

Barnich–Troessaert bracket as a Dirac bracket on the covariant phase space

Wolfgang Wieland* 

Institute for Quantum Optics and Quantum Information (IQOQI), Austrian Academy of Sciences, Boltzmannngasse 3, 1090 Vienna, Austria

E-mail: wolfgangmartin.wieland@oeaw.ac.at

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Abstract

The Barnich–Troessaert bracket is a proposal for a modified Poisson bracket on the covariant phase space for general relativity. The new bracket allows us to compute charges, which are otherwise not integrable. Yet there is a catch. There is a clear prescription for how to evaluate the new bracket for any such charge, but little is known how to extend the bracket to the entire phase space. This is a problem, because not every gravitational observable is also a charge. In this paper, we propose such an extension. The basic idea is to remove the radiative data from the covariant phase space. This requires second-class constraints. Given a few basic assumptions, we show that the resulting Dirac bracket on the constraint surface is nothing but the BT bracket. A heuristic argument is given to show that the resulting constraint surface can only contain gravitational edge modes.

Keywords: quantum gravity, covariant phase space for general relativity, quasi-local charges and boundary symmetries

1. Introduction

At null infinity, there is no conserved mass, because gravitational radiation carries energy [1–5]. An immediate consequence of this simple observation is that the BMS supertranslations are not integrable on the covariant phase space [6–10]. If $\Omega_M(\cdot, \cdot)$ is the (vastly degenerate) pre-symplectic two-form for a partial Cauchy surface M that intersects \mathcal{I}^+ at a cross section

* Author to whom any correspondence should be addressed.



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$\partial M = \mathcal{C} \subset \mathcal{I}^+$, the relevant equation reads

$$\Omega_M(\mathcal{L}_\xi, \delta) = \frac{1}{8\pi G} \oint_{\partial \mathcal{C}} \xi^a k_a \left[\kappa_{(\ell)} \delta \varepsilon - \frac{1}{2} \vartheta_{(\ell)} \delta \varepsilon - \varepsilon \delta \vartheta_{(\ell)} - i \sigma_{(\ell)} \bar{m} \wedge \delta \bar{m} + i \bar{\sigma}_{(\ell)} m \wedge \delta m \right] \neq -\delta[Q_\xi], \quad (1)$$

where ξ^a is a BMS supertranslation, ℓ^a denotes a null generator of \mathcal{I}^+ , $\kappa_{(\ell)}$ is its non-affinity, and m_a, \bar{m}_a is a $U(1)$ dyad on the null surface, such that $2m_{(a}\bar{m}_{b)}$ is the pull-back of the space-time metric g_{ab} to the null boundary. The canonical area element is $\varepsilon = -i m \wedge \bar{m}$, and $\vartheta_{(\ell)}$ and $\sigma_{(\ell)}$ are the shear and expansion of the null generator, while the one-form k_a is dual to it, i.e. $k_a \ell^a = -1$. Equation (1) holds for generic null surfaces [11–13]. Taking into account the fall-off and gauge-fixing conditions on \mathcal{I}^+ , the terms that are responsible for the non-integrability of Q_ξ are only contained in the second line, which depends on the time derivative $\dot{\sigma}^{(0)}(u, z, \bar{z})$ of the asymptotic shear via $\sigma_{(\ell)} = -\dot{\sigma}^{(0)}(u, z, \bar{z})/r + \mathcal{O}(r^{-2})$, see e.g. [12].

To compute the charges from the pre-symplectic two-form, we have to relax the requirement that Q_ξ is the Hamiltonian generator of the desired symmetries. This can be achieved by adding a counter term, which depends on the symplectic current J_{rad} of the radiative modes. This counter term was identified by Wald and Zoupas in [10]. Computing the resulting charge amounts to integrating the equation

$$\delta[Q_\xi] = -\Omega_M(\mathcal{L}_\xi, \delta) + \oint_{\mathcal{C}} \xi \lrcorner J_{\text{rad}}(\delta), \quad (2)$$

for all linearised solutions $\delta[\cdot]$ on the covariant phase space. More recent results have given prescriptions to extend these definitions to finite domains, see [12–15]. Equation (2) defines a charge, but now we face the problem that we cannot use covariant phase space methods to compute the resulting commutation relations $\{Q_\xi, Q_{\xi'}\}$, because the Hamiltonian vector field of Q_ξ , so it exists, does not coincide with the Lie derivative \mathcal{L}_ξ .

A proposal to resolve this issue was given by Barnich and Troessaert, who introduced a new bracket [16–18]. On the covariant phase space, it is defined as follows: if J_{rad} denotes the symplectic current for the radiative modes at null infinity, the new bracket is given by

$$\{Q_\xi, Q_{\xi'}\}_{\text{TB}} := \Omega_M(\mathcal{L}_\xi, \mathcal{L}_{\xi'}) - \oint_{\mathcal{C}} [\xi \lrcorner J_{\text{rad}}(\mathcal{L}_{\xi'}) - \xi' \lrcorner J_{\text{rad}}(\mathcal{L}_\xi)]. \quad (3)$$

Now we have a new bracket, but by changing the bracket, we also change the phase space. Therefore, a new set of questions arises. What is the phase space for which the Barnich–Troessaert bracket defines a (non-degenerate) symplectic two-form? Furthermore, if O and O' denote Dirac observables of the gravitational field, such as those defined in e.g. [19–22], what are their commutation relations with respect to the new bracket, i.e. what is $\{O, O'\}_{\text{TB}}$ for generic Dirac observables O and O' ?

In this note, we will reflect on these questions. Our main message will be that the Barnich–Troessaert bracket should be understood as an ordinary Dirac bracket for a large (in fact infinite) number of second-class constraints. The role of the second-class constraints is to simply remove the radiative data from the covariant phase space on a partial Cauchy surface M and replace them by auxiliary background fields (*c*-numbers). The resulting reduced phase space, which is now indexed by the background fields, is the phase space of gravitational edge modes alone. A different and more algebraic perspective is given in [23].

2. Covariant phase space and bulk-boundary dynamics

To begin with, let us first clarify the problem in full generality. The analysis will be based on covariant phase space methods for manifolds with boundaries [6–10, 12, 24, 25]. The general set up is a field theory on a d -dimensional manifold \mathcal{M} with a time-like or null boundary. Typically, its topology is that of an infinite¹ cylinder, i.e. $\mathcal{N} = S^{d-2} \times \mathbb{R}$, $\partial\mathcal{M} = \mathcal{N}$. The bulk and boundary field equations are derived from the variation of an action, which will have the following general form

$$S[\Phi, \varphi|\sigma] = \int_{\mathcal{M}} L[\Phi, d\Phi] + \int_{\mathcal{N}} l[\Phi, \varphi, d\varphi|\sigma], \quad (4)$$

where the \mathbb{R} -valued d -form $L[\Phi, d\Phi] \in \Omega^d(\mathcal{M}; \mathbb{R})$ is a Lagrangian in the bulk and $l[\Phi, \varphi, d\varphi|\sigma] \in \Omega^{d-1}(\mathcal{N}; \mathbb{R})$ is the boundary Lagrangian. The action (4) is a functional

$$S : \mathcal{F}_{\text{kin}} \rightarrow \mathbb{R}; (\Phi, \varphi|\sigma) \mapsto S[\Phi, \varphi|\sigma], \quad (5)$$

on the space of kinematical histories, i.e. the space of bulk and boundary field configurations for $(\Phi, \varphi|\sigma)$. The basic configuration variables are Φ and φ , which are tensor-valued² differential forms, i.e. $\Phi \in \Omega^{|\Phi|}(\mathcal{M}; \mathbb{V})$ and $\varphi \in \Omega^{|\varphi|}(\mathcal{N}; \mathbb{W})$ that take values in some unspecified target spaces \mathbb{V} and \mathbb{W} . The integer $|\Phi| = p$ is the degree of the p -form Φ . The bulk and boundary Lagrangians depend only on the fields and their exterior derivatives. In the absence of a metric or other background structures, the only available derivative that can operate on a p -form and commutes with the pull-back is the exterior derivative, which is nilpotent, i.e. $d^2\Phi = 0$. Hence, no second derivatives can ever appear in our Lagrangian³. Besides the configuration variables, the action also depends on external sources σ (c -numbers or background fields), which are unspecified tensor-valued p -forms on the boundary. Both the bulk and boundary Lagrangians are local in all fundamental variables. Examples for such bulk and boundary actions in three and four spacetime dimensions are plentiful, see e.g. [12, 25–30] and references therein.

Since we are considering gravity, the action is diffeomorphism invariant. For all $\alpha \in \text{Diff}(\mathcal{M}; \mathcal{M})$, we thus have⁴

$$L[\alpha^*\Phi, d(\alpha^*\Phi)] = L[\alpha^*\Phi, \alpha^*(d\Phi)] \approx (\alpha^*L)[\Phi, d\Phi], \quad (6)$$

$$l[\alpha^*\Phi, \alpha^*\varphi, d(\alpha^*\varphi)|\alpha^*\sigma] = l[\alpha^*\Phi, \alpha^*\varphi, \alpha^*(d\varphi)|\alpha^*\sigma] = (\alpha^*l)[\Phi, \varphi, d\varphi|\sigma], \quad (7)$$

where α^* denotes the pull-back.

To introduce the covariant Hamiltonian formalism, which equips \mathcal{F}_{kin} with a pre-symplectic structure, it is useful to define the *kinetic momenta*⁵,

$$\Pi_\Phi = d\Phi \in \Omega^{|\Phi|+1}(\mathcal{M}; \mathbb{V}), \quad (8)$$

$$\pi_\varphi = d\varphi \in \Omega^{|\varphi|+1}(\mathcal{M}; \mathbb{W}). \quad (9)$$

¹ This is in slight derogation from figure 1, where the null surface \mathcal{N} has itself a boundary.

² In the following, all tensor indices are suppressed, and ‘ d ’ denotes the exterior derivative.

³ Second derivatives can only appear by integrating out auxiliary fields. This happens when going from the Einstein–Palatini–Cartan action to the more familiar Einstein–Hilbert action, where we solve for the torsion-free condition and insert it back into the action.

⁴ We may assume that the bulk Lagrangian $L[\Phi, d\Phi]$ is invariant under diffeomorphisms only up to a total exterior derivative, but such exact forms should be reabsorbed into a redefinition of the boundary Lagrangian l .

⁵ The terminology is borrowed from electrodynamics. The kinetic momentum of a charged particle is its four-velocity \dot{x}^μ , the canonical momentum, on the other hand is $p_\mu = m\dot{x}_\mu - eA_\mu(x)$, where A_μ is the vector potential.

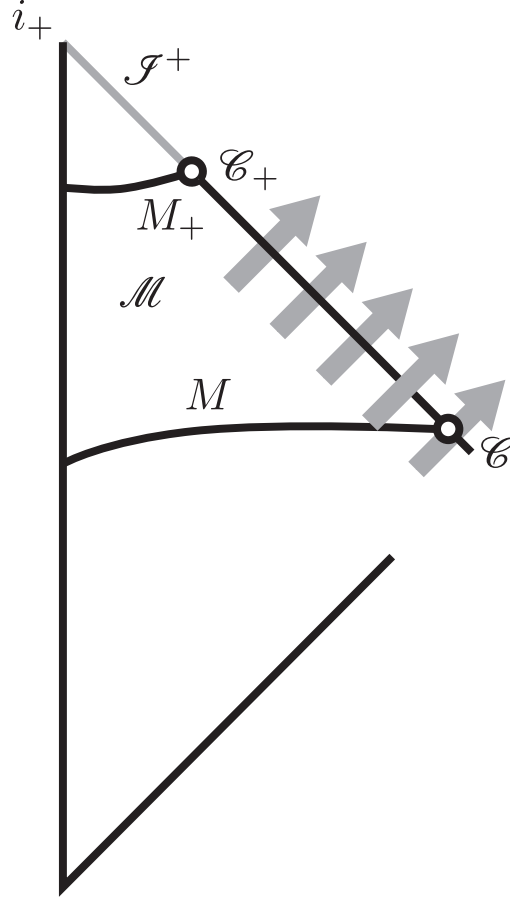


Figure 1. Setup of the problem. We consider an asymptotically flat spacetime. The three-manifolds M and M_+ are partial Cauchy hypersurfaces, which are bounded by consecutive cross sections $\mathcal{C} = \partial M$ and $\mathcal{C}_+ = \partial M_+$ of future null infinity \mathcal{I}^+ . The null surface \mathcal{N} is the portion of \mathcal{I}^+ between \mathcal{C} and \mathcal{C}_+ . We restrict ourselves to regions in phase space where \mathcal{C}_+ lies far enough ahead such that all radiation at \mathcal{I}^+ vanishes at and beyond the cross section \mathcal{C}_+ . Care needs to be taken with orientations. Our conventions are as follows. The orientation of \mathcal{N} is induced from the bulk, which is \mathcal{M} , whereas the orientation of the cross sections $\{\mathcal{C}, \mathcal{C}_+\}$ is induced from M and M_+ . The boundary of \mathcal{M} is $\partial\mathcal{M} = M \cup M_+^{-1} \cup \mathcal{N}$.

The field equations follow from the variation of the action. If $\delta \in T\mathcal{F}_{\text{kin}}$ denotes a tangent vector (variation) on field space, we have⁶

$$\delta[L] = \text{Tr} \left[\frac{\partial L}{\partial \Phi} \wedge \delta[\Phi] + \frac{\partial L}{\partial \Pi_\Phi} \wedge \delta[\Pi_\Phi] \right] = (\text{EOM})(\delta) + \text{d}[J_{\text{bulk}}(\delta)], \quad (10)$$

⁶ N.B.: if $\Phi \in \Omega^{|\Phi|}(\mathcal{M}, \mathbb{V})$ is a \mathbb{V} -valued differential form, the derivative $\frac{\partial L}{\partial \Phi}$ defines a $(d-p)$ -form that takes values in the dual vector space \mathbb{V}^* . Accordingly, the symbol Tr denotes the natural pairing between elements of \mathbb{V} and \mathbb{V}^* .

where we defined the following one-forms on field space \mathcal{F}_{kin} , namely

$$\text{EOM} = \text{Tr} \left[\left(\frac{\partial L}{\partial \Phi} + (-1)^{d-|\Phi|} \mathfrak{d} \left(\frac{\partial L}{\partial \Pi_\Phi} \right) \right) \wedge \mathfrak{d}\Phi \right], \quad (11)$$

$$J_{\text{bulk}} = (-1)^{d-|\Phi|-1} \text{Tr} \left[\frac{\partial L}{\partial \Pi_\Phi} \wedge \mathfrak{d}\Phi \right], \quad (12)$$

and \mathfrak{d} denotes the exterior derivative on the space of kinematical histories \mathcal{F}_{kin} . In the same way, we introduce the variation of the boundary Lagrangian,

$$\begin{aligned} \delta[l] &= \text{Tr} \left[\frac{\partial l}{\partial \Phi} \wedge \delta[\Phi] + \frac{\partial l}{\partial \varphi} \delta[\varphi] + \frac{\partial l}{\partial \pi_\varphi} \delta[\pi_\varphi] + \frac{\partial l}{\partial \sigma} \delta[\sigma] \right] \\ &= (\text{eom})(\delta) - \mathfrak{d}[j_{\text{edge}}(\delta)] - J_{\text{glue}}(\delta) + J_{\text{source}}(\delta), \end{aligned} \quad (13)$$

where we introduced the following one-forms on field space

$$\text{eom} = \text{Tr} \left[\left(\frac{\partial l}{\partial \varphi} + (-1)^{d-1-|\varphi|} \mathfrak{d} \left(\frac{\partial l}{\partial \pi_\varphi} \right) \right) \wedge \mathfrak{d}\varphi \right], \quad (14)$$

$$J_{\text{glue}} = - \text{Tr} \left[\frac{\partial l}{\partial \Phi} \wedge \mathfrak{d}\Phi \right], \quad (15)$$

$$J_{\text{source}} = - \text{Tr} \left[\frac{\partial l}{\partial \sigma} \wedge \mathfrak{d}\sigma \right], \quad (16)$$

$$j_{\text{edge}} = (-1)^{d-1-|\varphi|} \text{Tr} \left[\frac{\partial l}{\partial \pi_\varphi} \wedge \mathfrak{d}\varphi \right]. \quad (17)$$

At its saddle points, the coupled bulk plus boundary action is stationary under all variations $\delta \in T\mathcal{F}_{\text{kin}}$ that satisfy the boundary conditions, which are now given by

$$\int_{\mathcal{N}} J_{\text{source}}(\delta) = 0. \quad (18)$$

The resulting bulk and boundary field equations are $\text{EOM} = 0$, $\text{eom} = 0$ plus additional gluing conditions. The gluing conditions couple the boundary fields (i.e. φ and σ) to the pull-back (i.e. $\alpha_{\mathcal{N}}^* \Phi$) of the configuration variables in the bulk. The solutions to the bulk and boundary field equations and gluing conditions define the space of physical histories $\mathcal{F}_{\text{phys}} \hookrightarrow \mathcal{F}_{\text{kin}}$, where for all $\delta \in T\mathcal{F}_{\text{kin}}$,

$$(\text{EOM})(\delta)|_{\mathcal{F}_{\text{phys}}} = 0, \quad (19)$$

$$(\text{eom})(\delta)|_{\mathcal{F}_{\text{phys}}} = 0, \quad (20)$$

$$(\alpha_{\mathcal{N}}^* J_{\text{bulk}})(\delta)|_{\mathcal{F}_{\text{kin}}} - J_{\text{glue}}(\delta)|_{\mathcal{F}_{\text{phys}}} = 0, \quad (21)$$

where $\alpha_{\mathcal{N}}^* : T^*\mathcal{M} \rightarrow T^*\mathcal{N}$ denotes the pull-back of differential forms from the interior of the manifold to the boundary.

The pre-symplectic currents J_{bulk} and j_{edge} define the pre-symplectic potential, which, in turn, defines the pre-symplectic structure on the covariant phase space. Given a partial Cauchy

surface M , which is anchored at the boundary, i.e. $\partial M = \mathcal{C} \subset \mathcal{N}$, we obtain the pre-symplectic potential

$$\Theta_M = \int_M J_{\text{bulk}} + \oint_{\partial M} J_{\text{edge}}. \quad (22)$$

The pre-symplectic two-form Ω_M is the exterior derivative of (22). If δ_1 and δ_2 are vector fields (variations) on \mathcal{F}_{kin} , and $[\delta_1, \delta_2] \in T\mathcal{F}_{\text{kin}}$ denotes their Lie bracket, we have

$$\Omega_M(\delta_1, \delta_2) = \delta_1 [\Theta_M(\delta_2)] - \delta_2 [\Theta_M(\delta_1)] - \Theta_M([\delta_1, \delta_2]). \quad (23)$$

3. Boundary Hamiltonian and Hamiltonian flux

Next, we introduce a quasi-Hamiltonian on the space of physical histories. In gravity, diffeomorphisms are gauge symmetries. For every gauge symmetry, there is a corresponding conserved current, which is the exterior derivative of some charge aspect. The resulting total charge on a $(d-1)$ -dimensional surface M will vanish unless there is a co-dimension two boundary $\mathcal{C} = \partial M$. If there is such a boundary, the charge turns into a surface integral localised at \mathcal{C} . The intuitive reason why this is so is rather obvious: the introduction of the boundary breaks diffeomorphism invariance. At the boundary, there are auxiliary boundary sources σ , and the addition of these background fields breaks gauge invariance.

At the infinitesimal level, any diffeomorphism $\alpha_\xi : \mathcal{M} \rightarrow \mathcal{M}$, $\alpha_\xi = \exp \xi$ is generated by the Lie derivative $\mathcal{L}_\xi[\cdot] = \frac{d}{d\varepsilon} \big|_{\varepsilon=0} \alpha_{\varepsilon\xi}^*$, which defines a vector field on field space, i.e. $\mathcal{L}_\xi[\cdot] \in T\mathcal{F}_{\text{kin}}$. In terms of the exterior derivative ‘ d ’ and the interior product ‘ \lrcorner ’,⁷ the Lie derivative of any differential form can be written as

$$\mathcal{L}_\xi[\cdot] = d(\xi \lrcorner (\cdot)) + \xi \lrcorner (d(\cdot)). \quad (24)$$

In the following, we will always assume that the vector field $\xi^a \in T\mathcal{M}$ preserves the boundary, i.e.

$$\xi^a|_{\mathcal{N}} \in T\mathcal{N}, \quad (25)$$

such that the definition (24) naturally extends to all bulk and boundary fields $(\Phi, \varphi|\sigma) \in \mathcal{F}_{\text{kin}}$. Notice also that the vector field $\xi^a \in T\mathcal{M}$ may itself be field dependent, such that e.g. $[\delta, \mathcal{L}_\xi] = \mathcal{L}_{\delta\xi}$ for all $\delta \in T\mathcal{F}_{\text{kin}}$.

Given such a vector field $\xi^a \in T\mathcal{M}$, we may now *define* the corresponding Hamiltonian as the following functional on the space of physical histories,

$$Q_\xi[M] := \Theta_M(\mathcal{L}_\xi) - \int_M \xi \lrcorner L + \oint_{\partial M} \xi \lrcorner l. \quad (26)$$

Notice that we have not yet specified what the underlying phase space \mathcal{P} actually is, we only used the familiar definition $H[q, \dot{q}] = \frac{\partial L}{\partial \dot{q}} \dot{q} - L[q, \dot{q}]$, where L includes now both the bulk and boundary Lagrangian. Due to gauge redundancies, \mathcal{F}_{kin} is vastly larger than \mathcal{P} and it is not at all immediate to turn $H[q, \dot{q}]$ into a function on phase space (a true Hamiltonian).

The key point of this paper is to identify a candidate for a phase space, where the Hamiltonian is integrable. It is integrable if it satisfies the Hamiltonian field equations. This is to

⁷ If ω is a p -form, and ξ is a vector field, $(\xi \lrcorner \omega)$ is the $(p-1)$ -form that is defined via $(\xi \lrcorner \omega)(X_1, \dots, X_{p-1}) = \omega(\xi, X_1, \dots, X_{p-1})$ for all vector fields X_i .

say that there is a phase space \mathcal{P} and an embedding (or rather a family of gauge equivalent embeddings) $\alpha : \mathcal{P} \hookrightarrow \mathcal{F}_{\text{phys}}$ such that the pull-back $\alpha^* \Omega_M$ equips \mathcal{P} with a non-degenerate symplectic structure such that for all tangent vectors $\delta \in T\mathcal{P}$;

$$\delta [Q_\xi[M] \circ \alpha] = (\alpha_* \delta) [Q_\xi[M]] = \Omega_M(\alpha_* \delta, \mathcal{L}_\xi) = (\alpha^* \Omega_M)(\delta, \delta_\xi), \quad (27)$$

where $\delta_\xi \in T\mathcal{P}$ is a tangent vector on phase space such that the difference $\alpha_* \delta_\xi - \mathcal{L}_\xi$ is a null vector of the pre-symplectic potential Ω_M .

To get an idea for how to construct a proposal for such an embedding, let us go again on-shell, i.e. restrict our discussion to $\mathcal{F}_{\text{phys}}$ alone. Let then δ be a linearised solution of the bulk and boundary field equations, i.e. a tangent vector to $\mathcal{F}_{\text{phys}}$. Imposing the field equations (19), (20) and gluing conditions (21), and taking into account the variation of the bulk and boundary action, i.e. (10) and (13), we obtain the variation of the Hamiltonian

$$\begin{aligned} \delta [Q_\xi[M]] - Q_{\delta\xi}[M] &= \delta [\Theta_M(\mathcal{L}_\xi)] - \Theta_M(\mathcal{L}_{\delta\xi}) - \int_M \xi \lrcorner \delta[L] - \oint_{\partial M} \xi \lrcorner \delta[L] \\ &= \delta [\Theta_M(\mathcal{L}_\xi)] - \Theta_M([\delta, \mathcal{L}_\xi]) - \int_M \xi \lrcorner d[J_{\text{bulk}}(\delta)] + \\ &\quad - \oint_{\partial M} [\xi \lrcorner (d(j_{\text{edge}}(\delta))) + \xi \lrcorner J_{\text{glue}}(\delta) - \xi \lrcorner J_{\text{source}}(\delta)] \\ &= \delta [\Theta_M(\mathcal{L}_\xi)] - \Theta_M([\delta, \mathcal{L}_\xi]) - \int_M \mathcal{L}_\xi [J_{\text{bulk}}(\delta)] - \oint_{\partial M} \mathcal{L}_\xi [j_{\text{edge}}(\delta)] + \\ &\quad - \oint_{\partial M} [-\xi \lrcorner J_{\text{bulk}}(\delta) + \xi \lrcorner J_{\text{glue}}(\delta) - \xi \lrcorner J_{\text{source}}(\delta)] \\ &= \delta [\Theta_M(\mathcal{L}_\xi)] - \mathcal{L}_\xi [\Theta_M(\delta)] - \Theta_M([\delta, \mathcal{L}_\xi]) + \oint_{\partial M} \xi \lrcorner J_{\text{source}}(\delta). \end{aligned} \quad (28)$$

In other words,

$$\delta [Q_\xi[M]] - Q_{\delta\xi}[M] = \Omega_M(\delta, \mathcal{L}_\xi) + \oint_{\partial M} \xi \lrcorner J_{\text{source}}(\delta). \quad (29)$$

If we insist to use only field-independent diffeomorphisms, i.e. $\delta\xi^a = 0$, the Hamiltonian is integrable only on those surfaces in field space, where the variation of the source term $\oint_{\partial M} \xi \lrcorner J_{\text{source}}(\delta)$ is constrained to vanish. In general, $\oint_{\partial M} \xi \lrcorner J_{\text{source}}(\delta) \neq 0$ and the Hamiltonian is non-integrable. In three spacetime dimensions, this is not a big deal. Choosing e.g. conformal boundary conditions, see e.g. [26, 31], the source σ is essentially the conformal class $\tilde{q}^{ab} = d^2x \sqrt{q} q^{ab}$ of the fiducial boundary metric. In three-dimensions, the constraint $\delta[\tilde{q}^{ab}] = 0$ is no restriction to the space of solutions to the field equations in the bulk, which is the moduli space of flat connections in the interior. Infinitesimal boundary diffeomorphism that preserve the background structure \tilde{q}^{ab} are generated by conformal Killing vectors $\xi^a : \mathcal{L}_\xi \tilde{q}^{ab} = 0$. For every such conformal Killing ξ^a , there is a corresponding conserved charge (essentially a Virasoro generator). In higher dimensions, the situation is very different, because now gravity is no longer topological. There are gravitational waves, and for generic boundaries⁸, the imposition of $\oint_{\partial M} \xi \lrcorner J_{\text{source}}(\delta) = 0$ will be a very strong constraint on the solutions in the bulk.

⁸ If we restrict ourselves to spacelike infinity, the term $\oint_{\partial M \rightarrow i_0} \xi \lrcorner J_{\text{source}}(\delta) = 0$ will vanish for asymptotic symmetries due to the falloff and parity conditions at i_0 .

Let us close this section by considering the finite flux version of the infinitesimal variation (28) that we identified above. Consider thus two consecutive slices M and M_+ , with $\mathcal{N}_{M \rightarrow M_+}$ denoting the portion of \mathcal{N} between M and M_+ , such that M , M_+ and $\mathcal{N}_{M \rightarrow M_+}$ bound a d -dimensional region $\mathcal{M} \subset \mathcal{M} : \partial\mathcal{M} = M^{-1} \cup M_+ \cup \mathcal{N}_{M \rightarrow M_+}$, see figure 1. Going back to the variation of the bulk and boundary Lagrangian, i.e. (10) and (13), and taking into account that $\xi^a|_{\mathcal{N}} \in T\mathcal{N}$, we obtain

$$\begin{aligned}
Q_\xi[M_+] - Q_\xi[M] &= \int_{\partial\mathcal{M}^*} J_{\text{bulk}}(\mathcal{L}_\xi) - \int_{\mathcal{N}_{M \rightarrow M_+}} J_{\text{glue}}(\mathcal{L}_\xi) - \int_{\mathcal{N}_{M \rightarrow M_+}} d[j_{\text{edge}}(\delta)] + \\
&\quad - \int_{\partial\mathcal{M}} \xi \lrcorner L + \int_{\mathcal{N}_{M \rightarrow M_+}} \xi \lrcorner L + \oint_{\partial M_+} \xi \lrcorner l - \oint_{\partial M} \xi \lrcorner l \\
&= \int_{\mathcal{M}} \mathcal{L}_\xi[L] - \int_{\mathcal{N}_{M \rightarrow M_+}} J_{\text{glue}}(\mathcal{L}_\xi) + \int_{\mathcal{N}_{M \rightarrow M_+}} [\mathcal{L}_\xi[l] + J_{\text{glue}}(\mathcal{L}_\xi) - J_{\text{source}}(\mathcal{L}_\xi)] + \\
&\quad - \int_{\partial\mathcal{M}} \xi \lrcorner L + \int_{\mathcal{N}_{M \rightarrow M_+}} \xi \lrcorner L - \int_{\mathcal{N}_{M \rightarrow M_+}} \mathcal{L}_\xi[l] \\
&= - \int_{\mathcal{N}_{M \rightarrow M_+}} [J_{\text{source}}(\mathcal{L}_\xi) - \xi \lrcorner L] = - \int_{\mathcal{N}_{M \rightarrow M_+}} J_{\text{source}}(\mathcal{L}_\xi),
\end{aligned}$$

where we assumed that the gluing conditions and bulk and boundary field equations are satisfied. We thus have a balance law

$$F_\xi[M \rightarrow M_+] := Q_\xi[M_+] - Q_\xi[M] = - \int_{\mathcal{N}_{M \rightarrow M_+}} J_{\text{source}}(\mathcal{L}_\xi). \quad (30)$$

If the Lie derivative \mathcal{L}_ξ does not preserve the background fields (the sources σ), the Hamiltonian will not be conserved in general. A more simplified derivation of the flux law (30) and the variation of the charge (29) based on functional-differential techniques has been recently developed by Margalef-Bentabol and Villaseñor, see section 3 of [30].

All such derivations for charges and fluxes based on the covariant phase space approach hide an important subtlety. The existence of the BT bracket is a manifestation of this fact. The basic problem is that the covariant phase space approach is based on field space, but field space is much bigger than phase space. Phase space \mathcal{P} is a submanifold $\mathcal{P} \hookrightarrow \mathcal{F}_{\text{phys}} \hookrightarrow \mathcal{F}_{\text{kin}}$, where the pre-symplectic two-form has no null directions. There is no unique such phase space, because the embedding depends on various gauge-fixing, boundary and falloff conditions. If we have found an observable O , which is integrable⁹, we are still left with the difficult task to compute the resulting Poisson brackets $\{O, O'\}_{\mathcal{P}}$. To equate $\{\cdot, \cdot\}_{\mathcal{P}}$ with $\Omega(\mathbb{X}_O, \mathbb{X}_{O'})$ is only possible if the vector fields \mathbb{X}_O ($\mathbb{X}_{O'}$) lie tangential to \mathcal{P} . In general $\{O, O'\}_{\mathcal{P}} \neq \Omega(\mathbb{X}_O, \mathbb{X}_{O'})$. In the following, we will see that the BT bracket provides a specific example of this subtlety on a natural phase space attached to future infinity i_+ .

⁹ This is to say $\delta[O] = \Omega(\delta, \mathbb{X}_O)$ for a (Hamiltonian) vector field $\mathbb{X}_O \in T\mathcal{F}_{\text{phys}}$ and all variations δ that satisfy the boundary and falloff conditions.

4. Barnich–Troessaert bracket as a Dirac bracket

For simplicity and definiteness, we consider here the Palatini action in asymptotically flat spacetimes. The action is evaluated in a four-dimensional spacetime region \mathcal{M} that bounds future null infinity,

$$S[e, A] = \frac{i}{8\pi G} \left[\int_{\mathcal{M}} \Sigma_{AB} \wedge F^{AB} + \int_{\mathcal{N}} \eta_A \wedge \left(D - \frac{1}{2} \varkappa \right) \ell^A \right] + \text{c.c.} \quad (31)$$

The fields in the interior are the self-dual connection A^A_B , whose curvature is F^A_B , and the soldering forms $e_{AA'}$. The soldering forms determine the two-form Σ_{AB} (the Plebański two-form), which is the self-dual part of $e_{AA'} \wedge e_{BB'}$, i.e. $\Sigma_{AB} = -\frac{1}{2} e_{AC'} \wedge e_B^{C'}$. The natural $SL(2, \mathbb{C})$ covariant derivative at the boundary is D , which is $D = d + [\alpha_N^*, A, \cdot]$. The boundary fields are the null flag ℓ^A , the spinor-valued two-form η_A , and the abelian boost connection \varkappa , which defines the non-affinity of the null generators, see [11, 13].

At the saddle points, where the bulk and boundary field equations are satisfied, the variation of the action is determined by the pre-symplectic potentials Θ on the various components of the boundary.

$$\delta[S] \Big|_{\text{EOM}=0} = \Theta_M(\delta) - \Theta_{M_+}(\delta) + \Theta_{\mathcal{N}}(\delta), \quad (32)$$

where $\delta \in T\mathcal{F}_{\text{phys}}$ is a linearised solution of the vacuum Einstein equations for asymptotically flat boundary conditions. On M and M_+ , the pre-symplectic potential is the integral of the symplectic current, i.e.

$$\Theta_M = \frac{i}{8\pi G} \left[\int_M \Sigma_{AB} \wedge \mathbb{d}A^{AB} - \oint_{\partial M} \eta_A \mathbb{d}\ell^A \right] + \text{c.c.} \quad (33)$$

On the asymptotic boundary \mathcal{N} , the situation is more subtle [4, 10, 32]. We have to impose boundary and gauge fixing conditions to remove otherwise IR divergent terms¹⁰. Upon removing such divergencies, the radiative symplectic structure [4] is given by

$$\Theta_{\mathcal{N}} = -\frac{1}{8\pi G} \int_{\mathcal{N}} du \wedge d^2\Omega \left(\dot{\sigma}^{(0)} \delta \bar{\sigma}^{(0)} + \text{c.c.} \right), \quad (34)$$

where $\sigma^{(0)}(u, z, \bar{z})$ is the asymptotic shear and $d^2\Omega$ is the fiducial area element at \mathcal{I}^+ . One possibility to derive the symplectic structure (34) is to consider the pre-symplectic radiative structure on a finite null surface and perform an asymptotic $r \rightarrow \infty$ limit using an auxiliary double-null foliation, see [12, 13].

To realise the Barnich–Troessaert bracket as a Dirac bracket, we have to say what are the relevant second-class constraints. Our proposal is that the constraints *remove the entire radiative data* from the covariant phase space on a partial Cauchy surface M . In other words, we consider the following constraints on the radiative phase space

$$\forall (u, z, \bar{z}) \in \mathcal{N} : \Phi_\alpha \equiv \Phi[\sigma, h](u, z, \bar{z}) = \dot{\sigma}^{(0)}(u, z, \bar{z}) - \dot{h}^{(0)}(u, z, \bar{z}) \approx 0, \quad (35)$$

¹⁰ An example for such an IR divergence arises from the naive inclusion of conformal transformations $q_{ab}^o \rightarrow \omega^2 q_{ab}^o$ of the fiducial two-metric at \mathcal{I}^+ into the pre-symplectic potential. The constraints at null infinity impose that $\partial_u \delta \omega = 0$. Such u -independent terms (and their conjugate pairs) lead to IR divergent integrals at \mathcal{I}^+ .

where $h^{(0)}(u, z, \bar{z})$ is a background field (a c -number) of compact support on \mathcal{I}^+ that commutes with all other phase space variables and $\alpha, \beta, \gamma, \dots$ are (De Witt) multi-indices¹¹ and the symbol ‘ \approx ’ means that the equation is imposed as a constraint. The asymptotic shear $\sigma^{(0)}(u, z, \bar{z})$, or more precisely its time derivative, describes the outgoing radiation. Imposing that the constraint (35) is satisfied amounts to constraining the outgoing radiation on a portion \mathcal{N} of \mathcal{I}^+ . The constraints (35) are second-class. In fact, the only non-vanishing Poisson brackets among $\sigma^{(0)}$ and $\bar{\sigma}^{(0)}$ are given by

$$\{\sigma^{(0)}(x), \bar{\sigma}^{(0)}(y)\} = -4\pi G \Theta(x, y) \delta^{(2)}(x, y), \quad (36)$$

where Θ is the step function. Since the background fields $h^{(0)}(u, z, \bar{z})$ commute under the Poisson bracket, the Dirac matrix can be inferred immediately from (36).

The Dirac bracket defines a (vastly) degenerate pre-symplectic structure on the covariant phase space associated to M . Its pull-back to the constraint hypersurface $\forall \alpha : \Phi_\alpha = 0$ introduces a natural pre-symplectic structure thereon. The bracket is defined as follows, see e.g. [33]. First of all, we have the Dirac matrix

$$\Delta_{\alpha\beta} = \{\Phi_\alpha, \Phi_\beta\}. \quad (37)$$

Let then $\Delta^{\alpha\beta} : \Delta^{\alpha\mu} \Delta_{\mu\beta} = \delta^\alpha_\beta$ be its inverse such that we can define the resulting Dirac bracket

$$\{A, B\}^* = \{A, B\} - \{A, \Phi_\alpha\} \Delta^{\alpha\beta} \{\Phi_\beta, B\}. \quad (38)$$

Our goal is now to develop an argument to demonstrate that the Dirac bracket (38) for the constraints (35) returns the Barnich–Troessaert bracket provided a few basic assumptions are satisfied¹². The *first assumption* is that the algebra for the BMS symmetries at \mathcal{I}^+ as given by the Barnich–Troessaert bracket is non-anomalous. The *second assumption* is that we restrict ourselves to such vector fields ξ^a that have no functional dependence on the fundamental bulk and boundary fields¹³, i.e. $\delta[\xi^a] = 0$, $\xi^a \in T\mathcal{M}$. The *third assumption* is that the outgoing radiation at \mathcal{I}^+ is of compact support such that there exists a cross section \mathcal{C}^+ beyond which no further radiation is received (see figure 1 above). The *fourth assumption* is that on-shell (pull back to $\mathcal{F}_{\text{phys}}$) the pre-symplectic structure on M admits the block-diagonal decomposition

$$\Omega_M = \frac{1}{2} \Omega_{\text{rad}}^{\alpha\beta}[\sigma, \eta] \, \mathbb{d}\sigma_\alpha \wedge \mathbb{d}\sigma_\beta + \frac{1}{2} \Omega_{\text{edge}}^{\mu\nu}[\sigma, \eta] \, \mathbb{d}\eta_\mu \wedge \mathbb{d}\eta_\nu, \quad (39)$$

where the first term describes the radiative data on \mathcal{N} , but now expressed in terms of (Dirac) observables that are evaluated on M rather than \mathcal{I}^+ , whereas the second term describes all possible boundary degrees of freedom (edge modes η_μ) that are localised at the cross section \mathcal{C} . The fourth assumption implies, in other words, that there is a symplectomorphism that allows us to express the radiative modes on M in terms of radiative data recorded at \mathcal{N} , i.e.

$$\Omega_{\text{rad}} = \frac{1}{2} \Omega_{\text{rad}}^{\alpha\beta} \, \mathbb{d}\sigma_\alpha \wedge \mathbb{d}\sigma_\beta \simeq \Omega_{\mathcal{N}}, \quad (40)$$

¹¹ Summation and integration over repeated pairs of such indices is implicitly assumed, i.e. $\sum \int \Psi^\alpha \Phi_\alpha \equiv \Psi^\alpha \Phi_\alpha$.

¹² We expect that some of the assumption could be dropped or weakened. In the following, we consider, however, only the simplest possibility.

¹³ An example of a field-dependent vector field would be $\xi^a[g_{ab}] = \nabla^a(R_{bcdf}R^{bcdf})$, where R_{abcd} is the Riemann curvature tensor of the spacetime metric g_{ab} .

where the symbol \simeq indicates that the two phase spaces are symplectomorphic. In $2 + 1$ or $1 + 1$ dimensions, the decomposition (39) is trivial: gravity is topological and the only contributions to the symplectic structure are the edge modes alone [26, 34–42].

Since the symplectic structure factorises into edge modes and radiative modes (our fourth assumption), the inverse of the Dirac matrix is simply given by $\Delta^{\alpha\beta} \simeq \Omega_{\text{rad}}^{\alpha\beta}$, where the symbol ‘ \simeq ’ stands for *equality under an (possibly η -dependent) symplectomorphism*. By imposing the constraint (35), the corresponding pre-symplectic two-form for the Dirac bracket (38) is then only given by the contribution from the edge modes,

$$\Omega_M^* = \Omega_{\text{edge}} = \frac{1}{2} \Omega_{\text{edge}}^{\mu\nu} [h, \eta] \mathbb{d}\eta_\mu \wedge \mathbb{d}\eta_\nu. \quad (41)$$

Let us now compute charges with respect to the Dirac bracket (38). Since we have just identified the corresponding pre-symplectic two-form, we can immediately employ covariant phase space methods to evaluate the charge (provided our assumptions are satisfied). Consider thus a tangent vector δ to the radiative phase space, such that $[\delta, \mathcal{L}_\xi] = 0$. We now immediately get

$$\begin{aligned} \Omega_{\text{edge}}(\delta, \mathcal{L}_\xi) &= \Omega_M(\delta, \mathcal{L}_\xi) - \Omega_{\text{rad}}(\delta, \mathcal{L}_\xi) \\ &= \Omega_M(\delta, \mathcal{L}_\xi) - \int_{\mathcal{N}} [\delta[J_{\text{rad}}(\mathcal{L}_\xi)] - \mathcal{L}_\xi[J_{\text{rad}}(\delta)]] \\ &= \Omega_M(\delta, \mathcal{L}_\xi) + \oint_C \xi \lrcorner J_{\text{rad}}(\delta) - \int_{\mathcal{N}} \delta[J_{\text{rad}}(\mathcal{L}_\xi)]. \end{aligned} \quad (42)$$

Notice that the second term is precisely the counter term, which is added in the Wald–Zoupas framework to render the pseudo-charge $\mathbb{Q}_\xi := \Omega_M(\cdot, \mathcal{L}_\xi)$ integrable. In other words, there is a functional Q_ξ on covariant phase space such that

$$\delta [Q_\xi[C]] = \Omega_M(\delta, \mathcal{L}_\xi) + \oint_C \xi \lrcorner J_{\text{rad}}(\delta). \quad (43)$$

It is important to note that in integrating the charges via (43), the vector fields $\delta \in T\mathcal{F}_{\text{phys}}$ denote an arbitrary linearised solution of the field equations. It is not assumed, in particular, that they lie tangential to the constraint hypersurface (35), see also [12].

Thus, the first two terms of equation (42) reproduce the differential of the quasi-local charge (29). The third term on the right-hand side of (42) has an immediate interpretation as well, see equation (30) above. It determines the radiative flux associated to the asymptotic symmetry $\xi^a|_{\mathcal{N}} \in T\mathcal{N}$, which we assumed to be a BMS generator. Such a flux integral can be expressed entirely in terms of radiative modes. Can it be written as the difference of two Hamiltonian generators corresponding to the two consecutive cross-sections? On the radiative phase space, this is impossible [4]. From the perspective of the partial Cauchy hypersurface M , the situation is different. Now, there is a charge, and the flux is simply the difference of the charges at the two consecutive cross sections. In other words,

$$F_\xi[M \rightarrow M_+] = - \int_{\mathcal{N}} J_{\text{rad}}(\mathcal{L}_\xi) = Q_\xi[C_+] - Q_\xi[C] \quad (44)$$

Going back to (42), we obtain

$$\Omega_{\text{edge}}(\delta, \mathcal{L}_\xi) = \delta [Q_\xi[C_+]] =: \delta[Q_\xi^+]. \quad (45)$$

Finally, let us compute the Poisson algebra for the diffeomorphism charges under the Dirac bracket. If our assumptions are satisfied, the (vastly degenerate) pre-symplectic two-form for the Dirac bracket is given by $\Omega_{\text{edge}} = \Omega_M - \Omega_{\mathcal{N}} \simeq \Omega_M - \Omega_{\text{rad}}$. Consider then vector fields ξ^a, ξ'^a , whose restriction to \mathcal{I}^+ is an asymptotic BMS symmetry. We now have

$$\begin{aligned} \{Q_\xi^+, Q_{\xi'}^+\}^* &= \Omega_{\text{edge}}(\mathcal{L}_\xi, \mathcal{L}_{\xi'}) \\ &= \Omega_M(\mathcal{L}_\xi, \mathcal{L}_{\xi'}) - \Omega_{\text{rad}}(\mathcal{L}_\xi, \mathcal{L}_{\xi'}) \\ &= \Omega_M(\mathcal{L}_\xi, \mathcal{L}_{\xi'}) - \int_{\mathcal{N}} [\mathcal{L}_\xi[J_{\text{rad}}(\mathcal{L}_{\xi'})] - \mathcal{L}_{\xi'}[J_{\text{rad}}(\mathcal{L}_\xi)] - J_{\text{rad}}([\xi, \xi'])] \\ &= \Omega_M(\mathcal{L}_\xi, \mathcal{L}_{\xi'}) - \oint_{\mathcal{C}} [\xi \lrcorner J_{\text{rad}}(\mathcal{L}_{\xi'}) - \xi' \lrcorner J_{\text{rad}}(\mathcal{L}_\xi)] - F_{[\xi, \xi']}[\mathcal{N}]. \end{aligned} \quad (46)$$

Going from the second to the third line, we inserted the definition of the radiative flux (44) and used Stokes' theorem and the definition of the Lie derivative $\mathcal{L}_\xi[\cdot] = d[\xi \lrcorner \cdot] + \xi \lrcorner (d[\cdot])$ to express the second and third term as an integral over the corner. There is no contribution from \mathcal{C}_+ , because we have assumed that there is no gravitational radiation (at or) beyond \mathcal{C}_+ . Notice that integrals of the form $\int_{\mathcal{N}} \xi \lrcorner \varphi = 0$ vanish, if $\xi \in T\mathcal{N}$ and $\varphi \in \Omega^3(\mathcal{N})$. The meaning of equation (46) is immediate: the first three terms are nothing but the Barnich–Troessaert bracket at the cross section \mathcal{C} . If the resulting algebra for the diffeomorphism charges has no anomaly (our first assumption), we obtain

$$\Omega_M(\mathcal{L}_\xi, \mathcal{L}_{\xi'}) - \oint_{\mathcal{C}} [\xi \lrcorner J_{\text{rad}}(\mathcal{L}_{\xi'}) - \xi' \lrcorner J_{\text{rad}}(\mathcal{L}_\xi)] = -Q_{[\xi, \xi']}[\mathcal{C}]. \quad (47)$$

Let us now return back to equation (46). It differs from the Barnich–Troessaert bracket (47) by the flux integral that simply shifts the charges upwards along the null generators. Going back to (44), we obtain

$$\{Q_\xi^+, Q_{\xi'}^+\}^* = -Q_{[\xi, \xi']}[\mathcal{C}] - F_{[\xi, \xi']}[M \rightarrow M_+] = -Q_{[\xi, \xi']}^+. \quad (48)$$

We have thus given a simple argument to demonstrate that the charges Q_ξ^+ are integrable, but only on a reduced phase space, which is stripped off from all the radiative modes. The resulting charges satisfy the commutation relation (48). The corresponding Poisson bracket $\{\cdot, \cdot\}^*$ is nothing but the Dirac bracket on the covariant phase space. The constraints (35) remove the radiative data from the covariant phase space and turn them into auxiliary background fields on \mathcal{C} . Notice also that the constraints (35) will necessarily commute under the Dirac bracket, since the flux only depends on the radiative modes and will thus commute under the Dirac bracket, i.e. $\{F_\xi, \cdot\}^* = 0$.

5. Summary and conclusion

On the radiative phase space, it is straightforward to introduce Hamiltonian generators for asymptotic BMS symmetries [4, 32]. These generators are flux integrals. They determine the evolution of the BMS charge aspect due to gravitational radiation. Yet, the charge integrals themselves do not exist on the radiative phase space. To access the charges, we need a different phase space, such as the ADM phase space [43], which is associated to a complete Cauchy surface.

In this note, we pointed out that there is yet another (and perhaps more minimalistic) possibility to realise the charges as Hamiltonian generators. We considered the covariant phase space on a partial Cauchy surface M and removed the radiative data via the Dirac bracket. We argued that the resulting reduced phase space is the phase space of gravitational edge modes (Coulombic modes) alone. Given a few basic assumptions, we gave a heuristic argument, which allowed us to infer the resulting Dirac bracket. The result returned the Barnich–Troessaert bracket on a cross section of \mathcal{I}^+ plus an additional flux integral, which only depends on the radiative data, which commutes under the Dirac bracket (the flux depends only on the radiative modes). The role of the flux integral is to simply shift the charges upwards to future infinity (i^+). There are thus three distinct phase spaces. First of all, there is the ADM phase space on a complete Cauchy surface [44]. Next, there is the radiative phase space at \mathcal{I}^+ , which is slightly smaller. The difference between the two is also a phase space, which is the phase space of the edge modes alone, now localised at i^+ . The symplectic structure on i^+ can be inferred in two different ways: via the Barnich–Troessaert bracket shifted by an additional flux integral, or via the Dirac bracket (38).

To summarise, there are two distinct ways to consider null infinity from a Hamiltonian perspective. The first approach is to work on the usual radiative phase space, where we know the symplectic structure of the radiative data at the full non-perturbative level. On the radiative phase space, the BMS fluxes are Hamiltonian, but the charges are not. The second approach addresses this issue using a more holographic perspective. The radiative data is fixed via auxiliary boundary conditions. Imposing these boundary conditions amounts to introducing auxiliary second-class constraints such that the charges are integrable. These background fields are not part of the resulting phase space and commute under the Dirac bracket. The holographic viewpoint clearly resonates with results in lower dimensions, where there are no radiative modes to begin with, and the entire physical phase space consists of the edge modes alone. It is our opinion that both approaches are equally important, and simply represent different ways of splitting the ADM phase space into different Hamiltonian subsystems. The question for how to identify such subsystems is an important problem both from the perspective of holography as well as non-perturbative quantum gravity and quantum foundations [11–13, 24–28, 45–59].

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Data availability statement

No new data were created or analysed in this study.

ORCID iDs

Wolfgang Wieland  <https://orcid.org/0000-0003-1371-3432>

References

- [1] Bondi H, van der Burg M G J and Metzner A W K 1962 Gravitational waves in general relativity: VII. Waves from axi-symmetric isolated system *Proc. R. Soc. A* **269** 21–52
- [2] Sachs R K 1962 Gravitational waves in general relativity: VIII. Waves in asymptotically flat space-time *Proc. R. Soc. A* **270** 103–26
- [3] Horowitz G T and Perry M J 1982 Gravitational energy cannot become negative *Phys. Rev. Lett.* **48** 371
- [4] Ashtekar A 1987 *Asymptotic Quantization* (Based on the 1984 Naples Lectures) (Naples: Bibliopolis)
- [5] Ashtekar A 2015 Geometry and physics of null infinity *Surveys in Differential Geometry* (One hundred Years of General Relativity) ed L Bieri and S-T Yau vol 20 (Somerville: International Press of Boston, Inc.)
- [6] Peierls R E 1952 The commutation laws of relativistic field theory *Proc. R. Soc. A* **214** 143–57
- [7] Ashtekar A, Bombelli L and Reula O 1990 The covariant phase space of asymptotically flat gravitational fields *Mechanics, Analysis and Geometry: 200 Years after Lagrange* ed M Francaviglia and D Holm (Amsterdam: North-Holland)
- [8] Lee J and Wald R M 1990 Local symmetries and constraints *J. Math. Phys.* **31** 725–43
- [9] Iyer V and Wald R M 1994 Some properties of the Noether charge and a proposal for dynamical black hole entropy *Phys. Rev. D* **50** 846–64
- [10] Wald R M and Zoupas A 2000 A general definition of ‘conserved quantities’ in general relativity and other theories of gravity *Phys. Rev. D* **61** 084027
- [11] Wieland W 2017 New boundary variables for classical and quantum gravity on a null surface *Class. Quantum Grav.* **34** 215008
- [12] Wieland W 2020 Null infinity as an open Hamiltonian system *J. High Energy Phys.* **JHEP04(2021)095**
- [13] Wieland W 2021 Gravitational $SL(2, \mathbb{R})$ algebra on the light cone *J. High Energy Phys.* **JHEP07(2021)057**
- [14] Chandrasekaran V, Flanagan É É and Prabhu K 2018 Symmetries and charges of general relativity at null boundaries *J. High Energy Phys.* **JHEP11(2018)125**
- [15] Chandrasekaran V and Speranza A J 2021 Anomalies in gravitational charge algebras of null boundaries and black hole entropy *J. High Energy Phys.* **JHEP01(2021)137**
- [16] Barnich G and Troessaert C 2010 Symmetries of asymptotically flat four-dimensional spacetimes at null infinity revisited *Phys. Rev. Lett.* **105** 111103
- [17] Barnich G and Troessaert C 2011 BMS charge algebra *J. High Energy Phys.* **JHEP12(2011)105**
- [18] Barnich G and Troessaert C 2010 Aspects of the BMS/CFT correspondence *J. High Energy Phys.* **JHEP05(2010)062**
- [19] Dittrich B and Tambornino J 2007 A perturbative approach to Dirac observables and their spacetime algebra *Class. Quantum Grav.* **24** 757–83
- [20] Dittrich B 2006 Partial and complete observables for canonical general relativity *Class. Quantum Grav.* **23** 6155–84
- [21] Giddings S B, Marolf D and Hartle J B 2006 Observables in effective gravity *Phys. Rev. D* **74** 064018
- [22] Donnelly W and Giddings S B 2016 Observables, gravitational dressing, and obstructions to locality and subsystems (arXiv:1607.01025)
- [23] Freidel L, Oliveri R, Pranzetti D and Speziale S 2021 The Weyl BMS group and Einstein’s equations (arXiv:2104.05793)
- [24] Fernando Barbero G J, Margalef-Bentabol J, Varo V and Villaseñor E J S 2021 Covariant phase space for gravity with boundaries: metric vs tetrad formulations (arXiv:2103.06362)
- [25] Harlow D and Wu J-q 2020 Covariant phase space with boundaries *J. High Energy Phys.* **JHEP10(2020)146**
- [26] Wieland W 2018 Conformal boundary conditions, loop gravity and the continuum *J. High Energy Phys.* **JHEP10(2018)089**

- [27] Freidel L, Geiller M and Pranzetti D 2020 Edge modes of gravity: I. Corner potentials and charges *J. High Energy Phys.* **JHEP11(2020)026**
- [28] Freidel L, Geiller M and Pranzetti D 2020 Edge modes of gravity: II. Corner metric and Lorentz charges *J. High Energy Phys.* **JHEP11(2020)027**
- [29] Geiller M and Jai-akson P 2020 Extended actions, dynamics of edge modes, and entanglement entropy *J. High Energy Phys.* **JHEP09(2020)134**
- [30] Margalef-Bentabol J and Villaseñor E J S 2021 Geometric formulation of the covariant phase space methods with boundaries *Phys. Rev. D* **103** 025011
- [31] Witten E 2018 A note on boundary conditions in Euclidean gravity (arXiv:1805.11559)
- [32] Ashtekar A, Campiglia M and Laddha A 2018 Null infinity, the BMS group and infrared issues *Gen. Relativ. Gravit.* **50** 140–63
- [33] Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Princeton, NJ: Princeton University Press)
- [34] Balachandran A P, Momen A and Chandar L 1996 Edge states in gravity and black hole physics *Nucl. Phys. B* **461** 581–96
- [35] Strominger A 1998 Black hole entropy from near-horizon microstates *J. High Energy Phys.* **JHEP02(1998)009**
- [36] Banados M, Brotz T and Ortiz M E 1999 Boundary dynamics and the statistical mechanics of the $(2 + 1)$ -dimensional black hole *Nucl. Phys. B* **545** 340–70
- [37] Carlip S 2003 *Quantum Gravity in $2 + 1$ Dimensions* (Cambridge: Cambridge University Press)
- [38] Carlip S 2005 Conformal field theory, $(2 + 1)$ -dimensional gravity and the BTZ black hole *Class. Quantum Grav.* **22** R85–R123
- [39] Afshar H, Detournay S, Grumiller D, Merbis W, Perez A, Tempo D and Troncoso R 2016 Soft Heisenberg hair on black holes in three dimensions *Phys. Rev. D* **93** 101503
- [40] Compère G and Fiorucci A 2017 Asymptotically flat spacetimes with BMS_3 symmetry *Class. Quantum Grav.* **34** 204002
- [41] Namburi J C and Wieland W 2020 Deformed Heisenberg charges in three-dimensional gravity *J. High Energy Phys.* **JHEP03(2020)175**
- [42] Wieland W 2020 Twistor representation of Jackiw–Teitelboim gravity *Class. Quantum Grav.* **37** 195008
- [43] Arnowitt R, Deser S and Misner C 1962 *The Dynamics of General Relativity* (New York: Wiley) ch 7 pp 227–64 arXiv:gr-qc/0405109v1
- [44] Arnowitt R, Deser S and Misner C W 1961 Coordinate invariance and energy expressions in general relativity *Phys. Rev.* **122** 997–1006
- [45] Andrade T and Marolf D 2016 Asymptotic symmetries from finite boxes *Class. Quantum Grav.* **33** 015013
- [46] Giddings S B 2019 Gravitational dressing, soft charges, and perturbative gravitational splitting *Phys. Rev. D* **100** 126001
- [47] Donnelly W and Giddings S B 2018 Gravitational splitting at first order: quantum information localization in gravity *Phys. Rev. D* **98** 086006
- [48] Wieland W 2017 Fock representation of gravitational boundary modes and the discreteness of the area spectrum *Ann. Henri Poincaré* **18** 3695–717
- [49] Rovelli C 2020 Gauge is more than mathematical redundancy *Fundam. Theor. Phys.* **199** 107–10
- [50] Dittrich B, Goeller C, Livine E R and Riello A 2018 Quasi-local holographic dualities in non-perturbative 3D quantum gravity *Class. Quantum Grav.* **35** 13LT01
- [51] Dittrich B, Goeller C, Livine E R and Riello A 2019 Quasi-local holographic dualities in non-perturbative 3D quantum gravity: I. Convergence of multiple approaches and examples of Ponzano–Regge statistical duals *Nucl. Phys. B* **938** 807–77
- [52] Dittrich B, Goeller C, Livine E R and Riello A 2019 Quasi-local holographic dualities in non-perturbative 3D quantum gravity: II. From coherent quantum boundaries to BMS_3 characters *Nucl. Phys. B* **938** 878–934
- [53] Gomes H and Riello A 2019 The quasilocal degrees of freedom of Yang–Mills theory (arXiv:1910.04222)
- [54] Vanrietvelde A, Hoehn P A, Giacomini F and Castro-Ruiz E 2020 A change of perspective: switching quantum reference frames via a perspective-neutral framework *Quantum* **4** 225
- [55] Höhn P A, Smith A R and Lock M P 2019 The trinity of relational quantum dynamics (arXiv:1912.00033)

- [56] Castro-Ruiz E, Giacomini F, Belenchia A and Brukner Č 2020 Quantum clocks and the temporal localizability of events in the presence of gravitating quantum systems *Nat. Commun.* **11** 2672
- [57] Giacomini F, Castro-Ruiz E and Brukner Č 2019 Relativistic quantum reference frames: the operational meaning of spin *Phys. Rev. Lett.* **123** 090404
- [58] Giacomini F, Castro-Ruiz E and Brukner Č 2019 Quantum mechanics and the covariance of physical laws in quantum reference frames *Nat. Commun.* **10** 494
- [59] Krumm M, Höhn P A and Müller M P 2021 Quantum reference frame transformations as symmetries and the paradox of the third particle *Quantum* **5** 530