cdot J^2\gg|(Q-\frac{1}{2}B)\cdot B-Q_0|$, $|(Q-BP)\cdot J|$ is:

$$|Z(\Gamma,t)|=\frac{1}{2}P\cdot J^2+(Q-\frac{1}{2}BP)\cdot B-Q_0+\frac{((Q-BP)\cdot J)^2}{P\cdot J^2}+\mathcal{O}(J^{-2}). \tag{3.1}$$

All but the first term are homogeneous of degree 0 in J, and thus invariant under rescalings. The combination $\frac{((Q-BP)\cdot J)^2}{P\cdot J^2}$ is positive definite: $(Q-B)^2_+$. J has thus a natural interpretation as a point of the Grassmannian which parametrizes 1-dimensional subspaces on which D_{ab} is positive definite. It therefore determines a decomposition of $\Lambda\otimes\mathbb{R}$ into a 1-dimensional positive definite subspace and a (b_2-1)-dimensional negative definite subspace.

For $P^0=0$, $P\neq 0$, the transformations (2.1) act on the charges and moduli as

$$\begin{aligned}Q_0 &\to Q_0+k\cdot Q+\frac{1}{2}d_{abc}k^ak^bP^c,\\ Q_a &\to Q_a+d_{abc}k^bP^c,\\ t^a &\to t^a+k^a,\end{aligned}$$

with $k^a\in\Lambda$.

As mentioned in the Introduction, the microscopic explanation for the macroscopic entropy $S_{\mathrm{BH}}=\pi|Z|^2$ of a single center D4-D2-D0 black hole was given by Ref. (32) using M-theory. The black hole degrees of freedom are in this case those of an M5-brane which wraps the divisor in X times the torus T^2. The microscopic counting relied on a 2-dimensional $\mathcal{N}=(4,0)$ CFT, which can be obtained as the reduction of the M5-brane worldvolume theory to T^2. The magnetic charge P determines mainly the field content of the CFT, whereas the electric charges Q

and Q_0 are charges of states within the CFT. The BPS-indices of the single center black hole are the Fourier coefficients of the SCFT elliptic genus $\mathscr{Z}_{\rm CFT}(\tau,C,B)$ (4; 21; 31).

To test the compatibility of S-duality in supergravity with wall-crossing, one needs to consider the full supergravity partition function $\mathscr{Z}(\tau,C,t)$,[5] which captures the stability of BPS-states as a function of t. Properties of $\mathscr{Z}(\tau,C,t)$ are now briefly reviewed, tailored for the present discussion. It is defined by

$$\begin{aligned}\mathscr{Z}(\tau,C,t) = &\sum_{Q_0,Q} \mathrm{Tr}_{\mathscr{H}(P,Q,Q_0;t)}\, \frac{1}{2}(2J_3)^2(-1)^{2J_3+P\cdot Q} \\ &\times \exp\Big(-2\pi\tau_2|Z(\Gamma,t)| + 2\pi i\tau_1(Q_0 - Q\cdot B + B^2/2) + 2\pi i C\cdot(Q-B/2)\Big),\end{aligned}$$

with $\tau_2 = \frac{\beta}{g_s} \in \mathbb{R}^+$, $\tau_1 = C_1 \in \mathbb{R}$, $t = B + iJ \in \Lambda\otimes\mathbb{C}$ and $B, C \in \Lambda\otimes\mathbb{R}$. This function sums over Hilbert spaces with fixed magnetic charge and varying electric charges. This is in agreement with a microcanonical ensemble for magnetic charge and a canonical ensemble for electric charges, which is natural in the statistical physics of BPS black holes (37). After insertion of (3.1) one finds

$$\begin{aligned}\mathscr{Z}(\tau,C,t) = \exp(-\pi\tau_2 J^2) &\sum_{Q_0,Q} \mathrm{Tr}_{\mathscr{H}(P,Q,Q_0;t)}\, \frac{1}{2}(2J_3)^2(-1)^{2J_3+P\cdot Q} \\ &\times e\left(-\bar\tau\hat{Q}_{\bar 0} + \tau(Q-B)_+^2/2 + \bar\tau(Q-B)_-^2/2 + C\cdot(Q-B/2)\right),\end{aligned}$$

with $\hat{Q}_{\bar 0} = Q_{\bar 0} + \frac{1}{2}Q^2$, $Q_{\bar 0} = -Q_0$ and $e(x) = \exp(2\pi i x)$. The modular invariant prefactor $\exp(-\pi\tau_2 J^2)$ is omitted in the following. The partition function has an expansion

$$\begin{aligned}\mathscr{Z}(\tau,C,t) = &\sum_{Q_0,Q} \Omega(P,Q,Q_0;t)\,(-1)^{P\cdot Q} \\ &\times e\left(-\bar\tau\hat{Q}_{\bar 0} + \tau(Q-B)_+^2/2 + \bar\tau(Q-B)_-^2/2 + C\cdot(Q-B/2)\right).\end{aligned}$$

Note that the partition function depends in various ways on the Kähler moduli t: they appear in $\Omega(P,Q,Q_0;t)$, moreover B shifts the electric charges and J determines the decomposition of the lattice into a positive and negative definite subspace of $\Lambda\otimes\mathbb{R}$. The sum over Q_0 and Q is unrestricted and might at some point invalidate the estimate used for (3.1), even in the limit $J\to\infty$. To verify that this does not invalidate the analysis, we compute the term $\mathscr{O}(J^{-2})$. It is given by $-2(Q-B)_+^2\,(\hat{Q}_{\bar 0} - \frac{1}{2}(Q-B)_-^2)/P\cdot J^2$. It follows from the CFT analysis that $\hat{Q}_{\bar 0}$ is bounded below for a single constituent, therefore $\hat{Q}_{\bar 0} - \frac{1}{2}(Q-B)_-^2$ is as well. Moreover, the next section shows that $\hat{Q}_{\bar 0} - \frac{1}{2}(Q-B)_-^2$ is also bounded below for stable bound states of 2 constituents. If $(Q-B)_+^2\,(\hat{Q}_{\bar 0} - \frac{1}{2}(Q-B)_-^2)$ is $\mathscr{O}(J^2)$, then $|Z(\Gamma,t)| - \frac{1}{2}P\cdot J^2$ is at least $\mathscr{O}(J)$. Contributions to the partition function of states for which the approximations for Eq. (3.1) are not satisfied, are thus highly suppressed compared to the states for which they are satisfied, which shows that the analysis is not invalidated. Also for any given value of the charges, one can always increase J to sufficiently large values, such that the approximations are valid. It is

[5] The subscript "sugra" used in the Introduction will be omitted.

very well possible however, that not the whole partition function has a nice Fourier expansion.

It is well known that $\mathscr{Z}(\tau,C,t)$ contains a pole for $\tau\to i\infty$ and its $SL(2,\mathbb{Z})$ images. It is less clear at this point whether poles in B or C can appear in $\mathscr{Z}(\tau,C,t)$. Examples of CFT's where such poles appear, are the characters of massless representations of the $\mathcal{N}=4$ SCFT algebra (18), and the sigma model with the non-compact target space H_3^+ (24). The Fourier expansion of a partition function with poles depends on the integration contour. This is how the partition function of dyons in $\mathcal{N}=4$ supergravity (40) captures wall-crossing phenomena. However, the stability condition (2.5) for D4-branes on ample divisors show that no wall-crossing as a function of C is present. Moreover, the partition functions for bound states of two constituents, derived in the next section, are not directly suggestive for "wall-crossing by poles". Therefore, in the following it is assumed that no poles in B or C are present in $\mathscr{Z}(\tau,C,t)$.

The translations $\mathbf{K}(k)$ of the electric-magnetic duality group imply a symmetry for the partition function. Using (2.10) and assuming the Fourier expansion, one verifies easily that

$$\mathscr{Z}(\tau,C,t)\longrightarrow(-1)^{P\cdot k}\,e(C\cdot k/2)\,\mathscr{Z}(\tau,C,t),$$

under transformations by $\mathbf{K}(k)$. Also using (2.10) one can show a quasi-periodicity in B:

$$\mathscr{Z}(\tau,C,t+k)=(-1)^{P\cdot k}\,e(C\cdot k/2)\,\mathscr{Z}(\tau,C,t).$$

Additionally, $\mathscr{Z}(\tau,C,t)$ satisfies a quasi-periodicity in C:

$$\mathscr{Z}(\tau,C+k,t)=(-1)^{P\cdot k}\,e(-B\cdot k/2)\,\mathscr{Z}(\tau,C,t). \tag{3.2}$$

These translations are large gauge transformations of C. A theta function decomposition is not implied by the two periodicities since the Fourier coefficients $\Omega(\Gamma;t)$ explicitly depend on B, and generically $\Omega(\mathbf{K}(k)\Gamma;t)\neq\Omega(\Gamma;t)$.

A distinguishing property of the partition function for this class of BPS-states is that charges multiply either τ or $\bar\tau$, in contrast to, for example, D2- or D6-brane partition functions. Additionally, space-time S-duality suggests that the function transforms as a modular form, such that techniques of the theory modular forms can be usefully applied. Refs. (21; 22) present some coefficients $\Omega((0,1,Q,Q_0);t)$ for several Calabi-Yau 3-folds with $b_2=1$. These coefficients determine the whole partition function, and confirm modularity in a non-trivial way. However, stability phenomena do not occur in the limit $J\to\infty$ for these Calabi-Yau's, since $b_2=1$. The next section tests modularity, if wall-crossing is present.

The arguments from CFT for modularity are very robust. Refs. (4; 21; 33) derive that the action of the generators of $SL(2,\mathbb{Z})$ on $\mathscr{Z}_{\mathrm{CFT}}(\tau,C,t)$ is given by:

$$\begin{aligned} S:&\quad \mathscr{Z}(-1/\tau,-B,C+i|\tau|J)=\tau^{\frac12}\,\bar\tau^{-\frac32}\,\varepsilon(S)\,\mathscr{Z}(\tau,C,t),\\ T:&\quad \mathscr{Z}(\tau+1,C+B,t)=\varepsilon(T)\,\mathscr{Z}(\tau,C,t), \end{aligned} \tag{3.3}$$

where $\varepsilon(T)=e\left(-c_2(X)\cdot P/24\right)$ and $\varepsilon(S)=\varepsilon(T)^{-3}$ (12; 33). Here the analysis of (4; 21) is adapted to the supergravity point of view following (12). The next section gives evidence that the same transformation properties continue to hold for the full supergravity partition function.[6] Note that S-duality is consistent with the two periodicities mentioned above. The periodicities and the $SL(2,\mathbb{Z})$ form together a Jacobi group $SL(2,\mathbb{Z}) \ltimes (\mathbb{Z}^{b_2})^2$.

The partition function for single constituents can be decomposed in a vector-valued modular form and a theta function by arguments from CFT. The indices of the CFT are independent of the moduli at infinity: $\Omega_{\rm CFT}(\Gamma;t) = \Omega_{\rm CFT}(\Gamma) = \Omega(\Gamma)$, and obey the "spectral flow symmetry" $\Omega(\Gamma) = \Omega(\mathbf{K}(k)\Gamma)$. To see this, recall that the D2-brane charges appear in the CFT in a $U(1)^{b_2}$ current algebra, which can be factored out of the total CFT by the Sugawara construction, which implies that the indices satisfy $\Omega_{\rm CFT}(\Gamma) = \Omega_{\rm CFT}(\mathbf{K}(k)\Gamma)$ (4; 21; 31). The name "spectral flow" comes originally from the SCFT of superstrings. In the current context, one could see the flow as a flow of the B-field. As mentioned already after Eq. (2.10), no evidence exists that this is a symmetry of the full spectrum of 4-dimensional supergravity. In fact, Sect. 4 shows that wall-crossing is incompatible with this symmetry at generic points of the moduli space.

Since the spectral flow symmetry is present in the spectrum of a single D4-D2-D0 black hole, the theta function decomposition is reviewed here. We define the functions

$$h_{P,Q-\frac{1}{2}P}(\tau) = \sum_{Q_{\bar 0}} \Omega(P,Q,Q_0)\, q^{Q_{\bar 0}+\frac{1}{2}Q^2}. \tag{3.4}$$

Using that $\Omega(\Gamma) = \Omega(\mathbf{K}(k)\Gamma)$, one can show that the invariants $\Omega(P,Q,Q_0)$ depend only on $\hat{Q}_{\bar 0}$ and the conjugacy class μ of $Q \in \Lambda^*$, thus $\Omega(P,Q,Q_0) = \Omega_\mu(\hat{Q}_{\bar 0})$. Therefore, $h_{P,Q-\frac{1}{2}P}(\tau) = h_{P,Q-\frac{1}{2}P+k}(\tau)$ with $k \in \Lambda$. This allows a decomposition of $\mathscr{Z}_{\rm CFT}(\tau,C,t)$ into a vector-valued modular form $h_{P,\mu}(\tau)$ and a Siegel-Narain theta function $\Theta_\mu(\tau,C,B)$:

$$\mathscr{Z}_{\rm CFT}(\tau,C,t) = \sum_{\mu\in\Lambda^*/\Lambda} \overline{h_{P,\mu}(\tau)}\,\Theta_\mu(\tau,C,B), \tag{3.5}$$

with

$$\Theta_\mu(\tau,C,B) = \sum_{Q\in\Lambda+P/2+\mu} (-1)^{P\cdot Q} e\left(\tau(Q-B)_+^2/2 + \bar\tau(Q-B)_-^2/2 + C\cdot(Q-B/2)\right). \tag{3.6}$$

The dependence of $\Theta_\mu(\tau,C,B)$ on the Kähler moduli J is not made explicit. The transformation properties of $\Theta_\mu(\tau,C,B)$ are

$$\begin{aligned} S: \Theta_\mu(-1/\tau,-B,C) &= \frac{1}{\sqrt{|\Lambda^*/\Lambda|}}(-i\tau)^{b_2^+/2}(i\bar\tau)^{b_2^-/2} e(-P^2/4) \\ &\quad \times \sum_\nu e(-\mu\cdot\nu)\Theta_\nu(\tau,C,B), \\ T: \Theta_\mu(\tau+1,C+B,B) &= e\left((\mu+P/2)^2/2\right)\Theta_\mu(\tau,B,C). \end{aligned}$$

[6] Evidence exists that $\mathscr{Z}(\tau,C,t)$ does only transform as (3.3) under the full group $SL(2,\mathbb{Z})$ if P is prime. Otherwise it transforms as a modular form of a congruence subgroup, whose level is determined by the divisors of P. Consequently, the rest of the article assumes implicitly that P is prime, although it nowhere explicitly enters the calculations.

They satisfy in addition two periodicity relations for B and C with $k \in \Lambda$:

$$\Theta_\mu(\tau,C,B+k) = (-1)^{k\cdot P}\, e(C\cdot k/2)\,\Theta_\mu(\tau,C,B),$$
$$\Theta_\mu(\tau,C+k,B) = (-1)^{k\cdot P}\, e(-B\cdot k/2)\,\Theta_\mu(\tau,C,B).$$

All the dependence on τ and the "explicit" dependence on B, C and J of $\mathscr{Z}_{\rm CFT}(\tau,C,t)$ is captured by the $\Theta_\mu(\tau,C,B)$. Note that the $\Theta_\mu(\tau,C,B)$ are annihilated by $\mathscr{D} = \partial_\tau + \frac{i}{4\pi}\partial^2_{C_+} + \frac{1}{2}B_+\cdot\partial_{C_+} - \frac{1}{4}\pi i B_+^2$. $\mathscr{Z}(\tau,C,t)$ is also annihilated by $\mathscr{D}$, if holomorphic anomalies in $h_{P,\mu}(\tau)$ are ignored; these are known to arise in similar partition functions for 4-dimensional gauge theory (43).

The transformation properties of $\Theta_\mu(\tau,C,B)$ imply that $h_{P,\mu}(\tau)$ transforms as a vector-valued modular form:

$$\begin{aligned} S: \quad h_{P,\mu}(-1/\tau) &= -\frac{1}{\sqrt{|\Lambda^*/\Lambda|}}(-i\tau)^{-b_2/2-1}\varepsilon(S)^*e\left(-P^2/4\right) \\ &\quad\times \sum_{\delta\in\Lambda^*/\Lambda} e(-\delta\cdot\mu)h_{P,\delta}(\tau), \\ T: h_{P,\mu}(\tau+1) &= \varepsilon(T)^*e\left((\mu+P/2)^2/2\right)h_{P,\mu}(\tau). \end{aligned}$$

From the asymptotic growth of these Fourier coefficients follows the black hole entropy $S_{\rm BH} = \pi\sqrt{\frac{2}{3}(P^3+c_2(X)\cdot P)\hat{Q}_{\bar 0}}$ for $\hat{Q}_{\bar 0} \gg P^3 + c_2(X)\cdot P$.

4 Wall-Crossing in the Large Volume Limit

As explained in Sect. 3, the partition function is expected to exhibit the modular symmetry and electric-magnetic duality in the large volume limit $J\to\infty$. This section constructs the contribution $\mathscr{Z}_{P_1\leftrightarrow P_2}(\tau,C,t)$ of bound states of two primitive constituents with primitive D4-brane charges P_1 and $P_2 \neq \mathscr{O}$ to $\mathscr{Z}(\tau,C,t)$, and tests its modular properties. I take the following Ansatz for the contribution to the index of a bound state of two primitive constituents at a point t in the moduli space:

$$\begin{aligned} \Omega_{\Gamma_1\leftrightarrow\Gamma_2}(\Gamma;t) = &\frac{1}{2}\left(\mathrm{sgn}(\mathrm{Im}(Z(\Gamma_1,t)\bar{Z}(\Gamma_2,t))) + \mathrm{sgn}(\langle\Gamma_1,\Gamma_2\rangle)\right) \\ &\times(-1)^{\langle\Gamma_1,\Gamma_2\rangle-1}\langle\Gamma_1,\Gamma_2\rangle\,\Omega(\Gamma_1)\Omega(\Gamma_2). \end{aligned} \tag{4.1}$$

The first term of the first line ensures that this Ansatz reproduces the wall-crossing formula (2.9). The non-trivial part of the Ansatz is thus the term $\mathrm{sgn}(\langle\Gamma_1,\Gamma_2\rangle)$. This section explains that this is also in agreement with other important physical requirements. Based on the Ansatz, the generating function of the contribution to the index of the bound states is determined in Eq. (4.7). A study of the generating function leads to the following results:

- the generating function (4.7) is convergent,
- the generating function does not exhibit the modular properties of $\mathscr{Z}_{\rm CFT}(\tau,C,t)$ (the partition function of a single center black hole with magnetic charge $P_1 + P_2$), but it can be made so by the addition of a "modular completion" using techniques of mock modular forms. The "completed" generating function (4.10) is proposed as the contribution $\mathscr{Z}_{P_1\leftrightarrow P_2}(\tau,C,t)$ of 2-center bound states, which is thus compatible with S-duality.

– $\mathscr{Z}_{P_1\leftrightarrow P_2}(\tau,C,t)$ has the unexpected property that it is continuous as a function of the moduli, which is reminiscent of earlier work on wall-crossing (23; 29). The generating function is by construction a discontinuous function of the moduli.

The combination of the first and second property is essentially a unique consequence of the Ansatz. The agreement of the Ansatz with the supergravity picture is discussed later.

We continue now by taking a closer look at the walls of marginal stability. Specializing Eq. (2.5) gives for the walls at $J\to\infty$ (without $1/J$ corrections)

$$P_1\cdot J^2\,(Q_2-BP_2)\cdot J - P_2\cdot J^2\,(Q_1-BP_1)\cdot J = 0. \tag{4.2}$$

Note that this wall is independent of the D0-brane charges $Q_{0,i}$. And so states decay at this wall, independent of their D0-charge and of their distribution between the constituents. The condition for stability for this class of states is

$$P_1\cdot J^2(Q_2-BP_2)\cdot J - P_2\cdot J^2(Q_1-BP_1)\cdot J < 0,$$

if $\langle\Gamma_1,\Gamma_2\rangle > 0$. This stability condition is a natural generalization of slope stability for sheaves or bundles on surfaces (15), since $P\cdot J^2$ replaces the notion of rank. It can be derived from the stability for sheaves (28). When $1/J$ corrections are included, one finds that actually many physical walls merge with each other in the limit $J\to\infty$ (13). We define

$$\mathscr{I}(Q_1,Q_2;t) = \frac{P_1\cdot J^2\,(Q_2-BP_2)\cdot J - P_2\cdot J^2\,(Q_1-BP_1)\cdot J}{\sqrt{P_1\cdot J^2\,P_2\cdot J^2\,P\cdot J^2}}, \tag{4.3}$$

which is invariant under rescalings of J.

It is instructive to look at the symmetries of the wall (4.2). Clearly, it is invariant under the translations $\mathbf{K}(k)$ (2.1), if it acts both on the charges and the moduli. However, the wall is not invariant in general if only the charges are transformed. This is only the case for very special situations like $P_1||P_2$. The change in the index is therefore not consistent with the spectral flow symmetry. Indeed, already in Sect. 2 we argued that this symmetry is not natural from the supergravity perspective. The fact that the symmetry is broken has major implications for supergravity partition functions, since the decomposition into a vector-valued modular form and theta functions is not valid.

We can now see that Eq. (4.1) is in agreement with the supergravity picture. As mentioned before, the picture of stability in supergravity shows that only the single center solution exists if the moduli are chosen at the corresponding attractor point $t(\Gamma)$. Therefore, the index should equal the CFT-index at this point: $\Omega(\Gamma;t(\Gamma)) = \Omega_{\mathrm{CFT}}(\Gamma)$, which is consistent with the account of black hole entropy (32). More evidence for this idea comes from the conjectures in Refs. (1; 5), which suggest a one to one correspondence between connected components of the solution space of multi-centered asymptotic $\mathrm{AdS}_3\times S^2$ solutions and IIA attractor flow trees starting at $t(\Gamma)=\lim_{\lambda\to\infty}D^{-1}Q+i\lambda P$. Note that $\mathscr{Z}(\tau,C,t)$ does not depend on λ in the limit $J\to\infty$. By the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, this also suggests that $\Omega(\Gamma,t(\Gamma)) = \Omega_{\mathrm{CFT}}(\Gamma)$. If this is correct, (4.1) should not contribute to $\Omega(\Gamma;t(\Gamma))$. Indeed, computation of $\mathscr{I}(Q_1,Q_2;t(\Gamma))$ gives $\sqrt{\frac{P^3}{P_1P^2\,P_2P^2}}(P_1\cdot Q_2-P_2\cdot Q_1)$, and therefore $\mathrm{sgn}(\mathscr{I}(Q_1,Q_2;t(\Gamma)))-\mathrm{sgn}(\mathscr{P}\cdot$

$Q)=0$, such that there is never a contribution from bound states at the attractor point using this Ansatz. On the other hand, bound states with two constituents for charges $\tilde{\Gamma}\neq\Gamma$ might exist at $t(\Gamma)$, and consequently $\Omega(\tilde{\Gamma},t(\Gamma))\neq\Omega_{\mathrm{CFT}}(\tilde{\Gamma})$. Therefore, these considerations of BPS-configurations with two constituents show that generically $\mathscr{Z}(\tau,C,t)$ equals nowhere in the moduli space $\mathscr{Z}_{\mathrm{CFT}}(\tau,C,t)$.

A special choice of charges is $P_1=0$, i.e. $\Gamma_1=(0,0,Q_1,Q_{0,1})$. If one does not move the moduli outside the Kähler cone, then walls for this choice can not be crossed. To see this, recall that Q_1 represents now the support of a coherent sheaf and must therefore represent a holomorphically embedded D2-brane. Therefore, $Q_1\cdot J>0$ for $J\in C_X$. The stability condition for $(P,Q,Q_{0,1})\to(0,Q_1,Q_{0,1})+(P,Q_2,Q_{0,2})$ is given by

$$P\cdot Q_1\,Q_1\cdot J<0, \tag{4.4}$$

which is independent of the B-field. Equation (4.4) may or may not be satisfied for given charges. However, because $Q_1\cdot J$ cannot change its sign for $J\in C_X$, no walls of marginal stability are present in the large volume limit. It is thus consistent to consider only bound states of constituents with non-zero D4-brane charge.

To construct the generating function, it is covenient to introduce some notation. For constituent $i=1,2$ with charge Γ_i, the corresponding quadratic form is denoted by $(Q_i)_i^2$ and the conjugacy class of Q_i in Λ_i^*/Λ_i is μ_i. $\langle\Gamma_1,\Gamma_2\rangle$ can be written as an innerproduct of 2 vectors in $\Lambda_1\oplus\Lambda_2\otimes\mathbb{R}$. Define to this end the unit vector $\mathscr{P}=\frac{(-P_2,P_1)}{\sqrt{PP_1P_2}}\in\Lambda_1\oplus\Lambda_2\otimes\mathbb{R}$, then $(Q_1,Q_2)\cdot\mathscr{P}=Q\cdot\mathscr{P}=\langle\Gamma_1,\Gamma_2\rangle/\sqrt{PP_1P_2}$. In the Appendix, $\mathscr{I}(Q_1,Q_2;t)$ is also written as an innerproduct.

Since the wall is independent of the D0-brane charge, the index $\Omega(P,Q,Q_0;t)$ jumps irrespective of the D0-brane charge. For the partition function, we only want to keep track of the magnetic charge of the two constituents and sum over all the electric charge. Therefore, the contribution to the index $\Omega(P,Q,Q_0;t)$ from bound states of constituents whose D4-brane charges are P_1 and P_2 includes a sum over the D0- and D2-brane charge:

$$\begin{aligned}\Omega_{P_1\leftrightarrow P_2}(P,Q,Q_0;t) = &\sum_{(Q_1,Q_{0,1})+(Q_2,Q_{0,2})=(Q,Q_0)}\frac{1}{2}\left(\mathrm{sgn}(\mathscr{I}(Q_1,Q_2;t))-\mathrm{sgn}(\langle\Gamma_1,\Gamma_2\rangle)\right)\\ &\times(-1)^{\langle\Gamma_1,\Gamma_2\rangle}(P_1\cdot Q_2-P_2\cdot Q_1)\,\Omega(P_1,Q_1,Q_{0,1})\Omega(P_2,Q_2,Q_{0,2}).\end{aligned}$$

The generating function of $\Omega_{P_1\leftrightarrow P_2}(P,Q,Q_0;t)$ analogous to (3.4) is $h_{P_1\leftrightarrow P_2,Q-\frac{1}{2}P}(\tau)=\sum_{Q_0}\Omega_{P_1\leftrightarrow P_2}(P,Q,Q_0;t)\,q^{-Q_0+\frac{1}{2}Q^2}$. This can be expressed in terms of the vector-valued modular forms of the last section:

$$
\begin{aligned}
&h_{P_1\leftrightarrow P_2,Q-\frac{1}{2}P}(\tau)\,q^{-\frac{1}{2}Q^2}\\
&\quad=\sum_{\substack{(Q_1,Q_{0,1})+(Q_2,Q_{0,2})=(Q,Q_0)\\ Q_0}}(-1)^{P_1\cdot Q_2-P_2\cdot Q_1}\,(P_1\cdot Q_2-P_2\cdot Q_1)\,\Omega(\Gamma_1)\,\Omega(\Gamma_2)\\
&\qquad\times\frac{1}{2}\left(\mathrm{sgn}(\mathscr{I}(Q_1,Q_2;t))-\mathrm{sgn}(\langle\Gamma_1,\Gamma_2\rangle)\right)q^{Q_{\bar 0,1}+Q_{\bar 0,2}}\\
&\quad=\sum_{Q_1+Q_2=Q}\frac{1}{2}\left(\mathrm{sgn}(\mathscr{I}(Q_1,Q_2;t))-\mathrm{sgn}(\langle\Gamma_1,\Gamma_2\rangle)\right)(-1)^{P_1\cdot Q_2-P_2\cdot Q_1}\\
&\qquad\times(P_1\cdot Q_2-P_2\cdot Q_1)\,h_{P_1,\mu_1}(\tau)\,h_{P_2,\mu_2}(\tau)\,q^{-\frac{1}{2}(Q_1)_1^2-\frac{1}{2}(Q_2)_2^2}. \qquad (4.5)
\end{aligned}
$$

Note that the spectral flow symmetry is used here to write $h_{P_i,\mu_i}(\tau)$ instead of $h_{P_i,Q_i-P_i/2}(\tau)$. Equation (4.5) can be seen as a major generalization of a similar formula for rank 2 sheaves on a rational surface (27).

To obtain the full generating function, we have to multiply $\overline{h_{P_1\leftrightarrow P_2,Q-\frac{1}{2}P}(\tau)}$ by

$$
(-1)^{P\cdot Q}\,e\left(\tau(Q-B)_+^2/2+\bar\tau(Q-B)_-^2/2+C\cdot(Q-B/2)\right), \qquad (4.6)
$$

and sum over $Q\in\Lambda^*$. The various quadratic forms in the exponent combine to

$$
e\left(\tau(Q-B)_+^2/2+\bar\tau\left((Q-B)_{1\oplus 2}^2-(Q-B)_+^2\right)/2+C\cdot(Q-B/2)\right),
$$

where $Q_{1\oplus 2}^2=(Q_1)_1^2+(Q_2)_2^2$. See the Appendix for more explanation of the notation. The term $(Q-B)_{1\oplus 2}^2-2(Q-B)_+^2$, which multiplies $\pi\tau_2$ in the exponent is not negative definite, but has signature $(1,2b_2-1)$. An unrestricted sum over all $(Q_1,Q_2)\in\Lambda_1\oplus\Lambda_2$ is therefore clearly divergent. However, the presence of $\mathrm{sgn}(\mathscr{I}(Q_1,Q_2;t))-\mathrm{sgn}(\mathscr{P}\cdot Q)$ ensures that the function is convergent, which follows from Proposition 1 in the Appendix. Thus the stability condition implies that the quadratic form is negative definite, if evaluated for stable bound states. Performing the sum over Q, one obtains the generating series:

$$
\sum_{\mu_{1\oplus 2}\in\Lambda_{1\oplus 2}^*/\Lambda_{1\oplus 2}}\overline{h_{P_1,\mu_1}(\tau)}\,\overline{h_{P_2,\mu_2}(\tau)}\,\Psi_{\mu_{1\oplus 2}}(\tau,C,B), \qquad (4.7)
$$

where $\Lambda_{1\oplus 2}=\Lambda_1\oplus\Lambda_2$, $\mu_{1\oplus 2}=(\mu_1,\mu_2)\in\Lambda_{1\oplus 2}^*/\Lambda_{1\oplus 2}$ and

$$
\begin{aligned}
\Psi_{\mu_{1\oplus 2}}(\tau,C,B)=&\sum_{\substack{Q_1\in\Lambda_1+\mu_1+P_1/2\\ Q_2\in\Lambda_2+\mu_2+P_2/2}}(P_1\cdot Q_2-P_2\cdot Q_1)\,(-1)^{P_1\cdot Q_1+P_2\cdot Q_2}\\
&\times\frac{1}{2}\left(\mathrm{sgn}(\mathscr{I}(Q_1,Q_2;t))-\mathrm{sgn}(\mathscr{P}\cdot Q)\right)\\
&\times e\left(\tau(Q-B)_+^2/2+\bar\tau\left((Q-B)_{1\oplus 2}^2-(Q-B)_+^2\right)/2\right.\\
&\left.+C\cdot(Q-B/2)\right), \qquad (4.8)
\end{aligned}
$$

with $\mathscr{P}=\frac{(-P_2,P_1)}{\sqrt{PP_1P_2}}\in\Lambda_{1\oplus 2}\otimes\mathbb{R}$.

The test of S-duality is now reduced to testing modularity for (4.8). Since $\Psi_{\mu_{1\oplus 2}}(\tau,C,B)$ is not a sum over the total lattice $\Lambda_{1\oplus 2}$, it does not have the nice modular properties of the familiar theta functions. However, Ref. (46) explains that a real-analytic term can be added to a sum over a positive definite cone in an indefinite lattice with signature $(n-1,1)$, such that the resulting function transforms as a familiar theta function. Appendix A applies this technique to $\Psi_{\mu_{1\oplus 2}}(\tau,C,B)$, and explains in detail how it can be completed to a function $\Psi^*_{\mu_{1\oplus 2}}(\tau,C,B)$, which transforms as a Siegel-Narain theta function.[7] The essential idea of this procedure is to make the replacement

$$\mathrm{sgn}(z) \quad \longrightarrow \quad 2\int_0^{\sqrt{2\tau_2}z} e^{-\pi u^2} du, \tag{4.9}$$

which interpolates monotonically and continuously between -1 at $z=-\infty$ and 1 at $z=+\infty$. It approaches $\mathrm{sgn}(z)$ in the limit $\tau_2\to\infty$. To complete $\Psi_{\mu_{1\oplus 2}}(\tau,C,B)$ to a modular function, one also needs to replace $z\,\mathrm{sgn}(z)$ by an appriopriate continuous function as explained in the Appendix. Indefinite theta functions are prominent in the work on mock modular forms (46); $\Psi_{\mu_{1\oplus 2}}(\tau,C,B)$ is therefore appropriately called a "mock Siegel-Narain theta function".

By replacing $\Psi_{\mu_{1\oplus 2}}(\tau,C,B)$ with $\Psi^*_{\mu_{1\oplus 2}}(\tau,C,B)$ in Eq. (4.7), we obtain our final proposal of the contribution of 2-center bound states $\mathcal{Z}_{P_1\leftrightarrow P_2}(\tau,C,t)$ to $\mathcal{Z}(\tau,C,t)$:

$$\mathcal{Z}_{P_1\leftrightarrow P_2}(\tau,C,t) = \sum_{\mu_{1\oplus 2}\in\Lambda^*_{1\oplus 2}/\Lambda_{1\oplus 2}} \overline{h_{P_1,\mu_1}(\tau)}\,\overline{h_{P_2,\mu_2}(\tau)}\,\Psi^*_{\mu_{1\oplus 2}}(\tau,C,B). \tag{4.10}$$

From the transformation properties of the three functions it follows that $\mathcal{Z}_{P_1\leftrightarrow P_2}(\tau,C,t)$ transforms precisely as the CFT partition function $\mathcal{Z}_{\mathrm{CFT}}(\tau,C,t)$ of the single constituent with D4-brane charge P_1+P_2 (3.3)! To see that the weight agrees, note that the weight of $\Psi^*_{\mu_{1\oplus 2}}(\tau,C,B)$ is $\frac{1}{2}(1,2b_2+1)=\frac{1}{2}(1,2b_2-1)+(0,1)$, where $\frac{1}{2}(1,2b_2-1)$ is due to the lattice sum and $(0,1)$ is due to the insertion of $P_1\cdot Q_2-P_2\cdot Q_1$. Combining this with $2\cdot(0,-\frac{1}{2}b_2-1)$ of the vector-valued modular forms $\overline{h_{P_i,\mu_i}(\tau)}$, one precisely finds the weight $(\frac{1}{2},-\frac{3}{2})$ for $\mathcal{Z}_{P_1\leftrightarrow P_2}(\tau,C,t)$. A crucial detail is the grading by $(-1)^{P\cdot Q}$: $(-1)^{(P_1+P_2)\cdot(Q_1+Q_2)+(P_1\cdot Q_2-P_2\cdot Q_1)}=(-1)^{P_1\cdot Q_1+P_2\cdot Q_2}$, such that $\Psi^*_{\mu_{1\oplus 2}}(\tau,C,B)$ does transform conjugately to $\overline{h_{P_1,\mu_1}(\tau)}\,\overline{h_{P_2,\mu_2}(\tau)}$. Moreover, as was already mentioned above, coexistence of convergence and modularity is essentially a unique consequence of the Ansatz. In particular, the fact that $\mathcal{P}$ is independent of the moduli and satisfies $\mathcal{P}\cdot(J,J)=\mathcal{P}\cdot(B,B)=0$ is essential. We thus observe that all factors in (4.1) combine in a neat way such that $\mathcal{Z}_{P_1\leftrightarrow P_2}(\tau,C,t)$ has the same modular properties as $\mathcal{Z}_{P_1+P_2}(\tau,C,t)$.

One could of course object to correcting the partition function by hand and argue that an anomaly appeared for S-duality. However, the correcting factor could also arise automatically in a more physical derivation, for example by perturbative contributions. It is also not so surprising that corrections to the Fourier expansion (3.2) are necessary, since it was derived by assuming that the charges are finite and $J\to\infty$, which is clearly not the case everywhere in the Hilbert space. Note that a physical derivation might lead to a slightly different modular completion of the generating function, since one could always add a real-analytic function with

[7] Note that the Fourier expansion (3.2) is thus not modular.

the same transformation properties. This would however not change the crucial properties we have established.

Besides S-duality, there is another very appealing aspect in favor of the correction term. Equation (4.8) is not continuous as a function of the moduli B and J because of the terms $\mathrm{sgn}(\mathscr{I}(Q_1,Q_2;t))$. As discussed above, the correction term is essentially a replacement of the discontinuous functions $\mathrm{sgn}(z)$ and $z\,\mathrm{sgn}(z)$ by real analytic functions (which approach the original expression in the limit $|z| \to \infty$). The modular invariant partition function is therefore continuous in B and J. This might not be such a coincidence as it seems at first sight. Ref. (29) proposed a continuous and holomorphic generating function for Donaldson-Thomas invariants (or an extension thereof), which captures wall-crossing. Moreover, Ref. (23) describes that continuity of the metric g of the target manifold of a 3-dimensional sigma model, essentially implies the Kontsevich-Soibelman wall-crossing formula. Continuity of $\mathscr{Z}(\tau,C,t)$ is very intriguing from this perspective, and it would be interesting to investigate whether it plays here an as fundamental role as in these references.

The contribution of all 2-constituent BPS-states with primitive, ample charges is easily included in $\mathscr{Z}(\tau,C,t)$ by the sum $\sum_{\substack{P_1+P_2=P \\ \text{ample, primitive}}} \mathscr{Z}_{P_1\leftrightarrow P_2}(\tau,C,t)$. The above analyses give some evidence that modularity is also preserved if one of the charges is not ample.

5 Conclusion and Discussion

The consistency of wall-crossing with S-duality and electric-magnetic duality is tested by analyzing the BPS-spectrum of D4-D2-D0 branes on a compact Calabi-Yau 3-fold X. The stability of composite BPS-states with two primitive constituents is considered, in the large volume limit of the Kähler moduli space. The consistency of electric-magnetic duality with wall-crossing follows rather straightforwardly from the structure of the walls and the primitive wall-crossing formula. From the equations for the walls in the moduli space it can also be seen that wall-crossing is not compatible with the spectral flow symmetry, which appears in the microscopic description of a single D4-D2-D0 object by a CFT (32). S-duality is tested by the construction of a partition function (4.10) for two constituents, which captures the changes of the spectrum if walls of marginal stability are crossed. The essential building block is a "mock Siegel-Narain theta function", which might be of independent mathematical interest. The stability condition and the BPS-degeneracies combine in a very intricate way in order to preserve modularity, which is a confirmation of S-duality.

The results of this paper are applicable to various problems, for example those related to entropy enigmas (12). With these are meant BPS-configurations with multiple constituents, whose number of degeneracies is larger than the number of degeneracies of a single constituent with the same charge. Originally, the common thought was that wall-crossing would only have a subleading effect on the degeneracies. Ref. (1) has shown that enigmatic changes in the spectrum can also happen from D4-D2-D0 configurations with 2 constituents, which are considered in this paper. The present work shows that these enigmatic phenomena can be captured by modular invariant partition functions. This might prove useful in future studies on the entropy enigma. For example Eq. (4.10) shows that the leading entropy of two constituents (if their bound state exists) is $\pi\sqrt{\frac{2}{3}(P_1^3+P_2^3+c_2\cdot P)\left(Q_{\bar{0}}+\frac{1}{2}(Q_1)_1^2+\frac{1}{2}(Q_2)_2^2\right)}$ extremized with respect to Q_1 and Q_2, under the constraint $Q_1+Q_2=Q$. This should be compared with the single constituent entropy $\pi\sqrt{\frac{2}{3}(P^3+c_2\cdot P)(Q_{\bar{0}}+\frac{1}{2}Q^2)}$. Based on these equations, one can show the existence of enigmatic configurations, even in the regime $\sqrt{\frac{\hat{Q}_{\bar{0}}}{P^3}}\gg 1$, or large topological string coupling. This shows that $\mathscr{Z}_{\rm wc}(\tau,C,t)$ is not necessarily a small correction to $\mathscr{Z}_{\rm CFT}(\tau,C,t)$ in (1.1). A detailed analysis of the conditions for the first entropy to be larger than the second would be very instructive. This raises the question of the relation of the discussed partition functions in this paper and the OSV-conjecture, which relates the black hole partition function and the one of topological strings (37).

The D4-D2-D0 BPS-degeneracies are also related to mathematically defined invariants. In the large volume limit, the D4-D2-D0 index correspond to the Euler number (or a variant thereof) of the moduli space of coherent sheaves with support on the divisor of the Calabi-Yau. An explicit calculation of these Euler numbers is currently not feasible, but would be magnificent. It would for example provide a more rigorous test of modularity of the partition functions. A more tractable possibility for future work is to replace the index $\Omega(\Gamma;t)$ by a more refined quantity (14) by including the spin dependence $\Omega(\Gamma;t,y)=\mathrm{Tr}_{\mathscr{H}(\Gamma;t)}(-y)^{2J_3}$. This is not a protected quantity, but is nevertheless of interest. The corresponding partition function might still exhibit modular properties, and wall-crossing formulas do exist in the literature for $\Omega(\Gamma;t,y)$ in the context of surfaces (27; 45) and also physics (13). A generalization of Sect. 4 to include these refined invariants should therefore be possible. Another suggestion is to move away from the limit $J\to\infty$ by including finite size corrections. This would also leave the description of the BPS-states as coherent sheaves, and the relations with dualities probably become more intricate.

A limitation of this work is that it considers only primitive wall-crossing. One might continue in a similar fashion as Sect. 4 to construct partition functions for BPS-
configurations with more constituents, and test the compatibility of the semi-primitive wall-crossing formula (12) and S-duality in this way. Much more appealing would be a closed expression for the partition function, which does not sum over all possible decays. Such an expression might ultimately allow for a test of the generic Kontsevich-Soibelman wall-crossing formula with respect to S-duality, or even explain the KS-formula in $\mathcal{N}=2$ supergravity from physical considerations, as was done for $\mathcal{N}=2$ field theory (23). Although this paper took in some sense an opposite approach, some lessons might still be learned.

The requirement of the dualities implies non-trivial constraints for the indices and wall-crossing formulas. These do not seem constraining enough to deduce the KS-formula. For example, the appearance of mock modular forms instead of normal modular forms was a priori unknown. This can of course be seen as an anomaly for S-duality. On the other hand, it is really pretty close to modularity, and the functions can be made modular by a simple modification as explained in the Appendix. These modifications might appear in a more physical derivation of the partition function in order to preserve S-duality. The correction terms might be determined by a differential equation, similar to the holomorphic anomaly equation of topological strings (3). Proposition 5 gives the action of $\mathscr{D}$, defined in Sect. 3, on $\Psi^*_{\mu_{1\oplus 2}}(\tau,C,B)$. This shows that $\mathscr{D}\mathscr{Z}_{P_1\leftrightarrow P_2}(\tau,C,t)$ includes a term $\mathscr{Z}_{\mathrm{CFT},P_1}(\tau,C,B)\mathscr{Z}_{\mathrm{CFT},P_2}(\tau,C,B)$, which is suggestive and reminiscent of earlier work on holomorphic anomaly equations, see for example Ref. (34). Another consequence of the correction terms is that they make the function continuous as a function of the moduli, although it captures the changes of the spectrum under variations of the moduli. This is quite intriguing, since "continuity" was essential in the field theory derivation of the KS-formula in Ref. (23), more precisely the continuity of the metric of the target space of a 3-dimensional sigma model. The appearance of a continuous partition function in this paper suggests that continuity might be fundamental here too. More investigation is clearly necessary to find out to what extent continuity and the dualities can imply the generic wall-crossing formula (30) for BPS-invariants. Ref. (29) suggested earlier a continuous, holomorphic generating function for Donaldson-Thomas invariants, and its discussion resembles in some respects Ref. (23). However, $\mathscr{Z}_{P_1\leftrightarrow P_2}(\tau,C,t)$ does not seem to be holomorphic in t.

Note that the way $\mathscr{Z}_{P_1\leftrightarrow P_2}(\tau,C,t)$ captures stability is quite different from how the partition function of $\frac{1}{4}$-BPS states (or dyons) of $\mathscr{N}=4$ supergravity captures stability. That function captures wall-crossing in a very appealing way by poles (40) and a proper choice of the integration contour (8) to obtain Fourier coefficients. In this way, mock modular forms arise via meromorphic Jacobi forms (9).

Section 4 shows that the supergravity partition function is nowhere in moduli space equal to the CFT partition function (except for special cases like a Calabi-Yau with $b_2=1$). A natural question is: is the supergravity partition function related to the partition function of a lower dimensional theory, just as the spectrum of a single constituent is captured by the $\mathscr{N}=(4,0)$ SCFT? Ref. (5) (see also (6)) proposes that such a theory might be classically a 2-dimensional sigma model into the moduli space of supersymmetric divisors in the Calabi-Yau, whose "beta function does not vanish for Y [8] different from the attractor point and the Y undergo renormalization group flow till they reach the attractor point, an IR fixed point. Along the flow, the constituents of M5-M5 bound states decouple from each other; each of them has its own IR fixed point corresponding to an $\mathrm{AdS}_3\times S^2$." The structure of the partition function (4.10) shows the decoupled constituents. It is also in agreement with the suggestion that the theory is not a CFT, since the spectral flow symmetry is not present. On the other hand, $\mathscr{Z}_{\mathrm{sugra}}(\tau,C,t)$ does not equal $\mathscr{Z}_{\mathrm{CFT}}(\tau,C,t)$ at attractor points, which indicates that the microscopic theory (if it exists) is not a CFT, not even at these points. A better understanding of these issues is clearly desired. Another alternative for a microscopic theory is quiver quantum mechanics (10), which arises in the limit $g_{\mathrm{s}}\to 0$, and is known to capture bound states in 4 dimensions. A connection between this theory, the D4-D2-D0 bound states and their partition functions might lead to interesting insights.

An intriguing implication of the proposed function is wall-crossing as a function of the C-field for the BPS-states one obtains after S-duality. A D4-D2-D0 BPS-state becomes a D3-D1-D-1 instantonic BPS-state after performing a T-duality

[8] Y is the vector of normalized 5-dimensional Kähler moduli, which is proportional to J.

along the time circle. This does not yet change anything fundamental, stability of this configuration is still captured by B and J. However, S-duality transforms such a configuration to one with instanton D3-branes and fundamental string instantons. Moreover, B and C are interchanged, which implies that the degeneracies of these BPS-states jump as a function of C and J. This is quite interesting since the C-field is generically not considered as a stability parameter, and gives also evidence that B and C should be considered on a more equal footing. The K-theoretic description of the C-fields is however very different in nature than the description of the B-field.

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A Two Mock Siegel-Narain Theta Functions

This Appendix computes the transformation properties of the Siegel-Narain mock theta function which appear in Sect. 4. The proofs are similar to those given in (46). The dependence on the Grassmannian, which parametrizes 1-dimensional positive definite subspaces in the lattice Λ, however complicates the discussion. First, properties of a simpler mock Siegel-Narain theta function are analyzed before those of $\Psi^*_{\mu_{1\oplus 2}}(\tau,C,B)$.

Let Λ, Λ_1 and Λ_2 be three lattices with signature $(1,b_2-1)$. The quadratic forms of the lattices are determined by a cubic form d_{abc}: respectively $d_{abc}P^c$, $d_{abc}P_1^c$ and $d_{abc}P_2^c$. The vectors $P_{(i)}$ are characteristic vectors of the lattices and positive: $P_{(i)}^3>0$. They are related by $P=P_1+P_2$. The projection of a vector $x\in\Lambda\otimes\mathbb{R}$ on the positive definite subspace is determined by the vector $J\in\Lambda\otimes\mathbb{R}$: $x_+=(x\cdot J/P\cdot J^2)J$, $x_-=x-x_+$, and $x^2=x_+^2+x_-^2$. The positive definite combination $x_+^2-x_-^2$ is called the majorant associated to J. It is sufficient for this Appendix that J lies in the space

$$C_\Lambda:=\left\{J\in\Lambda\otimes\mathbb{R}: P_{(i)}\cdot J^2,\, P_{(i)}^2\cdot J>0,\, i=1,2\right\}.$$

J is thus positive in all three lattices.

The direct sum $\Lambda_1\oplus\Lambda_2$ is denoted by $\Lambda_{1\oplus 2}$ with quadratic form $Q_{1\oplus 2}^2=(Q_1)_1^2+(Q_2)_2^2$ for $Q=(Q_1,Q_2)\in\Lambda^*_{1\oplus 2}$. Vectors in $\Lambda_{1\oplus 2}$ are sometimes given the subscript $1\oplus 2$, and in Λ_i the subscript i. For example, $P_{1\oplus 2}=P_1+P_2\in\Lambda_{1\oplus 2}$. Similarly, $\mu_{1\oplus 2}=\mu_1+\mu_2\in\Lambda^*_{1\oplus 2}/\Lambda_{1\oplus 2}$, and $\mu=\mu_1+\mu_2\in\Lambda^*/\Lambda$ with $\mu_i\in\Lambda_i^*/\Lambda_i$. With a slight abuse of notation Q_+^2 denotes $((Q_1+Q_2)\cdot J)^2/P\cdot J^2$.

Define $\mathscr{I}(Q_1,Q_2;t)$ as in the main text by

$$\mathscr{I}(Q_1,Q_2;t)=\frac{P_1\cdot J^2(Q_2-P_2B)\cdot J-P_2\cdot J^2(Q_1-P_1B)\cdot J}{\sqrt{P_1\cdot J^2\,P_2\cdot J^2\,P\cdot J^2}}. \tag{A.1}$$

Define additionally the vector

$$\mathscr{P}=\frac{(-P_2,P_1)}{\sqrt{PP_1P_2}}\in\Lambda_{1\oplus 2}\otimes\mathbb{R}, \tag{A.2}$$

which satisfies $\mathscr{P}^2=1$.

Definition 1 *Let $t = B + iJ$, with $B \in \Lambda \otimes \mathbb{R}$, and $J \in C_\Lambda$. Then $\Phi^*_{\mu_{1\oplus 2}}(\tau, C, B)$ is defined by*:

$$\begin{aligned}\Phi^*_{\mu_{1\oplus 2}}(\tau,C,B) = &\frac{1}{2}\sum_{Q\in\Lambda_{1\oplus 2}+\mu_{1\oplus 2}+P_{1\oplus 2}/2} (-1)^{P_1\cdot Q_1+P_2\cdot Q_2}\\ &\times\left(E\left(\mathscr{I}(Q_1,Q_2;t)\sqrt{2\tau_2}\right)-E\left(\mathscr{P}\cdot Q\sqrt{2\tau_2}\right)\right)\\ &\times e\left(\tau(Q-B)_+^2/2+\bar\tau((Q-B)^2_{1\oplus 2}-(Q-B)_+^2)/2+(Q-B/2)\cdot C\right),\end{aligned} \tag{A.3}$$

with

$$E(z) = 2\int_0^z e^{-\pi u^2}du = \operatorname{sgn}(z)\left(1-\beta(z^2)\right),$$

where

$$\beta(x) = \int_x^\infty u^{-\frac{1}{2}}\, e^{-\pi u}\, du, \qquad x\in\mathbb{R}_{\geq 0}.$$

The moduli in the exponent of (A.3) *are determined by t. The " * " of $\Phi^*_{\mu_{1\oplus 2}}(\tau,C,B)$ distinguishes this function from $\Phi_{\mu_{1\oplus 2}}(\tau,C,B)$, which would be defined by replacing $E(z)$ by $\operatorname{sgn}(z)$ in the definition.*

Proposition 1 *$\Phi^*_{\mu_{1\oplus 2}}(\tau,C,B)$ is convergent for $J\in C_\Lambda$ and $B, C\in\Lambda\otimes\mathbb{R}$.*

Proof First consider the case $B = C = 0$. The term which multiplies τ_2 in the exponent, and thus determines the absolute value of the exponential is

$$Q_J^2 := Q_{1\oplus 2}^2 - 2\frac{((Q_1+Q_2)\cdot J)^2}{P\cdot J^2} = Q^2_{1\oplus 2} - 2Q_+^2. \tag{A.4}$$

The signature of this quadratic form is $(1, 2b_2-1)$ which is problematic for convergence.

To show convergence, note that $0 \leq \beta(x) \leq e^{-\pi x}$ for all $\mathbb{R}_{\geq 0}$ and that therefore the terms involving $\beta(x)$ in (A.3) are convergent. Consider next the terms with $\operatorname{sgn}(\mathscr{P}\cdot Q) - \operatorname{sgn}(\mathscr{I}(Q_1,Q_2;iJ))$. There are essentially two possibilities: $\operatorname{sgn}(\mathscr{P}\cdot Q)\operatorname{sgn}(\mathscr{I}(Q_1,Q_2;iJ)) < 0$ or > 0. Define the vector

$$s(J) = \frac{(-P_2\cdot J^2 J, P_1\cdot J^2 J)}{\sqrt{P_1\cdot J^2\, P_2\cdot J^2\, P\cdot J^2}} \in \Lambda_{1\oplus 2}\otimes\mathbb{R},$$

such that $Q\cdot s(J) = \mathscr{I}(Q_1,Q_2;iJ)$ and $s(J)^2 = 1$.

One can show that $\mathscr{P}\cdot s(J) = \sqrt{\frac{P\cdot J^2\,(P_1P_2J)^2}{PP_1P_2\,P_1\cdot J^2\,P_2\cdot J^2}} > 0$ and $\mathscr{P}_+ = s(J)_+ = 0$. The space $\operatorname{span}(\mathscr{P}, s(J))$ has signature $(1,1)$ in $\Lambda_{1\oplus 2}$ with inner product Q_J^2. Therefore

$$\begin{vmatrix} 1 & \mathscr{P}\cdot s(J)\\ \mathscr{P}\cdot s(J) & 1\end{vmatrix} = 1-(\mathscr{P}\cdot s(J))^2 < 0.$$

Take now a vector $Q\in\Lambda_{1\oplus 2}$, which is linearly independent of $\mathscr{P}$ and $s(J)$, then $\operatorname{span}(Q,\mathscr{P},s(J))$ is a space with signature $(1,2)$. Therefore,

$$\begin{vmatrix} Q_J^2 & Q\cdot\mathscr{P} & Q\cdot s(J)\\ Q\cdot\mathscr{P} & 1 & \mathscr{P}\cdot s(J)\\ Q\cdot s(J) & \mathscr{P}\cdot s(J) & 1\end{vmatrix} > 0.$$

From this follows directly

$$Q_J^2 + \frac{2\,\mathscr{P}\cdot s(J)}{1-(\mathscr{P}\cdot s(J))^2}\,Q\cdot\mathscr{P}\,Q\cdot s(J) < \frac{(Q\cdot\mathscr{P})^2+(Q\cdot s(J))^2}{1-(\mathscr{P}\cdot s(J))^2} < 0. \tag{A.5}$$

Therefore, if $\mathrm{sgn}(\mathscr{P}\cdot Q)\,\mathrm{sgn}(\mathscr{I}(Q_1,Q_2;iJ))<0$ then $Q_J^2<0$. If Q is a linear combination of $\mathscr{P}$ and $s(J)$, the determinant is zero. From this follows that $Q_J^2=0$ only for $Q=0$, and otherwise $Q_J^2<0$. The sum for $\mathrm{sgn}(Q\cdot\mathscr{P})\,\mathrm{sgn}(Q\cdot J)<0$ is therefore convergent.

What is left is the case >0. Then all the terms vanish identically, and therefore the whole sum is convergent. Inclusion of B and C does not alter the final conclusion.

Proposition 2 *$\Phi^*_{\mu_{1\oplus2}}(\tau,C,B)$ transforms under the generators S and T of $SL(2,\mathbb{Z})$ as:*

$$\begin{aligned}
S:\quad \Phi^*_{\mu_{1\oplus2}}(-1/\tau,-B,C) &= -\frac{i(-i\tau)^{1/2}(i\bar\tau)^{b_2-1/2}}{\sqrt{|\Lambda_1^*/\Lambda_1||\Lambda_2^*/\Lambda_2|}}\,e(-P^2_{1\oplus2}/4)\\
&\quad\times\sum_{\nu_{1\oplus2}\in\Lambda^*_{1\oplus2}/\Lambda_{1\oplus2}} e(-\mu_{1\oplus2}\cdot\nu_{1\oplus2})\,\Phi^*_{\nu_{1\oplus2}}(\tau,C,B),\\
T:\Phi^*_{\mu_{1\oplus2}}(\tau+1,B+C,B) &= e((\mu_{1\oplus2}+P_{1\oplus2}/2)^2_{1\oplus2}/2)\,\Phi^*_{\mu_{1\oplus2}}(\tau,C,B),
\end{aligned}$$

Proof The S-transformation is proven using $\sum_{k\in\Lambda}f(k)=\sum_{k\in\Lambda^*}\hat f(k)$, with $\hat f(k)$ the Fourier transform of $f(k)$. Therefore, one needs to determine the following Fourier transform:

$$\begin{aligned}
&\int_{\Lambda_{1\oplus2}\otimes\mathbb{R}} d^{2b_2}x\,E\left(\mathscr{I}(x_1,x_2;iJ)\sqrt{2\mathrm{Im}(-1/\tau)}\right)\\
&\quad\times\exp\left(\pi i\mathrm{Re}(-1/\bar\tau)x^2_{1\oplus2}+\pi\mathrm{Im}(-1/\bar\tau)(x^2_{1\oplus2}-2x_+^2)+2\pi i x\cdot y\right)\\
&=\int_{\Lambda_{1\oplus2}\otimes\mathbb{R}} d^{2b_2}x\,E\left(\mathscr{I}(x_1,x_2;iJ)\sqrt{2\mathrm{Im}(-1/\tau)}\right)\\
&\quad\times e\left(-x_+^2/2\tau-(x^2_{1\oplus2}-x_+^2)/2\bar\tau+x\cdot y\right),
\end{aligned}\tag{A.6}$$

and the one with $\mathscr{I}(x_1,x_2;iJ)$ replaced by $Q\cdot\mathscr{P}$. The following concentrates on the case with $\mathscr{I}(x_1,x_2;iJ)$, the derivation for $Q\cdot\mathscr{P}$ is completely analogous.

Let $Q\cdot s(J)=\mathscr{I}(Q_1,Q_2;iJ)$ as in Proposition 1, then the following definite quadratic forms can be defined:

$$Q^2_{1\oplus2+}=Q_+^2+(Q\cdot s(J))^2,\qquad Q^2_{1\oplus2-}=Q^2_{1\oplus2}-Q^2_{1\oplus2+},$$

since $(J,J)\cdot s(J)=0$. Using these quadratic forms, we write

$$e\left(-x_+^2/2\tau-(x^2_{1\oplus2}-x_+^2)/2\bar\tau\right)=e\left(-x_+^2/2\tau-(x\cdot s(J))^2/2\bar\tau-x^2_{1\oplus2-}/2\bar\tau\right).$$

The Fourier transform can be written in the form

$$\begin{aligned}
&=e\left(\tau y_+^2/2+\bar\tau\mathscr{I}(y_1,y_2;iJ)^2/2+\bar\tau y^2_{1\oplus2-}\right)\\
&\quad\times\int_{\Lambda_{1\oplus2}\otimes\mathbb{R}} d^{2b_2}x\,E\left(\mathscr{I}(x_1,x_2;iJ)\sqrt{2\mathrm{Im}(-1/\tau)}\right)\\
&\quad\times e\left(-(x-y\tau)_+^2/2\tau-\mathscr{I}(x_1-y_1\bar\tau,x_2-y_2\bar\tau;iJ)^2/2\bar\tau-(x-y\bar\tau)^2_{1\oplus2-}/2\bar\tau\right).
\end{aligned}$$

To proceed, one calculates the derivative of the integral

$$\begin{aligned}
&\frac{\partial}{\partial\mathscr{I}(y_1,y_2;iJ)}\int_{\Lambda_{1\oplus2}\otimes\mathbb{R}} d^{2b_2}x\,E\left(\mathscr{I}(x_1,x_2;iJ)\sqrt{2\mathrm{Im}(-1/\tau)}\right)\\
&\quad\times e\left(-(x-y\tau)_+^2/2\tau-\mathscr{I}(x_1-y_1\bar\tau,x_2-y_2\bar\tau;iJ)^2/2\bar\tau-(x-y\bar\tau)^2_{1\oplus2-}/2\bar\tau\right)\\
&=-\frac{i(-i\tau)^{1/2}(i\bar\tau)^{b_2-1/2}}{\sqrt{|\Lambda_1^*/\Lambda_1||\Lambda_2^*/\Lambda_2|}}\,\frac{\partial E\left(\mathscr{I}(y_1,y_2;iJ)\sqrt{2\tau_2}\right)}{\partial\mathscr{I}(y_1,y_2;iJ)}.
\end{aligned}$$

This is shown by replacing the derivative by $-\bar\tau\partial_{\mathscr{I}(x_1,x_2;iJ)}$, acting only on the exponent, and performing a partial integration. The equality is then easily established. Since (A.6) is an odd function of y, the integration constant is 0. Therefore (A.6) is equal to

$$-\frac{i(-i\tau)^{1/2}(i\bar{\tau})^{b_2-1/2}}{\sqrt{|\Lambda_1^*/\Lambda_1||\Lambda_2^*/\Lambda_2|}}E\left(\mathscr{I}(y_1,y_2;iJ)\sqrt{2\tau_2}\right)$$
$$\times e\left(\tau y_+^2/2+\bar{\tau}\mathscr{I}(Q_1,Q_2;iJ)^2/2+\bar{\tau}y_{1\oplus 2-}^2\right). \tag{A.7}$$

Using the standard techniques to include B- and C-field dependence etc., one finds the posed transformation law. Note that $\mathscr{P}\cdot(Q_1-BP_1,Q_2-BP_2)=\mathscr{P}\cdot(Q_1,Q_2)=\mathscr{P}\cdot Q$. The proof of the T-transformation is standard. □

Proposition 3 *Define* $\mathscr{D}=\partial_\tau+\frac{i}{4\pi}\partial_{C_+}^2+\frac{1}{2}B_+\cdot\partial_{C_+}-\frac{1}{4}\pi i B_+^2$, *then*

$$\tau_2^{1/2}\,\mathscr{D}\Phi_{\mu_{1\oplus 2}}(\tau,C,B)$$

is a modular form of weight $(2,b_2-1)$.

Proof The action of $\mathscr{D}$ on the exponents vanishes, and therefore only the derivative to τ on the functions $E(z\sqrt{2\tau_2})$ remains. The proposition follows easily from here. □

Definition 2 *With the same input as for Definition 1*:

$$\Psi_{\mu_{1\oplus 2}}^*(\tau,C,B)$$
$$=\frac{1}{2\pi\sqrt{2\tau_2}}\left(\sqrt{\frac{P\cdot J^2\,(P_1P_2J)^2}{P_1\cdot J^2\,P_2\cdot J^2}}\Theta_{\mu_1}(\tau,C,B)\,\Theta_{\mu_2}(\tau,C,B)\right.$$
$$\left.-\sqrt{PP_1P_2}\,\Theta_{\mu_{1\oplus 2}}(\tau,C,B,\mathscr{P})\right)$$
$$+\frac{1}{2}\sum_{Q\in\Lambda_{1\oplus 2}+\mu_{1\oplus 2}+P_{1\oplus 2}/2}(-1)^{P_1\cdot Q_1+P_2\cdot Q_2}\,(P_1\cdot Q_2-P_2\cdot Q_1)$$
$$\times\left(E\left(\mathscr{I}(Q_1,Q_2;t)\sqrt{2\tau_2}\right)-E\left(\mathscr{P}\cdot Q\sqrt{2\tau_2}\right)\right)$$
$$\times e\left(\tau(Q-B)_+^2/2+\bar{\tau}((Q-B)_{1\oplus 2}^2-(Q-B)_+^2)/2+(Q-B/2)\cdot C\right) \tag{A.8}$$

with $\Theta_{\mu_i}(\tau,C,B)$ *as defined by Eq.* (3.6)*, summing over* Λ_i. $\Theta_{\mu_{1\oplus 2}}(\tau,C,B,\mathscr{P})$ *is defined by*

$$\Theta_{\mu_{1\oplus 2}}(\tau,C,B,\mathscr{P})=\sum_{Q\in\Lambda_{1\oplus 2}+P_{1\oplus 2}/2+\mu_{1\oplus 2}}(-1)^{P_{1\oplus 2}\cdot Q}$$
$$\times e\left(\tau(Q-B)_+^2/2+\tau(\mathscr{P}\cdot Q)^2/2+\bar{\tau}(Q-B)_{1\oplus 2-}^2/2+C\cdot(Q-B/2)\right).$$

In the limit $\tau_2\to\infty$, $\Psi_{\mu_{1\oplus 2}}^*(\tau,C,B)$ *approaches* $\Psi_{\mu_{1\oplus 2}}(\tau,C,B)$*, which is defined in Eq.* (4.8)*. This series is convergent because* $\Phi_{\mu_{1\oplus 2}}^*(\tau,C,B)$ *is convergent.*

Proposition 4 $\Psi_{\mu_{1\oplus 2}}^*(\tau,C,B)$ *transforms under the generators* S *and* T *of* $SL(2,\mathbb{Z})$ *as*:

$$S:\Psi_{\mu_{1\oplus 2}}^*(-1/\tau,-B,C)=-\frac{(-i\tau)^{1/2}(i\bar{\tau})^{b_2+1/2}}{\sqrt{|\Lambda_1^*/\Lambda_1||\Lambda_2^*/\Lambda_2|}}e(-P_{1\oplus 2}^2/4)$$
$$\times\sum_{\nu_{1\oplus 2}\in\Lambda_{1\oplus 2}^*/\Lambda_{1\oplus 2}}e(-\mu_{1\oplus 2}\cdot\nu_{1\oplus 2})\,\Psi_{\nu_{1\oplus 2}}^*(\tau,C,B),$$
$$T:\Psi_{\mu_{1\oplus 2}}^*(\tau+1,B+C,B)=e((\mu_{1\oplus 2}+P_{1\oplus 2}/2)_{1\oplus 2}^2/2)\,\Psi_{\mu_{1\oplus 2}}^*(\tau,C,B).$$

Proof This is a continuation of the proof of Proposition 2. The following Fourier transform needs to be calculated:

$$\int_{\Lambda_{1\oplus 2}\otimes\mathbb{R}} d^{2b_2}x\,(P_1\cdot x_2 - P_2\cdot x_1)\,E\left(\mathscr{I}(x_1,x_2;iJ)\sqrt{2\mathrm{Im}(-1/\tau)}\right)$$
$$\times e\left(-(x_{1\oplus 2}^2 - x_+^2)/2\bar\tau - x_+^2/2\tau + x\cdot y\right), \tag{A.9}$$

and the one with $\mathscr{I}(x_1,x_2;iJ)$ replaced by $\mathscr{P}\cdot Q$. We again concentrate on the case with $\mathscr{I}(x_1,x_2;iJ)$. It is instructive to write $P_1\cdot x_2 - P_2\cdot x_1$ as $(-P_2,P_1)\cdot x^{\mathrm{T}}$ with $x=(x_1,x_2)$. The inner product $(-P_2,P_1)\cdot x_+$ with $x_+ = x\cdot JJ/P\cdot J^2$ vanishes. Therefore,

$$\begin{aligned}(-P_2,P_1)\cdot x^{\mathrm{T}} &= (-P_2,P_1)\cdot x_-^{\mathrm{T}} + (-P_2,P_1)\cdot s(J)^{\mathrm{T}}x\cdot s(J)\\ &= (-P_2,P_1)\cdot x_-^{\mathrm{T}} + \sqrt{\tfrac{P\cdot J^2\,(P_1P_2J)^2}{P_1\cdot J^2\,P_2\cdot J^2}}\,\mathscr{I}(x_1,x_2;iJ),\end{aligned}$$

with $s(J)\in\Lambda_{1\oplus 2}$ as in the proof of Proposition 1. This shows that the factor $P_1\cdot x_2 - P_2\cdot x_1$ can be replaced by $(2\pi i)^{-1}\left((-P_2,P_1)\cdot\partial_{y_-} + \sqrt{\frac{P\cdot J^2\,(P_1P_2J)^2}{P_1\cdot J^2\,P_2\cdot J^2}}\,\partial_{\mathscr{I}(y_1,y_2;iJ)}\right)$. Using Proposition 2, one finds that (A.9) equals

$$-\frac{(-i\tau)^{1/2}(i\bar\tau)^{b_2+1/2}}{\sqrt{|\Lambda_1^*/\Lambda_1||\Lambda_2^*/\Lambda_2|}}\left[(P_1\cdot y_2 - P_2\cdot y_1)\,E\left(\mathscr{I}(y_1,y_2;iJ)\sqrt{2\tau_2}\right)e\left(\tau y_+^2/2 + \bar\tau(y_{1\oplus 2}^2 - y_+^2)/2\right)\right.$$
$$\left. + \frac{\sqrt{2\tau_2}}{\pi i\bar\tau}\sqrt{\frac{P\cdot J^2\,(P_1P_2J)^2}{P_1\cdot J^2\,P_2\cdot J^2}}\;e\left(\tau y_+^2/2 + \tau\mathscr{I}(y_1,y_2;iJ)^2/2 + \bar\tau y_{1\oplus 2-}^2/2\right)\right].$$

Clearly, this Fourier transform leads to a shift in the modular transformation properties. This can be cured if one recalls the transformation properties of the second Eisenstein series: $E_2(-1/\tau) = \tau^2\left(E_2(\tau) - \frac{6i}{\pi\tau}\right)$. A correction term can be added to $E_2(\tau)$: $E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi\tau_2}$ which transforms as a modular form of weight 2. This leads precisely to the term with theta functions in the definition. This means that the discontinuous function $z\,\mathrm{sgn}(z)$, which appears in (4.8), is replaced in $\Psi^*_{\mu_{1\oplus 2}}(\tau,C,B)$ by the real analytic function $F(z) = zE(z) + \frac{1}{\pi}e^{-\pi z^2}$. $F(z)$ approaches $z\,\mathrm{sgn}(z)$ for $|z|\to\infty$. □

Proposition 5 *With $\mathscr{D}$ as in Proposition 3,*

$$\begin{aligned}\mathscr{D}\Psi^*_{\mu_{1\oplus 2}}(\tau,C,B) = &-\frac{i}{2\sqrt{2\tau_2}}\sqrt{\frac{P\cdot J^2\,(P_1P_2J)^2}{P_1\cdot J^2\,P_2\cdot J^2}}\,\Upsilon_{\mu_{1\oplus 2}}(\tau,C,B)\\ &+\frac{i}{4\pi(2\tau_2)^{3/2}}\left(\Theta_{\mu_1}(\tau,C,B)\Theta_{\mu_2}(\tau,C,B) - \Theta_{\mu_{1\oplus 2}}(\tau,C,B,\mathscr{P})\right),\end{aligned}$$

with

$$\begin{aligned}\Upsilon_{\mu_{1\oplus 2}}(\tau,C,B) = &\sum_{Q\in\Lambda_{1\oplus 2}+P_{1\oplus 2}/2+\mu_{1\oplus 2}} (-1)^{P_{1\oplus 2}\cdot Q}\,(-P_2,P_1)\cdot Q_-\,\mathscr{I}(Q_1,Q_2;t)\\ &\times e\left(\tau(Q-B)_+^2/2 + \tau\mathscr{I}(Q_1,Q_2;t)^2/2 + \bar\tau(Q-B)_{1\oplus 2-}^2/2 + C\cdot(Q-B/2)\right).\end{aligned}$$

Proof The proof is straightforward. Note that $\Theta_{\mu_i}(\tau,C,B)$ and $\Upsilon_{\mu_{1\oplus 2}}(\tau,C,B)$ are not mock modular forms. The weights are respectively $(1,b_2-1)$ and $(2,b_2)$, such that the weight of $\mathscr{D}\Psi^*_{\mu_{1\oplus 2}}(\tau,C,B)$ is $(5/2,(2b_2+1)/2)$ as expected.

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