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To cite this article: Vladimir Stukopin 2019 *J. Phys.: Conf. Ser.* **1194** 012103

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On the relationship between super Yangian and quantum loop superalgebra in the case Lie superalgebra $\mathfrak{sl}(1, 1)$

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Abstract. We construct isomorphism between super \hbar -Yangian $Y_{\hbar}(\mathfrak{sl}(1, 1))$ of special linear superalgebra $\mathfrak{sl}(1, 1)$ and quantum loop superalgebra $U_{\hbar}(L\mathfrak{sl}(1, 1))$.

1. Introduction

Lie Superalgebras are a working tool for physicists in the study of supersymmetric models of quantum field theory. Classification of simple Lie superalgebras of classical type was obtained by Victor Kac in the late 70-ies of the last century (see [14]). Lie superalgebras of classical type in this classification are divided into basic and strange Lie superalgebras, and the last unlike basic Lie superalgebras, are not contragredient Lie superalgebras (see [14], [10]). In the middle of the 80s of the last century V. Drinfeld had introduced quantum groups that are deformations of universal algebras of simple Lie algebras (see [6], [7], [8], [9], [3]). One of the most important examples of such deformations were the Yangians ([6], [7], [9], [3], [17]) associated with rational solutions of the quantum Yang-Baxter equation as well as quantized universal enveloping Kac-Moody algebras and quantum loop algebras. Somewhat later, the Yangians of Lie superalgebras (see [23], [19], and also, for example, [26], [28]) were also studied, which also found application in quantum field theory, in particular, in the quantum theory of superstrings ([2], [5], [22]).

The problems of representation theory of the infinite dimensional Quantum Algebras (see [17], [1], [20], [27], [29], [32], [31]) are the some of the most important problems of Quantum Algebras theory for applications in mathematical physics and for the theory of exactly solvable models of statistical mechanics and quantum field theory especially in connection with the problems of finding quantum and universal R -matrices for various classes of quantum algebras (see [6], [7],

¹ Research of the author was supported by “Domestic Research in Moscow” Interdisciplinary Scientific Center J.-V. Poncelet (CNRS UMI 2615) and Scoltech Center for Advanced Study

[8], [4], [16], [22], [15], [24], [25]). Two most important examples of infinite dimensional quantum algebras the Yangians and quantum affine algebras have a similar representations theory as well as in the case of superalgebras that which indicates a deep connection between these objects (see [6], [7], [27], [29], [32]). This relationship was clear already V. Drinfeld, who determined both these classes of Hopf algebras. But explicit results that formulate a connection between these objects have been published relatively recently (see [11], [12], [13], [30]).

Some words about structure of this article. In the second section we recall definition of the special linear Lie Superalgebra in terms of root generators and defining relations. After them we define of the Yangian of Lie Superalgebra and quantum loop superalgebra in terms of generators and defining relations. We define it in terms of current system of generators ([23], [27]). We formulate the theorem on classification of finite dimensional irreducible representations of Yangian and Quantum loop superalgebra. The main result we formulate in next section and prove it in last section.

We'll use the following notations. We denote by \mathbb{C} the field of complex numbers, by $K[u]$ the ring of polynomials with coefficients in ring K , by $K[[u]]$ the ring of polynomials and ring of formal power series, correspondingly, with coefficients from ring K ; by $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ the set of non negative integers, which is union of zero and set of natural numbers \mathbb{N} , and by \mathbb{Z} the set of integer numbers.

2. Definition of Lie superalgebra $A(m, n)$

2.1. Lie Superalgebra $A(m, n)$

Let me recall basic definitions from Lie Superalgebra theory, related to the Lie Superalgebra $A(m, n)$ (see also [10], [14]).

Let $C(m+1|n) = \mathbb{C}^{m+1} \oplus \mathbb{C}^n$ be a \mathbb{Z}_2 -graded $(m+1|n)$ dimensional vector space (superspace) over complex numbers field \mathbb{C} . Let's choose a standart basis in this superspace. We'll numerate the elements of this basis by integer numbers from segment $[1, n+m+1]$ (without zero). So, let $(e_1, e_2, \dots, e_{m+1}, e_{m+2}, \dots, e_{m+n+1})$ be a standart basis in $C(m+1|n)$, and $End(C(n|n))$ be a superalgebra of linear operators acting in $C(m+1|n)$. Basis in $End(C(m+1|n))$ is formed by matrices $E_{a,b}$, $1 \leq a, b, \leq m+n+1$, $ab \neq 0$ and parity p for $E_{a,b}$ is defines by formula:

$$p(E_{a,b}) = |a| + |b|, \quad (1)$$

where $|a| = p(a) = 0$, if $a \leq m+1$ and $|a| = p(a) = 1$, for $a > m+1$, $|a|, |b| \in \mathbb{Z}_2$.

Recall, that general linear Lie superalgebra $\mathfrak{gl}(n|n)$ is defined as vector space $End(C(n|n))$ with (super)bracket $[\cdot, \cdot]$ defined by formula :

$$[x, y] = x \cdot y - (-1)^{|x||y|} y \cdot x. \quad (2)$$

Super trace on $\mathfrak{gl}(n|n)$ defined by formula $str(E_{ab}) = \delta_{ab}(-1)^{1+|a|}$ on base elements and extends linearly to all other elements of the vector $\mathfrak{gl}(n|n)$.

Recall also a definition of special linear Lie superalgebra $\mathfrak{sl}(m+1|n)$:

$$\mathfrak{sl}(m+1|n) = \{A \in \mathfrak{gl}(n|n) | str(A) = 0\}. \quad (3)$$

I recall a definition of Lie superalgebra $A(m, n)$. There is a basic Lie superalgebra generated by generators $h_i, x_i^\pm, i \in \{1, 2, \dots, m, m+1, \dots, m+n+1\}$. Cartan matrix determines defining relations.

I recall that corresponding symmetric Cartan matrix has the following matrix elements: $a_{ii} = 2, i < m+1, a_{mm} = 0, a_{i,i} = -2, m+1 < i \leq m+n+1, a_{i,i+1} = a_{i+1,i} = -1$ for $i < m+1$, and $a_{i,i+1} = a_{i+1,i} = 1$ for $i \geq m+1$. All other matrix elements are equal zero. Unless otherwise stated, below we will use the Cartan symmetric matrix everywhere.

The system of defining relations looks as follows:

$$[h_i, h_j] = 0,$$

$$i \in I = \{1, 2, \dots, m, m+1, \dots, m+n+1\}; \quad (4)$$

$$[h_i, x_j^\pm] = \pm a_{ij} x_j^\pm, i, j \in I; \quad (5)$$

$$[x_i^+, x_j^-] = \delta_{ij} h_i, i, j \in I; \quad (6)$$

$$[x_{m+1}^\pm, x_{m+1}^\pm] = 0; \quad (7)$$

$$[x_i^\pm, x_j^\pm] = 0; |i-j| > 1; \quad (8)$$

$$[x_i^\pm, [x_i^\pm, x_j^\pm]] = 0, |i-j| = 1, i, j \in I; \quad (9)$$

$$[[x_m^\pm, x_{m+1}^\pm], [x_{m+1}^\pm, x_{m+2}^\pm]] = 0. \quad (10)$$

Let's note that the generators x_{m+1}^\pm are odd, and other generators are even, $p(x_{m+1}^\pm) = 1, p(x_i^\pm) = p(h_j) = 0$, where p be a parity function, $j \in I, i \in I \setminus \{m+1\}$ (see [14]). Let now $\mathfrak{g} = A(m, n) = \mathfrak{sl}(m+1, n+1)$.

2.2. The Lie superalgebra $\mathfrak{sl}(1, 1)$

We first consider an important special case of the Lie superalgebra $\mathfrak{sl}(1, 1)$.

It is easy to see that the Yangian $Y(\mathfrak{sl}(1, 1))$ is generated by the generators $h_n, e_n, f_n, n \in \mathbf{Z}_0$, which satisfy the following defining relations:

$$[h_k, h_l] = 0, \quad (11)$$

$$[h_k, e_l] = [h_k, f_l] = 0, \quad (12)$$

$$[e_k, e_l] = [f_k, f_l] = 0, \quad (13)$$

$$[e_k, f_l] = h_{k+l}. \quad (14)$$

3. Definition of Yangian and quantum loop superalgebra

3.1. Definition of Yangian

I recall definition of Yangian of special linear superalgebra (see [23], [24], [25]) for the case Lie superalgebra $\mathfrak{mathfrak{sl}(1, 1)$.

Definition 1 *The Yangian $Y(\mathfrak{sl}(1, 1))$ is generated by the generators $h_n, e_n, f_n, n \in \mathbf{Z}_0$, which satisfy the following defining relations:*

$$[h_k, h_l] = 0, \quad (15)$$

$$[h_k, e_l] = [h_k, f_l] = 0, \quad (16)$$

$$[e_k, e_l] = [f_k, f_l] = 0, \quad (17)$$

$$[e_k, f_l] = h_{k+l}. \quad (18)$$

3.2. Definition of quantum loop superalgebra

In this subsection we define quantum loop superalgebra.

We now explicitly formulate the definition of the quantum loop superalgebra $U_h(LA(m, n))$, slightly changing the notation, following G. Lusztig. In addition, we assume that $q = e^{\hbar/2}$. We use also the following standard notation:

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [n]_q \cdot [n-1]_q \cdot \dots \cdot [1]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! \cdot [k]_q!}.$$

Now, we define quantum universal enveloping superalgebra of Lie superalgebra $\mathfrak{g} = A(m, n)$.

Definition 2 Let $U_{\hbar}(\mathbf{Lg})$ be an associative superalgebra over $C[[\hbar]]$ topologically generated by generators $\{E_{i,k}, F_{i,k}, H_{i,k}\}_{i \in I, k \in \mathbb{Z}}$, which satisfy the following system of defining relations:

1) For every $i, j \in I$ and $r, s \in \mathbb{Z}$

$$[H_{i,r}, H_{j,s}] = 0. \tag{19}$$

2) For every $i, j \in I$ and $k \in \mathbb{Z}$

$$[H_{i,0}, E_{j,k}] = a_{i,j} E_{j,k}, \quad [H_{i,0}, F_{j,k}] = -a_{i,j} F_{j,k}. \tag{20}$$

3) For every $i, j \in I$ and $r, k \in \mathbb{Z} \setminus \{0\}$

$$[H_{i,r}, E_{j,k}] = \frac{[ra_{i,j}]_{q_i}}{r} E_{j,r+k}, \quad [H_{i,r}, F_{j,k}] = -\frac{[ra_{i,j}]_{q_i}}{r} F_{j,r+k}. \tag{21}$$

4) For every $i, j \in I$ and $k, l \in \mathbb{Z}$

$$\begin{aligned} E_{i,k+1} E_{j,l} - q_i^{a_{ij}} E_{j,l} E_{i,k+1} &= q_i^{a_{ij}} E_{i,k} E_{j,l+1} - E_{j,l+1} E_{i,k}, \\ F_{i,k+1} F_{j,l} - q_i^{-a_{ij}} F_{j,l} F_{i,k+1} &= q_i^{-a_{ij}} F_{i,k} F_{j,l+1} - F_{j,l+1} F_{i,k}. \end{aligned} \tag{22}$$

5) For every $i, j \in I$ and $k, l \in \mathbb{Z}$

$$[E_{i,k}, F_{j,l}] = \delta_{i,j} \frac{\psi_{i,k+l} - \varphi_{i,k+l}}{q_i - q_i^{-1}}. \tag{23}$$

6) Let $i \neq j \in I$ and let $m = 1 - a_{ij}$. For every $k_1, \dots, k_m \in \mathbb{Z}$ and $l \in \mathbb{Z}$

$$\sum_{\pi \in \mathfrak{S}_m} \sum_{s=0}^m (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} E_{i,k_{\pi(1)}} \cdots E_{i,k_{\pi(s)}} \cdot E_{j,l} \cdot E_{i,k_{\pi(s+1)}} \cdots E_{i,k_{\pi(m)}} = 0, \tag{24}$$

$$\sum_{\pi \in \mathfrak{S}_m} \sum_{s=0}^m (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} F_{i,k_{\pi(1)}} \cdots F_{i,k_{\pi(s)}} \cdot F_{j,l} \cdot F_{i,k_{\pi(s+1)}} \cdots F_{i,k_{\pi(m)}} = 0, \tag{25}$$

7)

$$[[E_{m,k}, E_{m+1,0}]_q, [E_{m+1,0}, E_{m+2,r}]_q]_q = 0, \tag{26}$$

$$[[F_{m,k}, F_{m+1,0}]_{q^{-1}}, [F_{m+1,0}, F_{m+2,r}]_{q^{-1}}]_{q^{-1}} = 0, \tag{27}$$

where elements $\psi_{i,r}, \varphi_{i,r}$ are defined the following formulas:

$$\psi_i(z) = \sum_{r \geq 0} \psi_{i,r} z^{-r} = \exp\left(\frac{\hbar d_i}{2} H_{i,0}\right) \exp\left((q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,s} z^{-s}\right),$$

$$\varphi_i(z) = \sum_{r \geq 0} \varphi_{i,-r} z^r = \exp\left(-\frac{\hbar d_i}{2} H_{i,0}\right) \exp\left(-(q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,-s} z^s\right),$$

and $\psi_{i,-k} = \varphi_{i,k} = 0$ for $k \geq 1$. In addition, $p(H_{i,r}) = 0$ for $i \in I, r \in \mathbb{Z}_+$, and $p(X_{i,r}^\pm) = 0$ for $i \in I \setminus \{m+1\}, r \in \mathbb{Z}$, $p(x_{m+1,r}^\pm) = 0$ for $r \in \mathbb{Z}$. We also use the notation $[a, b]_q := ab - (-1)^{p(a)p(b)} qba$ for q -(super)commutator.

We also denote by $U^0 \subset U_{\hbar}(\mathbf{Lg})$ a commutative subalgebra generated by generators $\{H_{i,r}\}_{i \in I, r \in \mathbb{Z}}$.

Definition 3 The quantum loop algebra $U_{\hbar}(L\mathfrak{sl}(1,1))$ is generated by the generators $\{E_n, F_n, H_n\}_{n \in \mathbb{Z}}$, which satisfy the following system of defining relations:

$$[H_r, H_s] = 0, \quad (28)$$

$$[H_r, E_s] = [H_r, F_s] = 0, \quad (29)$$

$$[E_r, E_s] = [F_r, F_s] = 0, \quad (30)$$

$$[E_r, F_s] = \frac{\psi_{r+s} - \varphi_{r+s}}{e^{\hbar/2} - e^{-\hbar/2}}, \quad (31)$$

for all $r, s \in \mathbb{Z}$. Here as above elements ψ_r, φ_r are defined by the following formulas:

$$\psi_i(z) = \sum_{r \geq 0} \psi_r z^{-r} = \exp\left(\frac{\hbar}{2} H_0\right) \exp\left((e^{\hbar/2} - e^{-\hbar/2}) \sum_{s \geq 1} H_s z^{-s}\right),$$

$$\varphi_i(z) = \sum_{r \geq 0} \varphi_r z^r = \exp\left(-\frac{\hbar}{2} H_0\right) \exp\left(-(e^{\hbar/2} - e^{-\hbar/2}) \sum_{s \geq 1} H_{-s} z^s\right).$$

4. Representation theory of super Yangian and quantum loop superalgebra

4.1. Representation theory of Yangian

For the reader's convenience, we formulate the main result of the paper [27] (see also [29], [32]), the classification theorem for finite-dimensional irreducible representations of the Yangian $Y(A(m, n))$ in the case $m = n = 0$. (We note that $Y(\mathfrak{sl}(1, 1)) = Y(A(0, 0))$ for $m \neq n$. We give its obvious modification for $Y_{\hbar}(A(m, n))$)

Theorem 1 1) Every irreducible finite-dimensional $Y_{\hbar}(\mathfrak{sl}(1, 1))$ -module V is a module with highest weight $d : V = V(d)$, i. e.,

$$h(u)v_0 = \left(1 + \hbar \sum_{k=0}^{\infty} h_k \cdot u^{-k-1}\right) v_0 = \left(1 + \hbar \sum_{k=0}^{\infty} d_k \cdot u^{-k-1}\right) v_0,$$

where v_0 is a highest vector.

2) The module $V(d)$ is finite-dimensional if and only if there exist polynomials P^d, Q^d , which satisfy the following conditions:

- all these polynomials with leading coefficients equal to 1 (or monic);
-

$$\frac{P^d(u)}{Q^d(u)} = 1 + \hbar \sum_{k=0}^{\infty} d_k \cdot u^{-k-1}. \quad (32)$$

4.2. Representation theory of Quantum Affine Superalgebras

We now formulate an analogue of the above result on the classification of finite-dimensional irreducible representations of the Yangian for the quantum affine Lie superalgebra $U_q(A^{(1)}(0, 0))$.

Let $\mathfrak{g} = A(m, n)$, V be a module over Yangian $U_{\hbar}(L\mathfrak{g}) = U_{\hbar}(LA(m, n))$.

Definition 4 Let V be a module over $U_{\hbar}(LA(m, n))$ of Lie Superalgebra $\mathfrak{g} = A(m, n)$, $\bar{\delta} = \{\delta_{i,r} | i \in I, r \in \mathbb{Z}_+\}$ be a collection of complex numbers. Let's denote by $V_{\bar{\delta}^{\pm}}$ and call weight subspace of module V the space

$$V_{\bar{\delta}^+} = \{v \in V : \psi_{i,r} v = \delta_{i,r}^+ v\}, \quad V_{\bar{\delta}^-} = \{v \in V : \varphi_{i,r} v = \delta_{i,r}^- v\}. \quad (33)$$

The module V is called *highest weight module* (or Verma module) if it is generated by primitive vector, that is $V = U_h(\mathfrak{g}) \cdot v$ for some primitive vector $v \in V_\delta$.

Let $V_0 = \{v \in V | x_{i,k}^+ v = 0, \forall k \in \mathbb{Z}_+\}$.

Easy to check that if module generated by highest vector is finite dimensional then we can find such polynomial $Q(z) = \sum_{s=0}^d a_s z^s \in 1 + z\mathbb{C}[z]$ such that $\sum_{s=0}^d a_s F_{d-s} v_+ = 0$.

Collection of numbers $\bar{\delta} = \{\delta_{i,r}\}$ is called weight of module over Quantum loop superalgebra. We also use the notation Δ for wight generating function:

$$Kv_+ = \Delta(K)v_+, \quad \sum_{n \in \mathbb{Z}} \frac{\psi_n - \varphi_n}{q - q^{-1}} z^n v_+ = \Delta \left(\sum_{n \in \mathbb{Z}} \frac{\psi_n - \varphi_n}{q - q^{-1}} z^n \right) v_+ = \sum_{n \in \mathbb{Z}} f_n z^n = f(z). \quad (34)$$

The vector $v \in V$ is called *primitive*, if $v \in V_\delta$ and $X_{i,r}^\pm \cdot v = 0$ for all $i \in I, r \in \mathbb{Z}_+$.

Easy to check that $Q(z)f(z) = 0$.

We need the following result (see [31]).

Definition 5 Define $\mathbb{R}_{m,n}$ to be set of tetrads (P, f, c, Q) such that:

1)

$$f(z) = \sum_{n \in \mathbb{Z}} f_n z^n \in z\mathbb{C}[z]$$

for all $1 \leq i \leq M + N - 1, i \neq M$,

2) $c \in \mathbb{C} \setminus \{0\}$ with $\frac{c - c^{-1}}{q - q^{-1}} = f_0$,

3) $P = (P_i)_{1 \leq i \leq M+N, i \neq M}$, $P_i(z) \in 1 + z\mathbb{C}[z]$ for all i ,

4) $Q(z) \in 1 + z\mathbb{C}[z]$ with $Q(z)f(z) = 0$.

Theorem 2 Finite-dimensional simple $U_q(\mathfrak{sl}(M, N))$ - modules are parametrized by their highest weights $\Delta \in \mathbb{R}_{M,N}$.

From this theorem 2 follows the following theorem.

Theorem 3 1) Every irreducible finite-dimensional $U_q(A^{(1)}(0, 0))$ -module V is a module with highest weight $\delta : V = V(\delta)$, i.e.

$$\psi(z)v_0 = \left(\sum_{k=0}^{\infty} \delta_k^+ \cdot z^{-k} \right) v_+, \quad \varphi(z)v_0 = \left(\sum_{k=0}^{\infty} \delta_k^- \cdot z^k \right) v_+,$$

where v_0 is a highest vector.

2) The module $V(\delta)$ is finite-dimensional if and only if there exist polynomials P^δ, Q^δ , which satisfy the following conditions:

a) all these polynomials with leading coefficients equal to 1 and non-zero free terms;

b)

$$\frac{P^\delta(z)}{Q^\delta(z)} = \sum_{k=0}^{\infty} \delta_k^+ \cdot z^{-k} = \sum_{k=0}^{\infty} \delta_{-k}^- \cdot z^k. \quad (35)$$

We note that $\mathbb{R}_{1,1}$ to be set of triples (f, c, P) . Let $\mathfrak{T}_{1,1}$ be the set of monic polynomials of the same degree with non-zero free terms, which classifies finite dimensional modules over $U_h(\mathfrak{sl}(1, 1))$ due theorem 3.

Lets define bijective correspondence between the set $\mathfrak{T}_{1,1}$ and the set $\mathbb{R}_{1,1}$, namely, $g : \mathfrak{T}_{1,1} \rightarrow \mathbb{R}_{1,1}$,

$$g : (P, Q) \mapsto \left(\frac{1}{q - q^{-1}} \mathbb{I}_+ \left(\frac{P(z)}{Q(z)} \right) - \frac{1}{q - q^{-1}} \mathbb{I}_- \left(\frac{P(z)}{Q(z)} \right), c, P(z) \right)$$

where $\mathbb{I}_\pm : \mathbb{C}[[z^\pm]] \rightarrow \mathbb{C}[[z, z^{-1}]]$ be a canonical inclusions of formal series, $c^{-2} = \lim_{z \rightarrow 0} \frac{P(z)}{Q(z)}$.

Thus, above defined mapping g defines an isomorphism between pairs of polynomials (P^δ, Q^δ) and the set $\mathbb{R}(1, 1)$, which proves the equivalence of the theorems 2 and 3.

5. Main result

Let $\mathfrak{g} = \mathfrak{sl}(1, 1)$. We denote (as above) by $L\mathfrak{g}$ the Lie algebra (Laurent polynomial) loops with values in the simple algebra (basic Lie superalgebra) of \mathfrak{g} . In this subsection we construct a homomorphism of the quantum loop (super)algebra $U_\hbar(L\mathfrak{g})$ into the quantum (super)algebra $Y_\hbar(\mathfrak{g})$ from which the Yangian $Y(\mathfrak{g})$ is obtained by specialization for $\hbar = 1$. In constructing, we confine ourselves to the special case of a basic Lie superalgebra of type $A(m, n)$, that is, we assume that $\mathfrak{g} = A(0, 0) = \mathfrak{sl}(1, 1)$. To construct this homomorphism, we need descriptions of quantum affine algebras (superalgebras) and Yangians in terms of generating functions of generators.

Let $\{E_r, F_r, H_r\}_{r \in \mathbb{Z}}$ be the loop generators of the quantum affine algebra $U_\hbar(L\mathfrak{g})$, and $\{e_k, f_k, h_k\}_{k \in \mathbb{Z}_+}$ are the generators of the Yangian $Y_\hbar(\mathfrak{g})$.

Let $\mathfrak{g} = \mathfrak{sl}(1, 1)$, $\widehat{Y_\hbar(\mathfrak{g})}$ be a completion of Yangian $Y_\hbar(\mathfrak{g})$ with respect to \hbar -adic topology, defined by natural filtration. I recall, that this filtration defined by N -grading, which, is defined as follows:

$$deg(h_k) = deg(x_k^\pm) = k, deg(\hbar) = 1.$$

Define the map

$$\Phi : U_\hbar(L\mathfrak{g}) \rightarrow \widehat{Y_\hbar(\mathfrak{g})}, \tag{36}$$

on generators by the following formulas:

$$\Phi(H_r) = \frac{\hbar}{q - q^{-1}} \sum_{k \geq 0} t_k \frac{r^k}{k!}, \tag{37}$$

$$\Phi(E_r) = e^{r\sigma} \sum_{m \geq 0} g_m^+ e_m, \tag{38}$$

$$\Phi(F_r) = e^{r\sigma} \sum_{m \geq 0} g_{i,m}^- f_m. \tag{39}$$

Here, as above, we use the following notations $q = e^{\hbar/2}$. We use also the system of logarithmic generators $\{t_r\}_{r \in \mathbb{N}}$ of commutative subalgebra $Y_\hbar(\mathfrak{h}) \subset Y_\hbar(\mathfrak{g})$ generated by generators $\{h_r\}_{r \in \mathbb{N}}$. These logarithmic generators are defined by the following equality for the generating functions:

$$\hbar \sum_{r \geq 0} t_r u^{-r-1} = \log \left(1 + \sum_{r \geq 0} h_r u^{-r-1} \right). \tag{40}$$

The elements $\{g_m^\pm\}_{m \in \mathbb{N}}$ lie in the completion of the algebra $Y_\hbar(\mathfrak{h})$ and are defined as follows. Consider the following formal power series:

$$\gamma(v) = \hbar \sum_{r \geq 0} \frac{t_r}{r!} \left(-\frac{d}{dv} \right)^{r+1} G(v).$$

Then

$$\sum_{m \geq 0} g_m^\pm v^m = \left(\frac{\hbar}{q - q^{-1}} \right)^{1/2} \exp \left(\frac{\gamma(v)}{2} \right).$$

Finally, σ_i^\pm are homomorphisms of sub-superalgebras

$$\sigma^\pm : Y_{\hbar}(\mathfrak{b}_\pm) (\subset Y_{\hbar}(\mathfrak{g})) \rightarrow Y_{\hbar}(\mathfrak{b}_\pm),$$

which are defined on the generators $\{h_r, e_r, f_r\}$ as follows. They leave the generators h_k fixed, and the other generators act as shifts:

$$\sigma^+ : e_r \rightarrow e_{r+1}, \quad \sigma^- : f_r \rightarrow f_{j,r+1}.$$

The elements $\{g_m^\pm\}_{m \in \mathbb{N}}$ belong to the completion of the algebra $Y_{\hbar}(\mathfrak{h})$ and are defined as follows. Consider the following formal power series:

$$G(v) = \log \left(\frac{v}{e^{v/2} - e^{-v/2}} \right) \in Q[[v]]$$

and define $\gamma \in Y^{\widehat{0}}[v]$ by formula:

$$\gamma(v) = \hbar \sum_{r \geq 0} \frac{t_r}{r!} \left(-\frac{d}{dv} \right)^{r+1} G(v).$$

Then

$$\sum_{m \geq 0} g_m^\pm v^m = \left(\frac{\hbar}{q - q^{-1}} \right)^{1/2} \exp \left(\frac{\gamma(v)}{2} \right).$$

Let σ^\pm subsuperalgebra homomorphisms

$$\sigma^\pm : Y_{\hbar}(\mathfrak{b}_\pm) (\subset Y_{\hbar}(\mathfrak{g})) \rightarrow Y_{\hbar}(\mathfrak{b}_\pm),$$

which are given on the generators $\{h_r, e_r, f_r\}$ as follows. They leave the generators h_k fixed, and to the other generators act as shifts: $\sigma^+ : e_r \rightarrow e_{j,r+1}$, $\sigma^- : f_r \rightarrow f_{j,r+1}$.

Let $\mathfrak{g} = \mathfrak{sl}(1, 1)$. Let also $\widehat{Y_{\hbar}(\mathfrak{g})}$ be the completion of the Yangian with respect to its N -grading. Consider the following mappings

$$c : U_{\hbar}(L\mathfrak{g}) \rightarrow U(L\mathfrak{g}), \quad (41)$$

which given by $\hbar \rightarrow 0$, and also map

$$d : U(L\mathfrak{g}) = U(\mathfrak{g}[z, z^{-1}]) \rightarrow U(\mathfrak{g}), \quad (42)$$

which given by formula $d(f(z)) = f(1)$. Then the kernel of the composition of these mappings

$$e = d \circ c : U_{\hbar}(L\mathfrak{g}) \rightarrow U(\mathfrak{g})$$

we denote by $I := \text{Ker}(e)$. We define the completion of a quantum loop algebra with respect to the I -adic topology given by the powers of the ideal I :

$$\widehat{U_{\hbar}(L\mathfrak{g})} := \lim_{n \rightarrow \infty} U_{\hbar}(L\mathfrak{g})/I^n. \quad (43)$$

We state the main result of this paper.

Theorem 4 *The map Φ , given by the formulas (37) – (39), defines a monomorphism of associative superalgebras over $\mathbb{Q}[[\hbar]]$:*

$$\Phi : U_{\hbar}(L\mathfrak{g}) \rightarrow \widehat{Y_{\hbar}(\mathfrak{g})}, \quad (44)$$

which extends to an isomorphism of completions

$$\widehat{\Phi} : \widehat{U_{\hbar}(L\mathfrak{g})} \rightarrow \widehat{Y_{\hbar}(\mathfrak{g})}. \quad (45)$$

6. Proof of the main result

First, we describe homomorphism $\Phi : U_{\hbar}(\mathfrak{sl}(1, 1)) \rightarrow Y_{\hbar}(\mathfrak{sl}(1, 1))$.

We first recall the definition of the Borel transformation (more precisely, the inverse of the Borel transform), which is a discrete analog of the Laplace transform. We denote by B the transformation associating the functions $f(u) \in A[[u]]$ with values in some associative algebra A , the function $B(f)(v) \in u^{-1}A[[u^{-1}]]$, defined as

$$f(u) = \sum_{k=0}^{\infty} f_k u^{-k-1} \rightarrow B(f)(v) = \sum_{r=0}^{\infty} \frac{f_r v^r}{r!}. \tag{46}$$

We note the following properties of the Borel transform of the generating function $t(u)$ of the logarithmic generators $\{t_k\}_{k=0}^{\infty}$.

Because the,

$$[h(u), e_k] = [h(u), f_k] = 0,$$

then it follows that u

$$[t(u), e_k] = [t(u), f_k] = 0,$$

which means that

$$[B(t(u))(v), e_k] = [B(t(u))(v), f_k] = 0. \tag{47}$$

Easy to check that

$$B(\log(1 - pu^{-1})) = \frac{1 - e^{pv}}{v}. \tag{48}$$

Indeed, using the Borel transformation properties, which coincide with the properties of the Laplace transform, we obtain the following equalities:

$$\begin{aligned} B(\log(1 - pu^{-1})) &= \frac{1}{v} B\left(\frac{d}{du}(\log(1 - pu^{-1}))\right) = \frac{1}{v} B\left(\frac{-pu^{-2}}{1 - pu^{-1}}\right) \\ &= \frac{1}{v} B\left(\frac{-p}{u(u-p)}\right) = \frac{1}{v} B\left(\frac{1}{u} - \frac{1}{u-p}\right) = \frac{1}{v}(1 - e^{pv}). \end{aligned}$$

To prove the theorem in this particular case, it suffices to verify the equivalence of the relations (18) and (23). We first note that

$$\Phi(E_r) = e^{r\sigma_+} \sum_{m \geq 0} g_m e_m = \sum_{m \geq 0} g_m^{(+,k)} e_m = e^{r\sigma_+} g(\sigma_+) e_0, \tag{49}$$

$$\Phi(F_r) = e^{r\sigma_-} \sum_{m \geq 0} g_m f_m = \sum_{m \geq 0} g_m^{(-,k)} f_m = e^{r\sigma_-} g(\sigma_-) f_0. \tag{50}$$

Let us prove an explicit calculation of equality

$$[\Phi(E_r), \Phi(F_l)] = \frac{\Phi(\psi_{r+s}) - \Phi(\varphi_{r+s})}{e^{\hbar/2} - e^{-\hbar/2}}. \tag{51}$$

Explicitly calculate the right and left sides of the equality to be proved (51). It is easy to see that

$$\Phi(E_r)\Phi(F_l) = e^{r\sigma_+} g(\sigma_+) e_0 e^{r\sigma_-} g(\sigma_-) f_0.$$

As

$$g(v) = \sum_{m \geq 0} g_m v^m = \sum_{m \geq 0} g_m v^m = \left(\frac{\hbar}{e^{\hbar} - e^{-\hbar/2}}\right)^{1/2} \exp(\gamma(v)/2)$$

$$= \left(\frac{\hbar}{e^{\hbar/2} - e^{-\hbar/2}} \right)^{1/2} \exp \left(\frac{1}{2} B(t(u)) \left(-\frac{d}{dv} \right) \left(\frac{d}{dv} \log \left(\frac{e^{v/2} - e^{-v/2}}{v} \right) \right) \right).$$

For brevity we use the notation $\partial_v := \frac{d}{dv}$. Then

$$g(v) = \sum_{m \geq 0} g_m v^m = \left(\frac{\hbar}{e^{\hbar/2} - e^{-\hbar/2}} \right)^{1/2} \exp \left(\frac{1}{2} B(t(u)) (-\partial_v) \left(\log \left(\frac{e^{v/2} - e^{-v/2}}{v} \right) \right)' \right).$$

It is easy to check that

$$g(\sigma_+) e_0 = \sum_{m \geq 0} g_m \sigma_+^m e_0 = \sum_{m \geq 0} g_m e_m.$$

Similarly,

$$g(\sigma_-) f_0 = \sum_{m \geq 0} g_m \sigma_-^m e_0 = \sum_{m \geq 0} g_m f_m.$$

Note also that if the variables u and v commute, then

$$g(u)g(v) = \left(\frac{\hbar}{e^{\hbar} - e^{-\hbar/2}} \right) \exp(\gamma(u)/2 + \gamma(v)/2) = \left(\frac{\hbar}{e^{\hbar} - e^{-\hbar/2}} \right) \exp \left(\frac{1}{2} B(t(u_1)) \left(-\frac{d}{du} \right) (\log(f_0(u)))' + \frac{1}{2} B(t(u_1)) \left(-\frac{d}{dv} \right) \frac{d}{dv} (\log(f_0))' \right),$$

where $f_0(u) = \frac{e^{u/2} - e^{-u/2}}{u}$.

Now we can calculate $\Phi(E_r)\Phi(E_l)$. Actually,

$$\Phi(E_r)\Phi(E_l) = e^{r\sigma_+} g(\sigma_+) e_0 e^{r\sigma_-} g(\sigma_-) e_0 f_0 = g(\sigma_+) g(\sigma_-) e_r f_l$$

Similarly,

$$\Phi(F_l)\Phi(E_r) = g(\sigma_+) g(\sigma_-) f_l e_r.$$

From this we immediately obtain that

$$\begin{aligned} [\Phi(E_r)\Phi(E_l), \Phi(E_l)\Phi(E_r)] &= [g(\sigma_+)g(\sigma_-)e_r f_l, g(\sigma_+)g(\sigma_-)f_l e_r] \\ &= \sum_{m \geq 0} \sum_{n \geq 0} g_m g_n (e_{m+r} f_{n+l} - f_{n+l} e_{m+r}) = \sum_{m \geq 0} \sum_{n \geq 0} g_m g_n h_{m+n+r+l} \end{aligned}$$

On the other hand,

$$\Phi(H_r) = \frac{B(t(u))(r)}{e^{\hbar/2} - e^{-\hbar/2}}.$$

Then

$$\Phi(\psi(z)) = \sum_{r \geq 0} \Phi(\psi_r) z^{-r} = \exp \left(\frac{\hbar \Phi(H_0)}{2} \right) \exp \left(\left(e^{\hbar/2} - e^{-\hbar/2} \right) \sum_{s \geq 1} \Phi(H_s) z^{-s} \right)$$

$$\begin{aligned}
&= \exp\left(\frac{\hbar\Phi(H_0)}{2}\right) \exp\left(\sum_{s \geq 1} B(t(u))(s)z^{-s}\right) \\
&= \exp\left(\frac{\hbar}{2(e^{\hbar/2} - e^{-\hbar/2})}t_0\right) \exp\left(\sum_{s \geq 1} B(t(u))(s)z^{-s}\right).
\end{aligned}$$

Similarly,

$$\Phi(\varphi(z)) = \sum_{r \geq 0} \Phi(\varphi_r)z^r = \exp\left(\frac{-\hbar\Phi(H_0)}{2}\right) \exp\left(\sum_{s \geq 1} B(t(u))(-s)z^s\right).$$

We describe the action of generators, respectively, of a quantum loop superalgebra and a Yangian, in finite-dimensional irreducible modules. We need the results of the papers [27] and [29], in which the classification of finite-dimensional irreducible Yangian modules of a special linear Lie superalgebra is given. We need an explicit definition of the action of the Cartan generators on the highest vectors of finite-dimensional irreducible modules over, respectively, the Yangian and the quantum loop algebra.

We denote by D^Y the Yangian morphism into a finite-dimensional irreducible module given by a family of Drinfeld polynomials. The finite-dimensional irreducible Yangian module is given by a pair of Drinfeld polynomials $P^d(u)$ and $Q^d(u)$ with leading coefficient equal to 1. In this case the polynomials are uniquely determined by their complex roots: a_1, \dots, a_n and b_1, \dots, b_n and the action of the generating function of the Yangian on the highest vector is given by the following formula (see [27]):

$$h(u)v_0 = \left(1 + \sum_{k \geq 0} d_k u^{-k-1}\right)v_0, \quad \frac{P^d(u)}{Q^d(u)} = 1 + \sum_{k \geq 0} d_k u^{-k-1}, \quad (52)$$

where as above $h(u) = 1 + \sum_{k \geq 0} h_k u^{-k-1}$ be a generating function of Cartan generators. In other words

$$h(u)v_0 = \frac{P^d(u)}{Q^d(u)}v_0 = R(m) = \frac{(u - a_1) \cdot \dots \cdot (u - a_n)}{(u - b_1) \cdot \dots \cdot (u - b_n)}v_0. \quad (53)$$

Let's define then

$$D^Y(h(u)) = \frac{P^d(u)}{Q^d(u)} = \frac{(u - a_1) \cdot \dots \cdot (u - a_n)}{(u - b_1) \cdot \dots \cdot (u - b_n)}. \quad (54)$$

Assuming the quantities $a_1, \dots, a_n, b_1, \dots, b_n$ as parameters, we obtain the mapping

$$D^Y : Y^0 \rightarrow \mathbf{C}[\hbar, a_1, \dots, a_n, b_1, \dots, b_n].$$

Cartan subalgebra of the Yangian into a commutative polynomial ring defined by the formula (54).

Similarly we define the map

$$D^U : U^0 \rightarrow S(m) = \mathbf{C}[q, A_1, \dots, A_n, B_1, \dots, B_n],$$

$$D^U(\psi(z)) = \frac{P^\delta(z)}{Q^\delta(z)} = \frac{(z - A_1) \cdot \dots \cdot (z - A_n)}{(z - B_1) \cdot \dots \cdot (z - B_n)}. \quad (55)$$

We now compute the action of the homomorphism $D^Y : Y(\mathfrak{sl}(1, 1)) \rightarrow \text{End}(V_{P,Q})$ onto $t(u)$, as well as the images of the generators h_k and t_k under the action of this mapping. The following proposition holds.

Proposition 6.1 *The following formulas are valid:*

$$D^U(\psi_r) = \sum_{p=1}^n B_p^r (B_p - A_p) \left(\prod_{p' \neq p} \frac{B_p - A_{p'}}{B_p - B_{p'}} \right), \quad (56)$$

$$D^U(\varphi_r) = \sum_{p=1}^n B_p^{-r} (A_p - B_p) \left(\prod_{p' \neq p} \frac{B_p - A_{p'}}{B_p - B_{p'}} \right), \quad (57)$$

$$D^U((q - q^{-1})H_k) = \frac{1}{p} \sum_{p=1}^n (B^p - A^p), \quad (58)$$

$$D^Y(h_r) = \sum_{p=1}^n b_p^r (b_p - a_p) \left(\prod_{p' \neq p} \frac{b_p - a_{p'}}{b_p - b_{p'}} \right), \quad (59)$$

$$D^Y(t_r) = \frac{1}{r+1} \sum_{p=1}^n \frac{b_p^{r+1} - a_p^{r+1}}{\hbar}, \quad (60)$$

$$D^Y(B(t(u))(v)) = \sum_{p=1}^n \frac{\exp(b_p v) - \exp(a_p v)}{v}. \quad (61)$$

Proof The proof is carried out by direct calculation, by expansion in powers of the variable u^{-1} of the left and right sides of the equalities to be proved.

□

From the sentence 6.1 it follows easily that

Proposition 6.2 *1) Homomorphisms*

$$D^Y : Y^0 \rightarrow \bigoplus_{n \geq 1} R(n), \quad D^U : U^0 \rightarrow \bigoplus_{n \geq 1} S(m) \quad (62)$$

are injective.

Now we can formulate and prove the most important auxiliary assertion

Lemma 1 *The following equality holds*

$$\Phi \left(\frac{\psi_k - \varphi_k}{e^{\hbar/2} - e^{-\hbar/2}} \right) = \frac{\hbar}{e^{\hbar/2} - e^{-\hbar/2}} e^{kv} \exp(\gamma(v))|_{v^n=h_n}. \quad (63)$$

Proof We first calculate the left-hand side of the equality.

$$\begin{aligned} \hbar e^{kv} \exp(B(-\partial)G'(v))|_{v^n=h_n} &= \hbar e^{kv} \exp \left(\sum_{p=1}^n \frac{e^{b_p(\partial)} - e^{a_p(\partial)}}{\partial} \right) G(v)|_{v^n=h_n} \\ &= \hbar e^{kv} \sum_{p=1}^n \exp(G(v + b_p) - G(v + a_p))|_{v^n=h_n} \\ &= \hbar e^{kv} \prod_{p=1}^n \frac{v - b_p}{v - a_p} \cdot \frac{e^{(v-a_p)/2} - e^{-(v-a_p)/2}}{e^{(v-b_p)/2} - e^{-(v-b_p)/2}}|_{v^n=h_n}. \end{aligned}$$

So, we obtain

$$D^Y \left(\hbar e^{kv} \prod_{p=1}^n \frac{v - b_p}{v - a_p} \cdot \frac{e^{(v-a_p)/2} - e^{-(v-a_p)/2}}{e^{(v-b_p)/2} - e^{-(v-b_p)/2}} \Big|_{v^n=h_n} \right).$$

A direct calculation proves the following assertion.

Proposition 6.3 *Let $F(v) = \sum_{k=0}^{\infty} f_k v^k$. Then*

$$F(v)|_{v^n=h_n} = \sum_{k=0}^{\infty} f_k h_k.$$

Taking into account the equality (59) and the sentence 6.3, we get that the last equality is

$$D^Y \left(\hbar e^{kv} \exp(\gamma v) \Big|_{v^n=h_n} \right) = D^Y \left(\hbar e^{kv} \prod_{p=1}^n \frac{v - b_p}{v - a_p} \cdot \exp\left(\frac{b_p - a_p}{2}\right) \cdot \frac{e^v - e^{a_p}}{e^v - e^{b_p}} \Big|_{v^n=h_n} \right).$$

Taking into account, that $\lim_{v \rightarrow b_p} \frac{v - b_p}{e^v - e^{b_p}} = e^{-b_p}$ we obtain, that last equality is equal

$$\begin{aligned} \hbar \sum_{p=1}^n e^{kb_p} \hbar \prod_{p' \neq p} \frac{b_p - b_{p'}}{b_p - a_{p'}} \cdot \frac{1}{b_p - a_p} \cdot \frac{e^{b_p} - e^{a_{p'}}}{e^{b_p} - e^{b_{p'}}} \cdot (e^{b_p} - e^{a_p}) \\ = \hbar \sum_{p=1}^n e^{kb_p} (e^{b_p} - e^{a_p}) \prod_{p' \neq p} \frac{e^{b_p} - e^{a_{p'}}}{e^{b_p} - e^{b_{p'}}}. \end{aligned}$$

By the formula (56), we see that

$$\left(\frac{\hbar}{e^{\hbar/2} - e^{-\hbar/2}} \right) D^U(\psi_r) = \hbar \sum_{p=1}^n B_p^r (B_p - A_p) \left(\prod_{p' \neq p} \frac{B_p - A_{p'}}{B_p - B_{p'}} \right),$$

that coincides after replacement $A_p = e^{a_p}$, $B_p = e^{b_p}$ with $D^Y(\hbar e^{rv} \exp(\gamma v)|_{v^n=h_n})$. Analogously, considering the expansion in a neighborhood of an infinitely, we obtain that $D^Y(\hbar e^{rv} \exp(\gamma v)|_{v^{-n-1}=h_n})$ coincides with $\left(\frac{\hbar}{e^{\hbar/2} - e^{-\hbar/2}} \right) D^U(\varphi_r)$.

Lemma is proved. \square

Easy to see that from lemma 1 and proposition 6.2 follows equality (51):

$$[\Phi(E_r), \Phi(F_l)] = \frac{\Phi(\psi_{r+s}) - \Phi(\varphi_{r+s})}{e^{\hbar/2} - e^{-\hbar/2}},$$

since, as was proved earlier, by the injectivity of the mapping D^U , which follows from the proposition 6.2, left-hand side of equality is equal to $e^{kv} \exp(\gamma v)|_{v^n=h_n}$. From the lemma 1 and the injectivity of the map D^Y , which also follows from the proposition 6.2, it follows that the right-hand side of the equality (51) also is equal to $e^{kv} \exp(\gamma v)|_{v^n=h_n}$.

Thus, the equality (51) is proved.

Equality (51) easily implies the equivalence of (2) and (3).

\square

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