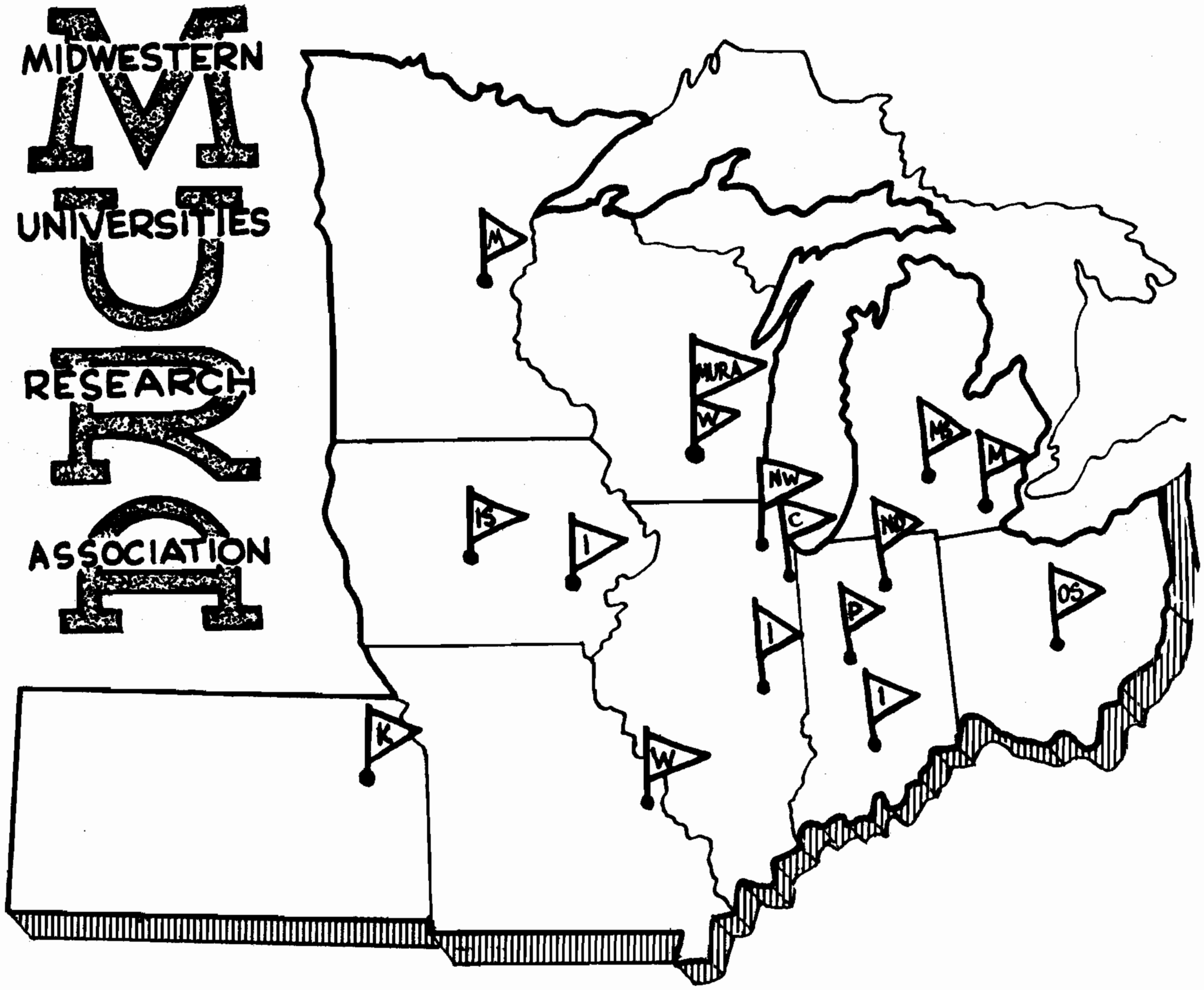

ANALYTICAL CALCULATION OF FFAG OPERATING POINTS AND ORBITS

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ABSTRACT

Among the reports on an analytic description of particle motion in FFAG accelerators,[1, 2, 3] the specific case of an integral structure machine is not treated in sufficient detail to yield quantitative conclusions starting from a set of general field coefficients. In parallel with computer studies, an analytical investigation of the unperturbed differential equations of a charged particle motion in a steady magnetic field has been made. A special technique is developed to yield the exact solution (in terms of power series) of the non-linear equation of motion in the median plane. Based on the results of the equilibrium orbits, betatron oscillation frequencies are obtained by the application of second-order perturbation theory of quantum mechanics. Analytical method has been programmed for the IBM 704 computer. A typical operating time of six minutes yields the betatron oscillation frequencies.

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I. EQUILIBRIUM ORBIT

The motion of a charged particle in the median plane of an accelerator in cylindrical coordinates[4] is

$$\frac{d}{d\theta} \frac{R'}{\sqrt{R^2 + R'^2}} - \frac{R}{\sqrt{R^2 + R'^2}} = \frac{eR}{pc} B_z \tag{1}$$

where B_z is the magnetic field in the plane and in general given by

$$B_z = - B_o \sum_n G_n (R) e^{inN\theta} {}^{*} \tag{2}$$

B_o is const.

G_n (R) is complex.

The solution of Eq. (1) is nearly the average radius of the equilibrium orbit. Let the solution be R

$$R = R_o e^{u(\theta)} \quad . \tag{3}$$

R_o is defined in such a way that $\int_o^{2\pi/N} u(\theta)\, d\theta = 0$. $u(\theta)$ is approximately equal to the deviation from the average radius of equilibrium orbit. We assume that the radial component of the particle's velocity in the equilibrium orbit shall be small compared with the longitudinal component, i.e.,

$$\left| \frac{1}{R} \frac{dR}{d\theta} \right| << 1 \tag{4}$$

or

$$\left| \frac{du}{d\theta} \right| << 1 \quad . \tag{5}$$

Let

$$\Theta = \tan^{-1} u' \qquad \text{where} \qquad u' = \frac{du}{d\theta} \quad . \tag{6}$$

Substituting Eqs. (2), (3), and (6) into Eq. (1) lead to

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*All sums whose limits are not given are to be taken to extend from $-\infty$ to ∞ .

$$\frac{d}{d\theta}\sin\Theta - \cos\Theta = -\lambda \sum_{n} e^{u}\, G_n\,(R\,(u))\, e^{inN\theta} \tag{7}$$

where

$$\lambda = \frac{e\, B_o\, R_o}{pc} \quad . \tag{8}$$

We introduce a new function

$$F_n\,(u) = e^{u}\, G_n\,(u)\,. \tag{9}$$

Then Eq. (7) becomes

$$\frac{d}{d\theta}\sin\Theta - \cos\Theta = -\lambda \sum_{n} F_n\,(u)\, e^{inN\theta} \quad . \tag{10}$$

The magnetic field is assumed not to vary much over the radial extent of the orbit. We assume that a Taylor expansion of the F_n (u)'s in powers of u exist: i.e.,

$$F_n\,(u) = F_n\,(o) + F_n'\,(o)\, u + F_n''\,(o)\,\frac{u^2}{2} + \cdots \quad . \tag{11}$$

For a normalized field, $F_o\,(o) = 1$, $F_n^{(j)} < 1$ when $n \neq 0$. Let the solution for the equilibrium orbit be represented as a Fourier series

$$u = \sum_{n} a_n\, e^{inN\theta} \quad . \tag{12}$$

Using the notation in Appendix A, the results are summarized in the following:

$$\sum_{n} F_n\,(u)\, e^{inN\theta} = \sum_{n,\,m} {}_{n-m}E_m\, e^{inN\theta} \tag{A12}$$

$$\sin\Theta = \sum_{n} S_n\, e^{inN\theta} \tag{A22}$$

$$\cos\Theta = \sum_{n} C_n\, e^{inN\theta} \tag{A-23}$$

where

$$ {}_{n}E_{m} = \sum_{j} \frac{F_n^{(j)} A_m^{j}(a)}{j\,!} \qquad j = 0,\ 1,\ 2,\ 3,\ \cdots \tag{A-6}$$

$$F_n^{(o)} = F_n(o), \quad F_n^{(1)} = F_n' = F_n'(o), \quad F_n^{(2)} = F_n''(o) \tag{A-2}$$

$$A_m^{j}(a) = \sum_{\substack{m_1 \\ m_1+m_2+\cdots+m_j = m}} \sum_{m_2} \cdots \sum_{m_j} a_{m_1} a_{m_2} \cdots a_{m_j} \tag{A-7}$$

$$S_n = \sum_{\bar{j}} (-1)^{(\bar{j}+3)/2} \frac{A_n^{\bar{j}}(T)}{\bar{j}\,!} \qquad \bar{j} = 1,\ 3,\ 5,\ \cdots \tag{A-24}$$

$$C_n = \sum_{\bar{\bar{j}}} (-1)^{\bar{\bar{j}}/2} \frac{A_n^{\bar{\bar{j}}}(T)}{\bar{\bar{j}}\,!} \qquad \bar{\bar{j}} = 0,\ 2,\ 4,\ \cdots \tag{A-25}$$

Therefore Eq. (10) becomes

$$\frac{d}{d\theta} \sum_n S_n e^{inN\theta} - \sum_n C_n e^{inN\theta} = -\lambda \sum_{n,m} {}_{n-m}E_m \, e^{inN\theta} \,. \tag{13}$$

The recurrence relation of Eq. (13) is

$$(inN)\, S_n - C_n = -\lambda \sum_m {}_{n-m}E_m \,. \tag{14}$$

The important relation Eq. (14) yields the exact solution of the equilibrium orbit. When $n = 0$,

$$C_o = \lambda \sum_m {}_{-m}E_m \,. \tag{15}$$

From Eqs. (A-30) and (A-6), Eq. (15) is equal to

$$1 - A_o^2(inNa) + \cdots = -\lambda \left\{ F_o^{(o)} + \sum_n F_n^{(1)} A_n^1(a) + \frac{1}{2!} \sum_n F_n^{(2)} A_n^2(a) + \cdots \right\} \tag{16}$$

Equation (16) gives the value of λ. Since $u' << 1$ and $F_n^{(j)} < F_o^{(o)}$ imply

$$\left| A_o^2(inNa) \right| << 1$$

$$\left| \sum_n F_{-n}^{(1)} A_n^1(a) \right| << F_o^{(o)}$$

$$\left| \sum_n F_{-n}^{(2)} A_n^2(a) \right| << F_o^{(o)} \qquad ,$$

we get

$$\lambda F_o^{(o)} = \left[1 - A_o^2\,(inNa) + \cdots\right] \Big/ \left\{1 + \frac{1}{F_o^{(o)}} \sum_n F_{-n}^{(1)} A_n^1\,(a) + \frac{1}{2!\,F_o^{(o)}} \sum_n F_{-n}^{(2)} A_n^2\,(a) + \cdots\right\}$$

$$= 1 - \frac{1}{F_o^{(o)}} \sum_n F_{-n}^{(1)} A_n^1\,(a) - A_o^2\,(inNa) - \frac{1}{2!\,F_o^{(o)}} \sum_n F_{-n}^{(2)} A_n^2\,(a) + \cdots .$$

Let

$$\lambda = \lambda^{(o)} + \lambda^{(1)} + \lambda^{(2)} + \cdots , \tag{17}$$

then

$$\lambda^{(o)} = 1/F_o^{(o)} \tag{18}$$

$$\lambda^{(1)} = -\frac{1}{\left(F_o^{(o)}\right)^2} \sum_n F_{-n}^{(1)} A_n^1\,(a) \tag{19}$$

$$\lambda^{(2)} = -\frac{1}{F_o^{(o)}} A_o^2\,(inNa) - \frac{1}{2!\,(F_o^{(o)})^2} \sum_n F_{-n}^{(2)} A_n^2\,(a) \tag{20}$$

For the normalized field, $\lambda^{(o)} = 1$. When n is any integer except o, the recurrence relation Eq. (14) gives

$$(inN)\left\{A_n^1\,(inNa) - \frac{5}{6} A_n^3\,(inNa) + \cdots\right\} - \left\{-A_n^2\,(inNa) + \frac{1}{3!} A_n^4\,(inNa) + \cdots\right\}$$

$$= -\lambda \left\{F_n^{(o)} + \sum_m F_{n-m}^{(1)} A_m^1\,(a) + \frac{1}{2!} \sum_m F_{n-m}^{(2)} A_m^2(a) + \cdots\right\} \tag{21}$$

or

$$n^2N^2 a_n + \sum_{\substack{n_1, n_2 \\ n_1 + n_2 = n}} N^2\, n_1\, a_{n_1} \cdot n_2\, a_{n_2} + \cdots = \lambda \left\{F_n^{(o)} + \sum_m F_{n-m}^{(1)} a_m + \frac{1}{2!} \sum_{n_1, n_2, n_3} F_{n_1}^{(2)} a_{n_2}\, a_{n_3} + \cdots\right.$$

$$n_1 + n_2 + n_3 = n \tag{22}$$

The coefficients a_n can be evaluated from Eq. (22). Let,

$$a_n = a_n^{(o)} + a_n^{(1)} + a_n^{(2)} + \cdots . \tag{23}$$

Then

$$a_n^{(o)} = \frac{F_n^{(o)}}{n^2 N^2 - \lambda F_o^{(1)}} \tag{24}$$

$$a_n^{(1)} = \frac{\lambda \sum_{m \neq n} F_{n-m}^{(1)} a_m^{(o)}}{n^2 N^2 - \lambda F_o^{(1)}} \tag{25}$$

$$a_n^{(2)} = \frac{1}{n^2 N^2 - \lambda F_o^{(1)}} \left\{ \frac{\lambda}{2!} \sum_{\substack{n_1, n_2, n_3 \\ n_1+n_2+n_3 = n}} F_{n_1}^{(2)} a_{n_2}^{(o)} a_{n_3}^{(o)} - \sum_{\substack{n_1, n_2 \\ n_1 + n_2 = n}} N^2 n_1 a_{n_1}^{(o)} \cdot n_2 a_{n_2}^{(o)} + \cdots \right\} \tag{26}$$

The solution for the equilibrium orbit of a charged particle is completed and summarized in the following:

1. The orbit: $R = R_o e^u$ and $u = \sum_n a_n e^{inN\theta}$.
2. R_o is defined such that $a_o = 0$.
3. a_n can be evaluated from Eqs. (23), (24), (25) and (26).
4. λ is a dimensionless constant and defined as

   $$\lambda = \frac{e B_o R_o}{pc}$$

   which can be evaluated from Eqs. (17), (18), (19), and (20).

   For a given value of B_o, the ratio R_o/p_c can be obtained from λ .

   The relation between kinetic energy and λ is given

   $$B_o R_o = \frac{\lambda}{0.3 z} (T + 2 T E_o)^{1/2}$$

   where T is kinetic energy in MeV

   E_o is rest mass ($m_o c^2$) in MeV

   B_o and R_o are in kilogauss and centimeters respectively

   z is the number of charges in a particle.

II. BETATRON OSCILLATIONS

A. Radial Betatron Oscillation

Radial oscillations obey the same equation as the equilibrium orbit in Eq. (1). The solution is written in a different form than Eq. (3)

$$R = R_{eq}\,(1 + v) = R_o\, e^{u}\,(1 + v) \quad . \tag{27}$$

The equation of motion is

$$\frac{d}{d\theta}\,\frac{(u' + v' + vu')}{\sqrt{(1+v)^2 + (u' + v' + vu')^2}} - \frac{1}{\sqrt{(1+v)^2 + (u' + v' + vu')^2}} = -\sum_n F_n(u+v)\, e^{inN\theta} \tag{28}$$

For linear oscillation, only two terms are needed on the right-hand side of Eq. (28), i.e.,

$$F_n\,(u+v) = F_n\,(u) + F_n'\,(u)\, v + \cdots \quad . \tag{29}$$

Equation (28) becomes

$$v'' + C_1 = -\lambda\, v \sum_n F_n'\,(u)\, e^{inN\theta} \tag{30}$$

where

$$C_1 = \frac{d}{d\theta}\,\frac{v\,u'}{\sqrt{1 + u'^2}} - \frac{u'^2\, v' + v^2\, u'}{(1 + u')^{3/2}} - \frac{u'\, v'}{1 + u'^2} \quad . \tag{31}$$

C_1 represents a correction term which turns out to be small and is neglected. Then Eq. (30) becomes

$$v'' + \lambda\, v \sum_n F_n'\,(u)\, e^{inN\theta} = 0 \quad . \tag{32}$$

From Appendix A, Eqs. (A-13) and (A-11) are

$$\sum_n F_n'\,(u)\, e^{inN\theta} = \sum_{n,m} {}_{n-m}E_m'\; e^{inN\theta}$$

$${}_nE_m' = \sum_j \frac{F_n^{(j+1)}\, A_m^j\,(a)}{j} \qquad j = 0, 1, 2, 3, \cdots$$

Substituting into Eq. (32),

$$v'' + \lambda\, v \sum_{n,m} {}_{n-m} E'_m \, e^{inN\theta} = 0 \quad . \tag{33}$$

Equation (33) is similar to the equation

$$\frac{d^2}{d\theta^2} v + (E - g(\theta))\, v = 0 \tag{35}$$

where

$$g(\theta) = \sum_{n \neq o} g_n \, e^{inN\theta} \quad .$$

This is the Floquet-type equation. It can be solved simply by the application of the perturbation theory of quantum mechanics. Let an operator

$$H_{op} = -\frac{d^2}{d\theta^2} + g(\theta) \quad . \tag{36}$$

Then Eq. (35) becomes

$$H_{op}\, v = Ev \quad . \tag{37}$$

H_{op} is Hermitian under the condition that

$$g_n^* = -g_n \quad .$$

We may consider H_{op} as a Hermitian operator and E is the eigenvalue of the eigenfunction v. When v is normalized, the average value of the operator H_{op} for the function v is

$$(H)_{ave} = \int v^* H_{op}\, v\, d\theta \quad . \tag{38}$$

If one of the true solutions of Eq. (3) can be expanded in terms of the complete orthogonal set of eigenfunctions v_n,

$$v = \sum_n b_n v_n \, . \tag{39}$$

Substituting into Eq. (37), we multiply on the left by v_n^* and integrate over 2π. We take advantage of the orthogonal character of v_n's using the notation

$$H_{nm} = \int v_n^* H_{op} v_m \, d\theta \tag{40}$$

and find

$$\sum_k b_k (H_{nk} - E \delta_{nk}) = 0 \ . \tag{41}$$

It is shown in algebra that Eq. (40) will have solutions if the determinant of coefficients vanishes

$$\det | H_{nk} - E \delta_{nk} | = 0 \ . \tag{42}$$

To determine the eigenvalue E from Eq. (41), we take the approximation that discards all nondiagonal matrix components H_{nk} for which n and k are both different from 0. Then the equations become

$$b_n (H_{nn} - E) + b_o H_{no} = 0 \qquad n \neq 0 \tag{43}$$

$$\sum_n b_n H_{on} - b_o E = 0 \ . \tag{44}$$

From Eq. (43),

$$\frac{b_n}{b_o} = \frac{H_{no}}{E - H_{nn}} \qquad n \neq 0 \ . \tag{45}$$

Substituting Eqs. (45) into (44), we have

$$E = H_{oo} + \sum_{n \neq o} \frac{H_{on} \cdot H_{no}}{E - H_{nn}} \ . \tag{46}$$

Substituting Eq. (45) into Eq. (39), the eigenfunction is equal to

$$v = v_o + \sum_{n \neq o} \frac{H_{no}}{E - H_{nn}} v_n \tag{47}$$

when $E \sim H_{oo}$, we replace E in the second term of Eq. (46) and Eq. (47) by H_{oo} and get the approximation

$$E = H_{oo} + \sum_{n \neq o} \frac{H_{on} \cdot H_{no}}{H_{oo} - H_{nn}} \tag{48}$$

$$v = v_o + \sum_{n \neq o} \frac{H_{no}}{H_{oo} - H_{nn}} v_n \ . \tag{49}$$

According to Floquet theorem, the solution of Eq. (35) is

$$v = \sum_n b_n e^{i(\nu+nN)\theta} \quad . \tag{50}$$

Comparing with Eq. (39), we have

$$v_o = e^{i\nu\theta}, \qquad b_o = 1$$

$$v_n = b_n e^{i(\nu+nN)\theta} \quad .$$

We get

$$\begin{aligned} H_{oo} &= \int e^{-i\nu\theta} H_{op} e^{+i\nu\theta} d\theta = \nu^2 \\ H_{on} &= \int e^{-i\nu\theta} H_{op} e^{i(\nu+nN)\theta} d\theta = g_{-n} \\ H_{no} &= \int e^{-i(\nu+nN)\theta} H_{op} e^{+i\nu\theta} d\theta = g_n \\ H_{nn} &= \int e^{-i(\nu+nN)\theta} H_{op} e^{i(\nu+nN)\theta} d\theta = (\nu + nN)^2 \quad . \end{aligned} \tag{51}$$

Therefore the eigenvalue and eigenfunction are

$$\begin{aligned} E &= \nu^2 + \sum_{n \neq o} \frac{g_n \cdot g_{-n}}{\nu^2 - (\nu + nN)^2} \\ &= \nu^2 - 2 \sum_{n \geqslant 1} \frac{g_n \cdot g_{-n}}{(nN)^2 - 4\nu^2} \end{aligned} \tag{52}$$

$$v = e^{i\nu\theta} + \sum_{n \neq o} \frac{g_n e^{i(\nu+nN)\theta}}{\nu^2 - (\nu + nN)^2} \quad . \tag{53}$$

If we use "smooth" approximation,[1] instead of Eq. (52), we get

$$E = \nu^2 - 2 \sum_{n \geqslant 1} \frac{g_n \cdot g_{-n}}{n^2 N^2} \quad . \tag{54}$$

If N is an even integer, Eq. (35) becomes Hill's equation. Under the conditions that

1. $\sum_{n \neq o} g_n$ should be absolutely convergent,

2. $E - (\nu + nN)^2 \neq 0$.

We get[5]

$$\sin^2\left(\frac{\pi}{2}\nu\right) = \Delta(0)\sin^2\left(\frac{\pi}{2}\sqrt{E}\right) \tag{55}$$

and

$$\Delta(0) = 1 - 2\sum_{n \geqslant 1} \frac{g_n \cdot g_{-n}}{(n^2N^2 - E)\, E} \quad . \tag{56}$$

The determinant Δ (0) is obtained by the approximation made in Eqs. (43) and (44). Δ (0) can be used to study the stability region of the operating points. To evaluate the numerical values of tunes, we prefer Eq. (52) since it is a better approximation than Eq. (54).

The radial tune ν_r is

$$\nu_r^2 = \lambda \sum_m {}_{-m}E_m' + 2\lambda^2 \sum_{n \geqslant 1} \sum_m \frac{{}_{n-m}E_m' \cdot {}_{n-m}E_m'}{n^2N^2 - 4\nu_r^2}$$

where

$$\sum_m {}_{-m}E_m' = F_o^{(1)} + \sum_n F_{-n}^{(2)} A_n^1(a) + \frac{1}{2!}\sum_n F_{-n}^{(3)} A_n^2 + \cdots \tag{57}$$

$${}_{n-m}E_m' = F_n^{(1)} + \sum_m F_{n-m}^{(2)} A_m^1(a) + \cdots \tag{58}$$

$${}_{-n-m}E_m' = F_{-n}^{(1)} + \sum_m F_{-n-m}^{(2)} A_{-m}^1(a) + \cdots \quad . \tag{59}$$

Including second-order terms, the tune is

$$\nu_r^2 = \lambda\left\{F_o^{(1)} + \sum_n F_{-n}^{(2)} A_n^1(a)\right\} + 2\lambda^2 \sum_{n \geqslant 1} \frac{F_n^{(1)} \cdot F_{-n}^{(1)}}{n^2N^2 - 4\nu_r^2} \, . \tag{60}$$

If we take

$$\lambda \simeq \lambda^{(o)} = \frac{1}{F_o^{(o)}} \qquad A_n^1(a) \simeq a_n^{(o)} = \frac{F_n^{(o)}/F_o^{(o)}}{n^2N^2 - F_o^{(1)}/F_o^{(o)}}$$

then

$$\nu_r^2 = \frac{F_o^{(1)}}{F_o^{(o)}} + \frac{1}{F_o^{(o)}} \sum_{n \neq o} \frac{F_{-n}^{(2)} \cdot F_n^{(o)}/F_o^{(o)}}{n^2N^2 - F_o^{(1)}/F_o^{(o)}} + \frac{2}{(F_o^{(o)})^2} \sum_{n \geqslant 1} \frac{F_n^{(1)} \cdot F_{-n}^{(1)}}{n^2N^2 - 4\nu_r^2} .$$

B. Vertical Betatron Oscillation

Vertical oscillations obey the equation[4]

$$\frac{d}{d\theta} \frac{z^2}{\sqrt{R^2 + R'^2 + z^2}} = \frac{e}{pc} (R' B_\theta - R B_r). \tag{61}$$

The field B_θ and B_r are given by the relation

$$\nabla \times \vec{B} = 0 .$$

For linear terms in z, B_θ and B_r are

$$B_\theta = \frac{\partial B_\theta}{\partial z} z = \frac{1}{R} \frac{\partial B_z}{\partial \theta} z = -\frac{B_o z}{R} \sum_n G_n(R)(inN) e^{inN\theta} \tag{62}$$

$$B_r = \frac{\partial B_r}{\partial z} z = z \frac{\partial B_z}{\partial R} = -B_o z \sum_n \frac{d G_n(R)}{d R} e^{inN\theta} . \tag{63}$$

Substitute Eqs. (9), (62), and (63) into Eq. (61) and let

$$z = R_{eq} y . \tag{64}$$

Then

$$y'' + C_2 = -\lambda y \left\{ \sum_n \frac{du}{d\theta} (inN) F_n(u) e^{inN\theta} - \sum_n \left(\frac{dF_n(u)}{du} - F_n(u) \right) e^{inN\theta} \right\} \tag{65}$$

where

$$C_2 = \frac{d}{d\theta} y \left[1 - \frac{1}{2} (u'^2 + y'^2) \right] . \tag{66}$$

C_2 represents a correction term which turns out to be small and is neglected.

With the aid of Eqs. (A-5), (A-6), and (A-13), Eq. (65) becomes

$$y'' + \lambda y \sum_{n,m} \left\{ \sum_{\substack{n_1, n_2, n_3 \\ n_1+n_2+n_3=n}} (in_1 N)(in_2 N)\, a_{n_2 n_3 - m} E_{m-n-m} E'_{m+n-m} E_m \right\} e^{inN\theta} = 0 \tag{67}$$

Similar to the treatment in radial betatron oscillations, we assume the solution for Eq. (67) is of the form

$$y = \sum_n y_n e^{i(\nu_z + nN)\theta} \tag{68}$$

and the vertical tune is

$$\nu_z^2 = -\lambda \left\{ F_o^{(1)} + \sum_n \left(F_{-n}^{(2)} - F_{-n}^{(1)} \right) A_n^1(a) - F_o^{(o)} \right\} + (nN)^2 \lambda \sum_n F_{-n}^{(o)} A_n^1(a)$$

$$+ 2\lambda^2 \sum_{n \geq 1} \frac{\left(F_n^{(o)} - F_n^{(1)} \right) \cdot \left(F_{-n}^{(o)} - F_{-n}^{(1)} \right)}{n^2 N^2 - 4\nu_z^2} . \tag{69}$$

If we take

$$\lambda \simeq \lambda^{(o)} = \frac{1}{F_o^{(o)}} , \quad A_n^1(a) \simeq a_n^{(o)} = \frac{F_n^{(o)}/F_o^{(o)}}{n^2 N^2 - F_o^{(1)}/F_o^{(o)}} ,$$

then

$$\nu_z^2 = 1 - \frac{F_o^{(1)}}{F_o^{(o)}} + \frac{1}{F_o^{(o)}} \sum_{n \neq o} \frac{\left(F_{-n}^{(2)} - F_{-n}^{(1)} \right) F_n^{(o)}/F_o^{(o)}}{n^2 N^2 - F_o^{(1)}/F_o^{(o)}}$$

$$+ \frac{(nN)^2}{F_o^{(o)}} \sum_{n \neq o} \frac{F_{-n}^{(o)} F_n^{(o)}/F_o^{(o)}}{n^2 N^2 - F_o^{(1)}/F_o^{(o)}} + \frac{2}{F_o^{(o)\,2}} \sum_{n \geq 1} \frac{F_n^{(o)} - F_n^{(1)} \cdot F_{-n}^{(o)} - F_{-n}^{(1)}}{n^2 N^2 - 4\nu_z^2} .$$

The results in Sections I and II are general. These can be applied to various accelerators. An example of this has shown in Appendex B for a spiral sector accelerator. Here we emphasize on the application to an integral structure machine.

III. INTEGRAL SCALING IN FFAG

Let N and M be the number of spiral sectors and radial cuts respectively in an integral structure machine. The magnetic field in the median plane of the structure is periodic in azimuthal angle θ and is expected to be of the form

$$G\ (R) \sum_{n,\,m} g_{n,\,m}\ e^{in\left[k \ln \frac{R}{r_o} - N\theta\right]}\ e^{imM\theta} \tag{70}$$

or

$$G\ (R) \sum_{L} a_L\ (R)\ e^{iLP\theta} \tag{71}$$

where

$$G\ (R) = -\ B_o \left(\frac{R}{r_o}\right)^{k}\ . \tag{72}$$

P is called the superperiod. It is the greatest common divisor of N and M.

We write

$$-\ nN + mM = P\ (-\ sn + rm). \tag{73}$$

Let

$$L = -\ sn + rm\ ; \tag{74}$$

s and r are the number of spiral sectors and radial cuts respectively in a superperiod. It is clear from Eq. (74) that for a given L, there are infinite pairs of (n, m). If a pair (n, m) satisfies (74) for + L, there will be a pair (- n, - m) for - L. The magnetic field is expected to be real. This reality condition leads to the assumption that if $n \neq o$, $m \neq o$

$$g_{nm} = (a_{nm} - d_{nm}) - i\ (b_{nm} + c_{nm}) = h_{nm}\ e^{-i\alpha_{nm}} \tag{75}$$

$$g_{-n,\,-m} = (a_{nm} - d_{nm}) + i\ (b_{nm} + c_{nm}) = h_{nm}\ e^{+i\alpha_{nm}} \tag{76}$$

$$g_{n,\,-m} = +\ (a_{nm} + d_{nm}) + i\ (b_{nm} - c_{nm}) = h_{-nm}\ e^{+i\alpha_{-nm}} \tag{77}$$

$$g_{-n,\,m} = +\ (a_{nm} + d_{nm}) - i\ (b_{nm} - c_{nm}) = h_{-nm}\ e^{-i\alpha_{-nm}} \tag{78}$$

where h_{nm} is real number. For $n = 0$, $m \neq 0$

$$g_{om} = a_{om} - i\, b_{om} \tag{79}$$

$$g_{o,-m} = a_{om} + i\, b_{om} \quad . \tag{80}$$

For $m = 0,\ n \neq 0$

$$g_{no} = + a_{no} - i\, c_{no} \tag{81}$$

$$g_{-no} = + a_{no} + i\, c_{no} \quad . \tag{82}$$

For both $n = 0,\ m = 0$

$$g_{oo} = a_{oo} \quad . \tag{83}$$

With the aid of the reality condition, the field can be expressed as the summation of the following terms:

$$\begin{aligned} & f\, a_{nm}\, G\,(R) \cos n\left[K \ln \frac{R}{r_o} - N\theta\right] \cos mM\theta \\ & f\, b_{nm}\, G\,(R) \cos n\left[K \ln \frac{R}{r_o} - N\theta\right] \sin mM\theta \\ & f\, c_{nm}\, G\,(R) \sin n\left[K \ln \frac{R}{r_o} - N\theta\right] \cos mM\theta \\ & f\, d_{nm}\, G\,(R) \sin n\left[K \ln \frac{R}{r_o} - N\theta\right] \sin mM\theta \end{aligned} \tag{84}$$

where

$$\begin{aligned} f &= 1 \text{ when } n = 0 \text{ and } m = 0 \\ f &= 2 \text{ when either } n = 0 \text{ or } m = 0 \\ f &= 4 \text{ when } n \neq 0 \text{ and } m \neq 0 \ . \end{aligned} \tag{85}$$

There exists a computer program (MURA F46) which calculates the coefficents a_{nm}, b_{nm}, c_{nm}, and d_{nm} . The main discussion here is to use these coefficients to study dynamic problems.

In Eq. (70), we would like to replace r_o which is picked for the convenience of the computer program, by R_o, the average radius of the equilibrium orbit.

$$- B_1 \left(\frac{R}{R_o}\right)^{k} \sum_{n,m} g_{nm}\, e^{inK\ln\frac{R}{r_o}}\, e^{inK\ln\frac{R}{R_o}}\, e^{i(mM - nN)\theta} \tag{86}$$

where

$$B_1 = B_o \left(\frac{R_o}{r_o}\right)^{k} \quad . \tag{87}$$

The coefficient a_L in Expression (72) is

$$a_L(R) = \sum_{(n,m)_L} g_{nm}\, e^{inK(u_o + u)} \quad . \tag{88}$$

The sum $\sum_{(n,m)_L}$ is over all the pairs of (n, m) for a given L.

$u_o = \ln \frac{R_o}{r_o}$ and $u = \ln \frac{R}{R_o}$. The function $F_L(u)$ is

$$\begin{aligned} F_L &= e^{u(k+1)} \sum_{(n,m)_L} g_{nm}\, e^{inK(u_o+u)} \\ &= e^{u(k+1)} \sum_{(n,m)_L} h_{nm}\, e^{i\left[nK(u_o+u) - \alpha_{nm}\right]} \quad . \end{aligned} \tag{89}$$

Therefore

$$F_L^{(o)} = \sum_{(n,m)_L} h_{nm}\, e^{i(nKu_o - \alpha_{nm})} \tag{90}$$

$$F_L^{(1)} = \sum_{(n,m)_L} (k + 1 + inK)\, h_{nm}\, e^{i(nKu_o - \alpha_{nm})} \tag{91}$$

$$F_L^{(2)} = \sum_{(n,m)_L} (k + 1 + inK)^2 h_{nm}\, e^{i(nKu_o - \alpha_{nm})} \quad . \tag{92}$$

Take the proposed 500 MeV machine[6] for example:

$N = 16$

$M = 72$

$P = 8$

$r = 9$

$s = 2$

$k = 8.2$

$K = 75$

The pairs of (n, m) for L = 0 are

$$\begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 9 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -9 \\ -2 \end{pmatrix}, \quad \cdots \tag{93}$$

Equations (90), (91), and (92) become

$$F_o^{(o)} = a_{oo} + 2\, h_{92} \cos\,(9\, K u_o - \alpha_{92}) + \cdots \tag{94}$$

$$F_o^{(1)} = (k + 1)\, a_{oo} + 2\left[(k + 1)^2 + (9K)^2\right]^{1/2} h_{92} \cos\,(9 K u_o - \alpha_{92} + \beta_9) \tag{95}$$

$$F_o^{(2)} = (k + 1)^2\, a_{oo} + 2\left[(k + 1)^2 + (9K)^2\right] h_{92} \cos\,(9\, K u_o - \alpha_{92} + 2\,\beta_9) \tag{96}$$

where

$$\beta_n = \tan^{-1} \frac{(k + 1)}{n\, K} \quad .$$

Equation (18) gives

$$\lambda^{(o)} = \frac{1}{F_o^{(o)}} = \frac{1}{a_{oo} + 2\, h_{92} \cos\,(9\, K u_o - \alpha_{92}) + \cdots}$$

$$= \frac{1}{a_{oo}} \quad 1 - 2\, \frac{h_{92}}{a_{oo}} \cos\,(9\, K u_o - \alpha_{92}) + \cdots \quad . \tag{97}$$

The magnetic field is scaled for a finite number of points in the radial direction and its shape repeats itself nine times (r = 9) in a superperiod. Therefore the dynamic quantities will be periodic with period of nine in that superperiod. λ and tunes are periodic functions of R_o or energy. For the whole energy range, their deviations are small. We may consider that the constant parts are due to the scaling field and the deviations are due to radial cuts in the spiral geometry. In general, the tunes and λ are expressed as follows: First, the constant parts:

$$\lambda^{(o)} = \frac{1}{a_{oo}} \tag{98}$$

$$a_L^{(o)} = \frac{\sum_{(n,m)_L} h_{nm}\, e^{i(nKu_o - \alpha_{nm})}}{a_{oo}\left[L^2 P^2 - (k + 1)\right]} \tag{99}$$

$$\nu_r^2 = (k + 1) + (\lambda^{(o)})^2 \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \frac{f\left[(k+1)^2 - (nK)^2\right]\left(a_{nm}^2 + b_{nm}^2 + c_{nm}^2 + d_{nm}^2\right)}{(-sn + rm)^2 P^2 - (k+1)}$$

except both $n = o$, $m = o$

$$+ (\lambda^{(o)})^2 \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \frac{f\left[(k+1)^2 + (nK)^2\right]\left(a_{nm}^2 + b_{nm}^2 + c_{nm}^2 + d_{nm}^2\right)}{(-sn + rm)^2 P^2 - 4\nu_r^2}{}^{*} \tag{100}$$

except both $n = 0$, $m = o$

$$\nu_z^2 = 1 - (k + 1) + (\lambda^{(o)})^2 \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \frac{f(-sn + rm)^2 P^2\left(a_{nm}^2 + b_{nm}^2 + c_{nm}^2 + d_{nm}^2\right)}{(-sn + rm)^2 P^2 - (k+1)}$$

except both $n = o$, $m = o$

$$- (\lambda^{(o)})^2 \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \frac{f\left[(k+1)k - (nK)^2\right]\left(a_{nm}^2 + b_{nm}^2 + c_{nm}^2 + d_{nm}^2\right)}{(-sn + rm)^2 P^2 - (k+1)}$$

except both $n = o$, $m = o$

$$+ (\lambda^{(o)})^2 \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \frac{f\left[k^2 + (nK)^2\right]\left(a_{nm} + b_{nm} + c_{nm} + d_{nm}\right)}{(-sn + rm)^2 P^2 - 4\nu_z^2} \tag{101}$$

except both $n = o$, $m = o$

where f is defined in Expression (85).

---

*In $\sum_L F_L^{(1)} \cdot F_{-L}^{(1)}$, if a pair (n, m) for + L, there will be a pair (- n, - m) for - L, and a pair (n, - m) for L$'$, there will be a pair (- n, m) for - L$'$. Therefore, $a_{nm}\, d_{nm}$ and $b_{nm}\, c_{nm}$ are canceled out. The summation is simply equal to $\sum_{n=o} \sum_{m=o} f\left[(k+1)^2 - (nK)^2\right]\left(a_{nm}^2 + b_{nm}^2 + c_{nm}^2 + d_{nm}^2\right)$.

except both $n = o$, $m = o$

Second, the maximum values of the deviations:

$$\Delta\lambda^{(o)} = \frac{2\, h_{rs}}{a_{oo}^2} \tag{102}$$

$$\Delta\nu_r^2 = \frac{2\,(k+1)\, h_{rs}}{a_{oo}^2} + \sum_{(n_1,m_1)\neq(n_2,m_2)} \frac{h_{n_1m_1}h_{n_2m_2}\left\{\left[(k+1)^2+(n_1K)^2\right]^{1/2}\left[(k+1)^2+(n_2K)^2\right]^{1/2}+\left[(k+1)^2+(n_1K)^2\right]\right\}}{(-sn_1+rm_1)^2\,P^2} \tag{103}$$

$$\Delta\nu_z^2 = -\frac{2\,(k+1)h_{rs}}{a_{oo}^2} + \sum_{(n_1,m_1)=(n_2,m_2)} h_{n_1m_1}\,h_{n_2m_2} + \sum_{(n_1,m_1)\neq(n_2,m_2)} \frac{h_{n_1m_1}\,h_{n_2m_2}\left\{\left[k^2+(n_1K)^2\right]^{1/2}\left[k^2+(n_2K)^2\right]^{1/2}+\left(\left[(k+1)^2+(n_1K)^2\right]+\left[(k+1)^2+(n_2K)^2\right]\right)^{1/2}\right\}}{(-sn_1+rm_1)^2\,P^2} \tag{104}$$

where the pairs (n_1, m_1) and (n_2, m_2) correspond to a given L. The last term in Eqs. (103) and (104) are much larger than the first or second term. The relation of the pairs (n_1, m_1) and (n_2, m_2) is

$$\begin{pmatrix} n_2 \\ m_2 \end{pmatrix} = \begin{pmatrix} n_1 \\ m_1 \end{pmatrix} + \ell \begin{pmatrix} r \\ s \end{pmatrix} \qquad \text{where } \ell \text{ is any integer .}$$

The harmonics **n** of the original spiral field should be taken larger than r. Cole and Morton[7] neglected all the harmonics n higher than $\frac{r}{2}$. The terms in Eqs. (103) and (104) disappear automatically. This will explain why they get the conclusion that ν_r and ν_z are independent of energy. It is found that the pairs (-1, 0) and (r - 1, s) yield the largest deviations. If we only take account of the interaction of pairs (-1, 0) and (r - 1, s), we get

$$\Delta\nu_r^2 \simeq \Delta\nu_z^2 \simeq 2\, h_{-1,0}\, h_{r-1,s} \left\{\left[(k+1)^2 + K^2\right]^{1/2}\left[(k+1)^2 + (r-1)^2K^2\right]^{1/2} + \left[(k+1)^2 + (r^2 - 2r + 2)K^2\right]\right\} / \left[(2P)^2 - (k+1)\right]$$

The maximum values of deviations are

$$\Delta\nu_r = \sqrt{\nu_r^2 + \Delta\nu_r^2} - \nu_r \simeq \frac{\Delta\nu_r^2}{2\nu_r} \tag{105}$$

$$\Delta\nu_z = \sqrt{\nu_z^2 + \Delta\nu_z^2} - \nu_z \simeq \frac{\Delta\nu_z^2}{2\nu_z} \quad . \tag{106}$$

The actual range of tunes are

radial, $\nu_r \pm \Delta\nu_r$ (107)

vertical, $\nu_z \pm \Delta\nu_z$. (108)

There is a program, MURA F46, which calculates the magnetic field coefficients a_{nm}, b_{nm}, c_{nm}, and d_{nm} . For the proposed 500 MeV injector, these coefficients are shown in Table I. Equations (100) and (101) have been programmed by Mr. Francis Murphy for the IBM 704 computer (MURA F147). The program, MURA F147, uses the magnetic field coefficients in Table I and yields the constant part of tunes:

$$\nu_r = 3.228$$

$$\nu_z = 2.370$$

and from Eqs. (105) and (106) the maximum values of the deviations are

$$\Delta\nu_r \leqslant 0.004$$

$$\Delta\nu_z \leqslant 0.006$$

TABLE I. MEDIAN PLANE COEFFICIENTS FOR THE PROPOSED 500 MEV INJECTOR

	m \ n	0	1	2	3	4	5	6	7	8	9	10
a_{nm}	0	1.19228	-0.41445	-0.02562	-0.00160	-0.00599	0.00267	0.00017	-0.00028	0.00003	-0.00005	0.00001
	1	-0.20740	0.09537	0.00805	0.00046	0.00214	-0.00013	-0.00006	0.00014	-0.00001	0.00002	-0.00000
	2	-0.07438	0.03517	0.00164	0.00062	0.00101	-0.00045	-0.00002	0.00002	-0.00001	0.00002	-0.00000
	3	-0.00345	0.00140	0.00008	-0.00001	0.00001	0.00010	-0.00001	-0.00001	-0.00000	-0.00001	0.00000
	4.	0.00816	-0.00433	-0.00004	-0.00020	-0.00022	0.00019	0.00000	-0.00001	0.00000	-0.00001	0.00000
b_{nm}	0	0	0	0	0	0	0	0	0	0	0	0
	1	0.02613	-0.00846	0.00116	-0.00063	-0.00034	0.00008	-0.00002	0.00001	0.00000	-0.00000	-0.00000
	2	0.00126	0.00180	0.00248	-0.00082	-0.00022	-0.00003	-0.00004	0.00003	0.00000	0.00000	0.00000
	3	-0.00126	0.00103	0.00036	-0.00005	0.00004	-0.00002	-0.00000	-0.00000	0.00000	0.00000	0.00000
	4	-0.00028	-0.00014	-0.00029	0.00024	0.00006	0.00000	0.00002	-0.00002	-0.00000	0.00000	-0.00000
c_{nm}	0	0	-0.14867	-0.08327	0.02047	0.00401	0.00012	0.00065	-0.00052	-0.00005	0.00006	-0.00002
	1	0	0.03570	0.02445	-0.00746	-0.00159	-0.00002	-0.00026	0.00022	0.00001	-0.00003	0.00001
	2	0	0.01616	0.00917	-0.00325	-0.00047	-0.00007	-0.00012	0.00009	0.00001	-0.00001	0.00000
	3	0	0.00068	0.00042	0.00021	0.00002	-0.00000	0.00000	-0.00003	0.00000	0.00000	-0.00000
	4	0	-0.00229	-0.00108	0.00083	0.00008	0.00002	0.00004	-0.00004	-0.00000	0.00000	0.00000
d_{nm}	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	-0.00648	-0.00258	0.00058	-0.00010	0.00010	0.00002	-0.00003	-0.00000	-0.00000	-0.00000
	2	0	-0.00462	-0.00058	-0.00011	-0.00037	0.00011	0.00001	-0.00001	0.00000	-0.00000	-0.00000
	3	0	-0.00022	0.00009	-0.00015	-0.00005	-0.00001	-0.00000	0.00001	-0.00000	-0.00000	-0.00000
	4	0	0.00073	0.00006	0.00001	0.00008	-0.00006	-0.00001	0.00000	-0.00000	0.00000	-0.00000

APPENDIX A

(1) If a complex function $F_n(u)$ has a Taylor expansion in powers of u,

$$F_n(u) = F_n(o) + F_n'(o)\,u + F_n''(o)\,\frac{u^2}{2!} + F_n'''(o)\,\frac{u^3}{3!} + \cdots \quad . \tag{A-1}$$

We use the notation

$$F_n^{(o)} = F_n(o),\; F_n^{(1)} = F_n'(o),\; F_n^{(2)} = F_n''(o),\; \cdots \quad . \tag{A-2}$$

Equation (A-1) becomes

$$F_n(u) = \sum_j \frac{F_n^{(j)}\, u^j}{j\,!} \qquad j = 0, 1, 2, 3, \cdots \quad . \tag{A-3}$$

When u is a periodic function of θ, it can be expressed as a Fourier series

$$u = \sum_n a_n\, e^{inN\theta} \quad . \tag{A-4}$$

Inserting this into Eq. (A-3) yields

$$F_n(u) = \sum_m {}_nE_m\, e^{imN\theta} \tag{A-5}$$

where

$${}_nE_m = \sum_j \frac{F_n^{(j)}\, A_m^j(a)}{j\,!} \qquad j = 0, 1, 2, 3, \cdots \tag{A-6}$$

$$A_m^j(a) = \sum_{m_1}\sum_{m_2}\cdots\sum_{m_j} a_{m_1} a_{m_2}\cdots a_{m_j} \qquad m_1 + m_2 + \cdots m_j = m \tag{A-7}$$

$$A_m^o = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases} \quad . \tag{A-8}$$

All sums whose limits are not given are to be taken to extend from $-\infty$ to ∞, as, for example, the sums over n in Eq. (A-4), m in Eq. (A-5) and m_1, m_2, $\cdots$ m_j in Eq. (A-7). The derivative with respect to u of Eq. (A-1) is

$$\frac{d}{du} F_n(u) = F_n'(o) + F_n''(o)\, u + F_n'''(a) \frac{u^2}{2!} + \cdots$$

$$= \sum_j \frac{F_n^{(j+1)} u^j}{j\,!} \qquad j = 0, 1, 2, 3, \cdots \tag{A-9}$$

$$= \sum_m {}_n E_m' \, e^{imN\theta} \tag{A-10}$$

where

$${}_n E_m' = \sum_j \frac{F_n^{(j+1)} A_m^j (a)}{j\,!} \qquad j = 0, 1, 2, 3, \cdots \quad . \tag{A-11}$$

Let us consider $F_n(u)$ as the coefficient of a Fourier series, then

$$\sum_n F_n(u)\, e^{inN\theta} = \sum_n \left(\sum_m {}_n E_m \, e^{imN\theta} \right) e^{inN\theta}$$

$$= \sum_{n,m} {}_{n-m} E_m \, e^{inN\theta} \quad . \tag{A-12}$$

Similarly,

$$\sum_n \frac{d}{du} F_n(u)\, e^{inN\theta} = \sum_n \left(\sum_m {}_n E_m' \, e^{imN\theta} \right) e^{inN\theta}$$

$$= \sum_{n,m} {}_{n-m} E_m' \, e^{inN\theta} \quad . \tag{A-13}$$

(2) In Eq. (6), Θ is defined as

$$\Theta = \tan^{-1} u' \quad . \tag{A-14}$$

Under the assumption that $u' << 1$,

$$\tan^{-1} u' = u' - \frac{1}{3} u'^3 + \frac{1}{5} u'^5 + \cdots \quad . \tag{A-15}$$

From Eq. (A-4)

$$u' = \frac{du}{d\theta} = \sum_n (i n N a_n)\, e^{inN\theta} \quad . \tag{A-16}$$

Let

$$T_n = \sum_{\gamma} (-1)^{(\gamma+3)/2} \frac{A_n^{\gamma} (i n N a)}{\gamma} \qquad \gamma = 1, 3, 5, 7, \cdots \tag{A-17}$$

where

$$A_n^{\mathcal{J}}(\text{inNa}) = \sum_{n_1} \sum_{n_2} \cdots \sum_{n_j} (in_1 N a_{n_1})(in_2 N a_{n_2}) \cdots (in_j N a_{n_j}) \quad . \tag{A-18}$$
$$n_1 + n_2 + \cdots n_j = n$$

Then Eq. (A-14) becomes

$$\Theta = \tan^{-1} u' = \sum_n T_n \, e^{inN\theta} \quad . \tag{A-19}$$

The assumption $u' << 1$ implies that $\Theta << 1$. Therefore

$$\sin \Theta = \Theta - \frac{\Theta^3}{3\,!} + \frac{\Theta^5}{5\,!} + \cdots \tag{A-20}$$

$$\cos \Theta = 1 - \frac{\Theta^2}{2\,!} + \frac{\Theta^4}{4\,!} + \cdots \quad . \tag{A-21}$$

Inserting Eq. (A-19) into Eqs. (A-20) and (A-21),

$$\sin \Theta = \sum_n S_n \, e^{inN\theta} \tag{A-22}$$

$$\cos \Theta = \sum_n C_n \, e^{inN\theta} \tag{A-23}$$

where

$$S_n = \sum_{\mathcal{J}} (-1)^{(\mathcal{J}+3)/2} \frac{A_n^{\mathcal{J}}(T)}{!} \qquad \mathcal{J} = 1, 3, 5, \cdots \tag{A-24}$$

$$C_n = \sum_{\mathcal{J}} (-1)^{\mathcal{J}/2} \frac{A_n^{\mathcal{J}}(T)}{!} \qquad \mathcal{J} = 0, 2, 4, \cdots \quad . \tag{A-25}$$

The function $A_m^{\mathcal{J}}(T)$ is very complicated, the first few terms are listed below:

$$A_n^{o}(T) = \begin{cases} 1 & \text{if } n = o \\ 0 & \text{if } n \neq 0 \end{cases} \tag{A-26}$$

$$A_n^1(T) = T_n = A_n^1(\text{inNa}) - \frac{1}{3} A_n^3(\text{inNa}) + \frac{1}{5} A_n^5(\text{inNa}) + \cdots \tag{A-27}$$

$$A_n^2(T) = \sum_{\substack{n_1 \\ n_1+n_2=n}} \sum_{n_2} T_{n_1} T_{n_2}$$

$$= \sum_{\substack{n_1 \\ n_1+n_2=n \\ \bar{\gamma}_1, \bar{\gamma}_2 = 1,3,5,\cdots}} \sum_{n_2} \sum_{\bar{\gamma}_1} \sum_{\bar{\gamma}_2} \frac{(-1)^{(\bar{\gamma}_1+\bar{\gamma}_2+6)/2}}{\bar{\gamma}_1 \cdot \bar{\gamma}_2} A_{n_1}^{\bar{\gamma}_1}(\mathrm{inNa})\, A_{n_2}^{\bar{\gamma}_2}(\mathrm{inNa})$$

$$= \sum_{n_1} \sum_{n_2} \left[(2 A_{n_1}^1 A_{n_2}^1) - \frac{1}{3}(A_{n_1}^1 A_{n_2}^3 + A_{n_1}^3 A_{n_2}^1) + \frac{2}{9}(A_{n_1}^3 A_{n_2}^3) + \cdots \right.$$

$$\simeq 2 A_n^2(\mathrm{inNa}) \tag{A-28}$$

$$A_n^3(T_n) = \sum_{\substack{n_1 \\ n_1+n_2+n_3=n}} \sum_{n_2} \sum_{n_3} T_{n_1} T_{n_2} T_{n_3}$$

$$= \sum_{n_1} \cdots \sum_{n_3} \sum_{\bar{\gamma}_1} \cdots \sum_{\bar{\gamma}_3} \frac{(-1)^{(\bar{\gamma}_1+\bar{\gamma}_2+\bar{\gamma}_3+9)/2}}{\bar{\gamma}_1 \cdot \bar{\gamma}_2 \cdot \bar{\gamma}_3} A_{n_1}^{\bar{\gamma}_1}(\mathrm{inNa})\, A_{n_2}^{\bar{\gamma}_2}(\mathrm{inNa})\, A_{n_3}^{\bar{\gamma}_3}(\mathrm{inNa})$$

$$\simeq \sum_{n_1} \cdots \sum_{n_3} 3 A_{n_1}^1 A_{n_2}^1 A_{n_3}^1$$

$$\simeq 3 A_n^3(\mathrm{inNa}) \tag{A-29}$$

$$C_o = \sum_{\gamma} (-1)^{\gamma/2} \frac{A_o^{\gamma}(T)}{\gamma!} \qquad \gamma = 0, 2, 4, \cdots$$

$$= A_o^o(T) - \frac{1}{2!} A_o^2(T) + \frac{1}{4!} A_o^4(T) + \cdots$$

$$= 1 - A_o^2(\mathrm{inNa}) + \frac{1}{3!} A_o^4(\mathrm{inNa}) + \cdots \tag{A-30}$$

$$C_n = A_n^o(T) - \frac{1}{2!} A_n^2(T) + \frac{1}{4!} A_o^4(T) + \cdots$$

$$= -A_n^2(\mathrm{inNa}) + \frac{1}{3!} A_n^4(\mathrm{inNa}) + \cdots \tag{A-31}$$

$$S_o = 0$$

$$\begin{aligned}
S_n &= \sum \frac{(-1)^{(\bar{\gamma}+3)/2}}{\bar{\gamma}!} A_m^{\bar{\gamma}}(T) \qquad \bar{\gamma} = 1, 3, 5, \cdots \\
&= A_n^1(T) - \frac{1}{3!} A_n^3(T) + \frac{1}{5!} A_n^5(T) + \cdots \\
&= \left\{ A_n^1(inNa) - \frac{1}{3} A_n^3(inNa) + \frac{1}{5} A_n^5(inNa) + \cdots \right\} - \frac{3}{3!} A_n^3(inNa) + \cdots \\
&\simeq A_n^1(inNa) - \frac{5}{6} A_n^3(inNa) + \cdots \quad .
\end{aligned} \tag{A-32}$$

APPENDIX B

The treatments in Sections I and II are general and applicable to most accelerators as long as the magnetic field in the median plane can be expressed as $B_z = \sum_n G_n (R) e^{inN\theta}$. Even the direct application of their results does not appear familiar. For the purpose of comparison, a spiral sector accelerator is taken as an example. We start with a brief derivation and show that the approximate differential equations for the equilibrium orbit and tunes are the same as in References 1 and 8. The difference is a matter of the approximation to obtain the tunes. It is evident after appropriate rearrangement of terms that the tunes which are obtained from an application of quantum perturbation theory are the same as those of the "smooth approximation."[1]

(1) The Orbit

The magnetic field in the median plane of a spiral sector accelerator is

$$B_z = - B_o \left(\frac{R}{R_o}\right)^k \left[1 + f \cos \left(N\theta - K \ln \frac{R}{R_o}\right)\right]. \tag{B-1}$$

The equation of the motion, Eq. (1), after expanding in power series of u, becomes

$$\begin{aligned} 1 - u'' - \frac{1}{2} u'^2 = \lambda \Big\{ & 1 + (k + 1) u + f \cos N\theta + (k + 1) f u \cos N\theta \\ & + f K u \cos (N\theta - \frac{\pi}{2}) + (k + 1) f K u^2 \cos (N\theta - \frac{\pi}{2}) \\ & - f K^2 u^2 \cos N\theta + \frac{(k + 1)^2}{2} u^2 \cos N\theta \Big\} . \end{aligned} \tag{B-2}$$

Equation (B-2) can be solved by successive approximations. When the power of u increases, its numerical value is expected to decrease rapidly. The first term on the right-hand side of Eq. (B-2) gives the zero-order approximation of the average radius of the equilibrium orbit, R_o ; the second and third terms, the amplitudes of oscillation, a_n, around R_o. The second term also gives the zero-order of the radial tune, ν_r; the fourth and fifth terms, the first-order of ν_r; the seventh and ninth terms, the second order of ν_r.

For the equilibrium orbit, the significant terms appear below:

$$1 - u'' = \lambda \left\{ 1 + (k + 1)\, u + f\, K\, u \cos\left(N\theta - \frac{\pi}{2}\right) + f \cos N\theta \right\} . \tag{B-3}$$

If the solution is of the form

$$u = \sum_{n \neq o} a_n\, e^{inN\theta} , \tag{B-4}$$

the contribution of the term $f\, K\, u \cos\left(N\theta - \frac{\pi}{2}\right)$ to the zero-harmonic vanishes. Therefore, we get

$$\lambda = 1 \tag{B-5}$$

and for the first harmonic

$$a_1 = a_{-1} = \frac{f/2}{N^2 - (k + 1)} . \tag{B-6}$$

Equations (18) and (24) are equivalent to Eqs. (B-5) and (B-6). The equilibrium orbit is approximately

$$R = R_o \left(1 + \frac{f}{N^2 - (k + 1)} \cos N\theta\right) \tag{B-7}$$

which is in agreement with References 1 and 8.

(2) The Tunes

The equations of oscillating particles in radial and vertical motion are

$$\begin{aligned} v'' + v \Big\{ & (k + 1) + (k + 1)\, f \cos N\theta + K\, f \cos\left(N\theta - \frac{\pi}{2}\right) + 2\,(k + 1)\, K\, f\, u \cos\left(N\theta - \frac{\pi}{2}\right) \\ & - K^2 f\, u \cos N\theta + (k + 1)^2 f\, u \cos N\theta + (k + 1)^2 u \Big\} = 0 \end{aligned} \tag{B-8}$$

$$\begin{aligned} y'' + y \Big\{ & u' \sqrt{N^2 + \left(\tfrac{f}{2}\right)^2} \cos\left(N\theta - \beta\right) - k + f \left[K^2 - k\,(k + 1)\right] u \cos N\theta \\ & - f \left[K \cos\left(N\theta - \frac{\pi}{2}\right) + k \cos N\theta\right] \Big\} = 0 \end{aligned} \tag{B-9}$$

where $\beta = \tan^{-1} \frac{2\,N}{f}$.

Substituting $u = 2 a_1 \cos N\theta$ into Eqs. (B-8) and (B-9), we get

$$v'' + v \left\{ (k+1) - a_1 f \left[K^2 - (k+1)^2 \right] + f \left[K \cos \left(N\theta - \frac{\pi}{2} \right) + (k+1) \cos N\theta \right. \right.$$
$$\left. \left. - a_1 f K^2 \cos 2 N\theta \right\} = 0 \qquad \text{(B-10)}$$

$$y'' + y \left\{ a_1 f N^2 - k + a_1 f \left[K^2 - k(k+1) \right] - f \left[K \cos \left(N\theta - \frac{\pi}{2} \right) + k \cos N\theta \right] \right.$$
$$\left. + a_1 f K^2 \cos 2 N\theta \right\} = 0 \; . \qquad \text{(B-11)}$$

In Eq. (B-11), the Kerst focusing term is mainly $a_1 f K^2$ which is desired to overcome the defocusing term $-k$, due to the rise of the magnetic field with increasing radius. It is evident that when K is larger than N, the Thomas focusing, $a_1 f N^2$, becomes smaller than the Kerst focusing. When K is much larger than the field index, k, we may neglect $a_1 f (k+1)^2$, $(k+1) \cos N\theta$, $a_1 f k (k+1)$, $k \cos N\theta$. Then Eqs. (B-10) and (B-11) are the same as Eqs. (8.8) and (8.9) in Reference 1. (There is $\pi/2$ difference in defining the θ-coordinate.)

By the application of quantum perturbation theory, the radial and vertical tunes are

$$\nu_r^2 = (k+1) - a_1 f \left[K^2 - (k+1)^2 \right] + 2 \left(\frac{f}{2} \right)^2 \frac{K^2 + (k+1)^2}{N^2 - 4 \nu_r^2} + 2 \times \frac{\left(\frac{a_1 f K^2}{2} \right)^2}{(2N)^2 - 4 \nu_r^2} \qquad \text{(B-12)}$$

$$\nu_z^2 = a_1 f N^2 - k + a_1 f \left[K^2 - k(k+1) \right] + 2 \times \left(\frac{f}{2} \right)^2 \frac{K^2 + k^2}{N^2 - 4 \nu_z^2} + 2 \times \frac{\left(\frac{a_1 f K^2}{2} \right)^2}{(2N)^2 - 4 \nu_z^2} \qquad \text{(B-13)}$$

or

$$\nu_r^2 = (k+1) - \frac{f^2}{2} \frac{K^2 - (k+1)^2}{N^2 - (k+1)} + \frac{f^2}{2} \frac{K^2 + (k+1)^2}{N^2 - 4 \nu_r^2} + \frac{2 \times \left(\frac{f}{2} K \right)^4}{\left[(2N)^2 - 4 \nu_r^2 \right] \left[N^2 - (k+1) \right]^2} \qquad \text{(B-14)}$$

$$\nu_z^2 = -k + \frac{f^2}{2} \frac{N^2}{N^2 - (k+1)} + \frac{f^2}{2} \frac{K^2 - k(k+1)}{N^2 - (k+1)} + \frac{f^2}{2} \frac{K^2 + k^2}{N^2 - 4 \nu_z^2} + \frac{2 \times \left(\frac{f K}{2} \right)^4}{\left[(2N)^2 - 4 \nu_z^2 \right] \left[N^2 - (k+1) \right]^2} \qquad \text{(B-15)}$$

These are the same expression as (60) and (69) in Section II. If we assume that

$$\frac{k+1}{N^2} << 1, \qquad \frac{4\nu_r^2}{N} << 1 \quad \text{and} \quad \frac{4\nu_z^2}{N} << 1,$$

the tunes can be rearranged in power series of N^{-2};

$$\nu_r^2 = (k+1) + f\left\{\frac{(k+1)^2}{N^2} + \frac{3}{2}\frac{(k+1)K^2}{N^4} + \frac{1}{36}\frac{K^4}{N^6} + \cdots\right\} \tag{B-15}$$

$$\nu_z^2 = -k + f^2\left\{\frac{1}{2} + \frac{K^2}{N^2} + \frac{5}{2}\frac{k^2K^2}{N^4} + \frac{1}{36}\frac{K^4}{N^6} + \cdots\right\}. \tag{B-16}$$

The last two terms in Eqs. (B-15) and (B-16) are obtained by assuming that $\nu_r^2 \sim (k+1)$ and $\nu_z^2 \sim k$. In Reference 1, Eq. (13.1) is only the first term of Eq. (B-15) and Eq. (13.2) is the first three terms of Eq. (B-16). For large N, we may say that the radial tune, ν_r, is practically independent of the variation of the flutter, f, and spiral parameter, K. However, the tunes of vertical motion, ν_z, is strongly dependent of the f and K.

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