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THE CONSTRUCTION OF LOCAL QUANTUM FIELDS DESCRIBING MANY MASSES AND SPINS*

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ABSTRACT

We discuss the possibility of constructing out of particle creation and destruction operators local quantum fields which transform as representations of the homogeneous Lorentz group. Our immediate goal is to write down a consistent local quantum field theory which can simultaneously describe many particles with different masses and spins. In the case that the field is a finite dimensional irreducible Lorentz tensor, we are able to carry through our program with no restrictions on the masses considered as functions of the spin, provided the usual connection between spin and statistics is satisfied. However, when the field transforms as a unitary irreducible representation of the homogeneous Lorentz group (an infinite dimensional representation), the requirement of locality, along with the physical assumption that the masses are bounded below, $m(j) \geq m_0 > 0$, leads to the restriction that the masses are independent of the spin. This property is shown to hold when the transformation law of the field is taken to be an irreducible finite dimensional representation $\otimes$ a unitary irreducible representation. The physical consequences of this result and possible methods for evading it are discussed. Finally an appendix is included where the related problem of orthogonality properties of timelike solutions to infinite component wave equations is examined. In particular, we show that when the solutions of such wave equations transform as unitary irreducible representations of the homogeneous Lorentz group, only the Majorana representations support a scalar product, which is orthogonal for different spins.

I. INTRODUCTORY REMARKS

Conventional studies of quantum field theories[1] have been directed toward the construction of fields which describe one particle of definite mass and spin. One has proceeded by examining finite dimensional spinors having appropriate properties under Lorentz transformations and then hastened to eliminate, in an invariant manner, any "extra" field components thus introduced. The traditional description of spin one fields, for example, introduces a four-vector object $A_\mu(x)$ and immediately subtracts off one degree of freedom by asking that $A_\mu(x)$ be divergence free. A notable exception to this ritual is the work of Weinberg[2], who constructs fields for any spin with the "correct" number of components and, thus, has none to throw away.

The intriguing idea of Regge poles[3], or more explicitly, the possibility that particles of different spins and masses may be connected, opens for our consideration quantum field theories where the fields describe a variety of particles with various masses and spins. Success in such studies would provide a compact, field-theoretic framework in which to examine the properties of Regge trajectories[4]. In this paper we investigate the structure of such field theories.

We have in mind using these fields to calculate a Lorentz invariant S-matrix via the Dyson-Wick prescription, so we shall require that they have simple properties under Lorentz transformations and be local in the sense of commuting or anti-commuting at spacelike separations. In particular we shall address ourselves to the question of whether it is possible to find local fields which transform as Lorentz tensors and are linear combinations of (Fourier transforms of) particle creation and destruction operators. The method we employ has been advocated by a number of authors[5] and avoids reference to Lagrangians and field equations by concentrating on the Lorentz transformation properties of the fields and the particle states they are to describe.

We will show below that in the case where the fields are finite dimensional irreducible tensors, we can answer the question posed in the affirmative with no restrictions on the masses as a function of spin. In the instances where the fields transform as unitary infinite dimensional irreducible representations of the homogeneous Lorentz group, we will show that locality is a non-trivial restriction on the theory and present arguments that when the masses are bounded below by some $m_0 > 0$ for all spin, then locality forces them to be independent of the spin. This latter result is also shown to be true for the circumstance where the field transforms as the direct product of an irreducible finite dimensional and an irreducible unitary representation of the homogeneous Lorentz group.

In the next section we review the procedure for building the fields from the physical creation and destruction operators. Section III is devoted to a discussion of finite dimensional fields, and in Section IV we direct our attention to the possibility of having local fields which transform according to unitary irreducible representations of the homogeneous Lorentz group. In Section V we consider a direct product of a finite dimensional and a unitary representation as the transformation rule of the field. Some discussion of the physical implications of the results will be found in the final paragraph. In an appendix we examine the related question of orthogonality properties of solutions to infinite component wave equations. In particular, we show that for the case of a unitary representation of the homogeneous Lorentz group, scalar products which vanish for two solutions of different spin are defined only in the Majorana representations (those which support a four vector Γ_μ - called the Majorana vector), only with the metric Γ_0, and only when the mass depends on spin j as a constant / 2j + 1. This is independent of the wave equation.

II. TRANSFORMATION PROPERTIES AND COMMUTATION RELATIONS

We begin with a description of the physical states, their behavior under Lorentz transformations[6], and the construction of field operators, which are irreducible Lorentz tensors, out of the creation and annihilation operators for these states. We follow here the authors of Ref. 5 and adopt, fairly closely, the notation of Weinberg[2].

The physical states $|\vec{p}\ m(j)\ j\sigma>$ are characterized by their three-momentum $\vec{p}$, the spin j, its projection on the z axis σ, and the mass m(j), which we allow to be a function of j. They are, of course, the basis states of Wigner[6] for a unitary irreducible representation of the Poincare group. The four-momentum of the state is such that $p^2 = p_0(j)^2 - \vec{p}^2 = m^2(j)$. These states can be obtained from the rest states $|m(j)\ j\sigma>$ by a unitary transformation $U[L(\vec{u})]$,

$$|\vec{p}\ m(j)\ j\sigma> = U[L(\vec{u})]\ |\ m(j)\ j\sigma> ,$$

where $\vec{u} = \vec{p}/m(j)$ is the "velocity" of the state and $L(\vec{u})$ is the pure Lorentz transformation, the "boost", which takes the four-vector $(\vec{0}, m(j))$

into $(\vec{p},\ p_0(j))$. Under a homogeneous Lorentz transformation Λ, we then have

$$U[\Lambda]\left|\vec{p}\ m(j)\ j\sigma\right> = \left|\vec{\Lambda p}\ m(j)\ j\sigma'\right> D^j_{\sigma'\sigma}\,[R_W]\ ,$$

with $R_W = L^{-1}(\vec{\Lambda u})\Lambda L(\vec{u})$ the Wigner rotation and $D^j[R_W]$ the usual rotation matrix for spin j.

Now we introduce creation and annihilation operators $a^*(j\vec{p}\sigma)$ and $a(j\vec{p}\sigma)$ such that operating on the vacuum $|0>$, $\left|\vec{p}\ m(j)j\sigma\right> = a^*(j\vec{p}\sigma)\left|0\right>$, and

$$[a(\vec{p}j\sigma),a^*(\vec{p}'j'\sigma')]_\epsilon = F^2(j)2u_0\delta^3(\vec{u}-\vec{u}')\delta_{jj'}\delta_{\sigma\sigma'}\ .$$

$\epsilon = \pm 1$ for an anti-commutator (commutator), $u_0 = p_0(j)/m(j)$, and $F(j)$ is a normalization factor. Other commutators or anti-commutators vanish. The Lorentz transformation properties of these operators are immediate

$$U[\Lambda]a^*(\vec{p}j\sigma)U[\Lambda]^{-1} = a^*(\vec{\Lambda p}\ j\sigma')D^j_{\sigma'\sigma}\,[R_W]\ ,$$

and

$$U[\Lambda]a(\vec{p}j\sigma)U[\Lambda]^{-1} = D^j_{\sigma\sigma'}\,[R_W^{-1}]a(\vec{\Lambda p}\ j\sigma').$$

By introducing a charge conjugation matrix $C^{(j)}$ satisfying[7]

$$D^j[R]^* = C^{(j)}D^j[R]C^{(j)^{-1}},$$

we can make these transformation rules quite similar. Define $\tilde{a}^*(\vec{p}j\sigma)$ by

$$\tilde{a}^*(\vec{p}j\sigma) \equiv \left(C^{(j)^{-1}}\right)_{\sigma\sigma'} a^*(\vec{p}j\sigma'),$$

then

$$U[\Lambda]\tilde{a}^*(\vec{p}j\sigma)U[\Lambda]^{-1} = D^j_{\sigma\sigma'}\,[R_W^{-1}]\tilde{a}^*(\vec{\Lambda p}\ j\sigma').$$

Thus far we have simply recounted the lessons of Wigner on the description of the single particle physical states. Now we turn to the construction of quantum fields out of which one may build an interaction Hamiltonian which via Dyson's formula enables one to compute the S-matrix describing transitions between the original physical states. The simplest way to guarantee that the interaction Hamiltonian be a scalar and commute with itself at spacelike separations is to make it an invariant polynomial in local fields. The usual method for making this invariant polynomial, which we adopt, is to choose the fields to be Lorentz tensors and then couple them to an invariant[8]. We will, at least for the moment, even take them to be irreducible tensors.

So now we construct from the $a(j\vec{p}\sigma)$'s field operators $\varphi(x)$, which transform covariantly under an irreducible representation of the homogeneous Lorentz group[9]. Let $\varphi_{j\sigma}(x)$ be the $(j\sigma)$ component of that field; $(j\sigma)$ are sufficient to label the components within one irreducible representation. We insist that both particles and anti-particles be included in the field operator and write[10]

$$\varphi_{j\sigma}(x) = \varphi_{j\sigma}^{(-)}(x) + \varphi_{j\sigma}^{(+)}(x)$$

where $\varphi_{j\sigma}^{(-)}(x)$ is the annihilation part for particles

$$\varphi_{j\sigma}^{(-)}(x) = \int \frac{d^3u}{2u_0} \sum_{j'\sigma'} D_{j\sigma,j'\sigma'}[L(\vec{u})]a(\vec{p}j'\sigma')e^{-im(j')u\cdot x}$$

and $\varphi_{j\sigma}^{(+)}(x)$ is the creation part for anti-particles

$$\varphi_{j\sigma}^{(+)}(x) = \eta \int \frac{d^3u}{2u_0} \sum_{j'\sigma'} D_{j\sigma,j'\sigma'}[L(\vec{u})]\tilde{b}^*(\vec{p}j'\sigma')e^{+im(j')u\cdot x} .$$

Anti-particles are created and destroyed by operators $b^*(\vec{p}j\sigma)$ and $b(\vec{p}j\sigma)$ which commute (anti-commute) among themselves as the a*'s and a's and commute or anti-commute with them. In general the particles and anti-particles are distinct; the case of self-conjugate particles does not change the results to follow. In the definition of the creation and annihilation parts of the field η is a phase factor $|\eta| = 1$, and

$$D_{j\sigma,j'\sigma'}[L(\vec{u})] = < j\sigma \left| e^{-i\vec{K}\cdot\hat{u}\theta(\vec{u})} \right| j'\sigma' >$$

is the representation matrix for the boost along the direction $\hat{u}$, of magnitude $\theta(\vec{u}) = \sinh^{-1}|\vec{u}|$, generated by the boost operator $\vec{K}$. $\vec{K}$ commutes with itself and the generator of rotations $\vec{J}$ in the usual way

$$[J_a, J_b] = i\epsilon_{abc} J_c,$$

$$[J_a, K_b] = i\epsilon_{abc} K_c,$$

and

$$[K_a, K_b] = -i\epsilon_{abc} J_c.$$

These fields transform covariantly under Lorentz transformations

$$U[\Lambda]\varphi_{j\sigma}^{(\pm)}(x)U[\Lambda]^{-1} = \sum_{j'\sigma'} D_{j\sigma,j'\sigma'}[\Lambda^{-1}]\varphi_{j'\sigma'}^{(\pm)}(\Lambda x).$$

The "wave function" for a particle state described by this field is just

$$< 0 \left| \varphi_{j\sigma}^{(-)}(x) \right| \vec{p}\ m(\ell)\ell\ell_3 > = D_{j\sigma,\ell\ell_3}[L(\vec{u})] e^{-im(\ell)u\cdot x} F^2(\ell),$$

and the ℓ values run over all the spins contained in a given representation of the Lorentz group. For a finite dimensional representation ℓ

will have a certain restricted range, while for an infinite dimensional representation, ℓ will begin from some minimum value j_0 and take the values j_0, j_0+1, j_0+2, ...

We have thus completed the task of building fields which describe particles of many masses and spins and have simple Lorentz transformation properties. It remains to discuss the local nature of our fields, before we can proceed to the computation of the S-matrix. Therefore, we ask whether they can be made to commute or anti-commute at spacelike separations. Noting that the unitarity of the charge conjugation matrix leads to

$$[\tilde{b}^*(j\sigma\vec{p}),\ \tilde{b}(j'\sigma'\vec{p}')]_\epsilon = \epsilon\, F^2(j) 2u_0 \delta^3(\vec{u}-\vec{u}')\delta_{jj'}\delta_{\sigma\sigma'}\ ,$$

we find for the commutation relations of the fields

$$[\varphi_{j\sigma}(x),\ \varphi^+_{j'\sigma'}(y)]_\epsilon = \int \frac{d^3u}{2u_0} \sum_{j''\sigma''} D_{j\sigma,j''\sigma''}[L(\vec{u})] D^+_{j''\sigma'',j'\sigma'}[L(\vec{u})]$$

$$F^2(j'')\left\{e^{-im(j'')u\cdot(x-y)} + \epsilon\, e^{+im(j'')u\cdot(x-y)}\right\},$$

with all other commutators vanishing. By a change of integration variables we can express this as

$$[\varphi_{j\sigma}(x),\ \varphi^+_{j'\sigma'}(y)]_\epsilon = \frac{1}{2}\int d^3p\ P_{j\sigma,j'\sigma'}(\vec{p})[e^{-ip\cdot(x-y)} + \epsilon\, e^{+ip\cdot(x-y)}],$$

where

$$P_{j\sigma,j'\sigma'}(\vec{p}) = \sum_{j''\sigma''} \frac{F^2(j'')}{m^2(j'')\sqrt{\vec{p}^2+m^2(j'')}} D_{j\sigma,j''\sigma''}[L(\vec{p}/m(j''))] D_{j''\sigma'',j'\sigma'}[L(\vec{p}/m(j''))]$$

Our aim in the following sections will be to determine under what conditions this commutator vanishes for x-y spacelike for $\varphi(x)$ transforming under various representations of the homogeneous Lorentz group.

III. FINITE DIMENSIONAL REPRESENTATIONS

The first case we are invited to address ourselves to is that where the quantum field transforms under the finite dimensional non-unitary representations of the homogeneous Lorentz group. These representations are most simply characterized by introducing the operators $\vec{A} = (\vec{J} + i\vec{K})/2$ and $\vec{B} = (\vec{J} - i\vec{K})/2$ which decouple the commutation relations of the rotation operators $(\vec{J})$ and the boosts $(\vec{K})$. $\vec{A}$ and $\vec{B}$ now form two independent angular momenta, and an irreducible representation [a,b] is labeled by $\vec{A}^2 = a(a + 1)$ and $\vec{B}^2 = b(b + 1)$ and has dimension $(2a + 1)(2b + 1)$. The operators $\vec{K}$ are anti-hermitian here and the function $P_{j\sigma,j'\sigma'}(\vec{p})$ from above becomes

$$P_{j\sigma,j'\sigma'}(\vec{p}) = \sum_{j''\sigma''} \frac{F^2(j'')}{m^2(j'')\sqrt{\vec{p}^2 + m^2(j'')}} < j\sigma \left| e^{-i\hat{p}\cdot\vec{K}\theta(j'')} \right| j''\sigma'' > < j''\sigma'' \left| e^{-i\hat{p}\cdot\vec{K}\theta(j'')} \right| j\sigma >$$

where $\sinh \theta(j'') = |\vec{p}|/m(j'')$.

Consider now the quantity $P^{(a,b)}_{j\sigma(J)j'\sigma'}(p)$ for the irreducible representation [a,b]:

$$P^{(a,b)}_{j\sigma(J)j'\sigma'}(p) \equiv \sum_{\sigma''} < j\sigma \left| e^{-i\hat{p}\cdot\vec{K}\theta(J)} \right| J\sigma'' > < J\sigma'' \left| e^{-i\hat{p}\cdot\vec{K}\theta(J)} \right| j'\sigma' > .$$

The sum $\sum_{\sigma''} |J\sigma''\rangle\langle J\sigma''|$ gives a projection operator on spin J which can be written as a certain integral over the rotation group. Using this form for the projection operator and the representation of homogeneous Lorentz transformations in terms of two by two complex matrices, it is straightforward to prove that $P^{(a,b)}_{j\sigma(J)j'\sigma'}(p)$ is a polynomial in the components of p_λ [11]. Now we will show that when $p_\lambda \to -p_\lambda$ it picks up a phase $(-1)^{2(a+b)}$. This will enable us to take $P^{(a,b)}_{j\sigma(J)j'\sigma'}(p)$ out of the locality integral and show the commutator becomes a finite number of derivatives on the usual causal function $\Delta(x-y)$ and, therefore, vanishes for x-y spacelike.

To proceed, note that $i\vec{K} = \vec{A} - \vec{B}$ and insert two complete sets of states $|\hat{p}\,\lambda_a\lambda_b\rangle$ where λ_a and λ_b are the eigenvalues of $\vec{A}\cdot\hat{p}$ and $\vec{B}\cdot\hat{p}$. Also observe that to take p_λ into $-p_\lambda$, set $\hat{p}\to-\hat{p}$ and $\theta(J)\to i\pi-\theta(J)$. These two operations result in

$$P^{(a,b)}_{j\sigma(J)j'\sigma'}(-p) = \sum_{\sigma''\lambda_a'\lambda_b'\lambda_a''\lambda_b''} \langle j\sigma| e^{-i\hat{p}\cdot\vec{K}\theta(J)} e^{i\pi(\lambda_a'-\lambda_b')} |\hat{p}\,\lambda_a'\lambda_b'\rangle$$

$$\langle\hat{p}\,\lambda_a'\lambda_b'|J\sigma''\rangle\langle J\sigma''|\hat{p}\,\lambda_a''\lambda_b''\rangle$$

$$\langle\hat{p}\,\lambda_a''\lambda_b''| e^{i\pi(\lambda_a''-\lambda_b'')} e^{-i\hat{p}\cdot\vec{K}\theta(J)} |j'\sigma'\rangle .$$

Since $\lambda_a' + \lambda_b' = \lambda_a'' + \lambda_b''$, $\lambda_a' - \lambda_b' + \lambda_a'' - \lambda_b'' = 2[(\lambda_a' + \lambda_b') - (\lambda_b' + \lambda_b'')]$, and

$$e^{2i\pi[(\lambda_a' + \lambda_b') - (\lambda_b' + \lambda_b'')]} = (-1)^{2(a+b)} ,$$

so one has

$$P_{j\sigma(J)j'\sigma'}(-p) = (-1)^{2(a+b)} P_{j\sigma(J)j'\sigma'}(p) .$$

The commutator of a field transforming as [a,b] and its adjoint can now be written as

$$[\varphi_{j\sigma}^{(a,b)}(x), \varphi_{j'\sigma'}^{(a,b)\dagger}(y)]_\epsilon = \sum_J \frac{F^2(J)}{m^2(J)} \int \frac{d^3p}{2p_0(J)} P_{j\sigma(J)j'\sigma'}^{(a,b)}(p) \times$$
$$\times [e^{-ip\cdot(x-y)} + \epsilon\, e^{ip\cdot(x-y)}]$$
$$= \sum_J \frac{F^2(J)}{m^2(J)} P_{j\sigma(J)j'\sigma'}^{(a,b)}(i\partial) \int \frac{d^3p}{2p_0(J)} \times$$
$$\times [e^{-ip\cdot(x-y)} + \epsilon\, (-1)^{2(a+b)} e^{ip\cdot(x-y)}] .$$

Taking the usual connection between spin and statistics, $\epsilon = -(-1)^{2(a+b)}$, this commutator becomes

$$[\varphi_{j\sigma}^{(a,b)}(x), \varphi_{j'\sigma'}^{(a,b)}(y)]_\epsilon = \sum_J \frac{F^2(J)}{m^2(J)} P_{j\sigma(J)j'\sigma'}^{(a,b)}(i\partial)[i(2\pi)^3 \Delta(x-y;m(J)^2)] ,$$

which vanishes for x-y spacelike and thus establishes locality for the finite dimensional case.

Hence, covariance and locality lead to no constraints on the mass spectrum as a function of spin when the usual connection between spin and statistics is taken and the fields are irreducible finite dimensional Lorentz tensors. Such a field theory may now be used to describe the interactions of a set of particles with an arbitrary mass spectrum.

We shall see in the next section that in the case the fields are irreducible unitary Lorentz tensors strong restrictions on the masses will follow from locality.

IV. UNITARY IRREDUCIBLE REPRESENTATIONS

Having tasted success in our attempts to build local fields which describe a finite number of particles of different mass and spin, we now turn to the unitary irreducible representations of the homogeneous Lorentz group as the transformation law of our quantum fields. These representations contain all spins j_0, $j_0 + 1$, ... greater than some minimum spin j_0, so the "Born approximation" written in terms of such a field would describe an infinite number of spins being exchanged and might then resemble a Regge pole[4]. We have made no secret of the fact that for local fields we find that the masses in such representations are required to be spin independent and, thus, this nice program loses its attractiveness; but now to the demonstration.

Naimark[9] shows that the irreducible representations of the homogeneous Lorentz group are determined by two numbers (j_0, c) with j_0, integer or half-integer, the minimum spin in the representation, and c complex. Each spin $j \geq j_0$ appears once and only once in an irreducible representation. For unitary representations

$$j_0 \neq 0 \, , \quad c = ir, \; r \text{ real} \qquad \text{(Principal Series)}$$

or

$$j_0 = 0 \, , \quad 0 < c^2 < 1 \qquad \text{(Secondary Series).}$$

In any representation (j_0, c) the action of $\vec{K}$ on the basis states $|(j_0, c)jm>$

(called $|jm>$ below) is

$$K_+|jm> = \sqrt{(j-m)(j-m-1)}\; b^j|j-1, m+1> + \sqrt{(j-m)(j+m+1)}\; c^j|j, m+1>$$
$$- \sqrt{(j+m+1)(j+m+2)}\; b^{j+1}|j+1, m+1> ,$$

$$K_-|jm> = -\sqrt{(j+m)(j+m-1)}\; b^j|j-1, m-1>$$
$$+ \sqrt{(j+m)(j-m+1)}\; c^j|j, m-1> + \sqrt{(j-m+1)(j-m+2)}\; b^{j+1}|j+1, m-1> ,$$

and

$$K_3|jm> = \sqrt{j^2 - m^2}\; b^j|j-1, m> + m\, c^j|j, m> + \sqrt{(j+1)^2 - m^2}\; b^{j+1}|j+1, m> ,$$

with

$$c^j = ij_0c/j(j+1) \quad \text{and} \quad b^j = \frac{1}{j}\sqrt{\frac{(j^2-j_0^2)(j^2-c^2)}{(2j+1)(2j-1)}} \quad .$$

In the unitary representations $\vec{J}$ and $\vec{K}$ are hermitian so the operator $P_{j\sigma,j'\sigma'}(\vec{p})$ in the commutator of a field and its adjoint is now

$$P_{j\sigma,j'\sigma'}(\vec{p}) = \sum_{j''\sigma''} \frac{F^2(j'')}{m^2(j'')\sqrt{\vec{p}^2 + m^2(j'')}} < j\sigma|e^{-i\hat{p}\cdot\vec{K}\theta(j'')}|j''\sigma''>< j''\sigma''|e^{i\hat{p}\cdot\vec{K}\theta(j'')}|j'\sigma'>$$

If all the masses were equal and $F^2(j) = F^2$ some constant, $P_{j\sigma,j'\sigma'}(\vec{p})$ becomes $\left(F^2/\left[m^2\sqrt{\vec{p}^2 + m^2}\right]\right)\delta_{jj'}\delta_{\sigma\sigma'}$ and we have local commutation relations only for $\epsilon = -1$, namely Bose statistics. This was observed some time ago by Feldman and Matthews[12]. We first investigate here what is the most general mass spectrum consistent with Bose commutation rules. Besides locality and covariance we make the physical assumption that the mass spectrum is bounded below: $m(j) \geq m_0 > 0$. Under these conditions we will show that the only mass spectrum allowed is that of equal masses!

Consider then the equal-time commutator

$$[\varphi_{j\sigma}(\vec{x},0),\ \varphi^{+}_{j'\sigma'}(\vec{y},0)] = i\int d^3p\ \sin\vec{p}\cdot(\vec{x}-\vec{y})P_{j\sigma,j'\sigma'}(\vec{p}) = \frac{i}{2}\int d^3p\ \sin\vec{p}\cdot(\vec{x}-\vec{y}) \times$$

$$\times\ [P_{j\sigma,j'\sigma'}(\vec{p}) - P_{j\sigma,j'\sigma'}(-\vec{p})]\ .$$

Since this commutator is well-defined only in the sense of a distribution, its vanishing or non-vanishing depends on the space of test functions on which one is allowed to apply it. Within the usual framework of quantum field theory[13] $P_{j\sigma,j'\sigma}(\vec{p}) - P_{j\sigma,j'\sigma'}(-\vec{p})$ must be a polynomial in the components of $\vec{p}$. Jaffe[14] has extended the notion of a local field to that of a <u>strictly localizable field</u> by introducing a cleverer set of test functions than is usually entertained. In this paper we shall restrict ourselves to the usual notion of a local field and return to Jaffe fields in the future.

We are invited then to imagine that the anti-symmetric combination

$$P_{j\sigma,j'\sigma'}(\vec{p}) - P_{j\sigma,j'\sigma'}(-\vec{p}) = \sum_{j''\sigma''} \frac{F^2(j'')}{m^2(j'')\sqrt{\vec{p}^2+m^2(j'')}}$$

$$\Big[< j\sigma\big|e^{-i\hat{p}\cdot\vec{K}\theta(j'')}\big|j''\sigma'' > < j''\sigma''\big|e^{i\hat{p}\cdot\vec{K}\theta(j'')}\big|j'\sigma' > -$$

$$< j\sigma\big|e^{i\hat{p}\cdot\vec{K}\theta(j'')}\big|j''\sigma'' > < j''\sigma''\big|e^{-i\hat{p}\cdot\vec{K}\theta(j'')}\big|j'\sigma' >\Big]$$

is a polynomial in $\vec{p}$ of degree, say, $2N + 1$. The case where the expression vanishes identically is included by $N = -1$, as will be clear in what follows. Now let $\vec{p}$ be infinitesimal and expand the right hand side in a power series in the components of $\vec{p}$. This expansion is certainly allowed for mass spectra bounded below and for $F^2(j)/m^3(j)$ bounded by a polynomial in j, which we henceforth assume. We consider the coefficient of $(\vec{p})^{2N+3+2n}$, which vanishes by assumption for n = 0,1,2,..., and choose $j' = j + 2(N+n) + 3$. To take j to this value of j' one needs at least $2(N+n) + 3$ powers of $\vec{K}$, but then we have $2(N+n) + 3$ powers of $\theta(j'')$ which give for infinitesimal $|\vec{p}|$, $[|\vec{p}|/m(j'')]^{2(N+n)+3}$. Hence, in all other factors we may set $\vec{p} = 0$. Combining all these steps results in

$$\sum_{\substack{j''\sigma'' \\ r}} \frac{F^2(j'')}{m^3(j'')} \left\{ < j\sigma \left| \frac{1}{r!} \left[\frac{i\vec{p}.\vec{K}}{m(j'')}\right]^r \right| j''\sigma'' > < j''\sigma'' \right| \frac{(-1)^{r+1}}{[2(N+n)+3-r]!}$$

$$\left[\frac{i\vec{p}\cdot\vec{K}}{m(j'')}\right]^{2(N+n)+3-r} \left| j+2(N+n)+3 \;\sigma' > - \; (\vec{p} \to -\vec{p}) \right\} = 0 \;,$$

for all n. The vector character of $\vec{K}$ means that for any r, only $j'' = j+r$ gives a non-vanishing contribution in the sum over j'', so we may do the sum and get

$$\left\{ \sum_{r=0}^{2(N+n)+3} \binom{2(N+n)+3}{r} (-1)^r \frac{F^2(j+r)}{[m(j+r)]^{2(N+n+3)}} \right\} \times$$

$$\times < j,\sigma \left| (i\vec{p}\cdot\vec{K})^{2(N+n)+3} \right| j+2(N+n)+3, \; \sigma' > = 0 \;.$$

Inspection of the matrix elements of $\vec{K}$ given above shows that the given matrix element of $\vec{p}\cdot\vec{K}$ does not vanish, therefore, the expression in curly brackets must be zero. This, however, is the $2(N+n)+3^{th}$ difference of the function $F^2(j)/[m(j)]^{2(N+n+3)}$, so that function must be a polynomial in j of order 2(N+n+1)

$$F^2(j)/[m(j)]^{2(N+n+3)} = \sum_{r=0}^{2(N+n+1)} a_r(n) j^r \ ,$$

for all n.

This means

$$m(j)^{2n} = \sum_{r=0}^{2(N+1)} a_r(0) j^r \bigg/ \sum_{r=0}^{2(N+n+1)} a_r(n) j^r \ .$$

Our assumption that $m(j) \geq m_0 > 0$ leads to the requirement $a_r(n) = 0$ for $n \geq 1$ and $r \geq 2(N+3)$; otherwise $m(j)^{2n}$ would go to zero for $j \to \infty$. With this observation we have

$$m(j)^{2n} = \sum_{r=0}^{2(N+1)} a_r(0) j^r \bigg/ \sum_{r=0}^{2(N+1)} a_r(n) j^r = c(n) \prod_{r=1}^{2(N+1)} (j - j_r(0)) \bigg/ \prod_{r=1}^{2(N+1)} (j - j_r(n))$$

which implies for all $n \geq 0$

$$c(1)^n \prod_{r=1}^{2(N+1)} (j - j_r(0))^n \bigg/ \prod_{r=1}^{2(N+1)} (j - j_r(1))^n = c(n) \prod_{r=1}^{2(N+1)} (j - j_r(0)) \bigg/ \prod_{r=1}^{2(N+1)} (j - j_r(n)).$$

From this it follows that

$$j_r(n) = j_r(0) \qquad \text{and} \qquad c(n) = c(1)^n \ ,$$

which immediately leads to $m(j) = m$ and determines $F^2(j)$ to be a polynomial in j of maximum degree 2(N+1)

$$F^2(j) = \sum_{r=0}^{2(N+1)} b_r j^r \ ,$$

which is the announced result.

In the case of Fermi statistics one has to consider the symmetric function $P_{j\sigma,j'\sigma'}(\vec{p}) + P_{j\sigma,j'\sigma'}(-\vec{p})$ and require that it be a polynomial of degree, say, 2N for locality. This again leads to the conclusion that all masses are equal and determines $F^2(j)$ to be a polynomial of degree 2N+1.

In general, then, for a field which transforms as a unitary irreducible representation of the homogeneous Lorentz group to also be a local field it must create only particles (and anti-particles) of equal mass, when the masses are bounded below. The wave function normalization functions $F^2(j)$ are also constrained to be polynomials in j.

It is possible to give explicit examples of this behavior for any N which are generalizations of the examples considered in Ref. 10 for N = -1 for bosons and N = 0 for fermions. The examples are constructed in the Majorana representations[9,15]: $(j_0, c) = (1/2, 0)$ or $(0, 1/2)$. These are the unitary irreducible representations which support a four vector Γ_λ - the Majorana vector -, namely

$$i[M_{\mu\nu}, \Gamma_\lambda] = g_{\mu\lambda}\Gamma_\nu - g_{\nu\lambda}\Gamma_\mu$$

with $J_k = \frac{1}{2}\epsilon_{kij}M_{ij}$ and $K_j = M_{0j}$. The action of Γ_λ on the basis states $|jm>$ is given by

$$\Gamma_0|jm> = (j + 1/2)|jm>,$$

$$\Gamma_+|jm> = \frac{i}{2}\left[\sqrt{(j-m)(j-m-1)}\,|j-1, m+1> + \sqrt{(j+m+1)(j+m+2)}\,|j+1, m+1>\right]$$

$$\Gamma_-|jm> = -\frac{i}{2}\left[\sqrt{(j+m)(j+m-1)}\,|j-1, m-1> + \sqrt{(j-m+1)(j-m+2)}\,|j+1, m-1>\right],$$

and

$$\Gamma_3|jm> = \frac{i}{2}\left[\sqrt{j^2 - m^2}\,|j-1,\ m> - \sqrt{(j+1)^2 - m^2}\,|j+1,\ m>\right] .$$

One may also show that

$$i[\Gamma_\mu, \Gamma_\nu] = M_{\mu\nu} .$$

$$\vec{J}\cdot\vec{\Gamma} = 0 ,$$

$$\vec{\Gamma}^2 = \vec{K}^2 ,$$

and $\Gamma_\mu \Gamma^\mu = \vec{J}^2 - \vec{K}^2 + 1/4 = -1/2$.

Now we choose $F^2(j) = (j + 1/2)^R$. It follows then for $m(J) = m$,

$$P_{j\sigma, j'\sigma'}(\vec{p}) = \sum_{j''\sigma''} \frac{(j'' + 1/2)^R}{m^2\sqrt{\vec{p}^2 + m^2}} < j\sigma|e^{-i\hat{p}\cdot\vec{K}\theta}|j''\sigma'' > < j''\sigma''|e^{i\hat{p}\cdot\vec{K}\theta}|j'\sigma' >$$

$$= \frac{1}{m^2 p_0} < j\sigma|e^{-i\hat{p}\cdot\vec{K}\theta}\ \Gamma_0^R\ e^{i\hat{p}\cdot\vec{K}\theta}|j'\sigma' >$$

$$= \frac{1}{m^2 p_0} < j\sigma|\left(\frac{p^\mu \Gamma_\mu}{m}\right)^R|j'\sigma' > = \frac{1}{p_0 m^{R+2}} < j\sigma|\left(p_0\Gamma_0 - \vec{p}\cdot\vec{\Gamma}\right)^R|j'\sigma' > .$$

For $R = 2(N+1)$, the anti-symmetric part in $\vec{p}$ is a polynomial of degree $2N+1$, and for $R = 2N+1$, the symmetric part is a polynomial of degree $2N$. The former then yields local commutators, and the latter, local anti-commutators. Similar examples can be constructed in unitary representations which support finite dimensional tensors $\Gamma_{\mu_1 \cdots \mu_N}$.

V. A REDUCIBLE REPRESENTATION

Irreducible representations of the homogeneous Lorentz group are interesting because they are the least complex structures in the representation theory of the group. It may happen, however, that certain reducible representations are as interesting because of their usefulness in physics. For finite demensional representations this is the case, for example, with the Dirac representation $[1/2,0] \oplus [0,1/2]$ and the Rarita-Schwinger representation $[1/2, 1/2] \otimes [[1/2,0] \oplus [0,1/2]]$. For infinite dimensional representations it may be that the Dirac $\otimes$ unitary: $[[1/2,0] \oplus [0,1/2]] \otimes (j_0,c)$ representation is relevant to the problems of current algebra[16]. Both for its physical significance, then, and as an example of a "simple" reducible representation we will treat here the case where the fields transform as the direct product of the finite dimensional representation $[1/2,0]$ and the unitary representation (j_0,c). The extension to the case Dirac $\otimes$ unitary $[(j_0,c)]$ or the more general $[[a,b] \oplus [b,a]] \otimes [(j_0,c) \oplus (j_0,-c)]$ - which includes parity - is straightforward. As we shall see again, only a trivial mass spectrum is consistent with locality, covariance, and boundedness of the masses from below.

In general, in the representation $(j_0,c) \otimes [a,b]$, the wave functions and field components may be labeled by the two pairs of indices $(j_1\sigma_1 j_2\sigma_2)$, where $(j_1\sigma_1)$ refers to (j_0,c) and $(j_2\sigma_2)$ to $[a,b]$. The wave function for a particle is characterized by three momentum $\vec{p}$, spin J, spin projection $J_z = \Sigma$, and a parameter ρ which tells how J was made out of j_1 and j_2. For a given J, the number of values of ρ is given by the number of different pairs j_1, j_2 such that $|j_1 - j_2| \leq J \leq j_1 + j_2$. The mass of a particle is now a function of both J and ρ, m(Jρ).

The wave function $\psi_{j_1\sigma_1 j_2\sigma_2}(\vec{p}J\Sigma\rho)$ is given by

$$\psi_{j_1\sigma_1 j_2\sigma_2}(\vec{p}J\Sigma\rho) = \sum_{j_1'\sigma_1' j_2'\sigma_2'} D^{(j_0,c)}_{j_1\sigma_1, j_1'\sigma_1'}[L(\vec{p}/m(J,\rho))] D^{(a,b)}_{j_2\sigma_2, j_2'\sigma_2'}[L(\vec{p}/m(J,\rho))] \times$$

$$\times < j_1'\sigma_1' j_2'\sigma_2' | J\Sigma > < j_1' j_2'(J) | \rho > .$$

$< j_1\sigma_1 j_2\sigma_2 | J\Sigma >$ is the usual Wigner coefficient, and $< j_1 j_2(J) | \rho >$ is a coefficient which tells how to make J from j_1 and j_2 and which we take to form a unitary matrix

$$\sum_{j_1 j_2} < \rho' | j_1 j_2(J) >^* < j_1 j_2(J) | \rho > = \delta_{\rho\rho'} ,$$

and

$$\sum_{\rho} < j_1' j_2'(J) | \rho >^* < \rho | j_1 j_2(J) > = \delta_{j_1 j_1'} \delta_{j_2 j_2'} .$$

This form for the wave function is motivated by analogy with solutions of a Lorentz covariant wave equation, which at rest ($\vec{p} = 0$) reduces to an hermitian operator diagonalizable by a unitary transformation.

The matrix $P(\vec{p})$ appearing in the commutator of one of our fields and its adjoint is now

$$P_{(j_1\sigma_1 j_2\sigma_2),(j_1'\sigma_1' j_2'\sigma_2')}(\vec{p}) = \sum_{\substack{J\Sigma\rho \\ j_1''\sigma_1''j_2''\sigma_2'' \\ \bar{j}_1\bar{\sigma}_1\bar{j}_2\bar{\sigma}_2}} \frac{F^2(J\rho)}{m^2(J\rho)\sqrt{\vec{p}^{\,2} + m^2(J\rho)}} \times$$

$$\times < J\Sigma \left| j_1''\sigma_1''j_2''\sigma_2'' \right> < J\Sigma \left| \bar{j}_1\bar{\sigma}_1\bar{j}_2\bar{\sigma}_2 \right> \times$$

$$\times < \rho \left| j_1''j_2''(J) \right> < \rho \left| \bar{j}_1\bar{j}_2(J) \right>^* \times$$

$$\times < j_1\sigma_1 \left| e^{-i\hat{p}\cdot\vec{K}\theta(J\rho)} \right| j_1''\sigma_1'' > < j_2\sigma_2 \left| e^{-i\hat{p}\cdot\vec{K}\theta(J\rho)} \right| j_2''\sigma_2'' >$$

$$\times < \bar{j}_1\bar{\sigma}_1 \left| e^{i\hat{p}\cdot\vec{K}\theta(J\rho)} \right| j_1'\sigma_1' > < \bar{j}_2\bar{\sigma}_2 \left| e^{-i\hat{p}\cdot\vec{K}\theta(J\rho)} \right| j_2'\sigma_2' > ,$$

where we have noted that $\vec{K}$ is hermitian for (j_0,c) and anti-hermitian for $[a,b]$. $\theta(J\rho)$ is, of course, defined as

$$\theta(J\rho) = \sinh^{-1}[|\vec{p}|/m(J,\rho)] .$$

As promised we deal here with the simplest of our class of reducible representations: $(j_0,c)\otimes[1/2,0]$. Furthermore we present the argument only for Bose commutation relations, since the arguments and the conclusions are similar for Fermi statistics, namely, the masses $m(J,\rho)$ must be independent of J.

Suppose then that $P(\vec{p}) - P(-\vec{p})$ is a polynomial of degree $2N+1$ in the components of $\vec{p}$. As before, we study the coefficient of $(\vec{p})^{2(N+n)+3}$ in the expansion of the boost operators for $n = 0,1,2, \ldots$. Also we choose $j_1' = j_1 + 2(N+n+2)$. Since $|j_1'' - \bar{j}_1| = 0,1$ the only non-zero contributions to the coefficient of $(\vec{p})^{2(N+n)+3}$ come from the terms with $j_1'' = j_1 + r$, $\bar{j}_1 = j_1 + r + 1$ for $0 \leq r \leq 2(N+n)+3$. We are thus lead to the condition

$$\sum_{\substack{J\Sigma\rho \\ \sigma_1''\bar{\sigma}_1}} \sum_{r=0}^{2(N+n)+3} \frac{F^2(J\rho)}{m^3(J\rho)} < J\Sigma \big| j_1+r \;\sigma_1'', \tfrac{1}{2}\;\sigma_2 > < J\Sigma \big| j_1+r+1 \;\bar{\sigma}_1, \tfrac{1}{2}\;\sigma_2' >$$

$$< \rho \big| j_1+r \;\tfrac{1}{2}(J) > < \rho \big| j_1+r+1 \;\tfrac{1}{2}(J) >^* < j_1\sigma_1 \Big| \frac{1}{r!} \left[\frac{i\vec{p}\cdot\vec{K}}{m(J,\rho)}\right]^r \Big| j_1+r \;\sigma_1'' >$$

$$< j_1+r+1 \;\bar{\sigma}_1 \Big| \frac{(-1)^{r+1}}{[2(N+n)+3-r]!} \left[\frac{i\vec{p}\cdot\vec{K}}{m(J,\rho)}\right]^{2(N+n)+3-r} \Big| j_1+2(N+n+2) \;\sigma_1' > = 0 \;.$$

To proceed we choose $\vec{p} = p\,\hat{e}_z$ and $\sigma_2' = \sigma_2$. This means $\sigma_1 = \sigma_1'$, or the expression vanishes identically. Observing that

$$< j_1+r \;\sigma_1 \big| K_3 \big| j_1+r+1 \;\sigma_1 > = \sqrt{(j_1+r+1)^2 - \sigma_1^2} \;\; b^{j_1+r+1}$$

and that

$$< j_1+r \;\sigma_1, \tfrac{1}{2}\;\sigma_2 \big| j_1+r+\tfrac{1}{2}, \;\sigma_1+\sigma_2 > < j_1+r+\tfrac{1}{2}, \;\sigma_1+\sigma_2 \big| j_1+r+1 \;\sigma_1, \tfrac{1}{2}\;\sigma_2 > =$$

$$(-1)^{\frac{1}{2}-\sigma_2} \sqrt{[(j_1+r+1)^2 - \sigma_1^2]/[4(j_1+r+1)^2 - 1]} \;\;,$$

we can cast our requirement for locality into

$$\sum_{\rho} \sum_{r=0}^{2(N+n)+3} \frac{(-1)^r}{r![2(N+n)+3-r]!} \; \frac{G(j_1+r+\frac{1}{2},\rho)}{[m(J\rho)]^{2n}} = 0 \;,$$

by defining

$$G(j+\tfrac{1}{2},\rho) \equiv F^2(j+\tfrac{1}{2},\rho) < \rho|j\ \tfrac{1}{2}\ (j+\tfrac{1}{2}) >< \rho|j+1\ \tfrac{1}{2}\ (j+\tfrac{1}{2}) >^*$$

$$\frac{j+1}{\sqrt{[(j+1)^2 - j_0^2][(j+1)^2 - c^2]}} \times \frac{1}{[m(j+\tfrac{1}{2},\rho)]^{2(N+3)}} ,$$

which we suppose is non-vanishing. This, of course, means that $\sum_{\rho} G(J,\rho)/[m(J,\rho)]^{2n}$ is a polynomial in J of maximum degree 2(N+n+1):

$$\sum_{\rho=1}^{2} G(J,\rho)/[m(J,\rho)]^{2n} = \sum_{r=0}^{2(N+n+1)} b_r(N+n)J^r .$$

Some straightforward manipulations show that in order to implement the boundedness assumption on the masses $m(J,\rho) \geq m_0 > 0$, one requires

$$b_r(n) = 0 \quad \text{for} \quad n \geq 2,\ r \geq 2(N+2) + 1.$$

This means

$$\sum_{\rho=1}^{2} G(J,\rho)/[m(J,\rho)]^{2n} = \sum_{r=0}^{2(N+2)} b_r(N+n)J^r \equiv P_n(J);\ [b_{2N+3}(N) = b_{2N+1}(N) = 0] .$$

Now, by examining the expression for $G(J,\rho)$, which is independent of n, one may solve for $[m(J,1)m(J,2)]^{-2n}$ and $m(J,1)^{-2n} + m(j,2)^{-2n}$ with the results

$$[m(J,1)m(J,2)]^{-2n} = \frac{P_{3n}(J)P_n(J) - P_{2n}^2(J)}{P_{2n}(J)P_0(J) - P_n^2(J)} ,$$

and

$$m(J,1)^{-2n} + m(J,1)^{-2n} = \frac{P_{3n}(J)P_0(J) - P_{2n}(J)P_n(J)}{P_{2n}(J)P_0(J) - P_n^2(J)} .$$

Thus the masses may be written as

$$m^{2n}(J,1) = \frac{A_n(J) + \sqrt{B_n(J)}}{C_n(J)} ,$$

and

$$m^{2n}(J,2) = \frac{A_n(J) - \sqrt{B_n(J)}}{C_n(J)} ,$$

where $A_n(J)$ and $C_n(J)$ are polynomials of degree $4(N+2)$ at most, and $B_n(J)$, of degree $8(N+2)$ at most. This form for the masses implies

$$\left[\frac{A_1(J) + B_1(J)}{C_1(J)}\right]^n = \frac{A_n(J) + \sqrt{B_n(J)}}{C_n(J)} ,$$

for all $n = 0,1,2, \ldots$. If there is a zero of $C_1(J)$ which is not a zero of $A_1(J) + \sqrt{B_1(J)}$, we cannot have this equation, since this zero is raised, on the left hand side of the last expression, to an arbitrary power n, while the degree of the polynomials on the right is independent of n. Thus, every zero of $C_1(J)$ is a zero of $A_1(J) + \sqrt{B_1(J)}$, which implies they are proportional, with a J-independent proportionality constant. This means the masses are independent of J:

$$m(J,1) = C_1 ,$$

and

$$m(J,2) = C_2 ,$$

whenever the $G(J,\rho)$ are non-zero. The $G(J,\rho)$ themselves are also found to be polynomials in J of maximum degree $2(N+2)$.

If $G(J,\rho)$ turns out to be zero, this implies that

$$< \rho \big| j \tfrac{1}{2} (j + \tfrac{1}{2}) > < \rho \big| j + 1 \tfrac{1}{2} (j + \tfrac{1}{2}) >^* = 0 ,$$

or that the two ways of making $j + \frac{1}{2}$ - from j and $\frac{1}{2}$ or from $j + 1$ and $\frac{1}{2}$ - do not mix. Again one examines the coefficient of $(\vec{p})^{2N+3+2n}$ in the matrix $P_{(j_1\sigma_1 j_2\sigma_2)(j_1'\sigma_1' j_2'\sigma_2')}(\vec{p})$ taking $\sigma_1 = \sigma_1'$, $\sigma_2 = \sigma_2'$, $\vec{p} = p\,\hat{e}_z$ and $j_1' = j_1 + 2(N+n) + 3$, and this time finds that the $F^2(J,\rho)/[m(J,\rho)]^{2(N+n)+3}$ for both $\rho = 1$ and 2 are polynomials in J of the same degree. Familiar arguments now lead us to conclude that even in this case the $m(J,\rho)$ are independent of J. $F^2(J,\rho)$ turn out to be polynomials of degree $2(N+1)$ at most.

Once again we have found that the requirements of covariance, locality and boundedness of the mass spectrum from below are severe enough, in the case where the quantum field transforms as $[1/2,0] \otimes (j_0,c)$, to imply a trivial mass spectrum, that is, masses independent of the spin J. We conjecture that this result holds for the more general case $[a,b] \otimes (j_0,c)$ also.

VI. SUMMARY AND OBSERVATIONS

We have, following the lead of the authors in Ref. 5, constructed quantum fields with well-defined transformation properties under the Lorentz group out of the annihilation and creation operators for physical states. In the case where the field is a finite dimensional irreducible Lorentz tensor, we found that locality placed no restrictions on the masses, considered as a function of spin, of the particles described by the field. However, when we chose the field to transform as a unitary irreducible representation (j_0, c) or a direct product $[1/2, 0] \otimes (j_0, c)$ and made the physical assumption that the masses were bounded below, $m(J, \rho) \geq m_0 > 0$, we were led to the conclusion that locality of the fields required that all the masses be independent of spin. This conclusion means that in a local field theory one can describe an infinite number of particles of spin j_0, $j_0 + 1$, $j_0 + 2$, ... by an irreducible unitary Lorentz tensor or a Lorentz tensor of the type $[1/2, 0] \otimes (j_0, c)$ only in the physically uninteresting case where all those particles have the same mass. Such a field clearly has little to do with a Regge trajectory.

It behooves us to inquire whether there is some way in which we can avoid this last conclusion, since the idea of describing an infinite number of particles by a single quantum field is not only attractive but may be imperative if field theory and hadron physics are to have anything to do with one another. There are at least two possible alternate paths which might lead us out of the limbo of equal masses; each requires an enlargement of our notion of a quantum field: (1) perhaps our requirement that the field be local is too restrictive. It might well be that allowing it to be one of Jaffe's strictly localizable fields[14] would give us sufficient extra freedom to have a physical mass spectrum again. (2) We have constructed our fields as linear combinations of (Fourier transforms of) particle creation and destruction operators. In so doing we have included only timelike momenta in the Fourier expansion of the field. Experience with infinite component wave equations[17], especially of the variety where the wave function transforms as a unitary irreducible representation (j_0, c) or as Dirac $\otimes (j_0, c)$, shows that solutions of these equations with spacelike frequencies are a general occurrence. Our conclusions may demonstrate simply that we have been in error in omitting such Fourier components in the construction of our fields. These and other ways out are the subjects of future research.

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APPENDIX

The problem of local commutation rules is related to the question of completeness of the timelike solutions to some wave equation. To be more precise, suppose we are considering a covariant wave equation whose solutions transform as an irreducible unitary representation of the homogeneous Lorentz group. In momentum space the wave functions, for timelike momenta, are the $D_{j\sigma,j'\sigma'}[L(\vec{p}/m(j'))]$. Imagine that the wave functions with timelike p span the whole Hilbert space and that a scalar product exists such that

$$\sum_{\substack{j_1\sigma_1 \\ j_2\sigma_2}} D^*_{j_1\sigma_1,j\sigma}[L(\vec{p}/m(j))]\ A_{j_1\sigma_1 j_2\sigma_2} D_{j_2\sigma_2,j'\sigma'}[L(\vec{p}/m(j'))]$$

$$= \delta_{jj'}\delta_{\sigma\sigma'}\sqrt{\vec{p}^2 + m^2(j)}[m^2(j)/F^2(j)],$$

with A the constant metric matrix. Then one finds

$$\sum_{j_1\sigma_1 j_2\sigma_2} P_{j\sigma,j_1\sigma_1}(\vec{p})A_{j_1\sigma_1,j_\sigma\sigma_2} D_{j_2\sigma_2,j'\sigma'}[L(\vec{p}/m(j'))] = D_{j\sigma,j'\sigma'}[L(\vec{p}/m(j'))]$$

and hence

$$\sum_{j_1\sigma_1} P_{j\sigma,j_1\sigma_1}(\vec{p})A_{j_1\sigma_1,j'\sigma'} = \delta_{jj'}\delta_{\sigma\sigma'}\ ,$$

because of the completeness of the wave functions with timelike p. This implies

$$P_{j\sigma,j'\sigma'}(\vec{p}) = (A^{-1})_{j\sigma,j'\sigma'}$$

and guarantees local commutation rules for Bose statistics and local anti-commutation rules for Fermions. Now we will demonstrate that such a scalar product is possible for unequal masses <u>only</u> in the Majorana representations, only with $A = \Gamma_0/\kappa$, only with a mass spectrum of the form $m(j) = \kappa/j + \frac{1}{2}$, and only with $F(j) = m^2(j)$. It is ironic that this is the best known case where there is a wave equations where the wave functions of timelike momenta are not complete[15], namely, the Majorana equation $(-i\partial^{\alpha}\Gamma_{\alpha} + \kappa)\psi(x) = 0$.

We address ourselves, then, to the question whether it is possible to have an orthogonality relation of the form

$$\sum_{\substack{j_1\sigma_1 \\ j_2\sigma_2}} D^*_{j_1\sigma_1, j\sigma}[L(\vec{p}/m(j))]A_{j_1\sigma_1 j_2\sigma_2}D_{j_2\sigma_2, j'\sigma'}[L(\vec{p}/m(j'))] = B(\vec{p},j)\delta_{jj'}\delta_{\sigma\sigma'}$$

in a unitary irreducible representation (j_0, c). First, note that it is immediate to obtain orthogonality for the case of equal world velocities, since if $\vec{p}/m(j) = \vec{p}\,'/m(j')$, then

$$D^*_{j_1\sigma_1, j\sigma}[L(\vec{p}/m(j))]D_{j_1\sigma_1, j'\sigma'}[L(\vec{p}\,'/m(j'))] = \delta_{jj'}\delta_{\sigma\sigma'} ,$$

because of the unitarity of the representation. This is actually very natural since the solutions of whatever wave equation one has in mind are taken to be orthogonal for distinct j at rest. The orthogonality at equal world velocity is then simply a statement of the Lorentz covariance of the equations since the boost operation takes one to a system of new velocity, not momentum. However, for our considerations regarding the properties of $P_{j\sigma,j'\sigma'}(\vec{p})$ we need the orthogonality for equal $\vec{p}$. This arose since the conjugate variable to space-time, in which we inquire about locality, is p = mu, not u.

By taking $\vec{p} = 0$ the orthogonality relation under examination reads

$$A_{j\sigma j'\sigma'} = B(0,j)\delta_{jj'}\delta_{\sigma\sigma'} \quad ,$$

hence with $B(0,j) \equiv B(j)$ it becomes at momentum p:

$$\sum_{j''\sigma''} D^{*}_{j''\sigma'',j\sigma}[L(\vec{p}/m(j))]B(j'')D_{j''\sigma'',j'\sigma'}[L(\vec{p}/m(j'))] = B(|\vec{p}|,j)\delta_{jj'}\delta_{\sigma\sigma'} \ .$$

The right hand side is a function of $|\vec{p}|$ only because of rotational invariance of the orthogonality relation.

One now expands the left hand side of this relation and examines it order by order in $\vec{p}$. This expansion does not require a boundedness assumption on m(j) since the j's in the arguments of the boost matrices are not summed over and may be fixed at an arbitrary finite value. The first order in $\vec{p}$ tells us

$$B(j) < j\sigma\left|\frac{i\vec{p}\cdot\vec{K}}{m(j')}\right|j'\sigma' > - B(j') < j\sigma\left|\frac{i\vec{p}\cdot\vec{K}}{m(j)}\right|j'\sigma' > = 0 \ ,$$

so

$$B(j)/B(j + 1) = m(j + 1)/m(j) \ ,$$

and implies we may choose

$$B(j) = 1/m(j)$$

by fixing an arbitrary scale factor to be one.

Second order in $\vec{p}$ is more interesting and has the form

$$\frac{1}{2} < j\sigma \left| (i\vec{p}\cdot\vec{K})^2 \right| j'\sigma' > \left[\frac{1}{m^2(j)m(j')} + \frac{1}{m^2(j')m(j)} \right]$$

$$- \sum_{j''\sigma''} < j\sigma \left| i\vec{p}\cdot\vec{K} \right| j''\sigma'' > < j''\sigma'' \left| i\vec{p}\cdot\vec{K} \right| j'\sigma' > \frac{1}{m(j)m(j'')m(j')} =$$

$$\frac{1}{2} |\vec{p}|^2 \left[\frac{d^2}{d|\vec{p}|^2} B(|\vec{p}|, j) \right]_{|\vec{p}|=0} \delta_{jj'} \delta_{\sigma\sigma'} .$$

For $j' = j + 2$ we obtain

$$\frac{1}{m(j)} + \frac{1}{m(j+2)} = \frac{2}{m(j+1)}$$

which implies $m(j) = 1/(a+bj)$. For $j' = j + 1$ we get

$$\frac{1}{2} < j\sigma \left| (i\vec{p}\cdot\vec{K})^2 \right| j{+}1\ \sigma' > [2a + b + 2bj]$$

$$= \sum_{\sigma''} < j\sigma \left| i\vec{p}\cdot\vec{K} \right| j{+}1\ \sigma'' > < j{+}1\ \sigma'' \left| i\vec{p}\cdot\vec{K} \right| j{+}1\ \sigma' > [a+bj+b]$$

$$+ \sum_{\sigma''} < j\sigma \left| i\vec{p}\cdot\vec{K} \right| j\sigma'' > < j\sigma'' \left| i\vec{p}\cdot\vec{K} \right| j{+}1\ \sigma' > [a + bj]$$

$$= (a + bj) < j\sigma \left| (i\vec{p}\cdot\vec{K})^2 \right| j{+}1\ \sigma' >$$

$$+ \sum_{\sigma''} b < j\sigma \left| (i\vec{p}\cdot\vec{K}) \right| j{+}1\ \sigma'' > < j{+}1\ \sigma'' \left| i\vec{p}\cdot\vec{K} \right| j{+}1\ \sigma' > ,$$

or if $b \neq 0$ [18]

$$< j\sigma \left| (i\vec{p}\cdot\vec{K})^2 \right| j+1\ \sigma' > = 2 \sum_{\sigma''} < j\sigma \left| i\vec{p}\cdot\vec{K} \right| j+1\ \sigma'' > < j+1\ \sigma'' \left| i\vec{p}\cdot\vec{K} \right| j+1\ \sigma' > ,$$

which in turn leads to

$$\sum_{\sigma''} < j\sigma \left| i\vec{p}\cdot\vec{K} \right| j\sigma'' > < j\sigma'' \left| i\vec{p}\cdot\vec{K} \right| j+1\ \sigma' >$$

$$= \sum_{\sigma''} < j\sigma \left| i\vec{p}\cdot\vec{K} \right| j+1\ \sigma'' > < j+1\ \sigma'' \left| i\vec{p}\cdot\vec{K} \right| j+1\ \sigma' > .$$

Take $\vec{p} = p\ e_z$, which requires $\sigma' = \sigma$ and

$$< j\sigma \left| K_3 \right| j\sigma > = \sigma C^j = < j+1\ \sigma \left| K_3 \right| j+1\ \sigma > = \sigma C^{j+1} ,$$

which is satisfied only if $C^j = 0$, or $j_0 c = 0$.

Finally choose $j' = j$ in the second order of expansion

$$1/m^2(j) \Big\{ < j\sigma \left| (i\vec{p}\cdot\vec{K})^2 \right| j\sigma' > \frac{1}{m(j)}$$

$$- \sum_{\sigma''} < j\sigma \left| i\vec{p}\cdot\vec{K} \right| j-1\ \sigma'' > < j-1\ \sigma'' \left| i\vec{p}\cdot\vec{K} \right| j\sigma' > \frac{1}{m(j-1)}$$

$$- \sum_{\sigma''} < j\sigma \left| i\vec{p}\cdot\vec{K} \right| j+1\ \sigma'' > < j+1\ \sigma'' \left| i\vec{p}\cdot\vec{K} \right| j\sigma' > \frac{1}{m(j+1)} \Big\}$$

$$= \frac{1}{2} |\vec{p}|^2 \left[\frac{d^2}{d|\vec{p}|^2} B(|\vec{p}|, j) \right]_{|\vec{p}| = 0} \delta_{\sigma\sigma'}$$

remembering that since $j_0 c = 0$, $\vec{K}$ does not have matrix elements between states of the same j. This is equivalent to

$$\frac{b}{m^2(j)}\left\{ \sum_{\sigma''} < j\sigma\left|i\vec{p}\cdot\vec{K}\right|j-1\ \sigma'' > < j-1\ \sigma''\left|i\vec{p}\cdot\vec{K}\right|j\sigma' > \right.$$

$$\left. - \sum_{\sigma''} < j\sigma\left|i\vec{p}\cdot\vec{K}\right|j+1\ \sigma'' > < j+1\ \sigma''\left|i\vec{p}\cdot\vec{K}\right|j\sigma' > \right\}$$

$$= \frac{|\vec{p}|^2}{2} B''(\vec{p} = 0, j)\delta_{\sigma\sigma'} .$$

Again take $\vec{p} = p\ e_z$, and choose $\sigma = \sigma'$, then we find that $\left|< j\sigma\left|K_3\right|j-1\,\sigma >\right|^2 - \left|< j\sigma\left|K_3\right|j+1\,\sigma >\right|^2$ is independent of σ, which is satisfied only when $j_0c = 0$ and $j_0^2 + c^2 = 1/4$. This implies that only the Majorana representations are allowed.

Armed with the knowledge that we need only consider the Majorana representations, we return to the orthogonality relation

$$\sum_{j''\sigma''} D^*_{j''\sigma'',j\sigma}[L(\vec{p}/m(j))](a+bj'')D_{j''\sigma'',j'\sigma'}[L(\vec{p}/m(j'))] = B(|\vec{p}|,j)\delta_{jj'}\delta_{\sigma\sigma'} ,$$

and use our knowledge of the Majorana vector Γ_α. Since

$$\Gamma_0 = e^{i\hat{p}\cdot\vec{K}\theta(J)}[\cosh\ \theta(J)\Gamma_0 + \hat{p}\cdot\vec{\Gamma}\ \sinh\ \theta(J)]\ e^{-i\hat{p}\cdot\vec{K}\theta(J)} ,$$

we may write

$$< j\sigma\left|e^{i\hat{p}\cdot\vec{K}\theta(j)}[\cosh\ \theta(j)\Gamma_0 + \hat{p}\cdot\vec{\Gamma}\ \sinh\ \theta(j)]\ e^{-i\hat{p}\cdot\vec{K}\theta(j')}\right|j'\sigma' >$$

$$= (j + \tfrac{1}{2}) < j\sigma\left|e^{-i\hat{p}\cdot\vec{K}(\theta(j')-\theta(j))}\right|j'\sigma' > .$$

Interchanging j and j' in the argument of $\theta(j)$, we can quickly find

$$[m(j)\cosh\,\theta(j)-m(j')\cosh\,\theta(j')] < j\sigma \left| e^{i\hat{p}\cdot\vec{K}\theta(j)}\Gamma_0 e^{-i\hat{p}\cdot\vec{K}\theta(j')}\right| j'\sigma' >$$

$$= [m(j)(j+\tfrac{1}{2})-m(j')(j'+\tfrac{1}{2})] < j\sigma \left| e^{-i\hat{p}\cdot\vec{K}[\theta(j')-\theta(j)]}\right| j'\sigma' > .$$

The scalar product for $j \neq j'$ can now be set in the form

$$\sum_{j''\sigma''} < j\sigma \left| e^{+i\hat{p}\cdot\vec{K}\theta(j)}\right| j''\sigma'' > [(a-\tfrac{b}{2}) + b(j''+\tfrac{1}{2})] < j''\sigma'' \left| e^{-i\hat{p}\cdot\vec{K}\theta(j')}\right| j'\sigma' >$$

$$= (a-\tfrac{b}{2}) < j\sigma \left| e^{-i\hat{p}\cdot\vec{K}(\theta(j')-\theta(j))}\right| j'\sigma' >$$

$$+ b < j\sigma \left| e^{i\hat{p}\cdot\vec{K}\theta(j)}\Gamma_0 e^{-i\hat{p}\cdot\vec{K}\theta(j')}\right| j'\sigma' > = 0 ,$$

which implies $a = b/2$, or

$$m(j) = \kappa/(j+\tfrac{1}{2})$$

For $j = j'$, the scalar product can be explicitly evaluated

$$\frac{1}{\kappa} < j\sigma \left| e^{i\hat{p}\cdot\vec{K}\theta(j)}\Gamma_0 e^{-i\hat{p}\cdot\vec{K}\theta(j)}\right| j\ \sigma' > = \frac{1}{\kappa} < j\sigma \left| \cosh\,\theta(j)\Gamma_0 - \sinh\,\theta(j)\hat{p}\cdot\vec{\Gamma}\right| j\sigma' >$$

$$= \left[\sqrt{\vec{p}^{\,2} + m^2(j)}\Big/ m^2(j)\right]\delta_{\sigma\sigma'} .$$

This is what we set out to prove.

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