Session 1

FUNDAMENTAL THEORETICAL QUESTIONS

M. Froissart, Rapporteur

Introduction

Before reviewing for you the main fields of activity regarding the fundaments of theory, let me show you a map (Fig. 1-1) of the different logical connections among these fields, because, judging from the attendance at the discussion sessions, I presume that many of you are not quite as up to date as you perhaps might be.

I have drawn a boundary separating the results relevant to off-mass-shell theories or field theories in a generalized sense, and the on-mass-shell results or S-matrix theory results. Ideally the basic axioms should be immediate translations of our experience. A very attractive set would be Lorentz invariance, causality, and unitarity. The trouble is that the only tractable form of causality up to recently was through locality, which had to be formulated in terms of fields. Hence the very prominent role of field theory up to this day in our understanding of the microscopic world. May I stress also the importance of the success of quantum electrodynamics as a reason to stick to field theory. However, in view of the very singular nature of fields, it has been thought wise to dismiss the idea of fields and to consider only integrals of fields, over regions of space time. This led to the theory of local observables.

Most experimentalists are really interested only in knowing what happens on the mass shell. This has put a heavy burden on the theorists. Field theory gave the dispersion relations, and a small domain of analyticity in momentum transfer (Lehmann ellipse).

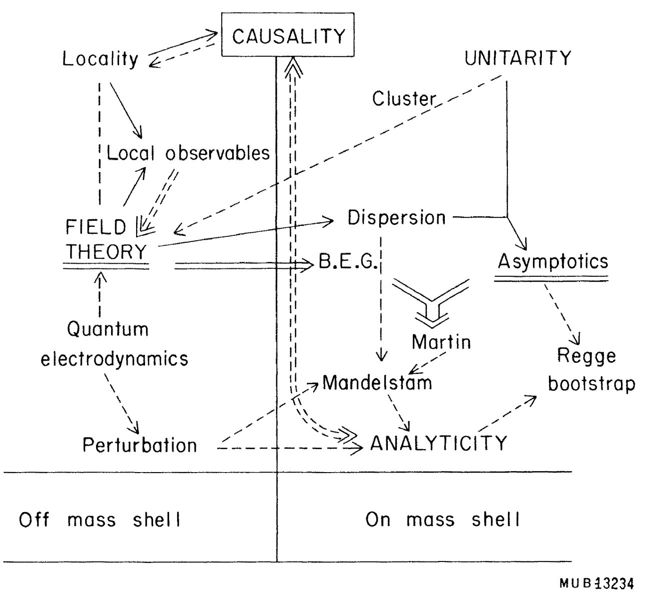

Fig. 1-1. Logical map of "Fundamental" concepts.
Continuous lines represent inferences, broken lines indications. Topics surveyed in this report are indicated by double lines.

When combined with unitarity, these results furnished bounds and relations on the asymptotic behavior of the amplitude. From then on, everything was until recently a matter of sophisticated guess. Mandelstam started the game by proposing a certain domain of analyticity compatible with perturbation theory to fourth order and dispersion theory. Then came different hypotheses: maximal analyticity, Regge behavior, bootstrap.

Field Theory

The main effort in field theory has been devoted to a study of the mathematical structure involved.[1] It is interesting to note that a number of tools have been developed, which are useful both for field theory and for statistical mechanics. The reason for this is clear: in both cases one has to do with systems with an infinite number of degrees of freedom. A very interesting example of the mathematical complications involved in the simplest cases has been discussed by Prof. Thirring[2] in connection with the B. C. S. theory of superconductivity. Although the model used is excessively simple and explicitly solvable in the case of a finite number of degrees of freedom, the passage to the limit exhibits a number of unexpected features, among which is the fact that some products of operators have a limit different from the product of the limits.

To study this kind of mathematical trouble, Jaffe and Lanford[3] have built model field theories with an artificially limited number of degrees of freedom. The object is to see which quantities have any likelihood of keeping a meaning as the cutoffs are removed. In an effort to remove the limitations imposed by the assumption of the existence of fields, and considering that the phenomena of ultraviolet divergence might make the fields more singular than they are usually assumed to be, Jaffe[4] has also developed a formalism capable of accommodating some nonrenormalizable theories. It appears that all results obtained up to date in conventional field theory hold true in this new version, except for the asymptotic behavior at high energy. Conventional dispersion relations may be proved, but might need an infinite number of subtractions if it were not for the results of Martin, which we shall discuss.

Much work has been devoted to the structure of the theory of local observables.[5] I understand that only technical difficulties prevent Borchers from deriving a concept very close to that of fields, even unobservable charged fields, from the theory of local observables. Epstein[6] has already proved the TCP theorem in this theory.

Analyticity on the Mass Shell

A sequence of very important results has broadened considerably the bridge between field theory and S-matrix theory. The papers of Bros, Epstein, and Glaser[7] have generalized the concept of crossing to all two-body amplitudes, whether dispersion relations are proved or not. Also, they have shown that the two-body amplitudes are all analytic in some small neighborhood of the physical region, this analyticity being in both variables s (energy variable) and t (momentum-transfer variable).

Using this information, the known dispersion relations, and unitarity, Martin[8] has been able to prove a very large number of results. In view of the importance of this work, let me go a little into the scheme of the proof.

We start from a conventional dispersion relation,

$$A(s,t) = \frac{s^N}{\pi} \int \frac{A_s(s',t)ds'}{s'^N(s'-s)}$$

$$+ \frac{u^N}{\pi} \int \frac{A_u(u',t)du'}{u'^N(u'-u)} + \text{subtractions.}$$

Unitarity is used then under the form

$$A_s(s,t) = \sum_l (2l+1) \operatorname{Im} a_l(s)\, P_l\left(1 + \frac{t}{2k^2}\right),$$

with $\operatorname{Im} a_l(s) > 0$, and with all derivatives of $P_l(Z)$ positive for $Z > 1$. This allows one to expand $A_s(s,t)$ in a Taylor series in t and to show that the radius of convergence is the same for all s, by interchange of the order of integration and summation of the series. Then the dispersion relation is valid for all values of t for which analyticity has been proved for fixed s. Typical values are[9]

$\pi\pi$	$\lvert t \rvert < 4\, m_\pi^2$,
$K\pi$	$\lvert t \rvert < 4\, m_\pi^2$,
KK	$\lvert t \rvert < 4\, m_\pi^2$,
πN	$\lvert t \rvert < 1.83\, m_\pi^2$.

That the dispersion relation stays valid as it is written leads to the consequence that, in these regions, the asymptotic behavior is nearly the same as in the forward direction. Hence, by an almost circular argument, using unitarity, one deduces that, even with dispersion relations à la Jaffe--with an infinite number of subtractions--to start with, the behavior of the amplitude is $s \ln^2 s$, therefore the dispersion relations are valid with two subtractions. An interesting feature is that the reasoning can be started again from the domain of analyticity just found, and a larger domain is obtained. The result for $\pi\pi$ scattering is represented on Fig. 1-2. The same operation can be performed for πK and πN scattering. The domain obtained in the latter case goes to larger values of -t than was proved in the classical proofs of dispersion relation. More work can undoubtedly be done by using, in the $\pi\pi$ case, the domains obtained in the three channels, and finding their holomorphy envelope. To date, the real part of the analyticity domain is given approximately by the cuts and the dashed region of Fig. 1-3. That is, it goes to the double spectral region except for a few cases. This gives a region of analyticity for the partial-wave amplitudes indicated on Fig. 1-4. Possible singularities are far enough away-- compared, say, with the mass of the ρ meson--so that calculations neglecting far-away singularities may be justified.

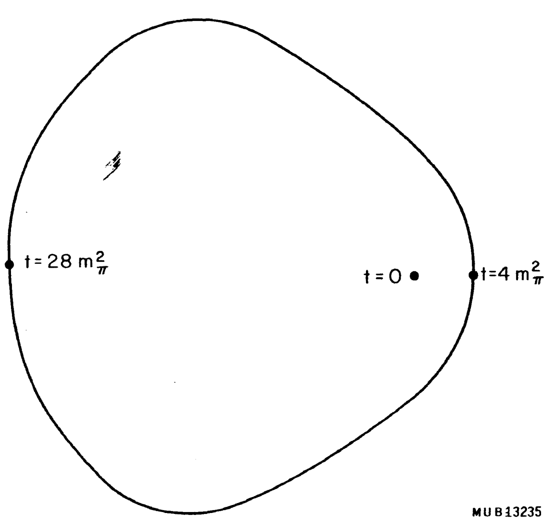

Fig. 1-2. Values of the momentum transfer variable t for which dispersion relations are proved in π-π scattering.

On-Mass-Shell Causality

Much work is also devoted to getting rid of the circuitous--and not completed--logical path connecting causality to analyticity. Many people think that an axiom like that of locality is probably as unphysical as an axiom of analyticity, in the sense that it can be checked only indirectly through its consequences. As Blokhintsev points out,[10, 11] it is quite feasible to build theories which are nonlocal on the microscopic scale, but which exhibit gross causality. These theories might perhaps only be distinguished by additional singularities. The works of the "Cambridge School" and others[12-19] have shown that an assumption of analyticity of the amplitude in some neighborhood of the physical region leads to uniquely deter-

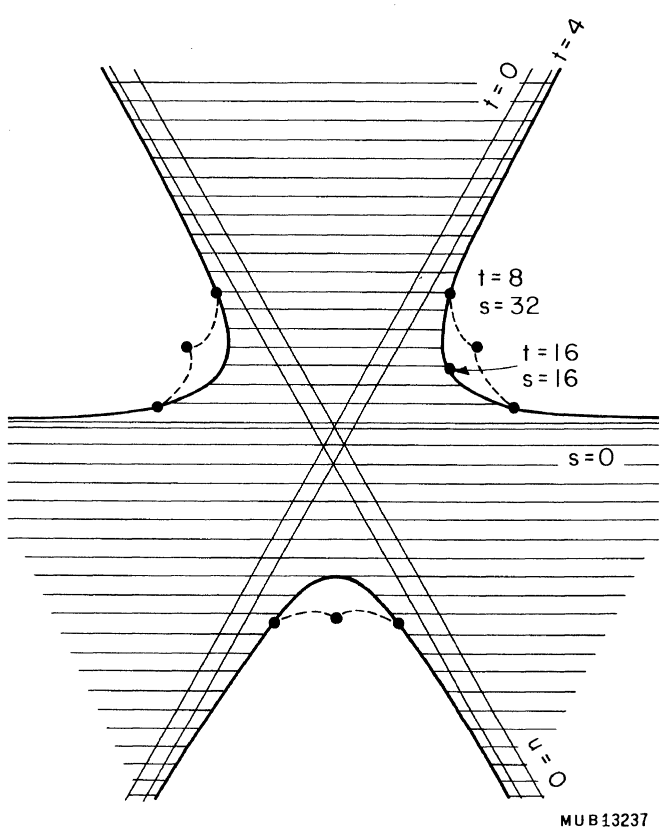

Fig. 1-3. The shaded region is the real trace of the analyticity domain of the π-π scattering amplitude. Within this region the cuts are à la Mandelstam. The dashed curves are the boundaries of the Mandelstam double spectral regions.

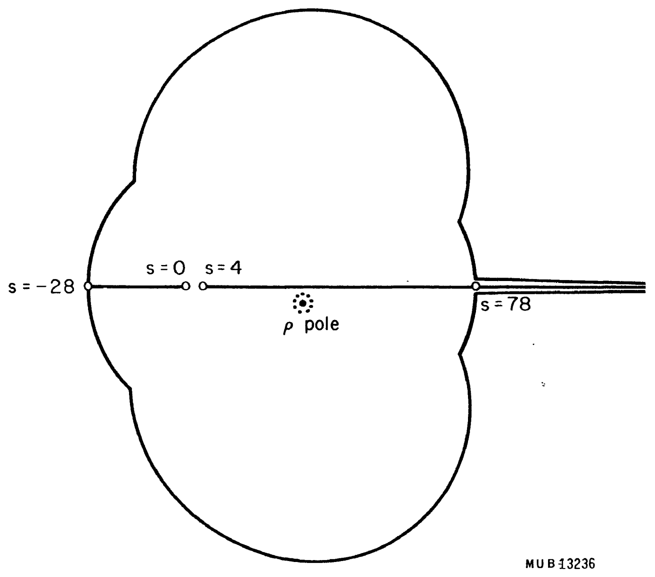

Fig. 1-4. Analyticity domain for π-π scattering partial waves. The thin neighborhood of the positive real axis is not drawn to scale.

mined singularities, and hence[20] to the usual description of successive interactions in multiple scattering. These arguments use only postulates of cluster decomposition[21] and unitarity.

The work of Stapp[22] assumes a weak condition on asymptotic decrease of transition probabilities when particles are taken away. The result is to prove the infinite differentiability of amplitudes on the physical region except at the points where singularities are expected. A condition of exponential decrease of transition probabilities allows Omnès[23] to derive a finite domain of analyticity for the two-body amplitudes in momentum transfer, apart from some technical difficulties. Some other authors[24-26] have also tried to attack this problem, but without any decisive success.

Asymptotic Theorems

We can distinguish between bounds and other theorems such as the Pomeranchuk theorem and the connections between the phase and the modulus of the amplitude. In the first field, no new result has been obtained. However, many bounds are now rigorously derived from field theory, and even from the generalized version due to Jaffe. This is of course due to the work of Martin. I would like to mention specially the bound on form factors $|F(t)| > \exp(-b\sqrt{-t})$ derived by Jaffe[27] rigorously, even for the cases for which no dispersion relations have been proved for the form factor. Tables 1-I, 1-II, and 1-III have been compiled by Martin, and give an up-to-date account of the situation.[28-43]

In other asymptotic theorems, the situation is still very confused. The Pomeranchuk theorem now holds for all processes, due to BEG,[7] but, as pointed out by Eden,[44] no one has yet succeeded in removing the extra hypothesis such as the existence of a limit of the amplitude--or at least some control over possible oscillations--on the one hand, and some control over the growth of the real part with respect to the imaginary part.

To obviate the need for these requirements, Khuri and Kinoshita[45,46] have found a number of inequalities which, unlike ordinary dispersion relations, allow one to test these ideas with measure-

Table 1-I. Upper bounds.

	Assumption	References
Forward scattering		
$\lvert F(s, \cos\theta = 1)\rvert < C\, s \log^2(s/s_0)$,		
$\sigma_{total} < C' \log^2(s/s_0)$ (arbitrary spins);	Axiomatic	7
$C' < 4\pi/\mu^2$, μ= pion mass for ππ, πk, kk,		
$C' < 12\pi/\mu^2$ for πN		28
Fixed transfer		
$\lvert F(s,t)\rvert < s(\log s)^{3/2}$, $t < -\epsilon < 0$;		29
$\lvert F(s,t)\rvert < s^{2-\epsilon}$, $0 < t < 4\mu^2$;	Axiomatic	28
$\lvert F(s,t)\rvert <$ const, $0 < t < 4\mu^2$		
if $\lvert F(s, 0)\rvert < s^{-\epsilon}$		30
Fixed angle		
$\lvert F(s, \cos\theta)\rvert < C(s^{3/4}(\log s)^{3/2})/(\sqrt{\sin\theta})$;	Axiomatic	29
$\lvert F(s, \cos\theta)\rvert < C(\log s)^{3/2}/(\sin^2\theta)$;	Mandelstam	31
$d\sigma/d\Omega < C(\log s)^3/(\sin^4\theta)\, s$		
(in this form valid for arbitrary spins)		32 - 34

Table 1-II. Diffraction peak.

	Assumption	References
$\frac{d}{dt} \log A_s(s,t)\|_{t=0} > \frac{1}{9} \frac{(\sigma_{tot})^2}{4\pi\sigma_{el}} - \frac{1}{k^2}$	None	35
$\frac{d}{dt} \log A_s(s,t)\|_{t=0} < C(\log s)^2$	Axiomatic	30
$\frac{\int_{-1}^{1} \|F(s,\cos\theta)\|^2 d\cos\theta}{\|F(s, \cos = 1)\|^2} \to 0$ if $A_s(s, \cos\theta = 1) > C/\log s$	Axiomatic	36 - 37

ments of the forward amplitude on a finite real interval. The information used on the behavior at infinity is the positivity of the imaginary part, and the established fact that there are at most two subtractions needed.

I would like to end by expressing my apologies to the many people who may have been treated unfairly, or whose work I may have misrepresented.

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Table 1-III. Lower bounds.

Elastic cross section	Assumption	References
$\sigma_{el} > C \dfrac{(\sigma_{tot})^2}{(\log s)^2}$	Axiomatic	38
Forward amplitude (strict for complex s; average sense for real s)		
(a) $\lvert F(s, t=0)\rvert > 1/s^2$ (no unphysical cut), σ total $> 1/(s^6 \log^2 s)$	Axiomatic	28
(b) if F(threshold) < 0 or is small compared with an integral over total cross sections at low energies only,	Axiomatic	28, 39
$\lvert F(s, t=0)\rvert >$ const > 0,	Axiomatic	
$\sigma_{total} > 1/(s^2 \log^2 s)$		
Fixed-angle amplitude		
$\lvert F(s, \cos\theta)\rvert > \exp[-\sqrt{s} \log s\, C(\theta)]$,	Mandelstam	40
$C(\theta)$ for small $\theta \sim \sqrt{\theta}$		41
Form factors		
$\lvert F(t)\rvert > \exp(-C \sqrt{\lvert t\rvert})$	Axiomatic	42, 27
Fixed-t amplitude, $t < 0$		
$\lvert F(s,t)\rvert > S^{-N}$	Axiomatic	
Fixed-u amplitude, $u < 0$		
$\lvert F(s,u)\rvert > S^{-N}$ if pole at $u=M^2$	Slightly more than axiomatic	43

Discussion

Nauenberg (Santa Cruz): Could you give some more details on what results have been obtained about Regge behavior from fundamental assumptions?

Froissart: From the fundamental side I haven't drawn any continuous arrow on the map. There are some inequalities which have been proved to hold asymptotically, and which are consistent with Regge behavior, but I don't think that anything more detailed has been proven.

Sudarshan (Syracuse): It was my impression that you said that Borchers had proved something about the connection between local observables and the existence of a vacuum.

Froissart: Yes, the proof suffers a little technical difficulty now, but I don't think it's very essential. He proves that in a theory of local observables there exists a vacuum state which has all good properties.

Sudarshan: I must have been misunderstood. If you took a free field, a Wick polynomial of second degree as Wightman and other people have done, then the states of this particular system split into two classes, those corresponding to an odd number of particles of the original field, and those corresponding to an even number of particles. If you consider that particular subset of states which consists only of odd numbers of particles of the original free field, the new field we have introduced seems to be able to connect only to other odd numbers of particles. In such a theory there seems to be no vacuum at all, and I don't quite understand how one could have proved the existence of a vacuum. This is a free-field theory; there are no divergences of any kind.

Froissart: The idea of Borchers is very simple. You start by taking any state, and then you essentially average out over the whole space; that is, if there are any particles to start with, then these particles should not contribute at all to the average state. That is the essence of Borchers's proof. I do not know which of the axioms the phenomena you are mentioning contradict.

Sudarshan: This assumes that that there exists a discrete mass spectrum of the field?

Wightman (Princeton): May I comment on that? The point is that Borchers is using a definition of physical equivalence which is appreciably weaker than that to which many people are accustomed, so that he would not have to assume that there was, a priori, a discrete point in the spectrum. He would count two theories equivalent if they gave arbitrarily close results for all measurements in bounded regions of space time. Presumably he will take your example and prove it equivalent to another example in which there was a vacuum state, but only in this weak sense. However, it has to be said that this weak sense is a very physical sense, because it corresponds to measurements in actual laboratories as best we know how to describe them.

Todorov (Dubna): I would like to make the following remark, concerning the derivation of Pomeranchuk's theorem.

It is not necessary to utilize the nontrivial results of Bros, Epstein, and Glaser about the analyticity domain of any two-body scattering amplitude in order to prove a Pomeranchuk-type theorem for it. It is sufficient to take into account the analyticity properties of the so-called asymptotic amplitude, introduced by Meiman (1964). (The exact conditions under which the physical amplitude has the same high energy behavior as the asymptotic one practically coincide with the smoothness assumption needed for the proof of Pomeranchuk's theorem.) The advantage of such a method results from the fact that it applies to production amplitudes too (cf. Logunov's report at the 12th International Conference on High Energy Physics, Dubna, 1964).

Sucher (Maryland): Could you extend the remarks you made about going from the local observable theory to field theory, that is, the possible existence of fields once you have the local observable theory?

Froissart: This is not really a rigorous connection. You have a number of stages of the theory. You have first the theory of local observables. Then you have fields à la Borchers, that is, things which are defined in the whole space but which are not quite local. They have some finite range, arbitrarily small, but finite. Otherwise, they enjoy all the properties of fields. Then you have fields à la Jaffe, where the fields are local but are not distributions. They are generalized functions of a higher order. And then you have fields à la Wightman, where the fields are distributions. Now, the step which has been accomplished is to show that local observables imply the existence of fields à la Borchers. Also, Jaffe has proved that fields à la Jaffe are just as good as fields à la Wightman for all practical purposes. So, we have only to cross now the gap between Borchers and Jaffe.

Lichtenberg (Indiana): Could you write down some of the asymptotic theorems that Martin has proved recently?

Froissart. No new asymptotic theorems have been proved since the Dubna Conference. The only refinements which have been proved is that the total cross sections are bounded so that the constant in front of the $(\log)^2$ term is now known for $\pi\pi$, πK, and KK scattering:

$$\sigma_{tot} < (4\pi/m_\pi^2)\log^2(s/s_0).$$

We still don't know s_0. For πN scattering the coefficient is $12\pi/m_\pi^2$. See Tables 1-I through 1-III.

Logunov (Serpukhov): What behavior at infinity of the form factor was assumed in the work of Jaffe in order to get the lower bound?

Froissart: He did it using his theory, with generalized functions of a higher order. See References 4 and 27.

Chew (Berkeley): Does the analyticity domain established by Martin suffice to allow the definition of the Froissart-Gribov continuation in angular momentum for large angular momentum?

Froissart: No, it does not go to infinity.

Martin (CERN): The only result on the Froissart-Gribov continuation is that you can do it for any s which is inside the region of analyticity (see Fig. 1-2). Of course, that is quite obvious because then you have dispersion relations with two subtractions. But for physical energies I have nothing at all.

Eden (Cambridge): Could you comment on what has been done on the relation between spin and statistics in S-matrix theory in the past two years?

Froissart: The relation between spin and statistics in S-matrix theory is largely a matter of semantics, because the arguments on analyticity are not very well standardized. The question is better understood now, in that we see better how this relation comes about, but it is very hard to say what has been derived and what has not been derived in S-matrix theory, in the sense that some authors take as postulates other authors' theorems, and conversely. The connection between spin and statistics has been proved now for four years by Stapp, using quite a number of axioms. Now, one by one these axioms have been removed, but the seam is not completely tight, I would say.

Stapp (Berkeley): There has been important progress in the last year on the S-matrix proof of the normal connection between spin and statistics. In the first place several technical assumptions that were important in the original proof have now been replaced by physical assumptions. In the second place the assumption of the earlier proof that the usual types of crossing relations are valid has now been shown to follow from assumptions that the singularities of the mass-shell scattering functions are Landau singularities with Cutkosky-type discontinuities. These improvements are given in recent papers by Lu and Olive, Froissart and Taylor, and myself.

Hepp (Princeton): I want to comment on the connection between spin and statistics for local observables. It has not been proved, up to now, for the particles connected with fields "à la Borchers," and it's a challenge for everybody. The TCP theorem has recently been proven by Epstein[6] under the framework of local observables, but only for the S matrix.

Blokhintsev (Dubna): Professor Froissart included in his diagram "The Field Theory." This gives me the opportunity to attract your attention to the work concerning nonlinear field theory done by B. Barbashev and N. Chernikov (Dubna). About thirty years ago M. Born and L. Infeld developed nonlinear field theory. The scalar field of the Born-Infeld Lagrangian is

$$L = \sqrt{1 - \left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2}$$

and coincides with the Lagrangian for the "soap film" in Minkowski space. They obtained an explicit and rigorous solution of the quantized nonlinear equations corresponding to this Lagrangian. Instead of x and t, they use the variables α, β, which only asymptotically coincide with x-t and x+t; but—for small x, t—α and β are quantum operators, so that space and time are automatically quantized in the region of high nonlinearity. This result seems to be very instructive from the mathematical point of view.

Appendix A

Recent Progress in Axiomatic S-Matrix Theory

Henry P. Stapp

There has been important recent progress in four areas: (a) S-matrix causality conditions and their consequences, (b) proof of the spin-statistics connection, (c) derivation of the crossing properties, and (d) derivation of discontinuity equations.

The first topic is S-matrix causality conditions. Here the aim is to formulate causality conditions that refer only to mass-shell quantities, and then to derive analyticity properties. Similar ideas have been discussed by Wanders,[1] Stapp,[2] Iagolnitzer,[3] Peres,[4] Pham,[5] and Chandler and Stapp.[6] The furthest development is by Pham and by the last two authors, whose work I now describe.

The basic quantity is the transition amplitude

$$S[\phi] \equiv \int S(k_i) \prod_i d^4k_i \phi_i(k_i). \qquad (1)$$

Here $S(k_i)$ is the scattering matrix, and ϕ_i is ψ_i or ψ_i^* for initial or final particles. The support of $\phi_i(k_i)$ lies on the mass shell. We shall take the $\phi_i(k_i)$ to have small compact support and to be infinitely differentiable in the mass shell. The important property of these functions is given by Ruelle's lemma, which states the following: Let the curve in Fig. 1-A1 represent the mass shell and let the small

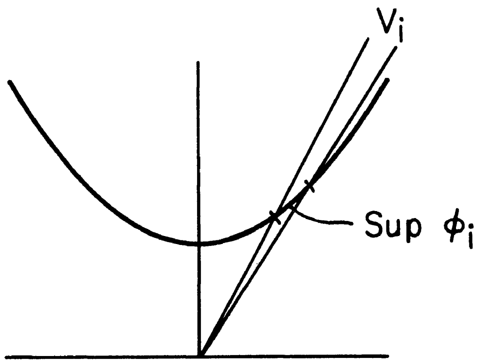

Fig. 1-A1. The velocity cone V_i.

segment on it represent the small compact support of ϕ_i. Let V_i be the smallest cone from the origin that contains the support of ϕ_i. The cone V_i is called the velocity cone of ϕ_i, and it is considered to be a cone also in space-time. Ruelle's lemma says that if the space-time point x is not in V_i then the Fourier transform of $\phi_i(k_i)$ satisfies the property

$$\tilde{\phi}_i(x\tau) \Rrightarrow 0. \qquad (2)$$

Here τ is a scale parameter and the double arrow means the limit is approached faster than any inverse power of τ as τ becomes infinite. Ruelle's lemma says that, in terms of the variable $\bar{x} = x/\tau$, the wave function collapses "rapidly" into the velocity cone as τ becomes infinite.

Using Ruelle's lemma, one easily proves that the integral over all space of the absolute value of the product of displaced wave functions satisfies

$$\int d^4x \, |\phi_i(x-u_i\tau)\, \phi_i(x-u_i\tau)| \Rrightarrow 0 \qquad (3)$$

unless the displaced velocity cones overlap (see Fig. 1-A2).

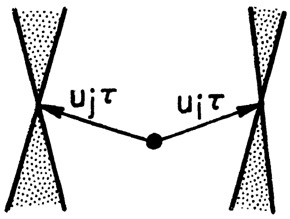

Fig. 1-A2. Displaced velocity cones.

Let $U \equiv \{U_i\}$ be a set of displacements. The quantity of interest is

$$S[\phi^{U\tau}] \equiv \int S(k_i) \prod_i d^4k_i \phi_i(k_i)\, e^{-ik_iu_i\tau}. \qquad (4)$$

The exponential factor $e^{-ik_iu_i\tau}$ displaces particle i by $u_i\tau$. We are interested in the behavior of $S[\phi^{U\tau}]$ as τ becomes infinite. Our causality requirements will be statements that under certain conditions

$$S[\phi^{U\tau}] \Rrightarrow S_0[\phi^{U\tau}], \qquad (5)$$

where $S_0[\phi]$ is the value that $S[\phi]$ would have if there were no scattering.

Two causality conditions have been formulated. The first, called weak asymptotic causality (WAC), says that Eq. 5 is satisfied if, for some $\tau\epsilon > 0$, the displaced initial-particle velocity cones do not intersect in $t \leqslant \epsilon\tau$ and the displaced final-particle velocity cones do not intersect in $t \geqslant -\epsilon\tau$. See Fig. 1-A3. The set of U such that these conditions are

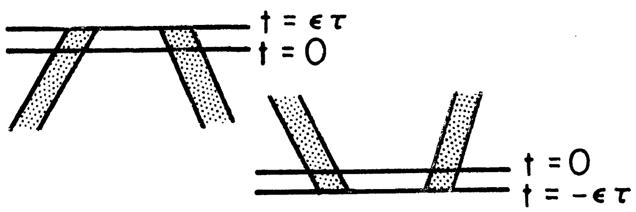

Fig. 1-A3. Initial and final displaced velocity cones well-separated in $t < \epsilon\tau$ and $t \geqslant -\epsilon\tau$ respectively.

satisfied is called the set of acausal displacements $A_\epsilon(\phi)$, and Eq. 5 is required to hold uniformly in U on compact subsets of $A_\epsilon(\phi)$, for fixed ϕ and ϵ.[7]

The WAC condition is justified by proving that it holds in any classical model with finite range interactions. The quantum-mechanical ϕ of Eq. 5 is correlated to the statistical probability function of the classical model.

The WAC condition is also very plausible quantum-mechanically. If U is in $A_\epsilon(\phi)$, then it follows from Ruelle's lemma that

$$\int_{-\infty}^{\epsilon\tau} d^4x \, |\phi_i(x-u_i\tau)\ \phi_j(x-u_j\tau)| \Rrightarrow 0 \qquad (6a)$$

for all initial $i \neq j$, and

$$\int_{-\epsilon\tau}^{\infty} d^4x \,|\phi_i(x-u_i\tau)\, \phi_j(x-u_j\tau)| \Rightarrow 0 \qquad (6b)$$

for all final $i \neq j$. That is, the initial-particle overlap in $t \leqslant \epsilon\tau$ and the final-particle overlap in $t \geqslant -\epsilon\tau$ both fall off rapidly. In fact if U is in $A_\epsilon(\phi)$, then the displaced initial-particle cones become infinitely far apart in $t < \epsilon\tau$ as τ becomes infinite, and the displaced final-particle velocity cones become infinitely far apart in $t > -\epsilon\tau$. But if the initial particles do not come close to each other in $t < \epsilon\tau$ then the initial particles should be well represented near $t = 0$ by the unperturbed initial-particle wave functions. And if the final particles do not come close to each other in $t > -\epsilon\tau$ then the final particles should be well represented near $t = 0$ by the unperturbed final-particle wave functions. Thus the transition amplitude should be well represented by the overlap of the unperturbed initial and final wave functions, which is just $S_0[\phi^{U\tau}]$.

The WAC condition has important consequences. If the two-body scattering function is analytic except for singularities in the energy variable at normal thresholds, then it follows from WAC that, apart from an infinitely differentiable function, this function is the limit of a function analytic in a strip lying in the upper-half energy plane. It is not assumed beforehand that the original functions in the various intervals between the normal thresholds are parts of a single analytic function. Thus what is proved is first that the functions in the different intervals are analytic continuations of each other, second that the path of continuation connecting them moves through the upper-half energy plane at the normal threshold singularity, and third that the integral over the physical function is obtained by taking the contour to run slightly above the threshold singularities. (Thus, for example, the principal value integral is not used.) Such rules for continuing around physical region singularities are called $i\epsilon$ rules. These considerations show how they can be derived from strictly mass-shell arguments. Singularities that possess finite derivatives of all orders with respect to real variations dE are not covered by the analysis, and hence $i\epsilon$ rules are not deduced for them. However, the usual pole, square root, and logarithmic singularities arising from changes in the form of the unitarity equations are covered.

It is worth noticing that the K matrix, like the S matrix, is analytic in the various intervals between the normal thresholds, but that these various functions are not analytically connected. Thus some principle is required to justify the assumption that it is S that has the nice analyticity property. The weak asymptotic causality condition serves this purpose, for the two-body scattering function.

One would like to obtain similar $i\epsilon$ rules for all physical region singularities of all (many-particle) scattering functions. The WAC is not strong enough for this. A stronger condition that is sufficient is the strong asymptotic causality (SAC). This condition expresses the idea that interactions are carried over infinite distances only by physical particles. More accurately, interactions not carried by physical particles are required to fall off faster than any inverse power of the Euclidian distance.

This condition is formulated as follows: Consider a $\phi \equiv \{\phi_i\}$ and a $U \equiv \{u_i\}$. The displaced velocity cones of certain initial particles may intersect somewhere (Region A of Fig. 1-A4). And the displaced velocity cones of certain final particles may intersect somewhere (B of Fig. 1-A4). And the displaced velocity cones of certain initial and final particles may intersect somewhere (C of Fig. 1-A4).

It may be possible that the initial particles having momenta in the support of the wave functions colliding at A can interact to produce particles that travel to regions B and C. The particle traveling to C may interact in C with particles having momenta in the support of the wave functions of the external particles that intersect at C to give a particle that travels to B and hits there the particle from A to give final particles with momenta in the supports of the final particles that intersect at B. The various particle momenta must satisfy the physical mass constraints, and the conservation laws must be satisfied at the vertices. Furthermore the internal particle velocities must be $v_i = k_i/m_i$. If one can find a set of internal-particle trajectories that satisfy simultaneously both the space-time conditions and the momentum-energy constraints, then U is said to be causal with respect to ϕ. Otherwise U is acausal with respect to ϕ. The set of U that are acausal with respect to ϕ is denoted by $A(\phi)$. The condition of strong asymptotic causality (SAC) asserts that Eq. 5 is satisfied uniformly on any compact subset of $A(\phi)$.[8]

From SAC it follows that, apart from an infinitely differentiable function, the scattering functions are analytic except on the closure of the positive-α Landau surfaces. In the neighborhood of an isolated Landau surface one obtains an $i\epsilon$ rule that is just the same as the one obtained in perturbation theory. Similar results have been obtained for points where several Landau surfaces intersect. Thus we have derived from a strictly mass-shell causality condition the result that, aside from singularities that have finite derivatives of all orders with respect to real variations, the analytic structure of each scattering function is just that given by perturbation theory. This supports the general idea of Landau that the perturbation-theory singularities have a significance that transcends that theory itself, and also provides a justification for the assumption that the $i\epsilon$ rules for continuing around singularities generated by the unitarity equations should agree with the $i\epsilon$ rules obtained from perturbation theory.

Footnotes and References (Appendix A)

1. G. Wanders, Nuovo Cimento 14, 168 (1959) and Helv. Phys. Acta 38, 142 (1965).
2. H. P. Stapp, Phys. Rev. 139, B257 (1965).
3. D. Iagolnitzer, J. Math. Phys. 6, 1576 (1956).
4. A. Peres, Ann. Phys. 37, 179 (1966).
5. F. Pham, Singularités des Processus de Dif-

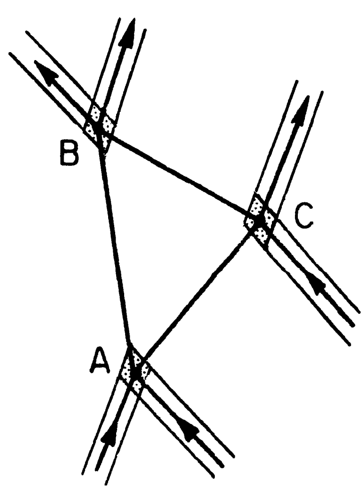

Fig. 1-A4. Displaced velocity cones and their intersection regions.

fusion Multiple (CERN preprint).

6. C. Chandler and H. P. Stapp, S-Matrix Causality Conditions and Physical-Region Analyticity Properties, in preparation.

7. This statement of the weak asymptotic causality condition is slightly oversimplified. We actually require the nonoverlap conditions to be satisfied also for slightly larger cones that contain the supports of the wave functions within their interiors.

8. Actually SAC asserts slightly less--it asserts falloff for wave functions ϕ' with support confined to the interior of the support of ϕ.

Appendix B

Proof of the Normal Connection Between Spin and Statistics in S-Matrix Theory

Henry P. Stapp

The original S-matrix proof of the normal connection between spin and statistics given by this author depended on an assumption that linear combinations of particle-antiparticle amplitudes were in principle observable. This assumption has no experimental basis, at least in the case of charged particles, and is therefore objectionable.

In that same paper the beginning of an alternative proof not depending on this assumption was given.[1] This proof was based upon an apparent conflict between abnormal statistics on the one hand and the crossing and Hermitian analyticity properties of scattering functions on the other. This argument has recently been developed in papers by Lu and Olive,[2] Froissart and Taylor,[3] and myself.[4]

The analyticity property needed in these works is indicated in Fig. 1-B1.

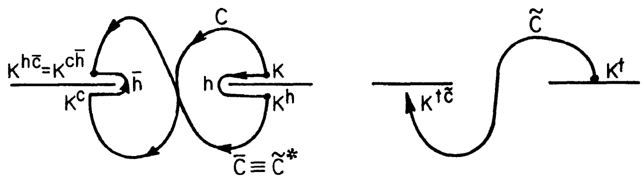

Fig. 1-B1. The path c represents the path connecting an original physical point K [of a scattering function $M_c(k)$] to the point K^c corresponding to a crossed process. The path h represents the path connecting K to the point K^h corresponding to the Hermitian conjugate process. The path $\bar{h}$ is the path from K^c to the point $K^{c\bar{h}}$ corresponding to the Hermitian conjugate of the crossed process. The path $\bar{c}$ is the path that takes the Hermitian conjugate function to the Hermitian conjugate function for the crossed reaction. This path $\bar{c}$ is the complex conjugate of the path $\tilde{c}$ that takes the function representing the transposed process into the function representing the transpose of the cross process.

The analyticity property needed for the new proofs of normal connection between spin and statistics is that the various (mass-shell) paths of crossing and Hermitian analytic do exist and that $c + \bar{h} = h + \bar{c}$. This last requirement is that the continuation of $M_c(K)$ first to K^h and then to $K^{h\bar{c}}$ take one to the same point on the Riemann surface of $M_c(K)$ as the continuation first to K^c and then to $K^{c\bar{h}}$. A proof that these analyticity properties do in fact hold if the singularities of the mass shell functions are Landau singularities with Cutkosky-type discontinuity formulas is given in Ref. 4.

Apart from this crucial analyticity question the main difficulty in the new proofs concerns the phase factor in the cluster decomposition equation. The cluster decomposition equation is written

$$S(K) = \sum_p a_p \prod_i S_c(K_{pi}), \qquad (1)$$

where the sum is over different ways of partitioning the external particles into groups, and $S_c(K_{pi})$ is the connected part of the S matrix for the ith group of pth partition. The phase factor a_p is, in field theory, the signature of the permutation of the order of fermion variables on one side of the equation relative to the order of these variables on the other side.

In field theory this phase emerges from certain manipulations with the field operators. The question therefore arises how one establishes this phase factor in an S-matrix theory. The point is that in an S-matrix theory one has, originally, certain functions that describe the scattering. A natural way to order the variables of these functions is, for instance, to write first the variables for the first type of particle, next the variables of the second type of particle, and so on. Under analytic continuation the order of two variables representing the same type of particle can become interchanged, and one may inquire into the relationship between these two functions. This leads to the spin-statistics question. But, from this standpoint, the notion of interchange of variables of difference particles does not naturally arise; analytic continuation does not interchange unlike variables. So why must one deal at all with functions having variables in other orders? Why is the phase in the cluster decomposition equation given in terms of this artificial idea of functions with variables in various possible orders?

The fundamental question here is how to determine the phases a_p in S-matrix theory. The simplest procedure is simply to borrow from field theory the usual creation-annihilation operator formalism for the free-particle states. If one does this then the connection between spin and statistics follows easily from the analyticity property described above. But this procedure is clearly unsatisfactory, for the question immediately arises whether it would be possible, within a pure S-matrix framework, to have an abnormal connection between spin and statistics if one relaxed these special phase assumptions. Since phase considerations are the essential part of the spin-statistics question one must obviously derive all important phases from physical considerations, not simply fix them by fiat.

This question was considered, and essentially resolved, in the recent work by Froissart and Taylor.[3] They show how certain "almost physical" requirements[5] allow the phases in the cluster decomposition principle to be adjusted to the phases obtained from the creation-annihilation operator formalism. Once the phases are fixed in this way the spin-statistics connection follows from the above-mentioned analyticity properties in a quite straightforward manner.[6]

The proofs by Froissart and Taylor and by Lu and Olive depend on the notion of interchanging variables of different types of particles. It is aesthetically somewhat disagreeable that one should have to bring into the spin-statistics question, which basically involves only the interchange of identical-particle variables, this extraneous notion; it would be preferable to avoid altogether the question of the phase change under the interchange of unlike variables. This can be done as follows.

One starts from unitarity, which can be written in bubble notation[7] as in Fig. 1-B2.

The two pole terms displayed in Fig. 1-B2 have δ-function singularities at $k^2 = m^2$, where k is the

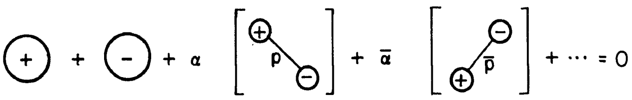

Fig. 1-B2. A unitarity equation with two pole terms exhibited. The external lines have been omitted, but the two little upper bubbles are both supposed to be connected to the same subset of the external lines and, similarly, the two little lower bubbles are both to be connected to the same subset of external lines. The three dots represent the other terms in the unitarity equation. The phases α and $\bar{\alpha}$ come from phases in the cluster decomposition equations. In this notation the plus bubbles represent the connected part of the scattering functions whereas the minus bubbles represent the complex conjugate of the connected part of the scattering function for the transposed (initial ⟶ final) process.

sum of the momentum-energy vectors of the final particles connected to the upper bubble, minus the sum of momentum-energy vectors of the initial particles connected to this bubble, and m is the mass of the particle p associated with the internal line. The manifold $k^2 = m^2$ intersects the physical region of the larger process in two disjoint regions corresponding to $k^0 > 0$ and $k^0 < 0$. The arguments of the M functions in these two regions are denoted K and $\bar{K}$, respectively.

The proof is carried through first for spinless particles. The pole-factorization theorem[8] gives for the residue of the pole at $k^2 = m^2$ the result shown in Fig. 1-B3.

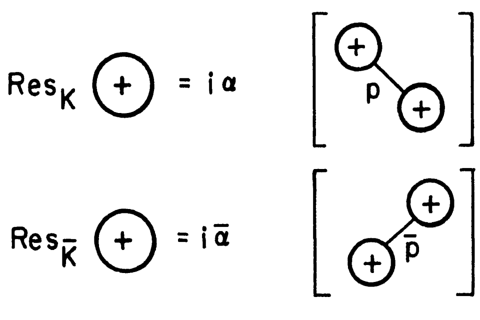

Fig. 1-B3. The bubble notation representation of the residue of the pole at $k^2 = m^2$ at points K and $\bar{K}$ in the physical region of the larger diagram. Notice that for these residue functions the two bubbles are both plus bubbles. For the residue of a minus bubble both little bubbles would be minus bubbles. Notice also that the phases α and $\bar{\alpha}$ are just the phases of the corresponding terms in the original unitarity equation.

The equations represented in Fig. 1-B3 are

$$\mathrm{Res}\ M(K) = i\,\alpha\, M(K_1)\, M(K_2) \qquad (2a)$$

and

$$\mathrm{Res}\ M(K) = i\,\bar{\alpha}\, M(\bar{K}_2)\, m(\bar{K}_1). \qquad (2b)$$

The factors on the right are ordinary functions and hence their ordering is immaterial. They have been written down in the order of the corresponding bubbles of Fig. 1-B3.

The crucial point of the argument is that the α and $\bar{\alpha}$ appearing in Eqs. 2 are precisely the factors that multiply the corresponding terms in Fig. 1-B2. This comes about because the minus bubbles on the right in (2) are, in the course of calculating the residue,[4,7] converted to plus bubbles by means of the unitarity equations

(+) + (-) = RHS.

However, the relative phase of the plus and minus bubbles in this equation is fixed.

In order to say something about the phases α and $\bar{\alpha}$, consider first the contributions to unitarity indicated in Fig. 1-B4.

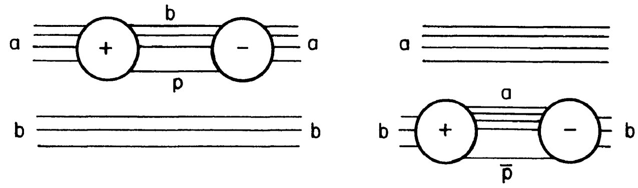

Fig. 1-B4. Two contributions to the forward-scattering unitarity equation for a + b → a + b. The intermediate particles in the first case consist of the sets 2b + p and in the second case of the sets 2a + $\bar{p}$. The a and b represent any two sets of two or more particles such that b + p → a.

Because the terms represented in Fig. 1-B4 are just the absolute-value-squared contributions to unitarity, the phases of these contributions are necessarily unity. This is a key point.

The intermediate lines of the first diagram of Fig. 1-B4 represent the initial particles of the reaction 2b + p → a + b. If one interchanges the particles of the two identical sets of initial-particle variables b of this reaction, and also does the analogous interchange for the second diagram, one obtains Fig. 1-B5.

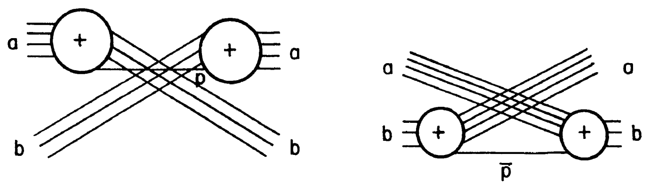

Fig. 1-B5. The result of interchanging identical initial particles of the reactions represented by the left-hand parts of the diagrams of Fig. 1-B4. The minus bubbles have been changed to plus bubbles, so that the diagrams represent the corresponding residues, apart from phase factors.

The residue equations corresponding to Fig. 1-B5 are

$$\mathrm{Res}\ M(K) = i\,\sigma_b\, M(K_1)\, M(K_1^{\,t}), \qquad (5a)$$

$$\mathrm{Res}\ M(\bar{K}) = i\,\sigma_a\, M(\bar{K}_1^{\,t})\, M(\bar{K}_1), \qquad (5b)$$

where

$$\sigma_b = \prod_{i\in b} \sigma_i$$

and

$$\sigma_a = \prod_{i\in a} \sigma_i .$$

Here σ_i is the sign change[9] induced by interchanging two variables corresponding to particles of type i. Equation 5 incorporates also the fact that the two factors of the residue correspond to transposed processes. This is a consequence of the special nature of the original terms shown in Fig. 1-B4, which involve scattering functions together with the complex conjugates of the scattering functions for the transpose reactions.

Now the crossing argument is introduced. In S-matrix theory the crossing property is deduced as follows. The two parts of the manifold $k^2 = m^2$ lying in the physical region of the larger process $a + b \rightarrow a + b$ are disjoint, as indicated in Fig. 1-B6.

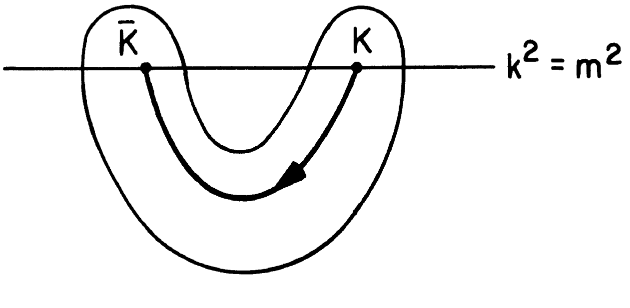

Fig. 1-B6. A diagram indicating that the manifold $k^2 = m^2$ intersects the physical region of $a + b \rightarrow a + b$ in two disjoint regions, in which lie K and $\bar{K}$, respectively. Also shown is a path in the physical region of $a + b \rightarrow a + b$ that leads from K to $\bar{K}$.

If the path in the physical region of $a + b \rightarrow a + b$ from K to $\bar{K}$ can be distorted into the manifold $k^2 = m^2$, without cutting across singularities having discontinuities that contribute to the residue function, then one can continue the residue function 5a to the residue function 5b. This would give

$$\sigma_b M(K_1^{c}) M(K_1^{t\tilde{c}}) = \sigma_a M(\bar{K}_1^{t}) M(\bar{K}_1), \quad (6)$$

where c and $\tilde{c}$ are the paths of continuations in the variables of the smaller processes. (The path of continuation for the process $a + b \rightarrow a + b$ originally depends on the variables of this larger reaction. But it must evidently reduce to a product of paths for the two individual functions, for the residue function of Eq. 5.

It was shown in Ref. 4 that, under reasonable assumptions about the mass-shell singularity structure, the distortions into the manifold $k^2 = m^2$ are possible, and that the paths c and $\tilde{c}$ are defined. The paths of Hermitian analyticity h and $\bar{h}$ are also defined, and the property $c + \bar{h} = h + \bar{c}$ holds true. Admitting these results, one proceeds as follows: The sets of external variables K_1^{c} and $\bar{K}_1$ in Eq. 5 are the same sets of variables. The function $M(K_1^{c})$ in Eq. 6 represents the continuation of $M(K_1)$ from K_1 to K_1^{c}. The function $M(\bar{K}_1)$, on the other hand, represents the actual physical function that describes the crossed process. This function $M(\bar{K}_1)$ does not necessarily have to be identical to $M(K_1^{c})$. All that is established by the crossing argument is Eq. 6, which allows one to conclude only that

$$M(K_1^{c}) = \lambda\, M(\bar{K}_1) \quad (7a)$$

and

$$M(K_1^{t\tilde{c}}) = \sigma_a \sigma_b \lambda^{-1} M(\bar{K}_1^{t}) \quad (7b)$$

for some constant factor λ, which is still to be determined.

Continuation along the path $\bar{h}$ takes Eq. 6 into

$$\sigma_b M(K_1^{c\bar{h}}) M(K_1^{tc}) = \sigma_a M(\bar{K}_1^{t}) M(\bar{K}_1^{\bar{h}})$$

$$= -\sigma_a M(\bar{K}_1^{t}) M^*(\bar{K}_1^{t}), \quad (8)$$

where we have used the fact that continuation along $\bar{h}$ takes $M(\bar{K}_1)$ to $-M^*(\bar{K}_1^{t})$. This is the Hermitian analyticity property. The important point is that apart from the factor $-\sigma_a$ the right-hand side is an absolute value squared. Using the analyticity property $c + \bar{h} = h + \bar{c}$, one obtains

$$M(K_1^{c\bar{h}}) = M(K_1^{h\bar{c}}). \quad (9)$$

But Hermitian analyticity gives

$$M(K_1^{h}) = -M^*(K_1^{t}), \quad (10)$$

from which it follows that

$$M(K_1^{h\bar{c}}) = -M^*(K_1^{t\tilde{c}}), \quad (11)$$

since $\bar{c} = \tilde{c}^*$. Substituting into Eq. 8, one obtains

$$\sigma_b M(K_1^{t\tilde{c}}) M^*(K_1^{t\tilde{c}}) = \sigma_a M(\bar{K}_1^{t}) M^*(\bar{K}_1^{t}), \quad (12)$$

from which one concludes that either $M(K_1^{t\tilde{c}})$ and $M(\bar{K}_1^{t})$ are both zero or

$$\sigma_a \sigma_b = \prod_{i \in a, b} \sigma_i = 1. \quad (13)$$

Multiplying by σ_p, the sign induced by an interchange of variables of type p, one gets

$$\prod_i \sigma_i = \sigma_p, \quad (14)$$

where the product now runs over all particles of the original process $b + p \rightarrow a$.

The point now is that this argument can be carried out with any one of the particles of $b + p \rightarrow a$ singled out as p. (Or $\bar{p}$ for a final particle.) Thus the product of all the σ_i is equal to any single one of them. This means either that the σ_i are all positive, or that they are all negative and that the total number of particles in the reaction $b + p \rightarrow a$ is odd.

The possibility that certain particles participate only in reactions involving odd numbers of particles is incompatible with unitarity. For such reactions would give contributions to the unitarity equations for forward scattering amplitudes, which involve even numbers of particles altogether. And all contributions to the forward scattering are of the same sign. Thus a particle cannot participate only in reactions involving odd numbers of particles, and hence

$$\sigma_p = 1 \quad (15)$$

for any particle p that scatters. [The argument given above requires a and b to consist of at least two particles. The possibility that the only connected parts involving p have single particles for a or b is ruled out by showing a conflict with the pole factorization property plus unitarity. (Unitarity guarantees the existence of a nonvanishing transpose process, which is needed to ensure a nonvanishing pole contribution to a process with more particles.)]

The extension of this result to particles having spin is trivial in the M-function formalism.[1,4,7] In place of Eq. 8 one obtains

$$\sigma_b M(K_1^{c\bar{h}}) \tilde{G}(p^{c}) M(K_1^{t\tilde{c}})$$

$$= \sigma_a M(\bar{K}_1^{\,t})\, \tilde{G}(\bar{p})\, M(\bar{K}_1^{\,\bar{h}}) , \qquad (16)$$

where $\tilde{G}(p)$ is a Hermitian relativistic spin matrix that satisfies

$$\tilde{G}(-p) = (-1)^{2s} \tilde{G}(p). \qquad (17)$$

Since the continued value p^c is the negative of the physical value $\bar{p}$ one obtains, in place of Eq. 14 , the equation

$$\prod_i \sigma_i = (-1)^{2s_p} \sigma_p ,$$

which can be written as

$$\prod_i \sigma_i' = \sigma_p' , \qquad (14')$$

where

$$\sigma_i' \equiv (-1)^{2s_i} \sigma_i . \qquad (18)$$

To get 14' we have used the fact that the total number of noninteger spin particles is conserved mod 2. From 14' one obtains in place of 15 the result

$$\sigma_p' = 1, \qquad (15')$$

which is the normal connection between spin and statistics.

Footnotes and References (Appendix B)

1. Henry P. Stapp, Phys. Rev. 125, 2139 (1962), Appendix I; see, also, The Decomposition of the S-Matrix and the Connection Between Spin and Statistics, UCRL-10289, June 1962.

2. E. Y. C. Lu and D. J. Olive, Spin and Statistics in S-Matrix Theory (Cambridge Preprint), Nov. 1965.

3. Marcel Froissart and John R. Taylor, Cluster Decomposition and the Spin Statistic Theorem in S-Matrix Theory, Princeton University Report, Aug. 1966.

4. Henry P. Stapp, Crossing, Hermitian Analyticity and the Connection Between Spin and Statistics, UCRL-16816, April 1966.

5. The "almost physical" requirement is that if the projection of the multiparticle wave functions f and g onto one-particle configuration space are nonoverlapping, then

$$|M(f, g; f', g')| = |M(f;f')|\, |M(g;g')| .$$

They use this requirement both in momentum space, where it is essentially an asymptotic physical cluster decomposition property, and in coordinate space, where its meaning is not so clear. If the coordinate space free-particle wave functions are strictly confined to some finite region at t = 0, then they spread over all space at any infinitesimally different time.

6. Once the phase questions are settled and the crossing and Hermitian analyticity properties established, the proof of the normal connection between spin and statistics as given in Ref. 3 requires only eight lines.

7. Henry P. Stapp, in High Energy Physics and Elementary Particles (IAEA, Vienna, 1965), Seminar Lectures at Trieste.

8. See Refs. 4 and 7 for proofs and further references.

9. That these sign changes are well defined is shown in Ref. 4.