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A Test of Gauge Invariant Canonical Angular Momentum in Landau Level Problem

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We investigate the orbital angular momentum (OAM) in Landau level problem, namely, the quantum mechanical system for an electron in a constant and uniform magnetic field. This system has two gauge invariant OAMs, the mechanical one and the pseudo one. The pseudo OAM can be regarded as a gauge invariant extension of the canonical OAM which is recently proposed in the nucleon spin problem. We take this system as a test ground of the gauge invariant canonical OAM. We show that the mechanical OAM describes the electron's physical OAM, whereas the pseudo OAM does not correspond to an observable at least in this system.

KEYWORDS: Angular momentum, Gauge theory, Landau level, Nucleon spin

1. Introduction

The decomposition of nucleon spin into the spin and orbital angular momentum (OAM) of quarks and gluons in keeping with the gauge-invariance is one of old problems in quantum chromodynamics (QCD). Currently, two types of the gauge invariant decomposition are accepted [1–4]. The first decomposition is the *mechanical* decomposition and the second one is *gauge invariant canonical* decomposition. These two decompositions seem to be acceptable, because they are both gauge invariant. However, there still remains unsatisfactory points in the physical interpretation of the two decompositions. Which OAM can describe the physical OAM ? Can both of the two OAMs describe the same quantity simultaneously ? It is important for us to answer these questions.

We choose a well-known system to test these two OAMs and see which decompositions can describe the physical OAM. In particular, we consider the Landau level problem [5] in quantum electrodynamics (QED), namely, the quantum mechanics for an electron under the presence of a constant and uniform magnetic field. The advantages of this system are: 1) analytically known solutions and 2) simpler gauge theory than QCD. Hence this system is a good test ground for the property of the gauge invariant extension of the canonical OAM.

In this system, there are two gauge invariant OAMs, namely the mechanical OAM which is manifestly gauge invariant and the "pseudo" OAM which is formally gauge invariant. This pseudo OAM can be regarded as a gauge invariant extension of the canonical OAM, because it is gauge invariant and it reduces to the conventional canonical OAM in taking a certain gauge. If there are two gauge invariant OAMs in the system, which OAM describes the electron's physical OAM ? The analysis of the pseudo OAM in the system will tell a flesh insight into the gauge invariant canonical OAM which is recently proposed in the nucleon spin decomposition. This is the purpose of our study.

This paper is organised in the following. First, we briefly review some basics of the Landau level system. In particular, we focus on two gauge invariant OAMs and compare the expectation values of the two gauge invariant OAMs in Sec. 2. We summarise this paper in Sec. 3.

2. Orbital angular momenta in Landau level problem

The eigen state of an electron in a constant and uniform magnetic field is described by the Hamiltonian H:

$$H\psi(x,y,z) = E\psi(x,y,z), \qquad H = \frac{\mathbf{\Pi}^2}{2m_e}, \qquad \mathbf{\Pi} = \mathbf{p} + \mathbf{eA}, \tag{1}$$

where E, m_e, and $-e$ $(e > 0)$ are the eigen energy, mass, and electric charge of the electron respectively. The vector potential $\mathbf{A}$ gives the magnetic field through the relation $\mathbf{\nabla} \times \mathbf{A} = \mathbf{B}$ and we choose the direction of the magnetic field as z-axis, namely, $\mathbf{B} \equiv (0, 0, B)$ in this paper. As is commonly known, the canonical momentum operator $\mathbf{p} = -i\mathbf{\nabla}$ is gauge-variant, whereas the mechanical (kinetic) momentum operator $\mathbf{\Pi}$ is gauge-invariant.

In the conventional approach written in textbooks, one has to fix a gauge to solve Eq. (1). The most popular choices are:

$$\text{1st Landau gauge} : \mathbf{A} = \mathbf{A}_{\mathrm{L}_1} = (-By, 0, 0)\,, \tag{2}$$

$$\text{2nd Landau gauge} : \mathbf{A} = \mathbf{A}_{\mathrm{L}_2} = (0, +Bx, 0)\,, \tag{3}$$

$$\text{symmetric gauge} : \mathbf{A} = \mathbf{A}_{\mathrm{S}} = \left(-\frac{By}{2}, +\frac{Bx}{2}, 0\right), \tag{4}$$

where the first and second Landau gauges are essentially same to each other and hence we simply call the second Landau gauge the Landau gauge in this paper. The solution of Eq. (1) in the Landau gauge is given in [6]:

$$\psi^{(\mathrm{L}_2)}_{n,k_y}(x,y) = \frac{N_n}{\sqrt{2\pi}} e^{ik_y y} e^{-\frac{\xi^2}{2}} H_n(\xi)\,, \quad \xi = \frac{x - x_0}{l_B}, \quad N_n = \frac{1}{\sqrt{\sqrt{\pi} 2^n n! l_B}}, \tag{5}$$

where $x_0 = -l_B^2\, k_y$ is the centre of the oscillation in x-direction, $l_B = 1/\sqrt{eB}$ is the magnetic length, and H_n is the n-th order Hermite polynomial. The solution of Eq. (1) in the symmetric gauge is given in [6]:

$$\psi^{(\mathrm{S})}_{n,m}(r,\phi) = \frac{N_{n,m}}{\sqrt{2\pi}} e^{im\phi} \rho^{\frac{|m|}{2}} e^{-\frac{\rho}{2}} L^{|m|}_{n-\frac{|m|+m}{2}}(\rho)\,, \quad \rho = \frac{r^2}{2l_B^2}, \quad N_{n,m} = (-1)^{n+\frac{|m|+m}{2}} \frac{\sqrt{\left(n - \frac{|m|+m}{2}\right)!}}{\sqrt{\left(n + \frac{|m|-m}{2}\right)!}}, \tag{6}$$

where $m \leq n$ and L_n^m is the associated Laguerre polynomial and the polar coordinates (r, ϕ) are related to the Cartesian coordinates (x, y) through the relations, $x = r\cos\phi, y = r\sin\phi$. These two solutions satisfy the eigen equations:

$$p_y \psi^{(\mathrm{L}_2)}_{n,k_y}(x,y) = k_y \psi^{(\mathrm{L}_2)}_{n,k_y}, \qquad L_z^{\mathrm{can}} \psi^{(\mathrm{S})}_{n,m}(r,\phi) = m\psi^{(\mathrm{S})}_{n,m}(r,\phi), \tag{7}$$

$$H\psi^{(\mathrm{L}_2)}_{n,k_y}(x,y) = E_n \psi^{(\mathrm{L}_2)}_{n,k_y}(x,y), \qquad H\psi^{(\mathrm{S})}_{n,m}(r,\phi) = E_n \psi^{(\mathrm{S})}_{n,m}(r,\phi), \tag{8}$$

$$E_n = \omega\left(n + \frac{1}{2}\right), \qquad \omega = \frac{eB}{m_e}, \tag{9}$$

where $p_y = -i\partial/(\partial y)$ is the canonical momentum of y-direction, $L_z^{\mathrm{can}} = -i\partial/(\partial\phi)$ is the canonical OAM, and the energy level E_n is known as the Landau level [5]. Of course, this energy level E_n is gauge independent.

At first glance, it seems to be that one can simply compare expectation values of OAMs in the two different gauges by using these two wave functions, $\psi^{(\mathrm{L}_2)}_{n,k_y}(x,y)$ and $\psi^{(\mathrm{S})}_{n,m}(r,\phi)$. However, we showed in

Ref. [7] that the expectation values for the mechanical OAM in the two different gauges give different results, despite its gauge invariance. This is because of the degeneracies in the two gauges and hence the naive comparison of the expectation values for the mechanical OAMs by the conventional solutions does not make any sense. This degeneracy is related to the issue of gauge transformations. The gauge transformation of the wave functions between two different gauges in the Landau level system are discussed in Refs. [8, 9] and Ref. [7].

To overcome the difficulty mentioned above, we find out gauge-independent solutions based on the DeWitt's gauge invariant formalism of QED [10]. One of the property of the application of this formalism to the Landau level system is that one does not need to fix a gauge to solve Eq. (1), however one needs to fix a path which has nothing to do with the motion of the electron. It is not difficult to check that the following wave function is one solution of Eq. (1),

$$\psi^{(\mathrm{C_I})}_{n,m}(r,\phi) \quad = \quad e^{-ie\int_0^r dr' A_r(r',\phi)} \times \frac{N_{n,m}}{\sqrt{2\pi}} e^{im\phi} \rho^{\frac{|m|}{2}} e^{-\frac{\rho}{2}} L^{|m|}_{n-\frac{|m|+m}{2}}(\rho)\,, \tag{10}$$

where the superscript "$\mathrm{C_I}$" stands for the path discussed in Fig.2 of Ref. [7] and A_r is the radial component of the gauge field. The gauge degree of freedom in the above solution is not fixed yet and hence one can use this gauge degree of freedom to check gauge dependencies in the expectation values of OAMs.

There are two gauge invariant OAMs in this system. The first one is the mechanical OAM L_z^{mech} defined by

$$L_z^{\mathrm{mech}} \equiv [\mathbf{r} \times (\mathbf{p} + e\mathbf{A})]_z = L_z^{\mathrm{can}} + [\mathbf{r} \times e\mathbf{A}]_z\,. \tag{11}$$

The second one is the pseudo OAM L_z^{ps} defined by

$$L_z^{\mathrm{ps}} \equiv [\mathbf{r} \times (\mathbf{p} + e\mathbf{A})]_z - \frac{eB}{2} r^2 = L_z^{\mathrm{mech}} - \frac{eB}{2} r^2. \tag{12}$$

This pseudo OAM is discussed in Ref. [11] many years ago and in Ref. [12] recently for roles of the conserved operator in the Landau level system. This wave function in Eq. (10) is the eigen state of the pseudo OAM, i.e. $L_z^{\mathrm{ps}} \psi^{(\mathrm{C_I})}_{n,m}(r,\phi) = m\psi^{(\mathrm{C_I})}_{n,m}(r,\phi)$ in arbitrary gauges and this relation reduces to $L_z^{\mathrm{can}} \psi^{(\mathrm{C_I})}_{n,m}(r,\phi) = m\psi^{(\mathrm{C_I})}_{n,m}(r,\phi)$ in the symmetric gauge. One can regard the pseudo OAM as a gauge invariant extension of the canonical OAM, because it is formally gauge invariant and it reduces to the canonical OAM in a certain gauge.

We compare the expectation values of the mechanical OAM L_z^{mech} and the pseudo OAM L_z^{ps} by using the solution in Eq. (10) with changing gauges. The results are summarised in Table I and we include the expectation values of the canonical OAM for reference. Although we used the different path $\mathrm{C_{II}}$ and corresponding wave function $\psi^{(\mathrm{C_{II}})}_{n,m}(r,\phi)$ for the comparison of OAMs in Table 3 of Ref. [7], we use the wave function in Eq. (10) to calculate expectation values in order to show the path-dependence of the results in this paper.

Our results in Table I show: 1) the canonical OAM shows the gauge dependence in the Bawin-Burnel (BB) gauge (multi-valued) [7, 13] and 2) both the mechanical and pseudo OAMs give the gauge invariant results. These results are expressed by two quantum numbers n, and m. Which OAM describes the electron's physical OAM ? The expectation value of the mechanical OAM is obviously an observable, because the result is expressed by the energy quantum number n. On the other hand, the expectation values of the pseudo OAM and of the canonical OAM are expressed by another quantum number m. The expectation values of the pseudo OAM coincide with those of the canonical OAM in $\mathrm{L_1}$, $\mathrm{L_2}$, and S gauges. However, this quantum number m is not an observable in this system, because the Landau level has the infinite number of the degeneracy and each degeneracy is specified by this quantum number $m \leq n$ for a given n. In addition, this quantum number is merely related to the conserved quantity reflecting the rotational symmetry of the electron's wave function around the origin.

	L_1	L_2	S	BB
$\langle L_z^{\mathrm{can}} \rangle$	m	m	m	$2n+1$
$\langle L_z^{\mathrm{mech}} \rangle$	$2n+1$	$2n+1$	$2n+1$	$2n+1$
$\langle L_z^{\mathrm{ps}} \rangle$	m	m	m	m

Table I. Expectation values of the three OAMs in different gauges. The abbreviations, L_1, L_2, S, and BB, stand for the first Landau, the second Landau, the symmetric, and the BB (multi-valued) gauges specified by $(A_r, A_\phi) = (-Br\phi, 0)$, respectively.

3. Conclusion

We investigated the Landau level system as a test ground of the gauge invariant canonical OAM which is recently proposed in the nucleon spin problem. In this Landau level system, there are two gauge invariant OAMs, i.e. the mechanical OAM and pseudo OAM. We found that the pseudo OAM in this system has the similar property to the gauge invariant canonical OAM in the nucleon spin problem. We used the wave functions based on the DeWitt's gauge invariant formalism and compared expectation values for these two OAMs. The expectation value of the mechanical OAM is related to the electron's energy quantum number, whereas that of the pseudo OAM is related to another quantum number which has nothing to do with an observable. Hence, we concluded that the electron's physical OAM in the system is described by the mechanical OAM. The pseudo OAM regarded as a gauge invariant extension of the canonical OAM in this system does not correspond to an observable.

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