Dirichlet branes, effective actions and supersymmetry

Martijn G.C. Eenink

The work described in this thesis was performed at the Centre for Theoretical Physics of the Rijks*universiteit* Groningen. This work is part of the research program of the 'Stichting voor Fundamenteel Onderzoek der Materie (FOM)', which is financially supported by the 'Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO)'.

Printed by Universal Press - Science Publishers / Veenendaal, The Netherlands.

Rijks*universiteit* Groningen

Dirichlet branes, effective actions and supersymmetry

Proefschrift

ter verkrijging van het doctoraat in de
Wiskunde en Natuurwetenschappen
aan de Rijks*universiteit* Groningen
op gezag van de
Rector Magnificus, dr. F. Zwarts,
in het openbaar te verdedigen op
vrijdag 28 oktober 2005
om 13.15 uur

door

Martijn Gerard Christian Eenink

geboren op 12 april 1978
te Winterswijk

Promotor:	Prof. dr. M. de Roo
Beoordelingscommissie:	Prof. dr. J. de Boer Prof. dr. M. Cederwall Prof. dr. A. Sevrin

ISBN: 90-367-2397-3

Contents

Introduction

This year we celebrate the World Year of Physics [1], marking the 100th anniversary of Einstein's 'miraculous year' 1905. In that year Einstein published his three remarkable papers which laid the groundwork for much of modern theoretical physics.

Einstein did his work in a period[1] when physics was facing what may be called a crisis. In the decade before 1905 many phenomena had been observed for which the physics canon (which consisted of classical mechanics, electromagnetism, atomic theory and statistical mechanics) did not have satisfactory explanations. Let us recall a few well-known examples. The classical equipartition theorem of statistical mechanics turned out to be inconsistent with results from spectroscopic measurements on diatomic gases, a problem which was only solved with the advent of quantum mechanics. The famous ether drift experiment of Michelson and Morley falsified the mechanical/hydrodynamical ether theory which physicists at that time considered to be a candidate for a 'theory of everything'.

This period also witnessed the discovery of a variety of rays in electric discharge experiments with vacuum tubes – see table 3.1 of [2] for a list – and the discovery of radioactivity. These were also phenomena for which the classical atomic theory had no explanation. Perhaps the most important of these discoveries was that of the electron by J. J. Thomson in 1897, an event which marks the birth of particle physics.

Although Einstein was certainly not the only person to contribute to the solution of the puzzles raised in the years before 1905, his work was arguably the most important. His explanation of the photoelectric effect marks the discovery of quantum theory and fueled the development which eventually led to the formulation of quantum mechanics in the 1920s. His work on the theory of Brownian motion was the definite confirmation of the atomic theory.

And there was the work for which Einstein is best known: the invention of the theory of relativity. Einstein realized that the classical distinction between space and time as reflected in the Galilean principle of relativity is incompatible with Maxwell's equations of electromagnetism. Instead of abandoning the principle of covariance, which states that all *inertial* observers are equivalent, Einstein replaced Galilean

[1] See [2] for a history of twentieth-century physics.

relativity with his own special relativity, in which space and time are treated on more or less the same footing as elements of spacetime. Again, this work marks only the beginning of a development which altered the way in which physicists understand spacetime and gravity, and which culminated in Einstein's publication of the theory of general relativity in 1915.

In the century that followed Einstein's miracle year, physics has made an almost unbelievable amount of progress and reached a degree of maturity that is unrivaled by any other field of science[2]. Twentieth century physics has provided answers to elementary questions as: why is the sky dark at night? Why is gravity always attractive? Why do metals conduct electricity and heat so well? Where does the sun get its energy from? The list goes on[3]. In experiments, physicists have probed distances as small as 10^{-18} m – a thousandth of the size of the hydrogen atom's nucleus –, and as big as 10^{26} m – the size of the observable universe. Many of the phenomena encountered in this phenomenal range of energy scales have a satisfactory explanation in terms of current physical theories. However, there are still many open questions in physics.

A lot of these questions are associated with situations in which the underlying physical theory is well established, but too difficult to handle in practice because of computational complications. Some associated buzzwords are nonlinearity (strong sensitivity to small variations in initial conditions), complexity (large number of degrees of freedom with no simple collective behavior) and nonperturbativity (absence of a small parameter in terms of which a meaningful approximation scheme can be set up). Although it is sometimes possible in these cases to capture at least some of the physics – which is often very interesting – in *effective* theories, most of these problems require a head-on approach. With the advent of powerful computers in the past decades, this has become more and more feasible.

But there are also physical phenomena that are not explained or expected to be explained by the currently established theories at all. It is not too surprising that such phenomena are associated with the extremely long or short distance scales we mentioned above. The theories that describe physics at these extreme length scales are the domain of theoretical high-energy physics. In the following we will discuss present theories and see where they fail. We will see that some of these issues have to do with situations in which quantum gravity effects are expected to be important.

A complete theory of quantum gravity has yet to be constructed, but at the moment there are two competing candidates – loop quantum gravity and string theory – that give at least a partial (but very different!) formulation of such a theory. This thesis will deal with a topic in string theory. At the end of this introduction we will therefore give a brief overview of this field, focusing on the aspects that are needed

[2]Note that we use a broad definition of physics, including for example a relatively small subject as quantum chemistry but also the entire field of astronomy.

[3]The reader is invited to make up his/her own list, it's fun!

to explain the title of this thesis.

The Standard Model: its achievements and its shortcomings

At the length scales that are probed by contemporary particle accelerators, the strength of the gravitational interaction is many orders of magnitude smaller than that of the other interactions. General relativity can thus be ignored at these scales and spacetime can be treated as flat. However, a description of the physics at these energies does require special relativity in addition to quantum mechanics. It turns out that this requirement implies that we have to treat particles as manifestations of quantum fields.

The study of the general structure of quantum field theories has taught us why there exists antimatter and why particles with integer and half-integer spin obey Bose and Fermi statistics, respectively. The particles with Bose statistics (the bosons) exhibit the kind of collective behavior that we associate with *force fields*, whereas the particles with Fermi statistics (the fermions) obey the Pauli exclusion principle and can form stable lumps of *matter*. Another remarkable consequence of quantum field theory is that the strength of the interactions between particles depends on the energy scale at which we investigate these particles. This is called renormalization group flow and we talk about *running coupling constants*.

The physics that we see at 10^{-18} m is governed by a renormalizable quantum field theory called the Standard Model which was constructed in the 1970s. The Standard Model describes three of the four 'fundamental' forces of the universe. These are the electromagnetic, weak and strong interactions. The electromagnetic interaction is long ranged, whereas the other two are short ranged and only manifest themselves on subatomic scales. These forces are mediated by spin 1 particles, the gauge bosons[4]. For the electromagnetic interaction this is the photon, for the weak interactions we have the W and Z bosons, and for the strong force these are the gluons. The Standard Model contains three 'generations' of spin $\frac{1}{2}$ fermions. Each of these generations consists of two quarks, an electron, a neutrino and their antiparticles and are identical as far as the three gauge interactions are concerned. All these particles are involved in the weak interactions, but only the quarks feel the strong interaction. The quarks and the electron are electrically charged and interact with the photon, but the neutrino is neutral. The generations differ only in their interactions with the Higgs boson, a massive scalar field. These interactions are quite important since they determine the mass of the fermions. The Higgs boson also gives a mass to the W and Z bosons, which explains why the weak force is short-ranged. The strong force is short-ranged for a completely different reason. As we lower the energy scale at which we study the strong interactions, the coupling constant increases and at a certain scale becomes

[4]The name derives from the fact that they have a field theoretical description in terms of Yang-Mills gauge theories, i.e. theories with local symmetries.

so large that we can no longer trust our analytical calculations which are based on a perturbative expansion in powers of the coupling constant. Experiments however indicate that at larger distances the quarks and gluons become confined into nucleons and pions, and numerical calculations seem to confirm these results.

The Standard Model agrees with experiment. For one, all the particles in the model have by now been observed, with the exception of the Higgs boson. Theoretical considerations give an upper bound of roughly $1\,\mathrm{TeV/c^2}$ for the mass of the Higgs. The Higgs should therefore be observed in the next generation of accelerators. The interactions between the particles have of course also been studied and found to agree with the renormalizable interactions of the model. These measurements are however of only a limited accuracy, and it has not been ruled out that the leading order nonrenormalizable corrections to the Standard Model Lagrangian lie 'just around the corner'. This is fortunate since, as we have mentioned, there are various other theoretical and observational reasons to expect physics 'beyond the Standard Model' to appear already in the next generation of accelerators. To build up the tension, we first discuss the theoretical reasons – which are already quite convincing – and save the most devastating blow that has been dealt to the Standard Model since it was conceived in the 1970s for later, after we have discussed general relativity.

Some obvious questions that immediately come to mind are: why three generations and not more or less? Why do only quarks feel the strong force? Why is the weak interaction parity asymmetric? Why only three forces and where do they come from? In addition, the Standard Model contains 25 dimensionless parameters – see e.g. [3] for a list. Though the values of these parameters can and have been obtained in experiments, it would be nice to actually calculate them in an underlying theory. Intriguingly, the Standard Model itself hints at a possible answer to at least some of these questions.

It turns out that the three running coupling constants that are associated with the Standard Model semisimple $SU(3)\times SU(2)\times U(1)$ gauge group become approximately equal at the tremendously high energy of roughly $10^{15}\,\mathrm{GeV}$. This *suggests* that at this energy the three forces become unified in a single 'grand unified theory' (GUT) based on a simple gauge group. As we will see below, recent cosmological observations also indicate that something drastic happens at the so-called GUT scale.

This raises an immediate issue. If it is true that the interactions and parameters of the Standard Model have their origin at the GUT scale, and are *naturally* expected to be of the same size at this scale, why then is the Higgs mass so small ($\sim 10^2\,\mathrm{GeV/c^2}$) at accelerator energies? One would expect it to be larger by 13 orders of magnitude (!) because the mass of a scalar field is a relevant operator. This is known as the *hierarchy problem*. One of the ways in which this problem can be avoided is if the mass of the Higgs is protected from quantum corrections by a symmetry. The most viable candidate is *supersymmetry*.

Supersymmetry differs from conventional internal and spacetime symmetries in that it mixes bosonic with fermionic fields. We will see in this thesis that it plays an important role in string theory, of which it is a necessary ingredient. But here we note that there are convincing reasons to expect to see supersymmetry already at low energies (10^2-10^3 GeV). In addition to solving the hierarchy problem, it predicts the existence of a whole new range of particles, since in supersymmetric extensions of the Standard Model (SSM), every particle has to be accompanied by its superpartner. Furthermore, in SSM's the gauge coupling constants can become exactly equal at the GUT scale, which typically lies at roughly 10^{16} GeV in these models.

The GUT scale lies tantalizingly close to another important scale in nature, the Planck scale, which is the scale at which quantum gravity effects become important:

$$E_{\text{Planck}} = \left(\frac{\hbar c^5}{8\pi G}\right)^{1/2} \sim 10^{18}\,\text{GeV}, \qquad \ell_{\text{Planck}} = \left(\frac{8\pi G\hbar}{c^3}\right)^{1/2} \sim 10^{-34}\,\text{m}.$$

This suggests that unification and quantum gravity are in fact closely related to each other. We will indeed see that in string theory the gravitational and gauge interactions are treated on the same footing. But before we turn to string theory, we first need to discuss general relativity and the impact of recent cosmological observations on our understanding of the universe.

General relativity: classical and quantum

At astronomical length scales, we do not see the forces of the Standard Model. The weak and strong interactions are short-ranged and the electromagnetic interactions cancel out since astronomical objects are electrically neutral. Gravity is therefore the dominant force, despite its relative weakness, since it is always attractive. We already mentioned that gravity is described by general relativity.

Just as the Standard Model, general relativity (GR) is a field theory. However, GR differs from other field theories in that it does not describe fields that live on a fixed background spacetime, but it in fact describes spacetime *itself* as a dynamical entity, the metric field[5]. In GR gravity manifests itself through the curvature of spacetime, which is in turn caused by the presence of mass and energy.

The experimental successes of GR are as impressive as those of the Standard Model. For instance, GR explains the bending of light by massive objects like our sun, and the orbits of systems of binary pulsars. GR predicts the existence of black holes, objects that are so massive that they are hidden behind an event horizon, a surface from which even light cannot escape (at least classically). Nowadays astronomers

[5] This is a deliberate oversimplification. In fact, spacetime does enter in GR as an independent mathematical construct, but it turns out that the local symmetries of GR, which go under the name of diffeomorphisms or *general covariance*, tell us that this background spacetime is not a gauge invariant quantity. What then actually are the observables of GR is a delicate question, one we do not want to into in any detail. See e.g. [4] for an interesting discussion.

seem to agree that black holes are indeed present throughout the universe, in all kinds of sizes [5,6]. GR also plays a pivotal role in contemporary cosmology, where it explains for instance the observed cosmological redshift of the light of distant galaxies as a consequence of the expansion of the universe. But GR is present in everyday life as well: the Global Positioning System would not work properly if the effects of the gravitational redshift had not been taken into account [7].

In the above example, GR is used as a classical field theory. There is however nothing wrong with treating it quantum mechanically as long as the vacuum expectation value of the metric field is that of flat spacetime. The perturbations of the metric field around this flat background are then described by a quantum field theory of a spin 2 particle, known as the graviton. Just as the Coulomb interaction of two charges follows from the exchange of photons in QED, Newton's law follows from the exchange of gravitons by (not too heavy) massive objects[6]. Since Newton's constant G is dimensionful, the usual effective field theory lore leads us to suspect that the perturbative description of gravity breaks down just below or at the Planck scale. A fundamental theory of quantum gravity should then take over.

It is important to realize that there are extreme situations where the perturbative description of gravity breaks down. In fact, there are powerful theorems[7] in classical GR which predict that spacetime singularities – i.e. points at which the gravitational field blows up – occur generically in nature, e.g. in black holes but also in cosmology[8]: if we trace back the time-evolution of current cosmological models, they break down at an *initial singularity*. In order to understand the physics that goes on at the length scales of these singularities, we certainly need a theory of quantum gravity.

But quantum gravity may even be relevant at length scales that are considerably larger than those associated with the singularities. Indeed, the high energy degrees of freedom that reside near[9] the singularities may sneak in to our low-energy effective description of nature by lowering their energy via the gravitational redshift (in the case of the Hawking radiation of black holes) and the cosmological redshift (in the early universe). It is thus not clear whether the effective field theory description of particle physics remains valid in the presence of strong gravitational fields. This important issue is the subject of much debate at the moment. See [11] for an introduction.

Recent cosmological observations

Even more reasons to doubt the validity of effective field theories in the presence of gravitational fields are provided by recent cosmological observations. Since we do not

[6] It is amusing to note that one also calculate quantum corrections to Newton's potential [8], taking the analogy with QED – the Lamb shift – even further.

[7] See [9] for an introduction and references.

[8] See [10] for a nice primer on cosmology.

[9] We use the word 'near' for both space- and timelike separations.

want to dive deeper into this subject than we really need to, we refer the reader to the excellent review [12], where these results are explained in more detail.

The current observational evidence – in particular the detailed measurements of the power spectrum of the cosmic microwave background – provide strong evidence for the claim that the universe underwent a period of accelerated expansion called *inflation* shortly after its conception. Though the mechanism that drives inflation is still unknown (there are many different proposals), inflation solves several important problems of the conventional Big Bang model (the horizon problem, the flatness problem and the problem of structure formation). It took place when the universe had a temperature of 10^{15}-10^{16} GeV (in natural units), which corresponds to the GUT scale! This is the first real experimental indication that there may be truth behind the hypothesized unification of the gauge interactions.

But there is more. Cosmologists now know the total present-day energy density of the universe. This energy density arises from three sources: baryonic matter (B), cold dark matter (DM) and dark energy (Λ), which contribute with the following fractions:

$$\Omega_{\mathrm{B}} = 0.04, \qquad \Omega_{\mathrm{DM}} = 0.26, \qquad \Omega_{\Lambda} = 0.70.$$

Now for the punchline: none of these numbers is understood in terms of Standard Model physics!

Baryonic matter may be *described* well by the Standard Model at accelerator energies, but it is still mysterious why there is a net amount of baryons in the universe to begin with. It is known that the Standard Model does not provide an answer to this question. But most of the matter in the universe comes in the form of cold (i.e. nonrelativistic) dark matter. Every known particle in the Standard Model has been ruled out as a candidate for this type of matter. It may very well turn out that at least a large part of the dark matter is comprised of the additional particles that are introduced in supersymmetric extensions of the Standard Model.

The dark energy is associated with the energy density of the vacuum, ρ_{vac}, which is unobservable in theories without gravity, but contributes to Einstein's general relativity in the form of the cosmological constant Λ. Nowadays, ρ_{vac} is thought to be very small, but nonzero nonetheless, $\rho_{\mathrm{vac}} \sim (10^{-3}\,\mathrm{ev})^4$. In quantum field theories, the vacuum energy density is proportional to the fourth power of cut-off of the theory. If we treat general relativity as an effective field theory with a cut-off which lies at the Planck scale, we get a theoretical prediction of $\rho_{\mathrm{vac}} \sim (10^{18}\,\mathrm{GeV})^4$! This is the 'mother of all discrepancies' as Zee [13] calls it, and goes under the name of the *cosmological constant problem*. Clearly, there is something crucial missing in our understanding of the vacuum[10].

[10] In theories with global supersymmetry, the vacuum energy is guaranteed to be zero. This lowers the expected value of Λ to the scale which is associated with the spontaneous breaking of supersymmetry, i.e. $\rho_{\mathrm{vac}} \sim (10^3\,\mathrm{GeV})^4$. This is still much too large.

String theory: perturbation theory and beyond

A century after Einstein's miracle year, physics again faces serious questions. Answers to these questions seem to require a revision of the way in which physicists describe nature at its smallest scales. What is needed is a theory of quantum gravity which is valid at energies beyond the Planck scale. String theory is just that.

In the first two chapters of this thesis we will review string theory in quite some detail, so will keep our discussion here short.

String theory is at the outset a *perturbative* theory of quantum gravity. Point particles arise as vibrations of little strings which propagate in *flat* spacetime. Remarkably, the graviton and nonabelian gauge fields are necessarily part of the spectrum. String theory thus naturally unifies the gravitational interaction with Yang-Mills theory. It turns out that the perturbative interactions of the strings are *finite*. There is no need for the introduction of an ultraviolet cut-off in the theory and the theory is consistent up to arbitrarily high energy scales and thus truly fundamental. String theory is necessarily supersymmetric and requires the existence of six additional spacelike dimensions beyond the ones we observe today.

There are actually five different consistent perturbative superstring theories. During the 1990s, it gradually became understood that these five theories actually arise as different limits in the parameter space of a *single* underlying theory, which has been named M-theory. The form that this theory will eventually take still remains shrouded in mystery.

It was realized that other extended objects than strings alone – known as p-branes, where p stands for their dimensionality – will play a fundamental role in the formulation of M-theory. p-branes are higher-dimensional generalizations of black holes and are generically associated with nonperturbative effects.

An important class of p-branes are the Dirichlet p-branes. These are special, since they do have a simple *microscopic* description in terms of string perturbation theory, where they arise as hyperplanes on which strings can end. At low energies, the strings manifest themselves as a gauge theory that lives on the D-brane.

Systems of multiple D-branes show quite remarkable behavior. They perceive spacetime in a way that is entirely different from what we are used to: according to D-branes, spacetime is *noncommutative*. Although this behavior is far from being understood completely, it provides important hints about the way in which spacetime and geometry will have to be described in M-theory.

In this thesis we will study the behavior of single and multiple D-branes from the point of view of their effective description in terms of supersymmetric gauge theories. We will see how the nonrenormalizable corrections to the leading order super-Yang-Mills actions provide interesting information about the 'stringy' aspects of D-brane physics. We will then explicitly construct some of these corrections, using the constraints of supersymmetry and information from perturbative string theory.

Chapter 1

Strings and D-branes

This long introductory chapter is aimed at an audience of non-specialists and will hopefully provide a sufficient amount of background material in order to justify the higher pace of the later chapters.

Reviews and books on string theory often start with a discussion of the bosonic string. This allows one to explain some of the basic concepts without the need to worry about the additional technical complications that one encounters when dealing with the more interesting superstrings. The present chapter will be no exception. A brief review of M-theory will be postponed to the next chapter.

We start with the free string and discuss the appearance of the critical dimension and gravity. We will be quite explicit. Our discussion of string interactions starts with a review of the concepts involved in interacting *field* theories. In this way we hope to explain the limitations of string perturbation theory and the usefulness of effective field theory techniques for the study of strings. Finally, we move on to D-branes and their effective field theory description, with an emphasis on T-duality.

The literature on string theory is vast. The following books [14–19] provide good introductions to the subject and contain extensive lists of references. In particular, the reader will find thorough discussions on string perturbation theory in these books which should supplement the brief review we provide here.

1.1 The free bosonic string

Our discussion starts with the classical theory of a relativistic string. A string is a one-dimensional object which sweeps out a two-dimensional surface (called the world-sheet Σ) as it moves through spacetime. Spacetime is considered to be flat d-dimensional

Minkowksi space[1] $\mathcal{M}$ with metric $\eta_{\mu\nu} = \text{diag}(-1, +1, \ldots, +1)$. The embedding of the world-sheet in spacetime $\imath : \Sigma \to \mathcal{M}$ is given by the functions[2] $X^\mu(\tau, \sigma)$, where $\sigma^\alpha = (\sigma^1, \sigma^2) = (\tau, \sigma)$ are dimensionless coordinates with which we parametrize the world-sheet. τ and σ are respectively time- and spacelike w.r.t. the induced metric h, which is given by:

$$h_{\alpha\beta} = (\imath^*\eta)_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{1.1.1}$$

The coordinates take the values $-\infty < \tau < \infty$ and $0 \leq \sigma \leq \ell$, where ℓ is a parameter which we will discuss later. The classical dynamics of the free string is governed by a generalization of Fermat's principle of least time: the area of the world-sheet is minimal. This leads us to the action of Nambu and Goto:

$$S_{\text{NG}}[X] = -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \, \sqrt{-\det h_{\alpha\beta}}. \tag{1.1.2}$$

The constant α' is known as the Regge slope parameter or simply as 'alpha-primed'. It has a mass dimension[3] of -2 and is related to the string's tension T by $T = 1/2\pi\alpha'$. For later convenience, we introduce yet another parameter called the *string length* by $\ell_s \equiv \sqrt{\alpha'}$.

The action (1.1.2) can be interpreted as defining a two-dimensional relativistic field theory for d scalar fields X^μ. From this point of view, we refer to spacetime as the *target space*. The appearance of the square root in the action makes this theory difficult to analyze. Fortunately, there is an alternative formulation which gives an action that is quadratic in the X^μ. First we introduce an auxiliary symmetric tensor field $\gamma_{\alpha\beta}(\tau, \sigma)$, which can be viewed as an intrinsic metric on the world-sheet. We claim that the following action is (at least classically) equivalent to (1.1.2):

$$S_{\text{P}}[\gamma, X] = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \, \sqrt{-\gamma} \, \gamma^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X^\mu. \tag{1.1.3}$$

This expression is known as the Howe-Tucker-Polyakov action, but we will simply refer to it as the Polyakov action. We obtain the following energy-momentum tensor

[1] We do not allow for spacetime metrics with more than one *time* dimension. Such spacetimes have closed timelike curves and give problems with causality already on the classical level.

[2] In general, we denote coordinates in spacetime by x^μ, but use capital letters X^μ when we refer to the embedding functions of extended objects such as strings.

[3] In principle string theory does not contain any fundamental dimensionful parameters. Indeed, we see that the factor $1/2\pi\alpha'$ can be removed by redefining the X^μ's. Nevertheless, in practice it is very convenient to be able to work with a system of units. We can manage this by keeping α' explicit and measure quantities in terms of powers of $1/\sqrt{\alpha'}$. To make contact with the way dimensions are usually defined in physics we note that the Hamiltonian for the free string is proportional to $1/\sqrt{\alpha'}$. Measuring units of mass is therefore equivalent to counting powers of $1/\sqrt{\alpha'}$. We could go on and introduce the well-known constants $\hbar$ and c, thereby defining units of length and time. This would allow us to make contact with the way the fundamental constants of nature are usually expressed. However, we will not do this and keep working in *natural* units, in which [length] = [time] = [mass]$^{-1}$.

from this action:

$$T_{\alpha\beta} := -\frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma^{\alpha\beta}} = \frac{1}{2\pi\alpha'}\left\{\partial_\alpha X_\mu \partial_\beta X^\mu - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\gamma\delta}\partial_\gamma X_\mu \partial_\delta X^\mu\right\}. \tag{1.1.4}$$

The equation of motion for $\gamma_{\alpha\beta}$ is simply $T_{\alpha\beta} = 0$, which is algebraic in $\gamma_{\alpha\beta}$. The Nambu-Goto action is recovered from (1.1.3) by eliminating $\gamma_{\alpha\beta}$ by means of its equation of motion.

The Polyakov action has the following local symmetries:

- General coordinate transformations or diffeomorphisms on the world-sheet:

$$\sigma^\alpha \to \sigma^\alpha - \xi^\alpha(\sigma). \tag{1.1.5}$$

  The presence of these symmetries is expected on physical grounds: physics should not depend on the way we choose to parametrize the world-sheet. The fields transform as

$$\delta X^\mu(\sigma) = \mathcal{L}_\xi X^\mu = \xi^\alpha \partial_\alpha X^\mu, \tag{1.1.6a}$$

$$\delta\gamma_{\alpha\beta}(\sigma) = (\mathcal{L}_\xi \gamma)_{\alpha\beta} = \xi^\gamma \partial_\gamma \gamma_{\alpha\beta} + \partial_\alpha \xi^\gamma \gamma_{\gamma\beta} + \partial_\beta \xi^\gamma \gamma_{\alpha\gamma}. \tag{1.1.6b}$$

  As usual, general covariance leads to the conservation of the energy-momentum tensor: $\nabla_\alpha T^{\alpha\beta} = 0$ when the equations of motion of the matter fields are satisfied.

- Local Weyl rescalings of the metric:

$$\begin{aligned} \delta X^\mu(\sigma) &= 0, \\ \delta\gamma_{\alpha\beta}(\sigma) &= \Lambda(\sigma)\gamma_{\alpha\beta}(\sigma). \end{aligned} \tag{1.1.7}$$

  As a consequence, the trace of the energy-momentum tensor vanishes identically: $\gamma^{\alpha\beta}T_{\alpha\beta} \equiv 0$.

We fix these symmetries by imposing the *conformal gauge*:

$$\gamma_{\alpha\beta} \equiv \eta_{\alpha\beta} = \mathrm{diag}\,(-1, +1), \tag{1.1.8}$$

The Polyakov action reduces to:

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\, \eta^{\alpha\beta}\partial_\alpha X_\mu \partial_\beta X^\mu \tag{1.1.9}$$

In the conformal gauge we have a free two-dimensional field theory for d scalars. Note however that X^0 appears in the action with the wrong sign. The equation of motion for the X^μ is simply the two-dimensional wave equation:

$$\Box X^\mu(\tau,\sigma) = \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu(\tau,\sigma) = 0, \tag{1.1.10}$$

with the general solution

$$X^\mu(\tau,\sigma) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma). \tag{1.1.11}$$

The subscripts R and L stand for right- and left-moving respectively. This needs to be supplemented with suitable boundary conditions in order that

$$\delta X^i \partial_\sigma X^i \Big|_{\sigma=0}^{\ell} = 0. \tag{1.1.12}$$

We will discuss the different boundary conditions below. The equation for the energy-momentum tensor, $T_{\alpha\beta} = 0$, has now become a constraint that needs to be imposed on solutions of the equations of motion for X^μ. The constraints can be expressed as:

$$(\partial_\tau X^\mu \pm \partial_\sigma X^\mu)^2 = 0. \tag{1.1.13}$$

1.1.1 The light-cone gauge

Actually, there are some symmetries that are not completely fixed by our gauge choice (1.1.8). They are given by

$$\tau \mapsto \tau' = f(\tau+\sigma) + g(\tau-\sigma), \tag{1.1.14a}$$
$$\sigma \mapsto \sigma' = f(\tau+\sigma) - g(\tau-\sigma), \tag{1.1.14b}$$

where f and g are arbitrary functions. We can use these symmetries to remove the unphysical scalar X^0 from (1.1.9). First we introduce *light-cone coordinates* on spacetime:

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^{d-1}). \tag{1.1.15}$$

In these coordinates we have the following inner product of coordinates and momenta

$$x \cdot p = \eta_{\mu\nu} x^\mu p^\nu = -x^+p^- - x^-p^+ + x^ip^i, \tag{1.1.16}$$

with $i = 1, \ldots, d-2$. Light-cone time is conventionally defined to be x^+. Hence p^- is the light-cone Hamiltonian in the target space. We then impose the *light-cone gauge*[4]:

$$X^+ = x_0^+ + \frac{2\pi\alpha'}{\ell} p^+ \tau. \tag{1.1.17}$$

[4] The gauge choice $X^+ \sim \tau$ essentially states that we can trade time evolution on the world-sheet for light-cone time evolution in the target space. To motivate the choice of the constant λ in $X^+ = \lambda\tau$, we note that the spacetime momentum carried by the string is $p^\mu = T\int_0^\ell \mathrm{d}\sigma\, \partial_\tau X^\mu$, where T is the tension. In particular $2\pi\alpha' p^+ = \int_0^\ell \mathrm{d}\sigma\, X^+ = \lambda\,\ell$.

The parameter x_0^+ sets the origin of time. This gauge can always be reached from a general $X^+(\tau,\sigma) = X_R^+(\tau-\sigma) + X_L^+(\tau+\sigma)$ by taking

$$2\pi\alpha' p^+/\ell \times f(\tau+\sigma) = X_L^+(\tau+\sigma) - a_L x_0^+, \tag{1.1.18a}$$

$$2\pi\alpha' p^+/\ell \times g(\tau-\sigma) = X_R^+(\tau-\sigma) - a_R x_0^+, \tag{1.1.18b}$$

in (1.1.14a), where a_L and a_R satisfy $a_L + a_R = 1$ but are otherwise arbitrary real numbers. From (1.1.14b) we have also that

$$\sigma' \sim X_L^+ - X_R^+ - (a_L - a_R)x_0^+. \tag{1.1.19}$$

Thus the light-cone gauge also fixes the spatial coordinate σ, but only up to an arbitrary constant shift $\sigma \to \sigma + a$.

With the light-cone gauge choice, we can easily solve the constraints (1.1.13) by expressing X^- in terms of X^+ and X^i. We will not need an explicit expression, though. Indeed, the Polyakov Lagrangian reduces to

$$L = -p^+\partial_\tau x^- - \frac{1}{4\pi\alpha'}\int_0^\ell \mathrm{d}\sigma\, \partial_\alpha X^i \partial^\alpha X^i, \tag{1.1.20}$$

where we defined

$$x^-(\tau) := \frac{1}{\ell}\int_0^\ell \mathrm{d}\sigma\, X^-(\tau,\sigma), \tag{1.1.21}$$

which is in accordance with (1.1.16), since we conclude from (1.1.20) that p^+ is the canonical momentum conjugate to x^-. We see that only the spatially constant part of X^- is dynamical. The Hamiltonian for our two-dimensional theory is given by

$$H = \int_0^\ell \mathrm{d}\sigma\, \mathcal{H} = \frac{1}{4\pi\alpha'}\int_0^\ell \mathrm{d}\sigma \left\{(\partial_\tau X^i)^2 + (\partial_\sigma X^i)^2\right\}. \tag{1.1.22}$$

So we end up with a Hamiltonian for $d-2$ free massless scalar fields X^i, which still satisfy the wave equation:

$$\Box X^i(\tau,\sigma) = 0. \tag{1.1.23}$$

Let us first discuss the relation between H and p^- before proceeding. The light-cone Hamiltonian p^- generates x^+ translations whereas the world-sheet Hamiltonian H generates τ translations. But x^+ and τ are related by (1.1.17), hence:

$$p^- = \frac{\ell}{2\pi\alpha' p^+}H. \tag{1.1.24}$$

We will use this equation to find the mass of the string.

We now discuss the possible boundary conditions that we can impose in order that

$$\delta X^i \partial_\sigma X^i \Big|_{\sigma=0}^{\ell} = 0. \tag{1.1.25}$$

First of all there are *closed strings*. They have no endpoints; hence there is a *periodic* boundary condition:

$$X^i(\tau, \sigma) = X^i(\tau, \sigma + \ell), \quad \forall\, i. \tag{1.1.26}$$

The shift symmetry $\sigma \to \sigma + a$ is unaffected by this condition. Its consequences will be discussed below when we quantize the closed string.

On the other hand there are *open strings*, which do have endpoints. Boundary conditions can be imposed on these endpoints in two ways. The strings can either move freely or have their endpoints confined to hyperplanes. The first case corresponds to imposing *Neumann* boundary conditions on all directions at both endpoints:

$$\partial_\sigma X^i(\tau, 0) = \partial_\sigma X^i(\tau, \ell) = 0, \quad \forall\, i. \tag{1.1.27}$$

In the second case we impose a *Dirichlet* condition on some direction(s) at one (or both) of the endpoints:

$$\delta X^i(\tau, 0 \text{ or } \ell) = 0 \;\Rightarrow X^i(\tau, 0 \text{ or } \ell) = \text{constant}, \quad \text{for some } i. \tag{1.1.28}$$

The hyperplanes on which open strings can end are called *D-branes* (where the 'D' stands for Dirichlet and 'brane' generalizes the concept of a membrane).

D-branes are an important class of extended objects in string theory and have played a crucial role in understanding the nonperturbative structure of string theory. Extended objects are in general called p-branes, where p stands for the number of *spatial* directions in the world-volume of these objects. So, for example, a particle is a 0-brane and a string a 1-brane. An open string that has both endpoints confined to the same Dp-brane satisfies Neumann boundary conditions in the $p+1$ directions tangent to the Dp-brane (including time) and Dirichlet conditions in the $d-p-2$ direction transverse to the brane[5]. We will discuss D-branes in more detail later in this chapter.

Note that the different open string boundary conditions explicitly break the shift symmetry $\sigma \to \sigma + a$.

[5]Note that in the light-cone gauge there are only $d-2$ directions X^i on which we can impose a Dirichlet boundary condition. So we can discuss Dp-branes for $p = 1, \ldots, d-1$. There also exists a D0-brane, the D-particle. This is a perfectly well-defined object within string theory, but in order to describe it we need a covariant gauge in which all the spatial X^μ's are kept. It is also possible to impose a Dirichlet condition on the time direction X^0 (thereby obtaining a spacelike D-brane or S-brane) or even on all directions in which case one obtains the D-instanton, which plays a similar role in string theory as the ordinary Yang-Mills instanton in gauge theories. We will not discuss these extended objects any further, though.

1.1.2 The open string spectrum

We now consider the open string with Neumann boundary conditions in all directions. The equations of motion and boundary conditions are solved by the following mode expansion:

$$X^i(\tau,\sigma) = x_0^i + \frac{2\pi\alpha'}{\ell}p^i\tau + i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\alpha_n^i\,\mathrm{e}^{-in\pi\tau/\ell}\cos\frac{n\pi\sigma}{\ell}. \tag{1.1.29}$$

We see that the left- and right-moving degrees of freedom have been related to each other by the Neumann condition. The σ-independent part describes the motion of the center of mass of the string. p^i is the momentum of the center of mass in the i-direction:

$$p^i = \frac{1}{2\pi\alpha'}\int_0^\ell \mathrm{d}\sigma\,\partial_\tau X^i. \tag{1.1.30}$$

The σ-dependent part describes the 'wiggling' of the string. Since the X^i are real, we have $\alpha_{-n}^i = (\alpha_n^i)^*$. The Hamiltonian becomes:

$$H = \frac{\pi\alpha'}{\ell}p^ip^i + \frac{\pi}{2\ell}\sum_{n\neq 0}\alpha_{-n}^i\alpha_n^i. \tag{1.1.31}$$

We arrive at the classical mass formula by using $p^2 = -2p^+p^- + p^ip^i = -M^2$ and $p^+p^- = \ell H/2\pi\alpha'$:

$$M^2 = \frac{1}{2\alpha'}\sum_{n\neq 0}\alpha_{-n}^i\alpha_n^i. \tag{1.1.32}$$

Let us turn to quantum mechanics. The canonical variables (x^-, p^+) and $(X^i, \Pi^j = \partial_\tau X^j/2\pi\alpha')$ are hermitian operators on a Hilbert space and satisfy the following equal time commutation relations:

$$[x^-, p^+] = -i, \tag{1.1.33a}$$

$$[X^i(\tau,\sigma), \Pi^j(\tau,\sigma')] = i\delta^{ij}\delta(\sigma-\sigma'). \tag{1.1.33b}$$

Some algebra leads to

$$[x_0^i, p^j] = i\delta^{ij}, \tag{1.1.34a}$$

$$[\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m+n}, \tag{1.1.34b}$$

where $\delta_{m+n} := \delta_{m,-n}$. Now $\alpha_{-n}^i = (\alpha_n^i)^\dagger$, hence the α_n^i with $n > 0$ are lowering operators and the α_{-n}^i raising operators (with an unconventional normalization). The mass-shell condition becomes:

$$M^2 = \frac{1}{\alpha'}\left\{\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i - a\right\} = \frac{1}{\alpha'}(N-a), \tag{1.1.35}$$

where we normal ordered the sum over the oscillators. The *level operator* N adds up the occupation numbers for the separate oscillators α^i_{-n}, weighted by the oscillators' level n. The constant a corresponds to a zero-point energy and diverges, as is usual in quantum field theories:

$$a = -\frac{1}{2}\sum_{n=1}^{\infty}[\alpha^i_n, \alpha^i_{-n}] = -\frac{d-2}{2}\sum_{n=1}^{\infty} n. \tag{1.1.36}$$

We can get a finite answer for the zero-point energy by means of a suitable regularization procedure. Zeta-function regularization [20–22] turns out to be particularly convenient.

We introduce a complex parameter s, which will play the role of a cut-off, and replace the divergent sum (1.1.36) by:

$$\sum_{n=1}^{\infty} n \;\rightarrow\; \sum_{n=1}^{\infty}\frac{1}{n^s}, \tag{1.1.37}$$

For $\mathrm{Re}\, s > 0$ this sum is actually a representation of the Riemann zeta-function $\zeta(s)$. It has a unique analytic continuation to negative values of s. In particular $\zeta(-1) = -1/12$. The zeta-function regularization method boils down to simply making the replacement $1+2+3+\ldots \;\rightarrow\; \zeta(-1) = -1/12$. Hence

$$a = \frac{d-2}{24}. \tag{1.1.38}$$

We see that the zero-point energy does not vanish. This is a Casimir energy associated with the finite size of the string world-sheet.

The ground state $|0,k\rangle$ is the unique state with light-cone momentum $k = (k^+, k^i)$ that satisfies $\alpha^i_n|0,k\rangle = 0$ for all positive n. It has $N = 0$ and $M^2 = -a/\alpha'$. It is a tachyon if $d > 2$. We can act on the ground state with raising operators to construct the complete spectrum of the open string.

There are $d-2$ states at level $N = 1$, with mass $M^2 = (1-a)/\alpha'$. They are given by $\alpha^i_{-1}|0,k\rangle$ and comprise a vector of $SO(d-2)$. Now we note something peculiar. According to Wigner's analysis of the irreducible representations of the Poincaré group, massless particle states should fall into multiplets of $SO(d-2)$, whereas massive states fall into multiplets of $SO(d-1)$. We see that for general values of a, the bosonic string does not have a Lorentz invariant spectrum.

This indicates that we need to take $a = 1$ and hence that $d = 26$. So the bosonic string is Lorentz invariant only in 26 spacetime dimensions[6]!

[6] This is only one of many ways of deriving the existence of a 'critical' dimension for the bosonic string. For instance, Polchinski [15] discusses seven different methods, with varying degrees of rigor. The trick with Riemann's ζ-function discussed in the text is probably the fastest way of arriving at the desired result. It may look like voodoo, but can be put on a firm basis and is equivalent to other regularization methods.

For a complete and rigorous proof of this remarkable result we refer to the literature. Let us however take a quick peek at level $N = 2$ to see how Lorentz invariance should come about for the massive excitations. We have the $(d-1)(d-2)/2$ states $\alpha^i_{-1}\alpha^j_{-1}|0,k\rangle$ with $i \leq j$ (the oscillators are bosonic), which provide us only with a (reducible) representation of $SO(d-2)$, whereas we expect these states to represent a massive particle since $M^2 = (2-a)/\alpha'$. Fortunately, due to the particular way in which N is defined there are also the $d-2$ states $\alpha^i_{-2}|0,k\rangle$. These states together have just the right number of degrees of freedom to describe the $(d+1)(d-2)/2$ dimensional rank-2 symmetric traceless tensor of $SO(d-1)$. So things work out nicely.

In the critical dimension the ground state $|0,k\rangle$ is a tachyon with $M^2 = -1/\alpha'$. It can be represented by a scalar field T_o. The first excited state can be represented by a massless vector field A_μ, with the usual $U(1)$ gauge invariance $A_\mu \to A_\mu + \partial_\mu \Lambda$. This gauge invariance is actually also present in string theory. We did not encounter it though, because it is fixed from the start by the light-cone gauge. It does show up in covariant gauges. It is actually possible to extend the abelian $U(1)$ gauge invariance to a nonabelian $U(n)$ invariance as we will discuss later. Besides the tachyon and the gauge field the open string spectrum contains an infinite tower of massive states of arbitrarily high spin.

The presence of the tachyon indicates that the bosonic string theory is unstable. For this reason (and others) we will eventually need to discard the bosonic string theory. It will turn out that the tachyon is absent in superstring theory and that in addition fermions are automatically incorporated in the spectrum.

1.1.3 The closed string spectrum

In the case of the closed string, the right- and left-moving degrees of freedom are not related to each other through the boundary conditions. They are not completely independent though, as we will see in a minute. We have:

$$X^i(\tau,\sigma) = X^i_R(\tau-\sigma) + X^i_L(\tau+\sigma), \tag{1.1.39}$$

with

$$X^i_R(\tau-\sigma) = \frac{1}{2}x^i_0 + \frac{\pi\alpha'}{\ell}p^i(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\alpha^i_n \mathrm{e}^{-2\pi i n(\tau-\sigma)/\ell}, \tag{1.1.40a}$$

$$X^i_L(\tau+\sigma) = \frac{1}{2}x^i_0 + \frac{\pi\alpha'}{\ell}p^i(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\tilde{\alpha}^i_n \mathrm{e}^{-2\pi i n(\tau+\sigma)/\ell}. \tag{1.1.40b}$$

In contrast with the open string, there are now two independent sets of oscillators with $\alpha^i_{-n} = (\alpha^i_n)^*$ and $\tilde{\alpha}^i_{-n} = (\tilde{\alpha}^i_n)^*$. The canonical quantization procedure leads to the same commutation relations for the center of mass coordinates and momenta as for the open string. For the oscillators we obtain:

$$[\alpha^i_m, \alpha^j_n] = m\delta^{ij}\delta_{m+n}, \qquad [\tilde{\alpha}^i_m, \tilde{\alpha}^j_n] = m\delta^{ij}\delta_{m+n}, \qquad [\alpha^i_m, \tilde{\alpha}^j_n] = 0. \tag{1.1.41}$$

The mass-shell condition reads :

$$M^2 = \frac{2}{\alpha'}(N + \tilde{N} - a - \tilde{a}), \tag{1.1.42}$$

where $a = \tilde{a} = (d-2)/24$ as before and

$$N = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n, \qquad \tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n. \tag{1.1.43}$$

We mentioned before that the shift symmetry $\sigma \to \sigma + b$ is preserved by the periodic boundary conditions. We still need to deal with it. Classically, the generator S of the shift symmetries is given by:

$$S = -\int_0^{\ell} \mathrm{d}\sigma\, \Pi^i \partial_\sigma X^i = \frac{\pi}{\ell}\sum_{n\neq 0}\big(\alpha^i_{-n}\alpha^i_n - \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n\big). \tag{1.1.44}$$

Indeed, using the Poisson bracket $\{X^i(\tau,\sigma), \Pi^j(\tau,\sigma')\}_{\mathrm{PB}} = \delta^{ij}\delta(\sigma - \sigma')$, one readily verifies that $\delta X^i = -b\,\partial_\sigma X^i \equiv -b\,\{S, X^i\}_{\mathrm{PB}}$. In the quantum theory we need to restrict the spectrum to states that are annihilated by S, since the shifts are gauge transformations. We therefore require that

$$S|N,\tilde{N},k\rangle = \frac{2\pi}{\ell}(N - \tilde{N} - a + \tilde{a})|N,\tilde{N},k\rangle \equiv 0, \tag{1.1.45}$$

where we had to normal order the oscillators as before. Since $a = \tilde{a}$, we need to impose the *level-matching condition* $N = \tilde{N}$, i.e. the number of left-movers is equal to the number of right-movers.

The already-mentioned ground state is $|0,0,k\rangle$ with $N = \tilde{N} = 0$ and $\alpha^i_n|0,0,k\rangle = \tilde{\alpha}^i_n|0,0,k\rangle = 0$ for $n > 0$. It is a scalar with mass $M^2 = -4a/\alpha'$.

The first excited state has $N = \tilde{N} = 1$ and $M^2 = 4(1-a)/\alpha'$ and consists of the $(d-2)^2$ states $\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0,0,k\rangle$. These states carry a reducible representation ζ^{ij} of $SO(d-2)$, which decomposes into a symmetric traceless 2-tensor, an antisymmetric 2-tensor and a scalar:

$$\zeta^{ij} = \frac{1}{2}\Big(\zeta^{ij} + \zeta^{ji} - \frac{2}{d-2}\delta^{ij}\,\mathrm{tr}\,\zeta\Big) + \frac{1}{2}\big(\zeta^{ij} - \zeta^{ji}\big) + \frac{1}{d-2}\delta^{ij}\,\mathrm{tr}\,\zeta. \tag{1.1.46}$$

We again conclude that we need to take $a = 1$ and $d = 26$ in order to get a Lorentz invariant theory. We obtain the same value for the critical dimension as with the open string. This is just as well, since we will see below that open string interactions necessarily give rise to closed strings.

So, apart from the infinite tower of massive fields, the closed string spectrum contains a scalar tachyon T_c with mass $M^2 = -4/\alpha'$ and three massless fields, of which the scalar Φ called the *dilaton*. The other two massless states are realized by fields of spin > 0, which therefore have an associated gauge invariance. The antisymmetric 2-tensor of $SO(d-2)$ can be obtained from a 2-form field $B_{\mu\nu}$ (often called the *Kalb-Ramond* field) with gauge transformation

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu. \tag{1.1.47}$$

It contains $(d-2)(d-3)/2 = 276$ on-shell degrees of freedom. The symmetric traceless tensor is obtained from a symmetric field $h_{\mu\nu}$ which transforms as

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{1.1.48}$$

and contains $d(d-3)/2 = 299$ on-shell degrees of freedom. This field is the *graviton* and is of course what got this whole business started in the first place.

1.1.4 World-sheet twists

Before we turn to string interactions we need to discuss one more thing. The Lagrangian (1.1.20) has an important discrete $\mathbb{Z}_2$ symmetry known as the world-sheet parity transformation $\sigma \mapsto \ell - \sigma$. It is realized by the *twist operator* Ω:

$$\Omega X^i(\tau, \sigma)\Omega^{-1} = X^i(\tau, \ell - \sigma). \tag{1.1.49}$$

Ω acts on the open string oscillators as:

$$\Omega \alpha^i_{-n} \Omega^{-1} = (-)^n \alpha^i_{-n}. \tag{1.1.50}$$

We assume that $\Omega|0, k\rangle = +|0, k\rangle$ (this is actually necessary in order that Ω be conserved in interactions) and see that Ω acts on the open string spectrum as $\Omega|N, k\rangle = (-)^N|N, k\rangle$. In particular, $T_o \to T_o$ and $A_\mu \to -A_\mu$.

For the closed string Ω acts differently:

$$\Omega \alpha^i_{-n} \Omega^{-1} = \tilde{\alpha}^i_{-n}, \tag{1.1.51a}$$

$$\Omega \tilde{\alpha}^i_{-n} \Omega^{-1} = \alpha^i_{-n}. \tag{1.1.51b}$$

Now $\Omega|N, \tilde{N}, k\rangle = |\tilde{N}, N, k\rangle$, again assuming that $\Omega|0,0,k\rangle = +|0,0,k\rangle$. In particular $T_c \to T_c$, $\Phi \to \Phi$ and $h_{\mu\nu} \to h_{\mu\nu}$, but $B_{\mu\nu} \to -B_{\mu\nu}$.

The existence of the parity symmetry Ω allows us to construct new *unoriented* string theories from the *oriented* theories that we discussed up to now. This simple procedure is called *twisting* and consists of truncating the spectrum to states that are *even* under Ω. This is consistent, since Ω is multiplicatively conserved in interactions. Unfortunately, this procedure does not get rid of the tachyons. We are left with a theory with either only closed strings, with massless fields Φ and $h_{\mu\nu}$, or with a theory that also contains open strings. It turns out that the yet-to-be-discussed $U(n)$ gauge invariance of the vector A_μ has to be replaced with the smaller $SO(n)$ or $USp(n)$.

1.2 String interactions

So far we have only described the behavior of a single open or closed string moving in flat spacetime. We continue our treatment of the bosonic string with a discussion of its interactions. We will only scratch the surface of this subject. First however, we will review some general properties of interacting field theory. Not only will this allow us to draw some useful analogies between field and string theories, but it actually turns out that some properties of string theories are (at present) only understood from the point of view of their low energy effective *field theoretic* approximation.

1.2.1 The S-matrix in field theory

In relativistic quantum theories, one typically studies scattering processes of particles. One is interested in experimentally measurable quantities such as scattering cross-sections and decay widths. These quantities can be obtained from a mathematical construct called the scattering or S-matrix. It contains all the information on the scattering process that is independent of the particular experimental setup[7].

The S-matrix for a relativistic quantum theory is defined as follows. We assume that we know the particle spectrum of the theory under consideration and that there are no long-range interactions between these particles[8] (this implies in particular that the theory is local). We can thus treat incoming ($t \to -\infty$) and outgoing ($t \to \infty$) particles as free – with the exception of self-interactions which give rise to mass and wavefunction renormalizations. A generic scattering process involves m incoming particles with momenta k_i and n outgoing particles with momenta p_j (we suppress additional quantum numbers for notational convenience). With the

[7] In this section we limit ourselves to field theories that are defined on Minkowski spacetime and consider inertial observers only. In this case there is a well-defined – i.e. observer independent – notion of particles and there exist asymptotic states and hence an S-matrix. For the important and interesting topic of field theories on general spaces the reader will have to look elsewhere (see e.g. [21]).

[8] In principle, this excludes theories with massless bosons (such as QED) since these give rise to Coulomb-like long-range interactions. The strategy in these cases is to ignore this issue at first and simply proceed with the calculations. The S-matrix will then suffer from infrared divergences that can be dealt with by standard methods. There is also the possibility that additional (metastable) bound states appear in the spectrum. These require special care.

incoming particles we associate asymptotic states: an *in*-state, which is obtained by taking tensor products of the free particle states $|m, \text{in}\rangle = |k_1\rangle \otimes \ldots \otimes |k_m\rangle$, and similarly an *out*-state $|n, \text{out}\rangle = |p_1\rangle \otimes \ldots \otimes |p_n\rangle$. The Hilbert spaces of in- and out-states are isomorphic. This isomorphism is given by the S-matrix, $S : \mathcal{H}_{\text{out}} \to \mathcal{H}_{\text{in}}$, i.e. $|n, \text{in}\rangle = S|n, \text{out}\rangle$. The matrix elements of the S-matrix are given by:

$$S_{nm} \equiv \langle n, \text{out}|m, \text{in}\rangle = \langle n, \text{out}|S|m, \text{out}\rangle. \tag{1.2.1}$$

In processes for which the out-states are identical to the in-states no actual scattering takes place. The S-matrix is then simply the identity operator. The interesting part of the S-matrix is called the transfer or T-matrix and is defined as follows:

$$S = 1 + iT, \tag{1.2.2}$$

It is well known that in quantum *field* theories, the particle spectrum and the T-matrix can be obtained from a study of the n-particle connected Green's functions $G^{(n)}(k_1, \ldots, k_n)$. Indeed, the particle spectrum is obtained from the poles of the 2-point function (this follows from the Källén-Lehmann spectral representation of the propagator), whereas the T-matrix for an n-particle process is related to the $G^{(n)}$ by means of the LSZ reduction formula. The Green's functions can in turn be obtained from the partition function $Z[J]$ (we consider a theory with a bunch of fields, collectively denoted by ϕ, and a generic action $S[\phi]$):

$$Z[J] = \int [\mathrm{d}\phi] \exp\Big\{iS[\phi] + i(J, \phi)\Big\} \tag{1.2.3a}$$

$$= Z[0] \sum_{k=0}^{\infty} \frac{i^k}{k!} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_k \, \langle \phi(x_1) \cdots \phi(x_k)\rangle \, J(x_1) \cdots J(x_k). \tag{1.2.3b}$$

The Green's functions $G^{(n)}(k_1, \ldots, k_n)$ in momentum space are obtained from

$$G^{(n)}(x_1, \ldots, x_n) \equiv \langle 0|T\phi(x_1) \cdots \phi(x_n)|0\rangle \equiv \langle \phi(x_1) \cdots \phi(x_k)\rangle \tag{1.2.4}$$

by a Fourier transformation.

Perturbation theory

In general, it is impossible to do the integral (1.2.3a) exactly. Hence one resorts to perturbation theory. This works as follows. We split the action in a Gaussian part (describing a free field theory) and a part describing the interaction between the particles. By locality, these interactions are given by a series of local operators $\mathcal{O}_i(\phi)$ (i.e. products of the fields and derivatives):

$$S[\phi] = S_{\text{free}}[\phi] + \sum_i \int \mathrm{d}^d x \, g_i \mathcal{O}_i(\phi). \tag{1.2.5}$$

If the coupling constants are small, $g_i \ll 1$, we can make a series expansion of the exponential. For example, the vacuum-to-vacuum amplitude is given by:

$$Z[0] = \sum_{k=0}^{\infty} \frac{i^k}{k!} \sum_i g_i^k \left\langle \left(\int \mathrm{d}^d x\, \mathcal{O}_i(\phi) \right)^k \right\rangle_{\mathrm{free}}. \tag{1.2.6}$$

The correlation functions on the RHS of (1.2.6) are calculated in the *free* theory. Similarly, we obtain the Green's functions by expanding $Z[J]$ for nonzero J and subsequently performing suitable functional differentiations. From the poles of the two point functions, one finds that the particle spectrum is that of the free field theory S_{free}, but with renormalized masses.

Solitons

There is however more to quantum field theory than its perturbative expansion. First of all, it might happen[9] that the coupling constants g_i are *not* small. Not only is the perturbation series then useless, but even the spectrum of the theory might be completely different from that of S_{free}. This is the case in QCD, for example. Second, even if $g_i \ll 1$, the perturbation series in general does not converge, but is at best an asymptotic series, in which every next order in the series provides a better approximation to the S-matrix, but for smaller and smaller values of the coupling constants. This is an indication that there are physical effects that are not captured by the perturbation series[10].

These so-called *nonperturbative* effects are typically related to the occurrence of nontrivial but localized solutions ϕ_{cl} to the classical field equations. These solutions owe their existence to the nonlinearities of the field equations. In quantum mechanics, they can be treated in perturbation theory by semiclassical methods. One expands the fields around the classical solution:

$$\phi = \phi_{\mathrm{cl}} + \delta\phi, \tag{1.2.7}$$

and quantizes the perturbations $\delta\phi$ under the assumption that the coupling constants are small. Note that ordinary perturbation theory can be interpreted as a semiclassical expansion around $\phi_{\mathrm{cl}} = 0$.

These particle-like solutions come in two kinds. There are finite-energy solutions to the Minkowskian field equations, known as *solitons*. They are interpreted as additional

[9] We will see below that the size of the coupling constants g_i actually depends on the energy scale at which we investigate a given theory.

[10] The occurrence of a diverging perturbation series should not be confused with the problem of ultraviolet divergences. These can occur at every order in perturbation theory and signify a breakdown of the field theory at high energies. In these cases we interpret the field theory under consideration as an effective description of some (possibly unknown) underlying theory. The divergences are dealt with by *regularizing* the field theory by means of an ultraviolet cut-off and subsequently *renormalizing* the S-matrix. We will come back to the use of effective field theories in a minute.

particles of the theory and should be included in the spectrum of asymptotic states[11]. Typically, solitons carry a conserved charge which prevents them from decaying into ordinary particles. The mass of a soliton is inversely proportional to the coupling constant and in principle renormalized by quantum corrections: $m \sim 1/g^2(1+\mathcal{O}(g))$. This means that at weak coupling, solitons are very massive and essentially behave as static classical potentials in which the particles scatter. At strong coupling however, one expects the solitons to become very light.

S-duality

Sometimes it happens that not only do the solitons become light as the coupling increases, but also (some of) the ordinary particles become very heavy. A remarkable phenomenon may then occur. There may exist a different *weakly coupled* field theory that describes exactly the same physics. The particles of this *dual* theory are the solitons of the theory we started with, whereas the particles of the original theory may appear as solitons or bound states of the dual theory[12]. This phenomenon – the existence of two different theories describing the same physics – is known in general as *duality*. The theories are said to be *dual* to each other. In the present case we talk about *S-duality* which exchanges the weak- and strong-coupling regimes of two theories. It is not surprising that S-duality only occurs in rare cases.

To actually *prove* an S-duality, one needs to solve a quantum field theory exactly, which is in general impossible. The best one can do is to check that certain necessary conditions for S-duality are indeed satisfied. For example, one of these conditions is that the quantum numbers of the particles (solitons) of the original theory need to match those of the solitons (particles) of the dual theory. This simple check is already difficult to perform in practice because of an important technical obstacle. The masses and charges of the particles and solitons were derived in perturbation theory and we are not allowed to simply extrapolate these results to large values of the coupling. Fortunately however, there are supersymmetric models for which there are powerful nonrenormalization theorems that constrain the quantum corrections

[11] When quantizing a theory around the soliton background, one typically obtains a ground state with an energy larger than the vacuum energy, which is interpreted as the quantum mechanical version of the classical soliton. In addition, one obtains an infinite tower of excited states. These are interpreted as excited states of the soliton itself, and as states of arbitrary numbers of ordinary particles (i.e. those obtained from S_{free}) scattering off the soliton [23]. As an analogy, one can think of photons (the particles) scattering off an atom (the soliton). The atom itself has different energy levels (corresponding to the electron orbitals) that can be excited by the absorption of a photon.

[12] A famous example of such behavior exists in $1+1$ dimensions: the sine-Gordon theory vs. the massive Thirring model [24]. The sine-Gordon theory consists of a single scalar field with a specific self-interaction. Besides the scalar particle, its spectrum contains a soliton. The Thirring model contains a single Dirac fermion field with self-interactions. Remarkably, it turns out that the fermion particle behaves in the same way as the soliton of the sine-Gordon theory, whereas the sine-Gordon particle behaves in the same way as a fermion-antifermion bound state of the Thirring model.

to the masses and charges of certain classes of solitons (the BPS states, see the following chapter). In these theories (and they include the superstring theories) we *are* allowed to extrapolate the results from perturbation theory to arbitrary values of the coupling and one indeed finds solid evidence for S-duality. Famous examples are the Montonen-Olive electric/magnetic duality [25] of the $\mathcal{N} = 4$ [26, 27] and $\mathcal{N} = 2$ [28, 29] supersymmetric Yang-Mills theories in $3 + 1$ dimensions.

Instantons

We mentioned that there are two kinds of particle-like solutions of classical field equations. Apart from solitons, there also exist *instantons*. These are obtained as finite-action solutions of the *Euclidean* field equations[13] and play an important role in the understanding of the vacuum structure of quantum field theories. Instantons correspond to tunneling processes and give nonperturbative corrections to scattering processes, i.e. contributions $\sim \mathrm{e}^{-1/g^2}$.

1.2.2 String perturbation theory

After this digression into field theory, let us pick up the thread and return to strings. As in field theory, we would like to compute the S-matrix. Now a serious problem immediately presents itself. We do not have a string theoretical analog of the field theoretical action $S[\phi]$ of the previous section[14]. Of course there is the action (1.1.2), but it describes only a *single* string.

Let us, for the sake of argument, imagine that we do have such an action. As in field theory, we would have a hard time identifying the physical degrees of freedom of this action. So we would resort to perturbation theory: we would find the particle spectrum and possibly bound states and solitons in a regime of the theory in which spacetime is flat and the coupling constants are small. We would then move on to calculate S-matrix elements.

[13] Recall that there is always an implicit Feynman $i\epsilon$-prescription in (1.2.3a), giving a small positive imaginary part to the time coordinate t. We can Wick rotate the time integral to the imaginary axis, which comes down to making the replacement $t \to -i\tau$ with τ real. The analytic continuation of the fields is defined according to the tensor transformation law. For example, for a gauge field A_μ this goes as $A_t \to iA_\tau$ and $A_x \to A_x$, where t denotes the timelike direction and x a spacelike direction. The result is a field theory on Euclidean spacetime, with an action defined by $S_{\mathrm{E}} \equiv -iS_{\mathrm{M}}$. One can calculate amplitudes in this formalism and at the end of the day analytically continue back to Minkowski signature.

[14] It should be mentioned that there does exist a *string field theory* (see [30] for a recent review). Through its use one can obtain nontrivial results beyond the perturbation series. It has for instance been applied successfully to the theory of tachyon condensation. Nevertheless, there are a number of important properties of string theories that are not covered by string field theory. It is fair to say that – at least until now – the effective action approach has proved to be a more useful tool for studying the nonperturbative structure of string theories.

Fortunately it turns out that we can carry out a part of this program *without* knowing the underlying action. Indeed, we already know what the particle spectrum looks like in this case. It is given by the excitations of open and closed strings, to which we devoted the first part of this chapter. Moreover, there exists a recipe, called *string perturbation theory*, that allows one to write down a perturbation series for the S-matrix involving these excited states. Not only is this S-matrix physically meaningful (i.e. unitary), it is even believed to be *finite* in the ultraviolet to every order in perturbation theory[15]. Since the spectrum of the closed string includes the graviton we argue that string theory gives a *finite* theory of perturbative quantum gravity.

The perturbation series (1.2.6) in field theory has a well-known graphical representation in terms of Feynman graphs: the world-lines of the particles are represented by lines, whereas the interactions are represented by vertices. Graphically, the expansion in powers of g_i is equivalent to an expansion in the number of loops in the Feynman diagrams (since to every vertex we associate a factor g_i). A similar representation exists for strings. However, since strings are two-dimensional objects, the different contributions to the S-matrix are not represented by graphs, but by compact punctured surfaces. Each puncture corresponds to a certain incoming or outgoing asymptotic free string state. What kind of surfaces we allow depends on the string theory in question. For oriented closed strings these are closed surfaces. For unoriented closed strings, closed surfaces with crosscaps are also allowed. For oriented closed and open strings we allow surfaces with boundaries[16]. The incoming/outgoing open string states are represented by punctures *on the boundaries*. Finally, for unoriented open and closed strings we allow for boundaries and crosscaps.

As in field theory, we can organize the series in terms of the topology of the diagrams. We introduce a coupling constant λ (more on this later), and associate a factor of $\mathrm{e}^{-\lambda\chi}$ with each diagram[17]. Furthermore, with each string state i with momentum[18] k^μ we associate a vertex operator $V_i(k)$, which is constructed from the operators of the two-dimensional field theory living on the world-sheet.

Up to this point our discussion of string perturbation theory has been completely

[15] The finiteness has been proved through all orders in perturbation theory in [31], but only in the light-cone gauge. In the case of the *covariantly* quantized superstring, finiteness has been proved for the one-loop amplitudes in the eighties (see e.g. [14, 15]) and, more recently, for the two-loop amplitudes (see [32] for a review). Though a complete proof is still lacking, there seems to be no reason to assume that the methods of [32] would not apply to higher orders in perturbation theory.

[16] A detailed investigation of the unitarity of the S-matrix reveals that a theory with open strings necessarily includes closed strings, whereas the opposite does not hold. Indeed, two open strings can join at their endpoints to form a closed string.

[17] χ is the Euler number of the surface under consideration. It depends only on the topology and is given by $\chi = 2 - 2g - b - c$, where g, b and c are the number of handles, boundaries and crosscaps of the surface, respectively.

[18] We employ the convention that for incoming states the momenta are positive, $k^\mu = (E, \vec{k})$, whereas for outgoing states they are negative $k^\mu = -(E, \vec{k})$.

general. In the following, we will limit ourselves to the bosonic string. The case of the superstrings is conceptually not very different, but is technically a lot more complicated.

In bosonic string theory the vertex operators are constructed using the X^μ fields of (1.1.3). We have for example

$$V_{\text{tachyon}}(k) \sim \int_{\partial\Sigma} \mathrm{d}s\, \mathrm{e}^{ik\cdot X}, \tag{1.2.8}$$

for an open string tachyon – note that the integral runs over a *boundary* of the world-sheet – and

$$V_{\text{graviton}}(k) \sim \frac{1}{\ell_s^2} \int_{\Sigma} \mathrm{d}^2\sigma\, \zeta_{\mu\nu} \sqrt{-\gamma}\, \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu\, \mathrm{e}^{ik\cdot X}, \tag{1.2.9}$$

for a graviton with polarization $\zeta_{\mu\nu}$. Actually, these expressions have to be normal ordered, but let us not worry about the details here. The relative normalization of the different vertex operators can be fixed by unitarity whereas the overall normalization is a convention. An n-particle T-matrix element is then given by the Polyakov path integral

$$iT_{i_1\cdots i_n}(k_1,\ldots,k_n) = \sum_{\text{topologies}} \mathrm{e}^{-\lambda\chi} \int \frac{[\mathrm{d}\gamma\, \mathrm{d}X]}{V_{\text{gauge}}} V_{i_1}(k_1)\cdots V_{i_n}(k_n)\, \mathrm{e}^{-S_P[\gamma,X]}, \tag{1.2.10}$$

where $S_P[\gamma, X]$ is the 'Euclideanized' version[19] of (1.1.3).

In the path integral we sum over all *physically distinct* geometries of the world-sheets and their embeddings in spacetime[20]. Hence we need to divide out the volume V_{gauge} of the diffeomorphism and Weyl gauge group. This is achieved by going to the conformal gauge, which is always possible locally as we have seen. However, there are complications due to the nontrivial topologies of the world-sheets. First of all, it turns out that it is impossible to bring the metric $\gamma_{\alpha\beta}$ to the form (1.1.8) *globally*[21] when $\chi \geq 0$. However, we are mainly interested in the tree-level diagrams – the sphere and disk topologies – and there this issue does not arise. More important for us is that

[19] We have Wick rotated the time direction of the world-sheet. The advantage of this is that we can use results from Riemannian geometry, such as the Gauss-Bonnet theorem. The Euclidean version of the path integral has been shown to be completely equivalent to treatments in Minkowski signature, see e.g. [14, 15].

[20] One also needs to integrate over the possible positions of the punctures on the world-sheet. This is the reason for the integrals in the definition of the vertex operators (1.2.8) and (1.2.9).

[21] It turns out that $\gamma_{\alpha\beta}$ can be fixed up to a certain finite number of parameters, the *moduli*. The path integral over $\gamma_{\alpha\beta}$ thus reduces to an integral over the *moduli space*. An important issue is then to find a measure on moduli space. This is relatively straightforward for the bosonic case, but in the case of the RNS superstring – where one considers supersurfaces – this is much more difficult. The one-loop case has been dealt with a long time ago and is treated in the textbooks, but the general case was only solved recently by d'Hoker and Phong (see again [32] for a review).

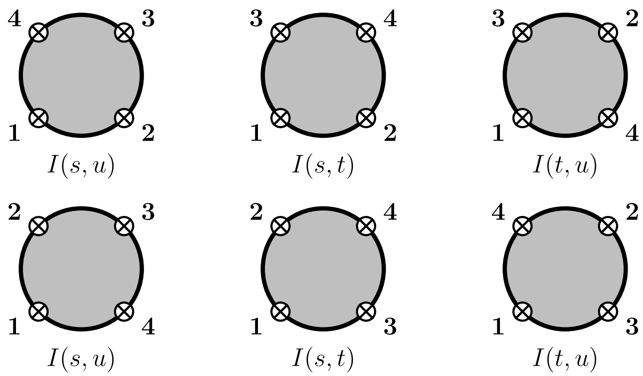

Figure 1.2.1. The six cyclically inequivalent orderings of the vertex operators on the boundary of the open string disk amplitude.

at tree-level[22] there are gauge symmetries that are not fixed by the conformal gauge choice. We encountered these before when we discussed the quantization of the free string. There we used them to go to the light-cone gauge. Now we can use them to fix the position of some of the punctures. For the sphere topology, we can fix three of the closed string vertex operators at arbitrary positions on the world-sheet. For the disk topology, we can fix the positions of three of the open string vertex operators on the boundary. However, it turns out that we can not fix the cyclic ordering of the vertices. So we still need to sum over the cyclically inequivalent orderings of the vertices.

Let us illustrate the above by treating an example of a string theory scattering process. We will focus on the ordering of the vertices and refer to chapter 6 of [15] for calculational details. We consider elastic scattering of two open string tachyons. The incoming tachyons have momenta k_1, k_2 and the outgoing tachyons k_3 and k_4. We define the conventional Mandelstam variables:

$$s = -(k_1 + k_2)^2, \qquad t = -(k_1 + k_3)^2, \qquad u = -(k_1 + k_4)^2. \tag{1.2.11}$$

They satisfy the relation

$$s + t + u = \sum_i m_i^2 = -\frac{4}{\alpha'}. \tag{1.2.12}$$

At tree level, this scattering process is associated with a disk diagram with four vertex operators on the boundary. Let us parametrize the boundary by an angular variable

[22] At one-loop ($\chi = 0$) there are also residual gauge symmetries. They act on the moduli. This *modular invariance* is responsible for the finiteness of the one-loop amplitudes.

$\theta \in [0, 2\pi)$. We fix the position of three vertex operators: #1 at θ_1, #2 at θ_2 and #3 at θ_3, with $\theta_1 < \theta_2 < \theta_3$. The integral over the position θ_4 of #4 splits in three pieces, $\theta_3 < \theta_4 < \theta_1$, $\theta_1 < \theta_4 < \theta_2$, and $\theta_2 < \theta_4 < \theta_3$. In addition, there are three contributions from interchanging #2 and #3. So we get a total of six cyclically inequivalent orderings of four open string vertex operators on the disk. It turns out that these six integrals give results that are equal up to an interchange of vertex operators. They are proportional to $I(s,t)$ with:

$$I(s,t) = \frac{\Gamma(-1-\alpha' s)\Gamma(-1-\alpha' t)}{\Gamma(-2-\alpha' s-\alpha' t)}, \tag{1.2.13}$$

an expression known as the *Veneziano amplitude*. The complete amplitude is

$$iT^{(4)}(k_1,\ldots,k_4) = 2i\,\mathrm{e}^{-\lambda}\,(2\pi)^{26}\delta^{(26)}(k_1+\ldots+k_4)\big(I(s,t)+I(t,u)+I(u,s)\big). \tag{1.2.14}$$

The first term comes from the ordering 1243 and 1342, the second from 1423 and 1324, the third from 1234 and 1432, see figure 1.2.1.

1.2.3 Low-energy approximation

Now that we have seen that it is possible to do perturbation theory of strings on flat spacetime, we would like to know more about the nonperturbative structure of string theories. In particular, we want to know what kind of solitons a given string theory contains. As explained above, this implies that we need solutions to the classical equations of motion of the strings. We do not know these equations, but we *do* know what these equations look like *at low energies*. They are field equations.

Because strings are extended objects, their interactions are intrinsically nonlocal. This nonlocality manifests itself in the infinite tower of massive string excitations, a phenomenon that does not occur in local field theories. At energies below ℓ_s^{-1} however, the massive modes of the strings are never produced in scattering processes, though they do appear as virtual particles and give rise to interactions between the massless modes. These interactions are short-ranged – i.e. distances smaller than ℓ_s – and are therefore effectively local at low energies. And thus, according to Weinberg's conjecture[23], string theories reduce to field theories at energies below the string scale ℓ_s^{-1}.

So instead of studying nonperturbative effects directly in string theory, we can use the low-energy effective field theory. Of course, because of the presence of the tachyon, it is not completely clear how we should interpret phrases like "low-energy limit" in the case of the bosonic string. We will simply ignore this issue and continue with our discussion of the bosonic string as if the tachyon did not exist at all. After all, it will not bother us in the case of the superstring, which is our real interest.

[23] Weinberg's conjecture states [33,34] that any Poincaré invariant quantum theory of particles with local interactions can be modeled by a quantum field theory. No counterexamples to this conjecture are known.

	High energy	Low energy
1.	GSW model (leptons, $W^{\pm}$, Z)	Fermi theory (leptons)
2.	QCD (quarks, gluons)	Chiral perturbation theory (mesons, hadrons)
3.	MSSM (superpartners)	Standard model
4.	Technicolor (composite Higgs)	Standard model
5.	General relativity on $\mathbb{R}^{1,3} \times S^1$	General relativity with $U(1)$ gauge field and scalar field on $\mathbb{R}^{1,3}$
6.	Superstring theories	Supergravity theories

Table 1.2.1. Some examples of effective field theories in high-energy physics. Cases 1 and 2 are realized in nature, whereas the others exist only in the theoretician's notebook – at least for now. Case 2 illustrates that the low-energy degrees of freedom may be completely different from those appearing in the underlying field theory. Case 3 and 4 illustrate that different theories can have the same low-energy limit. Case 5 is the famous scenario of Kaluza and Klein: space may be more than three-dimensional. Case 6 is the outcome of the first "superstring revolution" of the 1980's.

Wilsonian effective actions

Let us review some of the ideas behind the use of effective field theories. See [33] for a nice nontechnical review and e.g. [13, 35, 36] for elementary treatments. The typical situation is as follows: we are given a relativistic quantum theory with a characteristic energy scale Λ and are interested in the dynamics of this theory at energies below this scale. We then try to write down an effective theory that captures this dynamics.

For convenience, we analytically continue to Euclidean space, since then the condition $|k| < \Lambda$, where $|k|^2 = k_\mu k^\mu$, always implies small momenta and energies. We denote the momentum space components of the fields collectively by $\phi(k)$ and consider the following split into low- and high-frequency modes:

$$\phi(k) = \phi_{\mathrm{L}}(k) + \phi_{\mathrm{H}}(k), \tag{1.2.15}$$

where – using the Heaviside step function θ –

$$\phi_{\mathrm{L}}(k) = \phi(k)\,\theta(\Lambda - |k|), \qquad \phi_{\mathrm{H}}(k) = \phi(k)\,\theta(|k| - \Lambda), \tag{1.2.16}$$

If we interpret the bosonic string as a field theory with an infinite number of massive fields, the $\phi_{\mathrm{H}}(k)$ would include all these massive fields but also the high-frequency modes of the massless fields. We perform the path integral over the high-frequency modes

$$\int [\mathrm{d}\phi_{\mathrm{L}}(k)\,\mathrm{d}\phi_{\mathrm{H}}(k)]\,\mathrm{e}^{-S[\phi_{\mathrm{L}},\phi_{\mathrm{H}}]} \equiv \int [\mathrm{d}\phi_{\mathrm{L}}(k)]\,\mathrm{e}^{-S_{\mathrm{eff},\Lambda}[\phi_{\mathrm{L}}]}, \tag{1.2.17}$$

where the effective action is defined as

$$e^{-S_{\text{eff},\Lambda}[\phi_{\mathrm{L}}]} \equiv \int [\mathrm{d}\phi_{\mathrm{H}}(k)]\, e^{-S[\phi_{\mathrm{L}},\phi_{\mathrm{H}}]}. \tag{1.2.18}$$

By locality, we can expand the effective action in a series of local operators $\mathcal{O}_i(\phi)$ as in (1.2.5):

$$S_{\text{eff},\Lambda}[\phi] = S_{\text{free}}[\phi] + \sum_i \int \mathrm{d}^d x\, \Lambda^{d-\Delta_i}\, \lambda_i(\Lambda)\, \mathcal{O}_i(\phi), \tag{1.2.19}$$

where Δ_i is the dimension of $\mathcal{O}_i(\phi)$ in units of mass, and we defined dimensionless coupling constants $\lambda_i(\Lambda) \equiv \Lambda^{\Delta_i - d}\, g_i(\Lambda)$.

Since the effective field theory is defined with a cut-off Λ, it gives results that are *finite*[24]. This does not remove the need for renormalization, of course, since we still need to relate the bare parameters in the effective Lagrangian to quantities that we actually measure in experiments. Note that the Lagrangian contains an infinite number of nonrenormalizable operators. Before the 1970's it was thought that the appearance of such terms spells disaster for any field theory. For example, it was widely believed that a perturbative quantization of general relativity is impossible. Since the work of Wilson (and many others, see [37] for an early review) these issues have been better understood, as we will see in a minute[25].

For a process at energy E we expect on dimensional grounds that $\int \mathrm{d}^d x\, \mathcal{O}_i \sim E^{\Delta_i - d}$. From this estimate we obtain the classical scaling behavior of the λ_i's as we lower the energy[26]:

$$\lambda_i(E) = \lambda_i(\Lambda) \left(\frac{E}{\Lambda}\right)^{\Delta_i - d}. \tag{1.2.20}$$

We can recast this in the following form

$$E \frac{\mathrm{d}\lambda_i}{\mathrm{d}E} = (\Delta_i - d)\, \lambda_i(E) + \beta_i(\lambda_j), \tag{1.2.21}$$

where we added the renormalization group beta-functions $\beta_i(\lambda_j)$. They contain the quantum corrections to the classical scaling behavior, due to the effects of low-energy modes running in loops; the high energy modes were already taken into account in (1.2.18).

Now for the punchline: the effects of operators $\mathcal{O}_i$ for which $\Delta_i - d > 0$ become smaller and smaller as we lower the energy at which we investigate our system. These

[24] Except for the usual infrared divergences, which are harmless.

[25] Though the modern point of view on nonrenormalizable field theories has been around for over thirty years, it for some reason only made its way into a textbook at the advanced undergraduate level in 1995 [36].

[26] This is *renormalization*: the interchange of “bare” variables in the Lagrangian with variables that are natural for the energy scale under consideration.

$\Delta_i - d$	low-energy behavior	RG terminology	traditional term.
< 0	increasing	relevant	superrenormalizable
$= 0$	constant	marginal	renormalizable
> 0	decreasing	irrelevant	nonrenormalizable

Table 1.2.2. The low-energy behavior of an operator $\mathcal{O}_i$ is – in perturbation theory – dominated by its classical scaling, which is governed by the dimension Δ_i of the operator.

operators are called *irrelevant*. Operators for which $\Delta_i - d < 0$ are called *relevant*. Their effects become larger at low energies. The operators that do not scale classically are called *marginal*. Generically, they *do* scale as a result of the quantum corrections, but only with the logarithm $\log(E/\Lambda)$ of the ratio of energy scales and not a power.

At low energies, the physics of a perturbative quantum field theory is dominated by a *finite* number of operators (the relevant and marginal ones), or, in their absence, by the leading order irrelevant operators. Not only does this *explain* why the quantum field theories that are used in "low energy" particle physics are renormalizable, it also solves the traditional problems with nonrenormalizable interactions [38]. As is well known from renormalization theory, once we include a single nonrenormalizable operator in a theory, we need to include them all[27]. But now that we take the idea of a *physical* ultraviolet cut-off seriously, we see that we only need take into account a *finite* number of these interactions for the determination of physical quantities *up to a given accuracy*. The vast majority of these operators are suppressed by sufficiently high powers of E/Λ to render them unobservable.

Thus 'nonrenormalizable' theories like general relativity are predictive as long as we use them only at energies well below the cut-off[28]. In fact, they even predict where this cut-off lies. Consider for example two-graviton scattering in four dimensions. Since Newton's constant $G_{\rm N}$ has mass dimension -2 we find the following dependence of the amplitude $\mathcal{M}$ on the center of mass energy E:

$$\mathcal{M}(E) = G_{\rm N} + G_{\rm N}^2 E^2 + G_{\rm N}^3 E^4 + \ldots = G_{\rm N}(1 + G_{\rm N}E^2 + G_{\rm N}^2 E^4 + \ldots). \qquad (1.2.22)$$

At energies $E \sim 1/\sqrt{G_{\rm N}} \equiv M_{\rm Planck}$ the second term in this series becomes of the same size as the first: perturbation theory breaks down and the theory is cut off at the Planck mass $M_{\rm Planck}$. If a theory has more than one type of interactions, we deduce in this way the existence of a high-energy cut-off Λ_i for each of these interactions. Since the effective description breaks down already at the smallest of these Λ_i, we can rightfully call that particular Λ_i *the* cut-off Λ.

[27] This follows from the usual power counting for Feynman diagrams. As we go to ever higher orders in perturbation theory, we need to include ever more counterterms in the Lagrangian.

[28] The effective action approach has been used to calculate for example quantum corrections to Newton's law [8, 39]. We refer to [40] for a more complete list of references and to [11, 41, 42] for lectures on this topic.

In contrast, the relevant operators indicate a breakdown of the perturbative treatment of a theory at *low* energies, at a scale E_0 say[29]. As discussed above, one then expects new degrees of freedom that are associated with non-perturbative effects (solitons, bound states) to play the dominant role. Unless there are powerful non-renormalization theorems at work, we can no longer make any reliable calculations in the theory. The best one can hope for is that one can identify these new degrees of freedom (either by analyzing the original theory or by means of experimental input) and that they *are* weakly coupled, so that it is possible to write down a new effective theory valid at energies below E_0.

A last remark about the marginal operators. It useful to divide these further in three classes: the *marginally relevant*, *marginally irrelevant* and *exactly marginal* operators, which respectively increase or decrease in size due to quantum corrections, or stay exactly constant when one lowers the energy. The above discussion on the (ir)relevant operators also applies to the marginally (ir)relevant case, with the understanding that the scaling is now proportional to $\log(E/\Lambda)$.

Obtaining the effective action

It is often impractical or even impossible to do the integral in (1.2.18) explicitly. One relies rather on indirect methods. The most straightforward procedure is a *matching* calculation, which follows the following recipe:

1. Identify the low-energy degrees of freedom of the theory under consideration.
2. Identify the symmetries of the low-energy degrees of freedom[30].
3. Write down the most general Lagrangian with the fields of step 1 and the symmetries of step 2. This Lagrangian has the form of equation (1.2.5).
4. Calculate amplitudes in both the underlying theory and the effective theory and Taylor expand these amplitudes in powers of E/Λ. Compare the results to determine the coupling constants of (1.2.5).

[29] An exception are mass terms, since they can be treated exactly. Their presence leads to yet another issue that goes under the name of the *hierarchy problem*. When a mass term is not protected by a nonrenormalization theorem, it blows up at low energies. The result is that the associated field is integrated out, unless the original value $m(\Lambda)$ at the cut-off is fine-tuned to an unnaturally small value. Symmetries that can give rise to suitable nonrenormalization theorems are gauge invariance for spin ≥ 1, chirality for fields of spin $\frac{1}{2}$ and supersymmetry or the Goldstone mechanism for spin 0. We already mentioned in the introduction to this thesis that the hierarchy problem of the Higgs particle in the Standard Model is one of the major motivations for the introduction of low-energy supersymmetry. For a more detailed account of these and other arguments for low-energy supersymmetry see the following reviews [34, 43, 44].

[30] Some of these symmetries can be recognized immediately from the low-energy spectrum. For example, if the spectrum contains a massless spin 1 particle, the effective action needs to be gauge invariant. Similarly, a massless spin 2 particle implies general covariance and a massless spin $\frac{3}{2}$ particle local supersymmetry.

Obviously this method is particularly useful when the underlying theory is weakly coupled. The low energy degrees of freedom are then easy to recognize and amplitudes easy to calculate. This is for example the case for perturbative string theory, where the role of the cut-off is played by $1/\ell_s$. In chapters 3 and 4 we apply this method to the tree-level sector of the open superstring and discuss step 3 and 4 of the procedure in some detail. In this chapter however (section 1.2.4), we will discuss a different approach to the construction of effective actions in string theory.

Quantum effective actions

The Wilsonian effective action (WEA) is not the only "effective action" that occurs in quantum field theory. There also exists an object called the *quantum* effective action (QEA), which is in principle very different from the Wilsonian action, but can resemble it in certain special cases[31]. This often leads to confusion. It is for this reason that we devote the remainder of this section to a discussion of the QEA. The QEA is treated in any good modern book on quantum field theory. A particularly clear discussion can be found in [34]. A discussion on the relations between the QEA and WEA from a somewhat different viewpoint can be found in [45].

Let us consider again a generic field theory with fields ϕ and an action $S[\phi]$. The quantum effective action $\Gamma[\phi]$ is by definition the generating functional of the amputated one-particle irreducible diagrams (1PI) of this theory.

$$\Gamma[\phi] \equiv \sum_n \frac{1}{n!} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_n \, \Gamma^{(n)}(x_1, \ldots, x_n) \, \phi(x_1) \cdots \phi(x_n). \tag{1.2.23}$$

Γ can therefore be viewed as a *classical* field theory that encodes all the quantum information of the underlying field theory. The interaction vertices obtained from (1.2.23) are the 1PI diagrams of $S[\phi]$; Γ reproduces thus already at tree-level *all* the amplitudes of $S[\phi]$.

It is also possible to write down a QEA for only a subset φ of the fields ϕ, i.e. we simply only consider 1PI diagrams with φ's as external lines. This is for instance useful when studying anomalies, where one is only interested in the behavior of a certain restricted set of Feynman diagrams. The anomaly then manifests itself as the non-invariance of the QEA under a symmetry transformation of the fields φ.

Or we could restrict ourselves to the 1PI diagrams of the *light* fields only, like the massless sector of a perturbative string theory. The QEA for these fields is a horrible nonlocal object that is not known explicitly (for instance, not all amplitudes have been calculated in string theory). To make the situation more tractable, we can consider the low-energy expansion of these 1PI diagrams in powers of $\ell_s E$. We then

[31] Some field theory texts define yet a third type of effective action when discussing gauge theories. This action is simply defined as the ordinary gauge invariant action plus gauge fixing terms. Fortunately, this action is (almost) never confused with the WEA and QEA.

talk about the *low-energy* QEA for the massless fields, an object that takes a form similar to (1.2.5). Note that this object is *not* equal to the Wilsonian action! Indeed, the QEA incorporates the effects of loops of *all* fields (and is hence infrared divergent due to the massless particle loops), whereas the WEA only contains the effects of loops involving *massive* particles (the so-called *threshold corrections*).

When we stick to the tree-level or classical approximation however, we do not see the effects of loops. Therefore the tree-approximation of the Wilsonian effective action *is* equal to the low-energy approximation of the tree-level quantum effective action for the massless modes. This is quite a mouthful, so in practice one simply talks about the effective action[32].

Note that at leading order in the quantum corrections, $\Gamma[\phi]$ is given by just $S[\phi]$. We can thus simply read off the elementary vertices of S by functional differentiation, which sometimes provides a more efficient means of obtaining the Feynman rules than by expanding the partition function Z as in (1.2.6). We can not obtain the propagators in this way, though.

1.2.4 Strings on curved backgrounds

Up to now we only discussed strings in flat spacetime. What about strings in curved spaces? A possible starting point is the Polyakov action (1.1.3) with the replacement $\eta_{\mu\nu} \to g_{\mu\nu}(X)$. We can think of this as a string moving in a coherent background of gravitons. But the graviton is itself an excited state of the string, so we can generalize this by also turning on backgrounds for the other massless fields. The resulting action has the form of non-linear σ-model:

$$S_{\sigma\text{-model}} = S[g, B] + S[A] + S[\Phi], \tag{1.2.24a}$$

It is obtained by writing down the most general 2-dimensional action with at most two world-sheet derivatives and the right symmetries (i.e. general covariance on the world-sheet and in the target space, gauge invariance and local Weyl invariance). The graviton and the 2-form contribute as follows:

$$S[g, B] = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \, \sqrt{-\gamma} \big(\gamma^{\alpha\beta} g_{\mu\nu}(X) - \varepsilon^{\alpha\beta} B_{\mu\nu}(X)\big) \partial_\alpha X^\mu \partial_\beta X^\nu. \tag{1.2.24b}$$

The integral is over the entire world-sheet Σ, which reflects the fact that closed string vertex operators are inserted in the bulk of Σ. The open string gauge field A_μ couples to the boundary $\partial\Sigma$, hence:

$$S[A] = -\int_{\partial\Sigma} \imath^* A^{(1)} = \int_{-\infty}^{\infty} \mathrm{d}s \, \Big[A_\mu \partial_s X^\mu \big|_{\sigma=\ell} - A_\mu \partial_s X^\mu \big|_{\sigma=0} \Big]. \tag{1.2.24c}$$

[32]This sloppy use of terminology is probably the main cause of the confusion that sometimes arises.

This expression should be familiar: it describes the coupling of a point particle to an electromagnetic field. Thus the endpoints of open strings carry a charge under the gauge field A_μ. Similarly, from (1.2.24b) we see that the strings are charged w.r.t. the 2-form $B^{(2)}$. Finally, we have the coupling to the dilaton:

$$S[\Phi] = -\frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-\gamma}\,R[\gamma]\,\Phi(X) - \frac{1}{2\pi}\int_{\partial\Sigma}\mathrm{d}s\,K[\gamma]\,\Phi(X), \tag{1.2.24d}$$

where $R[\gamma]$ is the 2-dimensional curvature scalar. The boundary term involves the extrinsic curvature $K[\gamma]$ [46, 47] and is added to cancel the total derivative that is obtained by varying $R[\gamma]$. The various signs and factors in the above equations are conventional. This fixes the normalization of the massless fields in terms of the normalization of the associated vertex operators. The fields have the following dimensions in units of mass: $[G_{\mu\nu}] = [B_{\mu\nu}] = [\Phi] = 0$ and $[A_\mu] = +1$.

The invariance of (1.2.24) under the gauge transformations of the 2-form B is somewhat subtle. (1.2.24b) transforms as

$$-2\pi\alpha'\,\delta S[g,B] = \delta\int_\Sigma \imath^* B^{(2)} = \int_\Sigma \imath^*\mathrm{d}\Lambda^{(1)} = \int_\Sigma \mathrm{d}\imath^*\Lambda^{(1)} = \int_{\partial\Sigma}\imath^*\Lambda^{(1)}.$$

This is canceled by assigning a $\Lambda^{(1)}$ transformation to A. The invariance under the gauge transformation $\Lambda^{(0)}$ of A follows since "boundaries do not have boundaries", i.e. $\partial\partial\Sigma = \varnothing$. We thus have:

$$\delta B^{(2)} = \mathrm{d}\Lambda^{(1)}, \qquad \delta A^{(1)} = \mathrm{d}\Lambda^{(0)} - \frac{1}{2\pi\alpha'}\Lambda^{(1)}. \tag{1.2.25}$$

We define for later use the following gauge invariant expression (where $F = \mathrm{d}A$):

$$\mathcal{F} \equiv B + 2\pi\alpha' F. \tag{1.2.26}$$

The Euclidean version of (1.2.24) appears in the Polyakov path integral (1.2.10). By using the Gauss-Bonnet theorem (see e.g. [48])

$$\chi = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{\gamma}\,R + \frac{1}{2\pi}\int_{\partial\Sigma}\mathrm{d}s\,K \tag{1.2.27}$$

we see that if the dilaton background has a constant value Φ_0, the path integral is weighted by a factor

$$\mathrm{e}^{-\lambda\chi}\mathrm{e}^{-S_{\mathrm{E}}[\Phi_0]} = \mathrm{e}^{-(\lambda+\Phi_0)\chi}. \tag{1.2.28}$$

Thus it is actually the combination $\lambda + \Phi_0$ which plays the role of a coupling "constant". We put λ to zero by redefining the dilaton with a constant shift. Every string diagram now appears with a factor $g_s^{-\chi}$, where the *string coupling constant* g_s is determined by the vev of the dilaton:

$$g_s \equiv \mathrm{e}^{\Phi_0}. \tag{1.2.29}$$

Since the background fields depend on the scalars X^μ, the σ-model (1.2.24) defines an *interacting* 2-dimensional quantum field theory which we can no longer expect to solve exactly, except in certain special cases. Let us focus on closed strings and investigate the first term in (1.2.24b). As in our previous discussion on solitons we first pick a vev X_0^μ for $X^\mu(\tau,\sigma)$ and subsequently quantize small fluctuations x^μ around this vev, $X^\mu(\tau,\sigma) = X_0^\mu + x^\mu(\tau,\sigma)$. We expand $g_{\mu\nu}(X)$ in powers of x^μ. This is facilitated by using the target space diffeomorphisms to introduce Riemann normal coordinates:

$$g_{\mu\nu}(X) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\rho\nu\sigma}(X_0)x^\rho x^\sigma + \mathcal{O}(x^3). \tag{1.2.30}$$

The leading order term gives just the free field theory discussed earlier in this chapter. The corrections can be treated perturbatively. This is a valid approximation as long as $\ell_s/\ell_c \ll 1$, where $\ell_c = 1/\sqrt{R}$ is the radius of curvature of the target space. In this regime we can also use the low energy Wilsonian effective field theory with confidence[33]. The Wilsonian effective action can be obtained from the σ-model as follows. The symmetries of the free field theory action (1.1.3) are crucial in obtaining a consistent quantization of the string since they are responsible for the decoupling of unphysical degrees of freedom (we removed these by imposing the light-cone gauge). However, we are now dealing with an interacting field theory in which, as we have seen, the coupling constants are generically not invariant under changes of scale. Thus the Weyl symmetry is ruined unless the renormalization group β-functions for the couplings $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ vanish. We interpret these β-functions as classical field equations. At leading order in α' and g_s they are obtained by varying the following target space action (in $d = 26$):

$$S = \frac{1}{2\kappa_0^2}\int \mathrm{e}^{-2\Phi}\Big(R*1 + 4{*\mathrm{d}\Phi}\wedge \mathrm{d}\Phi - \frac{1}{2}{*H}\wedge H + \mathcal{O}(\alpha')\Big), \tag{1.2.31}$$

where $H = \mathrm{d}B$. The constant κ_0 is arbitrary: fixing it fixes the absolute normalization of the vertex operators. The overall power of $\mathrm{e}^{-2\Phi}$ comes from the fact that this is a closed string tree-level ($\chi = 2$) result. This action is written in the so-called *string frame*: the metric $g_{\mu\nu}^{\mathrm{S}}$ in (1.2.31) is the one that appears in (1.2.24b). It is related to the *Einstein frame* metric $g_{\mu\nu}^{\mathrm{E}}$ as follows (keeping d explicit):

$$g_{\mu\nu}^{\mathrm{E}} = \mathrm{e}^{-\frac{4}{d-2}\phi}g_{\mu\nu}^{\mathrm{S}}, \tag{1.2.32}$$

where we defined a shifted dilaton $\phi \equiv \Phi - \Phi_0$ such that its vev $\langle\phi\rangle = 0$ vanishes. In the Einstein frame the action (1.2.31) has the canonical Einstein-Hilbert term:

$$S = \frac{1}{2\kappa^2}\int \Big(R*1 - \frac{4}{d-2}{*\mathrm{d}\phi}\wedge \mathrm{d}\phi - \frac{1}{2}\mathrm{e}^{-\phi}\,{*H}\wedge H\Big). \tag{1.2.33}$$

[33]Actually the requirement that $\ell_s/\ell_c \ll 1$ is already implicit in (1.2.24), since we considered only background vevs for the massless fields.

Here $\kappa \equiv \kappa_0 g_s$ is related to the 26-dimensional Newton's constant[34] as $\kappa^2 = 8\pi G_{\mathrm{N}}$.

Now that we know the low-energy effective field theory of the closed string, we can look for classical solutions and see whether there are any solitons. We will quickly review how this works for the ten-dimensional supergravities in the next chapter. By incorporating the α'-corrections to the effective action one can obtain information about the "stringy" corrections to these classical solutions. That is, we can study the short-distance structure of a theory by means of effective field theory methods. In particular, the field equations of theories like electromagnetism and general relativity have physically interesting classical solutions that are singular at some point in spacetime (e.g. charged point particles and black holes). Such singularities simply indicate that the theory breaks down at such points; one expects that stringy effects take over and that the singularities are "smoothened out".

Although we will not investigate explicit α'-corrections to soliton solutions in this thesis, we will devote a considerable amount of time to the starting point of such an analysis: the determination of the α'-corrections to the effective action of open superstrings (and hence D-branes, see below) in chapters 3 and 4. For α'-corrections to the closed string effective action we refer to [49–51].

1.3 D-branes & T-duality

Usually, the soliton solutions of the effective field theory correspond to nonperturbative effects in string theory. These solitons can therefore not be described with the methods of perturbative string theory as developed in the previous sections. There is an important exception however: the D-branes. In the effective field theory regime, they arise as a certain class of p-brane solutions of the type II supergravities. They turn out to be the same objects as the hyperplanes that support open strings with Dirichlet boundary conditions. We will review the evidence for this remarkable fact in the following chapter.

In the remainder of this chapter we will treat D-branes in bosonic string theory. As before, we do this in the understanding that – if we ignore the tachyon – the results still hold in the case of the superstrings. There are some important aspects of supersymmetric D-branes that do not have a bosonic counterpart though, and we will mention these in the next chapter.

We will first look at a configuration with open strings ending on a single D-brane. We then move on to the simplest compactification scenario, i.e. one periodic dimension, and T-duality. Matters become considerably more complicated when we discuss configurations with multiple D-branes.

---

[34]Newton's constant is defined so that Einstein's equations read $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_{\mathrm{N}}\, T_{\mu\nu}$. Newton's law for the gravitational potential of a mass M in d dimensions then becomes $\Phi(r) = -\gamma M/R^{d-3}$, where $\gamma = 8\pi G_{\mathrm{N}}/(d-2)\,\mathrm{vol}\, S_{d-2}$. Here $\mathrm{vol}\, S_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of a $(d-1)$-sphere.

This section serves as the motivation and as background material for the work that we review in chapters 3 and 4.

1.3.1 Single D-brane

We consider a single Dp-brane, and align our coordinate axes such that the brane's world-volume is parallel to the $0, 1, \ldots, p$ directions and transverse to the other $d-p-1$ directions. It is located at position $x^a = \bar{x}^a$, $a = p+1, \ldots, d-1$. We single out the pth direction to define light-cone coordinates $x^\pm = (x^0 \pm x^p)$ and denote the remaining directions by x^i, $i = 1, \ldots, p-1$.

The quantization of an open string with both endpoints on the D-brane is similar to the quantization of the freely moving string. The only thing that changes are the boundary conditions in the a-directions:

$$X^a(\tau, 0) = X^a(\tau, \ell) = \bar{x}^a, \tag{1.3.1}$$

which give rise to the mode expansion

$$X^a(\tau, \sigma) = \bar{x}^a + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^a \mathrm{e}^{-in\pi\tau/\ell} \sin \frac{n\pi\sigma}{\ell}. \tag{1.3.2}$$

The sum still runs over the integers n and the oscillators satisfy $\alpha_{-n}^a = (\alpha_n^a)^\dagger$. In contrast to the x_0^i, however, the $\bar{x}^a$'s are not dynamical variables and should not be quantized. The directions along the brane (with Neumann boundary conditions) are treated exactly as before.

After some work, we obtain:

$$[\alpha_m^a, \alpha_n^b] = m\delta^{ab}\delta_{m+n}. \tag{1.3.3}$$

This is the same relation as that for the i-oscillators (1.1.34b), hence the normal ordering constant is unchanged:

$$a = \frac{1}{2}\sum_{n=1}^{\infty}[\alpha_n^i, \alpha_{-n}^i] + \frac{1}{2}\sum_{n=1}^{\infty}[\alpha_n^a, \alpha_{-n}^a] = [(p-1) + (d-p-1)] \times \frac{1}{2}\sum_{n=1}^{\infty} n, \tag{1.3.4}$$

i.e. $a = -(d-2)/24$. Since the string does not move in the a-directions, its momentum is zero in those directions, $p^a = 0$. Hence we still have $M^2 = 2p^+p^- - p^ip^i$, even though i runs only from 1 to $p-1$ instead of $d-2$, and thus:

$$M^2 = \frac{1}{\alpha'}(N_\parallel + N_\perp - a), \tag{1.3.5}$$

where

$$N_\parallel = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i, \qquad N_\perp = \sum_{n=1}^{\infty} \alpha_{-n}^a \alpha_n^a. \tag{1.3.6}$$

The ground state of the string is given by $|0,k\rangle$, where $k=(k^+,k^i)$. It is annihilated by all the lowering operators, $\alpha_n^i|0,k\rangle = \alpha_n^a|0,k\rangle = 0$ for all positive n, and has $N_\parallel = N_\perp = 0$, and mass $M^2 = -a/\alpha'$. Again, this is a tachyon since we will presently see that $a=1$.

Instead of a single level $N=1$ with $M^2=(1-a)/a'$, we now have two different possibilities for creating states with this mass, namely those with $(N_\parallel, N_\perp)=(1,0)$ or $N_\parallel, N_\perp)=(0,1)$. We thus act with the i-oscillators to obtain states $\alpha_{-1}^i|0,k\rangle$, which comprise a vector of $SO(p-1)$, and with the a-oscillators to obtain states $\alpha_{-1}^a|0,k\rangle$, which comprise a vector of $SO(d-p-1)$.

Because of the presence of the Dp-brane, the $SO(1,d-1)$ Lorentz invariance of the vacuum is broken to $SO(1,p)\times SO(d-p-1)$. As far as the center of mass motion of the open string is concerned, the $SO(1,p)$ factor is simply the Lorentz group, whereas the $SO(d-p-1)$ factor is an internal symmetry group. We therefore expect that massless particles fall into multiplets of $SO(p-1)\times SO(d-p-1)$ whereas massive particles are associated with $SO(p)\times SO(d-p-1)$.

So, just as before, we conclude that we need to take $a=1$ and hence $d=26$. The presence of D-branes does not change the critical dimension. This is a good thing, since the open strings can interact to produce closed strings, which are free to move anywhere in the bulk and necessarily live in 26 dimensions.

The lowest lying states of the open string on a Dp-brane are thus a scalar tachyon T_o with $M^2 = -1/a'$, a massless $U(1)$ vector field A_α, with $\alpha = 0,\ldots,p$ and a $(d-p-1)$-plet of massless scalars Φ_a.

These fields live on the world-volume of the Dp-brane. To actually prove this, one would need to study the interactions of these open string states with the closed strings in the bulk and show that these are indeed localized in the vicinity of the brane. A simple physical explanation is the following: since the open string endpoints are fixed at the brane, the string needs to be stretched in order to interact with closed strings that are located at a large distance from the brane. Because of the string's tension, such interactions will be highly suppressed at energies well below $1/\alpha'$. We can equivalently view the D-brane as an object with a thickness of roughly a string length ℓ_s.

Not only does there live a Maxwell-like gauge theory on the Dp-brane's world-volume, it turns out that the brane itself is a dynamical object in its own right. Indeed, the Dirichlet boundary conditions cause momentum to flow from the open strings into the D-brane. The D-brane thus possesses a non-zero energy density. It therefore couples to the graviton and is dynamical. Note the nice interplay between stringy arguments (boundary conditions) and field theoretic arguments (general covariance). The scalar fields Φ_a describe small fluctuations of the Dp-brane, transverse to its world-volume.

We are interested in the low-energy effective description of these fields. This will be a $(p+1)$-dimensional field theory, which also includes couplings to the massless closed string fields in the bulk *via* their pull-backs to the world-volume. Not surprisingly, this action is not known completely, since it involves an infinite series with arbitrarily high powers of the fields, their field strengths and curvatures, and derivatives. However, in the case that the fields are *slowly varying*, it turns out to be possible to resum the complete α'-expansion. One obtains the effective action in an elegant closed form, known as the Dirac-Born-Infeld action [52, 53]. Stated more precisely, if we parametrize the world-volume Σ of the brane with coordinates $\sigma^\alpha = (\sigma^0, \ldots, \sigma^p)$ and neglect all terms involving more than two world-volume derivatives ∂_α or one or more spacetime derivatives ∂_μ, the tree-level effective action for a Dp-brane is:

$$S_{\rm DBI}[A,X] = -T_p \int \mathrm{d}^{p+1}\sigma\, \mathrm{e}^{-\Phi} \sqrt{-\det\left(G_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right)}, \tag{1.3.7}$$

where the (string frame) metric and two-form are pulled back:

$$G_{\alpha\beta} \equiv (\imath^* g)_{\alpha\beta} = g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu, \qquad B_{\alpha\beta} \equiv (\imath^* B)_{\alpha\beta} = B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu. \tag{1.3.8}$$

The scalars X^μ describe the embedding of the brane in spacetime. The dependence on the dilaton follows because this is an open string tree-level result. Note the appearance of $\mathcal{F}_{\alpha\beta}$ as defined in (1.2.26). T_p is the tension of the Dp-brane as measured with the string frame metric. To determine it explicitly, one needs to compare a string theory amplitude with its effective field theory counterpart. See section 8.7 of [15] for an explicit calculation. The result is

$$\tau_p = \frac{\sqrt{\pi}}{16\kappa}(2\pi\ell_s)^{11-p}, \tag{1.3.9}$$

where we defined the physical value (i.e. Einstein frame) of the tension $\tau_p \equiv T_p \mathrm{e}^{-\Phi_0}$, where Φ_0 is the vev of the dilaton. The value of κ depends on the arbitrary overall normalization of the string vertex operators, for which we will introduce a convenient convention in the next chapter that will simplify the expression (1.3.9). The dependence on ℓ_s follows from dimensional analysis. Also, the ratio τ_p/τ_{p-1} does not depend on p. This is actually required by T-duality as we will see below. Note that although in the limit that $g_s \to 0$ the D-brane becomes infinitely massive ($\tau_p \to \infty$), the gravitational interaction of the brane vanishes, since this is governed by $\kappa^2\tau_p \sim g_s$. Thus the expression (1.3.9) is consistent with the description of D-branes as rigid hyperplanes in perturbative string theory.

The embedding fields X^μ are not all physical. We can use the world-volume diffeomorphisms $\delta\sigma^\alpha = \xi^\alpha$ to go to the *static gauge*, where we recover the scalar fields Φ^a:

$$X^\alpha = \sigma^\alpha, \qquad X^a = \Phi^a. \tag{1.3.10}$$

In flat spacetime – $g_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = 0$, $\Phi = \Phi_0$ – we get $G_{\alpha\beta} = \eta_{\alpha\beta} + \partial_\alpha\Phi^a\partial_\beta\Phi^a$ and thus obtain

$$S_{\rm DBI} = -\tau_p \int \mathrm{d}^{p+1}\sigma \sqrt{-\det\left(\eta_{\alpha\beta} + \partial_\alpha\Phi^a\partial_\beta\Phi^a + 2\pi\alpha' F_{\alpha\beta}\right)}. \tag{1.3.11}$$

In chapter 3 we will discuss derivative corrections to this action.

1.3.2 Compactification and T-duality

String theory predicts the existence of extra spatial dimensions beyond those we have observed up to today. If we are to obtain realistic physics from string theory, we have to find ways to effectively reduce the number of dimensions. One way to proceed is by *compactification*, i.e. one considers strings living on spacetimes with compact spatial directions. These backgrounds have to satisfy the field equations of (1.2.31). In the following we will consider the simplest scenario – the circle reduction. We refer the reader to [54] for a recent review of the (huge) field of string phenomenology and entry points into the literature.

That compactification works follows from an elementary property of quantum theories on compact spaces: the momenta along the compact directions are quantized. Consider a one-dimensional quantum system on a circle of radius R. The position eigenstates $|x\rangle$ and $|x+2\pi R\rangle$ represent the same physics, thus in particular $\langle x|p\rangle = \langle x+2\pi R|p\rangle$, with $|p\rangle$ a (bosonic) momentum eigenstate. Since $\langle x|p\rangle \sim \mathrm{e}^{ipx}$, p is quantized: $p = k/R$, with k integer. At energies below $1/R$ only the state with $k=0$ can be excited and the compactified dimension will be invisible. Above $1/R$ we see the infinite tower of *Kaluza-Klein* states. Note that the wave function $\langle x|p\rangle$ for the $k=0$ state is independent of x.

The discussion in this section will be limited to flat spacetime. The inclusion of gravity reveals additional interesting features that we will discuss in later chapters.

Let us consider the closed bosonic string in flat spacetime, but this time take spacetime to have the nontrivial topology $\mathbb{R}^{d-1}\times S^1$, i.e. x^{25} is periodically identified $x \cong x+2\pi R$, where R is the radius of the circle S^1. This solves the field equations of (1.2.31) trivially since the circle is flat. What distinguishes strings from field theory in this case is the fact that closed strings can wind around the compact direction. As we will see shortly, this has far-reaching consequences.

First, we need to modify the boundary conditions in order to account for the winding:

$$X^i(\tau,\sigma+\ell) = X^i(\tau,\sigma), \tag{1.3.12a}$$

$$X^{25}(\tau,\sigma+\ell) = X^{25}(\tau,\sigma) + 2\pi R w, \tag{1.3.12b}$$

where i now runs from 1 to $d-1$. The integer w is the *winding number*. It can be viewed as a charge for the closed string and is conserved in interactions.

The mode expansion for the 25-direction is modified to:

$$X_R^{25}(\tau-\sigma) = \frac{1}{2}x_0^{25} + \frac{\pi\alpha'}{\ell}\left(\frac{k}{R} - \frac{wR}{\alpha'}\right)(\tau-\sigma) + \text{oscillators}, \tag{1.3.13a}$$

$$X_L^{25}(\tau+\sigma) = \frac{1}{2}x_0^{25} + \frac{\pi\alpha'}{\ell}\left(\frac{k}{R} + \frac{wR}{\alpha'}\right)(\tau+\sigma) + \text{oscillators}. \tag{1.3.13b}$$

The part involving the oscillators is unchanged from (1.1.40).

The ground state $|0,0,p,(k,w)\rangle$ has $N=\tilde{N}=0$ and is labeled by the center of mass momentum $p=(p^+,p^i)$ along the non-compact directions, and by the Kaluza-Klein and winding charges (k,w). The mass-shell condition becomes

$$M^2 = \frac{k^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(N+\tilde{N}-2), \tag{1.3.14}$$

where M is the mass according to a 25-dimensional observer $\mathcal{O}$. The level-matching condition needs to be modified as well. Classically, the shift symmetry generator S is modified to

$$S = \frac{\pi}{\ell}\sum_{n\neq 0}\left(\alpha_{-n}^i\alpha_n^i - \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i\right) - \frac{2\pi kw}{\ell}. \tag{1.3.15}$$

Quantum mechanically, the action of S on the spectrum becomes

$$S|N,\tilde{N},p,(k,w)\rangle = \frac{2\pi}{\ell}(N-\tilde{N}-kw-a+\tilde{a})|N,\tilde{N},p,(k,w)\rangle. \tag{1.3.16}$$

So the various integers are constrained by $N-\tilde{N}-kw=0$.

Equations (1.3.14) and (1.3.16) are invariant under the following substitution:

$$R \to \frac{\alpha'}{R}, \qquad k \leftrightarrow w. \tag{1.3.17}$$

Hence the spectrum of a closed bosonic string theory on a circle of radius R is *identical* to that of a theory on a circle of radius α'/R; the Kaluza-Klein modes change places with the winding modes. This known as target space or *T-duality*. Though we have shown it only for free closed strings, it is also true at the interacting level [15].

We see that it does not really make sense to talk about radii smaller than the self-dual radius[35] ℓ_s. In particular, the limit of a vanishing circle $R\to 0$ is the same as the decompactification limit $R\to\infty$!

[35] For generic radii the massless states are those with $N=\tilde{N}=0$ and $k=w=0$, but for special values of R additional massless states appear in the spectrum. At the self-dual radius $R=\ell_s$ we have massless vector states with $N=1$, $\tilde{N}=0$, $k=-w=\pm1$ and $N=0$, $\tilde{N}=1$ and $k=w=\pm1$. There are also massless scalar states for $N=\tilde{N}=0$, with either $k=0$ at radii $R=2\ell_s/w$ or $w=0$ at $R=k\ell_s/2$. Since these latter states are tachyonic for $R<2\ell_s/w$ and $R>2k/\ell_s$, respectively, we do not expect them to occur at all in the case of the superstring and indeed they do not. The vector states at the self-dual radius however do and in principle need to be included in the low-energy effective action. In the following however we will always implicitly assume that $R\gg\ell_s$ so that we can safely ignore them.

The above holds for a flat background. T-duality acts nontrivially on nonzero background vevs for $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ – the Buscher rules [55, 56]. In the following we only need the transformation of the dilaton, which we obtain by means of the following argument. Let us call the closed string theories with radius R and α'/R theories A and B, respectively. Since they look identical to our observer $\mathcal{O}$, they must yield in particular the same gravitational coupling constant (in Einstein frame). Thus[36]

$$\frac{2\pi R}{2\kappa_{\mathrm{A}}^2} = \frac{2\pi\alpha'/R}{2\kappa_{\mathrm{B}}^2}. \tag{1.3.18}$$

Therefore the string coupling constant – or, equivalently, the dilaton – transforms under T-duality as

$$g_s^{\mathrm{B}} = g_s^{\mathrm{A}}\,\frac{\ell_s}{R}. \tag{1.3.19}$$

What happens to open strings under T-duality? They are distinguished from closed strings only by their boundary conditions. Consider the free open string of section 1.1.2, but now with a compactified 25-direction. This can equivalently be viewed as an open string ending on single D25-brane that is wrapped around the circle once. The T-duality transformation $k \leftrightarrow w$ of (1.3.17) acts on (1.3.13) as $X_L^{25} \to X_L^{25}$ and $X_R^{25} \to -X_R^{25}$, i.e. if the original theory is described by $X^{25} = X_L^{25} + X_R^{25}$, the T-dual theory is described by $X'^{25} = X_L^{25} - X_R^{25}$. It is not difficult to see that a Neumann boundary condition on X^{25} becomes a Dirichlet boundary condition on X'^{25} under T-duality:

$$\partial_\sigma X^{25} = 0 \quad \to \quad \partial_\tau X'^{25} = 0. \tag{1.3.20}$$

Hence the T-duality transforms the wrapped D25-brane into a unwrapped D24-brane. In general we can say that a T-duality transformation involving a world-volume direction transforms a Dp-brane into a D$(p-1)$-brane, whereas T-dualizing a transverse direction transforms it into a D$(p+1)$-brane. Free open strings can not wind around the compact direction, so in the D25-brane case we have only KK-modes. In the D24-brane case, the open strings are not free to move in the 25-direction, hence there are no KK-modes. But in this case there *is* winding. We can calculate the number of times that the string winds around the dual circle

$$X'^{25}(\ell) - X'^{25}(0) = \int_0^\ell \mathrm{d}\sigma\, \partial_\sigma X'^{25} = \int_0^\ell \mathrm{d}\sigma\, \partial_\tau X^{25} = 2\pi\alpha' p^{25} = 2\pi\alpha'\frac{k}{R} = 2\pi k R',$$

so T-duality exchanges $k \leftrightarrow w$ also for open strings.

What about the gauge fields A_μ and scalars Φ^a living on the Dp-brane world-volume? Suppose we T-dualize in a world-volume direction i. We will show below

[36] This result is obtained by dimensional reduction of the Einstein-Hilbert action. See B.1.2 for details. A similar result holds for the reduction of Yang-Mills theories on flat space as we will see in a minute.

that only the gauge field component in the i-direction is affected:

$$A_i \to -\frac{1}{2\pi\alpha'}\Phi^i. \tag{1.3.21}$$

If we T-dualize in a transverse direction, the above equation needs to be read backwards.

We will now show that the form of the DBI-action (1.3.7) is consistent with (1.3.21). Take a Dp-brane on flat space and wrap the p-direction around a circle of radius R (we use the static gauge). Suppose our observer $\mathcal{O}$ performs experiments at energies $< 1/R$ and thus only sees the $k = 0$ KK-mode. Keeping only the $k = 0$ mode is equivalent to *dimensional reduction*: take the $(p+1)$-dimensional Dp-brane action and split the fields into components along the p-direction and the other directions. Then take the fields to be independent of the p-direction and thus obtain an effective p-dimensional action. Denote the $(p+1)$-dimensional fields and indices with hats, i.e. $\hat{A}_{\hat{\mu}} = (A_\mu, A_p)$. Since T-duality in the p-direction does not involve the scalar fields Φ^a $(a = p+1, \ldots, 25)$, we will simply omit them in the following. The dimensional reduction of (1.3.11) then proceeds as follows

$$\begin{aligned}
S &= -\tau_p \int \mathrm{d}^{p+1}\sigma \sqrt{-\det\left(\hat{\eta}_{\hat{\mu}\hat{\nu}} + 2\pi\alpha' \hat{F}_{\hat{\mu}\hat{\nu}}\right)} \\
&= -\tau_p \int \mathrm{d}^{p+1}\sigma \sqrt{-\det\begin{pmatrix} \eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} & 2\pi\alpha'\partial_\mu A_p \\ -2\pi\alpha'\partial_\nu A_p & 1 \end{pmatrix}} \\
&= -2\pi R\tau_p \int \mathrm{d}^p\sigma \sqrt{-\det\left(\eta_{\mu\nu} + (2\pi\alpha')^2\partial_\mu A_p \partial_\nu A_p + 2\pi\alpha' F_{\mu\nu}\right)}.
\end{aligned}$$

$\mathcal{O}$ cannot distinguish between this wrapped Dp-brane and an unwrapped D$(p-1)$-brane in the T-dual theory, hence the above action should be identical to (1.3.11) for a D$(p-1)$-brane after substituting the T-duality rule $A_p \to -\Phi^p/2\pi\alpha'$. This is the case since the tensions are related by

$$\tau_{p-1} = 2\pi R\tau_p, \tag{1.3.22}$$

as follows from (1.3.9) and (1.3.19). For a discussion of T-duality and the DBI-action for curved backgrounds see [57].

1.3.3 Multiple D-branes

Configurations with multiple D-branes have some additional intriguing features. In this section we consider a setting with two parallel Dp-branes. This set-up is unstable in bosonic string theory, since the D-branes attract each other gravitationally. We will see however, that in type II superstring theory there is also a repulsive electric

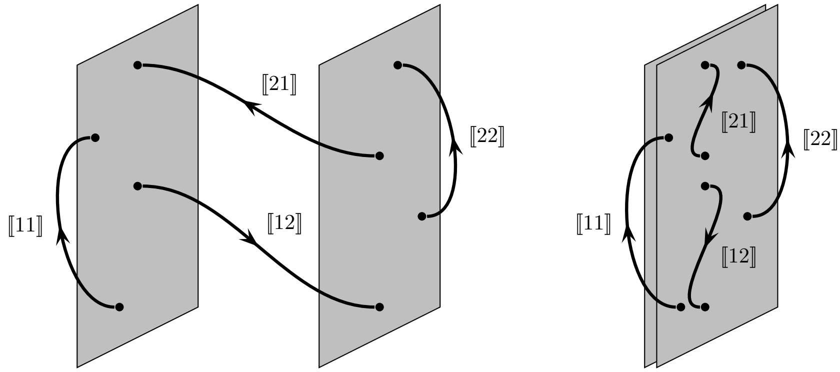

Figure 1.3.1. Two parallel D-branes. The ⟦11⟧ and ⟦22⟧ strings always contain a $U(1)$ gauge boson in their spectrum. The ⟦12⟧ and ⟦21⟧ strings give rise to additional gauge bosons when the D-branes coincide, thus enlarging the gauge group from $U(1)^2$ to $U(2)$.

interaction between the branes, resulting in configurations with parallel Dp-branes that are stable (at least perturbatively).

We label the two Dp-branes by 1 and 2. The have their world-volume directions in common, i.e. $x^0, \ldots, x^p$, but are located at different positions in the transversal directions, at respectively $\bar{x}_1^a$ and $\bar{x}_2^a$, say. In addition to open strings that have both of their endpoints confined to the same D-brane, we now also have strings that start on one brane and end on the other. For now, we discuss oriented strings, for which the $\sigma = 0$ and $\sigma = \ell$ endpoints are inequivalent. We then have four kinds of open strings that are distinguished by the location of their endpoints. We give these strings labels $[\![ij]\!]$, $i, j = 1, 2$, where i (equiv. j) denotes the brane on which the $\sigma = 0$ (equiv. $\sigma = \ell$) endpoint is located.

We can view these labels as nondynamical degrees of freedom or *charges* that live on the endpoints of the open strings. They are called *Chan-Paton* indices or charges.

The quantization of the ⟦11⟧ and ⟦22⟧ strings is identical to that of the previous section. The ⟦12⟧ and ⟦21⟧ strings give rise to intriguing new results. Let us consider the ⟦12⟧ string, for which:

$$X^a(\tau, 0) = \bar{x}_1^a, \qquad X^a(\tau, \ell) = \bar{x}_2^a. \tag{1.3.23}$$

These boundary conditions give rise to the following mode expansion:

$$X^a(\tau, \sigma) = \bar{x}_1^a + (\bar{x}_2^a - \bar{x}_1^a)\frac{\sigma}{\ell} + \sqrt{2\alpha'}\sum_{n \neq 0} \frac{1}{n}\alpha_n^a \mathrm{e}^{-in\pi\tau/\ell} \sin\frac{n\pi\sigma}{\ell}, \tag{1.3.24}$$

leading to the mass formula

$$M^2 = \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'} \right)^2 + \frac{1}{\alpha'}(N_\parallel + N_\perp - 1). \tag{1.3.25}$$

The same equation is valid for the $[\![21]\!]$ string. The first term is easy to interpret physically. It is the energy that is needed to create a classical static string of length $|\bar{x}_2 - \bar{x}_1|$ from the vacuum (remember that the tension of the string is given by $T = 1/2\pi\alpha'$).

For generic distances $|\bar{x}_2 - \bar{x}_1|$ between the two branes, all the states in the spectrum of the $[\![12]\!]$ and $[\![21]\!]$ string are massive. The only massless states are then two copies of the vector/scalar system that we discussed already in the previous section. This gives a gauge group $U(1) \times U(1)$. The effective action consists simply of two copies of the action (1.3.7), one for each brane.

However, if the two branes are put very close together, i.e. $|\bar{x}_2 - \bar{x}_1| \to 0$, the states for which either $N_\parallel = 1$ or $N_\perp = 1$ become *massless*. So we get two extra massless vector and two massless scalar fields, respectively. These additional massless degrees of freedom need to be included in the low-energy effective theory. The extra gauge fields together with the $U(1) \times U(1)$ gauge fields combine into a single gauge field for $U(2)$. The same holds for the scalar fields: we get a single scalar that is valued in the adjoint of $U(2)$. This is straightforwardly generalized to the case of n parallel Dp-branes. We get a $U(n)$ Yang-Mills theory coupled to adjoint scalars [58].

Our earlier discussion on vertex operators and string amplitudes requires a few modifications in the presence of D-branes. A general open string state $|N, k; \alpha\rangle$ is a superposition of the states $|N, k; ij\rangle$ that correspond to the $[\![ij]\!]$ strings:

$$|N, k; \alpha\rangle = \sum_{i,j}^{N} \lambda_{ij}^\alpha \, |N, k; ij\rangle. \tag{1.3.26}$$

The $n \times n$ matrices λ_{ij}^α are antihermitian and normalized as $\mathrm{Tr}\, \lambda^\alpha \lambda^\beta = -\delta^{\alpha\beta}$. Each vertex operator carries a factor λ_{ij}^α. Oriented open strings can interact only by joining a $\sigma = 0$ with a $\sigma = \ell$ endpoint and in addition these endpoints need to be confined to the same brane. As a result, every open string amplitude contains traces of products of the λ-matrices. Consider for example again the case of 4-tachyon scattering. The vertex operator for tachyon #1 now has a factor of λ^{α_1}, that of #2 a factor of λ^{α_2}, etc. We already saw that the scattering amplitude gets contributions from the six cyclically inequivalent orderings of the vertex operators. The 1234 ordering now gets an additional $\mathrm{Tr}\, \lambda^{\alpha_1} \lambda^{\alpha_2} \lambda^{\alpha_3} \lambda^{\alpha_4}$, the 1243 ordering $\mathrm{Tr}\, \lambda^{\alpha_1} \lambda^{\alpha_2} \lambda^{\alpha_4} \lambda^{\alpha_3}$, etc. The result is that the tachyon amplitude is modified to:

$$\begin{aligned} iT^{(4)}(k_1,\alpha_1;\ldots;k_4,\alpha_4) = i\,g_s^{-1}\,(2\pi)^{26}\delta^{(26)}(k_1+\ldots+k_4) \\ \times\Big[\big(\mathrm{Tr}\,\lambda^{\alpha_1}\lambda^{\alpha_2}\lambda^{\alpha_4}\lambda^{\alpha_3}+\mathrm{Tr}\,\lambda^{\alpha_1}\lambda^{\alpha_3}\lambda^{\alpha_4}\lambda^{\alpha_2}\big)\,I(s,t) \\ +\big(\mathrm{Tr}\,\lambda^{\alpha_1}\lambda^{\alpha_3}\lambda^{\alpha_2}\lambda^{\alpha_4}+\mathrm{Tr}\,\lambda^{\alpha_1}\lambda^{\alpha_4}\lambda^{\alpha_2}\lambda^{\alpha_3}\big)\,I(t,u) \\ +\big(\mathrm{Tr}\,\lambda^{\alpha_1}\lambda^{\alpha_2}\lambda^{\alpha_3}\lambda^{\alpha_4}+\mathrm{Tr}\,\lambda^{\alpha_1}\lambda^{\alpha_4}\lambda^{\alpha_3}\lambda^{\alpha_2}\big)\,I(u,s)\Big]. \end{aligned} \tag{1.3.27}$$

Adding a boundary to a Riemann surface now not only introduces an extra power of g_s, but also another trace of λ-matrices. As a result, the coupling strength for open strings is effectively $g_s n$ instead of g_s.

The effective field theory for a stack of n D-branes is less well understood than that of a single D-brane. One reason is technical in nature: there is no longer an unambiguous notion of slowly varying fields. This is because we are dealing with a Yang-Mills theory and need to use covariant instead of ordinary derivatives. Let us see what happens when one imposes $D_\mu F_{\nu\rho} = 0$. Acting on this with another derivative, taking the commutator and using that

$$[D_\mu, D_\nu]\,F_{\rho\sigma} = [F_{\mu\nu}, F_{\rho\sigma}], \tag{1.3.28}$$

we see that necessarily $[F_{\mu\nu}, F_{\rho\sigma}] = 0$, i.e. the field strengths are abelian.

The appearance of adjoint scalars Φ^a is intriguing. For a single D-brane, these scalars described the embedding of the brane in spacetime. Apparently, stacks of D-branes perceive spacetime in an unconventional way, since the embedding coordinates no longer commute, $[\Phi^a, \Phi^b] \neq 0$. This *non-commutative geometry* has interesting consequences for the physics of D-branes though it is probably fair to say that many aspects have yet to be understood. A thorough discussion of these topics is beyond the scope of this thesis. See [59] for a review.

The leading order contribution to the effective action for a stack of n D-branes can be obtained by T-duality. We start with a stack of n spacetime filling D25-branes in flat space. The massless spectrum consists of a gauge field $\hat{A}_{\hat{\mu}}$, there are no embedding scalars. The leading order in α' contribution to the effective action is uniquely determined by gauge invariance. It is $U(n)$ Yang-Mills theory[37]:

$$S = \frac{(2\pi\alpha')^2\tau_{25}}{4}\int \mathrm{d}^{26}\sigma\,\mathrm{Tr}\,\hat{F}_{\hat{\mu}\hat{\nu}}\hat{F}^{\hat{\mu}\hat{\nu}}. \tag{1.3.29}$$

We wrap the 25-branes around a torus T^{25-p}. The low-energy effective action for a $(p+1)$-dimensional observer $\mathcal{O}$ is obtained by dimensional reduction. We write $\hat{A}_{\hat{\mu}} =$

[37] We work with matrix-valued fields $A_\mu = A_\mu^\alpha\lambda^\alpha$ and $\Phi^a = \Phi^{a\alpha}\lambda^\alpha$, where $(\lambda^\alpha)^\dagger = -\lambda^\alpha$ and $\mathrm{Tr}\,\lambda^\alpha\lambda^\beta = -\delta^{\alpha\beta}$ as before. In particular, fields transforming in the adjoint of $U(1)$ are imaginary. This differs from the convention used in the previous sections, where the gauge field and scalar where taken to be real. Hence the appearance of extra factors of i in the equations of this section.

(A_μ, A_a), and take the fields to be independent of the a-directions. $\hat{F}_{\hat{\mu}\hat{\nu}}$ decomposes as follows:

$$\hat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \equiv F_{\mu\nu}, \tag{1.3.30a}$$

$$\hat{F}_{\mu a} = \partial_\mu A_a + [A_\mu, A_a] \equiv D_\mu A_a, \tag{1.3.30b}$$

$$\hat{F}_{ab} = [A_a, A_b], \tag{1.3.30c}$$

and plug this back into (1.3.29). The action for a stack of Dp-branes is obtained by applying the T-duality rules to the a-directions:

$$S = \frac{(2\pi\alpha')^2 \tau_p}{4} \int \mathrm{d}^{p+1}\sigma \, \mathrm{Tr} \Big(F_{\mu\nu} F^{\mu\nu} + \frac{2}{(2\pi\alpha')^2} D_\mu \Phi^a D^\mu \Phi^a + \frac{1}{(2\pi\alpha')^4} [\Phi^a, \Phi^b][\Phi^a, \Phi^b] \Big). \tag{1.3.31}$$

There is a potential for the scalars Φ^a, which is minimized when the scalars commute, $[\Phi^a, \Phi^b]$. We can then diagonalize the Φ's by a gauge transformation:

$$\Phi^a = i \, \mathrm{diag} \, (x_1^a, \ldots, x_n^a). \tag{1.3.32}$$

With this solution to the equations of motion of (1.3.31) we obtain a configuration of N parallel Dp-branes located at positions x_i^a.

The above illustrates a general strategy. To obtain the effective action for a stack of Dp-branes, we first calculate α'-corrections to the effective action for open strings with Neumann b.c.'s in all directions and non-trivial Chan-Paton factors. We subsequently obtain the effective action for general p by the requirements of T-duality. With this strategy in mind we will study α'-corrections to both the abelian and nonabelian D9-brane effective actions in chapters 3 and 4.

We finish this chapter with a proof of relation (1.3.21). We consider again the system of two parallel Dp-branes, but now with a compactified transverse direction X of radius R. We ignore the other target space directions in the following. The branes are located at $\bar{x}_1$ and $\bar{x}_2$, respectively. The mode expansion of a $[\![12]\!]$ string winding w times around the circle is:

$$\begin{aligned} X(\tau,\sigma) &= \bar{x}_1 + (\bar{x}_2 - \bar{x}_1 + 2\pi R w)\frac{\sigma}{\ell} + \text{oscillators} \\ &= \frac{\pi\alpha'}{\ell}\left(\frac{\bar{x}_2 - \bar{x}_1}{2\pi\alpha'} + \frac{wR}{\alpha'}\right)(\tau+\sigma) - \frac{\pi\alpha'}{\ell}\left(\frac{\bar{x}_2 - \bar{x}_1}{2\pi\alpha'} + \frac{wR}{\alpha'}\right)(\tau-\sigma) + \text{osc.} \end{aligned} \tag{1.3.33}$$

We apply the T-duality rules – the right-moving sector gets a sign, $R \to \alpha'/R$ and $w \leftrightarrow k$:

$$X'(\tau,\sigma) = \frac{2\pi\alpha'}{\ell}\left(\frac{\bar{x}_2 - \bar{x}_1}{2\pi\alpha'} + \frac{k}{R}\right)\tau + \text{oscillators} \tag{1.3.34}$$

The second term is quantized and thus corresponds to the *canonical* momentum associated with the center of mass movement of the open string along the dual circle. Because of the presence of the first term, the physical momentum of the CoM is no longer equal to its canonical momentum. This is familiar from electrodynamics: charged fields also show this behavior.

In the present case the endpoints of the open string are charged w.r.t. the background $U(2)$ gauge field that lives on the stack of the two D$(p+1)$-branes in the dual picture. The background field turns out to be a constant $U(1) \times U(1)$ Wilson line[38]:

$$A = i \,\mathrm{diag}\,(A_{11}, A_{22}) = -\frac{i}{2\pi\alpha'}\,\mathrm{diag}\,(\bar{x}_1, \bar{x}_2) \tag{1.3.35}$$

with A_{11} and A_{22} constant fields[39] on respectively brane #1 and #2. Indeed, the action for the $[\![12]\!]$-string in this background reads (in the conformal gauge):

$$\begin{aligned} S = -\frac{1}{4\pi\alpha'} &\int \mathrm{d}^2\sigma \Big[-(\partial_\tau X')^2 + (\partial_\sigma X')^2 \Big] \\ &+ \int \mathrm{d}\tau \Big[A_{22}\, \partial_\tau X' \big|_{\sigma=\ell} - A_{11}\, \partial_\tau X' \big|_{\sigma=0} \Big], \end{aligned} \tag{1.3.36}$$

since the $\sigma = 0$ endpoint ends on brane #1 and the $\sigma = \ell$ endpoint on brane #2. We read off the canonical momentum:

$$p_{\mathrm{can}} = \int_0^\ell \mathrm{d}\sigma\, \frac{\partial \mathcal{L}}{\partial \dot{X}} = \frac{\ell}{2\pi\alpha'}\, p_{\mathrm{phys}} + A_{22} - A_{11} \equiv \frac{k}{R}, \tag{1.3.37}$$

where we used $X = p_{\mathrm{phys}}\tau + \text{oscillators}$. Solving for p_{phys}, we recover (1.3.34) with the choice (1.3.35). Hence (1.3.21) gives the correct relation between the gauge fields A_μ and scalars Φ^a under T-duality.

1.3.4 Branes with fluxes

We will now investigate T-duality of Dp-branes that carry a nonzero electro-magnetic field $F_{\mu\nu}$ on their world-volume ($\mu = 0, \ldots p$). As in the previous sections, we consider

[38] A Wilson line is a vev for the gauge field with vanishing curvature, i.e. $\mathrm{d}A = 0$. Locally, A is exact and can be gauged away, $A = \mathrm{d}\chi$. However, in topologically nontrivial situations this χ may not exist globally. Consider a circle of radius R and take A constant. We can gauge A away locally with $\chi = -Ax$. This χ does not exist everywhere on the circle, since it is not periodic. In general, if $\oint A \neq 0$ there is no χ such that we can gauge away A. If there were we would have

$$\oint A = \oint \mathrm{d}\chi = \int_{\partial S^1} \chi = 0,$$

since the boundary of a circle is empty, $\partial S^1 = \varnothing$, which is a contradiction. The following conclusion is valid in general: along any 1-cycle we can turn on a vev for the 1-form A which does not give rise to a flux (since the 1-cycle is not a boundary) and which cannot be gauged away (since the 1-cycle does not have a boundary).

[39] The gauge field appearing in the non-linear σ-model (1.2.24) is real. Hence the factor of i.

a flat closed string background. The boundary conditions in the directions transverse are still the Dirichlet conditions (1.3.1), but from (1.2.24) we find that the boundary conditions for the directions along the brane need to modified as follows:

$$\partial_\sigma X^\mu - 2\pi\alpha' F^\mu{}_\nu \partial_\tau X^\nu = 0, \quad \text{at } \sigma = 0, \ell. \tag{1.3.38}$$

Instead of analyzing these b.c.'s in general, we will present a few simple examples that capture the important features.

The first example is that of a Dp-brane that carries an electric field $E^i = F^{0i}$ but no magnetic field. In this case the Born-Infeld action reduces to

$$S_{\text{DBI}} = -\tau_p \int \mathrm{d}^{p+1}\sigma \sqrt{1 - (2\pi\alpha')^2 \|\mathbf{E}\|^2}. \tag{1.3.39}$$

We conclude that there is an upper limit to the magnitude of the electric field. We have $\mathcal{E} \equiv 2\pi\alpha' \|\mathbf{E}\| \leq 1$. Now we rotate the brane such that the electric field points in the pth direction and wrap this pth direction once around a circle. (1.3.38) reduces to Neumann b.c.'s in all direction except x^0 and x^p:

$$\partial_\sigma X^0 - \mathcal{E}\, \partial_\tau X^p = 0, \qquad \partial_\sigma X^p - \mathcal{E}\, \partial_\tau X^0 = 0. \tag{1.3.40}$$

We perform a T-duality transformation on the x^p direction which replaces $\partial_\tau X^p \leftrightarrow \partial_\sigma X'^p$ and get

$$\partial_\sigma (X^0 - \mathcal{E} X'^p) = 0, \qquad \partial_\tau (X'^p - \mathcal{E} X^0) = 0. \tag{1.3.41}$$

These are the boundary conditions for a D$(p-1)$-brane that is boosted with a velocity $\mathcal{E}$ along the dual circle. The condition that $\mathcal{E} \leq 1$ thus translates under T-duality into the statement that nothing travels faster than the speed of light. Another way of seeing the above is to pick the gauge $A_p = F_{0p} x^0$ and apply the T-duality rule (1.3.21). The position of the D$(p-1)$-brane on the dual circle becomes $x'^p = -2\pi\alpha' A_p = \mathcal{E} x^0$.

It is not surprising that when electric fields correspond to boosts after T-duality, magnetic fields correspond to rotations. Consider for example a D2-brane wrapped once around a 2-torus with periods L_1 and L_2. We turn on a magnetic field F_{12} and get the boundary conditions

$$\partial_\sigma X^1 - \mathcal{B}\, \partial_\tau X^2 = 0, \qquad \partial_\sigma X^2 + \mathcal{B}\, \partial_\tau X^1 = 0, \tag{1.3.42}$$

where we defined $\mathcal{B} \equiv 2\pi\alpha' F_{12}$. What is interesting here is that we have a mixing of Neumann and Dirichlet boundary conditions. In particular, when $\mathcal{B} \to \infty$ the string coordinates X^1 and X^2 become Dirichlet. It is as if the D2-brane is filled with an infinite number of D0-branes, one for each value of (x^1, x^2). We will make this idea more precise in a minute.

After a T-duality transformation of the x^2-direction we get

$$\partial_\sigma (X^1 - \mathcal{B} X'^2) = 0, \qquad \partial_\tau (X'^2 - \mathcal{B} X^1) = 0. \tag{1.3.43}$$

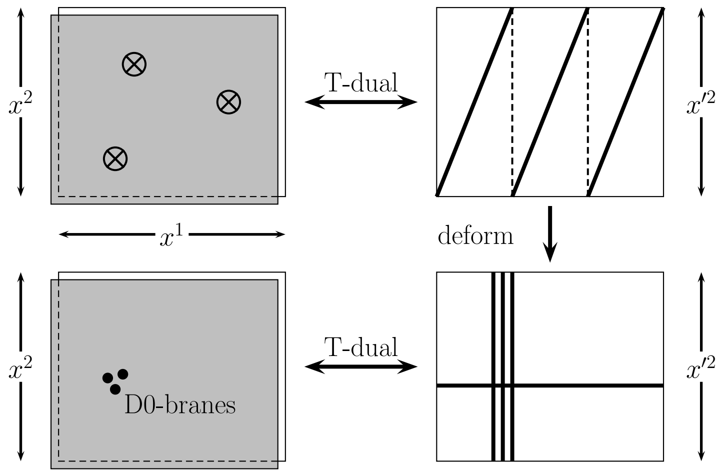

Figure 1.3.2. A D2-brane with three units of magnetic flux wrapped once around a torus with coordinates (x^1, x^2) is dual to a D1-brane that is wrapped 3 times around the dual torus (x^1, x'^2). This configuration can be deformed into one in which a single D1-brane wraps the x^1 direction and a stack of three D1-branes wraps the x'^2 direction, which is in turn T-dual to a wrapped D2-brane with three D0-branes. The tori are depicted as rectangles with opposite faces identified.

These are the boundary conditions for a D1-brane that has been rotated by an angle $\theta = \arctan \mathcal{B}$ w.r.t. the x^1-axis. This D1-brane winds around the x'^2-axis as we look at increasing values of x^1. Since the x^1 direction is periodic, the D1-brane needs to come back onto itself at $x^1 = L_1$. It can thus wind only an integer number n of times around the x'^2 direction – see figure 1.3.2. We have therefore that $\tan\theta = nL'_2/L_1$ or

$$\mathcal{B} = \tan\theta = 2\pi\alpha' \frac{2\pi n}{L_1 L_2}, \quad \text{with } n \in \mathbb{Z}, \tag{1.3.44}$$

i.e. the flux $\Phi \equiv F_{12} L_1 L_2$ through the original torus is quantized in units of $2\pi n$. The wrapped D1-brane has an energy

$$E_{\text{wrapped}} = \tau_{\text{D1}} \sqrt{L_1^2 + (nL'_2)^2}. \tag{1.3.45}$$

As shown in figure 1.3.2, we can continuously deform the D1-brane that wraps n times around the torus to a configuration with a stack of n D1-branes wrapped once around the x'^2 direction and a single D1-brane wrapped once around the x^1 direction. This costs some energy, since

$$E_{\text{unwrapped}} = \tau_{\text{D1}}(L_1 + nL'_2) \geq E_{\text{wrapped}}. \tag{1.3.46}$$

If we T-dualize x'^2 in the latter configuration, we recover our original torus, but with n D0-branes and a D2-brane *without* flux. This configuration has a higher energy than that of a D2-brane with n units of flux. So it is clear that we should interpret the latter case as a *bound state* of a D2-brane and n D0-branes.

The flux quantization can also be understood from the D2-brane's point of view. If we parallel transport the wave-function of a particle with a unit charge along a path $\mathcal{P}$, it will pick up a phase ϕ

$$\mathrm{e}^{i\phi} = \exp i \int_{\mathcal{P}} A. \tag{1.3.47}$$

This phase depends on the path taken. In particular, the phase factor that is obtained from the path $\mathcal{P}_1 : (0,0) \to (L_1, 0) \to (L_1, L_2)$ is in principle different from that of the path $\mathcal{P}_2 : (0,0) \to (0, L_2) \to (L_1, L_2)$. But on the torus T^2 these paths are the same and should therefore yield the same phase factor. We thus insist that

$$\exp i \int_{\mathcal{P}_1 - \mathcal{P}_2} A = \exp i \int_{T^2} F = \mathrm{e}^{i\Phi} \equiv 1.$$

The flux Φ is thus quantized, $\Phi = 2\pi n$. This is essentially a consequence of the fact that it is impossible to find a globally well-defined (i.e. continuous) gauge potential A on the torus. Indeed, if there had been a continuous potential, the gauge fields along the paths $\mathcal{P}_1$ and $\mathcal{P}_2$ would have been equal and hence also the phases. To show that it is indeed impossible to find a globally defined A, suppose the converse. We would then have

$$\int_{T^2} F = \int_{T^2} \mathrm{d}A = \int_{\partial T^2} A = 0, \quad \text{since } \partial T^2 = \varnothing,$$

where the use of Stokes' theorem is allowed since A is continuous. This is however in contradiction with a non-zero Φ, proving our statement.

The above construction can be generalized. For example, consider a D4-brane wrapped around a 4-torus T^4 with a constant magnetic field $F = F_{12}\mathrm{d}x^1 \wedge \mathrm{d}x^2 + F_{34}\mathrm{d}x^3 \wedge \mathrm{d}x^4$. By T-dualizing in the 2- and 4-directions we can argue that the flux is quantized as $F_{12}L_1L_2 = 2\pi m$, $F_{34}L_3L_4 = 2\pi n$ and that we have a bound state of a D4-brane, D2-branes and D0-branes. The number of D2-branes is given by the *first Chern class*:

$$c_1 = \frac{1}{2\pi} \sum_{\text{2-cycles}} \int F = m + n, \tag{1.3.48}$$

where the sum runs over the 2-cycles of T^4, and the number of D0-branes by the *second Chern class*:

$$c_2 = \frac{1}{8\pi^2} \sum_{\text{4-cycles}} \int F \wedge F = mn. \tag{1.3.49}$$

This should suffice to illustrate the general pattern. The number of D$(p-2n)$-brane charges immersed in the world-volume of a wrapped Dp-brane is given by the nth Chern-Class:

$$c_n = \frac{1}{(2\pi)^k k!} \sum_{2n\text{-cycles}} \int \underbrace{F \wedge \ldots \wedge F}_{n\times}. \tag{1.3.50}$$

1.3.5 Unoriented theories and T-duality

For our discussion of D-branes in superstring theories, we need to understand how T-duality works for the *unoriented* string.

Closed strings

Recall that the unoriented strings are obtained by restricting the spectrum to states with $\Omega = +1$. We will first investigate the closed strings to see the effect of this 'modding out' in the T-dual version of the theory. Writing $X' = TXT^{-1}$, we find the dual version Ω' as follows:

$$T(\Omega X \Omega^{-1})T^{-1} = (T\Omega T^{-1})(TXT^{-1})(T\Omega T^{-1})^{-1} \equiv \Omega' X' \Omega^{-1}. \tag{1.3.51}$$

So if we dualize over the 25-direction only, we have

$$\Omega' X'^i(\tau,\sigma)\Omega'^{-1} = X'^i(\tau, \ell-\sigma), \tag{1.3.52a}$$

$$\Omega' X'^{25}(\tau,\sigma)\Omega'^{-1} = -X'^{25}(\tau, \ell-\sigma). \tag{1.3.52b}$$

Thus in the dual version, the world-sheet parity transformation is accompanied by a spacetime parity transformation in the $25'$-direction. The points $x'^{25} = 0$ and $x'^{25} = \pi R'$ on this compact direction are special: they are fixed under the action of Ω'.

If we now restrict ourselves to states that are invariant under Ω' the resulting physics is quite remarkable. It is as if the fixed points $x'^{25} = 0, \pi R'$ act as some sort of mirror. The physics in the region $\pi R' < x'^{25} < 2\pi R'$ is completely determined by that of the region $0 \leq x'^{25} \leq \pi R'$ by reflection and reversing the orientation of the strings. Note that the physics in the 'fundamental' region $0 \leq x'^{25} \leq \pi R'$ is that of an *oriented* string theory.

The T-dual image of the unoriented theory on a circle of circumference $2\pi R$ is thus an oriented theory on an interval $S^1/\mathbb{Z}_2$ of length $\pi R'$. The restriction to Ω' states is known as an *orientifold* projection and the 'mirror planes' at the ends of the interval are called *orientifold planes* or O-planes. In the present example, these planes are 25-dimensional, i.e. O24-planes.

It is straightforward to generalize this construction to Op-planes for lower values of p – for T-duality over k directions we have 2^k O$(25-k)$-planes situated on the vertices of a k-dimensional hypercube. Moreover, we can reinterpret the original construction of unoriented strings as adding a spacetime-filling O25-plane to the theory of closed strings.

Open strings

First we investigate the effect of the Ω-projection on the open string theory with $U(n)$ Chan-Paton factors following section 6.5 of [15].

World-sheet parity Ω exchanges the endpoints of an open string and thus reverses the order of the Chan-Paton indices, $\Omega|N,k;ij\rangle = (-)^N|N,k;ji\rangle$. We can combine the action of Ω with a $U(n)$ rotation γ of the Chan-Paton indices as follows:

$$\Omega_\gamma\, \lambda^\alpha_{ij}|N,k;ij\rangle = (-)^N(\gamma\lambda^\alpha\gamma^{-1})_{ij}|N,k;ji\rangle. \tag{1.3.53}$$

For a general $U(n)$ rotation γ, Ω_γ does not define a multiplicatively conserved $\mathbb{Z}_2$ quantum number, but for certain subgroups of $U(n)$ it does. Indeed, Ω^2_γ acts on the λ matrix as

$$\lambda^\alpha \to (\gamma^{-1T}\gamma)\lambda^\alpha(\gamma^{-1T}\gamma)^{-1}. \tag{1.3.54}$$

Since the λ^α form an irreducible set we have by Schur's lemma that $\Omega^2_\gamma = 1$ if[40]

$$\gamma^T = \pm\gamma. \tag{1.3.55}$$

So there are *two* choices of a conserved Ω_γ by which we can mod out the open string spectrum. By changing the basis $\lambda^\alpha \to U\lambda^\alpha U^{-1}$, under which $\gamma \to U^{-1T}\gamma U^{-1}$, we can in both cases bring γ into a standard form.

In the case that γ is symmetric, we take for n even

$$\gamma = \delta_n \equiv \begin{pmatrix} 0 & I_{n/2} \\ I_{n/2} & 0 \end{pmatrix}, \tag{1.3.56}$$

with $I_{n/2}$ the $n/2 \times n/2$ identity matrix. For n odd we add an additional 1 to the diagonal. The result of the Ω_γ projection is then to keep the states with $\lambda^T = \delta\lambda\delta$ for N even and $\lambda^T = -\delta\lambda\delta$ for N odd. This means in particular that the massless gauge fields ($N = 1$) are the gauge bosons for the group $SO(n)$[41].

In the case that γ is antisymmetric, n must be even since γ is invertible. We take

$$\gamma = i\varepsilon_n \equiv i\begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}. \tag{1.3.57}$$

[40] Schur's lemma tells us that $\gamma^{-T}\gamma = cI$, i.e. $\gamma = c\gamma^T$. Taking the transpose of this relation and plugging the result back in yields $\gamma = c^2\gamma$, i.e. $c^2 = 1$.

[41] Usually, one takes the line element of $SO(2n)$ to be the unit matrix I_{2n}. Our choice for γ corresponds to a different embedding of $SO(2n)$ in $U(2n)$ and can be reached from the usual case by a $U(2n)$ rotation: $\delta_{2n} = U^{-1T}I_{2n}U^{-1}$, with $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$.

Modding out the spectrum, we keep the states with $\lambda^T = -\varepsilon\lambda\varepsilon$ for N even and those with $\lambda^T = +\varepsilon\lambda\varepsilon$ for N odd. This means that the gauge bosons are those of $USp(n)$.

We restrict ourselves to the case with $2n$ D25-branes for simplicity. We split the Chan-Paton indices into two groups $(i, \bar{\jmath})$, with $i, \bar{\jmath} = 1, \ldots, n$ such that

$$\lambda^\alpha = \begin{pmatrix} \lambda^\alpha_{ij} & \lambda^\alpha_{i\bar{\jmath}} \\ \lambda^\alpha_{\bar{\imath}j} & \lambda^\alpha_{\bar{\imath}\bar{\jmath}} \end{pmatrix} \tag{1.3.58}$$

We then have $\lambda^\alpha_{ij} = -\lambda^\alpha_{\bar{\jmath}\bar{\imath}}$ for both $SO(2n)$ and $USp(2n)$. For $SO(2n)$ in addition $\lambda^\alpha_{i\bar{\jmath}} = -\lambda^\alpha_{j\bar{\imath}}$ and $\lambda^\alpha_{\bar{\imath}j} = -\lambda^\alpha_{\bar{\jmath}i}$, whereas for $USp(2n)$ $\lambda^\alpha_{i\bar{\jmath}} = +\lambda^\alpha_{j\bar{\imath}}$ and $\lambda^\alpha_{\bar{\imath}j} = +\lambda^\alpha_{\bar{\jmath}i}$.

With these conventions it is straightforward to investigate the effects of a T-duality transformation along the 25-direction in the presence of a non-trivial Wilson line $A_{25} = A^\alpha_{25}\lambda^\alpha$. We take

$$A_{25} = -\frac{i}{2\pi\alpha'}\mathrm{diag}\,(\bar{x}_1, \ldots, \bar{x}_n, -\bar{x}_1, \ldots, -\bar{x}_n), \tag{1.3.59}$$

which breaks the gauge invariance from $SO(2n)$ or $USp(2n)$ to $U(1)^n$. In the T-dual picture there are n D24-branes on the interval $0 \leq X'^{25} \leq \pi R'$ – the branes labeled by i – and n at their image points under the orientifold identification – those labeled by $\bar{\imath}$. The n branes carry $U(1)$ gauge fields. When r branes become coincident, the gauge symmetry is enlarged to $U(r)$ according to the mechanism of section 1.3.3. However, if these branes lie at one of the orientifold planes, the strings that stretch between them and the mirror branes also become massless. This gives the additional states that are needed to fill out[42] the adjoint multiplets of $SO(2r)$ or $USp(2r)$. The maximal $SO(2n)$ or $USp(2n)$ is restored if all D24-branes are located at the same orientifold plane.

Before closing this chapter we would like to point out that we can view the original unoriented $SO(2n)$ and $USp(2n)$ open string theories as systems with n coincident D25-branes and a single O25-plane. Compactification over a k-torus and a subsequent T-duality transformation leaves us with a k-dimensional hypercube with 2^k O$(25-k)$-planes and n parallel D$(25-k)$-branes.

[42] The additional states correspond to the matrices $\lambda^\alpha_{i\bar{\jmath}}$ and $\lambda^\alpha_{\bar{\imath}j}$ in (1.3.58). For the $SO(2n)$ string there are thus $2 \times r(r-1)/2$ additional states, which give a we get a total of $r(2r-1)$ when combined with the r^2 states we already had. This is just the right amount of states to fill up the adjoint of $SO(2r)$. Similarly, we get $r(2r+1)$ states in the case of $USp(2n)$.

Chapter 2

Superstrings

In the previous chapter we saw that the bosonic string theory has a few obvious drawbacks. First of all the theory is unstable because of the presence of a tachyon in the spectrum. Second, the theory does not contain fermionic states, which rules out the bosonic string as a theory of nature. Third, the theory necessarily lives in a 26-dimensional spacetime.

The first two problems are solved by the superstring theories. We have seen in the previous chapter that, although it is not difficult to formulate a classical theory of bosonic strings, the laws of quantum mechanics act as a mathematical straitjacket. The requirement of unitarity not only led to the existence of a critical dimension, but also severely constrained the dynamics. Now it turns out that any attempt to introduce fermions in bosonic string theory requires *supersymmetry*[1]. Furthermore, there are only five distinct ways in which a superstring theory can be constructed on a flat background. All require a critical dimension $d = 10$.

There are two different, but physically equivalent, approaches to superstring theory: the Ramond-Neveu-Schwarz (RNS) and the Green-Schwarz (GS) superstrings. In both cases one starts from the bosonic string's Polyakov action $S[X^\mu, \gamma_{\alpha\beta}]$ (1.1.3). One then adds additional fields to the theory in order to obtain a supersymmetric system.

In the RNS approach supersymmetry is first achieved "on the world-sheet" by introducing fermionic partners to the bosonic fields. We add a spinor ψ^μ for each scalar X^μ and a gravitino χ_α for the metric $\gamma_{\alpha\beta}$. The result is a complicated interacting theory with a number of local symmetries: diffeomorphisms, local supersymmetry and super-Weyl invariance. As in the previous chapter, these symmetries can be fixed by going to the so-called superconformal gauge, in which the metric and gravitino are absent. The main advantage of the RNS approach lies in the fact that the su-

[1] See appendix A.2 for a condensed review of four-dimensional supersymmetry.

perconformal gauge gives rise to a *free* theory on the world-sheet without breaking the target space Lorentz invariance. In addition, the resulting theory possesses a large group of residual symmetries that preserve the conformal gauge. One can fix these symmetries by imposing the light-cone gauge condition at the price of breaking target space Lorentz invariance. This approach suffices if one wants to derive basic results like the spectrum of the superstring and the tree-level amplitudes – we will therefore proceed in this fashion in the next section. For calculations beyond the tree-level approximation however, it is much more convenient to quantize covariantly and deal with the symmetries later. There are also some disadvantages to the RNS approach. First of all, as we will see below, spacetime supersymmetry is not manifest and actually requires a truncation – the GSO projection – of the spectrum that one obtains from a straightforward quantization of the theory. Moreover, it turns out to be extremely complicated to describe strings on backgrounds of the Ramond-Ramond fields (more on these below). This is because of technical complications involving their vertex operators.

The Green-Schwarz (GS) formulation starts from an already spacetime *globally* supersymmetric action. The main disadvantage of this approach is that covariant quantization is more subtle in this case. Until recently, it was not known how to obtain a free field theory on the world-sheet without resorting to the light-cone gauge. This problem has however now been solved (see [60, 61] for reviews) and progress has been made in extending these methods beyond tree-level [62, 63], reproducing in part the results reviewed in [32] which were based on the RNS formalism.

We will use the RNS formalism in the following, since this is the method that we used in our paper [b].

A few remarks on spacetime supersymmetry are in order. It turns out that the maximum number of supercharges that a theory can possess is 32 (in any dimension). In ten dimensions we call this $\mathcal{N} = 2$ supersymmetry, since the charges can be represented by two Majorana-Weyl fermions. The minimum amount of supersymmetry is $\mathcal{N} = 1$ in ten dimensions, i.e. 16 supercharges. Both the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ case arise in superstring theory. This works roughly as follows. Remember from chapter 1 that the degrees of freedom in the two-dimensional world-sheet theory for the closed string split into independent left- and right-movers $X_L(\tau + \sigma)$ and $X_R(\tau - \sigma)$. In the RNS formulation we can add both left- and right-moving fermions to the theory. This yields $\mathcal{N} = 2$ supersymmetry. It turns out that we can perform the GSO projection in two ways, one giving rise to two spacetime supercharges of opposite chirality (the *type IIA* superstring), the other giving two charges of the same chirality (the *type IIB* string). The presence of open strings breaks the supersymmetry to $\mathcal{N} = 1$, since the Neumann or Dirichlet boundary conditions relate the left- and right-moving sectors. The result is the *type I* superstring. Alternatively, we can decide to add fermions only to the left-moving sector, but then it turns out that we need to modify the right-moving bosons in a particular way to guarantee the consistency of the procedure. The result are the $\mathcal{N}=1$ supersymmetric *heterotic* string theories.

In the following we will develop the type I and II string theories in the RNS approach. Details on the heterotic string are left to the literature. We then give the effective supergravity theories that describe these theories at low energies. After this, we move on to supersymmetric branes in general and D-branes in particular. Finally, we come to the famous dualities that relate all these different string theories to each other and briefly discuss the role that branes play in this story.

2.1 The RNS superstring

The action for the RNS superstring in the superconformal gauge is:

$$S_{\mathrm{RNS}} = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\big[\partial_\alpha X^\mu\partial^\alpha X_\mu + \bar{\psi}_\mu\rho^\alpha\partial_\alpha\psi^\mu\big]. \tag{2.1.1}$$

We denote the two-dimensional Dirac matrices by ρ^α. The spinors ψ^μ are Majorana, $\bar{\psi}^\mu = \psi^{\mu T}C$ where we use $C_- \equiv C$. See appendix A for an extensive review of the properties of Dirac matrices and spinors. The action (2.1.1) is invariant under the following rigid supersymmetry transformations:

$$\delta(\epsilon)X^\mu = \bar{\epsilon}\psi^\mu, \tag{2.1.2a}$$
$$\delta(\epsilon)\psi^\mu = \rho^\alpha\epsilon\,\partial_\alpha X^\mu, \tag{2.1.2b}$$

with ϵ a constant Majorana spinor. We verify that these are indeed supersymmetries by calculating the commutator of two of such transformations:

$$[\delta(\epsilon_1),\delta(\epsilon_2)]X^\mu = -2\,\bar{\epsilon}_1\rho^\alpha\epsilon_2\,\partial_\alpha X^\mu, \tag{2.1.3a}$$
$$[\delta(\epsilon_1),\delta(\epsilon_2)]\psi^\mu = -2\,\bar{\epsilon}_1\rho^\alpha\epsilon_2\,\partial_\alpha\psi^\mu + \bar{\epsilon}_1\rho^\alpha\epsilon_2\,\rho_\alpha\partial\!\!\!/\psi^\mu. \tag{2.1.3b}$$

As in our review of the Wess-Zumino model in appendix A.2, we obtain a closed algebra if the fermions satisfy their equations of motion $\partial\!\!\!/\psi^\mu = 0$.

In two dimensions, a Majorana spinor can be decomposed into two Majorana-Weyl spinors, $\psi^\mu = P_+\psi^\mu + P_-\psi^\mu$. It is convenient to work with an explicit basis for the ρ-matrices. We use:

$$\rho_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}. \tag{2.1.4}$$

The spinors ψ_- and ψ_+ are real Grassmann *numbers*. Note that $\rho_*\psi_\pm = \mp\psi_\pm$.

For the bosonic string, the conformal gauge did not fix all the local diffeomorphisms and Weyl rescalings. In the present case there are also residual symmetries. To analyze these, it is convenient to work with light-cone coordinates on the world-sheet $\sigma^\pm \equiv \tau \pm \sigma$. The action becomes[2]:

$$S = \frac{1}{2\pi\alpha'} \int d\sigma^+ d\sigma^- \left[\partial_+ X^\mu \partial_- X_\mu + \frac{i}{2}\psi_- \partial_+ \psi_- + \frac{i}{2}\psi_+ \partial_- \psi_+ \right]. \tag{2.1.5}$$

It is not difficult to check that (2.1.5) is invariant under

$$\delta X^\mu = \xi^+ \partial_+ X^\mu + \xi^- \partial_- X^\mu, \tag{2.1.6a}$$

$$\delta \psi^\mu_\pm = \xi^+ \partial_+ \psi^\mu_\pm + \xi^- \partial_- \psi^\mu_\pm + \frac{1}{2} \partial_\pm \xi^\pm \psi^\mu_\pm, \tag{2.1.6b}$$

with $\partial_+ \xi^- = \partial_- \xi^+ = 0$. These are the residual diffeomorphisms that we also encountered in (1.1.14). The extra terms in the variation of the fermions have their origin in the local Weyl and local Lorentz transformations of the underlying two-dimensional supergravity theory. In addition, there are now also residual 'quasi-local' supersymmetry transformations:

$$\delta X^\mu = i\epsilon_+ \psi^\mu_- - i\epsilon_- \psi^\mu_+, \tag{2.1.7a}$$

$$\delta \psi^\mu_\pm = \pm 2 \partial_\pm X^\mu \, \epsilon_\mp, \tag{2.1.7b}$$

with $\partial_+ \epsilon^+ = \partial_- \epsilon^- = 0$.

As before, we use the residual translational symmetries to impose the light-cone gauge condition (1.1.17) on X^+. We fix the residual supersymmetries by imposing[3]

$$\psi^+(\tau, \sigma) = 0. \tag{2.1.8}$$

In this gauge, the ψ^- drop out of the action entirely and we are left with the following Lagrangian:

$$L = -p^+ \partial_\tau x^- - \frac{1}{4\pi\alpha'} \int_0^\ell d\sigma \, \partial_\alpha X^i \partial^\alpha X^i + \frac{i}{2\pi\alpha'} \int_0^\ell d\sigma \left[\psi^i_- \partial_+ \psi^i_- + \psi^i_+ \partial_- \psi^i_+ \right]. \tag{2.1.9}$$

Boundary conditions

From (2.1.5) we derive the following equations of motion for the fermions:

$$\partial_+ \psi^i_- = 0, \qquad \partial_- \psi^i_+ = 0, \tag{2.1.10}$$

i.e. ψ^i_- and ψ^i_+ contain only right- and left-moving degrees, respectively. These equations need to be supplemented with suitable boundary conditions in order that

$$\left(\psi^i_- \delta \psi_{-i} - \psi^i_+ \delta \psi_{+i} \right) \Big|_{\sigma=0}^{\ell} = 0. \tag{2.1.11}$$

[2] We define $\partial_\pm \equiv \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$ such that $\partial_\pm \sigma^\pm = 1$ and $\partial_\pm \sigma^\mp = 0$. We have for the metric $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$ and $\eta^{+-} = \eta^{-+} = -2$ and for the integration measure $d\sigma^+ d\sigma^- = 2 d\tau d\sigma$.

[3] We define $\psi^\pm = (\psi^0 \pm \psi^{d-1})/\sqrt{2}$ as in (1.1.15).

This is equivalent to $\delta(\psi_+^i)^2|_{\sigma=0}^{\ell} = \delta(\psi_-^i)^2|_{\sigma=0}^{\ell}$. We want to obtain a Lorentz invariant spectrum. We therefore choose our boundary conditions such that they do not break the $SO(d-2)$ symmetry which is what remains from the target-space Lorentz symmetries in the light-cone gauge.

For open strings the boundary conditions relate the right- and left-movers at the endpoints. There are two possibilities that are known as the Neveu-Schwarz (NS) and Ramond (R) boundary conditions:

- Neveu-Schwarz: $\psi_+^i(\tau,0) = \varsigma\psi_-^i(\tau,0), \qquad \psi_+^i(\tau,\ell) = -\varsigma\psi_-^i(\tau,\ell),$ (2.1.12a)
- Ramond: $\psi_+^i(\tau,0) = \varsigma\psi_-^i(\tau,0), \qquad \psi_+^i(\tau,\ell) = +\varsigma\psi_-^i(\tau,\ell).$ (2.1.12b)

The overall sign $\varsigma = \pm 1$ depends on whether we consider Neumann or Dirichlet boundary conditions on the bosonic side. Since the boundary conditions relate left- and right-moving degrees of freedom, the presence of a boundary breaks some of the world-sheet supersymmetry. The $\sigma = 0$ boundary condition transforms under supersymmetry as

$$\delta[\psi_+^i(\tau,0) - \varsigma\psi_-^i(\tau,0)] = (\epsilon_- + \varsigma\epsilon_+)\partial_\tau X^i + (\epsilon_- - \varsigma\epsilon_+)\partial_\sigma X^i. \tag{2.1.13}$$

For the Neumann b.c. there is a preserved supersymmetry with $\epsilon_+ = -\varsigma\epsilon_-$, whereas the Dirichlet b.c. preserves $\epsilon_+ = \varsigma\epsilon_-$. We take $\varsigma = +1$ for the Neumann b.c. and $\varsigma = -1$ for the Dirichlet b.c. in order that the $\sigma = 0$ boundary preserves the world-sheet supersymmetry with $\epsilon_+ = -\epsilon_-$.

The $\sigma = \ell$ boundary preserves supersymmetries with $\epsilon_+ = +\epsilon_-$ in the NS case and $\epsilon_+ = -\epsilon_-$ in the R case. Open superstrings with NS boundary conditions will therefore not have a *world-sheet* supersymmetric spectrum, whereas the Ramond superstrings do. We will see that a *spacetime* supersymmetric spectrum can be achieved only if we include both NS and R strings and mod out the spectrum by a certain discrete symmetry.

For closed strings we can impose NS or R boundary conditions on the left- and right-movers independently:

- Neveu-Schwarz: $\psi_\pm^i(\tau,\sigma) = -\psi_\pm^i(\tau,\sigma+\ell),$ (2.1.14a)
- Ramond: $\psi_\pm^i(\tau,\sigma) = +\psi_\pm^i(\tau,\sigma+\ell).$ (2.1.14b)

There are therefore four kinds of closed strings: NS-NS, NS-R, R-NS and R-R. The R-R b.c.'s preserve all world-sheet supersymmetries, the NS-R and R-NS b.c.'s only half of them and the NS-NS break all. A *spacetime* supersymmetric spectrum can be achieved only if we include states from all the four types of closed strings.

Dirac quantization

The quantization of the fermions $\psi_\pm^i$ is somewhat subtle, since these are constrained fields – they are Majorana. We will use Dirac's method of quantization. We will not discuss the *why* of this method in any detail, but rather focus on the *how*. We refer to [34, 64–66] for reviews of this method and to [16] for an application to the problem at hand.

We derive the following canonical momentum densities[4]:

$$\pi^i_\pm(\tau,\sigma) \equiv \frac{\partial\mathcal{L}}{\partial\,\partial_\tau\psi^i_\pm} = -\frac{i}{4\pi\alpha'}\psi^i_\pm(\tau,\sigma). \tag{2.1.15}$$

There is thus a primary constraint

$$\Phi^i_\pm \equiv \pi^i_\pm + \frac{i}{4\pi\alpha'}\psi^i_\pm \approx 0. \tag{2.1.16}$$

The above equation defines a surface in phase space. We will now show that this constraint does not lead to any secondary constraints. First we calculate the Hamiltonian[5] $(A=+,-)$:

$$H_{\text{can}} \equiv \int_0^\ell \mathrm{d}\sigma\big(\dot\psi^i_A\pi^i_A\big) - L \approx -\frac{i}{4\pi\alpha'}\int_0^\ell \mathrm{d}\sigma\big[\psi^i_-\partial_\sigma\psi^i_- - \psi^i_+\partial_\sigma\psi^i_+\big]. \tag{2.1.17}$$

The $\approx$ means "equal on the constraint surface". Following Dirac, we extend the definition of the Hamiltonian off the primary constraint surface as follows:

$$H_* \equiv H_{\text{can}} + \int_0^\ell \mathrm{d}\sigma\, u^i_A(\tau,\sigma)\Phi^i_A(\tau,\sigma), \tag{2.1.18}$$

[4]We always take derivatives w.r.t. Grassmann variables from the left, i.e. $\delta f(\theta) = \delta\theta\frac{\partial f}{\partial\theta}$ for a function f of a Grassmann variable θ.

[5]For a system with Grassman Lagrangian $L(\theta^\alpha,\dot\theta^\alpha)$ we define the canonical momenta by $\pi^\alpha = \partial L/\partial\dot\theta^\alpha$ and the Hamiltonian by $H(\theta^\alpha,\pi^\alpha) \equiv \dot\theta^\alpha\pi^\alpha - L$. Hamilton's equations read $\dot\theta^\alpha = -\partial H/\partial\pi^\alpha$ and $\dot\pi^\alpha = -\partial H/\partial\theta^\alpha$. The Poisson bracket is defined such that $\mathrm{d}F/\mathrm{d}t = \{F,H\}_{\text{PB}} + \partial F/\partial t$ for any function F [65, 66]. For functions of the phase-space variables θ^α and π^α we have

$$\{F,G\}_{\text{PB}} \equiv (-)^{\varepsilon_F}\Big(\frac{\partial F}{\partial\theta^\alpha}\frac{\partial G}{\partial\pi^\alpha} + \frac{\partial F}{\partial\pi^\alpha}\frac{\partial G}{\partial\theta^\alpha}\Big),$$

where ε_F is 0 if F is bosonic and 1 if F is Grassmann. The 'fundamental' Poisson brackets thus read:

$$\{\theta^\alpha,\theta^\beta\}_{\text{PB}} = \{\pi^\alpha,\pi^\beta\}_{\text{PB}} = 0, \qquad \{\theta^\alpha,\pi^\beta\}_{\text{PB}} = -\delta^{\alpha\beta}.$$

In Dirac's approach, expressions involving the Poisson bracket of functions that do not depend on the phase space variables are manipulated by appealing to the algebraic properties of the bracket [64–66]:

$$\{F,G\}_{\text{PB}} = -(-)^{\varepsilon_F\varepsilon_G}\{G,F\}_{\text{PB}},$$
$$\{F,GH\}_{\text{PB}} = \{F,G\}_{\text{PB}}H + (-)^{\varepsilon_F\varepsilon_G}G\{F,H\}_{\text{PB}},$$
$$\{\{F,G\},H\}_{\text{PB}} + (-)^{\varepsilon_F(\varepsilon_G+\varepsilon_H)}\{\{G,H\},F\}_{\text{PB}} + (-)^{\varepsilon_H(\varepsilon_F+\varepsilon_G)}\{\{H,F\},G\}_{\text{PB}} = 0.$$

where the $u^i_A(\tau,\sigma)$ are as of yet arbitrary functions. Clearly $H_* \approx H_{\text{can}}$.

Using the fundamental Poisson brackets

$$\{\psi^i_A(\tau,\sigma),\psi^j_B(\tau,\sigma')\}_{\text{PB}} = \{\pi^i_A(\tau,\sigma),\pi^j_B(\tau,\sigma')\}_{\text{PB}} = 0, \tag{2.1.19a}$$

$$\{\psi^i_A(\tau,\sigma),\pi^j_B(\tau,\sigma')\}_{\text{PB}} = -\delta^{ij}\delta_{AB}\delta(\sigma-\sigma'), \tag{2.1.19b}$$

we get $\{\Phi^i_A(\tau,\sigma),H_{\text{can}}\}_{\text{PB}} = 0$ and thus

$$\{\Phi^i_A(\tau,\sigma),H_*\}_{\text{PB}} \approx -\frac{i}{2\pi\alpha'}u^i_A(\tau,\sigma).$$

This must vanish, since we demand that the constraints are constant in time. We achieve this by solving for the functions u^i_A, i.e. $u^i_A = 0$ and thus $H_* = H_{\text{can}}$. There are therefore no secondary constraints. Note that putting $\Phi^i_A \approx 0$ is inconsistent with the Poisson brackets: we can only impose the constraints *after* we have worked out all the Poisson brackets.

Our constraints are second class since they do not generate a closed algebra under the Poisson bracket:

$$\{\Phi^i_A(\tau,\sigma),\Phi^j_B(\tau,\sigma')\}_{\text{PB}} = -\frac{i}{2\pi\alpha'}\delta^{ij}\delta_{AB}\delta(\sigma-\sigma') \not\approx 0, \tag{2.1.20}$$

with $A,B=\pm$.

In our cookbook treatment of Dirac's method, the next step in the recipe is the following: in the presence of second class constraints, one needs to replace the Poisson bracket with a new *Dirac* bracket. It turns out that putting $\Phi^i_A = 0$ *is* consistent with the Dirac bracket.

In our case, the Dirac bracket is

$$\{f,g\}_{\text{DB}} \equiv \{f,g\}_{\text{PB}} - 2\pi i\alpha' \int_0^\ell \mathrm{d}\sigma\, \{f,\Phi^i_A(\tau,\sigma)\}_{\text{PB}}\{\Phi^i_A(\tau,\sigma),g\}_{\text{PB}}, \tag{2.1.21}$$

and we obtain

$$\{\psi^i_A(\tau,\sigma),\psi^j_B(\tau,\sigma')\}_{\text{DB}} = -2\pi i\alpha'\,\delta^{ij}\delta_{AB}\delta(\sigma-\sigma'). \tag{2.1.22}$$

We finally quantize by replacing the *Dirac* bracket by (anti)commutators $\{\cdot,\cdot\}_{\text{DB}} \to [\cdot,\cdot]_\pm/i$ and impose

$$\{\psi^i_A(\tau,\sigma),\psi^j_B(\tau,\sigma')\} = 2\pi\alpha'\,\delta^{ij}\delta_{AB}\delta(\sigma-\sigma'). \tag{2.1.23}$$

This differs an all-important factor 2 from the anticommutation relation for a Dirac spinor.

2.1.1 The open superstring

After having obtained the anticommutation relations (2.1.23), the rest of the quantization procedure is a straightforward exercise along the lines of chapter 1.

In the case of the open string with Neumann boundary conditions in all directions, the fermions have the following mode expansions:

$$\text{Neveu-Schwarz:} \qquad \psi^i_\pm(\tau,\sigma) = \sqrt{\frac{\pi\alpha'}{\ell}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^i_r \, \mathrm{e}^{-i\pi r(\tau\pm\sigma)/\ell}, \tag{2.1.24a}$$

$$\text{Ramond:} \qquad \psi^i_\pm(\tau,\sigma) = \sqrt{\frac{\pi\alpha'}{\ell}} \sum_{n\in\mathbb{Z}} d^i_n \, \mathrm{e}^{-i\pi n(\tau\pm\sigma)/\ell}. \tag{2.1.24b}$$

Here $(b^i_r)^* = b^i_{-r}$ and $(d^i_n)^* = d^i_{-n}$. As in the previous chapter, we obtain the classical mass formulas:

$$\text{Neveu-Schwarz:} \qquad M^2 = \frac{1}{2\alpha'} \sum_{n\in\mathbb{Z}\setminus\{0\}} \alpha^i_{-n}\alpha^i_n + \frac{1}{2\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} r b^i_{-r} b^i_r, \tag{2.1.25a}$$

$$\text{Ramond:} \qquad M^2 = \frac{1}{2\alpha'} \sum_{n\in\mathbb{Z}\setminus\{0\}} \alpha^i_{-n}\alpha^i_n + \frac{1}{2\alpha'} \sum_{n\in\mathbb{Z}} n d^i_{-n} d^i_n. \tag{2.1.25b}$$

Note that in the Ramond case the $n = 0$ mode drops out.

We impose (2.1.23) and find the following anticommutation relations for the Neveu-Schwarz oscillators:

$$\{b^i_r, b^j_s\} = \delta^{ij}\delta_{r+s}, \tag{2.1.26}$$

where $(b^i_r)^\dagger = b^i_{-r}$, and for the Ramond oscillators:

$$\{d^i_m, d^j_n\} = \delta^{ij}\delta_{m+n}, \tag{2.1.27}$$

where now $(d^i_n)^\dagger = d^i_{-n}$. These relations of course need to be supplemented with the commutators (1.1.34).

The quantum mechanical mass formulas are obtained after normal ordering:

- Neveu-Schwarz:

$$M^2 = \frac{1}{\alpha'}(N_{\mathrm{NS}} - a_{\mathrm{NS}}), \quad \text{with} \quad N_{\mathrm{NS}} = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{r=1/2}^{\infty} r b^i_{-r} b^i_r, \tag{2.1.28a}$$

- Ramond:

$$M^2 = \frac{1}{\alpha'}(N_{\mathrm{R}} - a_{\mathrm{R}}), \quad \text{with} \quad N_{\mathrm{R}} = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{n=1}^{\infty} n d^i_{-n} b^i_n. \tag{2.1.28b}$$

We calculate the normal ordering constants as follows:

$$a_{\rm NS} = -\frac{1}{2}\sum_{n=1}^{\infty}[\alpha_n^i, \alpha_{-n}^i] + \frac{1}{2}\sum_{r=1/2}^{\infty} r\{b_r^i, b_{-r}^i\} = -\frac{d-2}{2}\left[\sum_{n=1}^{\infty} n - \sum_{r=1/2}^{\infty} r\right]. \quad (2.1.29)$$

We need a slight generalization of the ζ-function procedure to regularize the second sum. The *Hurwitz zeta function* $\zeta(s,c)$ has the following representation for $\operatorname{Re} s > 0$

$$\zeta(s,c) = \sum_{n=0}^{\infty} \frac{1}{(n+c)^s}. \quad (2.1.30)$$

It has a unique analytic continuation to negative values of s. In particular

$$\zeta(-1,c) = -\frac{1}{12}(6c^2 - 6c + 1). \quad (2.1.31)$$

We thus replace $1/2 + 3/2 + 5/2 + \ldots \to \zeta(-1, 1/2) = 1/24$ and obtain:

$$a_{\rm NS} = \frac{d-2}{16}, \quad \text{and similarly} \quad a_{\rm R} = 0. \quad (2.1.32)$$

Spectrum: Neveu-Schwarz sector

The ground state $|0,k\rangle_{\rm NS}$ has light-cone momentum $k = (k^+, k^i)$ and is annihilated by all the lowering operators, i.e. $\alpha_n^i|0,k\rangle_{\rm NS} = b_r^i|0,k\rangle_{\rm NS} = 0$ for all positive n and r. It has $N_{\rm NS} = 0$ and $M^2 = -a_{\rm NS}/\alpha'$ and is tachyonic if $d > 2$. The ground state is unique, since all raising and lowering operators change the value of M^2. It is thus a spacetime scalar and therefore a boson.

Acting on $|0,k\rangle_{\rm NS}$ with a raising operator α_{-n}^i raises $N_{\rm NS}$ by n, whereas acting with b_{-r}^i raises $N_{\rm NS}$ by r. We thus have states with $N_{\rm NS} = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \ldots$. Since the raising operators carry spacetime vector indices and the ground state is bosonic, all excited states in the NS sector are spacetime bosons.

The lowest excited states are thus at level $N_{\rm NS} = 1/2$. They are given by $b_{-1/2}^i|0,k\rangle_{\rm NS}$ and comprise a vector of $SO(d-2)$. As for the bosonic string, Lorentz invariance requires that these states are massless, hence $a_{\rm NS} = 1/2$. The open RNS superstring thus requires $d = 10$.

Spectrum: Ramond sector

The Ramond sector differs from the previous cases in that the zero mode operators d_0^i commute with the mass operator M^2. As a consequence, the Ramond ground state is already degenerate. To account for this degeneracy we introduce an extra label α and denote the ground states by $|\alpha; 0, k\rangle_{\rm R}$. These states are massless and

$\alpha' M^2$	NS states	# d.o.f.	$\Gamma_{\rm NS}$	R states	# d.o.f.	$\Gamma_{\rm R}$
$-1/2$	$\vert 0\rangle$	1	-1			
0	$b^i_{-1/2}\vert 0\rangle$	8	$+1$	$\vert +\rangle$	8	$+1$
				$\vert -\rangle$	8	-1
$+1/2$	$\alpha^i_{-1}\vert 0\rangle$	8	-1			
	$b^i_{-1/2}b^j_{-1/2}\vert 0\rangle$	28	-1			
$+1$	$b^i_{-3/2}\vert 0\rangle$	8	$+1$	$d^i_{-1}\vert +\rangle$	64	-1
	$\alpha^i_{-1}b^j_{-1/2}\vert 0\rangle$	64	$+1$	$\alpha^i_{-1}\vert +\rangle$	64	$+1$
	$b^i_{-1/2}b^j_{-1/2}b^k_{-1/2}\vert 0\rangle$	56	$+1$	$d^i_{-1}\vert -\rangle$	64	$+1$
				$\alpha^i_{-1}\vert -\rangle$	64	-1

Table 2.1.1. The lowest lying states of the oriented open superstring in the Neveu-Schwarz and Ramond sectors. We list the number of degrees of the various states and their Γ eigenvalues. States with $\Gamma = +1$ survive the GSO projection. We used a simplified notation in which $|0\rangle \equiv |0,k\rangle_{\rm NS}$ and $|\pm\rangle \equiv |\alpha,\pm;0,k\rangle_{\rm R}$.

satisfy $\alpha^i_n|\alpha;0,k\rangle_{\rm R} = d^i_n|\alpha;0,k\rangle_{\rm R} = 0$ for all positive n. In addition, they carry a representation of the zero mode algebra

$$\{d^i_0, d^j_0\} = \delta^{ij}. \tag{2.1.33}$$

Up to a factor 2, this is just the $SO(d-2)$ real Clifford algebra. Thus

$$d^i_0|\alpha;0,k\rangle_{\rm R} = \frac{1}{\sqrt{2}}(\Gamma^i)_\beta{}^\alpha\,|\beta;0,k\rangle_{\rm R}, \tag{2.1.34}$$

where the Γ^i are the Dirac matrices. The Ramond ground state is thus a spacetime spinor and therefore a fermion. Taking $d = 10$, we need to consider the irreps of the $SO(8)$ Clifford algebra, which are carried by Majorana spinors. The Ramond ground state thus has 16 on-shell degrees of freedom.

Acting on this state with α^i_{-n} and d^i_{-n} raises $N_{\rm R}$ by n. We thus have states with $N_{\rm R} = 0,1,2,3,4,\ldots$. Since the raising operators carry only vector indices, all the states in the Ramond sector are spacetime fermions.

The GSO projection

It turns out that an interacting theory of open superstrings is only consistent at the one-loop level if we include states from both the NS and the R sector. Moreover, we cannot include all the states in the spectrum, but have to restrict ourselves to those states that are invariant under the action of a certain multiplicatively conserved $\mathbb{Z}_2$ quantum number Γ. This projection of the spectrum to Γ-invariant states is known as the Gliozzi-Scherk-Olive (GSO) projection.

For the NS sector Γ is defined as:

$$\Gamma_{\mathrm{NS}} = -(-)^{F_{\mathrm{NS}}}, \quad \text{with} \quad F_{\mathrm{NS}} = \sum_{r=1/2}^{\infty} b^i_{-r} b^i_r. \tag{2.1.35}$$

The operator F_{NS} counts the number of fermionic generators that were used to construct the state on which the operator acts. States with an even number of b-excitations therefore have $\Gamma_{\mathrm{NS}} = -1$ and those with an odd number have $\Gamma_{\mathrm{NS}} = +1$.

In the Ramond sector we can define a conserved Γ in two ways, differing up to a sign:

$$\Gamma_{\mathrm{R}} = \pm\Gamma_*(-)^{F_{\mathrm{R}}}, \quad \text{with} \quad F_{\mathrm{R}} = \sum_{n=1}^{\infty} d^i_{-n} d^i_n. \tag{2.1.36}$$

Here $\Gamma_* = \Gamma_1\Gamma_2 \cdots \Gamma_8 = 2^4\, d^1_0 d^2_0 \cdots\ d^8_0$ is the $SO(8)$ chirality matrix. Since Γ_* commutes with all the d^i_n of nonzero n, its eigenvalue on any state is determined by the chirality of the Ramond ground state. Indeed, from appendix A we know that Majorana-Weyl spinors exist for the $SO(8)$ Clifford algebra. We thus decompose the Ramond ground state in a direct sum of its positive and negative chirality parts:

$$|\alpha; 0, k\rangle_{\mathrm{R}} = |\alpha, +; 0, k\rangle_{\mathrm{R}} \oplus |\alpha, -; 0, k\rangle_{\mathrm{R}}. \tag{2.1.37}$$

If we take the + sign in (2.1.36), states with an even number of d-excitations have $\Gamma_{\mathrm{R}} = \pm 1$ if $\Gamma_* = \pm 1$, whereas states with an odd number of d-excitations have $\Gamma_{\mathrm{R}} = \pm 1$ if $\Gamma_* = \mp 1$.

The GSO projection comes down to restricting the spectrum to states with $\Gamma = +1$. We promised in chapter 1 that the tachyon would be absent in the case of the superstring and now we see it is. Moreover, the spectrum is actually $\mathcal{N} = 1$ spacetime supersymmetric. We leave a proof of this statement to the references. At the massless level we have a vector A_μ and a chiral spinor χ. Together these fields make up the $d = 10$, $\mathcal{N} = 1$ vector multiplet.

One last remark: the spectrum after the GSO projection is physically the same regardless of the sign we take in (2.1.36) – it differs only in the overall chirality of the fermions. This will be different in the case of the closed superstrings, to which we turn now.

2.1.2 The closed superstring

The mode expansion for the right-moving fermions $\psi_-^i(\tau-\sigma)$ is:

$$\text{Neveu-Schwarz:} \qquad \psi_-^i(\tau,\sigma) = \sqrt{\frac{2\pi\alpha'}{\ell}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^i \, \mathrm{e}^{-2\pi i r(\tau-\sigma)/\ell}, \tag{2.1.38a}$$

$$\text{Ramond:} \qquad \psi_-^i(\tau,\sigma) = \sqrt{\frac{2\pi\alpha'}{\ell}} \sum_{n\in\mathbb{Z}} d_n^i \, \mathrm{e}^{-2\pi i n(\tau-\sigma)/\ell}. \tag{2.1.38b}$$

For the left-moving fermions $\psi_+^i(\tau+\sigma)$ there are similar relations, with $b_r^i \to \tilde{b}_r^i$ and $d_n^i \to \tilde{d}_n^i$.

The level matching condition is obtained in the same way as in chapter 1. The shifts $\sigma \to \sigma + b$ are generated by

$$S = -\int_0^\ell \mathrm{d}\sigma \left[\Pi^i \partial_\sigma X^i + \frac{i}{4\pi\alpha'} \big(\psi_-^i \partial_\sigma \psi_-^i + \psi_+^i \partial_\sigma \psi_+^i \big) \right]. \tag{2.1.39}$$

In checking this, Dirac brackets need to be used when appropriate. Quantum mechanically, this operator depends on the particular sector of the closed string under consideration, since the normal ordering constants differ.

The condition that $d = 10$ follows from an investigation of the massless modes of the NS-NS sector in the same way as the condition for the closed bosonic string. In the end we obtain the following mass formulas and level-matching conditions:

$$\mathrm{NS}\otimes\mathrm{NS}: \qquad M^2 = \frac{2}{\alpha'}\big(N_{\mathrm{NS}} + \tilde{N}_{\mathrm{NS}} - 1\big), \qquad N_{\mathrm{NS}} - \tilde{N}_{\mathrm{NS}} = 0, \tag{2.1.40a}$$

$$\mathrm{NS}\otimes\mathrm{R}: \qquad M^2 = \frac{2}{\alpha'}\left(N_{\mathrm{NS}} + \tilde{N}_{\mathrm{R}} - \frac{1}{2}\right), \qquad N_{\mathrm{NS}} - \tilde{N}_{\mathrm{R}} - \frac{1}{2} = 0, \tag{2.1.40b}$$

$$\mathrm{R}\otimes\mathrm{NS}: \qquad M^2 = \frac{2}{\alpha'}\left(N_{\mathrm{R}} + \tilde{N}_{\mathrm{NS}} - \frac{1}{2}\right), \qquad N_{\mathrm{R}} - \tilde{N}_{\mathrm{NS}} + \frac{1}{2} = 0, \tag{2.1.40c}$$

$$\mathrm{R}\otimes\mathrm{R}: \qquad M^2 = \frac{2}{\alpha'}\big(N_{\mathrm{R}} + \tilde{N}_{\mathrm{R}}\big), \qquad N_{\mathrm{R}} - \tilde{N}_{\mathrm{R}} = 0. \tag{2.1.40d}$$

The number operators N_{NS} and N_{R} are as in (2.1.28). We thus have $N_{\mathrm{NS}}, \tilde{N}_{\mathrm{NS}} = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ and $N_{\mathrm{R}}, \tilde{N}_{\mathrm{R}} = 0, 1, 2, 3, 4, \ldots$.

The GSO projection: type IIA and IIB

As for the open superstring, a theory of closed superstrings requires the presence of states from the spectra of all the four different sectors. And also in this case we need a GSO projection: we keep only states that have $(\Gamma, \tilde{\Gamma}) = (+1, +1)$.

$\alpha' M^2$	sector	states	# d.o.f.	$(\Gamma, \tilde{\Gamma})_{\text{IIA}}$	$(\Gamma, \tilde{\Gamma})_{\text{IIB}}$
-2	NS-NS	$\|0\rangle \otimes \|\widetilde{0}\rangle$	1	$(-,-)$	$(-,-)$
0	NS-NS	$b^i_{-1/2}\|0\rangle \otimes \tilde{b}^j_{-1/2}\|\widetilde{0}\rangle$	64	$(+,+)$	$(+,+)$
0	NS-R	$b^i_{-1/2}\|0\rangle \otimes \|\widetilde{+}\rangle$	64	$(+,+)$	$(+,+)$
		$b^i_{-1/2}\|0\rangle \otimes \|\widetilde{-}\rangle$	64	$(+,-)$	$(+,-)$
0	R-NS	$\|+\rangle \otimes \tilde{b}^i_{-1/2}\|\widetilde{0}\rangle$	64	$(-,+)$	$(+,+)$
		$\|-\rangle \otimes \tilde{b}^i_{-1/2}\|\widetilde{0}\rangle$	64	$(+,+)$	$(-,+)$
0	R-R	$\|+\rangle \otimes \|\widetilde{+}\rangle$	64	$(-,+)$	$(+,+)$
		$\|+\rangle \otimes \|\widetilde{-}\rangle$	64	$(-,-)$	$(+,-)$
		$\|-\rangle \otimes \|\widetilde{+}\rangle$	64	$(+,+)$	$(-,+)$
		$\|-\rangle \otimes \|\widetilde{-}\rangle$	64	$(+,-)$	$(-,-)$

Table 2.1.2. The lowest lying states in the spectrum of the oriented closed superstring. We list the number of degrees of freedom of the various states and their Γ and $\tilde{\Gamma}$ eigenvalues, for the type IIA and IIB projectors. States with $(\Gamma, \tilde{\Gamma}) = (+,+)$ survive the GSO projection.

The main difference with the open string lies in the fact that there are now two physically inequivalent choices of the signs of Γ_{R} and $\tilde{\Gamma}_{\text{R}}$ in (2.1.36)[6] – two, because only the relative sign matters. We fix $\tilde{\Gamma}_{\text{R}} \equiv \tilde{\Gamma}_*(-)^{\tilde{F}_{\text{R}}}$ by convention and thus still have to choose the sign in

$$\Gamma_{\text{R}} = \pm\Gamma_*(-)^{F_{\text{R}}}, \tag{2.1.41}$$

The two choices give rise to two different consistent closed superstring theories with $\mathcal{N} = 2$ spacetime supersymmetry. If we take a $-$ sign in (2.1.41) we obtain the *type IIA* superstring, whereas for the + sign we have the *type IIB* superstring. Clearly, the fermionic spectrum of the type IIA theory is non-chiral, whereas that of the type IIB string is chiral.

Massless spectrum

The states in the NS-NS and R-R sectors are bosons, whereas those in the NS-R and R-NS sector are fermions.

The *NS-NS sector* is the same in the type IIA and type IIB theories. We are particularly interested in the massless fields: the metric $g_{\mu\nu}$, a 2-form $B_{\mu\nu}$ and the dilaton Φ, with 35, 28 and 1 degrees of freedom, respectively. These are the same

[6] $\tilde{\Gamma}_{\text{R}}$ is of course obtained by simply replacing d^i_n with $\tilde{d}^i_n$ everywhere in (2.1.36).

massless fields as for the closed bosonic string. It can be shown that these fields also behave in the same way, either by a calculation of their scattering amplitudes or by the background field methods of section 1.2.4. In particular, the expectation value of the dilaton again determines the string coupling constant and the strings are charged w.r.t. the 2-form $B_{\mu\nu}$.

The *NS-R sector* is also the same in both type II theories. These states form a reducible vector-spinor representation φ^i_α of $SO(8)$, where $\Gamma_*\varphi^i = +\varphi^i$. Indeed, the states $\Gamma^i\phi^i$ transform irreducibly under $SO(8)$. These 8 degrees of freedom can be represented by a right-handed (negative chirality) Majorana spinor field λ^- which is often referred to as a dilatino. The remaining 56 degrees of freedom $\bar{\varphi}^i_\alpha$ with $\Gamma^i\bar{\varphi}^i = 0$ can be represented by a left-handed (positive chirality) Majorana gravitino field[7] ψ^+_μ.

The *R-NS sector* for the IIB string is a copy of its NS-R sector: we have a dilatino λ^- and a gravitino ψ^+_μ. For the IIA string we have the same fields, but with a different chirality: a spinor λ^+ and a gravitino ψ^-_μ.

The massless modes in the *R-R sector* are obtained from the tensor product $\psi_\alpha\chi_\beta$ of two $SO(8)$ Majorana spinors, where ψ and χ correspond to the right- and left-moving Ramond ground state, respectively. We need to decompose this tensor product into irreps of $SO(8)$. To achieve this we first raise the spinor index[8] on χ and then use the Fierz decomposition of (A.1.45):

$$\psi_\alpha\chi^\beta = -\frac{1}{2^4}\sum_{k=0}^{8}\frac{1}{k!}\,\bar{\chi}\Gamma^{i_1\cdots i_k}\psi\,(\Gamma_{i_1\cdots i_k})_\alpha{}^\beta. \qquad (2.1.42)$$

Now we apply the GSO projection, after which χ becomes a Weyl spinor of positive chirality, and thus

$$\bar{\chi}\Gamma^{i_1\cdots i_k}\psi = \bar{\chi}\Gamma_*\Gamma^{i_1\cdots i_k}\psi = (-)^k\,\bar{\chi}\Gamma^{i_1\cdots i_k}\Gamma_*\psi = \pm(-)^k\,\bar{\chi}\Gamma^{i_1\cdots i_k}\psi,$$

where we have a $-$ sign for type IIA and a $+$ for IIB. So for IIA only the terms with k even are nonvanishing, whereas for IIB we have k odd. Moreover, from (A.1.19) we have

$$\Gamma^{i_1\cdots i_k} = \frac{1}{(8-k)!}\varepsilon^{i_1\cdots i_8}\,\Gamma_*\Gamma^{i_8\cdots i_{k+1}}. \qquad (2.1.43)$$

We can thus relate the terms in (2.1.42) with a high value of k to those with a low value $8-k$.

The massless degrees of freedom in the R-R sector are thus antisymmetric tensors of $SO(8)$. We write $\bar{\chi}\Gamma_{i_1\cdots i_k}\psi = c_{i_1\cdots i_k}$. The type IIA theory has c_i and c_{ijk}, with 8 and 56 degrees of freedom, respectively. The type IIB theory has c, c_{ij} and c^+_{ijkl},

[7] Remember that the graviton contains $d(d-3)/2$ degrees of freedom, a massless n-form $\binom{d-2}{n}$ and a gravitino $(d-3)\delta$, where δ is the dimension of a minimal spinor in d dimensions.

[8] As explained in appendix A.1 it makes no physical difference which C matrix we use for this in the case of $SO(8)$. For definiteness we will use C_+. We have $C_+\Gamma_* = \Gamma_*C_+$.

with $c_{ijkl} = \varepsilon_{ijklmnrs}c^{mnrs}/4!$. These contain respectively 1, 28 and 35 degrees of freedom. The $c^{(k)}$ can be represented by antisymmetric tensor fields $C^{(k)}_{\mu_1\cdots\mu_k}$ of the same rank. It turns out that the self-duality condition on $C^{+}_{\mu\nu\rho\sigma}$ has to be imposed on its field-strength.

Ramond-Ramond charges

We have seen in the previous chapter that the bosonic string is electrically charged w.r.t. the 2-form Kalb-Ramond field $B_{\mu\nu}$. Since the massless NS-NS fields behave in the same way as the massless fields of the closed bosonic string, we conclude that the IIA and IIB string are electrically charged w.r.t. the NS-NS 2-form $B^{(2)}$.

In contrast, in turns out that there are no states in the perturbative string spectrum that carry a charge w.r.t. the R-R forms $C^{(p)}$. This is not so easy to show in our light-cone gauge quantization, but if we had employed a covariant quantization procedure we would have seen that the states we constructed above actually describe the field strengths $G^{(p+1)}$ instead of the R-R fields $C^{(p)}$ themselves. The same holds for the vertex operators, hence string amplitudes involving an R-R field always vanish at zero momentum[9], indicating that the strings are not charged.

The absence of R-R charges in the perturbative string spectrum is somewhat of a riddle, since their presence is required by the S-duality of the type IIB string theory as we will see in section 2.4. This riddle was solved by Polchinski in 1995 [67]; he showed that D-branes are in fact the carriers of R-R charges.

IIA and IIB are T-dual

Consider for instance a compactified 9-direction. As for the bosonic degrees of freedom, T-duality acts as a parity operation on the right-moving fermions,

$$T\psi^9_- T^{-1} = -\psi^9_-, \qquad T\psi^9_+ T^{-1} = \psi^9_+. \tag{2.1.44}$$

We have in particular $Td^9_0 T^{-1} = -d^9_0$ and $T\tilde{d}^9 T^{-1} = \tilde{d}^9_0$. Hence T-duality interchanges the chirality Γ_* of the right-moving Ramond ground state, but leaves the chirality $\tilde{\Gamma}_*$ of the left-movers invariant. It follows that a T-duality transformation brings us from type IIA to IIB theory and vice versa.

2.1.3 Type I theory

As we know, a theory with interacting open strings necessarily includes closed strings as well. This coupling is not as straightforward in the case of the superstring as it

[9] Amplitudes involving a field strength G always have a factor of momentum – simply look at the Fourier transform $\tilde{G}_{\mu_1\ldots\mu_{p+1}} = (p+1)k_{[\mu_1}\tilde{C}_{\mu_2\ldots\mu_{p+1}]}$. The construction of R-R vertex operators is quite nontrivial. We refer to [15] for details.

was for the bosonic string, since one can not couple an $\mathcal{N}=1$ supersymmetric theory to one with $\mathcal{N}=2$ supersymmetry. We therefore need to bring down the number of supersymmetries in the closed string sector without ruining the consistency of the theory.

There is only one way in which this can be achieved and that is by modding out the spectrum with the world-sheet parity operator Ω. Ω acts on the closed string oscillators as

$$\Omega b^i_{-r}\Omega^{-1} = \tilde{b}^i_{-r}, \qquad \Omega\tilde{b}^i_{-r}\Omega^{-1} = -b^i_{-r}, \tag{2.1.45a}$$

$$\Omega d^i_{-n}\Omega^{-1} = \tilde{d}^i_{-n}, \qquad \Omega\tilde{d}^i_{-n}\Omega^{-1} = -d^i_{-n}, \tag{2.1.45b}$$

where the $-$ signs where introduced so that the product of a left- and right-moving fermionic oscillator is Ω-invariant. We see that Ω is not a symmetry of the IIA string, since it acts on the GSO projectors as $\Omega\Gamma_{\mathrm{R}}\Omega^{-1} = -\tilde{\Gamma}_{\mathrm{R}}$ and $\Omega\tilde{\Gamma}_{\mathrm{R}}\Omega^{-1} = -\Gamma_{\mathrm{R}}$.

The type IIB string on the other hand *is* left-right invariant. Keeping only Ω-invariant states, we have the following remaining massless fields. In the NS-NS sector there are the graviton $g_{\mu\nu}$ and the dilaton Φ whereas in the R-R sector the 2-form $C_{\mu\nu}$ survives[10]. Since Ω relates the NS-R to the R-NS sector, only a single gravitino ψ^+_μ and dilatino λ^- remain. So we obtain an unoriented closed string theory that indeed has only $\mathcal{N}=1$ supersymmetry.

The type I string theory is obtained by coupling the Ω-truncated type IIB string to unoriented open strings with $SO(32)$ Chan-Paton factors, since it turns out that $SO(32)$ is the only gauge group for which the theory is free of anomalies. We will briefly come back to this point below.

2.1.4 Superstring effective actions

We now present the low-energy effective actions of the type II and type I superstring theories that we discussed above. In addition, we give the effective actions for the heterotic strings and the 11-dimensional supergravity theory. The connection of these theories to the type I and II strings will be briefly pointed out later in this chapter.

[10]The Ramond ground state is fermionic, so we get an additional minus sign when interchanging the left- and right-movers: $\Omega(|+\rangle\otimes|\tilde{+}\rangle) = -|\tilde{+}\rangle\otimes|+\rangle$. Of the matrices $\Gamma^{(k)}C^{-1}$ with $k=0,2,4$ only the one with $k=2$ is antisymmetric. Thus only $C_{\mu\nu}$ survives the Ω-projection.

Type IIA theory

The leading order contribution to the effective action is determined by the $(1,1)$ supersymmetry[11]. The bosonic contributions are in string frame [71, 72]:

$$\begin{aligned} 2\kappa_0^2\,\mathcal{L}_{\rm IIA} &= \mathrm{e}^{-2\Phi}\left(R*1 + 4*\mathrm{d}\Phi\wedge\mathrm{d}\Phi - \frac{1}{2}*H^{(3)}\wedge H^{(3)}\right) \\ &\quad - \frac{1}{2}*G^{(2)}\wedge G^{(2)} - \frac{1}{2}*G^{(4)}\wedge G^{(4)} - \frac{1}{2}B^{(2)}\wedge\mathrm{d}C^{(3)}\wedge\mathrm{d}C^{(3)}, \end{aligned} \tag{2.1.46}$$

where used the following field strengths:

$$H^{(3)} = \mathrm{d}B^{(2)}, \qquad G^{(2)} = \mathrm{d}C^{(1)}, \qquad G^{(4)} = \mathrm{d}C^{(3)} - H^{(3)}\wedge C^{(1)}. \tag{2.1.47}$$

The type IIA theory is parity symmetric and hence anomaly-free.

Type IIB theory

The effective action is uniquely determined by $(2,0)$ supersymmetry. The bosonic contributions are in string frame [73, 74]:

$$\begin{aligned} 2\kappa_0^2\,\mathcal{L}_{\rm IIB} &= \mathrm{e}^{-2\Phi}\left(R*1 + 4*\mathrm{d}\Phi\wedge\mathrm{d}\Phi - \frac{1}{2}*H^{(3)}\wedge H^{(3)}\right) - \frac{1}{2}*G^{(1)}\wedge G^{(1)} \\ &\quad - \frac{1}{2}*G^{(3)}\wedge G^{(3)} - \frac{1}{4}*G^{(5)}\wedge G^{(5)} + \frac{1}{2}C^{(4)}\wedge\mathrm{d}C^{(2)}\wedge H^{(3)}. \end{aligned} \tag{2.1.48}$$

We used the following field strengths:

$$H^{(3)} = \mathrm{d}B^{(2)}, \qquad G^{(1)} = \mathrm{d}C^{(0)}, \qquad G^{(3)} = \mathrm{d}C^{(2)} - H^{(3)}\wedge C^{(0)}. \tag{2.1.49}$$

The field strength of the 4-form

$$G^{(5)} = \mathrm{d}C^{(4)} - \frac{1}{2}C^{(2)}\wedge H^{(3)} + \frac{1}{2}B^{(2)}\wedge\mathrm{d}C^{(2)}, \tag{2.1.50}$$

satisfies the following self-duality condition:

$$G^{(5)} \equiv *G^{(5)}, \tag{2.1.51}$$

which has to be imposed as a constraint[12] at the level of the equations of motion [76].

[11] $(1,1)$ supersymmetry allows for a more general *massive* IIA supergravity [68], which plays an important role in the theory of D8-branes [67, 69, 70]. This theory does not possess a Minkowski vacuum however.

[12] A Lorentz covariant action for self-dual p-forms which does not require the additional constraint (2.1.51) can only be constructed at the expense of introducing auxiliary fields. See e.g. [75].

The terms involving the R-R field strengths $G^{(n)}$ in (2.1.46) and (2.1.48) do not have a factor of $\mathrm{e}^{-2\Phi}$, even though they follow from closed string tree-level amplitudes. This convention was chosen in order to have the Bianchi identities for the R-R field strengths as simple as possible.

The chiral fields of the type IIB theory – the ψ_μ^+'s, λ^-'s and $C^+_{\mu\nu\rho\sigma}$ – give rise to gravitational, chiral and mixed anomalies. Remarkably, these anomalies exactly cancel against each other.

Type I theory

Since the Ω-projection of the type IIB string removes C, $C^+_{\mu\nu\rho\sigma}$ and one pair of the (ψ_μ^+, λ^-), the careful balance between the anomalies is upset and the consistency of the theory ruined. It turns out that in order to regain a consistent theory we *have* to add unoriented open strings with $SO(32)$ Chan-Paton factors to the theory.

At the massless level, the addition of open strings comes down to coupling the sugra to a $d = 10$, $\mathcal{N} = 1$ non-abelian vector multiplet (A_μ, χ^+) [77,78]. The fermion χ^+ is Weyl and gives rise to an *additional* chiral anomaly.

At this stage we seem to be in a lot of trouble, but it turns out that all the anomalies cancel exactly for the gauge groups $SO(32)$ and $E_8 \times E_8$. Of these, only $SO(32)$ can be obtained by means of Chan-Paton factors.

The low-energy effective Lagrangian of the type I string theory is:

$$2\kappa_0^2\,\mathcal{L}_{\mathrm{I}} \;=\; \mathrm{e}^{-2\Phi}\Big(R\,{*1} + 4\,{*\mathrm{d}\Phi}\wedge \mathrm{d}\Phi\Big) - \frac{1}{2}{*G^{(3)}}\wedge G^{(3)} + \frac{\kappa_0^2}{g_0^2}\,\mathrm{e}^{-\Phi}\,\mathrm{Tr}\,{*F}\wedge F. \qquad (2.1.52)$$

Supersymmetry demands that the field strength of the two-form be modified as follows[13]:

$$G^{(3)} = \mathrm{d}C^{(2)} - \frac{\kappa_0^2}{g_0^2}\,\omega^{(3)}, \qquad (2.1.53)$$

where $\omega^{(3)}$ is the Chern-Simons 3-form

$$\omega^{(3)} = \mathrm{Tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right). \qquad (2.1.54)$$

The heterotic theories

We mentioned already that the heterotic theories contain only closed strings. It turns out that the modification to the right-moving sector that we alluded to above gives rise to non-abelian gauge fields. In this case, both $SO(32)$ and $E_8\times E_8$ can be realized.

[13] We omit the gravitational Chern-Simons term, which is of higher order in α'.

The massless fields are similar to those of the type I theory, but the couplings are different. The action reads:

$$2\kappa_0^2\,\mathcal{L}_{\rm het} \;=\; {\rm e}^{-2\Phi}\left(R\,{*1} + 4\,{*{\rm d}\Phi}\wedge {\rm d}\Phi - \frac{1}{2}{*H^{(3)}}\wedge H^{(3)} + \frac{\kappa_0^2}{g_0^2}\,{\rm Tr}\,{*F}\wedge F\right). \quad (2.1.55)$$

Note the overall power of the dilaton: this action is a closed string tree-level result.

A few remarks about our conventions in equations (2.1.52) and (2.1.55) are in order. The Yang-Mills 1-forms $A = A_\mu {\rm d}x^\mu$ are matrix-valued fields $A_\mu = A_\mu^\alpha T_\alpha$, where the antihermitian matrices T_α represent the Lie algebra $\mathfrak{so}(32)$ or $\mathfrak{e}_8 \oplus \mathfrak{e}_8$. They are normalized by ${\rm Tr}\,T_\alpha T_\beta = -\delta_{\alpha\beta}$. The Yang-Mills field strength is defined by $F = {\rm d}A + A\wedge A$.

On dimensional grounds $\kappa_0^2/g_0^2 \sim \alpha'$. We will not need the precise value of this ratio. It depends on whether we are considering the type I, heterotic $SO(32)$ or $E_8\times E_8$ string and on the conventions – the normalization of the A_μ vertex operator and the representation T_α. See [15] for details.

Eleven-dimensional supergravity

The maximum number of dimensions in which one can construct a supergravity theory[14] turns out to be $d = 11$. This theory is unique. The fields are the metric $g_{\mu\nu}$, a 3-form $C_{\mu\nu\rho}$ and a Majorana gravitino ψ_μ. The bosonic part of the Lagrangian is [80]:

$$2\kappa_{11}^2\mathcal{L} = R\,{*1} - \frac{1}{2}{*H^{(4)}}\wedge H^{(4)} - \frac{1}{6}C^{(3)}\wedge H^{(4)}\wedge H^{(4)}, \quad (2.1.56)$$

where $H^{(4)} = {\rm d}C^{(3)}$.

2.2 Extended objects in supergravity

We will now consider some aspects of a specific class of supersymmetric solitons in supergravity theories: the p-branes. They are higher-dimensional versions of the well-known 4-dimensional Reissner-Nordström black holes. The p-branes have several properties in common: they carry charges w.r.t. a p-form gauge field and they preserve a certain fraction of the supersymmetries of the underlying supergravity theory. We will first review the RN black holes in some detail, since they already show most of these properties.

[14] This is for Minkowski signature spacetimes. If we allow for more than one timelike direction, the maximum number of dimensions turns out to be 12. See e.g. [79].

2.2.1 Reissner-Nordström black holes and BPS states

The RN black holes are electrically charged solutions of the Einstein-Maxwell theory:

$$2\kappa^2\,\mathcal{L} = R*1 - \frac{1}{2}*F\wedge F. \tag{2.2.1}$$

The solution reads in spherical coordinates (t, r, θ, ϕ):

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{2.2.2a}$$

$$A = -\frac{2Q}{r}\,\mathrm{d}t, \qquad \text{i.e.} \qquad F = -\frac{2Q}{r^2}\,\mathrm{d}t\wedge\mathrm{d}r. \tag{2.2.2b}$$

Here $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ is the metric on the round 2-sphere. The parameters Q and M are related to the electric charge q and the mass m of the black hole, respectively. Indeed, the charge q is proportional to the integrated flux of the electric field at spatial infinity:

$$q = \frac{1}{2\kappa^2}\int_{S^2} *F = \frac{4\pi Q}{\kappa^2}. \tag{2.2.3}$$

The mass m of the black hole can be read off from the behavior of the metric as $r\to\infty$:

$$g_{tt} \approx -1 - 2\Phi(r) = -1 + \frac{2\kappa^2}{(d-2)\,\mathrm{vol}\,S^{d-2}}\frac{m}{r^{d-3}} \quad\Rightarrow\quad m = \frac{8\pi M}{\kappa^2}. \tag{2.2.4}$$

Consider two such black holes. They attract each other due to the gravitational interaction and repel or attract each other due to the electric force. The force is given by[15]:

$$\vec{F} = (Q_1Q_2 - M_1M_2)\,\frac{1}{r^2}\,\hat{r}. \tag{2.2.5}$$

If both black holes satisfy $M = \pm|Q|$, the net force is zero and we have a stable configuration.

[15] To obtain this result, consider the equation of motion for a point particle of mass m_1 and charge q_1 in the background field (2.2.2) of the second black hole:

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = -\frac{q_1}{m_1}F^\mu{}_\nu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau},$$

and take the weak-field and low-velocity approximation. One obtains:

$$M_1\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = (Q_1Q_2 - M_1M_2)\,\frac{x^i}{r^3}.$$

The case $M = \pm|Q|$ is known as an *extremal* RN black hole and is interesting for a number of reasons. First of all, if $M > |Q|$ the solution (2.2.2) has two horizons at

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{2.2.6}$$

If $M < |Q|$ there are no horizons. This occurrence of a *naked singularity* is believed to be unphysical[16]. We thus have the bound $M \geq |Q|$, which is saturated in the extremal case. Furthermore, when $M = |Q|$ the two horizons coincide. Shifting the radial coordinate as $r \to r - M$ we rewrite (2.2.2) in *isotropic coordinates*:

$$\begin{aligned} \mathrm{d}s^2 &= -\left(1 + \frac{M}{r}\right)^{-2} \mathrm{d}t^2 + \left(1 + \frac{M}{r}\right)^2 (\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2) \\ &= -H^{-2}(r)\,\mathrm{d}t^2 + H^2(r)\,\mathrm{d}x^a \mathrm{d}x^a, \\ F &= 2\,\mathrm{d}t \wedge \mathrm{d}H^{-1}, \qquad H(r) = 1 + \frac{M}{r}. \end{aligned} \tag{2.2.7}$$

$H(r)$ is a harmonic function in the three-dimensional space transverse to the black hole. In this form the $\mathbb{R} \times SO(3)$ isometry of the solution is manifest. Note that the isotropic coordinates cover only the region outside the horizon[17].

Instead of a black hole with an electric charge Q_e, we could also have considered a black hole with a magnetic charge Q_m, or with both. The solution (2.2.2) still holds, but with $Q = (Q_e^2 + Q_m^2)^{1/2}$ and (2.2.2b) replaced by

$$F = -\frac{2Q_e}{r^2}\,\mathrm{d}t \wedge \mathrm{d}r + 2Q_m \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{2.2.8}$$

The extremal case is still given by $M = |Q|$. We remark that the charges are quantized according to the usual Dirac or Wu-Yang argument, i.e. $Q_e Q_m \sim n$, n integer.

The extremal case becomes even more interesting when one realizes that the Einstein-Maxwell Lagrangian (2.2.1) is in fact the bosonic part of the pure $\mathcal{N} = 2$, $d = 4$ supergravity theory [81]. In addition to the vielbein e_μ^a and the gauge field A_μ – which in this context is often called the graviphoton – the $\mathcal{N} = 2$ supergravity contains two Majorana gravitinos ψ_μ^i, $i = 1, 2$.

The supersymmetry transformations of the gravitinos read up to terms cubic in the fermions:

$$\delta_{\mathrm{Q}}(\epsilon)\, P_\pm \psi_\mu^i = D_\mu(\omega) P_\pm \epsilon^i - \frac{1}{4} F_{ab}^- \,\Gamma^{ab} \Gamma_\mu \varepsilon^{ij} P_\mp \epsilon^j, \tag{2.2.9}$$

[16] This is called the *cosmic censorship* conjecture.

[17] The *near-horizon limit* $r \to 0$ is interesting. We get

$$\mathrm{d}s^2 = -\frac{r^2}{M^2}\mathrm{d}t^2 + \frac{M^2}{r^2}\mathrm{d}r^2 + M^2 \mathrm{d}\Omega^2,$$

which is the product space $AdS_2 \times S^2$.

where $\varepsilon^{12} = +1$ and $F^{\pm} = \frac{1}{2}(F \pm i{*}F)$. The RN black hole is a purely bosonic solution of the $\mathcal{N} = 2$ supergravity field equations. For generic values of M and Q it preserves no supersymmetries. However, in the *extremal* case 1/2 of the supersymmetries *are* preserved [82]. To prove this, we need to show that the supersymmetry variations $\delta_{\mathrm{Q}}(\epsilon)$ of the fields are zero for suitably chosen ϵ. This is guaranteed for e_μ^a and A_μ, since their variations always involve at least one power of ψ_μ. We therefore only need to check that $\delta_{\mathrm{Q}}(\epsilon)\,\psi_\mu^I = 0$. The solutions of this equation are called *Killing spinors.* For the extremal RN black hole the Killing spinors turn out to be $\epsilon^i = H^{-1/2}\epsilon_0^i$, where ϵ_0^i are constant spinors that satisfy

$$P_{\pm}\epsilon_0^i = -\varepsilon^{ij}\Gamma_{\underline{0}}\,P_{\mp}\epsilon_0^j. \tag{2.2.10}$$

This condition determines exactly one half of the components of the ϵ_0^i in terms of the other half, demonstrating that the extremal RN solution indeed breaks 1/2 of the supersymmetries.

Configurations that preserve some fraction of the supersymmetry are called *BPS configurations.* There is another way of looking at these configurations that is very useful. In the previous chapter we mentioned the semiclassical approximation to a quantum theory in which one expands the fields around a classical solution of the equations of motion and obtains a field theory for the fluctuations. It turns out that the symmetries of the classical solution around which we expand become global symmetries of this resulting field theory. At large distances from the horizon ($r \to \infty$) the solution (2.2.2) reduces to Minkowski spacetime, which is actually a maximally supersymmetric solution of the $\mathcal{N} = 2$ supergravity theory. Physics very far from the black hole should thus be described by a field theory with global $\mathcal{N} = 2$ Poincaré supersymmetry. The RN black hole has an alternative interpretation as a state in the spectrum of this theory.

The interesting part of the $\mathcal{N} = 2$ superalgebra[18] reads

$$\{Q_\alpha^i, Q_\beta^j\} = 2\delta^{ij}(\Gamma^a C^{-1})_{\alpha\beta}P_a + 2\varepsilon^{ij}(C^{-1})_{\alpha\beta}\mathcal{Z}_e + 2i\varepsilon^{ij}(\Gamma_* C^{-1})_{\alpha\beta}\mathcal{Z}_m. \tag{2.2.11}$$

We have included on the right-hand side the central charges $\mathcal{Z}_e$ and $\mathcal{Z}_m$. They commute with all the other generators in the $\mathcal{N} = 2$ Poincaré superalgebra. It can be shown that these charges are just the electric and magnetic charges Q_e and Q_m that are carried by massive point particles.

We are interested in massive particle representations of this algebra. In the rest frame $P_a = (iM, 0, \ldots, 0)$. For the charges we have $\mathcal{Z}_e = iQ_e$, $\mathcal{Z}_m = iQ_m$. The tools of appendix A allow us to rewrite (2.2.11) as

$$\{Q, Q^\dagger\} = 2\begin{pmatrix} M & -Q_e\Gamma^0 - iQ_m\Gamma_*\Gamma^0 \\ Q_e\Gamma^0 + iQ_m\Gamma_*\Gamma^0 & M \end{pmatrix} \tag{2.2.12}$$

[18] Our conventions are such that the bosonic generators are antihermitian, i.e. $P_a^\dagger = -P_a$ and $\mathcal{Z}^\dagger = -\mathcal{Z}$.

The matrix on the RHS has eigenvalues $\lambda = 2(M \pm Q) \geq 0$. They are nonnegative since the operator on the LHS is positive semidefinite. This leads to the *Bogomolnyi-Prasad-Sommerfield (BPS) bound*:

$$M \geq |Q|, \qquad \text{with} \qquad Z = \sqrt{Q_e^2 + Q_m^2}. \tag{2.2.13}$$

Naked singularities are thus excluded by $\mathcal{N} = 2$ supersymmetry. When the bound is saturated, i.e. if $M = |Q|$, exactly half of the eigenvalues λ are zero (which ones depends on the sign of the charge). This means that this state is annihilated by 1/2 of the supersymmetry generators, in accordance with our earlier discussion on the Killing spinors of the extremal RN solution.

An important property of the BPS configurations is that they are stable: the charge Q is conserved and there are no states of lower mass with this charge. In addition, the BPS states fall into smaller multiplets than the states for which $M > |Q|$, because 1/2 of the supersymmetry generators are represented trivially. Since the number of states in a quantum theory can not change abruptly as we vary the parameters of the theory, the BPS relation $M = |Q|$ is not renormalized by quantum corrections [83]. It is hard to overemphasize the importance of this result. As mentioned in chapter 1, it allows one to extrapolate results from perturbation theory to the strong coupling regime of a theory and to test S-duality conjectures.

2.2.2 Supergravity p-branes

Let us reformulate the results of the previous section in the language of branes. We obtained a 0-brane solution of the field equations of $\mathcal{N} = 2$, $d = 4$ supergravity that was electrically or magnetically charged w.r.t. the 1-form A. The presence of these charges was crucial in obtaining a *supersymmetric* solution. Note that a magnetic charge corresponds to an electric charge w.r.t. to the dual $\tilde{A}$.

We can generalize this to arbitrary dimension d and rank q of the gauge field as follows. Consider the following Lagrangian for gravity coupled to a scalar ϕ and a q-form $C^{(q)}$:

$$2\kappa^2 \mathcal{L} = R *1 - \frac{1}{2} *\mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2}\, \mathrm{e}^{a\phi} *G^{(q+1)} \wedge G^{(q+1)}, \tag{2.2.14}$$

where a is a real constant and $G^{(q+1)} = \mathrm{d}C^{(q)}$. This action can be obtained as a consistent truncation from the full supergravity Lagrangians of the previous section for suitable choices of d, a and q after going to the Einstein frame using (1.2.32) and a suitable rescaling of ϕ – see e.g. [84,85] for details. With a consistent truncation we mean that any solution of the truncated theory is also a solution of the full theory.

We expect p-brane solutions that carry an electric charge Q_e for $p = q - 1$ and a magnetic charge Q_m for $p = \tilde{q} - 1$:

$$Q_e \sim \int_{S^{\tilde{q}+1}} \mathrm{e}^{a\phi} *G^{(q+1)}, \qquad Q_m \sim \int_{S^{q+1}} G^{(q+1)}, \tag{2.2.15}$$

where we defined $\tilde{q} = d - q - 2$ and the integrals are taken over spheres surrounding the branes in the directions transverse to their world-volumes. The charges are conserved due to the field equation and the Bianchi identity of $C^{(q)}$, respectively, and they are quantized, $Q_e Q_m \sim n$.

We now give the p-brane solutions in isotropic coordinates. We denote the coordinates of the world-volume by x^i with $i = 0, \dots, p$ and those of the space transverse to the brane by x^a with $a = p+1, \dots, d-1$. The metric is given by

$$\mathrm{d}s_{\mathrm{E}}^2 = H^{-4\tilde{q}/(d-2)\Delta}\, \eta_{ij} \mathrm{d}x^i \mathrm{d}x^j + H^{-4q/(d-2)\Delta}\, \delta_{ab} \mathrm{d}x^a \mathrm{d}x^b. \tag{2.2.16}$$

In this form the $ISO(1,p) \times SO(d-p-1)$ isometry is manifest. We defined the parameter Δ by

$$\Delta = a^2 + \frac{2q\tilde{q}}{d-2}. \tag{2.2.17}$$

For the electric solution we have in addition

$$\mathrm{e}^{\phi} = H^{2a/\Delta}, \qquad G_e^{(q+1)} = \frac{2}{\sqrt{\Delta}}\, \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^p \wedge \mathrm{d}H^{-1}, \tag{2.2.18a}$$

whereas for the magnetic solution

$$\mathrm{e}^{\phi} = H^{-2a/\Delta}, \qquad G_m^{(q+1)} = \frac{2}{\sqrt{\Delta}} * \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^p \wedge \mathrm{d}H. \tag{2.2.18b}$$

The function $H = H(r)$, with $r^2 = x^a x^a$, is harmonic on the transverse space, i.e. $\nabla_a \partial^a H = 0$. For $p < d-3$ we have

$$H = c + \frac{Q}{r^{d-p-3}}, \tag{2.2.19}$$

where if $a \neq 0$ the constant c is related to the asymptotic value of ϕ *via* $\mathrm{e}^{\phi_\infty} = c^{\pm 2a/\Delta}$. $(d-3)$-branes have a logarithmic harmonic and $(d-2)$-branes a linear harmonic.

In maximally supersymmetric theories it can be shown that solutions with $\Delta = 4/n$ preserve $32/2^n$ supersymmetries. All 1/2 BPS solutions in type IIA and IIB thus have $\Delta = 4$[19].

The charges again manifest themselves in the supersymmetry algebra[20], which

[19]Other values of Δ often give configurations that can be thought of as intersecting branes. The harmonic function (2.2.19) then needs to be modified.

[20]There are two ways of deriving this result. The first [86] makes use of the world-volume theory of the branes. In this approach the central terms in the algebra are a consequence of the fact that a Green-Schwarz type Lagrangian for branes is not supersymmetric but varies into a total derivative. This leads to additional terms in the definition of the supercharges that in turn give rise to the central charges. See [87] for a condensed review. In the second approach [88] one constructs the supercharges directly from the supergravity Lagrangian. One then calculates the commutator: the result involves the bosonic fields. Inserting the (asymptotic) solution finally yields the central charges.

reads schematically:

$$\{Q_\alpha, Q_\beta\} = 2(\Gamma^\mu C^{-1})_{\alpha\beta} P_\mu + 2\sum_p \frac{1}{p!}(\Gamma^{\mu_1\cdots\mu_p} C^{-1})_{\alpha\beta} \mathcal{Z}_{\mu_1\cdots\mu_p}. \tag{2.2.20}$$

Which p's are included in the sum depends on the particular theory under consideration. The 'central' charges $\mathcal{Z}$ are no longer central since they transform under Lorentz transformations.

2.2.3 BPS branes in string theory

We now focus our attention on the BPS p-brane solutions of the supergravity theories of section 2.1.4.

Type IIA and IIB

The type IIA and IIB theories both have the NS-NS 2-form $B^{(2)}$. We therefore the fundamental string or F1-brane solution and its magnetic dual, the Neveu-Schwarz five-brane or NS5-brane. The appearance of the F1-solution is not surprising, since we are considering the low-energy limit of string theories.

In addition the IIA and IIB theories contain the R-R forms. We mentioned already that the string theory Dirichlet branes carry these R-R charges. Here we find the supergravity description of these objects. In the type IIA theory one finds the D0-brane and the D2-brane and their magnetic duals, the D6-branes and the D4-brane. As mentioned before, there is also a D8-brane which can be obtained as a solution of the massive IIA supergravity. In the type IIB theory one finds the D1-brane and its magnetic partner the D5-brane. The D3-brane is self-dual. There is also a D7-brane, which carries a magnetic charge for the axion $C^{(0)}$. Its dual, the D(-1)-brane or D-instanton, can however be obtained only as a solution of the Euclidean version of (2.1.48).

We will show in the next section how the D-branes appear in superstring theory by means of an argument based on T-duality.

Type I

The type I theory does not contain an NS-NS 2-form and thus also does not have a BPS F1-brane solution. The fundamental string of the type I theory is thus not a stable object: it can decay, though this process is very slow at small values of the string coupling. The BPS solutions are the D1-brane and the D5-brane.

Heterotic

The heterotic $SO(32)$ and $E_8 \times E_8$ theories have a massless 2-form, leading to the heterotic string solution (F1) and the heterotic 5-brane (S5). The heterotic string theories do not have D-branes.

11-dimensional SUGRA

The 11-dimensional supergravity theory has a 3-form gauge field $C_{\mu\nu\rho}$ that leads to a membrane or M2-brane solution and its magnetic dual, the M5-brane.

Purely gravitational solutions

The theories considered above also possess BPS solutions that are not carried by a p-form field. They are the gravitational wave and the Kaluza-Klein monopole. These arise as purely gravitational solutions and are not of the form (2.2.16) – they have an off-diagonal metric. Though they play a role in string theory – they are related to the p-branes by dualities – treating them in detail would to us to far afield. We just remark here that the KK-monopole solution is a product of $(d-4)$-dimensional Minkowski space and the 4-dimensional Euclidean Taub-NUT space, which has a compact isometry direction. When performing a dimensional reduction along this compact direction, one obtains a $(d-5)$-brane that carries a magnetic charge w.r.t. the KK gauge field. Hence the name.

2.3 More on supersymmetric D-branes

As discussed in the previous chapter, we can view the type I theory as a system of 16 coincident D9-branes and an O9-plane. By T-duality we can go to configurations with lower-dimensional D-branes. Compactify for instance the 9-direction over a circle of radius R and introduce a $U(1)^{16}$ Wilson line. This does not break any of the 16 supersymmetries[21]. According to section 1.3.5, this configuration is T-dual to 16 parallel D8-branes situated on an interval $S^1/\mathbb{Z}_2$ with orientifold O8-planes at the endpoints. The unoriented type I string was obtained from the oriented type IIB theory. Therefore in the T-dual version the physics far away from the D8-branes is described by the oriented type IIA closed string theory.

We can continue like this and T-dualize over additional directions. We get 16 parallel Dp branes in the type IIB theory for p odd and in the type IIA theory for p even.

[21] An effective field theory argument for this is the following. The gauge field A_μ and gaugino χ transform as $\delta A_\mu = \bar{\epsilon}\Gamma_\mu\chi$ and $\delta\chi = \frac{1}{2}F^{\mu\nu}\Gamma_{\mu\nu}\epsilon$ under supersymmetry. Since for the Wilson line $F_{\mu\nu} = \chi = 0$, we have $\delta A_\mu = \delta\chi = 0$ for all ϵ.

We already mentioned several times before that the D-branes carry charges of the R-R forms. Polchinski showed this in [67] by an explicit string theory calculation. In addition, he showed that these charges satisfy the charge quantization relation $Q_e Q_m \sim n$ with $n = 1$. The D-branes thus carry the minimum R-R charge quantum. This strongly suggests that the D-branes of perturbative string theory and the Dp-brane solutions of the type II supergravities are indeed different descriptions of the same object.

Further evidence for this conjecture is obtained by comparing the tension of the D-branes with those calculated by means of the ADM-formula in supergravity. The BPS bound is satisfied also in perturbative string theory. A system of parallel supergravity Dp-branes is stable in the same way as a system of two RN black holes in four dimensions: the gravitational attraction is exactly matched by the electrostatic repulsion due to the R-R charges. In string theory, one obtains the same result from the exchange of closed strings between the D-branes (see e.g. [15]).

The D-brane's world-volume carries a supersymmetric Yang-Mills theory with 16 supercharges. In perturbative string theory this follows from dimensionally reducing the field content on a single D9-brane. These degrees of freedom can also be obtained from supergravity considerations, for example from the brane scan. See [79] for a review.

The consistency conditions of the type I string theory have a nice interpretation in terms of the R-R charges. The sixteen D9-branes act as sources for a R-R 10-form field. But the field strength of a 10-form vanishes identically in ten dimensions, hence the field equation of the 10-form $\mathrm{d}{*G} = {*J}$ is of the form $0 = 1$, unless the current density J vanishes. This is achieved by assigning a negative charge of -16 to the O9-plane. If we compactify and T-dualize k directions, the number of D-branes remains constant, but the number of O-planes increases to 2^k. A single Op-plane therefore carries -2^{p-5} units of R-R $(p+1)$-form charge. Whereas configurations with D-branes on compact spaces always have to be accompanied by objects with a negative charge because of flux conservation, D-branes can exist on their own in an uncompactified spacetime: the R-R field lines can move off to infinity. So it makes sense to consider individual D-branes as in section 2.2.

2.3.1 D-brane effective actions revisited

The type II string theories contain more massless fields than the bosonic string theory. The effective action for a D-brane in a curved background therefore contains additional terms. We again limit ourselves to *slowly-varying* fields. One can then show that the effective action is a sum of two terms:

$$S_{\mathrm{D}p\text{-brane}} = S_{\mathrm{DBI}} + S_{\mathrm{WZ}}. \tag{2.3.1a}$$

The Dirac-Born-Infeld action $S_{\rm DBI}$ is the same as before:

$$S_{\rm DBI} = -T_{{\rm D}p} \int {\rm d}^{p+1}\sigma \, {\rm e}^{-\Phi} \sqrt{-\det(G_{\alpha\beta} + \mathcal{F}_{\alpha\beta})}. \tag{2.3.1b}$$

In addition, there is the Wess-Zumino term $S_{\rm WZ}$ which describes the R-R charges of the Dp-brane:

$$S_{\rm WZ} = T_{{\rm D}p} \int i^* \sum_k C^{(k)} \wedge \exp \mathcal{F}. \tag{2.3.1c}$$

It is understood that in expanding the exponential in (2.3.1c) only the $(p+1)$-form is kept. The appearance of the tension $T_{{\rm D}p}$ in front of the Wess-Zumino term reflects the fact that the Dp-brane is a BPS state. If we use the following convention for Newton's constant

$$2\kappa_0^2 \equiv (2\pi)^7 l_s^8, \tag{2.3.2}$$

we get a nice expression for the tension:

$$\tau_{{\rm D}p} = \frac{1}{(2\pi)^p \ell_s^{p+1} g_s}. \tag{2.3.3}$$

Nice, because with this convention the ratio of the Einstein-frame tensions of the F-string and D-string becomes $\tau_{\rm F1}/\tau_{\rm D1} = {\rm e}^{\phi}$. The tensions of the F- and D-string are thus exchanged under S-duality (see below).

We see that when $\mathcal{F} = 0$ the Dp-brane couples only to its associated R-R form $C^{(p+1)}$. When $\mathcal{F}$ is switched on, the brane also couples to the R-R forms of lower rank. This agrees with our earlier discussion in section 1.3.4, where we argued that there are smeared branes of lower dimensionality present in the Dp-brane's world-volume when $\mathcal{F} \neq 0$. To see that the charges work out correctly, put $B_{\mu\nu} = 0$ and expand the exponential

$$S_{\rm WZ} = T_{{\rm D}p} \int \sum_k C^{(k)} \wedge \exp 2\pi\alpha' F = T_{{\rm D}p} \sum_k \frac{(2\pi\alpha')^k}{k!} \int C^{(p+1-2k)} \wedge F^k. \tag{2.3.4}$$

Using (2.3.3), we obtain:

$$S_{\rm WZ} = \sum_k T_{{\rm D}(p\,-\,2k)} \int C^{(p+1-2k)} \wedge \frac{1}{(2\pi)^k k!} F^k. \tag{2.3.5}$$

We infer from this that the number of lower-rank R-R charges is indeed equal to the Chern classes c_k of (1.3.50).

2.4 The web of dualities

In the 1990's it gradually became understood that the five different string theories all arise as different limits of an underlying unified theory, called *M-theory*. We end this chapter with a brief overview of these dualities.

Type IIA – Type IIB

As argued in the above, the type IIA and IIB theories are T-dual to each other. We have shown in section 1.3.2 that the D-branes transform as follows: if we perform the duality transformation in a direction orthogonal to the Dp-brane's world-volume we obtain a D(p+1)-brane, whereas for a direction tangent to the world-volume we obtain a D($p-1$)-brane. The same result can be obtained at the level of the supergravity solutions by applying the Buscher T-duality rules – see e.g. [89] for a review.

Similarly, it can be shown that the F1 solution does not transform under a T-duality transformation orthogonal to the world-sheet, but that a T-duality along the world-sheet results in the wave (W) solution. Vice versa: T-dualizing along the propagation direction of the wave yields the F1 string, whereas dualizing along a direction orthogonal to the propagation leaves the W solution invariant.

Finally, the NS5-branes and KK5-monopoles are related to each other by T-dualizing along a direction transverse to their world-volumes (in the case of the KK5-monopole we need to take the Taub-NUT isometry direction), but are invariant under duality transformation along their world-volume directions.

Het $SO(32)$ – Het $E_8 \times E_8$

The heterotic theories are also T-dual to each other when compactified on circles of radii R and α'/R respectively, but only when we turn on a Wilson lines that break the $SO(32)$ and $E_8 \times E_8$ symmetry to $SO(16) \times SO(16)$. The duality transformations for the F1, W, S5 and KK5 solutions are the same as those for the type II theories.

Type IIB – Type IIB

The type IIB supergravity action (2.1.48) reads in Einstein frame

$$\begin{aligned} 2\kappa^2\, \mathcal{L}^{\mathrm{E}}_{\mathrm{IIB}} \; &= R\,{*1} - \frac{1}{2}\,{*\mathrm{d}\phi} \wedge \mathrm{d}\phi - \frac{1}{2}\,\mathrm{e}^{2\phi}\,{*G^{(1)}} \wedge G^{(1)} - \frac{1}{2}\,\mathrm{e}^{-\phi}\,{*H^{(3)}} \wedge H^{(3)} \\ &\quad - \frac{1}{2}\,\mathrm{e}^{\phi}\,{*G^{(3)}} \wedge G^{(3)} - \frac{1}{4}{*G^{(5)}} \wedge G^{(5)} + \frac{1}{2}C^{(4)} \wedge \mathrm{d}C^{(2)} \wedge H^{(3)}. \end{aligned} \tag{2.4.1}$$

It can be shown that this action is invariant under the following $SL(2,\mathbb{R})$ transformation:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} C^{(2)} \\ B^{(2)} \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C^{(2)} \\ B^{(2)} \end{pmatrix}. \tag{2.4.2}$$

Here we defined a complex scalar $\tau \equiv C^{(0)} + i\mathrm{e}^{-\phi}$. The metric and the 4-form $C^{+}_{\mu\nu\rho\sigma}$ are inert. In a background where $C^{(0)}$ vanishes, the transformation with $a = d = 0$ and $b = -c = 1$ takes ϕ into $-\phi$. This suggests that the strong coupling regime of the type IIB string is in fact dual to its own weak coupling regime. This is an example of S-duality. It turns out that in quantum theory the $SL(2,\mathbb{R})$ symmetry is broken to $SL(2,\mathbb{Z})$ as a consequence of charge quantization.

Since the brane solutions of type IIB supergravity that we discussed are BPS, we can trust their properties also at strong coupling. One can argue in particular that the F1- and D1-string are transformed into each other under $\phi \to -\phi$. There actually exists an entire class of so-called (p,q)-strings that carry both NS and R-R charges and transform nontrivially under the complete $SL(2,\mathbb{Z})$. The NS5- and D5-branes are related in the same way, whereas the D3-brane is invariant. Also the W and KK5 solutions are invariant.

Type I – Het $SO(32)$

The strong coupling regime of the type I theory turns out to be dual to the weak coupling regime of the heterotic $SO(32)$ string and vice versa. As for the IIB string, this can be motivated by looking at the effective actions. The type I supergravity (2.1.52) reads in Einstein frame[22]

$$2\kappa^2\,\mathcal{L}_{\mathrm{I}}^{\mathrm{E}} \;=\; R\,{*1} - \frac{1}{2}{*\mathrm{d}\phi}\wedge\mathrm{d}\phi - \frac{1}{2}\mathrm{e}^{\phi}\,{*G^{(3)}}\wedge G^{(3)} + \frac{1}{2}\mathrm{e}^{\phi/2}\,\mathrm{Tr}\,{*F}\wedge F. \tag{2.4.3}$$

and the heterotic supergravity (2.1.55) becomes

$$2\kappa^2\,\mathcal{L}_{\mathrm{het}}^{\mathrm{E}} \;=\; R\,{*1} - \frac{1}{2}{*\mathrm{d}\phi}\wedge\mathrm{d}\phi - \frac{1}{2}\mathrm{e}^{-\phi}\,{*H^{(3)}}\wedge H^{(3)} + \frac{1}{2}\mathrm{e}^{-\phi/2}\,\mathrm{Tr}\,{*F}\wedge F. \tag{2.4.4}$$

These actions are related to each other as follows

$$\phi_{\mathrm{type\,I}} \leftrightarrow -\phi_{\mathrm{het}}, \qquad C^{(2)}_{\mathrm{type\,I}} \leftrightarrow B^{(2)}_{\mathrm{het}}. \tag{2.4.5}$$

So indeed the weak- and strong-coupling regimes are exchanged.

At strong coupling, the lifetime of the fundamental string of the type I theory becomes very short and this string disappears from the spectrum as a recognizable excitation. The D1-string however is stable, and it turns out that at strong coupling it behaves as the fundamental string of the $SO(32)$ heterotic string. Similarly, the D5-brane behaves as the S5-brane.

11-dim SUGRA – Type IIA & Het $E_8 \times E_8$

The strong coupling limits of the type IIA and the heterotic $E_8 \times E_8$ theories are arguably even more remarkable. When we increase the strength of the coupling, an eleventh dimension appears.

[22] We rescale the gauge field such that the factor κ_0^2/g_0^2 no longer multiplies the kinetic term for the gauge fields.

Let us motivate this for the IIA theory. We reduce the 11-dimensional supergravity theory (2.1.56) over a circle. We split our coordinates as $\hat{x}^{\hat{\mu}} = (x^\mu, z)$ and decompose the 11-dimensional fields as follows

$$\widehat{\mathrm{d}s}^2 = \mathrm{e}^{-2\Phi/3} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{4\Phi/3}(\mathrm{d}z - C^{(1)}_\mu \mathrm{d}x^\mu)^2, \tag{2.4.6a}$$

$$\widehat{C}^{(3)} = C^{(3)} + B^{(2)} \wedge (\mathrm{d}z + C^{(1)}). \tag{2.4.6b}$$

The reduction is achieved by taking the unhatted fields to be independent of z. We recognize the fields in (2.4.6) as the field content of the type IIA supergravity. It is then no surprise that after having plugged this Ansatz into (2.1.56) we eventually obtain:

$$2\kappa_{11}^2\, \mathcal{L}_{d=11} = 2\kappa_0^2\, \mathcal{L}_{\mathrm{IIA}} \wedge \mathrm{d}z, \tag{2.4.7}$$

where $\mathcal{L}_{\mathrm{IIA}}$ is the Lagrangian (2.1.46) of the type IIA theory.

From (2.4.6a) we see that the radius of the 11th dimension is related to the expectation value of the dilaton Φ. Introducing the type IIA string coupling constant g_s, we have from $\hat{g}_{zz}$ that $R_{11} \sim g_s^{2/3}$. This is the radius measured with the 11-dimensional Einstein metric. The closed strings of the IIA theory however experience the string metric, for which we have from (2.4.6a) that $g^{\mathrm{S}}_{\mu\nu} = \epsilon^{2\Phi/3}\hat{g}_{\mu\nu}$ and thus

$$R_{11} \sim g_s, \tag{2.4.8}$$

in string units. We see that the 11th dimension is invisible in perturbative string theory.

The above suggests that $d = 11$ supergravity is the low-energy effective description of a theory that is dual to the strongly coupled IIA string. A perturbative description of this dual theory is at present still lacking. The IIA supergravity is the low-energy limit of weakly coupled closed strings, and these strings are recovered as BPS F1-branes of the IIA supergravity. Similarly, by an inspection of the $d = 11$ BPS solutions, one would be inclined to argue that the $d = 11$ supergravity has to arise as the low-energy limit of a theory of M2-branes. Unfortunately, nobody has been able to make sense of a quantum theory of weakly coupled supermembranes[23] (or of any of the other p-branes with $p > 1$ for that matter).

One can however relate BPS solutions of the $d = 11$ theory to that of the IIA theory. The D0-branes arise as the KK momentum states of the $d = 11$ gravitational wave (note that the RR gauge field $C^{(1)}$ is the KK gauge field). The D2-branes are M2-branes and the NS5-branes are M5-branes. However, if we wrap the M2-branes and M5-branes around the circle, we obtain the F1-string and the D4-brane. Similarly, the $d = 11$ KK monopole gives rise to the $d = 10$ KK monopole and the D6-brane.

It has been shown [91,92] that the strong coupling regime of the $E_8 \times E_8$ heterotic string theory can be related to the 11-dimensional theory compactified on an interval.

[23] Another and at least partially successful attempt to construct an underlying theory for the $d = 11$ supergravity goes under the name of matrix-theory and is based on the quantum mechanics of D0-branes. See e.g. [90] for a review.

2.4.1 Toward M-theory

We have seen that the different string theories are dual to each other and to a mysterious 11-dimensional theory. We have summarized this web of dualities in figure 2.4.1. The picture that emerges from all of this is that the theories that we discussed in this chapter are in fact different limits in the parameter space of a single underlying theory, called *M-theory*.

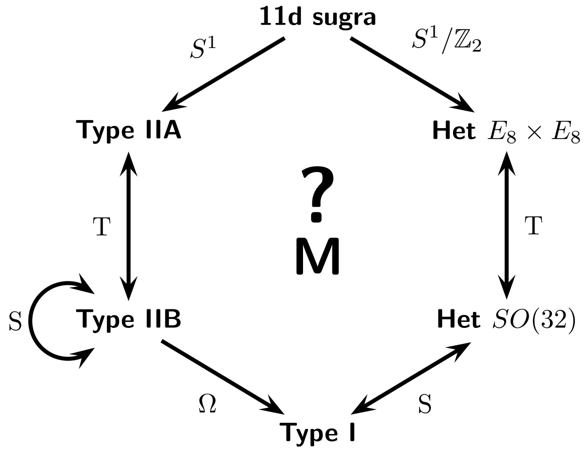

Figure 2.4.1. The M-theory duality web.

Perhaps the single most important theoretical challenge in this field is to find a non-perturbative formulation of M-theory. Though it is not at all clear what this final theory will look like, it is very likely that the different branes that we discussed will play a fundamental role in its formulation. In particular, the appearance of the non-commutative geometry of stacks of D-branes is a clear indication that spacetime itself and its group of symmetries will have to be treated very differently from what we are used to in general relativity.

Chapter 3

The D9-brane

We have seen in the previous chapters that D-branes have many interesting properties that can be deduced from their tree-level effective actions instead of deriving them directly in string perturbation theory. Some of these properties are already visible if we include only the leading order contribution to the effective action, e.g. T-duality in the Yang-Mills action. There are however other properties that are only visible when we include terms that are of higher order in α'. For example, to deduce the existence of a maximum value of the electric field we needed the Born-Infeld action. This involves arbitrarily high powers of α'. Another example is the Wess-Zumino action: the terms that describe the coupling of a Dp-brane to R-R charges of lower rank are of higher order in α'.

The above examples involve a single Dp-brane. It would be very interesting to extend these (and other cases) to a stack of Dp-branes. Unfortunately, there is no complete all-order result for the effective action for a stack of Dp-branes. As explained in chapter 1, this is partly due to the fact that covariant derivatives do not commute. As a result, there is no useful notion of *slowly varying fields* and one is forced to incorporate terms with derivatives when investigating α'-corrections to Yang-Mills theory.

In the following we will investigate derivative corrections to D-brane effective actions in a flat background. According to the discussion of section (1.3.3), it is sufficient to consider only the space-time filling D9-brane[1]. The actions for the lower-dimensional branes follow by dimensional reduction and the T-duality rules. The non-abelian case will be the topic of chapter 4. In this chapter we will discuss the abelian case, i.e. derivative corrections to the Born-Infeld action.

There are several reasons why it is useful to investigate the abelian case. First of

[1] Since we restrict ourselves to the *tree-level* effective action we do not have to worry about the fact that a single D9-brane is an inconsistent object in string theory.

all, the non-abelian case is very complicated and one can hope to gain some insight in the general structure of the derivative corrections in the simpler – but still nontrivial – abelian case. Second, as was argued in [93], the restriction to slowly varying fields actually implies that gravitational effects are large, invalidating the restriction to flat backgrounds. The argument of [93] goes roughly as follows. Negligible derivatives imply that the fields stay large over a vast region of spacetime. An estimate of the total energy and the corresponding volume indicates that under gravitational forces such a system would collapse to a black hole. To avoid this, fields have to fall off over a short distance, making derivatives large. Derivative corrections in the abelian case are thus important in their own right.

This chapter reviews results that were published in [c] and [d].

3.1 Corrections to the Born-Infeld action

Before turning to the derivative corrections, it is instructive to consider the α'-expansion of the Born-Infeld action itself. For a square matrix M we have $\log\det M = \operatorname{tr}\log M$, from which

$$\det(1+X) = \exp\left[\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\operatorname{tr}X^n\right].$$

Now if $\operatorname{tr}X^{2n+1}=0$:

$$\sqrt{\det(1+X)} = 1-\frac{1}{4}\operatorname{tr}X^2-\frac{1}{8}\Big(\operatorname{tr}X^4-\frac{1}{4}(\operatorname{tr}X^2)^2\Big)+\dots,$$

and thus[2]:

$$\begin{aligned} S_{\mathrm{D9}} &= -\tau_{\mathrm{D9}}\int \mathrm{d}^{10}x\,\sqrt{-\det(\eta_{ab}+2\pi\alpha' F_{ab})} \\ &= -\tau_{\mathrm{D9}}\int \mathrm{d}^{10}x\left[1+\frac{(2\pi\alpha')^2}{4}F_{ab}F_{ab}-\frac{(2\pi\alpha')^4}{8}\Big(\operatorname{tr}F^4-\frac{1}{4}(\operatorname{tr}F^2)^2\Big)+\mathcal{O}(\alpha'^6)\right]. \end{aligned}$$

From (2.3.3) we have $\tau_{\mathrm{D9}} = 1/g_s(2\pi)^9\ell_s^{10}$. The coupling constant g of the gauge theory living on the D9-brane is thus

$$g^2 = g_s(2\pi)^7\ell_s^6. \tag{3.1.1}$$

For a generic Dp-brane we have $g^2 = g_s(2\pi)^{p-2}\ell_s^{p-3}$. We drop the constant term to get

$$S=\frac{1}{g^2}\int \mathrm{d}^{10}x\Big[-\frac{1}{4}F_{ab}F_{ab}+\frac{(2\pi\alpha')^2}{8}\Big(\operatorname{tr}F^4-\frac{1}{4}(\operatorname{tr}F^2)^2\Big)+\dots\Big]. \tag{3.1.2}$$

[2] We use Latin indices for spacetime coordinates in chapter 3 and 4 and always write these as lower indices. This does *not* imply that we are working with a Euclidean signature.

The field strength has dimension $[F] = +2$ in units of mass and $[g] = -(d-4)/2$. In discussing the derivative corrections to the above action it is useful to introduce some notation. We write a generic term in the effective action schematically as

$$\mathcal{L}_{\text{eff}} = \frac{1}{g^2}\sum_{m,n}\mathcal{L}_{(m,n)}, \quad \text{with} \quad \mathcal{L}_{(m,n)} = \alpha'^m\big(\partial^n F^p + \partial^{n+1}F^{p-2}\bar{\chi}\Gamma\chi\big), \tag{3.1.3}$$

where we omit terms that are of quartic or higher order in the fermions. The powers in (3.1.3) are related by $2p - 2m + n - 4 = 0$. We will denote the terms at order α'^m and with n derivatives by (m, n).

The form of equation (3.1.3) already takes two of the symmetries of the underlying string theory into account: Poincaré invariance – from which follows that n is even – and the $U(1)$ gauge invariance of the massless fields. The following section is devoted to an investigation of the consequences of supersymmetry. Here we want to point out the consequences of the invariance of string theory under the world-sheet twist operation Ω. Since Ω changes the orientation of the world-sheet, it reverses the order of the vertex operators on the boundary of the disk amplitude. In addition, Ω acts on the vertex operators, giving an extra factor of $(-)^N$, where N counts the number of oscillators involved. For a primitive amplitude A with p external massless ($N = 1$) open string fields we thus have

$$A(1, 2, \ldots, p-1, p) = (-)^p A(p, p-1, \ldots, 2, 1). \tag{3.1.4}$$

The complete amplitude $\mathcal{A}$ is obtained by adding all noncyclic permutations:

$$\mathcal{A}(1, 2, \ldots, p) \equiv \sum_{\pi} A(1, \pi(2), \ldots, \pi(p)) = (-)^p \mathcal{A}(1, 2, \ldots, p), \tag{3.1.5}$$

where the latter equality follows from (3.1.4) and cyclic invariance. We conclude that the S-matrix elements involving an *odd* number of massless fields vanish and that terms with p odd in (3.1.3) are therefore absent in the open string effective action[3].

Most of the information on the abelian open string effective action concerns bosonic terms only. The terms $(m, 0)$ – i.e. the Born-Infeld action – was obtained by calculating the disk partition function [94]. In [95] (see also [96]) the contributions to $\mathcal{L}_{(2,0)}$ of quadratic order in the fermions were obtained in the context of the non-abelian theory, the quartic terms have been discussed in [97]. The supersymmetric completion of the entire Born-Infeld action was obtained by using κ-symmetry in an impressive paper [98] (see also [99–101]) and reads (in units where $2\pi\alpha' = 1$):

$$\mathcal{L} = -\tau_{\text{D9}}\sqrt{-\det\left(\eta_{ab} + F_{ab} - \bar{\chi}\Gamma_a\partial_b\chi + \frac{1}{2}\bar{\chi}\Gamma^c\partial_a\chi\,\bar{\chi}\Gamma_c\partial_b\chi\right)}. \tag{3.1.6}$$

[3] With the exception of terms that can be removed by a field redefinition, see below.

We will show in a minute that there are no $(m, 2m)$ and $(m, 2m-2)$ terms, i.e. terms with $p = 2$ and $p = 3$. In [52] it was shown that the bosonic terms $(m, 2)$ vanish for all m. Our calculation of $(4, 2)$ – which we will present below – confirms this and extends it to the corresponding fermionic terms. In [52] the bosonic part of $(4, 4)$ was constructed explicitly. More recently, Wyllard [102] obtained the bosonic $(m, 4)$ terms using the boundary state formalism. Further work has been done in [103, 104] with the Seiberg-Witten map and noncommmutativity.

In the following we will derive the terms $(m, 2m - 4)$ including all the fermionic contributions, i.e. the terms with $p = 4$. We will use the matching procedure based on the calculation of S-matrix elements that we discussed in chapter 1 (see also [49, 50, 93, 105, 106]). We are able to write down a closed form for the effective action because the open string four-point function factorizes in a product of two terms: the first term (K) depending on polarization vectors and wave functions, the second term ($\mathcal{G}$), proportional to the Veneziano amplitude, depending only on the momenta. The first term determines how the fields should appear in the effective action. The second term expands into an infinite series[4] in α', and determines how derivatives should be distributed over the fields. This structure applies to both the bosonic terms and the terms involving fermions.

Due to the factorization of the amplitude, supersymmetry of the effective action can be easily established. We mentioned already that the supersymmetry of the term $\mathcal{L}_{(2,0)}$ which reproduces the term K was demonstrated in [95]. The term $\mathcal{G}$, with momenta replaced by derivatives, acts on K in the full effective action, but we will show that the proof of supersymmetry still works "under the derivatives". Before we discuss these results, we will first review the Noether method and in particular the calculation of $\mathcal{L}_{(2,0)}$.

3.2 The Noether procedure

Our starting point is $\mathcal{L}_{(0,0)} \equiv \mathcal{L}_0$, i.e. the $d = 10$, $\mathcal{N} = 1$ supersymmetric Maxwell Lagrangian. χ is a Majorana-Weyl spinor and is inert under gauge transformations.

$$\mathcal{L}_0 = -\frac{1}{4} F_{ab} F_{ab} + \frac{1}{2} \bar{\chi} \partial\!\!\!/ \chi. \tag{3.2.1}$$

The equations of motion are simply

$$\partial_a F_{ab} = \partial\!\!\!/ \chi = 0 \,, \tag{3.2.2}$$

[4]In the absence of 3-point vertices, unitarity guarantees that the 4-point functions are analytic in the external momenta at $k_1 = k_2 = k_3 = k_4 = 0$.

supersymmetry is realized linearly on the fields:

$$\delta_0 A_a = \bar{\epsilon}\Gamma_a \chi, \tag{3.2.3a}$$

$$\delta_0 \chi = \frac{1}{2} F_{ab}\Gamma_{ab}\epsilon. \tag{3.2.3b}$$

Closure of the supersymmetry algebra requires the fields to be on-shell and involves a field dependent gauge transformation of the gauge field:

$$[\delta_{0\,\epsilon_1}, \delta_{0\,\epsilon_2}]A_a = 2\bar{\epsilon}_1 \not{\partial} \epsilon_2 A_a - \partial_a(2\bar{\epsilon}_1 \not{A} \epsilon_2), \tag{3.2.4a}$$

$$[\delta_{0\,\epsilon_1}, \delta_{0\,\epsilon_2}]\chi = 2\bar{\epsilon}_1 \not{\partial} \epsilon_2 \chi - \left(\frac{7}{8}\bar{\epsilon}_1 \Gamma_a \epsilon_2 \Gamma_a - \frac{1}{5!16}\bar{\epsilon}_1 \Gamma_{abcde}\epsilon_2 \Gamma_{abcde} \right) \not{\partial} \chi. \tag{3.2.4b}$$

This lowest order action also has a nonlinear supersymmetry:

$$\delta_0 A_a = 0, \tag{3.2.5a}$$

$$\delta_0 \chi = \eta. \tag{3.2.5b}$$

Supersymmetric deformations of the super-Maxwell theory can be obtained by the Noether method, which is an iterative procedure in α'. Every stage of the iteration consists of two steps. Let the $\mathcal{L}_k$ for $k < m$ be known[5]. The first step in obtaining the term $\mathcal{L}_m$ is to write down all possible terms of order α'^m, i.e., terms that have the correct dimension and are Lorentz and gauge invariant. We limit ourselves to terms that are at most of quadratic order in the fermions. Lagrangians are defined up to total derivatives and field redefinitions. The possibility for the latter arises when a term is proportional to the lowest order equation of motion (3.2.2) for one of the fields. If such a term is present in $\mathcal{L}_m$ it can be removed by a field redefinition of order m. The price one pays is that the contributions $\mathcal{L}_n$ with $n > m$ are modified. We deal with this ambiguity, at each order in α', by not allowing in the Lagrangian any terms that are proportional to the order α'^0 field equations, or terms that can be rewritten as such by means of a partial integration. Furthermore, we determine how the remaining terms are related by partial integrations and keep only an independent subset. This leaves us with a minimal Ansatz for $\mathcal{L}_0$ in which each term has an arbitrary coefficient that will be determined in the second step.

The second step is to vary the fields in this Ansatz with the supersymmetry transformation rules δ_0. In addition we need to vary the lower order terms in the Lagrangian, say $\mathcal{L}_k$, $k < m$, with the appropriate transformation rules δ_{m-k}; both were already obtained in a previous stage of the iterative procedure. Having done this, we are left with two types of variations. On the one hand there are terms which

[5] $\mathcal{L}_m$ is the contribution of order α'^m to the effective action. Similarly, δ_m indicate supersymmetry transformations of order α'^m. If we want to indicate the part of $\mathcal{L}_m$ with n derivatives we write $\mathcal{L}_{(m,n)}$, similarly for $\delta_{(m,n)}$.

are proportional to the lowest order field equation or that can be rewritten as such using a partial integration. On the other hand there are variations that cannot be rewritten in this way. The first set can be eliminated by new transformation rules δ_m of $\mathcal{L}_0$, the second set must be set to zero by solving the resulting equations for the unknown coefficients in the Ansatz.

In calculating the new transformation rules at order α'^m one will find that some variations may be quadratic in the lowest order equations of motion. In that case there is an ambiguity in the choice of the new transformations δ_m. Regardless of this choice, such variations always give rise to transformation rules that contain a lowest order equation of motion. Therefore these terms do not play a role in checking the closure of the supersymmetry algebra at order α'^m. If such transformations are applied to some $\mathcal{L}_k$ when constructing an invariant at order $m+k$, they give variations that can automatically be supersymmetrized. Their contribution to the order $m+k$ transformation rules need not contain a lower order equation of motion and therefore these terms are important when pursuing the Noether procedure to higher orders. This last issue does not yet play a role at the level of the 4-point function and will not bother us in the following.

In addition to terms that arise as higher order contributions to invariants that were already encountered at lower orders in α', the Noether procedure can at any given order in α' yield new leading order contributions to apparently independent superinvariants, all determined up to a multiplicative constant. However, some of these coefficients might be determined by pursuing the Noether procedure to even higher orders. We will come back to this issue at the end of this chapter, when we discuss the Noether procedure at order α'^4.

3.2.1 No corrections with $p = 2$ and $p = 3$

In this section we show that there are no terms in the effective action with $p = 2$ (with the exception of $\mathcal{L}_{(0,0)}$) or $p = 3$ that can not be removed by a field redefinition. This is not a deep result – at the level of amplitude calculations it is just kinematics – and we present it here to illustrate the kind of reasoning that we use to parametrize candidate terms in the effective action in general.

Consider first the bosonic terms. At $p = 2$ we have two F's with an arbitrary number of derivatives. By partial integration, we can bring any of these terms to the form $F_{ab}(\partial^n F)_{ab}$. By using the Bianchi identity in the form $\partial_a F_{bc} = -2\partial_{[b}F_{c]a}$ we write these terms as $F_{ab}\Box^{n/2}F_{ab}$. These can be removed by field redefinitions, unless $n = 0$.

For $p = 3$ we have three F's, and we can bring any term to the form $F_{ab}(\partial^k F \partial^l F)_{ab}$. Next we distribute a and b. If we put them both on the same factor $\partial^k F$ we get something of the form $F_{ab}\partial^k F_{ab}\partial^l F$ by using the Bianchi identity. Now terms of this form that contain contractions within a single factor can be removed by field redefinitions.

We thus require $k = l+2$ and get terms that read $F_{ab}\partial_c\partial_d\partial^l F_{ab}\partial^l F_{cd}$. But these vanish identically. So we have to put a and b on different factors to get something nontrivial. By using the Bianchi identity such terms can be reduced to $F_{ab}\partial^k F_{\cdot a}\partial^l F_{\cdot b}$, with $k = l$ by the same argument as before. These are of the form $F_{ab}\partial^{k-1}\partial_c F_{da}\partial^{k-1}\partial_c F_{db}$ and $F_{ab}\partial^{k-1}\partial_c F_{da}\partial^{k-1}\partial_d F_{cb}$. Both cases vanish identically. This concludes the proof for the bosonic terms.

Now for the fermionic terms. There are only terms with two fermions for $p = 2, 3$. By partial integration we can always remove any derivative on $\bar{\chi}$. At $p = 2$ we thus only have terms of the form $\bar{\chi}\Gamma_a\partial^n\partial_a\chi = \bar{\chi}\not{\partial}\Box^{n/2}\chi$ and these can be removed by a field redefinition. The terms with a $\Gamma^{(n)}$ with $n > 1$ vanish identically. We note here that terms with a $\Gamma^{(n)}$ for n even never appear anywhere in the effective action, since we are dealing with chiral fermions.

The argument for $p = 3$ is a bit more complicated. Terms with $\Gamma^{(1)}$ are necessarily of the form $\partial^k F_{ab}\,\bar{\chi}\Gamma_a\partial^k\partial_b\chi$, otherwise they vanish or contain contractions within the same factor. By using the antisymmetry in ab and the Γ-matrix identity $\Gamma_{abc} = \Gamma_{ab}\Gamma_c - \Gamma_a\delta_{bc} + \Gamma_b\delta_{ac}$ we can rewrite these terms as a sum of two expressions. The first is of the form $\partial^k F_{ab}\,\bar{\chi}\Gamma_{ab}\not{\partial}\chi$ and can be removed by a field redefinition. The second has the form $\partial^k F_{ab}\,\bar{\chi}\Gamma_{abc}\partial^k\partial_c\chi$, which happens to be the only term that we can write down with a $\Gamma^{(3)}$ according to our rules. We deal with this term by partially integrating all derivatives on F, which results in terms with contractions on the same factor and terms of the form $F_{ab}\,\partial^k\bar{\chi}\Gamma_{abc}\partial^k\partial_c\chi$. Now we use the that $\bar{\psi}\Gamma^{(3)}\chi = +\bar{\chi}\Gamma^{(3)}\psi$ to partially integrate ∂_c and obtain $\partial_c F_{ab}\,\partial^k\bar{\chi}\Gamma_{abc}\partial^k\chi$, which vanishes by the Bianchi identity. Terms with a $\Gamma^{(5)}$ or higher vanish identically.

3.2.2 The Noether procedure at α'^2

We will now perform the Noether procedure at order α'^2. The first step is to write down a minimal set of Lorentz and gauge invariant terms at this order. With minimal we mean terms that are not related – *via* partial integrations or the Bianchi identity – to each other or to terms that can be removed by a field redefinition. We already showed in the previous section that there are no such terms with $n > 0$, so we focus our attention on $\mathcal{L}_{(2,0)}$.

After some thought along the lines of the previous section[6] we arrive at the following Ansatz:

$$\begin{aligned}\alpha'^{-2}\mathcal{L}_{(2,0)} = a_1\, F_{ab}F_{bc}F_{cd}F_{da} + a_2\, F_{ab}F_{ab}F_{cd}F_{cd} \\ + a_3\, F_{ac}F_{bc}\,\bar{\chi}\Gamma_a\partial_b\chi + a_4\, F_{ad}\partial_d F_{bc}\,\bar{\chi}\Gamma_{abc}\chi.\end{aligned} \tag{3.2.6}$$

[6] In this case we are dealing with only a few fields and at most one explicit derivative, so the number of ways in which we can perform a partial integration or use the Bianchi identity is limited. In more complicated cases, it is best to proceed systematically and write down not only all possible terms but all possible total derivatives as well. One then works out the total derivatives to obtain relations between the terms.

We have given the terms undetermined coefficients a_i. The second step consist of varying these terms with the δ_0 of (3.2.3). We obtain

$$\begin{aligned}
a_1 :&\quad \delta\big[F_{ab}F_{bc}F_{cd}F_{da}\big] = -8\,\boxed{\text{ID}},\\
a_2 :&\quad \delta\big[F_{ab}F_{ab}F_{cd}F_{cd}\big] = -8\,\boxed{\text{IC}},\\
a_3 :&\quad \delta\big[F_{ac}F_{bc}\,\bar\chi\Gamma_a\partial_b\chi\big] = -\boxed{\text{IA}} - \boxed{\text{ID}} - \tfrac{1}{2}\boxed{\text{IIIA}} - \tfrac{1}{2}\boxed{\text{IIID}},\\
a_4 :&\quad \delta\big[F_{ad}\partial_d F_{bc}\,\bar\chi\Gamma_{abc}\chi\big] = 4\boxed{\text{IA}} + 2\boxed{\text{IB}} - 2\boxed{\text{IIIA}} + 2\boxed{\text{IIIC}} - \boxed{\text{VA}},
\end{aligned} \tag{3.2.7}$$

where we have used the Bianchi and Γ-matrix identities to write the result in terms of a set of "independent variations", i.e. terms that can not be related to each other by means of these identities. We choose the following set:

$$\begin{aligned}
\boxed{\text{IA}} &= F_{bd}F_{cd}\partial_b F_{ca}\,\bar\epsilon\Gamma_a\chi, & \boxed{\text{IIIA}} &= F_{ad}F_{de}\partial_e F_{bc}\,\bar\epsilon\Gamma_{abc}\chi,\\
\boxed{\text{IB}} &= F_{ab}F_{cd}\partial_b F_{cd}\,\bar\epsilon\Gamma_a\chi, & \boxed{\text{IIIB}} &= F_{ab}F_{de}\partial_c F_{de}\,\bar\epsilon\Gamma_{abc}\chi,\\
\boxed{\text{IC}} &= F_{ab}F_{cd}F_{cd}\,\bar\epsilon\Gamma_a\partial_b\chi, & \boxed{\text{IIIC}} &= F_{ad}F_{be}\partial_c F_{de}\,\bar\epsilon\Gamma_{abc}\chi,\\
\boxed{\text{ID}} &= F_{ac}F_{bd}F_{cd}\,\bar\epsilon\Gamma_a\partial_b\chi, & \boxed{\text{IIID}} &= F_{ab}F_{ce}F_{de}\,\bar\epsilon\Gamma_{abc}\partial_d\chi,\\
\boxed{\text{VA}} &= F_{ab}F_{cf}\partial_f F_{de}\,\bar\epsilon\Gamma_{abcde}\chi, & \boxed{\text{VB}} &= F_{ab}F_{cd}F_{ef}\,\bar\epsilon\Gamma_{abcde}\partial_f\chi.
\end{aligned}$$

The reader should be able to retrace the logic behind the construction of these terms by reading the indices in alphabetical order. None of these "variations" contains an order α'^0 equation of motion – $\partial_a F_{ab}$ or $\not{\partial}\chi$. Fortunately, certain linear combinations of the variations *can* be related to such terms by partial integrations.

We proceed by first constructing total derivatives which lead to contributions that have the same structure as the variations. We obtain the following list[7]:

$$\begin{aligned}
c_1 :&\quad \partial_a\big[F_{ab}F_{cd}F_{cd}\,\bar\epsilon\Gamma_b\chi\big] = \partial_a F_{ab}F_{cd}F_{cd}\,\bar\epsilon\Gamma_b\chi - 2\boxed{\text{IB}} - \boxed{\text{IC}},\\
c_2 :&\quad \partial_a\big[F_{ac}F_{bd}F_{cd}\,\bar\epsilon\Gamma_b\chi\big] = \partial_a F_{ab}F_{bd}F_{cd}\,\bar\epsilon\Gamma_c\chi + \boxed{\text{IA}} - \tfrac{1}{2}\boxed{\text{IB}} - \boxed{\text{ID}},\\
c_3 :&\quad \partial_a\big[F_{bc}F_{de}F_{de}\,\bar\epsilon\Gamma_{abc}\chi\big] = F_{ab}F_{de}F_{de}\,\bar\epsilon\Gamma_{ab}\not{\partial}\chi - 2\boxed{\text{IC}} + 2\boxed{\text{IIIB}},\\
c_4 :&\quad \partial_a\big[F_{bd}F_{cd}F_{de}\,\bar\epsilon\Gamma_{abc}\chi\big] = F_{ad}F_{be}F_{de}\,\bar\epsilon\Gamma_{ab}\not{\partial}\chi - 2\boxed{\text{ID}} + \boxed{\text{IIIA}} + \boxed{\text{IIIC}},\\
c_5 :&\quad \partial_a\big[F_{ae}F_{be}F_{cd}\,\bar\epsilon\Gamma_{bcd}\chi\big] = -\partial_a F_{ab}F_{bc}F_{de}\,\bar\epsilon\Gamma_{cde}\chi + \tfrac{1}{2}\boxed{\text{IIIB}} - \boxed{\text{IIIA}} + \boxed{\text{IIID}},\\
c_6 :&\quad \partial_a\big[F_{ab}F_{cd}F_{ef}\,\bar\epsilon\Gamma_{bcdef}\chi\big] = \partial_a F_{ab}F_{cd}F_{ef}\,\bar\epsilon\Gamma_{bcdef}\chi - 2\boxed{\text{VA}} - \boxed{\text{VB}},\\
c_7 :&\quad \partial_a\big[F_{bc}F_{de}F_{fg}\,\bar\epsilon\Gamma_{abcdefg}\chi\big] = F_{ab}F_{cd}F_{ef}\,\bar\epsilon\Gamma_{abcdef}\not{\partial}\chi - 6\boxed{\text{VB}}.
\end{aligned}$$

Next we *add* these total derivatives with undetermined coefficients c_j to the variation (3.2.7) of our Ansatz and demand that the coefficients that multiply the independent

[7] These are not all total derivatives. Not included are those which can be related to the ones we listed by means of the Bianchi or Γ-matrix identities since these do not contain any new information. In the general case we would also not include total derivatives that already contain an order α'^0 equation of motion under the derivative ∂_a since these do not help in reducing the set of variations.

variations $\boxed{\text{IA}}$, $\boxed{\text{IB}}$, ... vanish. This results in a set of *linear* equations:

$$\begin{aligned}
&\boxed{\text{IA}}: & -a_3+4a_4+c_2&=0, & \quad &\boxed{\text{IIIA}}: & \tfrac{1}{2}a_3+2a_4-c_4+c_5&=0,\\
&\boxed{\text{IB}}: & 2a_4-2c_1-\tfrac{1}{2}c_2&=0, & &\boxed{\text{IIIB}}: & 2c_3+\tfrac{1}{2}c_5&=0,\\
&\boxed{\text{IC}}: & 8a_2+c_1+2c_3&=0, & &\boxed{\text{IIIC}}: & 2a_4+c_4&=0,\\
&\boxed{\text{ID}}: & 8a_1+a_3+c_2+2c_4&=0, & &\boxed{\text{IIID}}: & -\tfrac{1}{2}a_3+c_5&=0,\\
&\boxed{\text{VA}}: & a_4+2c_6&=0, & &\boxed{\text{VB}}: & c_6+6c_7&=0.
\end{aligned}$$

Every independent solution of these equations corresponds to an independent supersymmetry invariant. The a_i determine the form of the Lagrangian, whereas the c_i give the modified supersymmetry transformation rules.

The above equations have a one-parameter family of solutions:

$$\begin{aligned}
&a_1=a, \quad a_2=-\tfrac{1}{4}a, \quad a_3=-2a, \quad a_4=\tfrac{1}{2}a, \quad c_1=\tfrac{3}{2}a, \quad c_2=-4a,\\
&c_3=\tfrac{1}{4}a, \quad c_4=-a, \quad c_5=-a, \quad c_6=-\tfrac{1}{4}a, \quad c_7=\tfrac{1}{24}a.
\end{aligned}$$

There is thus a unique supersymmetric deformation at order α'^2 which is given by (writing $a=a_{(2,0)}/8$):

$$\mathcal{L}_{(2,0)}=\frac{a_{(2,0)}\alpha'^2}{8}\Big[F_{ab}F_{bc}F_{cd}F_{da}-\frac{1}{4}\,F_{ab}F_{ab}F_{cd}F_{cd} - 2\,F_{ac}F_{bc}\,\bar{\chi}\Gamma_a\partial_b\chi+\frac{1}{2}\,F_{ad}\partial_dF_{bc}\,\bar{\chi}\Gamma_{abc}\chi\Big]. \tag{3.2.8}$$

The undetermined coefficient $a_{(2,0)}$ has to be fixed by other methods. We will see in the next section that the string theory 4-point function yields $a_{(2,0)}=(2\pi)^2$, which agrees with the expansion of the Born-Infeld action as in (3.1.2). It can be shown that the fermionic terms agree with the expansion of (3.1.6), but for this one needs to perform a field redefinition.

The modified supersymmetry transformation rules are for the boson:

$$\delta A_a=\bar{\epsilon}\Gamma_a\chi-\frac{a_{(2,0)}}{8}\Big[\frac{3}{2}\,F_{bc}F_{bc}\,\bar{\epsilon}\Gamma_a\chi+4\,F_{ac}F_{bc}\,\bar{\epsilon}\Gamma_b\chi + F_{ab}F_{cd}\,\bar{\epsilon}\Gamma_{bcd}\chi-\frac{1}{4}\,F_{bc}F_{de}\,\bar{\epsilon}\Gamma_{abcde}\chi\Big]+\ldots, \tag{3.2.9a}$$

and for the fermion

$$\delta\chi=\frac{1}{2}\,F_{ab}\,\Gamma_{ab}\epsilon-\frac{a_{(2,0)}}{8}\Big[F_{ab}F_{cd}F_{cd}\,\Gamma_{ab}\epsilon-F_{ac}F_{bd}F_{cd}\,\Gamma_{ab}\epsilon + \frac{1}{24}\,F_{ab}F_{cd}F_{ef}\,\Gamma_{abcdef}\epsilon\Big]+\ldots. \tag{3.2.9b}$$

An important check of the above calculation is to verify that the modified transformation rules still agree with the supersymmetry algebra. In particular, only the gauge transformation on the RHS of (3.2.4) can in principle receive corrections. Because of our quadratic fermion approximation we can only perform this check for the boson. In the present case even the gauge transformations should not receive any corrections because of the absence of derivatives in the transformation rules (3.2.9).

We could also have used the nonlinear supersymmetry (3.2.5) as the starting point of the Noether procedure. It turns out however that this does not provide any new information: in general nonlinear supersymmetry constrains the form of the effective action less than linear supersymmetry. The modified nonlinear transformations turn out to be

$$\delta A_a = \frac{a_{(2,0)}\alpha'^2}{8}\big(2F_{ab}\,\bar{\eta}\Gamma_b\chi - F_{bc}\,\bar{\eta}\Gamma_{abc}\chi\big), \tag{3.2.10a}$$

$$\delta\chi = \eta + \frac{a_{(2,0)}\alpha'^2}{32}\big(2F_{ab}F_{ab}\,\eta + F_{ab}F_{cd}\,\Gamma_{abcd}\eta\big). \tag{3.2.10b}$$

These do modify the order α'^0 algebra [97]:

$$[\delta_{\eta_1}, \delta_{\eta_2}]\,A_a = \frac{a_{(2,0)}\alpha'^2}{2}\big[\bar{\eta}_1\not{\partial}\eta_2 A_a - \partial_a(\bar{\eta}_1\not{A}\eta_2)\big]. \tag{3.2.11}$$

This is just the usual supersymmetry algebra, occurring at a higher order in α'. This *proves* that the nonlinear supersymmetry is indeed a supersymmetry. The mixed commutator is not modified at this order:

$$[\partial_\epsilon, \partial_\eta]\,A_a = 0. \tag{3.2.12}$$

We note here for future reference that the nonlinear supersymmetry persists to all orders in m for the terms $\mathcal{L}_{(m,0)}$, i.e. the action (3.1.6) is invariant under both linear and nonlinear supersymmetry transformations. This was shown in [98]. The linear and nonlinear supersymmetry are what remains of the underlying κ-symmetric formulation[8] after gauge fixing.

[8] κ-symmetry is a property of the effective action of D-branes formulated in an $\mathcal{N}=2$ supergravity background (for slowly varying fields). We have seen that the presence of the D-brane breaks half of the target-space supersymmetries. In the effective action this is achieved by fixing the κ-symmetry: this removes half of the fermionic degrees of freedom and breaks the supersymmetry to $\mathcal{N}=1$. The results of [98] were obtained starting from a flat background, whereas in [99–101] general supergravity backgrounds were considered.

3.3 The 4-point function

The open string tree-level 4-point function is given by [105]:

$$\mathcal{A}(1,2,3,4) = -16i\, g^{-2}\alpha'^2(2\pi)^{10}\delta^{(10)}(k_1+k_2+k_3+k_4)\times \\ \times\, \mathcal{G}(k_1,k_2,k_3,k_4)\, K(1,2,3,4) \quad (3.3.1)$$

$\mathcal{G}$ contains the α' dependence and is given by:

$$\begin{aligned}\mathcal{G}(k_1,k_2,k_3,k_4) &= G(s,t)+G(t,u)+G(u,s)\\ &= \frac{\Gamma(-\alpha's)\Gamma(-\alpha't)}{\Gamma(1-\alpha's-\alpha't)} + \frac{\Gamma(-\alpha't)\Gamma(-\alpha'u)}{\Gamma(1-\alpha't-\alpha'u)} + \frac{\Gamma(-\alpha'u)\Gamma(-\alpha's)}{\Gamma(1-\alpha'u-\alpha's)}.\end{aligned} \quad (3.3.2)$$

Here s, t, and u are the Mandelstam variables, satisfying $s+t+u=0$. They are defined in terms of the k_i only up to momentum conservation and the mass-shell condition. We choose to write them in such a way that $\mathcal{G}$ is manifestly symmetric in the k_i:

$$\begin{aligned} s &= -\,k_1\cdot k_2 - k_3\cdot k_4,\\ t &= -\,k_1\cdot k_3 - k_2\cdot k_4,\\ u &= -\,k_1\cdot k_4 - k_2\cdot k_3.\end{aligned} \quad (3.3.3)$$

As discussed in the above, $\mathcal{G}$ is regular as $k_i\to 0$, which one can verify by expanding (3.3.2) in α'. For now we just mention that

$$\mathcal{G}(k_1,k_2,k_3,k_4) = -\frac{\pi^2}{2} + \mathcal{O}(\alpha'^2), \quad (3.3.4)$$

and postpone a detailed discussion of the expansion to a later section. K involves not only the momenta of the external particles, but also their wave functions. For the 4-boson amplitude we have:

$$K(1,2,3,4) = t^{abcdefgh}k^1_a\zeta^1_b k^2_c\zeta^2_d k^3_e\zeta^3_f k^4_g\zeta^4_h, \quad (3.3.5)$$

where ζ^i is the polarization vector of the ith incoming photon. An explicit expression for the tensor t_8 is given for example in [105]. $t_{abcdefgh}$ is antisymmetric in the pairs (ab), (cd), etc., and is symmetric under the exchange of such pairs. It satisfies the following identity:

$$\begin{aligned} &t_{abcdefgh}M_1^{ab}M_2^{cd}M_3^{ef}M_4^{gh} = \\ &\quad = -2\big(\mathrm{tr}\, M_1M_2\,\mathrm{tr}\, M_3M_4 + \mathrm{tr}\, M_1M_3\,\mathrm{tr}\, M_2M_4 + \mathrm{tr}\, M_1M_4\,\mathrm{tr}\, M_2M_3\big)\\ &\qquad + 8\big(\mathrm{tr}\, M_1M_2M_3M_4 + \mathrm{tr}\, M_1M_3M_2M_4 + \mathrm{tr}\, M_1M_3M_4M_2\big), \end{aligned} \quad (3.3.6)$$

where the M_i are antisymmetric tensors.

The leading order contribution to the amplitude is just (3.3.5) times a constant and is reproduced by the following contribution to the effective action:

$$\begin{aligned} S_{(2,0)} &= \frac{1}{8}(2\pi\alpha')^2 \int \mathrm{d}^{10}x \, \frac{1}{24} t_{abcdefgh} F_{ab} F_{cd} F_{ef} F_{gh} \\ &= \frac{1}{8}(2\pi\alpha')^2 \int \mathrm{d}^{10}x \left(\mathrm{tr} F^4 - \frac{1}{4}(\mathrm{tr} F^2)^2 \right). \end{aligned} \tag{3.3.7}$$

This result agrees with the expansion of the Born-Infeld action in (3.1.2) and shows that we indeed need to take $a_{(2,0)} = (2\pi)^2$ in (3.2.8). We observe that every factor of momentum k_i in (3.3.5) is reproduced by a derivative acting on the appropriate field in (3.3.7).

The complete amplitude (3.3.1) differs from the leading order contribution by multiplication with $\mathcal{G}$, i.e. by extra factors of momentum. In order to reproduce these factors, we simply need to act with derivatives on the appropriate fields. This is implemented by first allowing the four fields to be "defined at different points in space-time", resulting in a non-local action. That is, we consider the fields $A_a(x_i)$, where $i = 1, \ldots, 4$, and then replace the momenta k_i in the amplitude by differentiations with respect to the appropriate coordinate in the effective action, i.e. $k_{i,a} \to -i\partial/\partial x_i^a$. We need to multiply the resulting expression by delta functions and then integrate over the x_i to make the action local.

Hence we define the following differential operator

$$D(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4}) \equiv \mathcal{G}(k_1, k_2, k_3, k_4)|_{k_i \to -i\partial_{x_i}} , \tag{3.3.8}$$

which we use to write down the effective action for the complete four-photon amplitude:

$$\begin{aligned} S_{\text{eff}}[A_a] = -\frac{1}{24} g^{-2} \alpha'^2 \int \mathrm{d}^{10}x \left\{ \prod_i \mathrm{d}^{10}x_i \, \delta^{(10)}(x - x_i) \right\} D(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4}) \\ \times \, t_{abcdefgh} F_{ab}(x_1) F_{cd}(x_2) F_{ef}(x_3) F_{gh}(x_4). \end{aligned} \tag{3.3.9}$$

D is understood as a Taylor expansion in α'. Then the multiple integral over the x_i factorizes into a product of integrals, each involving only one of the x_i and none of the others, which is necessary in order that the above expression is well defined. The actual proof that this action reproduces the amplitude (3.3.1) can be found in Appendix B.2.1.

As mentioned above, we choose to express s, t, u in terms of the k_i in such a way that $\mathcal{G}$ is manifestly symmetric in the momenta. This will turn out to be convenient in the following section. It is not difficult to see that a different prescription than (3.3.3) would result in modifications of the effective action (3.3.9) by total derivatives and/or the effects of field redefinitions. This follows from momentum conservation $k_1^a + k_2^a + k_3^a + k_4^a = 0$ and the mass-shell conditions $k_i^2 = 0$, respectively.

3.3.1 The fermionic contributions and supersymmetry

Equation (3.2.8) reproduces the four-point string amplitudes involving two fermions[9] to lowest order in α' [106]. It is then easy to guess what the effective action should be when fermionic interactions as well as higher derivative corrections are included:

$$\begin{aligned}S_{\text{eff}}[A_a,\chi] = & -g^{-2}\alpha'^2 \int \mathrm{d}^{10}x \left\{\prod_i \mathrm{d}^{10}x_i\, \delta^{(10)}(x-x_i)\right\} D(\partial_{x_1},\partial_{x_2},\partial_{x_3},\partial_{x_4}) \\ & \times \Big[F_{ab}(x_1)F_{bc}(x_2)F_{cd}(x_3)F_{da}(x_4) - \frac{1}{4}F_{ab}(x_1)F_{ab}(x_2)F_{cd}(x_3)F_{cd}(x_4) \\ & - 2F_{ab}(x_1)F_{ac}(x_2)\bar{\chi}(x_3)\Gamma_b\partial_c\chi(x_4) + F_{ab}(x_1)F_{cd}(x_2)\bar{\chi}(x_3)\Gamma_{abc}\partial_d\chi(x_4)\Big]. \qquad (3.3.10)\end{aligned}$$

It is not difficult to prove that this action is supersymmetric. As explained in the previous section, the operator D is symmetric in the ∂_{x_i}. This implies that, when we apply the Noether method, we can perform the same manipulations as the ones necessary to demonstrate the supersymmetry of (3.2.8).
Consider for example the variation of the first term in (3.2.8). It is given by

$$\begin{aligned}\delta\left(\operatorname{tr} F^4\right) &= \delta F_{ab}F_{bc}F_{cd}F_{da} + F_{ab}\delta F_{bc}F_{cd}F_{da} + F_{ab}F_{bc}\delta F_{cd}F_{da} + F_{ab}F_{bc}F_{cd}\delta F_{da} \\ &= 4\, F_{ab}F_{bc}F_{cd}\delta F_{da}.\end{aligned}$$

The last step is of course completely trivial in the local case, but essential for proving the supersymmetry. In the non-local case (3.3.10), this last step is not automatic. We see that it is the symmetry of D that allows us to perform it.

In addition to algebraic manipulations of the kind described above, it is also necessary to perform partial integrations to prove the supersymmetry. In the local case one encounters for example the following total derivative at an intermediate stage of the calculation: $\partial_a\left(F_{ab}\operatorname{tr} F^2\, \bar{\epsilon}\Gamma_b\chi\right)$. In the non-local case this term will manifest itself as

$$\left(\frac{\partial}{\partial x_1^a} + \frac{\partial}{\partial x_2^a} + \frac{\partial}{\partial x_3^a} + \frac{\partial}{\partial x_4^a}\right) F_{ab}(x_1)F_{cd}(x_2)F_{cd}(x_3)\, \bar{\epsilon}\Gamma_b\chi(x_4).$$

This still gives rise to a total derivative, since we can pull the $\sum_i \partial/\partial x_i^a$ out of the

[9] In [106] and [d] the terms with four fermions were also taken into account.

integration over the x_i:

$$\int \mathrm{d}^{10}x \left\{ \prod_i \mathrm{d}^{10}x_i\, \delta^{(10)}(x - x_i) \right\} D(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4})$$
$$\times \left(\sum_j \frac{\partial}{\partial x_j^a} \right) F_{ab}(x_1) F_{cd}(x_2) F_{cd}(x_3)\, \bar{\epsilon}\Gamma_b \chi(x_4)$$
$$= \int \mathrm{d}^{10}x \frac{\partial}{\partial x} \int \left\{ \prod_i \mathrm{d}^{10}x_i\, \delta^{(10)}(x - x_i) \right\} D(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4})$$
$$\times F_{ab}(x_1) F_{cd}(x_2) F_{cd}(x_3)\, \bar{\epsilon}\Gamma_b \chi(x_4).$$

Here the symmetry properties of D are not required. We conclude, that the fact that (3.3.10) is supersymmetric follows immediately from the supersymmetry of (3.2.8).

The above actually shows that when we replace D in (3.3.10) by *any* symmetric differential operator $\Delta(\partial_{x_1}, \ldots, \partial_{x_4})$, we obtain a supersymmetric action.

3.3.2 Derivative expansion

In this section we will consider the derivative expansion of the effective action (3.3.9). This will allow us to make contact with previously obtained results at order α'^4 as well as to present new results at order α'^5. But first let us discuss the form of the generic Lorentz invariant symmetric differential operator $\Delta(\partial_{x_1}, \ldots, \partial_{x_4})$ and determine the number of independent supersymmetric invariants that are possible at any given order in α'.

To find the form of $\Delta(\partial_{x_1}, \ldots, \partial_{x_4})$ we need the most general Lorentz invariant expression that is symmetric and regular in the momenta k_i, after which we substitute $k_i \to -i\partial_i$. In such an expression only combinations $k_i \cdot k_j$ and their products can enter[10]. Using momentum conservation and the mass-shell condition all such terms can be written as combinations of s, t, u. Any completely symmetric polynomial in s, t, u can be written as:

$$\sum_{k \le l \le m} \alpha'^{k+l+m}\, c_{k,l,m}\, \mathcal{P}(k,l,m), \tag{3.3.11}$$

where the $c_{k,l,m}$ are constants and

$$\mathcal{P}(k,l,m) = s^k t^l u^m + s^k t^m u^l + s^m t^k u^l + s^m t^l u^k + s^l t^m u^k + s^l t^k u^m. \tag{3.3.12}$$

Define

$$P(n) = s^n + t^n + u^n\,, \qquad Q = stu\,. \tag{3.3.13}$$

[10]We do not have to consider contractions with the ε-tensor, since all scalars that one can form by contracting it with the momenta k_i vanish.

$\mathcal{P}(k,l,m)$ can be expressed in terms of $P(n)$ and Q:

$$\mathcal{P}(k,l,m) = Q^k\big[P(l-k)P(m-k) - P(l+m-2k)\big]. \tag{3.3.14}$$

Furthermore, it follows from $P(1)P(n-1) = 0$ that

$$P(n) = \tfrac{1}{2}P(2)P(n-2) + QP(n-3). \tag{3.3.15}$$

We conclude that we can express (3.3.11) in powers of $P \equiv P(2)$ and Q:

$$\sum_{a,b} \alpha'^{2a+3b}\, d_{a,b}\, P^a Q^b, \tag{3.3.16}$$

where the $d_{a,b}$ are constants. The number $N_{P,Q}(m)$ of possible independent combinations of P and Q, at order α'^m in the above expansion, is given by

$$N_{P,Q}(m) = \begin{cases} [m/6]+1, & \text{if} \quad m \neq 6\times[m/6]+1 \\ [m/6]\,, & \text{if} \quad m = 6\times[m/6]+1, \end{cases} \tag{3.3.17}$$

where $[x]$ denotes the largest integer smaller than x.

This implies that, for a given m, there are $N_{P,Q}(m)$ independent supersymmetric contributions to the open string tree-level effective action that contain terms of the form $\partial^{2m}F^4$.

We now turn to the derivative expansion of (3.3.10). We use the Taylor expansion for $\log\Gamma(1+z)$,

$$\log\Gamma(1+z) = -\gamma z + \sum_{m=2}^{\infty}(-1)^m\zeta(m)\frac{z^m}{m}, \tag{3.3.18}$$

where $\zeta(n)$ is the Riemann zeta-function, γ the Euler-Mascheroni constant, to obtain the following expression for $G(s,t)$:

$$\alpha'^2\, G(s,t) = \frac{1}{st}\exp\left[\sum_{m=2}^{\infty}\alpha'^m\frac{\zeta(m)}{m}(s^m+t^m-(s+t)^m)\right]. \tag{3.3.19}$$

This expression can be used to calculate the α' expansion of $\mathcal{G}(k_1,\ldots,k_4)$. We give here the first terms in this expansion, expressed in P and Q:

$$
\begin{aligned}
\mathcal{G}(k_1,\dots,k_4) = {} \\
& -\tfrac{1}{2}\pi^2 - \tfrac{1}{48}\alpha'^2\pi^4\,P - \tfrac{1}{2}\alpha'^3\pi^2\zeta(3)\,Q \\
& -\tfrac{1}{960}\alpha'^4\pi^6\,P^2 - \tfrac{1}{48}\alpha'^5\pi^2\,\big[\pi^2\zeta(3)+12\,\zeta(5)\big]PQ \\
& -\tfrac{1}{967680}\alpha'^6\Big[51\pi^8\,P^3 + 8\pi^2\big[31\pi^6+30240\,\zeta(3)^2\big]\,Q^2\Big] \\
& -\tfrac{1}{960}\alpha'^7\pi^2\big[\pi^4\zeta(3)+10\,\pi^2\zeta(5)+120\,\zeta(7)\big]\,P^2Q \\
& -\tfrac{1}{58060800}\alpha'^8\Big[155\pi^{10}\,P^4 \\
& \qquad + 32\pi^2\big[67\pi^8+18900\,\pi^2\zeta(3)^2+453600\,\zeta(3)\zeta(5)\big]PQ^2\Big] \\
& -\tfrac{1}{967680}\alpha'^9\Big[\pi^2\big[51\pi^6\zeta(3)+504\,\pi^4\zeta(5)+5040\,\pi^2\zeta(7)+60480\,\zeta(9)\big]P^3Q \\
& \qquad + 8\,\pi^2\big[31\pi^6\zeta(3)+10080\,(\zeta(3)^3+2\,\zeta(9))\big]Q^3\Big] + \dots
\end{aligned}
\tag{3.3.20}
$$

We see that, at least to this order, all possible combinations of P and Q indeed appear. String theory thus seems to make use of all available superinvariants. By substituting derivatives for momenta in the above expansion and inserting the resulting expression in (3.3.10), one can straightforwardly construct the contribution to the effective action at any desired order in α'. We demonstrate this for the bosonic terms at order α'^4 and α'^5. At order α'^4 we obtain:

$$
\begin{aligned}
\mathcal{L}_{(4,4)} &= \tfrac{1}{288}\pi^4\alpha'^4\,t_{abcdefgh}\,\partial_kF_{ab}\partial_kF_{cd}\partial_lF_{ef}\partial_lF_{gh} \qquad (3.3.21)\\
&= \tfrac{1}{36}\pi^4\alpha'^4\Big[\big(\partial_kF_{ab}\partial_lF_{bc}\partial_kF_{cd}\partial_lF_{da} + 2\,\partial_kF_{ab}\partial_kF_{bc}\partial_lF_{cd}\partial_lF_{da}\big) \\
&\qquad - \tfrac{1}{4}\big(\partial_kF_{ab}\partial_kF_{ab}\partial_lF_{cd}\partial_lF_{cd} + 2\,\partial_kF_{ab}\partial_lF_{ab}\partial_kF_{cd}\partial_lF_{cd}\big)\Big].
\end{aligned}
$$

This expression is consistent with results obtained previously by different methods [52, 107]. The terms bilinear in the fermions agree with those obtained from a direct application of supersymmetry methods – see below and also [108]. We disagree however with [93].

At order α'^5 we obtain the following result:

$$
\begin{aligned}
\mathcal{L}_{(5,6)} &= -\tfrac{1}{6}\pi^2\zeta(3)\alpha'^5\,t_{abcdefgh}\,\partial_k\partial_l\partial_mF_{ab}\partial_kF_{cd}\partial_lF_{ef}\partial_mF_{gh} \qquad (3.3.22)\\
&= -4\pi^2\zeta(3)\alpha'^5\Big[\partial_k\partial_l\partial_mF_{ab}\partial_kF_{bc}\partial_lF_{cd}\partial_mF_{da} \\
&\qquad - \tfrac{1}{4}\,\partial_k\partial_l\partial_mF_{ab}\partial_kF_{ab}\partial_lF_{cd}\partial_mF_{cd}\Big].
\end{aligned}
$$

The bosonic part of this expression can be compared with a conjecture in [103]. Wyllard conjectured that all derivative corrections to the Born-Infeld action follow from the corrections to the Wess-Zumino term. He applied this conjecture in [103] using the results for the Wess-Zumino term of [102] as input. We have taken the six-derivative corrections given in formula (4.16) of [103], and extracted the terms of fourth order in F. We find:

$$\mathcal{L}_{(5,6)\,\mathrm{Wyllard}} = -4\pi^2\zeta(3)\,g^2\alpha'^5\Big[\partial_k\partial_l F_{ab}\partial_k\partial_m F_{bc}\partial_l\partial_m F_{cd}F_{da} - \tfrac{1}{4}\,\partial_k\partial_l F_{ab}\partial_k\partial_m F_{ab}\partial_l\partial_m F_{cd}F_{cd}\Big]. \quad (3.3.23)$$

This agrees, up to field redefinitions, with our result (3.3.22). However, this agreement should be interpreted with care. First of all the procedure of [103] involves an infinite series involving functional derivatives of the Born-Infeld action with respect to the field strength F. The conjecture requires an ordering prescription for these functional derivatives. For our comparison we have taken the simplest solution to this ordering ambiguity. Secondly, the corrections to the Wess-Zumino term in [102] are not complete. Other corrections, such as those evaluated in [104, 109, 110], will contribute as well. On applying Wyllard's proposal to these extra terms, further six-derivative corrections to the Born-Infeld term might be generated. Our agreement with [103] indicates that these extra terms do not give rise to new six-derivative F^4 terms in the Born-Infeld action.

3.4 The Noether procedure at α'^4 and higher

From the point of view of the Noether procedure, the terms $\mathcal{L}_{(m,2m-4)}$ that we constructed in the previous section constitute – together with the super-Maxwell action – the leading order contributions to genuine superinvariants that extend to all orders in the number of fields. We established that the number of these terms grow linearly with n. Two questions now come to mind.

The first question is whether the terms constructed using the symmetric non-local operator $\Delta(\partial_{x_1}, \ldots, \partial_{x_4})$ indeed give *all* possible superinvariants. We do not have a definite answer to this question; clearly methods beyond the Noether procedure are required. What we *can* say however is that even if there are any such invariants, string theory does not seem to make use of them. Moreover, we will pursue the Noether procedure to order α'^4 [c] and see that at least at this order there is only the term $\mathcal{L}_{(4,4)}$ that we already found in (3.3.21).

The second question is one that we already posed in the introduction of section 3.2: do the terms $\mathcal{L}_{(m,2m-4)}$ constitute the leading-order contributions to *separate* superinvariants, or does the Noether procedure relate these terms to each other at higher orders? Again, we have no definite answers, but is still interesting to speculate how this might come about.

3.4.1 Technical complications

Before turning our attention to the results of the Noether procedure at α'^4 and the general structure of the procedure at even higher orders, we pause to discuss the technical obstacles that one encounters when pursuing the method to higher orders.

First of all, the number of terms grows rapidly with the order of α'. For example, the problem of counting the number of bosonic terms in $\mathcal{L}_{(m,0)}$ – those terms with $(m+2)$ F's and no ∂'s – is equivalent to finding all integer partitions of $(m+2)/2$. There is a famous formula by Hardy and Ramanujan which provides an asymptotic formula for the number of partitions $p(n)$ of the integer n:

$$p(n) \sim \frac{\exp \pi \sqrt{2n/3}}{4n\sqrt{3}}, \quad \text{as } n \to \infty. \tag{3.4.1}$$

The possible number of terms with derivatives and/or fermions is even larger.

Another issue is that up to this point we have not used the fact that we are working in $d = 10$ (with the exception of the properties of the fermions of course). For example, when we wrote down the Ansatz (3.2.6) we assumed that $\operatorname{tr} F^4$ and $(\operatorname{tr} F^2)^2$ are independent. This is true in $d = 10$, but consider $d = 2$. We have:

$$\begin{aligned}
\operatorname{tr} F^2 &= F_{ab}F_{ab} = F_{01}F_{10} + F_{10}F_{01} = -2(F_{01})^2, \\
\operatorname{tr} F^4 &= F_{ab}F_{bc}F_{cd}F_{da} = F_{01}F_{10}F_{01}F_{10} + F_{10}F_{01}F_{10}F_{01} = 2(F_{01})^4,
\end{aligned}$$

and thus $\operatorname{tr} F^4 = \frac{1}{2}(\operatorname{tr} F^2)^2$ in $d = 2$. Similar identities exist in higher dimensions, but for higher values of m. Obtaining them by writing out the summations is quite tedious, but fortunately there is a nice trick. Consider again $d = 2$. Instead of working out the traces we consider expressions that are antisymmetrized over 3 indices and hence identically zero. For $m = 2$ there is only one expression that contains nontrivial information:

$$\begin{aligned}
0 &\equiv 2F_{[ab}F_{c]d}F_{ab}F_{cd} \\
&= F_{ab}F_{cd}F_{ab}F_{cd} + F_{bc}F_{ad}F_{ab}F_{cd} + F_{ca}F_{bd}F_{ab}F_{cd} = (\operatorname{tr} F^2)^2 - 2\operatorname{tr} F^4.
\end{aligned}$$

In order to construct such *Schouten identities* for general d we need terms with at least $2(d+1)$ indices. One can show that this implies for the bosonic terms $m \geq d-1$, whereas for the terms bilinear in the fermions Schouten identities already exist for $m \geq (d+1)/2$ for d odd and $m \geq (d+2)/2$ for d even. For $d = 10$ this means that we would have to include additional identities when pursuing the Noether procedure at α'^6 and higher. In this thesis we will only go as high as α'^4, though.

The ABRA program

Clearly at some point the use of a computer becomes unavoidable. For example, where the number of terms at intermediate stages of the calculation is in the order of 10^2 for the α'^2 case, we need to manipulate 10^4 terms in the α'^4 case.

We used the computer program ABRA developed by M. de Roo for the calculations at order α'^3 (nonabelian) and α'^4 (abelian). ABRA is a computer algebra system which was designed for calculations in component supersymmetry and was originally written for the work of [111, 112]. Basically, the program provides an environment in which the user can perform calculations in much the same way as on a piece of paper. There is a list of terms which are manipulated by the program. Some of these manipulations are performed automatically. For instance, a very time-consuming part of calculations with tensors is the relabeling of dummy indices at each step of the calculation. ABRA contains an algorithm which takes care of this, taking into account the symmetry properties of the tensors. Other manipulations are performed only at the express command of the user, for instance partial integrations, the 'flipping' of Majorana fermions, the use of the Γ-matrix identities (A.1.43), (A.1.44), (A.1.46), and the use of the Bianchi identity $\partial_a F_{bc} = -2\partial_{[b}F_{c]a}$. ABRA provides a crude but effective sorting mechanism which allows the user to reorganize the terms in the list and to perform manipulations on groups of terms of the same structure.

The construction of lists of terms and identities as in section 3.2.2 is still done by hand.

3.4.2 Order α'^3

At order α'^3 there are no supersymmetric contributions. This might be inferred by taking the abelian limit of the result of chapter 4 or [113]. However, since it is not obvious that every supersymmetric abelian action allows a nonabelian supersymmetric extension, it is important to check this directly in the abelian context. This has been done in [114] by superspace methods, and we have verified this result by an independent calculation using the Noether procedure.

3.4.3 Order α'^4

There are three nontrivial sectors in the Ansatz at order α'^4: $\mathcal{L}_{(4,0)}$, $\mathcal{L}_{(4,2)}$ and $\mathcal{L}_{(4,4)}$, since – according to section 3.2.1 – the structures with more derivatives are removable by field redefinitions. It turns out that this is also the case for the bosonic terms in $\mathcal{L}_{(4,2)}$ (the terms $\partial^2 F^5$) but not for the terms with fermions, which are of the form $\partial^3 F^3 \bar{\chi}\Gamma\chi$.

In applying the Noether method to the case $(4,0)$ we need the variations $\delta_0\mathcal{L}_{(4,0)}$ as well as $\delta_2\mathcal{L}_{(2,0)}$. In the cases $(4,2)$ and $(4,4)$ only the variation δ_0 is needed.

The results of the Noether procedure are the following: in the sector $\mathcal{L}_{(4,0)}$ with F^6 and $\partial F^4\bar{\chi}\Gamma\chi$ the only terms allowed by supersymmetry are those needed for the 'continuation' of the invariant of order α'^2, i.e. the Born-Infeld invariant [115]. There does not appear a new, independent invariant. Furthermore, the fermionic terms in $\mathcal{L}_{(4,2)}$ are not supersymmetrizable and thus $\mathcal{L}_{(4,2)} = 0$. Finally, in the sector $\mathcal{L}_{(4,4)}$

we find indeed a single unique superinvariant as promised. The action at order α'^4 is given by:

$$\mathcal{L}_4 = \mathcal{L}_{(4,0)} + \mathcal{L}_{(4,4)}, \tag{3.4.2a}$$

with

$$\begin{aligned}\mathcal{L}_{(4,0)} = \frac{(a_{(2,0)})^2\alpha'^4}{384}\Big[&- 32\, F_{ab}F_{bc}F_{cd}F_{de}F_{ef}F_{af} - 12\, F_{ab}F_{bc}F_{cd}F_{ad}F_{ef}F_{ef} \\ &- F_{ab}F_{ab}F_{cd}F_{cd}F_{ef}F_{ef} - 12\,\partial_a F_{bc}F_{de}F_{af}F_{be}\,\bar\chi\Gamma_{cdf}\chi \\ &+ 72\, F_{ab}F_{cd}F_{be}F_{de}\,\bar\chi\Gamma_a\partial_c\chi + 18\,\partial_a F_{bc}F_{de}F_{ef}F_{af}\,\bar\chi\Gamma_{bcd}\chi \\ &+ 12\,\partial_a F_{bc}F_{de}F_{bf}F_{ae}\,\bar\chi\Gamma_{cdf}\chi\Big],\end{aligned} \tag{3.4.2b}$$

and

$$\begin{aligned}\mathcal{L}_{(4,4)} = a_{(4,4)}\alpha'^4\Big[&- 8\, F_{ab}F_{bc}\partial_d\partial_e F_{af}\partial_d\partial_e F_{cf} - 8\, F_{ab}\partial_c F_{ad}\partial_e F_{bf}\partial_c\partial_e F_{df} \\ &+ 32\, F_{ab}\partial_c F_{ad}\partial_e F_{bf}\partial_d\partial_e F_{cf} + 16\, F_{ab}\partial_c F_{de}\partial_a F_{ef}\partial_d\partial_f F_{bc} \\ &+ 4\,\partial_a\partial_b F_{cd}\partial_a\partial_b F_{ce}\,\bar\chi\Gamma_d\partial_e\chi - 4\,\partial_a F_{bc}\partial_a\partial_d F_{ef}\,\bar\chi\Gamma_{bef}\partial_c\partial_d\chi \\ &+ 4\, F_{ab}\partial_c\partial_d F_{ef}\,\bar\chi\Gamma_{abe}\partial_c\partial_d\partial_f\chi + 8\, F_{ab}\partial_c\partial_d F_{ae}\,\bar\chi\Gamma_b\partial_c\partial_d\partial_e\chi \\ &+ 2\,\partial_a F_{bc}\partial_a\partial_d\partial_e F_{bc}\,\bar\chi\Gamma_d\partial_e\chi\Big].\end{aligned} \tag{3.4.2c}$$

We have also determined the modified transformation rules for both linear at nonlinear supersymmetry at this order and verified the closure of the supersymmetry algebra. See [c] for details.

3.4.4 Higher orders

Our present knowledge of the bosonic contributions to the open string effective action is summarized in figure 3.4.1. The sectors (m, n) for which the bosonic terms are known are indicated with a black dot. The sectors indicated by white dots are known not to contain any bosonic terms. The gray dots correspond to sectors which are known to be nonempty but of which the explicit form is unknown. They contain the higher-derivative contributions of the string $2k$-point functions with $k \geq 3$.

For the sectors $(m, 0)$ and $(m, 2m-4)$, which correspond to the Born-Infeld action (3.1.6) and the 4-point effective action (3.3.9) respectively, the fermionic terms are also known and supersymmetry has been established. For the sectors $(m, 4)$ with $m \geq 6$ and even, the fermionic contributions have yet to be constructed.

The absence of fermionic terms for the diagonal lines $(m, 2m - 2p + 4)$ with p odd, follows from the invariance of the effective action under Ω whereas for the line $(m, 2m)$ with $m > 0$ we showed this in section 3.2.1. Note however that the absence of contributions to $(3, 0)$ and $(4, 2)$ is also required by the Noether procedure. It

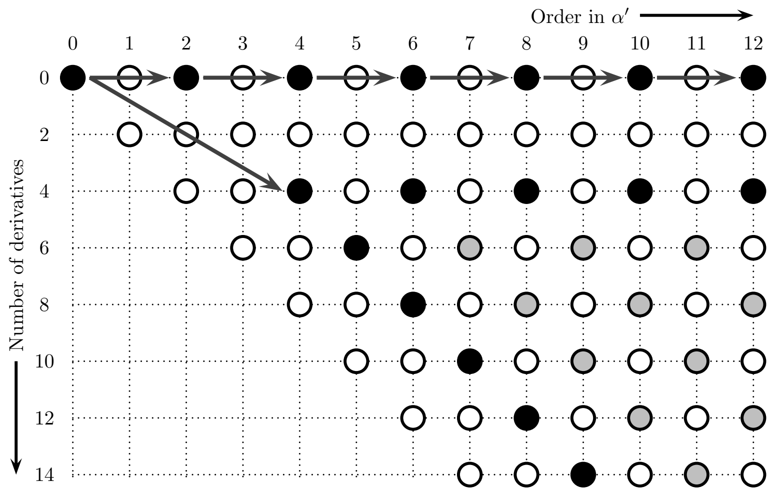

Figure 3.4.1. The structure of the abelian open superstring tree-level effective action. Depicted are the bosonic contributions. See the main text for an explanation of this figure.

would be interesting to see whether supersymmetry excludes also the other terms with $p = 3$ (or even the terms with p odd in general), or that the invariance under Ω is a necessary ingredient.

Though the absence of bosonic terms for the points $(m, 2)$ with m was established in [52], it is not known whether these sectors are also devoid of fermionic contributions – this with the exception of $(3, 2)$ for which we showed this using the 4-point function. It is in this light an interesting question whether the absence of bosonic terms implies in general the absence of fermionic contributions, either by pursuing the Noether procedure to higher orders or by some direct string theory calculation.

We now focus our attention on the supersymmetry transformations that connect the dots in figure 3.4.1. We have drawn arrows to indicate some of the known supersymmetry transformations. As a first example we consider the terms $(m, 0)$, m even, i.e., the Born-Infeld invariant. These terms are invariant under the transformations $\delta_0, \delta_{(2,0)}, \delta_{(4,0)}, \ldots$, depending only on the single parameter $a_{(2,0)}$. Note that we indicate these transformations by a repeated addition of the same arrow, and *not* by drawing new arrows from $(0, 0)$ to $(m, 0)$ for each m. In this way we denote that all these terms contribute to the same invariant. Similarly, the point $(4, 4)$ is the leading term in a new sequence of supersymmetry transformations that continues to the points $(4k, 4k)$, involving the parameter $a_{(4,4)}$. Similarly, all points on the diagonal

$(m, 2m-4)$ will lead to new arrows, involving new parameters $a_{(m,2m-4)}$[11] The question is now, whether these 'independent' invariants will remain independent when the Noether procedure is pursued to higher orders. Consider for example the point $(8, 8)$. This point can be reached from $(0, 0)$ by applying the arrow $(4, 4)$ twice, but also by applying the arrow $(2, 0)$ and then $(6, 8)$ (or vice versa). These contributions need to be canceled by the δ_0 variation of $\mathcal{L}_{(8,8)}$. In principle, there are now two possibilities: they can either be canceled separately, or not. In the latter case we need both contributions at the same time, and then there must be relations between the coefficients $a_{(4,4)}a_{(4,4)}$ and $a_{(2,0)}a_{(6,8)}$. So it is indeed possible that a priori independent invariants are related to each other at higher orders in the iteration. Note however that at least $a_{(2,0)}$ and $a_{(4,4)}$ will remain independent to all orders. The reason is that $a_{(2,0)}$ and $a_{(4,4)}$ can be changed independently by rescaling α' and the extra derivatives, respectively[12].

Now we invoke our knowledge of the string tree-level 4-point function. The invariants corresponding to P^m with m even in (3.3.20) have coefficients $a_{(m,2m-4)}$ which contain factors π^m. The relations between these coefficients alluded to in the previous paragraph are therefore in principle possible.

For the invariants involving powers of Q the situation is different, since the coefficients of these invariants involve $\zeta(n)$ for n odd. Consider for example the point $(9, 10)$. We can reach it for instance with supersymmetry transformations through the Q-invariant at $(5, 6)$ by applying $\delta_{(4,4)}$ and from the P-invariant at $(4, 4)$ by applying $\delta_{(5,6)}$. This gives rise to terms proportional to $a_{(4,4)}a_{(5,6)} \sim \pi^6\zeta(3)$. But $(9, 10)$ can also be reached from the PQ-invariant at $(7, 10)$ by applying $\delta_{(2,0)}$ or vice versa, yielding terms proportional to $a_{(2,0)}a_{(7,10)} \sim \pi^4(\pi^2\zeta(3) + \zeta(5))$. Since there are no (known) relations between the values of the Riemann ζ-function for odd values of its argument, this implies that the Noether procedure can never determine the coefficient $a_{(7,10)}$ in terms of $a_{(2,0)}$, $a_{(4,4)}$ and $a_{(5,6)}$.

We thus conclude the following: although it might be that certain terms, which appear to be independent superinvariants at a given order in the iteration, are related to each other at higher orders in the Noether procedure, there will always remain an infinite number of all-order invariants, corresponding to contributions which have factors of $\zeta(2n+1)$ for different n in the 4-point function. Such terms appear at any odd order in α'.

[11] The number of these parameters is given in (3.3.17).

[12] This is no longer true in the nonabelian case, since $[D, D]F = [F, F]$. In this case one would expect that the coefficients $a_{(2,0)}$ and $a_{(4,4)}$ are related to each other, as is indeed the case [107]. We will see another example of this at order α'^3 in the next chapter .

3.5 Outlook

The results of the previous section hardly look promising. Not only is it not feasible to continue the Noether procedure much further due to the rapidly increasing number of terms in the Lagrangian at higher orders in α', but the observation that there appear *new* independent all-order superinvariants at any odd order in α' also seems to exclude the possibility of using clever recursive arguments based on results obtained at low orders in α'.

In order to make further progress other methods are clearly required. One thing that immediately comes to mind is a further investigation of the string theory tree-amplitudes. For example, a detailed study of the six-point function should provide answers to many of the questions we raised in the previous section. Another possibility is to reorganize the Noether procedure by setting it up in $d = 10$ $\mathcal{N} = 1$ on-shell superspace. A clear advantage of this setting is that field redefinition ambiguities do not arise, since all fields are constrained to satisfy their lowest order equations of motion.

Finally, the persistence of the non-linear supersymmetry in the higher-derivative terms is a strong indication that a κ-symmetric formulation of the all-order effective action exists. Given the success of κ-symmetry in clarifying the structure of the supersymmetric Born-Infeld action [98–101], it is conceivable, if not likely, that it will yield similar striking results when applied to this problem.

We end this chapter with some wishful thinking. We have shown that the effective action for the 4-point function is given by a differential operator acting on the F^4 term and that supersymmetry followed “under the derivatives”. The best-case scenario would be that this structure generalizes to the *full* effective action, i.e. that we would obtain the complete effective action as some non-local derivative operator acting on an expression derived from the Born-Infeld action and that supersymmetry (and κ-symmetry) would follow from the supersymmetry of the Born-Infeld action.

Chapter 4

Multiple D9-branes

We now turn to the effective action of a stack of D9-branes. After a brief review we turn our attention toward our own contribution: the supersymmetric completion of the α'^3-terms [a]. We then test the fermionic contributions of this result against string theory [b]. We will only consider flat closed string backgrounds in this chapter.

4.1 The non-abelian Born-Infeld action

The work of Polchinski [67] and Witten [58] caused a renewed interest in the structure of the nonabelian open superstring tree-level effective action. A considerable step forward was made by Tseytlin [116] (see also [117]). Previous to [116] only terms at α'^2 were known completely [49,95] and partial results had been obtained at α'^3 [118].

Recall that the identity $[D_a, D_b]F_{cd} = [F_{ab}, F_{cd}]$ implies that the notion of slowly varying fields is well defined only when the commutators of the field strength are very small. This approximation is only of limited use. When one is interested in e.g. the equations of motion or the spectrum of small fluctuations around a fixed background, one needs to vary the action once or twice w.r.t. A_a and there is no reason to include only terms without derivatives. Consider for example the terms F^6 and $DFDFF^3$. These both give contributions of the form F^4DF when varied once.

However, regardless of its usefulness for practical applications, the $DF = 0$ limit has one big virtue: it allows one to obtain an all-order result, just as the $\partial F \to 0$ limit allowed this in the abelian case. Tseytlin showed in [116] that, *in the limit* $D_aF_{bc} = [F_{ab}, F_{cd}] = 0$, the bosonic part of the open string effective action is given by the following non-abelian generalization of the Born-Infeld action:

$$S_{\rm NBI} = -\tau_{\rm D9}\int {\rm d}^{10}x\,{\rm STr}\,\sqrt{-\det\left(\eta_{ab} + 2\pi\alpha' F_{ab}\right)}. \qquad (4.1.1)$$

This tree-level effective action arises from the disk diagram in string theory and hence contains a single trace over the $U(N)$ generators λ^A. Since the field strengths are effectively abelian, the generators are necessarily ordered symmetrically inside the trace:

$$\mathrm{STr}\,\lambda^{A_1}\cdots\lambda^{A_n} \equiv \frac{1}{n!}\sum_{\pi}\mathrm{Tr}\,\lambda^{A_{\pi(1)}}\cdots\lambda^{A_{\pi(n)}}. \tag{4.1.2}$$

The symmetrized trace is imposed after working out the expansion of the square root and determinant. The first three terms read (dropping the constant term):

$$\begin{aligned} S_{\mathrm{NBI}} = \frac{1}{g^2}\int \mathrm{d}^{10}x \Bigg[&\frac{1}{4}\mathrm{Tr}\,F_{ab}F_{ab} + (2\pi\alpha')^2\,\mathrm{STr}\left(\frac{1}{8}\mathrm{tr}\,F^4 - \frac{1}{32}(\mathrm{tr}\,F^2)^2\right) \\ &+ (2\pi\alpha')^4\,\mathrm{STr}\left(\frac{1}{12}\mathrm{tr}\,F^6 - \frac{1}{32}\mathrm{tr}\,F^4\,\mathrm{tr}\,F^2 + \frac{1}{384}(\mathrm{tr}\,F^2)^3\right) + \mathcal{O}(\alpha'^6)\Bigg]. \end{aligned} \tag{4.1.3}$$

The terms at α'^2 agree with those obtained from the string theory 4-point function in [49], see also [93, 106]. The supersymmetrization of these terms was obtained in [95, 97, 119]. The terms at α'^4 agree with those obtained in [107].

Although the action (4.1.1) reproduces the leading order contribution to the string theory 4-point function, it was already mentioned in [116] that this is no longer the case for the 5-point function[1], i.e. at order α'^3. Moreover, it was shown in [120] that the spectrum of small fluctuations around a fixed background predicted by (4.1.1) disagrees with the string theory spectrum at α'^4.

The first corrections to the action (4.1.1) appear at order α'^3. In the following we will derive the bosonic terms and the terms that are quadratic in the fermions that contribute at this order. We will use the Noether procedure to show that there is one unique superinvariant at this order [a], which is – as usual – determined up to a multiplicative constant. This constant can be determined from the string theory 4-point function. The bosonic terms at α'^3 had been obtained already in [113,121,122] by indirect methods, whereas partial results had been obtained in [93] from the string theory 4-point function.

More work on the nonabelian case was done after our paper [a] appeared. In [123] all the bosonic terms at α'^3 were obtained directly from the string 5-point function, improving on earlier partial results [118]. The bosonic terms at order α'^4 were obtained in [107], see [108] for a discussion of these terms from the point of view of $\mathcal{N}=4$, $d=4$ super-Yang-Mills theory.

In the previous chapter we derived an all-order result from the string theory 4-point function in the abelian case [d]. A similar result was subsequently derived in [124] for the terms of the form $D^{2n}F^4$. Since the ordering of the covariant derivatives is not fixed by the 4-point function, the authors of [124] made a *choice* for this

[1] The amplitudes involving an *odd* number of fields do not vanish in the nonabelian case, except for the three-point function, which still vanishes on-shell.

ordering. This choice affects the form of the terms with more F's through the identity $[D,D]F=[F,F]$. The result of [124] was subsequently extended to the terms of the form $D^{2n}F^5$ (but not their fermionic counterparts) by means of the 5-point function in [125]. The result of [125] is quite complicated, but agrees with the α'^3 contributions that were obtained earlier. It would be interesting to see whether this agreement still holds for the terms at α'^4 of [107].

In the following we will not discuss these recent results in any detail, but focus on our work in [a]. We will first delve a bit deeper into the complications that arise in the non-abelian case and spend some time on the string theory 4-point function. After that we proceed with the Noether procedure at α'^3.

4.2 α'-corrections to super-Yang-Mills theory

We will use the same notation as in the previous chapter and write terms in the effective action as

$$\mathcal{L}_{\text{eff}} = \frac{1}{g^2}\sum_{m,n}\mathcal{L}_{(m,n)}, \quad \text{with} \quad \mathcal{L}_{(m,n)} = \alpha'^m \operatorname{Tr}\left(D^n F^p + D^{n+1}F^{p-2}\bar{\chi}\Gamma\chi\right), \tag{4.2.1}$$

with a *single* trace over the Yang-Mills generators and $2p-2m+n-4=0$. We restrict ourselves to terms that are at most quadratic in the fermions throughout this chapter. The differences with the abelian case are essentially threefold.

First of all, we already mentioned several times that the terms $\mathcal{L}_{(m,n)}$ with m fixed are not independent, because the identity

$$[D_a, D_b]F_{cd} = [F_{ab}, F_{cd}]. \tag{4.2.2}$$

allows us to trade terms that contain covariant derivatives for terms without derivatives. We resolve this ambiguity by always using this relation as "LHS→RHS". In calculations we thus first deal with the terms that involve covariant derivatives. We treat the derivatives as if they commuted, but keep track of the $[F,F]$ commutators. Only after all the necessary manipulations on the terms with derivatives have been done, do we proceed to the terms without derivatives. We will see below that this rule allows us to remove the terms with $p=2$ and $p=3$ – with the exception of $\operatorname{Tr} F^3$, see below – just as we did in section 3.2.1 for the abelian case. Another consequence of the relation (4.2.2) is that terms which formed *independent* superinvariants in the abelian case, may become part of a single superinvariant in the nonabelian case. There may thus be less superinvariants in the nonabelian case. This becomes relevant at α'^4, where we found two independent superinvariants in the previous chapter, one with the coefficient $a_{(2,0)}$ and the other with $a_{(4,4)}$. The Noether procedure might put $a_{(4,4)} \sim a_{(2,0)}^2$ in the nonabelian case. We have not pursued the Noether procedure through α'^4, so we do not have a definite answer to this question. However, in [107] the bosonic terms at α'^4 were obtained by a different iterative procedure and found to depend only on $a_{(2,0)}$. It is thus very likely that the Noether procedure will give the same result.

Secondly, we need to worry about the ordering of the Yang-Mills generators inside the trace. We will see below how this information can be obtained from the 4-point function for the terms with $p = 4$.

Finally, because the covariant derivative contains the gauge field, it is no longer the case that terms of a given p contribute only to the 1PI p-point function. For example, the $d = 10$, $\mathcal{N} = 1$ supersymmetric Yang-Mills Lagrangian $\mathcal{L}_{(0,0)} \equiv \mathcal{L}_0$, which reads

$$\mathcal{L}_0 = -\mathrm{Tr}\Big[-\frac{1}{4}F_{ab}F_{ab} + \frac{1}{2}\bar{\chi}\not{D}\chi\Big], \tag{4.2.3}$$

contributes not only a 2-point function, but also a 1PI 3-point vertex and a 1PI 4-point vertex. In fact, the *entire* string theory 3-point function is already reproduced by $\mathcal{L}_0$ [14, 15, 105].

Before we proceed with the α' corrections to $\mathcal{L}_0$ we note that the lowest order equations of motion read

$$D_a F^A_{ab} - \frac{1}{2}f^{ABC}\,\bar{\chi}^B\Gamma_b\chi^C = 0, \qquad \text{and} \qquad \not{D}\chi^A = 0, \tag{4.2.4}$$

and that $\mathcal{L}_0$ is invariant under supersymmetry transformations:

$$\delta_0 A_a = \bar{\epsilon}\Gamma_a\chi, \qquad \text{and} \qquad \delta_0\chi = \frac{1}{2}F_{ab}\,\Gamma_{ab}\epsilon, \tag{4.2.5}$$

which satisfy the supersymmetry algebra:

$$[\delta_{0\epsilon_1}, \delta_{0\epsilon_2}]A_a = 2\bar{\epsilon}_1\not{\partial}\epsilon_2 A_a - D_a\left(2\bar{\epsilon}_1\not{A}\epsilon_2\right), \tag{4.2.6a}$$

$$[\delta_{0\epsilon_1}, \delta_{0\epsilon_2}]\chi = 2\bar{\epsilon}_1\not{D}\epsilon_2\chi - \left(\frac{7}{8}\bar{\epsilon}_1\Gamma_a\epsilon_2\Gamma_a - \frac{1}{5!16}\bar{\epsilon}_1\Gamma_{a_1\cdots a_5}\epsilon_2\Gamma_{a_1\ldots a_5}\right)\not{D}\chi. \tag{4.2.6b}$$

Note that the RHS of (4.2.6b) involves a gauge transformation of the fermion, this in contrast to (3.2.4b).

4.2.1 The terms with $p = 2$ and $p = 3$

We now show that there are indeed no terms with $p = 2$ or $p = 3$ in the nonabelian D9-brane effective action if we use our rule $[D, D]F \to [F, F]$. Contracted covariant derivatives acting on the same field can be removed by a field redefinition, modulo terms with fewer derivatives. We have in particular (with $\Box_{\mathrm{cov}} \equiv D_a D_a$):

$$\Box_{\mathrm{cov}} F_{ab} = D_a(D_c F_{cb}) - D_b(D_c F_{ca}) + 2[F_{ac}, F_{bc}],$$

$$\Box_{\mathrm{cov}}\chi = \not{D}\not{D}\chi - \frac{1}{2}[F_{ab}, \Gamma_{ab}\chi].$$

We conclude from (4.2.4) that terms which contain a factor $D_a F_{ab}$ can removed by a field redefinition at the expense of modifying the terms which are of higher order in the fermions. When we discuss Ansätze for the Noether procedure we can ignore this issue, since it only replaces one undetermined coefficient by another.

The absence of the terms with $p = 2$ is the established in the same way as in the abelian case. The terms with $p = 3$ are more complicated, since we now need to take into account the different inequivalent orderings of the Yang-Mills generators.

First we deal with the fermionic terms, which are of the form $\mathrm{Tr}\, D^{n+1} F \bar{\chi} \Gamma \chi$. Any derivative on F can be partially integrated, the only remaining terms that do not contain lowest-order equations of motion are then

$$\mathrm{Tr}\, \lambda^A \lambda^B \lambda^C \; F^A_{ab}\, D^k \bar{\chi}^B \Gamma_a D^k D_b \chi^C,$$
$$\mathrm{Tr}\, \lambda^A \lambda^C \lambda^B \; F^A_{ab}\, D^k \bar{\chi}^B \Gamma_a D^k D_b \chi^C.$$

Note that according to our rule the covariant derivatives can be treated as commuting. These terms can both be related to terms that are "on-shell" by means of partial integrations. To see this consider the following total derivatives:

$$\partial_a \big[\, \mathrm{Tr}\, F_{ab}\, D^k \bar{\chi} \Gamma_b D^k \chi \big] = \mathrm{Tr}\, \lambda^A [\lambda^B, \lambda^C]\, F^A_{ab}\, D^k \bar{\chi}^B \Gamma_a D^k D_b \chi^C + \ldots,$$
$$\partial_a \big[\, \mathrm{Tr}\, F_{bc}\, D^k \bar{\chi} \Gamma_{abc} D^k \chi \big] = \mathrm{Tr}\, \lambda^A \{\lambda^B, \lambda^C\}\, F^A_{ab}\, D^k \bar{\chi}^B \Gamma_a D^k D_b \chi^C + \ldots,$$

where the dots indicate terms that can be removed by field redefinitions and terms with a lower number of derivatives. Note that we constructed these total derivatives such that they do not give rise to terms with uncontracted derivatives on F. All fermionic terms with $p = 2$ can thus be removed from the action.

Now for the bosonic terms. As in the abelian case, we can always reduce these to the form $F_{ab} D^k D_c F_{da} D^k D_c F_{db}$ or $F_{ab} D^k D_c F_{da} D^k D_d F_{cb}$. In the abelian case these vanished by antisymmetry but here we need to take into account the ordering of the generators. The different orderings are:

$$\mathrm{Tr}\, F_{ab} [D^k D_c F_{da}, D^k D_c F_{db}] = \mathrm{Tr}\, F_{ab} [D_e D^{k-1} D_c F_{da}, D_e D^{k-1} D_c F_{db}],$$
$$\mathrm{Tr}\, F_{ab} [D^k D_c F_{da}, D^k D_d F_{cb}] = \mathrm{Tr}\, F_{ab} [D_e D^{k-1} D_c F_{da}, D_e D^{k-1} D_d F_{cb}].$$

Now use $2[D_a \Phi, D_a \Psi] = \Box[\Phi, \Psi] - [\Box\Phi, \Psi] - [\Phi, \Box\Psi]$ and partially integrate the first term that is obtained by means of this identity to see that all bosonic terms with $p = 3$ can be removed, with the exception of

$$\mathrm{Tr}\, F_{ab} F_{bc} F_{ca} = -\frac{1}{2} f^{ABC} F^A_{ab} F^B_{bc} F^C_{ca}. \tag{4.2.7}$$

This remaining term would contribute to a 3-point function. We however mentioned that the string theory 3-point function is reproduced already completely by $\mathcal{L}_0$. It turns out that (4.2.7) is in fact inconsistent with supersymmetry. Indeed, varying this term w.r.t. (4.2.5) yields

$$-3f^{ABC}F^A_{ac}F^B_{cb}\,\bar\epsilon\Gamma_a D_b\chi^C.$$

One can show – either by inspection or by explicit construction of the total derivatives – that this term cannot be related by means of partial integrations solely to terms that contain either $\not{D}\chi$ or D_aF_{ab}. It can therefore not be canceled by a modification of the supersymmetry transformation rules[2] of order α'.

4.2.2 The 4-point function, revisited

We now focus our attention toward the string theory 4-point function. As in the abelian case we will be able to extract useful information about the structure of the effective action from the 4-point function. In the nonabelian case the 4-point function takes on the following form:

$$\begin{aligned}\mathcal{A}_4 = -8i\,g^{-2}\alpha'^2(2\pi)^{10}\delta^{(10)}(k_1+k_2+k_3+k_4)\,K(1,2,3,4)\times \\ \times\left(T_1^{ABCD}G(s,u)+T_2^{ABCD}G(s,t)+T_3^{ABCD}G(t,u)\right).\end{aligned} \quad (4.2.8)$$

See section 3.3 for an explanation of the different factors appearing in this expression. As explained in chapter 1, the only difference with the abelian amplitude (3.3.1) are the traces over the Chan-Paton factors:

$$\begin{aligned}T_1^{ABCD} &= \mathrm{Tr}\,\lambda^A\lambda^B\lambda^C\lambda^D + \mathrm{Tr}\,\lambda^A\lambda^D\lambda^C\lambda^B,\\ T_2^{ABCD} &= \mathrm{Tr}\,\lambda^A\lambda^B\lambda^D\lambda^C + \mathrm{Tr}\,\lambda^A\lambda^C\lambda^D\lambda^B,\\ T_3^{ABCD} &= \mathrm{Tr}\,\lambda^A\lambda^C\lambda^B\lambda^D + \mathrm{Tr}\,\lambda^A\lambda^D\lambda^B\lambda^C.\end{aligned} \quad (4.2.9)$$

The last factor in A_4 can be written as a sum of terms that are proportional to $T_1+T_2+T_3$ (which is the symmetric trace) and T_i-T_j (which can be written in terms of structure constants only):

$$\begin{aligned}&\tfrac{1}{3}\big(T_1-T_2\big)\big[G(s,u)+G(t,u)-2G(s,t)\big]\\ &\quad+\tfrac{1}{3}\big(T_1-T_3\big)\big[G(s,u)+G(s,t)-2G(t,u)\big]\\ &\quad\quad+\tfrac{1}{3}\big(T_1+T_2+T_3\big)\big[G(s,u)+G(t,u)+G(s,t)\big].\end{aligned} \quad (4.2.10)$$

We recall the α'-expansion of $G(s,t)$:

$$\begin{aligned}\alpha'^2G(s,t) = \frac{1}{st}-\frac{1}{6}\alpha'^2\pi^2-\alpha'^3\zeta(3)(s+t)-\frac{1}{360}\alpha'^4\pi^4(4s^2+st+4t^2)\\ +\alpha'^5\left[\frac{1}{6}\pi^2\zeta(3)st(s+t)-\zeta(5)(s+t)(s^2+st+t^2)\right]+\ldots.\end{aligned} \quad (4.2.11)$$

[2]Note that, in general, terms that contain D_aF_{ab} would have to combine with terms of higher order in the fermions into the lowest order equation of motion (4.2.4). We do not see this in our quadratic fermion approximation, though.

In contrast to the abelian case, there is a contribution from the 4-point function at α'^0. This contribution is reproduced by completely by $\mathcal{L}_0$, *via* the A^4 vertex and a reducible diagram involving three-point vertices. The terms of higher order in α' are always reproduced by terms of the form $\alpha'^m D^{2m-4} F^4$.

For the abelian case we were able to show by using the 4-point function that the terms with $p = 4$ contain the leading order contributions to an infinite number of superinvariants. We speculated that some of these superinvariants might be related to each other at higher orders in the Noether procedure, but argued that there will always remain an infinite number of all-order invariants, corresponding to contributions which have factors of $\zeta(2n+1)$ for different n in the 4-point function. These terms appeared at any odd order in α'. We also mentioned earlier in this section that certain terms which contribute to *independent* superinvariants in the abelian case, will become part of a single superinvariant in the nonabelian case because of relation (4.2.2).

We might therefore have hoped that the Noether procedure is actually simpler in the nonabelian case. Sadly, this is not the case. There is in fact an additional infinite number of superinvariants, which vanish in the abelian case. We can already see this in the 4-point function, which contains terms that can be written entirely in terms of structure constants, i.e. the terms with $T_i - T_j$ in (4.2.10). These terms appear in particular at every *odd* order in α'. Indeed, the leading term in the expansion of $G(s,t)$ that contains a factor $\alpha'^n \zeta(n)$ for given n is proportional to $(s^n + t^n - (s+t)^n)/st$, see (4.2.11) and (3.3.19). For n *odd* this is of the form

$$(s+t)P_n, \qquad \text{with} \qquad P_n = -\frac{s^n + t^n + u^n}{stu}. \tag{4.2.12}$$

Now in (4.2.10) the symmetric trace is proportional to

$$G(s,t) + G(s,u) + G(t,u). \tag{4.2.13}$$

If we expand this in α', the leading term with the coefficient for n odd gives a factor:

$$(s+t)P_n + (s+u)P_n + (t+u)P_n = 2(s+t+u)P_n \equiv 0. \tag{4.2.14}$$

Therefore, at every α'^n for n odd there appears a new superinvariant, of which the leading order contribution can be expressed in terms of structure constants only and therefore vanishes in the abelian limit.

4.2.3 The Noether procedure at α'^3

We have seen that at α' there are no contributions to the effective action and that the first nontrivial corrections to $\mathcal{L}_0$ appear at α'^2. These corrections are essentially of the form (3.2.8) but involve covariant instead of ordinary derivatives and a symmetrized trace over the gauge group. There are thus also nontrivial corrections to the supersymmetry transformations rule, but because of the absence of contributions at order α' these α'^2 corrections cannot contribute to the order α'^3 variations.

In the previous section we concluded that the trace over the Yang-Mills generators of the terms of the form $\alpha'^3 D^2 F^4$ and their fermionic partners can be written entirely in terms of structure constants. Not only does the Noether procedure confirm this, it turns out that the trace structure of the terms of the form $\alpha'^3 F^5$ can also be written entirely in terms of structure constants.

As we mentioned before, the Ansatz for the Noether procedure is not unique because of (4.2.2). We use our aforementioned rule and obtain an Ansatz with 13 bosonic terms and 110 terms involving fermions: $7+18$ terms of the form $(DF)^2F^2$ and fermionic partners, and $6+92$ of type F^5 with partners.

After simplifying the resulting variations with ABRA[3] – see section 3.4.1 – there remain 128 linear equations from the sector with four fields, and 320 equations from the sector with five fields. These equations must be solved for the 123 coefficients from the Ansatz and the 182 coefficients that parametrize total derivatives having the same structure as the variations.

The Noether procedure yield *one* unique deformation of $d=10$, $\mathcal{N}=1$ supersymmetric Yang-Mills theory at order α'^3, which is determined up to a single multiplicative constant a_3:

$$
\begin{aligned}
a_3^{-1}\mathcal{L}_3 = f^{XYZ}f^{VWZ}\Big[& 2\,F^X_{ab}F^W_{cd}\,D_eF^V_{bc}D_eF^Y_{ad} - 2\,F^X_{ab}F^W_{ac}\,D_dF^V_{be}D_dF^Y_{ce} \\
& + F^X_{ab}F^W_{cd}\,D_eF^V_{ab}D_eF^Y_{cd} \\
& - 4\,F^W_{ab}\,D_cF^Y_{bd}\,\bar\chi^X\Gamma_aD_dD_c\chi^V - 4\,F^W_{ab}\,D_cF^Y_{bd}\,\bar\chi^X\Gamma_dD_aD_c\chi^V \\
& + 2\,F^W_{ab}\,D_cF^Y_{de}\,\bar\chi^X\Gamma_{ade}D_bD_c\chi^V + 2\,F^W_{ab}\,D_cF^Y_{de}\,\bar\chi^X\Gamma_{abd}D_eD_c\chi^V\Big] + \\
+\; f^{XYZ}f^{UVW}f^{TUX}\Big[& 4\,F^Y_{ab}F^Z_{cd}F^V_{ac}F^W_{be}\,F^T_{de} + 2\,F^Y_{ab}F^Z_{cd}F^V_{ab}F^W_{ce}\,F^T_{de} \\
& - 11\,F^Y_{ab}F^Z_{cd}F^V_{cd}\,\bar\chi^T\Gamma_aD_b\chi^W + 22\,F^Y_{ab}F^Z_{cd}F^V_{ac}\,\bar\chi^T\Gamma_bD_d\chi^W \\
& + 18\,F^Y_{ab}F^V_{cd}F^W_{ac}\,\bar\chi^T\Gamma_bD_d\chi^Z + 12\,F^T_{ab}F^Y_{cd}F^V_{ac}\,\bar\chi^Z\Gamma_bD_d\chi^W \\
& + 28\,F^T_{ab}F^Y_{cd}F^V_{ac}\,\bar\chi^W\Gamma_bD_d\chi^Z - 24\,F^Y_{ab}F^V_{cd}F^T_{ac}\,\bar\chi^W\Gamma_bD_d\chi^Z \\
& + 8\,F^T_{ab}F^Y_{cd}F^Z_{ac}\,\bar\chi^V\Gamma_bD_d\chi^W - 12\,F^T_{ab}F^Y_{ac}D_bF^V_{cd}\,\bar\chi^Z\Gamma_d\chi^W \\
& - 8\,F^Y_{ab}F^T_{ac}D_bF^V_{cd}\,\bar\chi^Z\Gamma_d\chi^W + 22\,F^V_{ab}F^Y_{ac}D_bF^T_{cd}\,\bar\chi^Z\Gamma_d\chi^W \\
& - 4\,F^Y_{ab}F^T_{cd}D_eF^V_{ac}\,\bar\chi^Z\Gamma_{bde}\chi^W + 4\,F^Y_{ab}F^T_{ac}D_cF^V_{de}\,\bar\chi^Z\Gamma_{bde}\chi^W \\
& + 4\,F^T_{ab}F^Y_{cd}F^V_{ce}\,\bar\chi^Z\Gamma_{abd}D_e\chi^W - 8\,F^Y_{ab}F^T_{cd}F^V_{ce}\,\bar\chi^Z\Gamma_{abd}D_e\chi^W \\
& + 6\,F^V_{ab}F^Y_{cd}F^W_{ce}\,\bar\chi^Z\Gamma_{abd}D_e\chi^T + 5\,F^V_{ab}F^W_{cd}\,F^Y_{ce}\,\bar\chi^Z\Gamma_{abd}D_e\chi^T
\end{aligned}
$$

[3] The number of terms at intermediate stages of the calculation reaches 10^4, so the use of a computer was clearly unavoidable.

$$
\begin{aligned}
&+ 6\, F^Y_{ab} F^T_{ac} F^V_{de}\, \bar\chi^Z \Gamma_{bcd} D_e \chi^W - 2\, F^Y_{ab} F^T_{ac} F^Z_{de}\, \bar\chi^V \Gamma_{bcd} D_e \chi^W \\
&+ 4\, F^Y_{ab} F^V_{ac} F^Z_{de}\, \bar\chi^W \Gamma_{bcd} D_e \chi^T + 4\, F^T_{ab} F^V_{cd} F^Y_{ce}\, \bar\chi^Z \Gamma_{abd} D_e \chi^W \\
&- 4\, F^Y_{ab} F^V_{cd} F^W_{ce}\, \bar\chi^Z \Gamma_{abd} D_e \chi^T \\
&+ \tfrac{1}{2} F^Y_{ab} F^T_{cd} F^V_{ef}\, \bar\chi^Z \Gamma_{abcde} D_f \chi^W + \tfrac{1}{2} F^Y_{ab} F^T_{cd} F^Z_{ef}\, \bar\chi^V \Gamma_{abcde} D_f \chi^W \Big]. \qquad (4.2.15)
\end{aligned}
$$

This result holds for an *arbitrary* compact gauge group $\mathcal{G}$. To make contact with string theory, we take $\mathcal{G} = U(N)$ and read off the value of a_3 from the 4-point function:

$$a_3 = -\frac{\zeta(3)}{2}\alpha'^3. \qquad (4.2.16)$$

Our results agree with those of [93, 113, 121, 122] on the bosonic terms. The higher derivative terms with fermions agree with [93]. We can be quite certain our result is also correct for the other fermionic contributions: we needed to solve 448 linear equations for 305 unknowns. If we had made a mistake in our calculation, we would almost certainly not have obtained any solution to these equations. Nevertheless, it would be nice to have an independent check of these results and we will present one in the following section (see also [b]).

An interesting implication of the fact that the trace structure can be expressed in terms of structure constants only is the following. If the group $\mathcal{G}$ contains a $U(1)$ factor, the corresponding $U(1)$ fields – which are certainly present at order α'^0 and α'^2 – do not occur in the α'^3 action. Another implication is that the action (4.2.15) is trivially invariant under the nonlinear supersymmetry present at order α'^0 and α'^2. The nonlinear transformation acts at order[4] α'^0 only on χ:

$$\delta_0 \chi^A = \eta^A, \qquad (4.2.17)$$

where η is a constant spinor, satisfying $f^{ABC}\eta^C = 0$. This implies that η commutes with all group generators, and must therefore be in a $U(1)$ factor. The invariance of (4.2.15) under (4.2.17) is then obvious.

4.3 A test: the spectrum of small fluctuations

We mentioned already the paper [120] in which it was shown that the spectrum of small fluctuations around a fixed background predicted by (4.1.1) disagrees with the string theory spectrum at α'^4. The test was further developed in [126, 127], and applied to the bosonic terms at α'^3 [128] and α'^4 [129]. In the following we will apply the method to the fermionic contributions and show that our result (4.2.15) passes the test.

[4] At order α'^2 there are modifications [97, 119].

The test works as follows. We start with a stack of two D$2p$-branes and wrap these around a p-dimensional torus and turn on a constant magnetic background field. In section 1.3.4 we explained what happens to a *single* wrapped D$2p$-brane under T-duality in the presence of magnetic fluxes: the T-dual picture contains a *tilted* Dp-brane that is tilted w.r.t. the T-dualized direction with angles that are related to the magnetic fluxes. Now we have two D$2p$-branes that will both give rise to a tilted Dp-brane under T-duality. Generically, these angles need not be the same for both Dp-branes and we get two *intersecting* branes [130] instead of a stack of Dp-branes.

String theory allows for the calculation of the spectrum of the strings that stretch between the two intersecting branes [130–132] in the T-dual picture. This spectrum should be reproduced by the mass spectrum of the off-diagonal field fluctuations in the effective action for the stack of D$2p$-branes. This is the essence of the test [120].

We will now develop this test for the terms quadratic in the gauginos χ^A. Throughout the remainder of this chapter we work in units where $2\pi\alpha' \equiv 1$.

4.3.1 The spectrum from string theory

We consider a constant magnetic background on two coincident D$2p$-branes,

$$\mathcal{F}_{2a-1\,2a} = i \begin{pmatrix} f_a & 0 \\ 0 & -f_a \end{pmatrix}, \tag{4.3.1}$$

with $a \in \{1, 2, \cdots, p\}$ and $f_a \in \mathbb{R}$, $f_a > 0$. We choose a gauge such that $\mathcal{A}_{2a-1} = 0$, $\forall a$, and T-dualize in the 2, 4, ..., $2p$ directions. We end up with two intersecting Dp-branes. We want to calculate the spectrum of open strings stretching between the two branes. We take the first brane located along the 1, 3, ..., $2p-1$ directions. The other brane has been rotated with respect to the first one over an angle θ_1 in the 12 plane, over an angle θ_2 in the 34 plane, ..., over an angle θ_p in the $2p-1\,2p$ plane. The angles are determined by the magnetic fields as in section (1.3.4):

$$\theta_a = 2 \arctan f_a, \quad \forall a \in \{1, 2, \cdots, p\}. \tag{4.3.2}$$

Inspired by [130] we introduce,

$$\hat{X}^{2a-1} = \cos\theta_a X^{2a-1} + \sin\theta_a X^{2a}, \quad \hat{X}^{2a} = -\sin\theta_a X^{2a-1} + \cos\theta_a X^{2a}, \tag{4.3.3a}$$
$$\hat{\psi}_\pm^{2a-1} = \cos\theta_a \psi_\pm^{2a-1} + \sin\theta_a \psi_\pm^{2a}, \qquad \hat{\psi}_\pm^{2a} = -\sin\theta_a \psi_\pm^{2a-1} + \cos\theta_a \psi_\pm^{2a}. \tag{4.3.3b}$$

We impose the following boundary conditions:

$$\sigma = 0: \quad \partial_\sigma X^{2a-1} = 0, \quad \partial_\tau X^{2a} = 0, \quad \psi_+^{2a-1} = \psi_-^{2a-1}, \quad \psi_+^{2a} = -\psi_-^{2a}; \tag{4.3.4a}$$
$$\sigma = \pi: \quad \partial_\sigma \hat{X}^{2a-1} = 0, \quad \partial_\tau \hat{X}^{2a} = 0, \quad \hat{\psi}_+^{2a-1} = \eta\hat{\psi}_-^{2a-1}, \quad \hat{\psi}_+^{2a} = -\eta\hat{\psi}_-^{2a}, \tag{4.3.4b}$$

where $\eta = +1$ or $\eta = -1$ in the Ramond and the Neveu-Schwarz sector, respectively. Upon solving the equations of motion and implementing the boundary conditions we get the following expansion for the bosons:

$$X^{2a-1} = \frac{i}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \left(\frac{\alpha_{n_{+a}}}{n_{+a}} e^{-in_{+a}\tau} \cos n_{+a}\sigma + \frac{\alpha_{n_{-a}}}{n_{-a}} e^{-in_{-a}\tau} \cos n_{-a}\sigma \right), \qquad (4.3.5a)$$

$$X^{2a} = \frac{i}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \left(\frac{\alpha_{n_{+a}}}{n_{+a}} e^{-in_{+a}\tau} \sin n_{+a}\sigma - \frac{\alpha_{n_{-a}}}{n_{-a}} e^{-in_{-a}\tau} \sin n_{-a}\sigma \right), \qquad (4.3.5b)$$

where we introduced

$$\varepsilon_a \equiv \frac{\theta_a}{\pi}, \qquad n_{\pm a} \equiv n \pm \varepsilon_a \text{ with } n \in \mathbb{Z}. \qquad (4.3.6)$$

In the Ramond sector (we will not need the Neveu-Schwarz sector), we get

$$\begin{aligned} \psi_{\pm}^{2a-1} &= \frac{1}{2} \sum_{n\in\mathbb{Z}} \left(d_{n_{+a}} e^{-in_{+a}(\tau\pm\sigma)} + d_{n_{-a}} e^{-in_{-a}(\tau\pm\sigma)} \right), \\ \psi_{\pm}^{2a} &= \pm\frac{i}{2} \sum_{n\in\mathbb{Z}} \left(d_{n_{+a}} e^{-in_{+a}(\tau\pm\sigma)} - d_{n_{-a}} e^{-in_{-a}(\tau\pm\sigma)} \right). \end{aligned} \qquad (4.3.7)$$

The non-vanishing (anti-)commutation relations are

$$\begin{aligned} [\alpha_{m_{+a}}, \alpha_{n_{-b}}] &= m_{+a}\delta_{m+n}\delta_{ab}, \\ \{d_{m_{+a}}, d_{n_{-b}}\} &= \delta_{m+n}\delta_{ab}. \end{aligned} \qquad (4.3.8)$$

Both X^{2a-1} and X^{2a} contribute to the vacuum energy (in units where $2\pi\alpha' = 1$) by $-\pi/12 + \pi\varepsilon_a(1-\varepsilon_a)/2$ which is precisely canceled by the contribution of the Ramond fermions. So just as for the case without magnetic fields, the vacuum energy vanishes in the Ramond sector. The (light-cone) states which in the absence of magnetic fields reduce to the gauginos are of the form

$$\prod_{a=1}^{p} (\alpha_{-\varepsilon_a})^{m_a} (d_{-\varepsilon_a})^{l_a} |0\rangle, \qquad (4.3.9)$$

where $m_a \in \mathbb{N}$ and $l_a \in \{0,1\}$, $\forall a \in \{1, \cdots, p\}$ and $|0\rangle$ carries a chiral spinor representation of $Spin(8-2p)$. Their masses are given by

$$M^2 = \sum_{a=1}^{p} 2\,(m_a + l_a)\,\theta_a. \qquad (4.3.10)$$

4.3.2 The spectrum from the effective action

The leading term

To set the stage we will first review some of the results of [133, 134] and [135]. Our starting point is the $U(2)$ $d = 10$, $\mathcal{N} = 1$ supersymmetric Yang-Mills theory[5] with the action $\mathcal{L}_0$ as given in equation (4.2.3). For simplicity we will put $g = 1$ in the following. We compactify $2p$ dimensions on a torus and introduce complex coordinates for the compact directions, $z^\alpha = (x^{2\alpha-1} + ix^{2\alpha})/\sqrt{2}$, $\bar{z}^{\bar{\alpha}} = (z^\alpha)^*$, $\alpha \in \{1, \cdots, p\}$. We switch on constant magnetic background fields in the compact directions $\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\bar{\alpha}\bar{\beta}} = 0$, $\mathcal{F}_{\alpha\bar{\beta}} = 0$ for $\alpha \neq \beta$ and[6]

$$\mathcal{F}_{\alpha\bar{\alpha}} = i \begin{pmatrix} f_\alpha & 0 \\ 0 & -f_\alpha \end{pmatrix}, \tag{4.3.11}$$

where the f_α, $\alpha \in \{1, \cdots, p\}$ are imaginary constants such that $if_\alpha > 0$. We only consider the off-diagonal components of the fermions,

$$\chi = i \begin{pmatrix} 0 & \chi^+ \\ \chi^- & 0 \end{pmatrix}, \tag{4.3.12}$$

since the diagonal fluctuations probe the abelian part of the action [126]. Using the previous choices, we can rewrite the second term in (4.2.3) as

$$\mathcal{L}_{\text{fermion}} = \bar{\chi}^- (\partial\!\!\!/_{\text{NC}} + \mathcal{D}\!\!\!\!/\,)\chi^+, \tag{4.3.13}$$

where the subindex NC denotes operators acting in the non-compact directions only and $\mathcal{D} \equiv \partial + 2i\mathcal{A}$, with $\mathcal{A}$ the background gauge fields. The background covariant derivatives satisfy

$$[\mathcal{D}_\alpha, \mathcal{D}_{\bar{\beta}}] = 2i\delta_{\alpha\beta} f_\alpha. \tag{4.3.14}$$

The equations of motion readily follow from (4.3.13):

$$(\partial\!\!\!/_{\text{NC}} + \mathcal{D}\!\!\!\!/\,)\chi^+ = 0. \tag{4.3.15}$$

We square the kinetic operator in (4.3.15) and use (4.3.14) to obtain

$$\left(\Box_{NC} + 2\sum_{\alpha=1}^{p} \{\mathcal{D}_\alpha \mathcal{D}_{\bar{\alpha}} - if_\alpha - if_\alpha \Gamma_{\alpha\bar{\alpha}}\} \right) \chi^+ = 0, \tag{4.3.16}$$

where $\Gamma_{\alpha\bar{\alpha}} \equiv (\Gamma_\alpha \Gamma_{\bar{\alpha}} - \Gamma_{\bar{\alpha}} \Gamma_\alpha)/2$ and $(\Gamma_{\alpha\bar{\alpha}})^2 = 1$. Once a complete set of eigenfunctions is constructed for the second part in (4.3.16), we can bring the relation

[5] The calculation of the spectrum only probes $U(2)$ sub-sectors of the full $U(n)$ theory [135].

[6] We do not sum over repeated indices corresponding to *complex* coordinates, unless indicated otherwise.

above in the form $(\Box - M^2)\chi = 0$ and read off the mass M. Such eigenfunctions are obtained from a spinor $|0\rangle$ satisfying $\mathcal{D}_{\bar{\alpha}}|0\rangle = 0$, $\forall\alpha$, that has been explicitly constructed in [133, 134] and [135]. We introduce the complete set of functions $|\{(m_1, n_1), (m_2, n_2), \ldots, (m_p, n_p)\}\rangle$, with $m_1, m_2, \ldots, m_p \in \mathbb{N}$ and $n_1, n_2, \ldots, n_p \in \{-1, +1\}$ by

$$\begin{aligned} &|\{(m_1, n_1),\, (m_2, n_2),\, \cdots, (m_p, n_p)\}\rangle \equiv \\ &\quad \frac{1}{2}(1 + n_1\Gamma_{1\bar{1}})\frac{1}{2}(1 + n_2\Gamma_{2\bar{2}}) \cdots \frac{1}{2}(1 + n_p\Gamma_{p\bar{p}})\mathcal{D}_1^{m_1}\mathcal{D}_2^{m_2} \cdots \mathcal{D}_p^{m_p}|0\rangle, \end{aligned} \tag{4.3.17}$$

and expand the fermion in this complete set:

$$\chi^+(y, z, \bar{z}) = \sum_{\{(m,n)\}} \chi^+_{\{(m,n)\}}(y)|\{(m, n)\}\rangle, \tag{4.3.18}$$

where $\{(m, n)\} \equiv \{(m_1, n_1), (m_2, n_2), \ldots, (m_p, n_p)\}$ and y collectively denotes the non-compact coordinates. Using this, one gets from eq. (4.3.14) and eq. (4.3.16) that the mass of $\chi^+_{\{(m,n)\}}(y)$ is given by

$$M^2 = 2i \sum_{\alpha=1}^{p} (2m_\alpha + 1 + n_\alpha)\, f_\alpha. \tag{4.3.19}$$

Replacing f_α by $\operatorname{arctanh}(f_\alpha)$ in (4.3.19) yields the stringy result (4.3.10). As expected, we only get agreement for very small magnetic background fields. The higher order terms in the effective action should add to this such that the string result gets reproduced. In particular one notices from this that only even orders in α' contribute to the spectrum.

The $\mathcal{O}(\alpha'^2)$ contribution to the spectrum

Modulo field redefinitions and up to terms terms quartic in the fermions, the effective action through $\mathcal{O}(\alpha'^2)$ is given by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_2$ where $\mathcal{L}_0$ was given (4.2.3) and $\mathcal{L}_2$ is given by [49, 95, 97, 106, 119, 136]:

$$\begin{aligned} \mathcal{L}_2 = \mathrm{STr}\Big[&x_1 F_{ab}F_{ab}F_{cd}F_{cd} + x_2 F_{ab}F_{bc}F_{cd}F_{da} \\ &+ x_3 F_{ab}F_{ac}\,\bar{\chi}\Gamma_b D_c\chi + x_4 F_{ab}D_a F_{cd}\,\bar{\chi}\Gamma_{bcd}\chi\Big], \end{aligned} \tag{4.3.20a}$$

with

$$x_1 = -\frac{1}{32}, \qquad x_2 = \frac{1}{8}, \qquad x_3 = -\frac{1}{4}, \qquad x_4 = -\frac{1}{16}. \tag{4.3.20b}$$

Again we want to calculate the fermionic spectrum through this order. It is clear that only the term proportional to x_3 will contribute since the background magnetic fields are covariantly constant. Following exactly the same strategy as above, we get the equations of motion:

$$\left(\partial\!\!\!/_{\rm NC} + D\!\!\!\!/ - \frac{2x_3}{3}\sum_{\alpha=1}^{p} f_\alpha^2\,(\Gamma_\alpha \mathcal{D}_{\bar\alpha} + \Gamma_{\bar\alpha}\mathcal{D}_\alpha)\right)\chi^+ = 0. \tag{4.3.21}$$

We square the kinetic operator:

$$\left(\Box_{\rm NC} + 2\sum_{\alpha=1}^{p}\left[1 - \frac{4x_3}{3}f_\alpha^2\right](\mathcal{D}_\alpha\mathcal{D}_{\bar\alpha} - if_\alpha - if_\alpha\Gamma_{\alpha\bar\alpha})\right)\chi^+ = 0, \tag{4.3.22}$$

where we ignored terms proportional to f^4 as they are of higher order in α'. However such terms will be relevant for a test of the – as of yet still unknown – fermionic contributions to the effective action at α'^4. It is clear that this gives the same spectrum as in (4.3.19), but with f_α replaced by

$$f_\alpha \to f_\alpha - \frac{4x_3}{3}f_\alpha^2. \tag{4.3.23}$$

Consistency with the string spectrum requires that $x_3 = -1/4$ which indeed agrees with (4.3.20).

It was shown in [127] that the bosonic part of the effective action at α'^2 is fixed completely by the requirements that the spectrum of the gauge fields is reproduced correctly and that the effective action has the correct abelian limit. It is clear from the above that this is not the case for the fermionic terms. The spectral test is thus weaker for the fermionic contributions than for the purely bosonic terms.

Testing the $\mathcal{O}(\alpha'^3)$ terms

We saw earlier that the terms with odd powers of α' should not contribute to the spectrum, so in particular we should not get any contributions from $\mathcal{L}_3$.

The bosonic terms at α'^3 were already investigated in [128] and we thus focus our attention on the fermionic terms in (4.2.15). The terms which contain a covariant derivative on a field strength F will not contribute to the spectrum, since we have chosen our background field to be covariantly constant. Furthermore, any term having two field-strengths contracted with a single f-symbol can be ignored as well since we took the background field strength in the Cartan subalgebra of $SU(2)$. Having discarded these terms, we note that – with our particular choice of background – the remaining terms have a group theoretical factor such that the Lie algebra indices on the gauginos are antisymmetrized. This implies that all terms involving a $\Gamma^{(1)}$ or $\Gamma^{(5)}$ will vanish as well up to a total derivative. The only terms which can potentially contribute to the spectrum are thus

$$x_1\,F^T_{ab}F^Y_{cd}F^V_{ce}\,\bar{\chi}^Z\Gamma_{abd}D_e\chi^W + x_2\,F^Y_{ab}F^T_{cd}F^V_{ce}\,\bar{\chi}^Z\Gamma_{abd}D_e\chi^W \\ + x_3\,F^Y_{ab}F^T_{ac}F^V_{de}\,\bar{\chi}^Z\Gamma_{bcd}D_e\chi^W + x_4\,F^T_{ab}F^V_{cd}F^Y_{ce}\,\bar{\chi}^Z\Gamma_{abd}D_e\chi^W\,, \quad (4.3.24)$$

with

$$x_1 = +4, \qquad x_2 = -8, \qquad x_3 = +6, \qquad x_4 = +4. \qquad (4.3.25)$$

We rewrite (4.3.24) in terms of the background field and the off-diagonal fermions and obtain

$$(x_1 + x_2 + x_4)\sum_{\alpha=1}^{p}\sum_{\beta=1}^{p} f_\beta f_\alpha^2 \left(\bar{\chi}^-\Gamma_{\bar{\beta}\beta\alpha}\mathcal{D}_{\bar{\alpha}}\chi^+ + \bar{\chi}^-\Gamma_{\bar{\beta}\beta\bar{\alpha}}\mathcal{D}_\alpha\chi^+\right). \qquad (4.3.26)$$

We see from (4.3.25) that this vanishes as required.

4.4 Conclusion

Though the spectral test is not as restrictive for the fermionic terms as it was for the purely bosonic terms, it is still gratifying to see that the fermionic terms pass it as well. At this point there is no doubt left that we have the correct supersymmetrization of the non-abelian D-brane effective action through order α'^3.

Afterword

The main topic of this thesis was the study of D-brane effective actions. The first two chapters provided an introduction to this subject. We explained that D-branes can be described in two complementary ways. On one side, they arise as hyperplanes that support open strings in perturbative string theory. On the other side, they are higher dimensional generalizations of extremal Reissner-Nordström black holes in supergravity theories. We reviewed the role of D-branes in the dualities that relate the various string theories to each other.

One important aspects of D-branes is the fact that they carry gauge fields on their world-volumes. We have argued in chapter 1 that many interesting aspects of the physics of D-branes can be investigated by studying the low-energy effective field theories of these gauge fields and the string-theoretical α'-corrections to these theories. As an example, we discussed the behavior of open strings under T-duality in the presence of electric and magnetic background fields and showed how this information is encoded in the Dirac-Born-Infeld and Wess-Zumino contributions to the effective action of a single Dp-brane.

Another important property of D-branes that we discussed in chapter 1 is the appearance of extra massless degrees of freedom when two or more Dp-branes coincide. Not only do we get non-abelian gauge fields, but also the scalar fields that describe the embedding of the stack of D-branes in spacetime become matrix-valued. A thorough understanding of these matrix-valued coordinates and the associated noncommutative geometry will very likely bring us a lot further on the road toward a formulation of M-theory.

With these things in mind, we studied derivative corrections to the effective actions of single and multiple D-branes in chapter 3 and 4, respectively. The approach we followed was to consider only flat closed string backgrounds, since in this case one does not have to deal with the conceptual difficulties involving the nonabelian coordinates. We limited ourselves to D9-branes, since – as argued in chapter 1 – the actions for the lower-dimensional branes follow by T-duality. We then used the Noether procedure and information from the string theory 4-point function to construct α'-corrections to the super-Yang-Mills action.

Because of our approach to the α'-corrections, we did not discuss the behavior of D-branes in nontrivial closed string backgrounds. It turns out however that this behavior is quite interesting, for single D-branes and especially for multiple D-branes. In order to make up for our disregard of these issues in the preceding chapters, we will now briefly mention some of the phenomena associated with nontrivial backgrounds and provide entry points into the literature. As we will see, these phenomena involve noncommutative geometry in one way or the other.

The first case that we will consider applies to both single and multiple D-branes, and is that of a nonzero background $\mathcal{F}$-field, where $\mathcal{F} = B + 2\pi\alpha' F$, something we encountered already before in our discussions on T-duality. What we did not mention there, was that the quantization of the open strings yields something remarkable. Consider for instance our earlier example of a D2-brane with a flux $\mathcal{F}_{12} \equiv \mathcal{F}$. It can be shown using Dirac's method of quantization that the coordinates which describe the position of the endpoints of the open string on the D2-brane do not commute [137,138]

$$[X^1(\tau,\sigma), X^2(\tau,\sigma')] = \begin{cases} \pm 2\pi i\alpha'\mathcal{F}/(1+\mathcal{F}^2), & \text{for } \sigma=\sigma'=0 \text{ and } \sigma=\sigma'=\pi, \\ 0, & \text{otherwise,} \end{cases}$$

with the $+$ and $-$ signs for the $\sigma = 0$ and $\sigma = \pi$ endpoints, respectively. According to the open strings, the geometry on the D2-brane's *world-volume* is noncommutative. A heuristic explanation of this is the following. We explained that when $\mathcal{F} \neq 0$ we actually have a bound state of a D2-brane with D0-branes. The open string endpoints are thus not just confined to the D2-brane but also to the D0-branes which are immersed in the world-volume of the D2-brane. But the D0-branes are completely delocalized and this is reflected in the non-zero commutation relations of the open string endpoints.

The above suggests that there exists an effective description for the low-energy degrees of freedom on a D-brane with nonvanishing $\mathcal{F}$ in terms of a *noncommutative* gauge theory in which the ordinary commutative product of functions is replaced by a noncommutative $\star$-product, i.e. $fg \to f \star g$, known as the Moyal product [139,140]. But we know that we can describe the low-energy degrees of freedom that live on the D-brane as an ordinary gauge theory. The description in terms of a noncommutative gauge theory should thus be related to the ordinary one by a change of variables. That this is indeed the case was shown in [141] and the change of variables has become known as the Seiberg-Witten map. The Seiberg-Witten map puts restrictions on the form of the D-brane effective action. It was shown in [141] that the Born-Infeld action satisfies these restrictions. The constraints of the Seiberg-Witten map on the nonabelian case were studied in [142].

The second case we want to discuss is that of nonzero RR-backgrounds. In [143] an attempt was made to construct the nonabelian Dp-brane effective action in the presence of background fields by taking the D9-brane as a starting point and using T-duality as discussed in chapter 1. It is not our goal here to review Myers' result in detail, but we do want to make a few remarks. Myers only considered the symmetrized trace action (4.1.1). His result is therefore only valid up to α'^3, but nevertheless has some interesting properties. For instance, the bulk fields depend on the nonabelian scalars Φ^a and the pull-back of the bulk fields to the D-brane world-volume is done with covariant instead of ordinary derivatives. So there are additional couplings between the world-volume and bulk fields that were not there in the abelian case.

Probably the most important aspect of Myers' result is the Wess-Zumino term. Recall that a single Dp-brane couples not only to the RR $(p+1)$-form, but also to the RR-forms of lower rank. Myers showed that the nonabelian WZ-term also involves the RR-forms of *higher* rank. A consequence of this is the *Myers dielectric effect*: when multiple Dp-branes are put together in an RR-background they become polarized. A concrete example is given by N D0-branes in a background of the RR 3-form $C^{(3)}$. One can show that the energetically most favorable configuration is one in which the D0-branes expand into a noncommutative or "fuzzy" two-sphere, which can be interpreted as a D0/D2-brane bound state. See [18, 59] for further information and references.

The third and final case we want to point out is that of a curved gravitational background. A major issue here is generalization of general coordinate invariance to $U(N)$ the matrix-valued coordinates, i.e. 'D-geometry' [144]. Significant progress was made in [145] (see [146–150] for earlier interesting work), where a formalism was developed that allows one to write actions for multiple D-branes with general covariance. A generic feature of these actions is that they *necessarily* involve derivatives of the metric in order for general covariance to be realized[7]. So we see that the breakdown of the notion of slowly varying fields in the nonabelian case is not limited to the world-volume fields, but also involves the bulk fields.

In [145] only the bosonic sector was investigated. A natural question to ask is what the consequences of the nonabelian coordinates are for the local supersymmetry of the bulk fields and κ-symmetry. κ-symmetry seems particularly important, since it played a crucial role in the understanding of the supersymmetry of the Born-Infeld action in the abelian case. A nonabelian generalization of κ-symmetry will perhaps provide similar insights. In [152] a κ-symmetric action was found for a system of coincident D0-branes. It would be interesting to see how this action relates to the results of [145, 149]. Hopefully the results of [145] will also shed some light on the failed attempt of [153] to construct κ-symmetric α'-corrections to $\mathcal{N} = 1$, $d = 10$ super-Yang-Mills theory.

Finally, it is interesting to think about possible application of the α'-corrections that we considered in this thesis. For example, it still remains to bee seen how the α'-corrections modify the behavior of classical solutions of the Yang-Mills equations of motion that are more complicated than the flat backgrounds that we considered in

[7]These additional couplings result in a gravitational version of the Myers effect [151].

chapter 4. One may expect for instance, that the instanton solutions of $\mathcal{N} = 4$, $d = 4$ Yang-Mills theory receive α'-corrections. In this context the fermionic contributions to the effective action are relevant for a study of the fermionic collective coordinates of the instanton solution.

Nevertheless, in the end one would definitely like to have a closed expression for the nonabelian D-brane effective action. In chapter 1 we saw an explicit example where the upper limit for the electric field on a D-brane could only be obtained by using the complete infinite collection of α'-corrections of the Born-Infeld action. In addition, it turns out that the Born-Infeld action also provides a finite self-energy for the electric point particle solution. In the nonabelian case, answers to these and other questions that are related to the resolution of classical singularities by α'-corrections, will have to wait until we have an all-order result.

Appendix A

Supersymmetry technicalities

The first part of this appendix is devoted to a discussion of the technical apparatus needed in supersymmetric field theories. The second part is a self-contained (but highly condensed) review of four-dimensional supersymmetry.

A.1 Spinors and Dirac matrices

In this section we discuss the properties of spinors in an arbitrary *even* number of dimensions, $d = 2n$. Though we mostly need the properties of spinors in ten-dimensional Minkowski space in the main text, we present an analysis for arbitrary signature. It is the author's opinion that a general discussion is helpful in keeping a clear distinction between intrinsic properties and mere conventions.
There are many different approaches to the theory of Clifford algebras and spinors and the literature on the subject is vast. The present discussion aims to stay close to the presentation that is usually encountered in the supergravity and string theory literature. Classic references are [154–158]. For applications to the theory of extended super-Poincaré algebras see e.g. [159, 160]. Useful reviews can be found in e.g. [15, 34, 47, 161–164]. For a more mathematical approach see [165, 166] and references therein. The topic of invariant actions is somewhat controversial. For a more extensive discussion than the one presented here, we refer to [167] and references therein.

A.1.1 The Poincaré group

The Poincaré group $\mathcal{P}$ is the isometry group of Minkowski space $\mathcal{M} = (\mathbb{R}^d, \eta)$. It consists of those transformations $(\Lambda, t) : x^a \to x'^a = \Lambda^a{}_b x^b + t^a$ that leave the line element $\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b$ invariant. We use the mostly-plus convention. Representations U of $\mathcal{P}_+^\uparrow$ (i.e. those elements of $\mathcal{P}$ for which $\Lambda^0{}_0 \geq 1$ and $\det\Lambda = +1$) are obtained from the Lie algebra

$$[P_a, P_b] = 0, \tag{A.1.1a}$$
$$[M_{ab}, P_c] = -\eta_{ac}P_b + \eta_{bc}P_a, \tag{A.1.1b}$$
$$[M_{ab}, M_{cd}] = -\eta_{ac}M_{bd} + \eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{bd}M_{ac}, \tag{A.1.1c}$$

by exponentiation:

$$U = \exp\frac{1}{2}\omega^{ab}M_{ab}\ \exp\tau^c P_c. \tag{A.1.2}$$

Infinitesimal Poincaré transformations are thus given by $\delta x^a = \omega^a{}_b x^b + \tau^a$ with $\omega^{ab} := \omega^a{}_c\eta^{cb} = -\omega^{ba}$. Moreover, when we represent the Poincaré group by unitary operators on the Hilbert space of physical states, the Lie algebra is represented by antihermitian matrices. The generators P_a of translations are related to the conventional momentum operators by $P_a = -iP_a^{\mathrm{conv}}$, where $P_a^{\mathrm{conv}} = (-E, \vec{p})$. The M_{ab} generate the Lorentz transformations of $\Lambda_+^\uparrow$.

In physics one actually needs to consider the covering group of $\Lambda_+^\uparrow$. In addition to the bosonic representations of $\Lambda_+^\uparrow$ (which can all be obtained by taking tensor products of the vector representation), the cover $\bar{\Lambda}_+^\uparrow$ also allows for fermionic representations. These can be obtained as follows. First one constructs a representation of the Clifford algebra, which is generated by Γ_a:

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}. \tag{A.1.3}$$

The elements of the Clifford algebra are thus given by $\Gamma^{(k)}$:

$$\Gamma_{a_1\cdots a_k} = \Gamma_{[a_1}\cdots\Gamma_{a_k]} = \frac{1}{k!}\sum_\pi (-)^\pi \Gamma_{\pi_{a_1}}\cdots\Gamma_{\pi_{a_k}}. \tag{A.1.4}$$

The *spinor* representation of the Lorentz group is then given by:

$$M_{ab} = \frac{1}{2}\Gamma_{ab}. \tag{A.1.5}$$

A.1.2 Dirac matrices

In the following we will keep the signature of the metric arbitrary:

$$\eta_{ab} = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{\times t}, \underbrace{+1, \ldots, +1}_{\times s}). \tag{A.1.6}$$

t and $s = 2n - t$ stand for the number of time- and spacelike dimensions, respectively[1]. First we construct an explicit representation for the Clifford algebra of $SO(2n)$ in terms of the complex Dirac matrices:

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}. \tag{A.1.7}$$

We restrict ourselves to unitary representations, hence the Γ-matrices are hermitian, $\Gamma_a^\dagger = \Gamma_a$. We define:

$$a_i = \frac{1}{2}(\Gamma_{2i-1} + i\Gamma_{2i}), \tag{A.1.8a}$$

$$a_i^\dagger = \frac{1}{2}(\Gamma_{2i-1} - i\Gamma_{2i}). \tag{A.1.8b}$$

These are fermionic creation and annihilation operators:

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0, \tag{A.1.9a}$$

$$\{a_i, a_j^\dagger\} = \delta_{ij}. \tag{A.1.9b}$$

In particular, the 2-dimensional Clifford algebra is given by

$$\{a, a^\dagger\} = 1, \quad a^2 = (a^\dagger)^2 = 0. \tag{A.1.10}$$

We define the "vacuum" $|-\rangle$ by $a|-\rangle \equiv 0$ and get a single excited state $|+\rangle = a^\dagger|-\rangle$, hence the Clifford algebra has a two-dimensional irreducible representation (up to equivalence) that acts on $\mathbb{C}^2$. If we represent a vector $|v\rangle = v^-|-\rangle + v^+|+\rangle$ by the column matrix $(v^-, v^+)^T$, we have

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{A.1.11}$$

There is only one irreducible representation of the $SO(2n)$ Clifford algebra. Its carrier space is obtained by taking tensor products of the states discussed above. Elements of this carrier space are called *Dirac spinors*. We define the vacuum $|\Omega\rangle$ by

$$|\Omega\rangle = |-\rangle \otimes \ldots \otimes |-\rangle, \quad n \text{ times}, \tag{A.1.12}$$

on which we act with the raising operators $a_i^\dagger$ to obtain $2^n = 2^{d/2}$ states $|\pm\rangle\otimes\ldots\otimes|\pm\rangle$. We obtain

$$a_i = \sigma_3 \otimes \ldots \otimes \sigma_3 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes 1 \otimes \ldots \otimes 1, \tag{A.1.13a}$$

$$a_i^\dagger = \sigma_3 \otimes \ldots \otimes \sigma_3 \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes 1 \otimes \ldots \otimes 1, \tag{A.1.13b}$$

[1] In this appendix, we number the dimensions from 1 to $2n$. For Minkowski space, $t = 1$, it is conventional to denote the timelike dimension by 0 and the spacelike dimensions by $1, \ldots, 2n-1$. This is the convention followed in the main text.

with 2×2 matrix in the ith place. Hence

$$\begin{aligned} \Gamma_1 &= \sigma_1 \otimes 1 \otimes 1 \otimes 1 \ldots, \quad \Gamma_3 = \sigma_3 \otimes \sigma_1 \otimes 1 \otimes 1 \ldots, \\ \Gamma_2 &= \sigma_2 \otimes 1 \otimes 1 \otimes 1 \ldots, \quad \Gamma_4 = \sigma_3 \otimes \sigma_2 \otimes 1 \otimes 1 \ldots, \text{ etc.} \end{aligned} \tag{A.1.14}$$

Here $\{\sigma_i\}$, $i = 1, 2, 3$, are the standard Pauli matrices. It can be shown that this 2^n-(complex)dimensional "standard" representation is unique up to an equivalence transformation:

$$\Gamma'_a = U \Gamma_a U^{-1}, \tag{A.1.15}$$

where U an arbitrary unitary matrix − unitary, since we want to preserve the hermiticity properties of the Γ-matrices. Γ-matrices for $SO(t, 2n-t)$ are obtained by simply multiplying the first t matrices by i. We have

$$\Gamma_a^\dagger = \begin{cases} -\Gamma_a, & \text{if } a = 1, \ldots, t \\ \Gamma_a, & \text{if } a = t+1, \ldots, 2n, \end{cases} \tag{A.1.16}$$

which can be written as

$$\Gamma_a^\dagger = (-)^t A \Gamma_a A^{-1}, \quad A \equiv \Gamma_1 \ldots \Gamma_t. \tag{A.1.17}$$

The *chirality matrix* Γ_* is defined by

$$\Gamma_* \equiv (-i)^{n+t} \Gamma_1 \cdots \Gamma_{2n}. \tag{A.1.18}$$

It anticommutes with the Γ_a, $\{\Gamma_*, \Gamma_a\} = 0$, is hermitian, $\Gamma_*^\dagger = \Gamma_*$, and squares to one, $(\Gamma_*)^2 = 1$. In the standard representation $\Gamma_* = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \ldots$. Note that

$$\Gamma_{a_1 \cdots a_k} = \frac{1}{(d-k)!} i^{d/2+t} \, \varepsilon_{a_1 \cdots a_d} \Gamma_* \Gamma^{a_d \cdots a_{k+1}}. \tag{A.1.19}$$

C-matrices The matrices $\{\Gamma_a^T\}$ and $\{-\Gamma_a^T\}$ also represent the Clifford algebra and must thus be equivalent to the Γ_a, since there is only one inequivalent representation. Hence there are *charge conjugation matrices*[2] C_+ and C_- that satisfy

$$\Gamma_a^T = \pm C_\pm \Gamma_a C_\pm^{-1}. \tag{A.1.20}$$

In the standard representation they are given by the following unitary matrices:

$$C_+ = \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \ldots, \tag{A.1.21a}$$

$$C_- = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \ldots, \tag{A.1.21b}$$

[2] This is an unfortunate misnomer. Changing the sign of the electric charge is related to complex conjugation of fields. So it are actually the B-matrices that give rise to charge conjugation. Indeed, given the Dirac equation $(\partial\!\!\!/ - m + eA\!\!\!/)\psi = 0$, one can show that the charge conjugated spinor $\psi^C = B_+^{-1}\psi^*$ satisfies $(\partial\!\!\!/ - m - eA\!\!\!/)\psi^C = 0$.

$\Gamma^{(k)}C_+^{-1}$:

$n\backslash k$	0	1	2	3
0	+	+	−	−
1	+	+	−	−
2	−	−	+	+
3	−	−	+	+

$\Gamma^{(k)}C_-^{-1}$:

$n\backslash k$	0	1	2	3
0	+	−	−	+
1	−	+	+	−
2	−	+	+	−
3	+	−	−	+

Table A.1.1. Behavior of $\Gamma^{(k)}C_\pm^{-1}$ under transposition. A + denotes symmetry, a − antisymmetry. The tables are (mod 4) in n and k.

regardless of the signature of the metric. We fixed an arbitrary phase. From

$$C_+^T = \sigma_1 \otimes -\sigma_2 \otimes \sigma_1 \otimes -\sigma_2 \otimes \dots,$$
$$C_-^T = -\sigma_2 \otimes \sigma_1 \otimes -\sigma_2 \otimes \sigma_1 \otimes \dots,$$

we obtain the symmetry properties of the C-matrices:

$$C_\pm^T = \epsilon_\pm C_\pm, \quad \text{with } \epsilon_\pm = (-)^{n(n\mp1)/2}. \tag{A.1.22}$$

We define the C matrices in other representations (A.1.15) by $C \to C' = U^{-1T}CU^{-1}$. It follows that their unitarity and symmetry properties are independent of the basis we choose in spinor space.
One can straightforwardly derive the following important relation for the matrices $\Gamma^{(k)}C^{-1}$:

$$(\Gamma_{a_1\cdots a_k}C_\pm^{-1})^T = (-)^{[n(n\mp1)+k(k\mp1)]/2}\Gamma_{a_1\cdots a_k}C_\pm^{-1}, \tag{A.1.23}$$

again regardless of the signature of the metric. The matrices $C\Gamma^{(k)}$ also satisfy this relation. The C-matrices are related to each other via the chirality matrix:

$$C_\pm = (-i)^n(-)^{n(n\pm1)/2}\, C_\mp\Gamma_*. \tag{A.1.24}$$

The matrices $\Gamma_*\Gamma^{(k)}C_\pm^{-1}$ therefore have the same symmetry properties as $\Gamma^{(k)}C_\mp^{-1}$.

B-matrices The behavior of the Γ-matrices under complex conjugation does depend on the signature. Since $\{\pm\Gamma_a^*\}$ represent the Clifford algebra, there are matrices $B_\pm$ such that:

$$\Gamma_a^* = \pm B_\pm\Gamma_a B_\pm^{-1}. \tag{A.1.25}$$

They can be related to the $C_\pm$ by means of A. Since

$$\Gamma_a^* = (\Gamma_a^\dagger)^T = (-)^t(A\Gamma_a A^{-1})^T = \pm(-)^t(C_\pm A^{-1})^T\Gamma_a(C_\pm A^{-1})^{-1T},$$

we have (fixing an arbitrary phase)

$$B_\pm^T = \begin{cases} C_\pm A^{-1}, & \text{if } t \text{ is even,} \\ C_\mp A^{-1}, & \text{if } t \text{ is odd.} \end{cases} \tag{A.1.26}$$

$B_+B_+^*$:

$n\backslash t$	0	1	2	3
0	$\oplus$	$-$	$\ominus$	$+$
1	$+$	$\oplus$	$-$	$\ominus$
2	$\ominus$	$+$	$\oplus$	$-$
3	$-$	$\ominus$	$+$	$\oplus$

$B_-B_-^*$:

$n\backslash t$	0	1	2	3
0	$\oplus$	$+$	$\ominus$	$-$
1	$-$	$\oplus$	$+$	$\ominus$
2	$\ominus$	$-$	$\oplus$	$+$
3	$+$	$\ominus$	$-$	$\oplus$

Table A.1.2. The sign of BB^*. A $+$ or $-$ sign indicates the existence of (pseudo)-Majorana and symplectic-(pseudo)-Majorana spinors, respectively. An encircled $\oplus$ or $\ominus$ indicates Majorana-Weyl and symplectic-Majorana-Weyl spinors, respectively. In the latter case, one can *choose* which B one wants to use. The tables are (mod 4) in n and t.

The B's are unitary since A and the C's are unitary. These relations make sense in any representation if we let the B's transform as $B \to B' = U^* B U^{-1}$. Also the B-matrices are related to each other:

$$B_\pm = \begin{cases} i^n (-)^{n(n\pm 1)/2}\, B_\mp \Gamma_*, & \text{if } t \text{ is even,} \\ i^n (-)^{n(n\mp 1)/2}\, B_\mp \Gamma_*, & \text{if } t \text{ is odd.} \end{cases} \tag{A.1.27}$$

The following equation will be used below:

$$B_\pm B_\pm^* = \begin{cases} \epsilon_\pm (-)^{t(t+1)/2}, & \text{if } t \text{ is even,} \\ \pm\epsilon_\mp (-)^{t(t+1)/2}, & \text{if } t \text{ is odd.} \end{cases} \tag{A.1.28}$$

Finally we give relations for the complex conjugates of the C-matrices. These are needed e.g. when discussing the reality properties of spinor bilinears:

$$C_\pm^* = \begin{cases} B_\pm^{-1T} C_\pm B_\pm^{-1} = (-)^n B_\mp^{-1T} C_\pm B_\mp^{-1}, & \text{if } t \text{ is even,} \\ \mp B_\mp^{-1T} C_\pm B_\mp^{-1} = \mp(-)^n B_\pm^{-1T} C_\pm B_\pm^{-1}, & \text{if } t \text{ is odd.} \end{cases} \tag{A.1.29}$$

Tables for all the above properties can be readily obtained. The following relations come in handy: $(-)^{n(n-1)/2} = (-)^{[n/2]}$ and $(-)^{n(n+1)/2} = (-)^{[(n+1)/2]}$. Here $[n/2]$ denotes the largest integer $\leq n/2$.

A.1.3 Irreducible spinors

When we defined the raising/lowering operators from which we constructed the Dirac matrices in the previous section, we took linear combinations of the Γ-matrices with *complex* coefficients. This means that up to now we have actually considered the complex extension of the Clifford algebra. Even though the Γ-matrices are irreducible representations of the complex extension of the Clifford algebra, they are in general not irreps of its real form. In addition, they are certainly not irreducible under the Lorentz group, which is only a subalgebra of the Clifford algebra. To arrive at irreps of the Lorentz group, we need to find subspaces in the 2^n-dimensional complex spinor space that are invariant under the action of $\frac{1}{2}\Gamma_{ab}$. This boils down to imposing certain constraints on the spinors. These constraints come in two kinds: there are *reality* conditions, which yield irreps of the real Clifford algebra, and *chirality* conditions, which yield irreps of the Lorentz group only. Before we proceed let us first mention that we denote spinors by a column matrix ψ_α, which are the components of the spinor w.r.t. some fixed basis in spinor space. We usually do not write the indices explicitly so we drop the α and simply write ψ. We also mention here that we use the convention that complex conjugation of Grassmann variables interchanges their order, e.g. $(\psi\chi)^* = \chi^*\psi^*$.

Chiral spinors Using the chirality matrix Γ_*, we define projection operators $P_\pm$:

$$P_\pm \equiv \frac{1}{2}(1 \pm \Gamma_*), \tag{A.1.30}$$

that satisfy $P_\pm P_\pm = P_\pm$, $P_\pm P_\mp = 0$, $P_\pm^\dagger = P_\pm$ and $P_+ + P_- = 1$. We can decompose any Dirac spinor ψ into positive and negative chirality parts:

$$\psi = P_+\psi + P_-\psi, \quad \text{where } \Gamma_* P_\pm\psi = \pm P_\pm\psi. \tag{A.1.31}$$

This decomposition is invariant with respect to Lorentz transformations. Indeed, $P_\pm\Gamma_a = \Gamma_a P_\mp$, which implies that $[\Gamma_{ab}, P_\pm] = 0$. However, physical spinor *fields* satisfy the Dirac equation

$$(\Gamma^a\partial_a - m)\psi = 0. \tag{A.1.32}$$

If we impose a chirality constraint on ψ, i.e. $\psi \equiv P_\pm\psi$, we can act on (A.1.32) with $P_\pm$ to get

$$0 = P_\pm(\Gamma^a\partial_a - m)\psi = (\Gamma^a\partial_a P_\mp - mP_\pm)\psi = -m\psi,$$

so $\psi = 0$, unless $m = 0$. Hence chiral spinor fields are necessarily massless. Chiral spinors are also called *Weyl spinors.*

Majorana spinors Given a Dirac spinor ψ, we can try to impose a reality condition on it by restricting ourselves to a subspace of spinor space in which $\psi^* = \tilde{B}\psi$, for some matrix $\tilde{B}$. This subspace is invariant under Lorentz transformations if $\Gamma_{ab}^*\tilde{B} = \tilde{B}\Gamma_{ab}$, i.e. if $[\Gamma_{ab}, B_\pm^{-1}\tilde{B}] = 0$. Therefore we try the following *Majorana* conditions:

$$\psi^* \equiv \alpha B_\pm\psi, \tag{A.1.33}$$

where $\alpha \in \mathbb{C}$. There is an important additional constraint. We demand that $(\psi^*)^* = \psi$, which is equivalent to $|\alpha|^2 B_\pm B_\pm^* = 1$. Thus the reality condition (A.1.33) is consistent only when $|\alpha|^2 = 1$ and

$$B_\pm B_\pm^* = +1. \tag{A.1.34}$$

For spinor *fields* we once again investigate the Dirac equation. Assuming ψ satisfies (A.1.33), we take the complex conjugate of (A.1.32) and act on it with $B_{\pm}^{-1}$:

$$\begin{aligned} 0 &= B_{\pm}^{-1}(\Gamma^{a*}\partial_a - m)\psi^* \\ &= B_{\pm}^{-1}(\pm B_{\pm}\Gamma^a B_{\pm}^{-1}\partial_a - m)B_{\pm}\psi = (\pm)(\Gamma^a\partial_a \mp m)\psi, \end{aligned}$$

If we use B_+ in (A.1.33) there is no problem. However, if we use B_-, we conclude that ψ vanishes, unless $m = 0$. So we have two kinds of real spinors[3]. Those for which

$$\psi^* \equiv \alpha B_+\psi \tag{A.1.35}$$

are called *Majorana spinors.* Spinors that satisfy

$$\psi^* \equiv \alpha B_-\psi \tag{A.1.36}$$

are called *pseudo-Majorana spinors.* They are necessarily massless. The arbitrary phase α can be fixed on physical grounds. One can demand e.g. that there exists a hermitian action or that the supersymmetry algebra implies a positive definite Hamiltonian.

Majorana-Weyl spinors Sometimes it is possible to impose both a chirality and a reality condition on a spinor, i.e. ψ satisfies $\psi = P_{\pm}\psi$ and $\psi^* = \alpha B_{\pm}\psi$. Now

$$P_{\pm}^* = \begin{cases} B_{\pm}P_{\pm}B_{\pm}^{-1}, & \text{if } n+t = \text{even}, \\ B_{\pm}P_{\mp}B_{\pm}^{-1}, & \text{if } n+t = \text{odd}. \end{cases} \tag{A.1.37}$$

Since $\psi^* = (P_{\pm}\psi)^* = \alpha B_{\pm}P_{\mp}\psi = 0$ if $n+t = \text{odd}$, we conclude that *Majorana-Weyl spinors* (pseudo, symplectic) exist only if $n+t = \text{even}$.
The above observation allows us to solve a small riddle. Sometimes $B_+B_+^* = B_-B_-^* = \pm 1$, so that we can use both B_+ and B_- to impose a (symplectic) reality condition. The question is then which one to use. Remarkably, the cases in which this happens are also given by $n+t = \text{even}$! Now $B_{\pm} \propto B_{\mp}\Gamma_*$ by (A.1.27), so on chiral spinors a reality condition with $B_{\pm}$ can be traded for one with $B_{\mp}$ by simply redefining the phase α. We conclude that in the case of (symplectic)-Majorana-Weyl spinors there is a *choice* of B-matrix.

Dirac and Majorana conjugates Using (A.1.26) we can write (A.1.33) as

$$\psi^\dagger A\alpha^{-1} = \begin{cases} \psi^T C_{\pm}, & \text{if } t \text{ is even}, \\ \psi^T C_{\mp}, & \text{if } t \text{ is odd}. \end{cases} \tag{A.1.38}$$

[3] Actually, when $BB^* = -1$ it is still possible to impose a reality condition. However, we need a pair of spinors, ψ and χ. We then demand $\psi^* = B\chi$, $\chi^* = -B\psi$. This is called a *symplectic* (pseudo)-Majorana condition.

We define the *Dirac conjugate* of a spinor by $\bar{\psi}_D = \psi^\dagger A\alpha^{-1}$. “The” *Majorana conjugate* is defined by $\bar{\psi}_M = \psi^T C$; note that there are actually two of these and that one always needs to specify which particular C is being used. With these definitions, (A.1.33) simply reads:

$$\bar{\psi}_D \equiv \bar{\psi}_M. \tag{A.1.39}$$

Invariant actions A complex (i.e. Dirac) spinor field satisfies (A.1.32). The hermitian conjugate of (A.1.32) reads

$$\bar{\psi}_D(\Gamma^a \overleftarrow{\partial}_a - (-)^t m) = 0. \tag{A.1.40}$$

This depends explicitly on t, hence we have to be careful when we try to write down an action for the Dirac fermion. Since the variational principle works with *real* functions, we treat ψ and $\bar{\psi}$ as independent fields in the action $S[\psi, \bar{\psi}]$. Varying the action w.r.t. $\bar{\psi}$ will give us (A.1.32) whereas varying w.r.t. ψ will give us (A.1.40). Hence for $t =$ odd we obtain the well-known Lagrangian density (dropping the D on $\bar{\psi}$)

$$\mathcal{L} = \bar{\psi}(\Gamma \cdot \partial - m)\psi, \tag{A.1.41}$$

whereas for $t =$ even the following expression does the trick:

$$\mathcal{L} = \bar{\psi}\Gamma_*(\Gamma \cdot \partial - m)\psi. \tag{A.1.42}$$

By making a suitable choice for the phase α, both these actions can be seen to be hermitian. We have $\alpha = \pm 1$ for $t = 0, 3 \pmod 4$ and $\alpha = \pm i$ for $t = 1, 2 \pmod 4$. The sign of α can be fixed by requiring that the Hamiltonian is bounded from below. In Minkowski spacetimes we get $\alpha = +i$ (see e.g. section A.2.1).

Unfortunately, the actions derived from these Lagrangians sometimes vanish when considering Majorana spinors. By taking the transpose of these actions and using (A.1.23) and (A.1.28), one can show that (A.1.41) is a total derivative when $t = 3 \pmod 4$ and (A.1.42) when $t = 0 \pmod 4$, if ψ is Majorana. Hence the actions for Majorana spinors exist only when $t = 1, 2 \pmod 4$. Fortunately this includes the all-important case $t = 1$. Crucial in the above discussion is that one treats the fermion fields ψ as anticommuting *Grassmann numbers*, by means of which one incorporates the spin-statistics theorem already at the classical level.

Note that, when considering Majorana spinors, ψ and $\bar{\psi}$ are no longer independent. Finally a small remark on normalization conventions: for Dirac spinors we always take the actions (A.1.41) and (A.1.42) as they stand, but for Majorana spinors we normalize them with a factor of $\frac{1}{2}$.

Spinor indices When explicitly using indices on spinors, they get a lower index, e.g. ψ_α. The Dirac matrices read $(\Gamma_a)_\alpha{}^\beta$. Though it makes sense to talk about the symmetry properties of the Γ-matrices in a particular representation, one does not expect these properties to be valid in general. Now the C-matrices do not actually transform as matrices under a change of basis in spinor space, but rather as bilinear forms. In particular $C^{-1} \to C'^{-1} = UC^{-1}U^T$. It is therefore natural to assign lower indices to C^{-1}, i.e. we write $(C^{-1})_{\alpha\beta}$. Thus we get $(\Gamma^{(k)}C^{-1})_{\alpha\beta}$, hence the symmetry properties of these matrices are independent of the particular representation used, as discussed above. Similarly, we assign upper indices to C, i.e. $C^{\alpha\beta}$. We get for example $\lambda^\alpha = (\bar{\lambda}_M)^\alpha = (\lambda^T C)^\alpha = \lambda_\beta C^{\beta\alpha}$.

A.1.4 Clebsch, Gordan & Fierz

In this section we present several identities involving Γ-matrices that are indispensable in practical calculations.

One often needs to work out products of the form $\Gamma^{(i)}\Gamma^{(j)}$. The following examples serve to illustrate the general pattern:

$$\begin{aligned}
\Gamma_a\Gamma_b &= \Gamma_{ab} + \eta_{ab}, \\
\Gamma_{ab}\Gamma_c &= \Gamma_{abc} + \eta_{bc}\Gamma_a - \eta_{ac}\Gamma_b, \\
\Gamma_{ab}\Gamma_{cd} &= \Gamma_{abcd} + \eta_{bc}\Gamma_{ad} - \eta_{bd}\Gamma_{ac} - \eta_{ac}\Gamma_{bd} + \eta_{ad}\Gamma_{bc} + \eta_{bc}\eta_{ad} - \eta_{bd}\eta_{ac}.
\end{aligned}$$

Hence the product $\Gamma^{(i)}\Gamma^{(j)}$ gives rise to terms of the form $\eta^k\Gamma^{(i+j-2k)}$. There are $k!\binom{i}{k}\binom{j}{k}$ of such terms, i.e. the number of ways in which one can form k distinct pairs, each consisting of one i- and one j-index. These factors appear explicitly in the decomposition when one introduces antisymmetrization brackets on the right-hand side of these relations, e.g.:

$$\Gamma_{ab}\Gamma^{cd} = \Gamma_{ab}{}^{cd} + 4\delta_{[b}^{[c}\Gamma_{a]}{}^{d]} + 2\delta_{[b}^{[c}\delta_{a]}^{d]}.$$

Hence the Clebsch-Gordan decomposition of two Γ-matrices is given by:

$$\Gamma_{a_1\cdots a_i}\Gamma^{b_1\cdots b_j} = \sum_{k=0}^{\min(i,j)} k!\binom{i}{k}\binom{j}{k}\delta_{[a_i}^{[b_1}\cdots\delta_{a_{i-k+1}}^{b_k}\Gamma_{a_1\cdots a_{i-k}]}{}^{b_{k+1}\cdots b_j]}. \tag{A.1.43}$$

Another important relation is the following contraction identity:

$$\begin{aligned}
\Gamma_{a_1\cdots a_k}\Gamma_{b_1\cdots b_l}\Gamma^{a_1\cdots a_k} &= c_{k,l}\Gamma_{b_1\cdots b_l}, \\
c_{k,l} &= k!(-)^{[k/2]}(-)^{kl}\sum_{\substack{p=\max\\(0,k+l-d)}}^{\min(k,l)} (-)^p\binom{d-l}{k-p}\binom{l}{p}.
\end{aligned} \tag{A.1.44}$$

This can be proved as follows. First note that

$$\Gamma_{a_1\cdots a_k}\Gamma_{b_1\cdots b_l}\Gamma^{a_1\cdots a_k} = k!(-)^{k(k-1)/2}\Gamma_{|a_1\cdots a_k|}\Gamma_{b_1\cdots b_l}\Gamma^{|a_k\cdots a_1|},$$

where $a_1 < \ldots < a_k$. We split the sum into separate cases, according to the number of indices (which we call p) that $\Gamma^{(k)}$ has in common with $\Gamma^{(l)}$. Then:

$$\Gamma_{\hat{a}_1\cdots\hat{a}_k}\Gamma_{b_1\cdots b_l}\Gamma^{\hat{a}_k\cdots\hat{a}_1} = (-)^{kl}(-)^p\Gamma_{b_1\cdots b_l};$$

the hats indicate that the indices are not summed over. What remains is a combinatorial problem: we need to determine the number of terms in each case p. First, there are $\binom{l}{p}$ ways in which we can select p of the b-indices. These are put equal to p of the a-indices. The remaining $k-p$ a-indices have to be chosen from the $d-l$ numbers that are not a b-index. This can be done in $\binom{d-l}{k-p}$ ways.

Finally, we discuss a Fierz relation. The matrices

$$\{1, \Gamma_a, \Gamma_{a_1a_2}, \ldots, \Gamma_{1\cdots 2n}\}, \quad \text{with } a_1 < a_2 < \ldots, \tag{A.1.45}$$

form a complete set in the space of $2^n \times 2^n$ complex matrices. This means that we can expand any matrix $M_\alpha{}^\beta$ in spinor space as follows:

$$M_\alpha{}^\beta = \frac{1}{2^n}\sum_{k=0}^{2n}(-)^{[k/2]}\frac{1}{k!}\,\mathrm{tr}\,(\Gamma^{a_1\cdots a_k}M)\,(\Gamma_{a_1\cdots a_k})_\alpha{}^\beta. \tag{A.1.46}$$

This can be derived by means of the following trace identity:

$$\mathrm{tr}\,\Gamma_{a_1\cdots a_i}\Gamma^{b_1\cdots b_j} = \delta_{ij}(-)^{[i/2]}\,\delta^{b_1\cdots b_i}_{a_1\cdots a_i}\,\mathrm{tr}\,1, \tag{A.1.47}$$

which in turn follows from (A.1.43), after using that $\mathrm{tr}\,\Gamma^{(j)} = 0$ unless $j = 0$. Indeed, for j even we have (with the a_i all different):

$$\mathrm{tr}\,\Gamma_{a_1}\cdots\Gamma_{a_j} = -\mathrm{tr}\,\Gamma_{a_j}\Gamma_{a_1}\cdots\Gamma_{a_{j-1}} = -\mathrm{tr}\,\Gamma_{a_1}\cdots\Gamma_{a_j} \equiv 0.$$

We used the commutation relations and the cyclicity of the trace. For j odd:

$$\begin{aligned}\mathrm{tr}\,\Gamma_{a_1}\cdots\Gamma_{a_j} &= \mathrm{tr}\,\Gamma_{a_1}\cdots\Gamma_{a_j}\Gamma_*\Gamma_* = -\mathrm{tr}\,\Gamma_*\Gamma_{a_1}\cdots\Gamma_{a_j}\Gamma_* \\ &= -\mathrm{tr}\,\Gamma_{a_1}\cdots\Gamma_{a_j}\Gamma_*\Gamma_* = -\mathrm{tr}\,\Gamma_{a_1}\cdots\Gamma_{a_j} \equiv 0,\end{aligned}$$

which holds even when the a_i are not all different.

A.2 Supersymmetry

This section serves as an introduction to some of the basic properties of globally and locally supersymmetric field theories and sets the stage for the more complicated cases that are discussed in the main text. We will treat the classic example of a globally supersymmetric field theory, the Wess-Zumino model. In this relatively simple setting, we will see the spinor machinery of section A.1 at work.

A.2.1 The Wess-Zumino model

The simplest four-dimensional supersymmetric field theory on Minkowski spacetime is the Wess-Zumino model [168]. It is a free field theory consisting of a single massless fermion field χ and massless (pseudo)scalars[4]. According to section A.1, the minimal spinor in four dimensions has two real on-shell degrees of freedom and is either Majorana or Weyl. We will choose a Majorana spinor, i.e. $\chi^* = iB_+\chi$ or $\bar{\chi} = \chi^T C$, dropping the suffix on C_-. Hence we need two real scalar fields A and B to get a matching number of bosonic and fermionic degrees of freedom. The Lagrangian for the Wess-Zumino model reads:

$$\mathcal{L}_{\mathrm{WZ}} = -\frac{1}{2}\partial_a A\partial^a A - \frac{1}{2}\partial_a B\partial^a B - \frac{1}{2}\bar{\chi}\Gamma^a\partial_a\chi \tag{A.2.1}$$

It is not difficult to show that this Lagrangian is invariant (up to a total derivative) under the following *supersymmetry transformations*:

$$\delta_{\mathrm{Q}}(\epsilon)\, A = \bar{\epsilon}\chi, \tag{A.2.2a}$$

$$\delta_{\mathrm{Q}}(\epsilon)\, B = i\,\bar{\epsilon}\Gamma_*\chi, \tag{A.2.2b}$$

$$\delta_{\mathrm{Q}}(\epsilon)\, \chi = \partial_a A\,\Gamma^a\epsilon + i\,\partial_a B\,\Gamma^a\Gamma_*\epsilon \tag{A.2.2c}$$

Here ϵ is a *constant* anticommuting Majorana spinor, hence this is a *global* or *rigid* supersymmetry. The factors i are needed to preserve the reality of B and χ. We see from (A.2.2b) that B is indeed a pseudoscalar.

We are interested in whether these symmetry transformations form a closed algebra. The commutator of two of these transformations on the scalars is easily calculated using the flip properties for the Majorana spinors:

$$[\delta_{\mathrm{Q}}(\epsilon_1), \delta_{\mathrm{Q}}(\epsilon_2)]\, A = -2\,\bar{\epsilon}_1\Gamma^a\epsilon_2\,\partial_a A, \tag{A.2.3a}$$

$$[\delta_{\mathrm{Q}}(\epsilon_1), \delta_{\mathrm{Q}}(\epsilon_2)]\, B = -2\,\bar{\epsilon}_1\Gamma^a\epsilon_2\,\partial_a B \tag{A.2.3b}$$

This is an interesting result: the commutator of two supersymmetry transformations is proportional to the action of a spacetime translation $\tau^a = \bar{\epsilon}_1\Gamma^a\epsilon_2$ on the fields[5]. We are thus led to suspect that the supersymmetry transformations form a closed algebra that contains the Poincaré algebra as an ideal. We will see in a minute that this is indeed the case, but that the algebraic structure is not that of an ordinary Lie algebra.

For the fermion χ, the calculation is a bit more involved. We obtain

$$[\delta_{\mathrm{Q}}(\epsilon_1), \delta_{\mathrm{Q}}(\epsilon_2)]\, \chi = \Gamma^a(\epsilon_2\bar{\epsilon}_1 - \epsilon_1\bar{\epsilon}_2)\partial_a\chi - \Gamma^a\Gamma_*(\epsilon_2\bar{\epsilon}_1 - \epsilon_1\bar{\epsilon}_2)\Gamma_*\partial_a\chi$$

[4]We limit ourselves to the massless case for simplicity, though we could have certainly considered massive fields. However, as we will see below, supersymmetry then demands that the fields all have the same mass.

[5]Under a translation $x^a \mapsto x^a + t^a$, fields Φ transform as $\Phi(x) \mapsto \Phi'(x) \equiv \Phi(x - t)$. For an infinitesimal translation τ we get $\delta_{\mathrm{T}}(\tau)\Phi = \Phi'(x) - \Phi(x) = -\tau^a\partial_a\Phi \equiv \tau^a P_a$. Thus a translation P_a is realized on fields by $-\partial_a$.

We would like an expression similar to that for the bosons, i.e. one containing $\bar{\epsilon}_1\Gamma^a\epsilon_2$. We use the following Fierz relation to achieve this:

$$\psi\bar{\chi} = -\tfrac{1}{4}\left[(\bar{\chi}\psi)1 + (\bar{\chi}\Gamma_*\psi)\Gamma_* + (\bar{\chi}\Gamma^a\psi)\Gamma_a - (\bar{\chi}\Gamma_*\Gamma^a\psi)\Gamma_*\Gamma_a - \tfrac{1}{2}(\bar{\chi}\Gamma^{ab}\psi)\Gamma_{ab}\right]. \quad \text{(A.2.4)}$$

Several cancellations occur and in the end we obtain:

$$[\delta_{\mathrm{Q}}(\epsilon_1), \delta_{\mathrm{Q}}(\epsilon_2)]\,\chi = -2\,\bar{\epsilon}_1\Gamma^a\epsilon_2\,\partial_a\chi + (\bar{\epsilon}_1\Gamma^a\epsilon_2)\Gamma_a\not{\partial}\chi \qquad \text{(A.2.5)}$$

The first term is again a translation, but the second term does not correspond to any known symmetry transformation. Luckily, it vanishes when the field equation for χ is satisfied. This is a general feature of supersymmetric field theories: *the supersymmetry algebra only closes on-shell.* This is an obvious consequence of the fact that the Dirac equation acts as a projection operator, thereby eliminating one-half of the fermionic degrees of freedom. So although the number of bosonic and fermionic degrees of freedom match on-shell, there is a mismatch off-shell[6].

We write a supersymmetry transformation on a generic field Φ as $\delta_{\mathrm{Q}}(\epsilon)\Phi = (\bar{\epsilon}Q)\Phi = \epsilon^\alpha Q_\alpha\Phi$, where Q_α is the generator of supersymmetries or *supercharge* (see appendix A for our conventions on spinor indices). Since $\delta_{\mathrm{Q}}(\epsilon)$ preserves the reality of the fields and ϵ is a Majorana spinor, the supercharge Q is also Majorana. Moreover, since $\delta_{\mathrm{Q}}(\epsilon)$ is an ordinary bosonic (Grassmann-even) operator, and ϵ is Grassmann-odd, the supercharges need to be Grassmann-odd as well. Hence:

$$\begin{aligned}[\delta_{\mathrm{Q}}(\epsilon_1), \delta_{\mathrm{Q}}(\epsilon_2)] &= (\epsilon_1^\alpha Q_\alpha)(\epsilon_2^\beta Q_\beta) - (\epsilon_2^\beta Q_\beta)(\epsilon_1^\alpha Q_\alpha) \\ &= -\epsilon_1^\alpha\epsilon_2^\beta(Q_\alpha Q_\beta + Q_\beta Q_\alpha) = -\epsilon_1^\alpha\epsilon_2^\beta\{Q_\alpha, Q_\beta\}\end{aligned}$$

On the other hand $\bar{\epsilon}_1\Gamma^a\epsilon_2 = \epsilon_1^T C\Gamma^a\epsilon_2 = -\epsilon_1^T C\Gamma^a C^{-1}(\epsilon_2^T C)^T = -\epsilon_1^\alpha\epsilon_2^\beta(\Gamma^a C^{-1})_{\alpha\beta}$ and thus:

$$\{Q_\alpha, Q_\beta\} = 2(\Gamma^a C^{-1})_{\alpha\beta}P_a. \qquad \text{(A.2.6)}$$

The supersymmetry transformations δ_ϵ of (A.2.2) commute with translations δ_τ and Lorentz transformations δ_ω. Now

$$\begin{aligned}[\delta_{\mathrm{L}}(\omega), \delta_{\mathrm{Q}}(\epsilon)] &= \frac{1}{2}[\omega^{ab}M_{ab}, \bar{\epsilon}Q] = -\frac{1}{4}\omega^{ab}\,\bar{\epsilon}\Gamma_{ab}Q + \frac{1}{2}\omega^{ab}\,\epsilon^\alpha[M_{ab}, Q_\alpha], \\ [\delta_{\mathrm{T}}(\tau), \delta_{\mathrm{Q}}(\epsilon)] &= [\tau^a P_a, \bar{\epsilon}Q] = \tau^a\epsilon^\alpha[P_a, Q_\alpha].\end{aligned}$$

[6]Sometimes it is possible to extend the field content with so-called auxiliary fields in such a way that the algebra does close off-shell. These fields then need to have an algebraic equation of motion, so that they can be eliminated on-shell. This is for example possible for the Wess-Zumino model and the $\mathcal{N} = 1$ supergravity theory in four dimensions. However, for the theories with extended supersymmetry and in particular the ten-dimensional theories in which we are interested, it is not known how to obtain a suitable set of auxiliary fields. We will just have to make do without.

Thus the supercharge Q_α is invariant under translations and a spinor under Lorentz transformations:

$$[P_a, Q_\alpha] = 0, \tag{A.2.7a}$$

$$[M_{ab}, Q_\alpha] = \frac{1}{2}(\Gamma_{ab}Q)_\alpha. \tag{A.2.7b}$$

The equations (A.2.6) and (A.2.7), together with (A.1.1), constitute a so-called *superalgebra*[7]. The particular algebra under consideration is known as the $\mathcal{N}=1$, $d=4$ super-Poincaré algebra, where $\mathcal{N}$ stands for the number of supercharges.

We end our discussion of the Wess-Zumino model with a few observations. First of all, $P^2 = P_a P^a$ is a Casimir operator of the entire super-Poincaré algebra. This implies that all fields in a single *supermultiplet* have the same mass. More interesting is the following. From (A.2.6) we obtain (dropping the spinor indices):

$$\{Q, Q^*\} = \alpha\{Q, Q\}B^T = 2\alpha\Gamma^a A^{-1} P_a, \tag{A.2.8}$$

where we used (A.1.26) and reinstated the constant α of (A.1.35). We have taken α to be $+i$ and now we see why. The LHS of (A.2.8) is a positive semi-definite operator. Taking a trace over the spinor indices, we obtain:

$$\mathrm{tr}\,\{Q, Q^*\} = 2\alpha\, P_a\, \mathrm{tr}\,\Gamma^a\Gamma^0 = -2\alpha\, P_0\, \mathrm{tr}\,1 = -8i\alpha\, H \geq 0.$$

This implies that (with $\alpha = +i$!), the Hamiltonian H is not only bounded from below but even nonnegative:

$$H \geq 0. \tag{A.2.9}$$

This inequality is saturated by those states $|\cdot\rangle$ that are invariant under all the supersymmetries: $Q|\cdot\rangle = 0$. From (A.2.6) we see that the only state in the theory that satisfies this condition is invariant under the translations P_a, i.e. the vacuum.

[7] A superalgebra is basically a Lie algebra that is extended by adding anticommuting generators to it and a suitable redefinition of the bracket operation and the Jacobi identity. It can be shown on quite general physical grounds that the only way in which one can nontrivially combine an internal symmetry algebra and the Poincaré transformations in a single algebraic structure, is by embedding these algebras in a superalgebra [169, 170]. We could have used this as the starting point of our discussion, i.e. first construct the super-Poincaré algebra, then find its representations *à la* Wigner and finally construct supersymmetric field theories. This is indeed the approach that is adopted in many treatments, but it would have taken us too far afield. For more information see e.g. [34, 163, 165, 171–173].

Appendix B

Miscellanea

In this appendix we present additional background material for some points in the main text. We also discuss our conventions.

B.1 Conventions and basic results

No work in string theory is complete without a discussion of the conventions. We already discussed fermions in appendix A, so this section will be entirely about bosons.

B.1.1 Indices & differential forms

First, a warning about indices. We do *not* use a consistent nor an unambiguous set of conventions for our indices throughout this thesis. In particular, the notation in chapters 1 and 2 differs from that in chapters 3 and 4. We explain in the text what conventions we are using at a given time.

Here we summarize our conventions for the differential form notation which we use throughout chapters 1 and 2. For a p-form $C^{(p)}$ we write

$$C^{(p)} = \frac{1}{p!} C_{\mu_1 \cdots \mu_p} \, \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}, \tag{B.1.1}$$

where $C_{\mu_1 \cdots \mu_p}$ denotes the components of the p-form w.r.t. a coordinate basis. We define antisymmetrization with 'weight one', i.e. $C_{\mu_1 \cdots \mu_p} = C_{[\mu_1 \cdots \mu_p]}$. The wedge product of the p basis 1-forms $\mathrm{d}x^\mu$ is defined by

$$\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} \equiv \sum_\pi \varsigma(\pi) \, \mathrm{d}x^{\mu_{\pi(1)}} \otimes \ldots \otimes \mathrm{d}x^{\mu_{\pi(p)}}, \tag{B.1.2}$$

where $\varsigma(\pi) = +1, -1$ if the permutation π is even or odd, respectively. The wedge product $A^{(p)} \wedge B^{(q)}$ of a p-form $A^{(p)}$ and a q-form $B^{(q)}$ is defined by

$$A^{(p)} \wedge B^{(q)} = \frac{1}{p!q!} A_{\mu_1 \cdots \mu_p} B_{\nu_1 \cdots \nu_q}\, \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} \wedge \mathrm{d}x^{\nu_1} \wedge \ldots \wedge \mathrm{d}x^{\nu_q}, \qquad \text{(B.1.3)}$$

hence $(A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]}$. The exterior derivative $\mathrm{d}C^{(p)}$ is defined by

$$\mathrm{d}C \equiv \frac{1}{p!} \partial_\nu C_{\mu_1 \cdots \mu_p}\, \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}, \qquad \text{(B.1.4)}$$

hence $(\mathrm{d}C)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\partial_{[\mu_1} C_{\mu_2 \cdots \mu_{p+1}]}$. Note that $\mathrm{d}^2 = 0$.

We denote the space of p-forms on a manifold M by $\Omega^p(M)$. In particular $\Omega^1(M) = T^*(M)$, i.e. 1-forms live in the cotangent bundle of M. Consider a submanifold Σ of M. The embedding of Σ in M is given by the inclusion map $\imath : \Sigma \to M$ and can be described in local coordinated by the functions $x^\mu(\sigma^\alpha)$, where x^μ are local coordinates on M and σ^α on Σ. We can use $\imath$ to 'pull-back' p-form fields $C^{(p)}$ that live on M to p-form fields that live on Σ. We have the map $\imath^* : \Omega^p(M) \to \Omega^p(\Sigma)$ which acts on p-forms $\imath^* : C^{(p)} \mapsto \imath^* C^{(p)}$ as

$$(\imath^* C)_{\alpha_1 \cdots \alpha_p} = C_{\mu_1 \cdots \mu_p} \partial_{\alpha_1} x^{\mu_1} \cdots \partial_{\alpha_p} x^{\mu_p}. \qquad \text{(B.1.5)}$$

We always use the 'mostly-plus' metric:

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \eta_{ab}\, e^a \otimes e^b, \qquad \text{(B.1.6)}$$

with $\eta_{ab} = \mathrm{diag}(-1, \ldots, -1, +1, \ldots, +1)$ as in (A.1.6) and where $e^a = e^a_\mu \mathrm{d}x^\mu$ is an orthonormal frame of 1-forms. e^a_μ is known as the vielbein, we denote its inverse by E^μ_a. It is customary to denote the μ-indices by 'curved' and the a-indices by 'flat', even though the μ-coordinates could in principle be Cartesian coordinates in Minkowksi space, say. We can use the vielbein to switch from curved to flat indices and vice versa, e.g. $A^\mu{}_\nu = E^\mu_a e^b_\nu A^a{}_b$.

The Levi-Civita *tensor* ε is defined by:

$$\varepsilon_{\mu_1 \cdots \mu_d} = \sqrt{|g|}\, \tilde{\varepsilon}_{\mu_1 \cdots \mu_d}, \qquad \varepsilon^{\mu_1 \cdots \mu_d} = \frac{1}{\sqrt{|g|}}\, \tilde{\varepsilon}^{\mu_1 \cdots \mu_d}, \qquad \text{(B.1.7)}$$

where the Levi-Civita *symbol* $\tilde{\varepsilon}$ is defined by

$$\tilde{\varepsilon}_{12 \cdots d} = +1, \qquad \tilde{\varepsilon}^{12 \cdots d} = (-)^t. \qquad \text{(B.1.8)}$$

The Hodge duality operation maps p-forms to q-forms with $q = d - p$, $* : \Omega^p(M) \to \Omega^{d-p}(M)$. It is defined by

$$*(\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}) \equiv \frac{1}{q!} \varepsilon_{\nu_1 \cdots \nu_q}{}^{\mu_1 \cdots \mu_p}\, \mathrm{d}x^{\nu_1} \wedge \ldots \wedge \mathrm{d}x^{\nu_q}. \qquad \text{(B.1.9)}$$

Hence $(*)^2 = (-)^{pq+t}$ and

$$(*A)_{\mu_1\cdots\mu_q} = \varepsilon_{\mu_1\cdots\mu_q}{}^{|\nu_1\cdots\nu_p|} A_{|\nu_1\cdots\nu_p|}. \tag{B.1.10}$$

In particular

$$*1 = \sqrt{|g|}\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^d \equiv \mathrm{d}^d x\,\sqrt{|g|}. \tag{B.1.11}$$

Finally we note that

$$*A \wedge B = \frac{1}{p!} A_{\mu_1\cdots\mu_p} B^{\mu_1\cdots\mu_p} *1. \tag{B.1.12}$$

B.1.2 Einstein-Cartan, Weyl and Kaluza-Klein

We use the Einstein-Cartan formulation of gravity (see e.g. [9, 46, 48, 174]). We start from

$$\mathrm{d}\epsilon^a = \frac{1}{2}\Omega^a{}_{bc} e^b \wedge e^c, \tag{B.1.13}$$

with the Ricci coefficients $\Omega^a{}_{bc} = E^\mu_b E^\nu_c(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu)$. We then have for the components of $\mathrm{d}A$ w.r.t. an orthonormal frame (A is a 1-form):

$$(\mathrm{d}A)_{ab} = \partial_a A_b - \partial_b A_a + \Omega^c{}_{ab} A_c, \tag{B.1.14}$$

where we define $\partial_a = E^\mu_a \partial_\mu$. We convert all tensors to flat indices by using (inverse) vielbeins and define spacetime covariant derivatives:

$$\nabla_\mu V^a = \partial_\mu V^a + \omega^a_{\mu b} V^b, \qquad \nabla_\mu A_a = \partial_\mu A_a - \omega^b_{\mu a} A_b. \tag{B.1.15}$$

The connection $\omega^a_{\mu b}$ is metric compatible: $\nabla_\mu \eta_{ab} = 0$. This implies that $\omega_{\mu ab} = -\omega_{\mu ba}$, where $\omega_{\mu ab} = \eta_{bc}\omega^c_{\mu a}$. We define connection 1-forms by $\omega^a{}_b = \omega^a_{\mu b}\mathrm{d}x^\mu$. The torsion 1-form $\mathcal{T}^a$ and curvature 2-form $\mathcal{R}^a{}_b$ are defined by the Maurer-Cartan equations:

$$\mathcal{T}^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b, \tag{B.1.16a}$$

$$\mathcal{R}^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{B.1.16b}$$

There is a unique metric-compatible and torsion-free (i.e. $\mathcal{T}^a = 0$) connection known as the Levi-Civita connection. It is given by:

$$\omega_{abc} = -\frac{1}{2}(\Omega_{abc} - \Omega_{bca} - \Omega_{cab}), \tag{B.1.17}$$

where we defined $\omega_{abc} = E^\mu_a \omega_{\mu ab}$. The Riemann tensor $R^a{}_{bcd}$ can be read off from $\mathcal{R}^a{}_b = \frac{1}{2!}R^a{}_{bcd} e^a \wedge e^b$. The Ricci tensor is defined as $R_{ab} = R^c{}_{acb}$ and the curvature scalar – from which we build the Einstein-Hilbert action – by $R = \eta^{ab}R_{ab}$. The following identity is very useful:

$$R = -2\partial_a\omega^a + \omega_{abc}\omega^{cab} - \omega_a\omega^a, \tag{B.1.18}$$

where $\omega_a = \eta^{bc}\omega_{bca}$. Indeed, the components of the curvature 2-form are $R^{ab}{}_{cd} = (\mathrm{d}\omega^{ab})_{cd} + (\omega^{ae} \wedge \omega_e{}^b)_{cd}$. The desired result follows after using:

$$(\mathrm{d}\omega^{ab})_{cd} = \partial_c \omega_d{}^{ab} - \partial_d \omega_c{}^{ab} + \Omega^e{}_{cd}\omega_e{}^{ab} = \partial_c \omega_d{}^{ab} - \partial_d \omega_c{}^{ab} + (\omega_{cde} + \omega_{dec})\omega^{eab},$$

and contracting indices, $R = R^{ab}{}_{ab}$.

Weyl rescalings

The expression (B.1.18) provides an efficient shortcut for the calculation of the curvature scalar. We now show how this works for the Weyl rescaling of the metric – something we have used several times in chapters 1 and 2.

We are given the metric $\bar{g}_{\mu\nu}$ and define a new metric $g_{\mu\nu}$ by

$$\bar{g}_{\mu\nu} = \mathrm{e}^{2\alpha\phi} g_{\mu\nu}, \tag{B.1.19}$$

where α is a number. We will show that

$$\bar{R}[\bar{g}] = \mathrm{e}^{-2\alpha\phi}\Big[R[g] - \alpha^2(d-1)(d-2)\partial_\mu\phi\partial^\mu\phi - 2\alpha(d-1)\nabla_\mu\partial^\mu\phi\Big]. \tag{B.1.20}$$

First we construct the vielbein 1-forms. We have $\bar{e}^a = \mathrm{e}^{\alpha\phi}e^a$. Now

$$\mathrm{d}\bar{e}^a = \mathrm{d}\mathrm{e}^{\alpha\phi} \wedge e^a + \mathrm{e}^{\alpha\phi}\mathrm{d}e^a = \alpha \mathrm{e}^{\alpha\phi}\delta^a_c\partial_b\phi\, e^b \wedge e^c + \tfrac{1}{2}\mathrm{e}^{\alpha\phi}\Omega^a{}_{bc}\, e^b \wedge e^c,$$

from which we read off $\bar{\Omega}^a{}_{bc} = \mathrm{e}^{-\alpha\phi}\big[\Omega^a{}_{bc} - \alpha(\delta^a_b\partial_c\phi - \delta^a_c\partial_b\phi)\big]$. The spin connection follows straightforwardly:

$$\bar{\omega}_{abc} = \mathrm{e}^{-\alpha\phi}\big[\omega_{abc} + \alpha(\eta_{ab}\partial_c\phi - \eta_{ac}\partial_b\phi)\big], \qquad \bar{\omega}_a = \mathrm{e}^{-\alpha\phi}\big[\omega_a + \alpha(d-1)\partial_a\phi\big].$$

We have to be a bit careful with the derivatives ∂_a since they contain vielbeins. We have $\bar{\partial}_a = \mathrm{e}^{-\alpha\phi}\partial_a$ and obtain

$$\begin{aligned}
\bar{\partial}_a\bar{\omega}^a &= \mathrm{e}^{-2\alpha\phi}\big[\partial_a\omega^a - \alpha^2(d-1)\partial_a\phi\partial^a\phi - \alpha\,\omega^a\partial_a\phi + \alpha(d-1)\partial_a\partial^a\phi\big],\\
\bar{\omega}_{abc}\bar{\omega}^{cab} &= \mathrm{e}^{-2\alpha\phi}\big[\omega_{abc}\omega^{cab} - 2\alpha\,\omega^a\partial_a - \alpha^2(d-1)\partial_a\phi\partial^a\phi\big],\\
\bar{\omega}_a\bar{\omega}^a &= \mathrm{e}^{-2\alpha\phi}\big[\omega_a\omega^a + 2\alpha(d-1)\omega^a\partial_a\phi + \alpha^2(d-1)^2\partial_a\phi\partial^a\phi\big].
\end{aligned}$$

The desired result follows upon recognizing that $\nabla_a V^a = (\partial_a + \omega_a)V^a$.

Kaluza-Klein reduction

We now consider the Kaluza-Klein reduction of the curvature scalar from $(d+1)$ to d dimensions. We denote $\hat{x}^{\hat{\mu}} = (x^\mu, z)$ and write higher dimensional fields and indices with a hat. The metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ decomposes into the lower-dimensional metric $g_{\mu\nu}$, a gauge field A_μ and a modulus ϕ as follows

$$\widehat{\mathrm{d}s}^2 = \mathrm{e}^{2\alpha\phi} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2\beta\phi}(\mathrm{d}z - A_\mu \mathrm{d}x^\mu)^2, \tag{B.1.21}$$

with α and β are real parameters. This parametrization guarantees that the d-dimensional fields have the correct transformation properties under the d-dimensional general coordinate transformations. The reduction is achieved by taking the unhatted fields to be independent of the z coordinate, i.e. $\partial_z g_{\mu\nu} = \partial_z A_\mu = \partial_z \phi = 0$.

The reduction of the Einstein-Hilbert action starts with picking a basis of vielbein 1-forms:

$$\hat{e}^a = \mathrm{e}^{\alpha\phi} e^a, \qquad \hat{e}^{\underline{z}} = \mathrm{e}^{\beta\phi}(\mathrm{d}z + A). \tag{B.1.22}$$

We underline the z-index to emphasize that it is a *flat* index. We have used local Lorentz transformations to gauge away $\hat{e}^a_{\underline{z}}$. A calculation similar the one we presented above then yields:

$$\begin{aligned} \sqrt{|\hat{g}|}\widehat{R} = \sqrt{|g|}\, \mathrm{e}^{[(d-2)\alpha+\beta]\phi} \Big[R - \frac{1}{4}\mathrm{e}^{-2(\alpha-\beta)\phi} F_{\mu\nu}F^{\mu\nu} - 2\big[\alpha(d-1)+\beta\big]\nabla_\mu \partial^\mu \phi \\ - \big[\alpha^2(d-1)(d-2) + 2\alpha\beta(d-2) + 2\beta^2\big]\partial_\mu\phi\partial^\mu\phi \Big], \end{aligned} \tag{B.1.23}$$

with $F = \mathrm{d}A$. We consider now the reduction of pure gravity in the Einstein frame over a circle. We take the following coordinate interval $z \in [0, 2\pi\rho)$, where ρ is a parameter with the dimension of length. It is convenient to take $\alpha = 0$ and $\beta = \frac{1}{2}$ and we obtain:

$$S[\hat{g}] = \frac{1}{2\hat{\kappa}^2}\int \mathrm{d}^{d+1}\hat{x}\, \sqrt{|\hat{g}|}\widehat{R} = \frac{2\pi\rho}{2\hat{\kappa}^2}\int \mathrm{d}^d x\, \sqrt{|g|}\, \mathrm{e}^{\phi}\Big[R - \frac{1}{4}\mathrm{e}^{2\phi}F_{\mu\nu}F^{\mu\nu} - 2\,\mathrm{e}^{-\phi}\Box \mathrm{e}^{\phi}\Big].$$

We can drop the last term since it is a total derivative, $\sqrt{|g|}\nabla_\mu V^\mu = \partial_\mu(\sqrt{|g|}V^\mu)$. We now expand the modulus ϕ around its vacuum expectation value and write $\varphi = \phi - \langle\phi\rangle$ and perform a Weyl rescaling that gets rid of the e^{φ} in front of R. We obtain

$$S[g, A, \varphi] = \frac{1}{2\kappa^2}\int \mathrm{d}^d x\, \sqrt{|g|}\Big[R - \frac{1}{4}\mathrm{e}^{2\varphi}F_{\mu\nu}F^{\mu\nu} - \frac{d-1}{d-2}\partial_\mu\varphi\partial^\mu\varphi\Big], \tag{B.1.24}$$

where we again dropped a total derivative. We defined the d-dimensional gravitational coupling constant by

$$\frac{1}{2\kappa^2} \equiv 2\pi\rho\, \mathrm{e}^{\langle\phi\rangle}\frac{1}{2\hat{\kappa}^2}. \tag{B.1.25}$$

This is the relation we used in equation (1.3.18), with $R \equiv \rho\, \mathrm{e}^{\langle\phi\rangle}$ the physical radius of the circle.

B.1.3 Yang-Mills theory

Here we review the standard construction of Yang-Mills theories in order to fix our conventions. Consider fields Φ^i that carry a matrix representation $(\mathsf{T}_\alpha)^i{}_j$ of a Lie algebra $\mathfrak{g}$:

$$[\mathsf{T}_\alpha, \mathsf{T}_\beta] = f^\gamma{}_{\alpha\beta}\mathsf{T}_\gamma, \tag{B.1.26}$$

i.e. $\delta\Phi^i = \epsilon^\alpha(\mathsf{T}_\alpha)^i{}_j\Phi^j$, where the parameters ϵ^α are real and are allowed to vary over spacetime. The structure constants $f^\gamma{}_{\alpha\beta}$ are also real and satisfy $f^\gamma{}_{\alpha\beta} = -f^\gamma{}_{\beta\alpha}$. We define the gauge covariant derivative on Φ as

$$(D_\mu\Phi)^i \equiv \partial_\mu\Phi^i + gA^\alpha_\mu(\mathsf{T}_\alpha)^i{}_j\Phi^j. \tag{B.1.27}$$

We will drop the parentheses and simply write $D_\mu\Phi^i$ in the following. g is the Yang-Mills coupling constant. The covariant derivative transforms as $\delta D_\mu\Phi^i = \epsilon^\alpha(\mathsf{T}_\alpha)^i{}_jD_\mu\Phi^j$ under gauge transformations if we suppose that the gauge fields A^α_μ transform as

$$\delta A^\alpha_\mu = -\frac{1}{g}\partial_\mu\epsilon^\alpha + f^\alpha{}_{\beta\gamma}\epsilon^\beta A^\gamma_\mu. \tag{B.1.28}$$

The field strength $F^\alpha_{\mu\nu}$ is defined as the commutator of two covariant derivatives:

$$[D_\mu, D_\nu]\Phi^i \equiv gF^\alpha_{\mu\nu}(\mathsf{T}_\alpha)^i{}_j\Phi^j, \tag{B.1.29}$$

from which

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\nu + gf^\alpha{}_{\beta\gamma}A^\beta_\mu A^\gamma_\nu. \tag{B.1.30}$$

A straightforward calculation yields $\delta F^\alpha_{\mu\nu} = f^\alpha{}_{\beta\gamma}\epsilon^\beta F^\gamma_{\mu\nu}$ after using the Jacobi identity

$$f^\delta{}_{\epsilon\alpha}f^\epsilon{}_{\beta\gamma} + f^\delta{}_{\epsilon\beta}f^\epsilon{}_{\gamma\alpha} + f^\delta{}_{\epsilon\gamma}f^\epsilon{}_{\alpha\beta} = 0. \tag{B.1.31}$$

The *adjoint* representation is defined by the action of the algebra on itself:

$$\mathrm{Ad}\,\mathsf{T}_\alpha|T_\beta\rangle \equiv |[\mathsf{T}_\alpha, \mathsf{T}_\beta]\rangle = f^\gamma{}_{\alpha\beta}|\mathsf{T}_\gamma\rangle. \tag{B.1.32}$$

The matrix elements of the adjoint representation are thus given by $(\mathsf{T}^{\mathrm{adj}}_\alpha)^\beta{}_\gamma = f^\beta{}_{\alpha\gamma}$. We see that the field strength transforms in the adjoint, whereas the gauge field has an inhomogeneous term. The *coadjoint* representation is given by $(\mathsf{T}^{\mathrm{coadj}}_\alpha)_\beta{}^\gamma = -f^\gamma{}_{\alpha\beta}$. The contraction of a field A^α in the adjoint with a field B_α in the coadjoint is invariant under gauge transformations: $\delta(A^\alpha B_\alpha) = \epsilon^\gamma f^\alpha{}_{\gamma\beta}A^\beta B_\alpha - A^\alpha\epsilon^\gamma f^\beta{}_{\gamma\alpha}B_\beta = 0$.

Since $F^\alpha_{\mu\nu}$ transforms covariantly under gauge transformations, the expression $D_\mu F^\alpha_{\nu\rho}$ is well defined. It is straightforward to show that $F^\alpha_{\mu\nu}$ satisfies the *Bianchi identity*

$$D_\mu F^\alpha_{\nu\rho} + D_\nu F^\alpha_{\rho\mu} + D_\rho F^\alpha_{\mu\nu} \equiv 0, \tag{B.1.33}$$

using either the definition (B.1.29) or the Jacobi identity (B.1.31). Note that we can write (B.1.28) as $\delta A^\alpha_\mu = -\frac{1}{g} D_\mu \epsilon^\alpha$.

It follows from the Jacobi identity that $\delta f^\alpha{}_{\beta\gamma} = \epsilon^\kappa (f^\alpha{}_{\kappa\delta} f^\delta{}_{\beta\gamma} - f^\delta{}_{\kappa\beta} f^\alpha{}_{\delta\gamma} - f^\delta{}_{\kappa\gamma} f^\alpha{}_{\beta\delta}) = 0$, i.e the structure constants are invariant tensors of the gauge algebra. The tensor

$$K_{\alpha\beta} \equiv \mathrm{Tr}\, \mathsf{T}^{\mathrm{adj}}_\alpha \mathsf{T}^{\mathrm{adj}}_\beta = f^\gamma{}_{\alpha\delta} f^\delta{}_{\beta\gamma} \tag{B.1.34}$$

is thus also invariant. This symmetric tensor is known as the Cartan-Killing metric of the Lie algebra (B.1.26). Contract the Jacobi identity with $f^\gamma{}_{\kappa\delta}$ and rename indices to obtain

$$K_{\alpha\delta} f^\delta{}_{\beta\gamma} = -f^\kappa{}_{\lambda\alpha} f^\lambda{}_{\mu\beta} f^\mu{}_{\kappa\gamma} + f^\kappa{}_{\lambda\beta} f^\lambda{}_{\mu\alpha} f^\mu{}_{\kappa\gamma}.$$

Hence

$$K_{\alpha\delta} f^\delta{}_{\beta\gamma} = -K_{\beta\delta} f^\delta{}_{\alpha\gamma}. \tag{B.1.35}$$

So the structure constants with only lower indices, $f_{\alpha\beta\gamma} \equiv K_{\alpha\delta} f^\delta{}_{\beta\gamma}$, are completely antisymmetric in all three indices.

We can use the Cartan-Killing metric to construct an invariant Lagrangian for the gauge field:

$$\mathcal{L}[A^\alpha_\mu] = \frac{1}{4} K_{\alpha\beta} F^\alpha_{\mu\nu} F^{\beta\mu\nu}. \tag{B.1.36}$$

Since $K_{\alpha\beta}$ is real and symmetric, it has real eigenvalues and can be diagonalized. An important physical requirement is the absence of ghosts, i.e. the absence of fields which have the wrong sign in front of their kinetic terms in the Lagrangian. Furthermore, we would like to have a nondegenerate Lagrangian. These requirements are fulfilled if the eigenvalues of $K_{\alpha\beta}$ are negative definite. This is the case if $\mathfrak{g}$ is the Lie algebra of a *compact, semisimple* Lie group $\mathcal{G}$ (see e.g. [175, 176]), which is what we will suppose from now on. By rescaling the generators, we can always make $K_{\alpha\beta} = \mathrm{Tr}\, \mathsf{T}^{\mathrm{adj}}_\alpha \mathsf{T}^{\mathrm{adj}}_\beta = -\delta_{\alpha\beta}$. It can be shown that $\mathrm{Tr}\, \mathsf{T}_\alpha \mathsf{T}_\beta = -c\, \delta_{\alpha\beta}$ for the other representations of $\mathfrak{g}$, where c is a positive real number that depends on the representation in question.

The finite dimensional representations U of a compact group $\mathcal{G}$ are all unitary and hence $(\mathsf{T}_\alpha)^\dagger = -\mathsf{T}_\alpha$ (we write $\mathsf{U} = \exp \epsilon^\alpha \mathsf{T}_\alpha$). Invariant terms for the fields Φ^i are now easily constructed. We write Φ^i as a column vector Φ. We have $\delta\Phi = \epsilon^\alpha \mathsf{T}_\alpha \Phi$ and thus $\delta\Phi^\dagger = -\Phi^\dagger \mathsf{T}_\alpha \epsilon^\alpha$, $\delta(D_\mu\Phi)^\dagger = -(D_\mu\Phi)^\dagger \mathsf{T}_\alpha \epsilon^\alpha$, etc. Possible invariants are thus for instance $\Phi^\dagger\Phi$ and $(D_\mu\Phi)^\dagger D^\mu\Phi$.

We can also introduce a matrix notation for the gauge field and other fields that are valued in the adjoint. Take any finite dimensional representation $(\mathsf{T}_\alpha)^i{}_j$ and define *matrix-valued* fields:

$$(\mathsf{A}_\mu)^i{}_j \equiv A^\alpha_\mu (\mathsf{T}_\alpha)^i{}_j, \quad (\mathsf{F}_{\mu\nu})^i{}_j \equiv F^\alpha_{\mu\nu} (\mathsf{T}_\alpha)^i{}_j, \quad (\mathsf{D}_\mu\Phi)^i{}_j \equiv D_\mu \Phi^\alpha (\mathsf{T}_\alpha)^i{}_j \tag{B.1.37}$$

and so on. We drop the indices i, j from now on. In terms of these fields we have for instance

$$\mathsf{D}_\mu \Phi = \partial_\mu \Phi + g[\mathsf{A}_\mu, \Phi], \tag{B.1.38}$$

and

$$[\mathsf{D}_\mu, \mathsf{D}_\nu]\Phi = g[\mathsf{F}_{\mu\nu}, \Phi], \quad \text{with} \quad \mathsf{F}_{\mu\nu} = \partial_\mu \mathsf{A}_\nu - \partial_\nu \mathsf{A}_\mu + g[\mathsf{A}_\mu, \mathsf{A}_\nu]. \tag{B.1.39}$$

The finite versions of the infinitesimal gauge transformations $\delta\Phi = [\epsilon, \Phi]$ now read $\Phi \to \mathsf{U}\Phi\mathsf{U}^\dagger$, where U is in the same representation as T_α. Gauge invariant objects can be formed by taking traces, e.g. $\mathrm{Tr}\,\mathsf{F}_{\mu\nu}\mathsf{D}^\mu\Phi\mathsf{D}^\nu\Phi$. We have in particular

$$\mathcal{L}[\mathsf{A}_\mu] = \frac{1}{4c}\mathrm{Tr}\,\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}. \tag{B.1.40}$$

Now we redefine the generators as $\widetilde{\mathsf{T}}_\alpha = \mathsf{T}_\alpha/\sqrt{c}$, so that $\mathrm{Tr}\,\widetilde{\mathsf{T}}_\alpha\widetilde{\mathsf{T}}_\beta = -\delta_{\alpha\beta}$ (this of course changes the normalization of the trace in the other representations as well), the coupling constant as $\widetilde{g} = g\sqrt{c}$, and the gauge field as $\widetilde{A}^\alpha_\mu = \widetilde{g}A^\alpha_\mu$. The Lagrangian becomes

$$\mathcal{L}[\widetilde{\mathsf{A}}_\mu] = \frac{1}{4\widetilde{g}^2}\mathrm{Tr}\,\widetilde{\mathsf{F}}_{\mu\nu}\widetilde{\mathsf{F}}^{\mu\nu}, \tag{B.1.41}$$

with $\widetilde{\mathsf{A}}_\mu = \widetilde{A}^\alpha_\mu\widetilde{\mathsf{T}}_\alpha$ and

$$\widetilde{\mathsf{F}}_{\mu\nu} = \partial_\mu\widetilde{\mathsf{A}}_\nu - \partial_\nu\widetilde{\mathsf{A}}_\mu + [\widetilde{\mathsf{A}}_\mu, \widetilde{\mathsf{A}}_\nu]. \tag{B.1.42}$$

This is the convention that we used in the main text. Note also that in the main text we often write not only the adjoint indices as upper indices, but also the coadjoint indices. In chapters 3 and 4 we indicate these indices with $A, B, \ldots$, and the generators with λ^A. For instance, the terms $F^\alpha_{\mu\nu}\mathsf{T}_\alpha$ and $f_{\alpha\beta\gamma}\mathrm{tr}\,F^\alpha F^\beta F^\gamma$ would read $F^A_{\mu\nu}\lambda^A$ and $f^{ABC}\mathrm{tr}\,F^A F^B F^C$, respectively, in the main text.

B.2 Amplitudes & the 4-point function

Here we list our conventions for the amplitude calculations in chapter 3. As explained in chapter 1, the quantum effective action for the massless modes and the Wilsonian effective action actually coincide at tree-level. We will calculate the amplitudes in the formalism of the QEA, since this is more convenient.

The QEA is by definition the generator of 1PI diagrams:

$$S_{\text{eff}}[A_a] \equiv \sum_n \frac{1}{n!}\int \mathrm{d}^{10}x_1 \cdots \mathrm{d}^{10}x_n\, \Gamma^{(n)}_{a_1\cdots a_n}(x_1, \ldots, x_n)A^{a_1}(x_1)\cdots A^{a_n}(x_n), \tag{B.2.1}$$

hence

$$\Gamma^{(n)}_{a_1\cdots a_n}(x_1, \ldots, x_n) = \left.\frac{\delta^n S_{\text{eff}}[A_a]}{\delta A^{a_1}(x_1)\ldots\delta A^{a_n}(x_n)}\right|_{A_a=0}. \tag{B.2.2}$$

We define the momentum space amplitudes as follows:

$$(2\pi)^{10}\delta^{(10)}(k_1+\ldots+k_n)\Gamma^{(n)}_{a_1\cdots a_n}(k_1,\ldots,k_n)$$
$$\equiv \int \prod_{i=1}^{n} \mathrm{d}^{10}x_i\, \mathrm{e}^{ik_i\cdot x_i}\, \Gamma^{(n)}_{a_1\cdots a_n}(x_1,\ldots,x_n). \quad \text{(B.2.3)}$$

An n-photon interaction gives the following contribution to the S-matrix:

$$\mathcal{A}(1,\ldots,n) = i(2\pi)^{10}\delta^{(10)}(k_1+\ldots+k_n)\,\zeta^1_{a_1}\cdots\zeta^n_{a_n}\,\Gamma^{(n)}_{a_1\cdots a_n}(k_1,\ldots,k_n). \quad \text{(B.2.4)}$$

B.2.1 Proof of equation (3.3.1)

In order to reproduce (3.3.1), we have to obtain the following 1PI four-point function from (3.3.9):

$$\Gamma^{(4)}_{klmn}(k_1,k_2,k_3,k_4) = -16(g\alpha')^2\, t_{akblcmdn} k_1^a k_2^b k_3^c k_4^d\, \mathcal{G}(k_1,k_2,k_3,k_4). \quad \text{(B.2.5)}$$

First we calculate the four-point function in position space:

$$\Gamma^{(4)}_{klmn}(y_1,\ldots,y_4) = \left.\frac{\delta^4 S_{\text{eff}}[A_a]}{\delta A^k(y_1)A^l(y_2)A^m(y_3)A^n(y_4)}\right|_{A_a=0}$$
$$= -4!2^4\frac{1}{24}(g\alpha')^2\int \mathrm{d}^{10}x\left\{\prod_i \mathrm{d}^{10}x_i\,\delta(x-x_i)\right\}D(\partial_{x_1},\ldots,\partial_{x_4})t_{akblcmdn}$$
$$\times\,\partial^a_{x_1}\delta(x_1-y_1)\partial^b_{x_2}\delta(x_2-y_2)\partial^c_{x_3}\delta(x_3-y_3)\partial^d_{x_4}\delta(x_4-y_4). \quad \text{(B.2.6)}$$

The factor of 2^4 arises from substituting $F_{ab} = \partial_a A_b - \partial_b A_a$, the factor 4! from the distributive property of the functional derivative. To arrive at the result we renamed dummy variables and made use of the fact that D is symmetric in its arguments. In momentum space this becomes:

$$-\frac{1}{16(g\alpha')^2}(2\pi)^{10}\delta(k_1+k_2+k_3+k_4)\Gamma^{(4)}_{klmn}(k_1,k_2,k_3,k_4)$$
$$= t_{akblcmdn}\int \mathrm{d}^{10}x\Big\{\prod_i \mathrm{d}^{10}x_i \mathrm{d}^{10}y_i\,\delta(x_i-x)\mathrm{e}^{ik_i\cdot y_i}\Big\}D(\partial_{x_1},\ldots,\partial_{x_4})$$
$$\times\,\partial^a_{x_1}\delta(x_1-y_1)\partial^b_{x_2}\delta(x_2-y_2)\partial^c_{x_3}\delta(x_3-y_3)\partial^d_{x_4}\delta(x_4-y_4)$$
$$= t_{akblcmdn}\int \mathrm{d}^{10}x\Big\{\prod_i \mathrm{d}^{10}x_i\,\delta(x_i-x)\Big\}$$
$$\times\, D(\partial_{x_1},\ldots,\partial_{x_4})\,\partial^a_{x_1}\partial^b_{x_2}\partial^c_{x_3}\partial^d_{x_4}\Big\{\prod_j\int \mathrm{d}^{10}y_j\,\mathrm{e}^{ik_j\cdot y_j}\delta(x_j-y_j)\Big\}$$

$$
\begin{aligned}
&= t_{akblcmdn} \int \mathrm{d}^{10}x \Big\{ \prod_i \mathrm{d}^{10}x_i \, \delta(x_i - x) \Big\} \\
&\qquad \times \mathcal{G}(-i\partial_{x_1}, \ldots, -i\partial_{x_4}) \partial^a_{x_1} \partial^b_{x_2} \partial^c_{x_3} \partial^d_{x_4} \Big\{ \prod_j \mathrm{e}^{ik_j \cdot x_j} \Big\} \\
&= t_{akblcmdn} \int \mathrm{d}^{10}x \Big\{ \prod_i \mathrm{d}^{10}x_i \, \delta(x_i - x) \, \mathrm{e}^{ik_i \cdot x_i} \Big\} \mathcal{G}(k_1, \ldots, k_4) \, k_1^a k_2^b k_3^c k_4^d \\
&= t_{akblcmdn} \, \mathcal{G}(k_1, \ldots, k_4) \, k_1^a k_2^b k_3^c k_4^d \times (2\pi)^{10} \delta(k_1 + k_2 + k_3 + k_4).
\end{aligned}
$$

This completes the proof.

Publications

[a] A. Collinucci, M. de Roo and M.G.C. Eenink,
Supersymmetric Yang-Mills theory at order α'^3,
JHEP **06** (2002) 024, `hep-th/0205150`.

[b] M. de Roo, M.G.C. Eenink, P. Koerber and A. Sevrin,
Testing the fermionic terms in the non-abelian D-brane effective action through order α'^3,
JHEP **08** (2002) 011, `hep-th/0207015`.

[c] A. Collinucci, M. de Roo and M.G.C. Eenink,
Derivative corrections in 10-dimensional super-Maxwell theory,
JHEP **01** (2003) 039, `hep-th/0212012`.

[d] M. de Roo and M.G.C. Eenink,
The effective action for the 4-point functions in open abelian superstring theory,
JHEP **08** (2003) 036, `hep-th/0307211`.

[e] M. de Roo, M.G.C. Eenink, S. Panda and D.B. Westra,
Group manifold reduction of dual N=1 d=10 supergravity,
JHEP **06** (2005) 077, `hep-th/0503059`.

Bibliography

[1] `http://www.physics2005.org/`

[2] H. Kragh, *Quantum generations: A history of physics in the twentieth century*, Princeton Univ. Pr. (1999)

[3] `http://math.ucr.edu/home/baez/constants.html`

[4] C. Rovelli, *Quantum gravity*, Cambridge Univ. Pr. (2004)

[5] M. J. Rees, *Astrophysical evidence for black holes*, `astro-ph/9701161`

[6] A. Celotti, J. C. Miller and D. W. Sciama, *Astrophysical evidence for the existence of black holes*, Class. Quant. Grav. **16** (1999) A3, `astro-ph/9912186`

[7] N. Ashby, *Relativity in the Global Positioning System*, Living Rev. Relativity **6** (2003) 1, `http://www.livingreviews.org/`

[8] J. F. Donoghue, *General relativity as an effective field theory: The leading quantum corrections*, Phys. Rev. **D50** (1994) 3874, `gr-qc/9405057`

[9] R. M. Wald, *General relativity*, Univ. of Chicago Pr. (1984)

[10] `http://pancake.uchicago.edu/∼carroll/cfcp/primer/`

[11] C. P. Burgess, *Quantum gravity in everyday life: General relativity as an effective field theory*, Living Rev. Rel. **7** (2004) 5, `gr-qc/0311082`

[12] S. M. Carroll and M. Trodden, *TASI lectures: Introduction to cosmology*, `astro-ph/0401547`

[13] A. Zee, *Quantum field theory in a nutshell*, Princeton Univ. Pr. (2003)

[14] M. B. Green, J. H. Schwarz and E. Witten, *Superstring theory*, Cambridge Univ. Pr. (1987), in two volumes

[15] J. Polchinski, *String theory*, Cambridge Univ. Pr. (1998), in two volumes

[16] D. Lüst and S. Theisen, *Lectures on string theory*, Springer-Verlag (1989)

[17] E. Kiritsis, *Introduction to superstring theory*, `hep-th/9709062`

[18] C. V. Johnson, *D-branes*, Cambridge Univ. Pr. (2003)

[19] B. Zwiebach, *A first course in string theory*, Cambridge Univ. Pr. (2004)

[20] S. W. Hawking, *Zeta function regularization of path integrals in curved space-time*, Commun. Math. Phys. **55** (1977) 133

[21] N. D. Birrell and P. C. W. Davies, *Quantum fields in curved space*, Cambridge Univ. Pr. (1982)

[22] P. Ramond, *Field theory: a modern primer*, Front. Phys. **74** (1989) 1

[23] R. Rajaraman, *Solitons and instantons*, North-Holland (1982)

[24] S. Coleman, *Quantum sine-Gordon equation as the massive Thirring model*, Phys. Rev. **D11** (1975) 2088

[25] C. Montonen and D. I. Olive, *Magnetic monopoles as gauge particles?*, Phys. Lett. **B72** (1977) 117

[26] A. Sen, *Strong-weak coupling duality in four-dimensional string theory*, Int. J. Mod. Phys. **A9** (1994) 3707, `hep-th/9402002`

[27] A. Sen, *Dyon-monopole bound states, selfdual harmonic forms on the multi-monopole moduli space, and $SL(2,\mathbb{Z})$ invariance in string theory*, Phys. Lett. **B329** (1994) 217, `hep-th/9402032`

[28] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $\mathcal{N}=2$ supersymmetric Yang-Mills theory*, Nucl. Phys. **B426** (1994) 19, `hep-th/9407087`

[29] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $\mathcal{N}=2$ supersymmetric QCD*, Nucl. Phys. **B431** (1994) 484, `hep-th/9408099`

[30] W. Taylor and B. Zwiebach, *D-branes, tachyons, and string field theory*, in *Boulder 2001 (TASI): Strings, branes and extra dimensions*, `hep-th/0311017`

[31] S. Mandelstam, *The n-loop string amplitude: explicit formulas, finiteness and absence of ambiguities*, Phys. Lett. **B277** (1992) 82

[32] E. D'Hoker and D. H. Phong, *Lectures on two-loop superstrings*, `hep-th/0211111`

[33] S. Weinberg, *What is quantum field theory, and what did we think it is?*, `hep-th/9702027`

[34] S. Weinberg, *The quantum theory of fields*, Cambridge Univ. Pr. (1995,1996,2000), in three volumes

[35] J. Polchinski, *Effective field theory and the Fermi surface*, `hep-th/9210046`

[36] M. E. Peskin and D. V. Schroeder, *An introduction to quantum field theory*, Addison-Wesley (1995)

[37] K. G. Wilson and J. G. Kogut, *The renormalization group and the epsilon expansion*, Phys. Rep. **12** (1974) 75

[38] J. Polchinski, *Renormalization and effective Lagrangians*, Nucl. Phys. **B231** (1984) 269

[39] J. F. Donoghue, *Leading quantum correction to the Newtonian potential*, Phys. Rev. Lett. **72** (1994) 2996, `gr-qc/9310024`

[40] N. E. J. Bjerrum-Bohr, *Quantum gravity, effective fields and string theory*, PhD thesis, NBI Copenhagen (2004), `hep-th/0410097`

[41] J. F. Donoghue, *Introduction to the effective field theory description of gravity*, `gr-qc/9512024`

[42] C. P. Burgess, *An ode to effective Lagrangians*, `hep-ph/9812470`

[43] S. P. Martin, *A supersymmetry primer*, `hep-ph/9709356`

[44] D. J. H. Chung, L. L. Everett, G. L. Kane, S. F. King, J. Lykken and L.-T. Wang, *The soft supersymmetry-breaking Lagrangian: Theory and applications*, `hep-ph/0312378`

[45] C. P. Burgess, J. P. Derendinger, F. Quevedo and M. Quiros, *On gaugino condensation with field-dependent gauge couplings*, Annals Phys. **250** (1996) 193, `hep-th/9505171`

[46] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman (1973)

[47] T. Ortín, *Gravity and strings*, Cambridge Univ. Pr. (2004)

[48] T. Frankel, *The geometry of physics: an introduction*, Cambridge Univ. Pr. (1997)

[49] D. J. Gross and E. Witten, *Superstring modifications of Einstein's equations*, Nucl. Phys. **B277** (1986) 1

[50] D. J. Gross and J. H. Sloan, *The quartic effective action for the heterotic string*, Nucl. Phys. **B291** (1987) 41

[51] J. H. Suelmann, *Supersymmetry and string effective actions*, PhD thesis, Rijksuniversiteit Groningen (1994)

[52] O. D. Andreev and A. A. Tseytlin, *Partition function representation for the open superstring effective action: cancellation of Möbius infinities and derivative corrections to Born-Infeld Lagrangian*, Nucl. Phys. **B311** (1988) 205

[53] R. G. Leigh, *Dirac-Born-Infeld action from Dirichlet σ-model*, Mod. Phys. Lett. **A4** (1989) 2767

[54] E. Kiritsis, *D-branes in standard model building, gravity and cosmology*, Fortsch. Phys. **52** (2004) 200, `hep-th/0310001`

[55] T. H. Buscher, *A symmetry of the string backround field equations*, Phys. Lett. **B194** (1987) 59

[56] T. H. Buscher, *Path integral derivation of quantum duality in nonlinear sigma models*, Phys. Lett. **B201** (1988) 466

[57] E. Bergshoeff and M. de Roo, *D-branes and T-duality*, Phys. Lett. **B380** (1996) 265, `hep-th/9603123`

[58] E. Witten, *Bound states of strings and p-branes*, Nucl. Phys. **B460** (1996) 335, `hep-th/9510135`

[59] R. C. Myers, *Nonabelian phenomena on D-branes*, Class. Quant. Grav. **20** (2003) S347, `hep-th/0303072`

[60] N. Berkovits, *ICTP lectures on covariant quantization of the superstring*, `hep-th/0209059`

[61] P. A. Grassi, G. Policastro and P. van Nieuwenhuizen, *An introduction to the covariant quantization of superstrings*, Class. Quant. Grav. **20** (2003) S395, `hep-th/0302147`

[62] N. Berkovits, *Covariant multiloop superstring amplitudes*, Comptes Rendus Physique **6** (2005) 185, `hep-th/0410079`

[63] N. Berkovits, *Super-Poincare covariant two-loop superstring amplitudes*, `hep-th/0503197`

[64] P. A. M. Dirac, *Lectures on quantum mechanics*, Dover Publications (2001)

[65] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton Univ. Pr. (1994)

[66] K. Sundermeyer, *Constrained dynamics*, Springer-Verlag (1982)

[67] J. Polchinski, *Dirichlet branes and Ramond-Ramond charges*, Phys. Rev. Lett. **75** (1995) 4724, `hep-th/9510017`

[68] L. J. Romans, *Massive $N = 2a$ supergravity in ten dimensions*, Phys. Lett. **B169** (1986) 374

[69] E. Bergshoeff, M. B. Green, G. Papadopoulos and P. K. Townsend, *The IIA super-eightbrane*, `hep-th/9511079`

[70] E. Bergshoeff, M. de Roo, M. B. Green, G. Papadopoulos and P. K. Townsend, *Duality of Type II 7-branes and 8-branes*, Nucl. Phys. **B470** (1996) 113, `hep-th/9601150`

[71] F. Giani and M. Pernici, *$N = 2$ supergravity in ten dimensions*, Phys. Rev. **D30** (1984) 325

[72] I. C. G. Campbell and P. C. West, *$N = 2$, $D = 10$ nonchiral supergravity and its spontaneous compactification*, Nucl. Phys. **B243** (1984) 112

[73] J. H. Schwarz, *Covariant field equations of chiral $N = 2$, $D = 10$ supergravity*, Nucl. Phys. **B226** (1983) 269

[74] P. S. Howe and P. C. West, *The complete $N = 2$, $d = 10$ supergravity*, Nucl. Phys. **B238** (1984) 181

[75] P. Pasti, D. P. Sorokin and M. Tonin, *On Lorentz invariant actions for chiral p-forms*, Phys. Rev. **D55** (1997) 6292, `hep-th/9611100`

[76] E. Bergshoeff, C. M. Hull and T. Ortín, *Duality in the type II superstring effective action*, Nucl. Phys. **B451** (1995) 547, `hep-th/9504081`

[77] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, *Ten-dimensional Maxwell-Einstein supergravity, its currents and the issue of its auxiliary fields*, Nucl. Phys. **B195** (1982) 97

[78] G. F. Chapline and N. S. Manton, *Unification of Yang-Mills theory and supergravity in ten dimensions*, Phys. Lett. **B120** (1983) 105

[79] M. J. Duff, *Supermembranes*, `hep-th/9611203`

[80] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, Phys. Lett. **B76** (1978) 409

[81] S. Ferrara and P. van Nieuwenhuizen, *Consistent supergravity with complex spin $\frac{3}{2}$ gauge fields*, Phys. Rev. Lett. **37** (1976) 1669

[82] G. W. Gibbons and C. M. Hull, *A Bogomolny bound for general relativity and solitons in $\mathcal{N} = 2$ supergravity*, Phys. Lett. **B109** (1982) 190

[83] E. Witten and D. I. Olive, *Supersymmetry algebras that include topological charges*, Phys. Lett. **B78** (1978) 97

[84] J. P. van der Schaar, *String theory limits and dualities*, PhD thesis, University of Groningen (2000)

[85] R. Halbersma, *Geometry of strings and branes*, PhD thesis, University of Groningen (2002)

[86] J. A. de Azcarraga, J. P. Gauntlett, J. M. Izquierdo and P. K. Townsend, *Topological extensions of the supersymmetry algebra for extended objects*, Phys. Rev. Lett. **63** (1989) 2443

[87] D. Youm, *Black holes and solitons in string theory*, Phys. Rept. **316** (1999) 1, `hep-th/9710046`

[88] A. Dabholkar, G. W. Gibbons, J. A. Harvey and F. Ruiz Ruiz, *Superstrings and solitons*, Nucl. Phys. **B340** (1990) 33

[89] B. Janssen, *Dualities of string and branes*, PhD thesis, University of Groningen (1998)

[90] W. Taylor, *M(atrix) theory: Matrix quantum mechanics as a fundamental theory*, Rev. Mod. Phys. **73** (2001) 419, `hep-th/0101126`

[91] P. Horava and E. Witten, *Heterotic and type I string dynamics from eleven dimensions*, Nucl. Phys. **B460** (1996) 506, `hep-th/9510209`

[92] P. Horava and E. Witten, *Eleven-Dimensional Supergravity on a Manifold with Boundary*, Nucl. Phys. **B475** (1996) 94, `hep-th/9603142`

[93] A. Bilal, *Higher-derivative corrections to the non-abelian Born- Infeld action*, Nucl. Phys. **B618** (2001) 21, `hep-th/0106062`

[94] E. S. Fradkin and A. A. Tseytlin, *Nonlinear electrodynamics from quantized strings*, Phys. Lett. **B163** (1985) 123

[95] E. Bergshoeff, M. Rakowski and E. Sezgin, *Higher-derivative super Yang-Mills theories*, Phys. Lett. **B185** (1987) 371

[96] R. R. Metsaev and M. A. Rakhmanov, *Fermionic terms in the open superstring effective action*, Phys. Lett. **B193** (1987) 202

[97] M. Cederwall, B. E. W. Nilsson and D. Tsimpis, $d = 10$ *super-Yang-Mills at* α'^2, JHEP **07** (2001) 042, `hep-th/0104236`

[98] M. Aganagic, C. Popescu and J. H. Schwarz, *Gauge-invariant and gauge-fixed D-brane actions*, Nucl. Phys. **B495** (1997) 99, `hep-th/9612080`

[99] M. Cederwall, A. von Gussich, B. E. W. Nilsson and A. Westerberg, *The Dirichlet super-three-brane in ten-dimensional type IIB supergravity*, Nucl. Phys. **B490** (1997) 163, `hep-th/9610148`

[100] M. Cederwall, A. von Gussich, B. E. W. Nilsson, P. Sundell and A. Westerberg, *The Dirichlet super-p-branes in ten-dimensional type IIA and IIB supergravity*, Nucl. Phys. **B490** (1997) 179, `hep-th/9611159`

[101] E. Bergshoeff and P. K. Townsend, *Super D-branes*, Nucl. Phys. **B490** (1997) 145, `hep-th/9611173`

[102] N. Wyllard, *Derivative corrections to D-brane actions with constant background fields*, Nucl. Phys. **B598** (2001) 247, `hep-th/0008125`

[103] N. Wyllard, *Derivative corrections to the D-brane Born-Infeld action: Non-geodesic embeddings and the Seiberg-Witten map*, JHEP **08** (2001) 027, `hep-th/0107185`

[104] S. R. Das, S. Mukhi and N. V. Suryanarayana, *Derivative corrections from noncommutativity*, JHEP **08** (2001) 039, `hep-th/0106024`

[105] J. H. Schwarz, *Superstring theory*, Phys. Rept. **89** (1982) 223

[106] E. A. Bergshoeff, A. Bilal, M. de Roo and A. Sevrin, *Supersymmetric non-abelian Born-Infeld revisited*, JHEP **07** (2001) 029, `hep-th/0105274`

[107] P. Koerber and A. Sevrin, *The non-abelian D-brane effective action through order* α'^4, JHEP **10** (2002) 046, `hep-th/0208044`

[108] J. M. Drummond, P. J. Heslop, P. S. Howe and S. F. Kerstan, *Integral invariants in* $N = 4$ *SYM and the effective action for coincident D-branes*, JHEP **08** (2003) 016, `hep-th/0305202`

[109] S. Mukhi, *Star products from commutative string theory*, Pramana **58** (2002) 21, `hep-th/0108072`

[110] P. Grange, *Derivative corrections from boundary state computations*, Nucl. Phys. **B649** (2003) 297, `hep-th/0207211`

[111] M. de Roo, H. Suelmann and A. Wiedemann, *Supersymmetric* R^4 *actions in ten dimensions*, Phys. Lett. **B280** (1992) 39

[112] M. de Roo, H. Suelmann and A. Wiedemann, *The supersymmetric effective action of the heterotic string in ten dimensions*, Nucl. Phys. **B405** (1993) 326, `hep-th/9210099`

[113] P. Koerber and A. Sevrin, *The non-Abelian Born-Infeld action through order* α'^3, JHEP **10** (2001) 003, `hep-th/0108169`

[114] M. Cederwall, B. E. W. Nilsson and D. Tsimpis, *Spinorial cohomology of abelian* $d = 10$ *super-Yang-Mills at* α'^3, JHEP **11** (2002) 023, `hep-th/0205165`

[115] R. R. Metsaev, M. A. Rakhmanov and A. A. Tseytlin, *The Born-Infeld action as the effective action in the open superstring theory*, Phys. Lett. **B193** (1987) 207

[116] A. A. Tseytlin, *On non-abelian generalisation of the Born-Infeld action in string theory*, Nucl. Phys. **B501** (1997) 41, `hep-th/9701125`

[117] A. A. Tseytlin, *Born-Infeld action, supersymmetry and string theory*, `hep-th/9908105`

[118] Y. Kitazawa, *Effective Lagrangian for open superstring from five point function*, Nucl. Phys. **B289** (1987) 599

[119] M. Cederwall, B. E. W. Nilsson and D. Tsimpis, *The structure of maximally supersymmetric Yang-Mills theory: constraining higher-order corrections*, JHEP **06** (2001) 034, `hep-th/0102009`

[120] A. Hashimoto and I. Taylor, W., *Fluctuation spectra of tilted and intersecting D-branes from the Born-Infeld action*, Nucl. Phys. **B503** (1997) 193–219, `hep-th/9703217`

[121] A. Refolli, A. Santambrogio, N. Terzi and D. Zanon, *F^5 contributions to the nonabelian Born Infeld action from a supersymmetric Yang-Mills five-point function*, Nucl. Phys. **B613** (2001) 64, Erratum, *Nucl. Phys.* **B648** (2003) 453, `hep-th/0105277`

[122] D. T. Grasso, *Higher order contributions to the effective action of $N = 4$ super Yang-Mills*, JHEP **11** (2002) 012, `hep-th/0210146`

[123] F. T. Brandt, F. R. Machado and R. Medina, *The open superstring 5-point amplitude revisited*, JHEP **07** (2002) 071, `hep-th/0208121`

[124] O. Chandia and R. Medina, *4-point effective actions in open and closed superstring theory*, JHEP **11** (2003) 003, `hep-th/0310015`

[125] L. A. Barreiro and R. Medina, *5-field terms in the open superstring effective action*, JHEP **03** (2005) 055, `hep-th/0503182`

[126] F. Denef, A. Sevrin and J. Troost, *Non-Abelian Born-Infeld versus string theory*, Nucl. Phys. **B581** (2000) 135, `hep-th/0002180`

[127] A. Sevrin, J. Troost and W. Troost, *The non-abelian Born-Infeld action at order F^6*, Nucl. Phys. **B603** (2001) 389, `hep-th/0101192`

[128] P. Koerber and A. Sevrin, *Testing the α'^3 term in the non-abelian open superstring effective action*, JHEP **09** (2001) 009, `hep-th/0109030`

[129] A. Sevrin and A. Wijns, *Higher order terms in the non-Abelian D-brane effective action and magnetic background fields*, JHEP **08** (2003) 059, `hep-th/0306260`

[130] M. Berkooz, M. R. Douglas and R. G. Leigh, *Branes intersecting at angles*, Nucl. Phys. **B480** (1996) 265–278, `hep-th/9606139`

[131] A. Abouelsaood, J. Callan, Curtis G., C. R. Nappi and S. A. Yost, *Open strings in background gauge fields*, Nucl. Phys. **B280** (1987) 599

[132] J. Callan, C. G., C. Lovelace, C. R. Nappi and S. A. Yost, *Loop corrections to superstring equations of motion*, Nucl. Phys. **B308** (1988) 221

[133] P. van Baal, *Some results for $SU(N)$ gauge fields on the hypertorus*, Commun. Math. Phys. **85** (1982) 529

[134] P. van Baal, *$SU(N)$ Yang-Mills solutions with constant field strength on T^4*, Commun. Math. Phys. **94** (1984) 397

[135] J. Troost, *Constant field strengths on T^{2n}*, Nucl. Phys. **B568** (2000) 180, `hep-th/9909187`

[136] A. A. Tseytlin, *Vector field effective action in the open superstring theory*, Nucl. Phys. **B276** (1986) 391

[137] C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, Nucl. Phys. **B550** (1999) 151, `hep-th/9812219`

[138] C.-S. Chu and P.-M. Ho, *Constrained quantization of open string in background B field and noncommutative D-brane*, Nucl. Phys. **B568** (2000) 447, `hep-th/9906192`

[139] A. Connes, M. R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: Compactification on tori*, JHEP **02** (1998) 003, `hep-th/9711162`

[140] M. R. Douglas and C. M. Hull, *D-branes and the noncommutative torus*, JHEP **02** (1998) 008, `hep-th/9711165`

[141] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP **09** (1999) 032, `hep-th/9908142`

[142] L. Cornalba, *On the general structure of the non-Abelian Born-Infeld action*, Adv. Theor. Math. Phys. **4** (2002) 1259, `hep-th/0006018`

[143] R. C. Myers, *Dielectric-branes*, JHEP **12** (1999) 022, `hep-th/9910053`

[144] M. R. Douglas, *D-branes and matrix theory in curved space*, Nucl. Phys. Proc. Suppl. **68** (1998) 381, `hep-th/9707228`

[145] D. Brecher, K. Furuuchi, H. Ling and M. Van Raamsdonk, *Generally covariant actions for multiple D-branes*, JHEP **06** (2004) 020, `hep-th/0403289`

[146] M. Van Raamsdonk, *Blending local symmetries with matrix nonlocality in D- brane effective actions*, JHEP **09** (2003) 026, `hep-th/0305145`

[147] I. Taylor, W. and M. Van Raamsdonk, *Multiple D0-branes in weakly curved backgrounds*, Nucl. Phys. **B558** (1999) 635, `hep-th/9904095`

[148] I. Taylor, W. and M. Van Raamsdonk, *Multiple Dp-branes in weak background fields*, Nucl. Phys. **B573** (2000) 703, `hep-th/9910052`

[149] J. de Boer and K. Schalm, *General covariance of the non-Abelian DBI-action*, JHEP **02** (2003) 041, `hep-th/0108161`

[150] J. de Boer, K. Schalm and J. Wijnhout, *General covariance of the non-Abelian DBI-action: Checks and balances*, Annals Phys. **313** (2004) 425, `hep-th/0310150`

[151] J. de Boer, E. Gimon, K. Schalm and J. Wijnhout, *Evidence for a gravitational Myers effect*, Annals Phys. **313** (2004) 402, `hep-th/0212250`

[152] S. Panda and D. Sorokin, *Supersymmetric and kappa-invariant coincident D0-branes*, JHEP **02** (2003) 055, `hep-th/0301065`

[153] E. A. Bergshoeff, M. de Roo and A. Sevrin, *Non-Abelian Born-Infeld and kappa-symmetry*, J. Math. Phys. **42** (2001) 2872, `hep-th/0011018`

[154] J. Scherk, *Extended supersymmetry and extended supergravity theories*, in *Cargèse 1978: Recent developments in gravitation*

[155] P. van Nieuwenhuizen, *Six lectures at the Trieste 1981 summer school on supergravity*, in *Trieste 1981: Supergravity*

[156] P. van Nieuwenhuizen, *An introduction to simple supergravity and the Kaluza-Klein program*, in *Les Houches 1983: Relativity, groups and topology, II*

[157] T. Kugo and P. K. Townsend, *Supersymmetry and the division algebras*, Nucl. Phys. **B221** (1983) 357

[158] A. D. Kennedy, *Clifford algebras in two omega dimensions*, J. Math. Phys. **22** (1981) 1330

[159] J. W. van Holten and A. Van Proeyen, $N = 1$ *Supersymmetry algebras in* $d = 2, 3, 4$ *mod* 8, J. Phys. **A15** (1982) 3763

[160] J. Strathdee, *Extended Poincaré supersymmetry*, Int. J. Mod. Phys. **A2** (1987) 273

[161] A. Van Proeyen, *Tools for supersymmetry*, `hep-th/9910030`

[162] P. C. West, *Supergravity, brane dynamics and string duality*, `hep-th/9811101`

[163] L. Castellani, R. d'Auria and P. Fré, *Supergravity and superstrings: a geometric perspective*, World Scientific (1991), in three volumes

[164] Y. Tanii, *Introduction to supergravities in diverse dimensions*, `hep-th/9802138`

[165] P. G. O. Freund, *Introduction to supersymmetry*, Cambridge Univ. Pr. (1986)

[166] R. d'Auria, S. Ferrara, M. A. Lledo and V. S. Varadarajan, *Spinor algebras*, J. Geom. Phys. **40** (2001) 101, `hep-th/0010124`

[167] P. van Nieuwenhuizen and A. Waldron, *On Euclidean spinors and Wick rotations*, Phys. Lett. **B389** (1996) 29, `hep-th/9608174`

[168] J. Wess and B. Zumino, *A Lagrangian model invariant under supergauge transformations*, Phys. Lett. **B49** (1974) 52

[169] S. R. Coleman and J. Mandula, *All possible symmetries of the S-matrix*, Phys. Rev. **159** (1967) 1251

[170] R. Haag, J. T. Lopuszanski and M. Sohnius, *All possible generators of supersymmetries of the S-matrix*, Nucl. Phys. **B88** (1975) 257

[171] M. F. Sohnius, *Introducing supersymmetry*, Phys. Rep. **128** (1985) 39

[172] P. C. West, *Introduction to supersymmetry and supergravity*, World Scientific (1990)

[173] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton Univ. Pr. (1992)

[174] M. Nakahara, *Geometry, topology and physics*, IOP Publishing (2003)

[175] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, Wiley-Interscience (1974)

[176] J. Fuchs and C. Schweigert, *Symmetries, Lie algebras and representations: A graduate course for physicists*, Cambridge Univ. Pr. (1997)

Samenvatting

De Nederlandse vertaling van de titel van dit proefschrift luidt 'Dirichlet branen, effectieve acties en supersymmetrie'. Dirichlet branen, afgekort D-branen, zijn objecten in snaartheorie en hebben een rol gespeeld in bijna alle belangrijke ontwikkelingen binnen dit vakgebied in de laatste tien jaar. De snaartheorie is een onderdeel van de theoretische hoge-energie fysica[1], de tak van de natuurkunde die zich bezighoudt met de wiskundige beschrijving van de fundamentele bouwstenen van de natuur. Ik zal eerst een beknopt overzicht geven van een aantal aspecten van dit vakgebied alvorens verder in te gaan op snaren, D-branen en de rol van effectieve acties en supersymmetrie in de wiskundige beschrijving van deze objecten.

Aan het eind van de 19e eeuw werd het gaandeweg duidelijk dat de klassieke mechanica van Newton slechts bij benadering een goede beschrijving levert van de natuur. Newton's mechanica volstaat (meestal) voor de beschrijving van fysische systemen die een groot aantal deeltjes bevatten. Systemen met slechts een klein aantal deeltjes gedragen zich echter volgens de wetten van de quantummechanica. Een belangrijk aspect van de quantummechanica is dat zij slechts uitspraken doet over de *kans* dat bepaalde processen plaatsvinden, hetgeen in schril contrast staat met het deterministische karakter van de klassieke mechanica. Een ander belangrijk gegeven binnen de quantummechanica is dat het bestuderen van almaar kleinere structuren steeds hogere energieën vergt[2]. Zo is bijvoorbeeld de ordening van atomen in een metaal niet te achterhalen met behulp van zichtbaar licht, maar wel door het metaal te beschieten met fotonen met een hogere energie (Röntgenstraling).

Een ander noodzakelijk ingrediënt van theorieën die deeltjes beschrijven is de speciale relativiteitstheorie: de hoge energieën gaan gepaard met snelheden die dicht bij de snelheid van het licht liggen. Volgens de speciale relativiteitstheorie zijn ruimte en tijd niet twee afzonderlijke concepten zoals ze dat in de klassieke mechanica zijn, maar zijn ze verenigd in *ruimtetijd*. De klassieke mechanica vertelt ons dat de energie E van een deeltje met massa m gerelateerd is aan zijn impuls p volgens $E = p^2/2m$. Als

[1] Ook wel bekend als de elementaire deeltjesfysica.

[2] Vandaar de naam hoge-energie fysica voor het vakgebied dat de kleinste bouwstenen van de natuur onderzoekt.

het deeltje in rust is ($p = 0$), is de energie nul. Volgens de speciale relativiteitstheorie geldt echter de volgende relatie tussen energie en impuls: $E^2 = m^2c^4 + p^2c^2$ (∗), met c de lichtsnelheid. Als het deeltje in rust is, geldt dus $E = mc^2$. Een consequentie van deze relatie is dat als lichte deeltjes botsen met voldoende hoge energie, er nieuwe, zwaardere deeltjes geproduceerd kunnen worden. Dit noemen we een *interactie*. Zo *kunnen* (quantummechanica!) twee voldoende energetische fotonen als ze botsen omgezet worden in een elektron en zijn antideeltje, het positron. Deze laatste deeltjes hebben beide een massa van 511 keV/c^2. De totale energie van de fotonen moet dus tenminste 1022 keV zijn om dit proces te laten plaatsvinden.

De theorie die de wisselwerking van elektrisch geladen (anti)deeltjes met fotonen beschrijft heet de quantumelektrodynamica (QED) en is een zogenaamde relativistische quantumveldentheorie. In quantumveldentheorieën worden deeltjes beschreven als manifestaties van velden. Zo wordt in QED het foton beschreven door het elektromagnetische veld en het elektron en positron door het Dirac veld. De gevestigde theorieën in de hoge-energie fysica zijn allemaal quantumveldentheorieën.

De algemene structuur van quantumveldentheorieën verklaart waarom er antideeltjes bestaan en waarom er twee typen deeltjes zijn in de natuur: *bosonen* en *fermionen*. De bosonen hebben altijd een heeltallige spin[3] en vertonen het soort collectieve gedrag dat we associëren met *krachtvelden*. Fermionen hebben een halftallige spin en vertonen in het algemeen juist geen collectief gedrag vanwege het uitsluitingprincipe van Pauli. *Materie* is opgebouwd uit fermionen. Een andere generieke eigenschap van quantumveldentheorieën is dat de sterkte van de interacties tussen deeltjes onder meer afhangt van de energie van de deeltjes. Zo blijkt de elektromagnetische interactie steeds sterker te worden op korte afstanden.

De wiskundige beschrijving van een veldentheorie wordt vaak gebaseerd op een zogenaamde *actie* – aangegeven met een S. Er bestaat een recept om, gegeven de actie van een veldentheorie, te achterhalen wat het 'spectrum' van deeltjes is dat beschreven wordt door de veldentheorie in kwestie en hoe de interacties tussen deze deeltjes verlopen. Het blijkt helaas meestal onmogelijk om de vergelijkingen die in dit recept een rol spelen exact door te rekenen. We zijn dus genoodzaakt benaderingen te maken. De belangrijkste benaderingsmethode is de *storingsrekening* of *perturbatietheorie*. In deze methode wordt de actie opgesplitst in een deel waarvoor het recept wel exact doorgerekend kan worden, S_{vrij}, en een resterend deel, S_{int}. We schrijven $S = S_{\text{vrij}} + S_{\text{int}}$. Voor S_{vrij} kunnen we het deeltjesspectrum precies bepalen. Het blijkt dat ieder veld dat in S_{vrij} voorkomt precies één type deeltje beschrijft en dat deze deeltjes *vrij* zijn, d.w.z. geen interacties met elkaar hebben. Als we vervolgens S_{int} behandelen als een *correctie* op S_{vrij}, dan volgt dat S_{int} op zijn beurt de *interacties* of *koppeling* tussen de uit S_{vrij} verkregen deeltjes beschrijft. De sterkte van de verschillende koppelingen

[3]Spin is een zogenaamd *quantumgetal*. Verschillende typen deeltjes onderscheiden zich van elkaar door middel van hun quantumgetallen. Andere voorbeelden van quantumgetallen zijn massa en elektrische lading. De spin van een deeltje is altijd een veelvoud van $\frac{1}{2}$.

tussen de deeltjes wordt gegeven door de zogenaamde *koppelingsconstante*[4]. In QED, bijvoorbeeld, is de koppelingsconstante de elektrische lading van het elektron.

De storingsrekening is alleen bruikbaar zolang het gerechtvaardigd is om S_{int} als een *kleine* verstoring te behandelen, d.w.z. bij energieën waarbij de koppelingsconstanten klein zijn. Als de koppelingsconstanten groot worden, is het in algemeen niet langer mogelijk om veldentheorieën analytisch door te rekenen[5]. Het is dan vaak niet eens mogelijk het deeltjesspectrum van de theorie te bepalen. Het bekendste voorbeeld is quantumchromodynamica (QCD), de theorie van de sterke kernkracht. Deze theorie bevat twee velden, het gluonveld en het quarkveld. Bij hoge energieën is de koppeling klein en bestaat het spectrum van de theorie dus uit quarks en gluonen. Het blijkt echter dat de koppelingsconstante van QCD steeds groter wordt naarmate de energie lager wordt. Uit experimenten weten we dat bij die lage energieën de quarks en gluonen niet meer voorkomen als afzonderlijke deeltjes, maar dat het dan protonen, neutronen en pionen zijn die een rol spelen in de sterke kernkracht. Deze deeltjes kunnen gezien worden als gebonden toestanden van quarks en gluonen, een verschijnsel dat bekend is onder de naam *confinement*. Het vinden van een theoretische beschrijving van confinement is één van de belangrijkste onopgeloste problemen in dit vakgebied.

Er zijn echter ook theorieën waarbij het *wel* mogelijk is om met behulp van analytische methoden informatie te verkrijgen over het *nonperturbatieve* regime van de theorie. Het idee is grofweg om bepaalde grootheden te berekenen in het perturbatieve regime en de resultaten te extrapoleren naar het regime waarin de koppelingsconstante groot is. Dit kan alleen maar als de grootheden 'beschermd' zijn door zogenaamde nonrenormalisatie stellingen. Deze stellingen zijn over het algemeen het gevolg van bepaalde symmetrieën die deze theorieën bezitten[6]. In snaartheorie blijkt het de supersymmetrie te zijn die leidt tot nonrenormalisatie stellingen.

Voor het beschrijven van zwak gekoppelde deeltjes is het gebruik van de veldentheoretische actie handig, maar niet noodzakelijk. Het blijkt namelijk mogelijk de interacties van de deeltjes te beschrijven zonder een beroep te doen op veldentheorie. Het extrapoleren van resultaten behaald bij zwakke koppeling naar het nonperturbatieve regime is echter gebaseerd op de zogenaamde *semi-klassieke methode*, waarbij het gebruik van een veldentheoretische actie essentieel is. We zullen zien dat dit belangrijke consequenties heeft voor snaartheorie.

Er is nog één veldentheoretisch begrip dat we moeten bespreken voordat we toe zijn aan de snaartheorie. Het komt vaak voor dat we alleen maar geïnteresseerd zijn in het gedrag van een veldentheorie bij lage energieën. We beschouwen als voorbeeld weer de verstrooiing van twee fotonen. We kunnen besluiten de fotonen alleen te

[4] Deze is niet echt constant, omdat de interactiesterktes energieafhankelijk zijn.

[5] We zijn dan genoodzaakt numerieke benaderingsmethoden m.b.v. de computer te gebruiken.

[6] Een theorie bezit een symmetrie, als een verandering van de variabelen waarmee we de theorie beschrijven de vorm van de actie niet verandert.

bestuderen bij energieën ver onder de 1022 keV. Hoewel de fotonen dan niet meer omgezet kunnen worden in een elektron-positron paar, kunnen ze volgens QED wel botsen met elkaar doordat ze onderling *virtuele* elektronen en positronen uitwisselen[7]. Het eindproduct van de botsing bestaat dus uit twee of meer[8] fotonen, maar bevat geen elektronen en positronen. Feitelijk zijn de fotonen dus de enige deeltjes die we nog echt waarnemen bij lage energieën en maken de elektronen en positronen hun bestaan alleen maar kenbaar via de interacties die de fotonen onderling met elkaar hebben.

We kunnen hier op een alternatieve manier tegen aan kijken. Stel hypothetisch dat we alleen metingen zouden kunnen doen aan licht bij lage energieën, die wel zo nauwkeurig zijn dat we de verstrooiing van fotonen kunnen waarnemen. We hebben dus geen weet van het bestaan van elektronen en positronen. Om de fotonen te beschrijven stellen we een veldentheorie op. De interacties tussen de fotonen zijn zwak, dus we kunnen storingsrekening gebruiken. Omdat we alleen fotonen zien, moet S_{vrij} het elektromagnetisch veld bevatten, maar *niet* het Dirac veld. Het deel van de actie dat de interacties beschrijft, S_{int}, bevat zodoende ook alleen het elektromagnetische veld; de fotonen koppelen *direct* aan elkaar. We nemen immers geen elektronen en positronen waar en zouden nooit op het idee komen deze als virtuele deeltjes te gebruiken in ons model.

Het bovenstaande is een voorbeeld van het gebruik van een *laagenergetische effectieve veldentheorie*. We spreken in dit verband dan ook over *effectieve acties*. Zoals uiteengezet in de inleiding en het eerste hoofdstuk van dit proefschrift, zijn er doorslaggevende theoretische en experimentele aanwijzingen dat de veldentheorieën die de op dit moment bekende deeltjes en wisselwerkingen beschrijven slechts effectieve theorieën zijn.

Een bijzondere eigenschap van veel veldentheorieën is dat ze hun eigen gebied van geldigheid *voorspellen*. Dit werkt als volgt. Een belangrijke eigenschap van quantummechanische theorieën is *unitariteit*, hetgeen zoveel wil zeggen als dat de kans dat een gegeven proces plaatsvindt nooit kleiner kan zijn dan 0 en nooit groter dan 1. Deze ogenschijnlijk flauwe eigenschap blijkt in de praktijk erg belangrijk te zijn, omdat de acties van veldentheorieën aan een aantal stringente mathematische eisen moeten voldoen, willen ze een unitaire theorie opleveren. Daarnaast blijkt dat bepaalde soorten interacties (de zogenaamde niet-renormaliseerbare interacties) slechts unitaire theorieën leveren bij *lage* energieën. In het geval van ons voorbeeld van de effectieve theorie voor de fotonen binnen QED uit zich dit als volgt. We kunnen met die theorie de kans op een bepaald verstrooiingsproces bij een gegeven energie van de fotonen berekenen. Bij de lage energieën waarvoor de effectieve theorie ontworpen was

[7]Virtuele deeltjes zijn deeltjes die niet voldoen aan de relatie $(*)$, dit in tegenstelling tot *fysische* deeltjes. Virtuele deeltjes kunnen nooit optreden als eindproduct van een botsingsproces.

[8]Omdat fotonen massaloos zijn, kan er een arbitrair aantal van geproduceerd worden. Het eindproduct bevat wel altijd een *even* aantal fotonen, een resultaat dat volgt uit QED en dat bekend staat als Furry's stelling.

gaat dit goed. Als we echter – op papier (!) – de energie opvoeren, dan zien we dat bij precies 1022 keV de theorie onzinnige resultaten produceert en niet langer unitair is. We zien dus dat de effectieve theorie *zelf* voorspelt dat ze slechts een effectieve beschrijving levert van een andere, onderliggende theorie – in ons voorbeeld QED. Hoe die onderliggende theorie er dan precies uitziet, kunnen we niet achterhalen op basis van de effectieve theorie alleen, maar we kunnen de effectieve theorie wel gebruiken om bepaalde aspecten van de onderliggende theorie te bestuderen.

Een ander beroemd voorbeeld van een niet-renormaliseerbare quantumveldentheorie is de Algemene Relativiteitstheorie (ART), die het zwaartekrachtsveld beschrijft. Zoals bij andere quantumveldentheorieën, is in de ART de waarde van koppelingsconstante – Newton's zwaartekrachtconstante G – afhankelijk van de energieschaal waarop we de theorie bestuderen. In de 'alledaagse' hoge-energie fysica die bestudeerd wordt in deeltjesversnellers is de zwaartekracht veel zwakker dan de andere drie bekende natuurkrachten en kan zij verwaarloosd worden[9]. Echter, bij extreem hoge energieën wordt de zwaartekracht net zo sterk als de andere interacties en moet zij op gelijke voet worden behandeld. En, omdat zwaartekracht niet-renormaliseerbaar is, komt er een moment dat de ART niet langer unitair is en vervangen moet worden door een onderliggende theorie. Dit gebeurt bij de zogenaamde Planck energie, 10^{18} GeV.

Er zijn in het verleden vele pogingen ondernomen om een theorie op te stellen die de ART en de andere quantumveldentheorieën zou kunnen vervangen bij energieën boven de Planck energie[10]. De meeste van deze pogingen waren weinig succesvol. Op dit moment zijn er slechts twee kandidaten voor zo'n theorie en van deze twee is alleen de snaartheorie in staat *alle* bekende interacties te beschrijven[11].

Snaartheorie is in beginsel een perturbatieve theorie. Dat wil zeggen dat *vrije* snaren als uitgangspunt genomen worden[12] en vervolgens bekeken wordt of er een consistente (d.w.z. unitaire) manier bestaat waarop de snaren met elkaar kunnen interageren. Dit blijkt op slechts *vijf* verschillende manieren te kunnen.

De vijf verschillende perturbatieve snaartheorieën vereisen allemaal het bestaan van zes extra ruimtelijke dimensies naast de drie die we heden ten dage waarnemen. Daarnaast zijn deze theorieën allemaal supersymmetrisch[13].

---

[9] De reden dat de zwaartekracht toch zo belangrijk is voor bijvoorbeeld astronomische verschijnselen, is dat de andere drie krachten effectief onzichtbaar zijn. De zwakke en sterke wisselwerkingen werken alleen over afstanden kleiner dan een atoomkern en astronomische objecten zijn elektrisch neutraal.

[10] Deze theorie zou dan met recht de 'theorie van alles' genoemd mogen worden.

[11] De zogenaamde *loop quantum gravity* beschrijft alleen de zwaartekracht en niet de andere drie natuurkrachten. Daarnaast lijkt deze theorie op dit moment nog niet in staat om de laagenergetische processen zoals die beschreven worden door de ART correct te reproduceren.

[12] De andere ingrediënten van snaartheorie zijn quantummechanica en de speciale relativiteitstheorie, net als in de deeltjesfysica.

[13] Supersymmetrie verschilt van andere symmetrieën in dat het bosonische en fermionische variabelen met elkaar mengt.

Net als de snaar van een gitaar hebben de 'supersnaren' een oneindige, maar aftelbare, hoeveelheid eigentoestanden. De verschillende trillingstoestanden van dezelfde snaar kunnen opgevat worden als verschillende typen deeltjes. Alle supersnaren hebben een eindig aantal massaloze deeltjes en een oneindig aantal massieve. Onder de massaloze deeltjes bevindt zich er in iedere snaartheorie één dat zich bij lage energieën[14] precies zo gedraagt als het graviton – het deeltje dat geassocieerd is met het zwaartekrachtsveld uit de ART. Andere van de massaloze bosonische deeltjes lijken in hun gedrag erg op de wisselwerkingdeeltjes van de andere drie natuurkrachten en bepaalde fermionische deeltjes op de ons bekende materie.

De situatie in snaartheorie is anders dan bij de veldentheorieën. Daar begonnen we met een actie en waren vervolgens genoodzaakt een benadering toe te passen. Het resultaat is een set regels om de interacties tussen deeltjes te beschrijven. In snaartheorie *beginnen* we met die set regels. Een onderliggende actie waar die regels uit afgeleid kunnen worden is niet bekend, m.a.w. de *niet-perturbatieve formulering* van snaartheorie is nog een raadsel. We hebben een paar bladzijden geleden gezien dat deeltjes manifestaties van velden zijn. Eén van de grote vragen binnen de snaartheorie is zodoende wat er op de plaats van de stippellijn moet komen in het volgende: "snaren zijn manifestaties van ...".

Toch is het mogelijk om informatie over het niet-perturbatieve regime van de snaartheorieën te verkijgen. Bij lage energieën worden de massieve trillingstoestanden van de snaar niet geproduceerd in botsingsprocessen en kunnen we ons – zoals bij QED – beperken tot een effectieve beschrijving van de massaloze toestanden alleen. Deze gedragen zich dan in essentie hetzelfde als puntdeeltjes en zodoende kunnen we voor de massaloze toestanden van de snaar *wel* een actie opschrijven. Op die actie kunnen we dan vervolgens de eerder genoemde semi-klassieke methode loslaten om niet-perturbatieve informatie te verkrijgen.

Zonder in detail te treden (zie hoofdstuk 2 van dit proefschrift voor meer informatie) leert dit ons het volgende: de vijf perturbatieve supersnaartheorieën zijn verschillende limieten van één unieke niet-perturbatieve overkoepelende theorie. In afwachting van haar uiteindelijke formulering is ze alvast *M-theorie* gedoopt.

Het formuleren van M-theorie is onder andere interessant om de volgende reden. Om contact te maken met de gevestigde theorieën in de hoge-energie fysica moeten de zes extra dimensies effectief onzichtbaar zijn bij relatief lage energieën. Eén manier om dit te bereiken is te veronderstellen dat de extra dimensies compact zijn en een heel klein volume hebben – bijvoorbeeld 'opgerold' tot kleine cirkeltjes. Echter, het aantal manieren waarop deze *compactificatie* gerealiseerd kan worden is zeer groot. Slechts een beperkt aantal van de mogelijke scenario's geeft resultaten die in meer dan alleen grote lijnen overeenkomen met de gevestigde theorieën. Het formuleren van M-theorie zal zonder meer leiden tot een beter begrip van het mechanisme achter compactificatie.

[14]D.w.z. 'laag' in vergelijking met de Planck energie.

Het blijkt dat M-theorie zich niet alleen maar manifesteert als snaren maar ook als andere hoger-dimensionale objecten. Deze worden aangeduid als p-branen, waar p staat voor de dimensionaliteit van deze objecten. Zo is een 0-braan een puntdeeltje, een 1-braan een snaar en een 2-braan een membraan. Deze p-branen zijn hoger-dimensionale generalisaties van zwarte gaten en worden in het algemeen geassocieerd met niet-perturbatieve effecten[15].

De best begrepen klasse van p-branen zijn de D-branen. Dit is omdat de D-branen in tegenstelling tot de andere branen beschreven kunnen worden met behulp van de perturbatieve snaartheorie als hypervlakken waar open snaren aan bevestigd zijn. Bij lage energieën manifesteren de open snaren zich als een veldentheorie die 'leeft' op de braan. Deze veldentheorie vertoont overigens sterke overeenkomsten met de effectieve beschrijving van de fotonen in QED die we eerder bespraken.

Systemen met meerdere D-branen vertonen een zeer opmerkelijk gedrag. Ze ervaren ruimtetijd op een manier die op een ingrijpende manier verschilt van wat we gewend zijn: volgens D-branen is ruimtetijd *niet-commutatief* (zie hoofdstuk 1 van dit proefschrift). Dit gedrag is op dit moment nog niet volledig begrepen, maar is belangrijk omdat het ons iets vertelt over hoe ruimtetijd uiteindelijk behandeld zal moeten gaan worden in M-theorie.

In de eerste twee hoofdstukken van dit proefschrift ga ik uitgebreid in op de formulering van snaartheorie en haar laagenergetische beschrijving in termen van superzwaartekrachttheorieën. Ik behandel D-branen zowel vanuit hun rol als hypervlakken in de perturbatieve snaartheorie als vanuit hun rol als zwarte gaten in superzwaartekrachttheorieën. Vervolgens bespreek ik kort hoe de vijf snaartheorieën aan elkaar gerelateerd zijn via dualiteiten.

Ik concentreer me in dit proefschrift in het bijzonder op de beschrijving van systemen met één of meerdere D-branen in termen van de laagenergetische veldentheorieën die leven op de D-branen. Ik laat zien hoe de niet-renormaliseerbare interacties in deze *deeltjes*theorieën gebruikt kunnen worden om typisch *snaar*theoretisch gedrag te onderzoeken.

In hoofdstuk 3 beschouw ik een systeem met één D-braan en vat ik samen wat er in de literatuur bekend is over de effectieve theorie die leeft op de D-braan. Vervolgens presenteer ik nieuwe tot dus ver onbekende bijdragen aan die effectieve theorie en beschrijf ik hoe deze resultaten verkregen zijn met een mix van veldentheoretische en snaartheoretische methoden. Ik benadruk het gebruik van supersymmetrie.

In hoofdstuk 4 beschouw ik systemen met meerdere D-branen. Over de niet-renormaliseerbare interacties in de effectieve theorie voor meerdere D-branen is veel minder bekend. Wederom behandel ik de literatuur en presenteer ik nieuwe bijdragen. Daarbij bespreek ik een snaartheoretische test voor deze resultaten.

Uiteindelijk vat ik in het nawoord het werk van hoofdstuk 3 en 4 samen en vergelijk

[15] p-branen blijken zogenaamde BPS objecten te zijn, dat zijn objecten die beschermd worden door de eerder vermelde supersymmetrische nonrenormalisatie stellingen.

ik de invalshoek die ik in dit proefschrift gekozen heb met andere methoden.

Dit proefschrift wordt gecompleteerd door twee appendices. De eerste appendix bevat een uitgebreide behandeling van de mathematische machinerie die onontbeerlijk is voor berekeningen aan hoger-dimensionale supersymmetrische theorieën. De tweede appendix bevat de afleidingen van een aantal kleinere resultaten die ik op verschillende plaatsen in de tekst gebruik.

Dankwoord

Ik wil mijn werkgever, Stichting FOM, niet alleen bedanken voor de financiële ondersteuning gedurende de laatste vier jaar, maar ook voor de ruimhartige scholingsmogelijkheden in de vorm van bezoeken aan conferenties in binnen- en buitenland en de Nyenrode managementtraining. Ik ben de Rijksuniversiteit Groningen en in het bijzonder het Centrum voor Theoretische Natuurkunde zeer erkentelijk voor de gelegenheid mijn promotieonderzoek aan deze instelling te verrichten en voor het onderwijs dat ik in de vijf daaraan voorafgaande jaren genoten heb. De Landelijke Onderzoekschool voor Theoretische Natuurkunde bedank ik voor Nijmegen en Dalfsen.

Dit proefschrift zou nooit tot stand gekomen zijn zonder de uitstekende begeleiding van mijn promotor Mees de Roo. In een vakgebied dat soms wel geregeerd lijkt te worden door hypes en waar de druk om te publiceren erg hoog is, is het prettig om een nuchtere en geduldige promotor zoals jij te hebben, die zijn promovendi de ruimte gunt om zich de basisbeginselen van het vak in alle rust eigen te maken. Daarnaast wil ik jou en Eric Bergshoeff bedanken voor de informele en stimulerende sfeer binnen de snaartheoriegroep in Groningen. Eric, bedankt voor de incidentele *peptalk* en nuttige kanttekeningen bij mijn onderzoek.

Het werk dat beschreven wordt in dit proefschrift en in mijn publicaties van de laatste vier jaar is tot stand gekomen in plezierige en produktieve samenwerking met Andrés Collinucci, Paul Koerber, Sudhakar Panda, Mees de Roo, Alexander Sevrin en Dennis Westra. De leden van mijn beoordelingscommissie, de professoren Jan de Boer, Martin Cederwall en Alexander Sevrin, wil ik bedanken voor het opofferen van een deel van hun zomer om mijn manuscript te lezen.

Ik wil de promovendi en postdocs van theoretische hoge-energiefysica bedanken voor de gezelligheid en uitdagende discussies tijdens maar ook na werktijd en in binnen- en buitenland. Wissam Chemissany, Andrés Collinucci, Ulf Gran, Rein Halbersma, Jelle Hartong, Sven Kerstan, Román Linares, Mikkel Nielsen, André Ploegh, Thomas Van Riet, Diederik Roest, Dennis Westra en Tim de Wit: zonder jullie als collega's waren de laatste vier jaar onvoorstelbaar minder interessant en amusant geweest. Ook de andere regelmatige bezoekers van de koffietafel in kamer 5111.0134 verdienen het hier genoemd te worden: Aernout van Enter, Maxim Mostovoy, Karel Netočný, Elisabetta Pallante, Hendrikjan Schaap, Jetze Sikkema en Rudolf Uitham. Dit geldt ook voor Ynske Joustra en Iris de Roo-Kwant, die ik daarnaast wil bedanken voor de nodige secretariële ondersteuning.

Ik wil mijn (schoon)familie en vrienden bedanken voor de vele gezellige uurtjes in Groningen en elders, en voor jullie belangstelling. Het logeerbed in Leiden staat altijd klaar!

Ten slotte wil ik mijn ouders, Henk Eenink en Mieke Eenink-Tijdink, en met name mijn vriendin, Marian Wilts, bedanken voor hun steun en bijna oneindige geduld tijdens mijn vele motivatiedipjes gedurende de promotie en de schrijfstress van het afgelopen jaar.