STACKY FORMULATIONS OF EINSTEIN GRAVITY

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ABSTRACT

STACKY FORMULATIONS OF EINSTEIN GRAVITY

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This is a thesis on higher structures in geometry and physics. Indeed, the current work involves an extensive and relatively self-contained investigation of higher categorical and *stacky* structures in (vacuum) Einstein gravity with vanishing cosmological constant. In the first three chapters of the thesis, we shall provide a realization of the moduli space of Einstein's field equations as a certain higher space *(a stack)*. In this part of the thesis, the first aim is to present the construction of *the moduli stack of vacuum Einstein gravity with vanishing cosmological constant* in an n-dimensional setup. In particular, we shall be interested in the moduli space of 3D Einstein gravity on specific Lorentzian spacetimes. With this spirit, the second goal of this part is to show that once it exists, the equivalence of 3D quantum gravity with gauge theory in a particular setup, in fact, induces *an isomorphism between the corresponding moduli stacks* where the setup involves Lorentzian spacetimes of the form $\Sigma \times \mathbb{R}$ with Σ being a closed Riemann surface of genus $g > 1$. For our purposes, we shall employ a particular treatment that is essentially based on a formulation of stacks in the language of homotopy theory. The remainder of the thesis, on the other hand, is designed as a detailed survey on formal moduli problems, and it is particularly devoted to formalizing specific Einstein gravities in the language of formal moduli problems and $\mathcal{L}_\infty$-algebras. Such an approach allows us to encode further higher structures in the theory if needed. To be more precise, this leads to the realization of the space of fields as a certain higher/derived stack *(a formal moduli problem)* endowed with more sensitive higher structure (encoding the possible higher symmetries/equivalences in the theory) once we ask the theory to possess higher symmetries. As a particular example, we use this approach to formulate specific 3D Einstein-Cartan-Palatini gravity. In addition, using local models for such higher structures and the algebra of functions on these higher spaces, we intend to study the algebraic structure of observables of 3D Einstein-Cartan-Palatini gravity as well.

Keywords: Derived/homotopical algebraic geometry, category theory, higher structures, higher spaces, stacks, derived stacks, formal moduli problems, classical/quantum Einstein gravity in 3D.

ÖZ

EİNSTEİN GRAVİTASYON KURAMININ STAKSAL FORMÜLASYONLARI

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Bu tez, geometri ve fizikte ortaya çıkan çeşitli yüksek geometrik ve cebirsel yapıları ele almaktadır. Özel olarak, bu mevcut araştırma, vakum Einstein gravitasyon kuramında (kozmolojik sabit sıfır alınmaktadır) ortaya çıkan yüksek kategori teorisel ve "$stacky$" yapıları araştırarak, bu yapılar yardımıyla ortaya çıkan alternatif formülasyonları inceleyen bir çalışmalar bütünüdür. Bu tezin ilk üç bölümünde temel olarak, Einstein denklemlerinin moduli uzayının, özel bir yüksek uzay *(stak)* olarak nasıl yorumlanabileceği tartışılmaktadır. Stak dilini kullanan bir takım çalışmalardan yola çıkarak, bu tezde Einstein denklemlerinin moduli uzayı için benzer sonuçlar gösterilmiştir. Bu bağlamda, ilk olarak, bahsi geçen Einstein gravitasyonu için n boyutlu durumdaki ilgili moduli stakın inşası verilecektir. Bununla birlikte, $n = 3$ durumu özel olarak tartışılacaktır. Bu yeni formülasyonla birlikte, özellikle üç boyutlu kuantum gravitasyonu ve ayar kuramı arasındaki özel bir durumdaki denkliğin, aslında ilgili kuramlar için inşa edilen çözüm uzaylarının *stak olarak izomorfik* olmalarına yol açtığı gösterilmektedir. Buradaki özel durum, $\Sigma \times \mathbb{R}$ tipinde Lorentz uzayzamanlarını

kapsamaktadır (Σ, genus $g > 1$ kapalı Riemann yüzeyidir). Öte yandan tezin geri kalan bölümleri, formal moduli problemleri ve $\mathcal{L}_\infty$-cebirleri üzerine detaylı bir literatür taraması içerecek şeklinde tasarlanmış olup, buradaki bölümlerde bu kavramların Einstein kuramı ile ilişkisi incelenmektedir. Bu yaklaşım sayesinde kuramların çözüm uzayları, kuramlarda ortaya çıkabilecek muhtemel yüksek simetrileri/denklikleri tespit etme konusunda daha hassas yüksek yapılara sahip olan, bir takım özel "$derived$" uzaylar *(formal moduli problemleri)* şeklinde görülebilmektedir. Tezin son bölümlerinde bu yaklaşımın özel bir durumu ve örneği olarak, 3D Einstein-Cartan-Palatini gravitasyon kuramının bu nesneler aracılığıyla formülasyonu üzerinde durulmuştur. Ayrıca bu bölümde yüksek uzayların yerel modelleri ve bu nesneler üzerindeki fonksiyonlar cebiri incelenerek, 3D Einstein-Cartan-Palatini gravitasyon kuramındaki *gözlenebilirlerin* cebirsel yapısı da çalışılmaktadır.

Anahtar Kelimeler: "$Derived$"/ homotopik cebirsel geometri, kategori teorisi, yüksek geometrik ve cebirsel yapılar, yüksek uzaylar, üç boyutlu klasik/kuantum Einstein gravitasyon kuramı, staklar, "$derived$" staklar, formal moduli problemleri.

To my father & mother

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Having enough formalities, now I would like to talk about more on my graduate journey with Özgür and Bayram hocas: Instead of a traditional mentor-apprentice relation in the old Jedi Order, they always treat me as their colleague, family, and friend. They always encourage me to lead the way during my graduate studies. Indeed, they always let me take the stage and rule the show. They always guide me and share their experiences whenever I need them, independent of the subject, but never restrict me in a specific direction. They always give me enough freedom all along the way. I learned uncountably many different aspects of mathematics and physics from them. Whenever I ask any question, they always try to help me as much as possible, and they share their insights, knowledge, and suggestions. Sometimes we learned together, and there were indeed times we taught together, which was an invaluable experience for me. As a person, I have learned a lot from them: not just how an exemplary academician is supposed to be, but how a good person, a family, and a friend are supposed to be. I will always be thankful to them for giving me such an opportunity to join their alliance... May the force always be with them! Doing graduate research with Özgür Kişisel and Bayram Tekin was a lifetime experience for me. Of course, this chapter of my life must come to an end sooner or later, but I believe this is just the beginning of lifetime friendships.

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Long story short, all of these efforts lead to the preparation of this document. There are many other untold stories and people to thank. Outside METU, I have benefited from the friendship and conversations of many mathematicians and physicists; I especially thank all my friends from Boğaziçi, Bilkent, and Koç Universities. It seems that it is unfortunately impossible to list them all here, so I would like to apologize in advance for any inconvenience. Please let me cross my fingers and thank all friends, family, loved ones, and the warm and supportive community of the Mathematics and Physics Departments at METU, and anyone who has supported my journey and be with me all along the way.

Last but not least, let me mention that stress is an inevitable element of any graduate study. The life of a graduate student is sometimes like riding a roller coaster. The stress may fill the air with despair. This may lead to fear, which is the path to the Dark Side. I want to thank all people in my life who give me the strength, love, and will to keep myself away from the Dark Side. With all gratitude, I thank my mother for her unconditional love and support. Without her effords and dedication, none of this would have been possible and meaningful. Also, I thank my wife, Melis, for her unlimited passion, encouragement, joy, and support all along the way. Her presence in my life has allowed me to sustain such strength and give me “a new hope”. Finally, I will always be grateful to Zekayi & Serpil Hazır for their love, guidance, and support. They had always treated me like family when I needed them the most. I have always seen Zekayi Hazır as my intellectual role model and mentor. He is a physicist, and I learned a lot from him. I have always admired his way of teaching, thinking and his unique sense of humor. Remembering his quirky jokes and joyful gestures always makes me smile every time. He will always be missed.

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CHAPTER 1

INTRODUCTION

1.1 Motivation and the Setup

It is an undoubted fact that moduli theory plays a significant role in analyzing field theories. Throughout the discussion, we shall focus on the classical side of the story, indeed. The reason why one may prefer to adopt a moduli-theoretic approach is relatively simple: A classical field theory can be described by a piece of data (M, F_M, S, G) where F_M denotes the space of fields on some base manifold M, $\mathcal{S}$ is a smooth action functional on F_M, and G is a certain group encoding the symmetries of the system. Then the standard folklore suggests that the key information about the system is encoded in *the critical locus* $crit(\mathcal{S})$ of $\mathcal{S}$, modulo symmetries. Therefore, the problem of interest boils down to the analysis of the properties of this moduli space. Of course, moduli theory has some natural questions related to "bad quotients" or "bad intersections". There are some classical techniques to deal with these sorts of problems, but nowadays some people prefer to use relatively new technology, namely *derived algebraic geometry* (DAG) [1, 2]. DAG combines certain higher categorical objects and homotopy theory with many tools from homological algebra. Hence, it can be considered as a higher categorical/homotopy theoretical refinement of classical algebraic geometry. Consequently, DAG suggests new and alternative perspectives in physics as well. In that respect, the formulation of certain gauge theories in the language of derived algebraic geometry, developed by Costello and Gwilliam [3, 4], provides new and fruitful insights to encode the formal geometry of the associated moduli space of the theory. Adopting such an approach, we would like to present a similar type of analysis in the case of Einstein gravity.

Before explaining the *derived* formulation of a classical field theory in a general set-up [3], we shall first briefly recall how to define a classical field theory in Lagrangian formalism [24]:

Definition 1.1.1. *A classical field theory* on a manifold M consists of the following data:

(i) The space F_M of *fields* of the theory, which is defined to be the space $\Gamma(M, \mathcal{F})$ of sections of a particular *sheaf* $\mathcal{F}$ on M,

(ii) The action functional $\mathcal{S} : F_M \longrightarrow k$ ($\mathbb{R}$ *or* $\mathbb{C}$) that captures the behavior of the system under consideration.

Furthermore, if we want to describe a quantum system, as a third component, we need to employ the path integral formalism. This part, however, is beyond of the scope of the current discussion. Instead, we refer to [24, 47, 51].

Remark 1.1.1. As briefly mentioned above, to encode the dynamics of the system in a well-established manner, we need to study *the critical locus* $crit(\mathcal{S})$ of $\mathcal{S}$. One can determine $crit(\mathcal{S})$ by employing variational techniques for the functional $\mathcal{S}$. This leads to defining $crit(\mathcal{S})$ to be the space of solutions to the Euler-Lagrange equations modulo symmetries. Therefore, a classical field theory can be thought of as *a study of the moduli space* $\mathcal{EL}$ *of solutions to E-L equations (the equations of motion).*

In the language of derived algebraic geometry, on the other hand, we would like to define the notion of a classical field theory in the following way [3]:

Definition 1.1.2. Let F_M be the space of fields for some base manifold M, and $\mathcal{S} : F_M \to k$ a smooth action functional as above. ***A (perturbative) classical field theory*** is a sheaf of *derived stacks* of solutions to the equations of motion on M equipped with a symplectic form of cohomological degree -1.

For a complete discussion of all concepts mentioned in the Definition 1.1.2, see Appendix of [8] or Chapter 3 of [3]. For a brief introduction to the notion of a derived stack, see [17]. In general, we work in the context of Toën and Vezzosi's DAG (or Homotopical Algebraic Geometry [2]), and T. Pantev, B. Toën, M. Vaquié, and G.

Vezzosi's theory of *shifted symplectic geometry*[20]. An accessible overview of the basics of this structure can be found in [21] or [22]. As stressed in [3], one can *unpackage* the Definition 1.1.2 as follows:

i. Describing a classical field theory sheaf theoretically [24] boils down to the study of the moduli space $\mathcal{EL}$ of solutions to the Euler-Lagrange equations *(and hence the critical locus of the action functional)* which is in fact encoded by a certain moduli functor.

ii. As stressed in [25], a moduli functor, however, would not be representable in a generic situation due to certain problems, such as the existence of degenerate critical points or non-freeness of the action of the symmetry group on the space of fields [23]. In order to avoid problems of this kind (and to capture the perturbative behavior at the same time), one may adopt the language of derived algebraic geometry. Hence, one may need to replace the naïve notion of a moduli problem by a so-called *formal moduli problem* in the sense of Lurie [10].

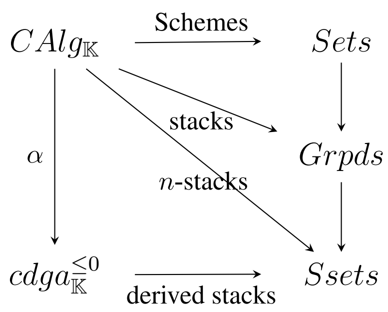
Formal moduli problems are in fact particular *derived stacks.* Roughly speaking, derived stacks are also higher spaces similar to stacks, but they are indeed more sensitive algebro-geometric objects to encode higher symmetries of the theory. In the first three chapters of the thesis, stacks are good enough for our purposes, and so we are interested in stacks. More detailed discussions on formal moduli theoretic constructions for Einstein gravity can be found in the upcoming chapters of the thesis, indeed.

To be more precise, let $cdga^{\leq 0}_{\mathbb{K}}$ denote the category of commutative differential graded $\mathbb{K}$-algebras in non-positive degrees, and $dSt_{\mathbb{K}}$ denote the ∞-category of derived stacks [17, 28]. Here, for instance, an object of $dSt_{\mathbb{K}}$ is given, in the functor of points perspective, as a certain simplicial presheaf

$$X:\ cdga^{\leq 0}_{\mathbb{K}} \longrightarrow\ sSets \tag{1.1.1}$$

where $sSets$ denote the ∞-category of simplical sets. Indeed, objects in $dSt_{\mathbb{K}}$ are simplicial presheaves preserving weak equivalences and possessing the decent/local-to-global property w.r.t. the site structure. For more details, we refer to [2].

Long story short, thanks to the Yoneda embedding, one can also realize algebro-geometric objects (like schemes, stacks, derived "spaces", etc...) as *certain functors* in addition to the standard ringed-space formulation. We have the following enlightening diagram from [17] encoding such a functorial interpretation:



One way of interpreting this diagram is as follows: In the case of schemes, for instance, such a functorial description implies that the points of a scheme form a *set*. Likewise, it implies that the collection of points of a stack has the structure of a *groupoid* and not that of a set. These kinds of interpretations, in fact, suggest the name "functor of points". Furthermore, since any commutative $\mathbb{K}$-algebra A can be realized as an object in $cdga_{\mathbb{K}}^{\leq 0}$ concentrated in degree 0 with the trivial differential, the morphism α is indeed an embedding and encodes the change in the local algebraic model of higher spaces. With the same spirit, the right hand side of the diagram captures the level of symmetries and leads to the different ways of organizing the moduli data. That is, the RHS is essentially about how to test two objects being the same. It also encodes the structure of points as discussed above.

Higher Structures and Einstein gravity. Having adopted the suitable language, our intentions are *(i)* to show that, in the case of a particular Einstein gravity on Lorentzian spacetimes of dimension n, one can view the moduli space $\mathcal{EL}$ of solutions to field equations as a suitable *moduli stack*, and *(ii)* to upgrade, once exists, the equivalence of 3D quantum gravity with gauge theory in a particular scenario to *a stack isomorphism* between the aforementioned theories. Moreover, we wish to elaborate *(iii)* the construction of the formal moduli problem of 3D Einstein gravity (in Cartan-Palatini formalism), and the algebraic structure of observables of the theory in the context of derived algebraic geometry.

Before focusing on each task mentioned above, let us start with an overview of vacuum 3D Einstein gravity with vanishing cosmological constant, which is the main situation of interest:

1. Starting with the usual metric formalism in 2+1 dimensions, the vacuum Einstein field equations with cosmological constant $\Lambda = 0$ read as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{1.1.2}$$

   Then, notice that after contracting with $g^{\mu\nu}$, one has $R = 0$. Therefore, it follows directly from substituting back into equation 1.1.2 that the moduli space $\mathcal{EL}$ of solutions to those field equations turns out to be *the moduli space $\mathcal{E}(M)$ of Ricci-flat ($R_{\mu\nu} = 0$) Lorentzian metrics on M*. In the 2+1 dimensional case, on the other hand, Weyl tensor is identically zero. Then the Riemann tensor can locally be expressed in terms of R and $R_{\mu\nu}$, and so we locally have $R_{\mu\nu\sigma\rho} = 0$ as well. That is, any solution of the vacuum Einstein field equations in 3-dimensions with vanishing cosmological constant is locally *flat*. In other words, it is just the moduli of *flat geometric structures on M*. With this interpretation in hand, it follows that Lorentzian spacetime is locally modeled on $(ISO(2,1), \mathbb{R}^{2+1})$ where $\mathbb{R}^{2+1}$ denotes the usual Minkowski spacetime [11, 33]. Thus, the metric is locally equivalent to the standard Minkowski metric $\eta_{\mu\nu}$. From a more physical point of view, on the other hand, the vanishing of $R_{\mu\nu\sigma\rho}$ means that 3D spacetime does not have any local degrees of freedom: there are no gravitational waves in the classical theory, and no gravitons in the quantum theory. For details, see [11, 12].

2. For the case of $\Lambda \neq 0$, $\mathcal{EL}$ turns out to be the moduli of *Lorentzian metrics of constant curvature* where the sign of this constant curvature depends on that of Λ. In fact, the field equations, in this case, read as

$$R_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R. \tag{1.1.3}$$

   Hence, the corresponding spacetime is locally modeled as either $(SO(2,2), dS_3)$ for $\Lambda > 0$ or $(SO(3,1), AdS_3)$ for $\Lambda < 0$. Analyzing the cases when $\Lambda \neq 0$, however, is beyond the scope of the current discussion.

3. Employing Cartan's formalism [11, 12, 5], one can reinterpret 2+1 gravity in the language of gauge theory. The basics of this interpretation will be briefly discussed in Section C.1. Roughly speaking, assuming the special case where M is topologically of the form $\Sigma \times \mathbb{R}$ and Σ is a closed Riemann surface of genus $g > 1$, the study of 2+1 gravity in fact boils down to that of $ISO(2,1)$ Chern Simons theory on M with the action functional $CS : \mathcal{A} \longrightarrow S^1$ of the form

$$CS[A] = \int_M \langle A, \mathrm{d}A + \frac{2}{3} A \wedge A \rangle \tag{1.1.4}$$

where $\langle \cdot, \cdot \rangle$ is a certain bilinear form on the Lie algebra $\mathfrak{g}$ of $ISO(2,1)$, and the gauge group $\mathcal{G}$ is locally of the form $Map(U, ISO(2,1))$. This group acts on the space $\mathcal{A}$ of $ISO(2,1)$-connections on M in a natural way: For all $\rho \in \mathcal{G}$ and $A \in \mathcal{A}$, we set

$$A \bullet \rho := \rho^{-1} \cdot A \cdot \rho + \rho^{-1} \cdot \mathrm{d}\rho. \tag{1.1.5}$$

The corresponding E-L equation, in this case, turns out to be

$$F_A = 0, \tag{1.1.6}$$

where $F_A = \mathrm{d}A + A \wedge A$ is *the curvature two-form* on M. Furthermore, under gauge transformations, the curvature 2-form F_A behaves as follows:

$$F_A \longmapsto F_A \bullet \rho := \rho^{-1} \cdot F_A \cdot \rho \;\; for\ all\ \rho \in \mathcal{G}. \tag{1.1.7}$$

Notice that this case involves the *non-compact* gauge group $ISO(2,1)$, and hence the required treatment is slightly different from the case of *compact* gauge groups. Recall that in the case of $G := SU(2)$, in particular, one has a unique Ad-invariant bilinear form on $\mathfrak{g}$, which is in fact *the Killing form* (up to a scaling constant). As outlined in [31] (see Ch. 25), the compact case $G = SU(2)$ can be analyzed by means of highly non-trivial theorems of Atiyah and Bott [48], but this is a rather different and deep story *per se*. Thus, the analysis of the compact case is beyond scope of the current discussion.

4. Using gauge theoretic interpretation, on the other hand, *the physical phase space* of 2+1 dimensional gravity on $M = \Sigma \times (0, \infty)$ (with $\Lambda = 0$) can

be realized as the moduli space $\mathcal{M}_{flat}$ of flat $ISO(2,1)$-connections on Σ [5] in a way that we have a map

$$\phi : \mathcal{E}(M) \longrightarrow \mathcal{M}_{flat} \tag{1.1.8}$$

sending *a flat pseudo-Riemannian metric* g to *the corresponding (flat) gauge field* A^g, where A^g is a Lie algebra $iso(2,1)$-valued 1-form. It means that any flat metric indeed defines a corresponding flat gauge connection, and hence there is an induced canonical map ϕ, which *need not to be invertible* in the first place. Indeed, this 1-from is constructed by combining the *vierbein* e^a and *the spin connection* ω^a as follows:

$$A_i := P_a e_i^a + J_a \omega_i^a \tag{1.1.9}$$

where $A^g = A_i(x)dx^i$ in a local coordinate chart $x = (x^i)$ such that P_a and J_a correspond to *translations* and *Lorentz generators* for the Lie algebra of the Poincaré group $ISO(2,1)$ for $a = 1, 2, 3$. For an introduction to the gauge theoretic interpretation, see [5, 11, 12] or Section 6 of [9]. As indicated above, the associated Chern-Simons theory has a non-compact gauge group $ISO(2,1)$, and hence the analysis of Atiyah and Bott is no longer available. But, instead, one will have the following identification [65]:

$$\mathcal{M}_{flat} \cong T^*(Teich(\Sigma)) \tag{1.1.10}$$

where $Teich(\Sigma)$ denotes the Teichmüller space associated to the closed surface Σ of genus $g > 1$. In that case, $\mathcal{M}_{flat}$ becomes a $12g - 12$ dimensional symplectic manifold with the standard symplectic structure on the cotangent bundle. The identification allows to employ the canonical/geometric quantization [72, 74] of the cotangent bundle in order to manifest the quantization of the phase space (even if this manifestation is by no means unique) [5].

Remark 1.1.2. In genuine quantum gravity, one seeks for the construction of a quantum Hilbert space by quantizing the “honest" moduli space $\mathcal{E}(M)$ of solutions to the vacuum Einstein field equations on M. In the gauge theoretic formulation, on the other hand, one can actually quantize the phase space $\mathcal{M}_{flat}$ of the Chern-Simons theory associated to 2+1 dimensional gravity in the sense of the naïve discussion above ([9], Sec. 6). That is, to construct a quantum theory of gravity, a possible strategy we may have is as follows: First, we translate everything into gauge theoretical framework, and view everything as a gauge theory. Then, one may try to "quantize" the corresponding gauge theory. When $\Lambda = 0$, as discussed above, the 2+1 gravity corresponds to the Chern-Simons theory with gauge group $G = ISO(2, 1)$. When $\Lambda \neq 0$, on the other hand, one has a Chern-Simons theory with either $G = SL(2, \mathbb{R}) \times SL(2; \mathbb{R})$ for $\Lambda < 0$ or $G = SL(2, \mathbb{C})$ for $\Lambda > 0$. As indicated before, analyzing the cases when $\Lambda \neq 0$ is beyond the scope of the current discussion. In any case, however, we end up with the following question: Are the resulting theories equivalent (in some sense)?

Definition 1.1.3. We say that the quantum gravity is **equivalent** to gauge theory in the sense of the canonical formalism if the map ϕ above is an isomorphism:

$$\phi : \mathcal{E}(M) \xrightarrow{\sim} \mathcal{M}_{flat}. \tag{1.1.11}$$

Remark 1.1.3.

1. As indicated in Section 6 of [9], one has the equivalence of quantum gravity with gauge theory (in the sense of Definition 1.1.3) in the case of a vacuum Einstein gravity on M with $\Lambda = 0$ where $M = \Sigma \times (0, \infty)$ and Σ is a closed Riemann surface of genus $g > 1$.

2. Indeed, the aforementioned result (and much more) were established via the works of Mess [33], Goldman [34] et al. For details, please see Appendix C. The main idea behind the construction is, roughly speaking, as follows: there is a one-to-one correspondence between the moduli space $\mathcal{M}_{flat}$ of flat G-connections on Σ and the moduli space

$$Hom(\pi_1(\Sigma), G)/G \tag{1.1.12}$$

of representations of $\pi_1(\Sigma)$ in G where G acts on $Hom(\pi_1(\Sigma), G)$ by conjugation [36]. The construction of such an isomorphism in Definition 1.1.3 is essentially based on the Teichmüller theoretic treatment [35] of representations of the surface group $\pi_1(\Sigma)$, and the global topology of the space $Hom(\pi_1(\Sigma), G)/G$ in the cases where $G = ISO(2, 1)$ or $G = PSL(2, \mathbb{R}) \cong SO_0(2, 1)$ [34].

Key achievements of the current work. [6] describes a certain "$stacky$" formulation of Yang–Mills fields on Lorentzian manifolds. The construction employs certain techniques naturally appearing in the homotopy theory of stacks [7]. [3, 4], on the other hand, provide a formulation of classical field theories in the language of formal moduli problems, and analyze the algebraic structure of observables in some classical field theories. In that respect, our main intention is in fact to provide similar constructions in the case of a certain Einstein gravity. Consequently, we upgrade the equivalence mentioned above to the "$stacky$" level. The main results of this paper can be outlined as follows:

1. *In the first three chapters of the thesis*, we employ the techniques in [6, 7] to show that
   (a) There is a suitable *moduli stack of Einstein gravity* (Theorem 3.1.1).
   (b) The isomorphism ϕ, in fact, induces *an isomorphism of the corresponding stacks* (Theorem 3.2.1).
2. *The remainder of the thesis* is essentially designed as a detailed survey on formal moduli problems, the structure of observables and factorization algebras. The appearances of these concepts in various classical field theories are also discussed. In that respect, it also includes the realizations of these concepts in the case of a particular 2+1 Einstein gravity. This part mainly relies on the derived geometric constructions especially in the case of Chern-Simons theory. Therefore, we first revisit basic constructions in [3, 4], and then show that 3D Einstein gravity arises as a natural example of these constructions. Our plan is as follows:
   (a) We present an obvious formal moduli problem in the case of 3D Cartan theory of the vacuum Einstein gravity with/without cosmological constant.
   (b) In this 3D scenario, we revisit the algebraic structure of observables in the language of factorization algebras as well.

   It should be noted that, in this part, the constructions related to 3D Einstein gravity are just the particular cases of the constructions given for Chern-Simons theory [3, 4, 14, 40].

1.2 Organization of the thesis

This work is organized into six main chapters together with a number of appendices. The main results and an outline of the remainder of this work are as follows:

In Chapter 1, which serves as an introductory discussion, we briefly explain the motivation and the setup.

In Chapter 2, we shall introduce certain preliminary notions naturally appearing in the context of algebraic geometry by using the functor of points approach. Having employed enough algebraic structures, we shall present Proposition 2.1.1 and hence define a particular pre-stack of Einstein gravity. The rest of Chapter 2 will be devoted to introducing basic concepts related to the homotopy theory of stacks.

In Chapter 3, adopting the homotopical perspective presented at the end of Chapter 2, we will prove Theorem 3.1.1, and introduce the desired stack of flat Lorentzian structures (see Definition 3.1.1) associated to moduli of Einstein gravity. Furthermore, we shall extend the equivalence of specific 3D quantum gravity with gauge theory in the case of $\Lambda = 0$ and $M = \Sigma \times (0, \infty)$ as addressed above to the context of stacks and establish an appropriate stack isomorphism via Theorem 3.2.1.

In Chapter 4, the basics of derived moduli spaces are revisited. We shall, in particular, investigate the notion of *(a sheaf of) formal moduli problems*, and the suitable derived spaces associated to the particular Chern-Simons theory (Lemma 4.1.1). Then, as an immediate observation, we also establish *the derived moduli space* or *formal moduli problem* encoding 2+1 Einstein gravity.

In Chapter 5, we shall discuss the notion of factorization algebra and particular constructions by which one can formalize the structure of observables of Einstein gravity in the context of derived algebraic geometry. Chapter 6, on the other hand, provides an epilogue for the thesis. It in fact serves as a brief summary.

The appendices, on the other hand, involve a variety of topics. Appendix A covers elementary aspects of Cartan geometry; Appendix B gives background on derived algebraic geometry and argues its role in the current discussion; Appendix C is about the theory behind the aforementioned equivalence of 3D quantum gravity and gauge theory; Appendix D is on the basics of moduli theory and stacks.

A disclaimer. Some of the material that we present throughout the text already appear in the literature and are very well-known to the experts. In some parts of the work, therefore, we prefer to present some objects of interest in a rather intuitive manner and encourage the readers to visit the indicated references. We in fact cross our fingers and sweep some technical material under the carpet without losing the spirit of the items of interest. We, on the other hand, apologize in advance for the omission of some references related to the current discussion if any. The author is the only person who is responsible with possible errors of all kinds (conceptually, technically or typographical) throughout the text if any.

CHAPTER 2

TOWARDS THE STACKY FORMULATION

2.1 Pre-stacky formulation

Before discussing the stacky formulation of Einstein gravity, we first recall, in a very brief and expository fashion, the notion of a moduli problem, and try to explain why one requires to employ the stacky refinement of the theory. For more details, we refer to Appendices D and B.2.2

A moduli problem is a problem of constructing a classifying space (or a moduli space $\mathcal{M}$) for certain geometric objects (such as manifolds, algebraic varieties, vector bundles etc...) and the families of objects up to their intrinsic symmetries. In the language of category theory, a moduli problem can be formalized as a certain functor

$$\mathcal{F} : \mathcal{C}^{op} \longrightarrow Sets \tag{2.1.1}$$

which is called ***a moduli functor*** where $\mathcal{C}^{op}$ is the *opposite* category of the category $\mathcal{C}$ and $Sets$ is the category of sets. In order to make the argument more transparent, we take $\mathcal{C}$ to be the category Sch of k-schemes. Note that for each scheme $U \in Sch$, $\mathcal{F}(U)$ is the *set* of isomorphism classes parametrized by U, and for each morphism $f : U \to V$ of schemes, we have a morphism $\mathcal{F}(f) : \mathcal{F}(V) \to \mathcal{F}(U)$ of sets. Together with the above formalism, the existence of a fine moduli space corresponds to the *representability* of the moduli functor $\mathcal{F}$ in the sense that

$$\mathcal{F} = Hom_{Sch}(\cdot, \mathcal{M}) \;\; for\ some\ \mathcal{M} \in Sch. \tag{2.1.2}$$

If this is the case, then we say that $\mathcal{F}$ *is represented by* $\mathcal{M}$.

In many cases, however, the moduli functor is *not* representable in the category Sch of schemes. This is essentially where the notion of *stack* comes into play. The notion of *stack* can be thought of as a first instance such that the ordinary notion of category *no longer* suffices to define such an object. To make sense of this new object in a well-established manner and enjoy the richness of this *new structure*, we need to introduce a higher categorical notion, namely a *2-category* [25, 28]. The theory of stacks, therefore, employs higher categorical techniques and notions in a way that it provides a mathematical treatment for the representability problem by re-defining the moduli functor as a stack, *a particular groupoid-valued pseudo-functor with local-to-global properties*,

$$\mathcal{X} : C^{op} \longrightarrow Grpds \tag{2.1.3}$$

where $Grpds$ denotes the 2-category of groupoids with objects being categories $\mathcal{C}$ in which all morphisms are isomorphisms (these sorts of categories are called *groupoids*), 1-morphisms being functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between groupoids, and 2-morphisms being natural transformations $\psi : \mathcal{F} \Rightarrow \mathcal{F}'$ between two functors. For more concrete definitions, we again refer to Appendix B.2.2.

Definition 2.1.1. [25] Let $\mathcal{C}$ be a category. **A prestack** $\mathcal{X} : \mathcal{C}^{op} \longrightarrow Grpds$ consists of the following data.

1. For any object U in $\mathcal{C}$, an object $\mathcal{X}(U)$ in $Grpds$. That is,

$$U \mapsto \mathcal{X}(U) \tag{2.1.4}$$

   where $\mathcal{X}(U)$ is a groupoid, i.e. *a category in which all morphisms are isomorphisms.*

2. For each morphism $U \xrightarrow{f} V$ in $\mathcal{C}$, a contravariant *functor* of groupoids

$$\mathcal{X}(V) \xrightarrow{\mathcal{X}(f)} \mathcal{X}(U). \tag{2.1.5}$$

   Note that $\mathcal{X}(f)$ is indeed *a functor of categories*, and hence one requires to provide an action of $\mathcal{X}(f)$ on objects and morphisms of $\mathcal{X}(V)$ in a compatible fashion in the sense of [49], ch. 1.

3. Given a composition of morphisms $U \xrightarrow{f} V \xrightarrow{g} W$ in $\mathcal{C}$, there is an invertible natural transformation between two functors

$$\phi_{g,f} : \mathcal{X}(g \circ f) \Rightarrow \mathcal{X}(f) \circ \mathcal{X}(g) \tag{2.1.6}$$

such that the following diagram commutes *(encoding the associativity).*

$$\begin{array}{ccc} \mathcal{X}(h \circ g \circ f) & \xrightarrow[\phi_{h,g\circ f}]{} & \mathcal{X}(g \circ f) \circ \mathcal{X}(h) \\ \Big\Downarrow \phi_{h\circ g,f} & & \Big\Downarrow \phi_{g,f} \star id_{\mathcal{X}(h)} \\ \mathcal{X}(f) \circ \mathcal{X}(h \circ g) & \xrightarrow[id_{\mathcal{X}(f)} \star \phi_{h,g}]{} & \mathcal{X}(f) \circ \mathcal{X}(g) \circ \mathcal{X}(h) \end{array} \tag{2.1.7}$$

Due to the Condition 3., the prestack $\mathcal{X}$ is indeed an object in $PFunc(\mathcal{C}, Grpds)$, and it is *a pseudo-functor* (See Appendix D). Or, equivalently it is also called *a presheaf of groupoids.*

Inspired by [6], we have the following observation encoding the pre-stacky formulation of Einstein gravities:

Proposition 2.1.1. *Given a Lorentzian n-manifold M, let $\mathcal{C}$ be the category of open subsets of M that are diffeomorphic to the Minkowski space $\mathbb{R}^{(n-1)+1}$ with morphisms being canonical inclusions between open subsets whenever $U \subset V$. Then, the following assignment*

$$\mathcal{E} : \mathcal{C}^{op} \longrightarrow Grpds \tag{2.1.8}$$

defines a prestack where

1. *For each object U of $\mathcal{C}$, $\mathcal{E}(U)$ is a groupoid of Ricci-flat pseudo-Riemannian metrics on U where objects form the set*

$$FMet(U) := \big\{ g \in \Gamma(U, Met_M) : Ric(g) = 0 \big\}. \tag{2.1.9}$$

Here Met_M denotes metric "bundle" on M, and the set of morphisms is defined by

$$Hom_{\mathcal{E}(U)}(g, g') := \{\varphi \in Diff(U) : g' = \varphi^* g\} \tag{2.1.10}$$

where $Diff(U)$ denotes the group of diffeomorphisms of U and it naturally acts on $FMet(U)$ by pull-back due to the fact that $Ric(\varphi^* g) = \varphi^* Ric(g)$ for any diffeomorphism φ. We denote this action by $g \cdot \varphi := \varphi^* g$ and a morphism $g \xrightarrow{\sim} g' = \varphi^* g$ in $Hom_{\mathcal{E}(U)}(g, g')$ by (g, φ). The composition of two morphisms is given as

$$(g \cdot \varphi) \cdot \psi = g \cdot (\varphi \circ \psi) \tag{2.1.11}$$

where $(g \cdot \varphi) \cdot \psi = \psi^* \varphi^* g = (\varphi \circ \psi)^* g = g \cdot (\varphi \circ \psi)$ *for any* $\varphi, \psi \in Diff(U)$.

2. *To each morphism* $U \xrightarrow{f} V$ *in* $\mathcal{C}$, *i.e.* $f : U \hookrightarrow V$ *with* $U \subset V$, *one assigns*

$$\mathcal{E}(V) \xrightarrow{\mathcal{E}(f)} \mathcal{E}(U). \tag{2.1.12}$$

Here $\mathcal{E}(f) \in Fun(\mathcal{E}(V), \mathcal{E}(U))$ *is a functor of categories whose action on objects and morphisms of* $\mathcal{E}(V)$ *is given as follows:*

(a) *For any object* $g \in Ob(\mathcal{E}(V)) = FMet(V)$,

$$g \xrightarrow{\mathcal{E}(f)} f^* g \tag{2.1.13}$$

where

$$\mathcal{E}(f)(g) := f^* g \in FMet(U). \tag{2.1.14}$$

Notice that the pullback of a Ricci-flat metric, in general, may no longer be Ricci-flat. But, in the case of particular canonical inclusions $f : U \hookrightarrow V$ *with* U, V *open subsets, if a metric* g *is Ricci-flat on* V, *so is* $f^* g$ *on* U. *This is because* $f^* g$ *is just the restriction* $g|_U$ *of* g *to the open subset* U.

(b) *For any morphism* $(g, \varphi) \in Hom_{\mathcal{E}(V)}(g, g')$ *with* $\varphi \in Diff(V)$ *such that* $g' = \varphi^* g$, *when restricted to* U, *both* g *and* g' *does still lie in the orbit space of* g *with* $\varphi \circ f =: f^* \varphi$ *being just the restriction of* g *to the smaller open subset* U *in* V, *and hence we set*

$$\left(g \xrightarrow[(g,\varphi)]{\sim} g' = \varphi^* g \right) \xrightarrow{\mathcal{E}(f)} \left(g|_U \xrightarrow[(f^* g, \varphi \circ f)]{\sim} \varphi^* g|_U = f^* \varphi^* g = (\varphi \circ f)^* g \right) \tag{2.1.15}$$

where

$$\mathcal{E}(f)(g, \varphi) := (f^* g, \varphi \circ f) = (g|_U, \varphi|_U) \tag{2.1.16}$$

is a morphism in $\mathcal{E}(U)$ *as* $\varphi|_U$ *gives a diffeomorphism of* U. *By using* $f^* \varphi := \varphi \circ f = \varphi|_U$, *we indeed have*

$$(g, \varphi) \xrightarrow{\mathcal{E}(f)} (f^* g, f^* \varphi) \in Hom_{\mathcal{E}(U)}(f^* g, f^* g'). \tag{2.1.17}$$

3. *Given a composition of morphisms* $U \xrightarrow{f} V \xrightarrow{h} W$ *in* $\mathcal{C}$, *there exists an invertible natural transformation (arising naturally from properties of pulling-back)*

$$\phi_{h \circ f} : \mathcal{E}(h \circ f) \Rightarrow \mathcal{E}(f) \circ \mathcal{E}(h) \tag{2.1.18}$$

together with the compatibility condition 2.1.6.

Proof. Besides the construction instructed in (1) and (2), we need to show that

(i) Given a composition of morphisms $U \xrightarrow{f} V \xrightarrow{h} W$ in $\mathcal{C}$, that is

$$U \xrightarrow[f]{} V \xrightarrow[h]{} W, \qquad U \xrightarrow{h \circ f} W \tag{2.1.19}$$

there is an invertible natural transformation $\psi_{h,f} : \mathcal{E}(h \circ f) \Rightarrow \mathcal{E}(f) \circ \mathcal{E}(h)$ given schematically as

$$\mathcal{E}(W) \underset{\mathcal{E}(f) \circ \mathcal{E}(h)}{\overset{\mathcal{E}(h \circ f)}{\Downarrow \psi_{h,f}}} \mathcal{E}(U) \tag{2.1.20}$$

(ii) Given a composition diagram of morphisms $U \xrightarrow{f} V \xrightarrow{h} W \xrightarrow{p} Z$ in $\mathcal{C}$, the associativity condition holds in the sense that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{E}(p \circ h \circ f) & \xRightarrow[\psi_{p,h\circ f}]{} & \mathcal{E}(h \circ f) \circ \mathcal{E}(p) \\
\Big\Downarrow \psi_{p\circ h,f} & & \Big\Downarrow \psi_{h,f} \star id_{\mathcal{E}(p)} \\
\mathcal{E}(f) \circ \mathcal{E}(p \circ h) & \xRightarrow[id_{\mathcal{E}(f)} \star \psi_{p,h}]{} & \mathcal{E}(f) \circ \mathcal{E}(h) \circ \mathcal{E}(p)
\end{array} \tag{2.1.21}$$

Proof of (i): First, we need to analyze objectwise: for any object $g \in Ob(\mathcal{E}(W)) = FMet(W)$, we have the following *strong condition* by which the rest of the proof will become rather straightforward:

$$\mathcal{E}(h \circ f)(g) = (h \circ f)^* g = f^* h^* g = \big(\mathcal{E}(f) \circ \mathcal{E}(h)\big)(g) \in FMet(U). \tag{2.1.22}$$

As we have identical metrics $\mathcal{E}(h \circ f)(g) = \mathcal{E}(f) \circ \mathcal{E}(h)(g)$ for any $g \in FMet(W)$, there is, by construction, an unique identity map

$$\big(\mathcal{E}(h \circ f)(g), id_U\big) \in Hom_{FMet(U)}(\mathcal{E}(h \circ f)(g), \mathcal{E}(f) \circ \mathcal{E}(h)(g)) \tag{2.1.23}$$

such that

$$\mathcal{E}(h\circ f)(g) \xrightarrow[\left(\mathcal{E}(h\circ f)(g), id_U\right)]{\sim} \mathcal{E}(f)\circ\mathcal{E}(h)(g) = id_U^*\big(\mathcal{E}(h\circ f)(g)\big) = \mathcal{E}(h\circ f)(g). \tag{2.1.24}$$

Thus, one has the natural choice of a collection of morphisms $\{m_g : \mathcal{E}(h\circ f)(g) \longrightarrow \mathcal{E}(f)\circ\mathcal{E}(h)(g)\}$ defined as

$$m_g = \big(\mathcal{E}(h\circ f)(g), id_U\big) \ \ for\ all\ g \in FMet(W), \tag{2.1.25}$$

for which the following diagram commutes: Just for the sake of notational simplicity, we let

$$\mathcal{F} := \mathcal{E}(h\circ f)\ \ and\ \ \mathcal{G} := \mathcal{E}(f)\circ\mathcal{E}(h), \tag{2.1.26}$$

then for each morphism $g \xrightarrow[(g,\phi)]{\sim} g'$ in $\mathcal{E}(W)$, we have

$$\begin{array}{ccc} \mathcal{F}(g) & \xrightarrow{\mathcal{F}((g,\phi))} & \mathcal{F}(g') \\ {\scriptstyle m_g}\downarrow & & \downarrow{\scriptstyle m_{g'}} \\ \mathcal{G}(g) & \xrightarrow[\mathcal{G}((g,\phi))]{} & \mathcal{G}(g') \end{array} \tag{2.1.27}$$

In fact , the commutativity follows from the following observation (thanks to the strong condition 2.1.22 we obtained above): By using the definition of action of the functor $\mathcal{E}(f)$ on $\mathcal{E}(W)$, we get

$$\begin{aligned} \mathcal{F}((g,\phi)) &= \mathcal{E}(h\circ f)((g,\phi)) \\ &= \big((h\circ f)^* g, (h\circ f)^*\phi\big) \\ &= \big(f^*\circ h^*(g), f^*\circ h^*(\phi)\big) \\ &= \big(\mathcal{E}(f)\circ\mathcal{E}(h)(g), f^*\circ h^*(\phi)\big) \\ &= \mathcal{E}(f)\circ\mathcal{E}(h)((g,\phi)) \\ &= \mathcal{G}((g,\phi)), \end{aligned} \tag{2.1.28}$$

which implies the commutativity of the diagram. Furthermore, it is clear from the strong condition 2.1.22, and hence the construction that $\psi_{h,f} : \mathcal{E}(h\circ f) \Rightarrow \mathcal{E}(f)\circ\mathcal{E}(h)$

is in fact invertible. In other words, we have $\mathcal{E}(h\circ f) \cong \mathcal{E}(f)\circ\mathcal{E}(h)$ up to an invertible natural transformation. In the language of 2-categories, on the other hand, these $\psi_{h,f}$'s are called *2-isomorphims* (for a complete treatment, see [25, 28]). This completes *the proof of (i)*.

Proof of (ii): Now, associativity follows directly from the following observations: If $U \xrightarrow{f} V \xrightarrow{h} W$ in $\mathcal{C}$ is a commuting diagram (i.e. *a composition of morphisms*), then what we have shown so far are as follows.

$$(1) \quad \mathcal{F}(g) = \mathcal{G}(g) \;\; for\ any \;\; g \in Ob(\mathcal{E}(W)) = FMet(W) \tag{2.1.29}$$

$$(2) \quad \mathcal{F}((g,\phi)) = \mathcal{G}((g,\phi)) \;\; for\ any\ g \xrightarrow[(g,\phi)]{\sim} g' \;\; in\ \mathcal{E}(W). \tag{2.1.30}$$

where $\mathcal{F} := \mathcal{E}(h\circ f)$ and $\mathcal{G} := \mathcal{E}(f)\circ\mathcal{E}(h)$. Now, let $U \xrightarrow{f} V \xrightarrow{h} W \xrightarrow{p} Z$ be a commutative diagram in $\mathcal{C}$, then it suffices to show that associativity condition in the sense introduced above holds both *objectwise* and *morphismwise* in a compatible manner:

- Let $g \in Ob(\mathcal{E}(Z)) = FMet(Z)$, then we have

$$\mathcal{E}(p\circ(h\circ f))(g) = \mathcal{E}(h\circ f)\circ\mathcal{E}(p)(g) \quad from\ 2.1.29\ with\ \psi_{p,h\circ f} \tag{2.1.31}$$

$$= \mathcal{E}(f)\circ\mathcal{E}(h)\circ\mathcal{E}(p) \quad from\ 2.1.29\ with\ \psi_{h,f}\star id_{\mathcal{E}(p)} \tag{2.1.32}$$

$$= \mathcal{E}(f)\circ\mathcal{E}(p\circ h)(g) \quad from\ 2.1.29\ with\ id_{\mathcal{E}(f)}\star\psi_{p,h} \tag{2.1.33}$$

$$= \mathcal{E}((p\circ h)\circ f)(g), \quad from\ 2.1.29\ with\ \psi_{p\circ h,f} \tag{2.1.34}$$

which gives the commutativity of the diagram objectwise.

- Let $g \xrightarrow[(g,\phi)]{\sim} g'$ in $\mathcal{E}(Z)$, then we have

$$\mathcal{E}(p \circ (h \circ f))((g,\phi)) = \mathcal{E}(h \circ f) \circ \mathcal{E}(p)((g,\phi)) \qquad from\ 2.1.30\ with\ \psi_{p,h\circ f} \tag{2.1.35}$$

$$= \mathcal{E}(f) \circ \mathcal{E}(h) \circ \mathcal{E}(p)((g,\phi)) \quad from\ 2.1.30\ with\ \psi_{h,f} \star id_{\mathcal{E}(p)} \tag{2.1.36}$$

$$= \mathcal{E}(f) \circ \mathcal{E}(p \circ h)((g,\phi)) \qquad from\ 2.1.30\ with\ id_{\mathcal{E}(f)} \star \psi_{p,h} \tag{2.1.37}$$

$$= \mathcal{E}((p \circ h) \circ f)((g,\phi)), \qquad from\ 2.1.30\ with\ \psi_{p\circ h,f} \tag{2.1.38}$$

which completes the proof.

□

2.2 Homotopy theory of stacks

We will study the stacky nature of the prestack $\mathcal{E}$ in Proposition 2.1.1 in the language of homotopy theory as discussed in [6, 7]. This homotopy theoretical approach is essentially based on *the model structure* [50] on the (2-) category $Grpds$ of groupoids and the category $Psh(\mathcal{C}, Grpds)$ of presheaves in groupoids. Furthermore, one also requires to adopt certain simplicial techniques and some practical results from [6, 7] in order to establish the notion of a stack in the language of homotopy theory. For an introduction to simplicial techniques, see [4], Appendix A.

2.2.1 A digression on (co-)simplicial objects

The following discussion is based on [4, 28]. Let Δ denote *the category of finite ordered sets* with objects being finite ordered sets

$$[n] := \{0 < 1 < 2 < \cdots < n\} \tag{2.2.1}$$

together with the morphisms $f : [n] \to [m]$ being non-decreasing functions. Note that the set $[n]$ corresponds to Δ^n, the usual *n-simplex* in $\mathbb{R}^{n+1}$, given as a set

$$\Delta^n := \Big\{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1 \ \ and \ \ 0 \leq x_k \leq 1 \ \ for\ all\ k\Big\}, \tag{2.2.2}$$

and hence the map $f : [n] \to [m]$ induces a linear map

$$f_* : \Delta^n \to \Delta^m, \ \ e_k \mapsto e_{f(k)}, \tag{2.2.3}$$

where $e_0 = (1, 0, 0, \cdots, 0)$ and $e_k = (0, \cdots, 0, 1, 0, \cdots, 0)$ is the k^{th} basis vector with 1 being at the $(k+1)^{th}$ slot. The ordering of $[n]$ defines a path along the edges of Δ^n, from e_0 to e_n. As addressed in [4], each map $f : [n] \to [m]$ can be factored into a surjection followed by an injection such that

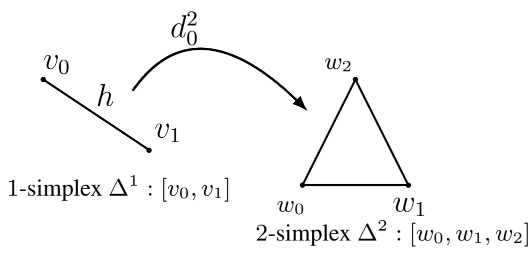
1. Any injection can also be factored into a sequence $\{d_i^n\}$ of *coface maps* where $d_i^n : [n-1] \to [n]$ is a map of simplicies given as

$$d_i^n(j) = \begin{cases} j & \text{if } j \leq i \\ j+1 & \text{else} \end{cases} \tag{2.2.4}$$

2. Each surjection can also be factored into a sequence $\{s_i^n\}$ of *codegeneracy maps* where $s_i^n : [n+1] \to [n]$ is a map of simplicies given as

$$s_i^n(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{else} \end{cases} \tag{2.2.5}$$

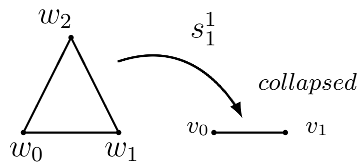
Remark 2.2.1. Geometrically speaking, the coface map d_i^n in fact "injects" the i^{th} $(n-1)$-simplex into an n-simplex depicted, for instance, as follows:



(2.2.6)

where $w_0 = (d_0^2)_*(v_0)$ and $w_2 = (d_0^2)_*(v_1)$. The codegeneracy map s_i^n, on the other hand, does collapse an edge $[i, i+1]$ of an $(n+1)$-simplex, and hence it projects an

$(n+1)$-simplex onto an n-simplex:



(2.2.7)

where $v_0 = (s_1^1)_*(w_0)$ and $v_1 = (s_1^1)_*(w_1) = (s_1^1)_*(w_2)$.

Definition 2.2.1. Let $\mathcal{C}$ be a category. **A (co-)simplicial object in** $\mathcal{C}$ is a (covariant) contravariant functor

$$X_\bullet : \Delta \longrightarrow \mathcal{C}. \tag{2.2.8}$$

If $\mathcal{C} = Sets$, then $X_\bullet \in Fun(\Delta, Sets)$ is called *a simplicial set,* and the image $X_\bullet([n])$ of $[n]$ is called the *set* of n-simplicies and is denoted by X_n. That is, we have

$$[n] \xrightarrow{X_\bullet} X_\bullet([n]) =: X_n, \tag{2.2.9}$$

and for each morphism $f : [n] \to [m]$, one has

$$X_m \xrightarrow{X_\bullet(f)} X_n. \tag{2.2.10}$$

Lemma 2.2.1. *[28] Any morphism $f \in Hom_\Delta([n],[m])$ can be written as a composition of the coface d_i^n and codegeneracy s_j^n maps such that the following relations hold:*

$$(1) \quad d_j^{n+1} \circ d_i^n = d_i^{n+1} \circ d_{j-1}^n \quad if \ \ 0 \le i < j \le n+1. \tag{2.2.11}$$

$$(2) \quad s_j^{n-1} \circ d_i^n = d_i^{n-1} \circ s_{j-1}^{n-2} \quad if \ \ 0 \le i < j \le n-1. \tag{2.2.12}$$

$$(3) \quad s_j^{n-1} \circ d_j^n = s_j^{n-1} \circ d_{j+1}^n = id_{[n-1]} \quad if \ \ 0 \le j \le n-1. \tag{2.2.13}$$

$$(4) \quad s_j^{n-1} \circ d_i^n = d_{i-1}^{n-1} \circ s_j^{n-2} \quad if \ \ n \ge i > j+1 > 0. \tag{2.2.14}$$

$$(5) \quad s_j^{n-1} \circ s_i^n = s_i^{n-1} \circ s_{j+1}^n \quad if \ \ 0 \le i \le j \le n-1. \tag{2.2.15}$$

Furthermore, given a cosimplicial object $X_\bullet$ in $\mathcal{C}$, *a covariant functor from Δ to $\mathcal{C}$*, for any object $[n]$ in Δ, one has a sequence of objects in $\mathcal{C}$

$$X_n := X_\bullet([n]) \tag{2.2.16}$$

together with the morphisms

$$X_\bullet(d_i^n) : X_{n-1} \longrightarrow X_n \quad and \quad X_\bullet(s_i^n) : X_{n+1} \longrightarrow X_n$$

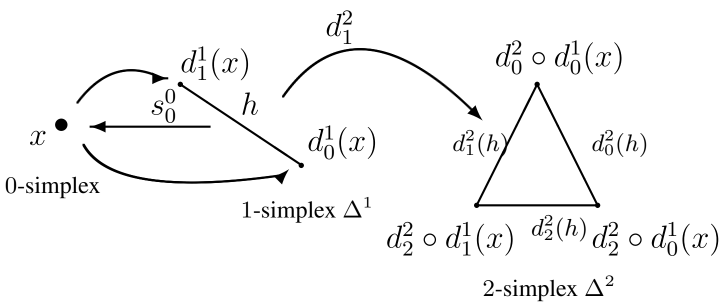
such that by abusing the notation and using d_i^n and s_i^n in places of the maps $X_\bullet(d_i^n)$ and $X_\bullet(s_i^n)$ respectively, Lemma 2.2.1 relating d_i^n and s_j^n's is also viable, and hence we can introduce the following diagram, namely *the cosimplicial diagram in* $\mathcal{C}$, which encodes the structure of the cosimplicial object $X_\bullet$ in terms of its simplicies X_n along with the corresponding coface and codegeneracy maps:

$$X_\bullet = \left(X_0 \;\begin{matrix} \xrightarrow{d_0^1} \\ \xleftarrow{s_0^0} \\ \xrightarrow{d_1^1} \end{matrix}\; X_1 \;\begin{matrix} \xrightarrow{d_0^2} \\ \xleftarrow{s_0^1} \\ \xrightarrow{d_1^2} \\ \xleftarrow{s_1^1} \\ \xrightarrow{d_2^2} \end{matrix}\; X_2 \cdots \right). \tag{2.2.17}$$

To simplify the notation, in general, we omit the codegeneracy maps and write the cosimplicial diagram in $\mathcal{C}$ in a rather compact way:

$$X_\bullet = \left(X_0 \rightrightarrows X_1 \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; X_2 \;\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\; \cdots \right). \tag{2.2.18}$$

Note that the cosimplicial diagram for $X_\bullet$ can also have a geometric interpretation in terms of the usual simplicies as follows: In order to make the geometric realization more transparent we assume $\mathcal{C} := Sets$, and hence consider $X_\bullet$ as a cosimplicial object in *Sets*. Let x be an object in X_0, $h : d_1^1(x) \to d_0^1(x)$ a morphism in X_1. Then x and h can be represented as 0- and 1-simplicies in $X_\bullet$ respectively such that, by using the properties of d_i^n and s_j^n, we pictorially have



(2.2.19)

Remark 2.2.2. Note that the existence of such a geometric interpretation above requires the following algebraic conditions:

$$(a)\ \ s^0_0(h) = id_x,\ \ and\ \ (b)\ d^2_0 \circ d^2_2(h) = d^2_1(h). \tag{2.2.20}$$

Indeed, condition (a) follows from the property 2.2.13 in the Lemma 2.2.1, namely

$$s^0_0 \circ d^1_1(x) = s^0_0 \circ d^1_0(x) = id_0(x) = x. \tag{2.2.21}$$

and hence $s^0_0(h) : s^0_0 \circ d^1_1(x) \longrightarrow s^0_0 \circ d^1_0(x)$ is just the morphism $id_x : x \to x$. Furthermore, condition (b) corresponds to the commutativity of the following diagram together with the property 2.2.11 in the Lemma 2.2.1:

$$\begin{array}{ccccc} d^2_2 \circ d^1_1(x) & \xrightarrow{d^2_2(h)} & d^2_2 \circ d^1_0(x) = d^2_0 \circ d^1_1(x) & \xrightarrow{d^2_0(h)} & d^2_0 \circ d^1_0(x) \\ \Big\downarrow {\scriptstyle \text{``}=\text{''}\ \text{by } 2.2.11} & & & & \Big\downarrow {\scriptstyle \text{``}=\text{''}\ \text{by } 2.2.11} \\ d^2_1 \circ d^1_1(x) & & \xrightarrow{\qquad d^2_1(h) \qquad} & & d^2_1 \circ d^1_0(x) \end{array} \tag{2.2.22}$$

2.2.2 Main ingredients of model categories

The notion of a model structure, which was originally defined by Quillen [60], serves as a particular mathematical treatment for abstracting homotopy theory in a way that one can localize the given category $\mathcal{C}$ by formally inverting a special class of morphisms, namely *the weak equivalences*, such that this extra structure formally encodes the localization procedure. In that respect, naïvely speaking, a model structure consists of three distinguished classes of morphisms, namely *weak equivalences* $\mathcal{W}$, *fibrations* $\mathcal{F}$ and *cofibrations* $\mathcal{CF}$ along with certain axioms and compatibility conditions. This structure eventually leads to localization $\mathcal{W}^{-1}\mathcal{C}$ of the given category $\mathcal{C}$, also called *the homotopy category* $Ho(\mathcal{C})$ of $\mathcal{C}$. We manifestly follow the treatment of the subject as discussed in [50].

Given a category $\mathcal{C}$, denote by $Map(\mathcal{C})$ *the category of morphisms of* $\mathcal{C}$ with objects being morphisms f in $\mathcal{C}$, and morphisms between f and g being a pair

$$(\phi_f, \phi_g) \in Ob(Map(\mathcal{C})) \times Ob(Map(\mathcal{C})) \tag{2.2.23}$$

such that the following diagram commutes: for any two morphism $f : A \to B$ and $g : C \to D$ in $\mathcal{C}$, we have

$$\begin{array}{ccc} A & \xrightarrow{\phi_f} & C \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow[\phi_g]{} & D \end{array} \tag{2.2.24}$$

Definition 2.2.2. Let $\mathcal{C}$ be a category.

1. A morphism $f : A \to B$ in $\mathcal{C}$ (or an object $f \in Ob(Map(\mathcal{C}))$) is called ***a retract of a morphism*** $g : C \to D$ in $\mathcal{C}$ if there exists a retraction of objects in the sense that one has retractions $r : C \to A$ and $r' : D \to B$ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{\imath} & C & \xrightarrow{r} & A \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ B & \xrightarrow{\imath'} & D & \xrightarrow{r'} & B \end{array} \tag{2.2.25}$$

   where $r \circ \imath = id_A$ and $r' \circ \imath' = id_B$ are retractions of objects.

2. ***A functorial factorization*** is an ordered pair $(\mathcal{F}, \mathcal{G})$ of functors

$$\mathcal{F}, \mathcal{G} : Map(\mathcal{C}) \to Map(\mathcal{C}) \tag{2.2.26}$$

   such that for any morphism f in $\mathcal{C}$, we have

$$f = \mathcal{G}(f) \circ \mathcal{F}(f). \tag{2.2.27}$$

3. Let $i \in Hom_{\mathcal{C}}(A, B)$ and $p \in Hom_{\mathcal{C}}(C, D)$. We say that i has **the left lifting property** with respect to p, and p has **the right lifting property** with respect to i if for any commutative diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & D \end{array} \tag{2.2.28}$$

there is a lift $h : B \to C$ commuting the following diagram.

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\big\downarrow & \overset{\exists h}{\nearrow} & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & D \end{array} \tag{2.2.29}$$

Note that in the language of standard homotopy theory, it is equivalently said that *the map $p : C \to D$ has the homotopy lifting property*. It is then called a *fibration*. Similarly, we say that *the map $i : A \to B$ has the homotopy extension property*. Such map i, on the other hand, is called a *cofibration.* Inspired by these notions naturally emerging in standard homotopy theory (on the category Top of topological spaces [32]), we have the following abstraction which allows us to make sense of homotopy theory on an arbitrary category in a rather axiomatic way [50].

Definition 2.2.3. *A model structure* on a category $\mathcal{C}$ (in which both initial and final objects exist) consists of three subcategories of $Map(\mathcal{C})$, so-called *weak equivalences* $\mathcal{W}$, *fibrations* $\mathcal{F}$ and *cofibrations* $\mathcal{CF}$, and two functorial factorizations (α, β) and (γ, δ) along with certain axioms given as

(i) *(2-out-3 axiom of weak equivalences)* If f and g are morphisms in $\mathcal{C}$ such that $g \circ f$ is defined and any two morphisms in the set $\{f, g, g \circ f\}$ are weak equivalences, then so is the third.

(ii) If f and g are morphisms in $\mathcal{C}$ such that f is a retract of g and g is a weak equivalence (cofibration or fibration respectively), then so is f.

(iii) $f \in Map(\mathcal{C})$ is called a *trivial cofibration (or fibration respectively)* if $f \in \mathcal{CF} \cap \mathcal{W}$ (or $f \in \mathcal{F} \cap \mathcal{W}$ resp.) Then we have

- For all $i \in \mathcal{CF} \cap \mathcal{W}$ and $p \in \mathcal{F}$, i has the left lifting property with respect to p. That is, trivial cofibrations have the left lifting property w.r.t fibrations.
- For all $i \in \mathcal{CF}$ and $p \in \mathcal{F} \cap \mathcal{W}$, i has the left lifting property with respect to $p \in \mathcal{F} \cap \mathcal{W}$. That is, cofibrations have the left lifting property w.r.t trivial fibrations.

(iv) (The existence of weak factorization system) For all $f \in Map(\mathcal{C})$, we have

$$(\alpha, \beta) \in \mathcal{CF} \times (\mathcal{F} \cap \mathcal{W}) \ \ and \ \ (\gamma, \delta) \in (\mathcal{CF} \cap \mathcal{W}) \times \mathcal{F}. \tag{2.2.30}$$

Then, a category $\mathcal{C}$ (in which both initial and final objects exist) is called **a model category** if it admits a model structure.

As we stressed before, we intend to introduce a (localization) functor

$$\mathcal{C} \longrightarrow \mathcal{W}^{-1}\mathcal{C} \tag{2.2.31}$$

such that all elements of $\mathcal{W}$ become invertible in $\mathcal{W}^{-1}\mathcal{C}$, where $\mathcal{W}^{-1}\mathcal{C}$ is called *the homotopy category of* $\mathcal{C}$ and is denoted by $Ho(\mathcal{C})$. The complete treatment will not be given here, but instead we shall introduce a prototype example which in fact captures the essence of the item. For a concrete construction, we refer to [28, 50].

Example 2.2.1. [32] Let Top be the category of topological spaces with morphisms being continuous functions $f : X \to Y$ between topological spaces. Then, it admits a model structure where we set

$$\mathcal{W} = \{f : X \to Y : \ \pi_i X \xrightarrow{f_*} \pi_i Y \ is\ an\ isomorphism\ for\ all\ i\}. \tag{2.2.32}$$

$$\mathcal{F} = \{f : X \to Y : \ f\ has\ homotopy\ lifting\ property\ w.r.t\ \mathbb{D}^n\} \tag{2.2.33}$$

$$= \{Serre's\ fibrations\} \tag{2.2.34}$$

Hence, in the case of compactly generated topological spaces, we have

$$Ho(Top) \simeq CW \tag{2.2.35}$$

where CW denotes the category of CW-complexes. For more examples, see [32].

Similarly, the 2-category $Grpds$ of groupoids will have a model structure for which the weak equivalences are set to be *the equivalences of groupoids*, namely *fully faithfully essentially surjective functors between groupoids.* Therefore, the next task will be to elaborate the model structures on $Grpds$ and $Psh(\mathcal{C}, Grpds)$, and then provide an alternative definition of a stack with the aid of such model structures [6, 7].

Theorem 2.2.1. *The category $Gpds$ admits a model structure where*

1. *A morphism $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between two groupoids (i.e. a functor between two particular categories) is said to be a weak equivalence if it is fully faithful and essentially surjective. See Definition D.1.2 for fully faithfulness and being essentially surjective.*

2. *A morphism $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between two groupoids is called a fibration if for each object A in $\mathcal{C}$ and and each morphism $\phi : \mathcal{F}(A) \xrightarrow{\sim} D$ in $\mathcal{D}$, there exist an object B and a morphism $f : A \xrightarrow{\sim} B$ in $\mathcal{C}$ such that $\mathcal{F}(f) = \phi$. That is,*

$$\begin{array}{ccc} A & \xrightarrow{\mathcal{F}} & \mathcal{F}(A) \\ {\scriptstyle f}\,\vdots & & \big\downarrow {\scriptstyle \phi = \mathcal{F}(f)} \\ B & \overset{}{\underset{\mathcal{F}}{\dashrightarrow}} & D = \mathcal{F}(B) \end{array} \qquad (2.2.36)$$

3. *A morphism $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between two groupoids is called cofibration if it is injective on objects.*

A model structure on $PSh(\mathcal{C}, Grpds)$, on the other hand, can also be defined in a similar fashion. In fact, it admits so-called a *global model structure* given as follows [6, 7]:

Lemma 2.2.2. *The category $PSh(\mathcal{C}, Grpds)$ admits a model structure where*

1. *A morphism $\phi : \mathcal{X} \Longrightarrow \mathcal{Y}$ in $PSh(\mathcal{C}, Grpds)$, which is indeed a natural transformation between two functors given schematically as*

$$\mathcal{C}^{op} \underset{\mathcal{Y}}{\overset{\mathcal{X}}{\rightrightarrows}} Grpds, \quad \Downarrow \phi \qquad (2.2.37)$$

is called a weak equivalence if for each object A in $\mathcal{C}$, the morphism

$$\phi(A) : \mathcal{X}(A) \longrightarrow \mathcal{Y}(A) \qquad (2.2.38)$$

is a weak equivalence in $Grpds$.

2. *A morphism $\phi : \mathcal{X} \Longrightarrow \mathcal{Y}$ in $PSh(\mathcal{C}, Grpds)$ is called a fibration if for each object A in $\mathcal{C}$, the morphism*

$$\phi(A) : \mathcal{X}(A) \longrightarrow \mathcal{Y}(A) \tag{2.2.39}$$

is a fibration in $Grpds$.

3. *A morphism $\phi : \mathcal{X} \Longrightarrow \mathcal{Y}$ in $PSh(\mathcal{C}, Grpds)$ is called a cofibration if it has the left lifting property w.r.t. all trivial fibrations ($p \in \mathcal{F} \cap \mathcal{W}$). That is, as we indicated before, there is a lift $h : \mathcal{Y} \to \mathcal{X}'$ commuting the following diagram.*

$$\begin{array}{ccc} \mathcal{X} & \Longrightarrow & \mathcal{X}' \\ \phi \Big\Downarrow & \overset{\exists h}{\nearrow} & \Big\Downarrow p \\ \mathcal{Y} & \Longrightarrow & \mathcal{Y}' \end{array} \tag{2.2.40}$$

Furthermore, as addressed in [6, 7], by using a suitable localization of $PSh(\mathcal{C}, Grpds)$, one can also define another model structure on $PSh(\mathcal{C}, Grpds)$, namely *a local model structure*. This structure indeed allows us to encode the local-to-global process. In other words, studying the local model structure instead of the global one allows us to make sense of gluing properties of presheaves $\mathcal{X} \in PSh(\mathcal{C}, Grpds)$. This essentially will lead to the description of stacks in the language of model categories and homotopy theory.

Theorem 2.2.2. *[7]. Let $\mathcal{C}$ be a site. There exists a model structure on $PSh(\mathcal{C}, Grpds)$ which is obtained by localizing the global model structure with respect to the morphisms of the form*

$$S := \{hocolim_{PSh(\mathcal{C},Grpds)}(\underline{U_\bullet}) \to \underline{U} : \{U_i \to U\} \; is \; a \; covering \; family \; of \; U\} \tag{2.2.41}$$

where $\underline{U} := Hom_{\mathcal{C}}(\cdot, U)$ is the standard Yoneda functor and $\underline{U_\bullet}$ denotes the simplicial diagram in $PSh(\mathcal{C}, Grpds)$. That is,

$$\underline{U_\bullet} := \Big(\coprod_i U_i \leftleftarrows \coprod_{ij} U_{ij} \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, \coprod_{ijk} U_{ijk} \cdots \Big). \tag{2.2.42}$$

2.2.3 Homotopy-theoretical definition of a stack

Unfortunately, we are not able to provide either a self-contained mathematical treatment or a proof of Theorem 2.2.2 or Theorem 2.2.1 in this thesis. But, instead we refer to [7]. As stressed in [6], moreover, we have the following definition/theorem which allows us to formulate the classical notion of a Deligne-Mumford stack [39] in the language of homotopy theory.

Definition 2.2.4. Let $\mathcal{C}$ be a site. **A stack** is a prestack $\mathcal{X} : \mathcal{C}^{op} \longrightarrow Grpds$ such that for each covering family $\{U_i \to U\}$ of U the canonical morphism

$$\mathcal{X}(U) \longrightarrow holim_{Grpds}\big(\mathcal{X}(U_\bullet)\big) \tag{2.2.43}$$

is a weak equivalence (and hence an equivalence of categories) in $Grpds$ where

$$\mathcal{X}(U_\bullet) := \Big(\prod_i \mathcal{X}(U_i) \rightrightarrows \prod_{ij} \mathcal{X}(U_{ij}) \ \substack{\rightarrow\\ \rightarrow\\ \rightarrow}\ \prod_{ijk} \mathcal{X}(U_{ijk}) \ \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\ \cdots\Big) \tag{2.2.44}$$

is the cosimplicial diagram in $Grpds$ (cf. Diagram 2.2.18) and $U_{i_1 i_2 \ldots i_m}$ denotes the fibered product of U_{i_n}'s in U, that is

$$U_{i_1 i_2 \ldots i_m} := U_{i_1} \times_U U_{i_2} \times_U \cdots \times_U U_{i_m}. \tag{2.2.45}$$

Remark 2.2.3.

1 The weak equivalences in Definition 2.2.4 are the ones introduced in Theorem 2.2.1, namely those morphisms in $Grpds$ which are fully faithful and essentially surjective.

2 We haven't discussed the notion of "$ho(co)lim_{Grpds}$" in detail. For a complete construction of this item we refer to section 2 of [7]. The following lemma ([7], corollary 2.11), on the other hand, does provide an explicit characterization of "$holim_{Grpds}(\cdot)$" as a particular groupoid.

Lemma 2.2.3. *Given a cosimplicial diagram $X_\bullet$ in $Grpds$ of the form*

$$X_\bullet = \Big(X_0 \rightrightarrows X_1 \ \substack{\rightarrow\\ \rightarrow\\ \rightarrow}\ X_2 \ \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\ \cdots\Big) \tag{2.2.46}$$

where each X_i is a groupoid, then $holim_{Grpds}(X_\bullet)$ is a groupoid for which *(i)* ***objects*** *are the pairs* (x, h) *where* x *is an object in* X_0, $h : d_1^1(x) \to d_0^1(x)$ *a morphism in* X_1 *such that*

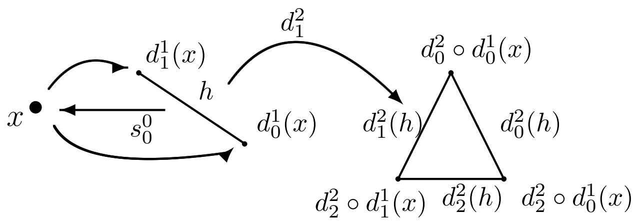
$$(a)\ \ s_0^0(h) = id_x, \tag{2.2.47}$$

$$(b)\ \ d_0^2 \circ d_2^2(h) = d_1^2(h). \tag{2.2.48}$$

Note that as we discussed in Lemma 2.2.1 and Remark 2.2.2, x *and* h *can be realized as 0- and 1-simplicies in* $X_\bullet$ *respectively such that, by using the properties of* d_i^n *and* s_j^n*, those conditions correspond to the commutativity of the diagram*

$$\begin{array}{ccccc} d_2^2 \circ d_1^1(x) & \xrightarrow{d_2^2(h)} & d_2^2 \circ d_0^1(x) = d_0^2 \circ d_1^1(x) & \xrightarrow{d_0^2(h)} & d_0^2 \circ d_0^1(x) \\ \downarrow \text{“ = ” by Lemma 2.2.1} & & & & \downarrow \text{“ = ” by Lemma 2.2.1} \\ d_1^2 \circ d_1^1(x) & & \xrightarrow{\qquad d_1^2(h) \qquad} & & d_1^2 \circ d_0^1(x) \end{array} \tag{2.2.49}$$

and hence we pictorially have



(2.2.50)

(ii) ***morphisms*** *are the arrows of pairs* $(x, h) \to (x', h')$ *that consist of a morphism* $f : x \to x'$ *in* X_0 *such that the following diagram commutes.*

$$\begin{array}{ccc} d_1^1(x) & \xrightarrow{d_1^1(f)} & d_1^1(x') \\ h \downarrow & & \downarrow h' \\ d_0^1(x) & \xrightarrow[d_0^1(f)]{} & d_0^1(x') \end{array} \tag{2.2.51}$$

Here, d_i^n*'s are in fact covariant functors between groupoids.*

Remark 2.2.4. Given a cosimplicial diagram $X_\bullet$ in $Grpds$, Lemma 2.2.3 indeed serves as an equivalent definition of $holim_{Grpds}(X_\bullet)$. Therefore, throughout the discussion, for those who are not comfortable with the construction of $holim_{Grpds}(X_\bullet)$

presented in [7] -involving homotopy theory, model structures etc...- we simply define the homotopy limit $holim_{Grpds}(X_\bullet)$ of a cosimplicial diagram $X_\bullet$ in $Grpds$

$$X_\bullet = \left(X_0 \rightrightarrows X_1 \substack{\rightarrow\\\rightarrow\\\rightarrow} X_2 \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right) \tag{2.2.52}$$

as *a particular groupoid with the properties outlined in Lemma 2.2.3.*

CHAPTER 3

STACKY FORMULATION

3.1 Stack of Einstein Gravity on Lorentzian Manifolds

Assume that $\mathcal{C}$ is the category described in Proposition 2.1.1. Let

$$\mathcal{E} : \mathcal{C}^{op} \longrightarrow \ Grpds \tag{3.1.1}$$

be a prestack defined in Proposition 2.1.1 encoding the moduli space of solutions to the vacuum Einstein field equations with $\Lambda = 0$ on a Lorentzian manifold M. Now, inspired by the approach presented in [6, 7], we shall provide the stacky structure on $\mathcal{E}$ in accordance with Definition 2.2.4 and Lemma 2.2.3.

Theorem 3.1.1. *$\mathcal{E} : C^{op} \longrightarrow Grpds$ is a stack.*

Proof. As in the case of [6], we first endow $\mathcal{C}$ with an appropriate Grothendieck topology τ (cf. Definition B.2.3) by defining the covering families $\{U_i \to U\}_i$ of U in $\mathcal{C}$ to be "good" open covers $\{U_i \subseteq U\}$ meaning that the fibered products

$$U_{i_1 i_2 \ldots i_m} := U_{i_1} \times_U U_{i_2} \times_U \cdots \times_U U_{i_m} \tag{3.1.2}$$

corresponding to the intersection of those open subsets U_i's in U are either empty or open subsets diffeomorphic to $\mathbb{R}^{(n-1)+1}$ (and hence lie in $\mathcal{C}$) where the morphisms

$$U_i \hookrightarrow U \tag{3.1.3}$$

are the canonical inclusions (and hence morphisms in $\mathcal{C}$) for each i. Therefore, we clearly have the same (or even simpler) site structure on $\mathcal{C}$ discussed in [6]. Let U be an object in $\mathcal{C}$. Given $\{U_i \subseteq U\}$ a covering family for U, one has the following cosimplicial diagram in $Grpds$

$$\mathcal{E}(U_\bullet) := \left(\prod_i \mathcal{E}(U_i) \rightrightarrows \prod_{ij} \mathcal{E}(U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{ijk} \mathcal{E}(U_{ijk}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right) \tag{3.1.4}$$

where $U_{i_1 i_2 \ldots i_m}$ denotes the fibered product of U_{i_n}'s in U, that is

$$U_{i_1 i_2 \ldots i_m} := U_{i_1} \times_U U_{i_2} \times_U \cdots \times_U U_{i_m}, \tag{3.1.5}$$

which, in that case, corresponds to the usual intersection of U_{i_n}'s in U. Note that for a family

$$\{g_i\} \text{ in } \prod_i \mathcal{E}(U_i), \tag{3.1.6}$$

where $\mathcal{E}(U_i) = FMet(U_i)$, the coface maps d_0^1 and d_1^1 correspond to the suitable restrictions of each component, namely

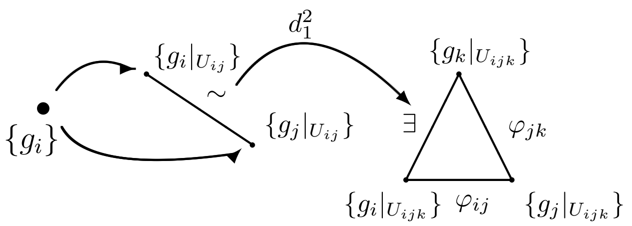
$$g_i|_{U_{ij}} \ \ and \ \ g_j|_{U_{ij}}. \tag{3.1.7}$$

Now, it follows from the Lemma 2.2.3 that $holim_{Grpds}(\mathcal{E}(U_\bullet))$ is indeed a particular groupoid and can be defined as follows:

1. *Objects* are the pairs (x, h) where

   $$x := \{g_i\} \in \prod_i \mathcal{E}(U_i), \tag{3.1.8}$$

   i.e. a family of flat pseudo-Riemannian metrics on U_i's, and hence we pictorially have the following observation:

   (3.1.9)

   where $g_j|_{U_{ij}} = \varphi_{ij}^* \, g_i|_{U_{ij}}$ for some $\varphi_{ij} \in Diff(U_{ij})$. Notice that for all i, j, k we have

   $$\begin{aligned} g_k|_{U_{ijk}} &= \varphi_{jk}^* g_j|_{U_{ijk}} \\ &= \varphi_{jk}^* \varphi_{ij}^* g_i|_{U_{ijk}} \\ &= (\varphi_{ij} \circ \varphi_{jk})^* g_i|_{U_{ijk}}. \end{aligned} \tag{3.1.10}$$

which means that there exists a morphism $\varphi_{ik} : g_i|_{U_{ijk}} \xrightarrow{\sim} g_k|_{U_{ijk}}$. Therefore, we define the morphism h in $\prod \mathcal{E}(U_{ij})$ as a family

$$h := \left\{ g_i|_{U_{ij}} \xrightarrow[(g_i|_{U_{ij}}, \varphi_{ij})]{\sim} g_j|_{U_{ij}} : g_j|_{U_{ij}} = \varphi_{ij}^* g_i|_{U_{ij}} \; with \; \varphi_{ij} \in Diff(U_{ij}) \right\} \tag{3.1.11}$$

where $g_k|_{U_{ijk}} = (\varphi_{ij} \circ \varphi_{jk})^* g_i|_{U_{ijk}}$ and $s_0^0(h) : \{g_i\} \to \{g_i\}$, which is just the identity morphism. As a remark, the conditions in the definition of the family $\{h\}$ correspond to those in Lemma 2.2.3 (2.2.47 and 2.2.48). Therefore, the objects of $holim_{Grpds}(\mathcal{E}(U_\bullet))$ must be of the form

$$(x, h) = \Big(\{g_i \in FMet(U_i)\}, \{\varphi_{ij} \in Diff(U_{ij})\} \Big) \tag{3.1.12}$$

where $\{g_i\}$ is an object in $\prod \mathcal{E}(U_i)$, and for each i, j, $\varphi_{ij} := (g_i|_{U_{ij}}, \varphi_{ij})$ is a morphism in $\prod \mathcal{E}(U_{ij})$ satisfying

$$(i) \;\; g_j|_{U_{ij}} = \varphi_{ij}^* g_i|_{U_{ij}} \; with \; \varphi_{ij} \in Diff(U_{ij}), \tag{3.1.13}$$

$$(ii) \;\; On \; U_{ijk}, \;\; \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik} \;\; (cocycle\ condition), \tag{3.1.14}$$

$$(iii) \;\; s_0^0(h) : \{g_i\} \to \{g_i\}, \; the \; identity \; morphism. \tag{3.1.15}$$

2. *A morphism* in $holim_{Grpds}(\mathcal{E}(U_\bullet))$

$$(x, h) \to (x', h') \tag{3.1.16}$$

consists of the following data:

(a) a morphism $x \xrightarrow{f} x'$ in $\prod \mathcal{E}(U_i)$, that is,

$$\{g_i\} \xrightarrow{\sim} \{g_i'\} \tag{3.1.17}$$

where $g_i, g_i' \in FMet(U_i)$ such that $g_i' = \varphi_i^* g_i$ for some $\varphi_i \in Diff(U_i)$,

(b) together with the commutative diagram

$$\begin{array}{ccc} g_i|_{U_{ij}} & \xrightarrow{\varphi_i|_{U_{ij}}} & g_i'|_{U_{ij}} \\ {\scriptstyle h = \varphi_{ij}} \downarrow & & \downarrow {\scriptstyle h' = \varphi_{ij}'} \\ g_j|_{U_{ij}} & \xrightarrow[\varphi_j|_{U_{ij}}]{} & g_j'|_{U_{ij}} \end{array} \tag{3.1.18}$$

In fact, it follows from the fact that $g_j|_{U_{ij}} = \varphi_{ij}^* g_i|_{U_{ij}}$ and $g'_j|_{U_{ij}} = \varphi'^*_{ij} g'_i|_{U_{ij}}$ we have

$$\begin{aligned}\big(g_i|_{U_{ij}} \cdot \varphi_i|_{U_{ij}}\big) \cdot \varphi'_{ij} &= \varphi'^*_{ij} \varphi_i^*|_{U_{ij}} g_i|_{U_{ij}} \\ &= \varphi'^*_{ij} g'_i|_{U_{ij}} \\ &= g'_j|_{U_{ij}}.\end{aligned} \tag{3.1.19}$$

and, on the other hand, one also has

$$\begin{aligned}\big(g_i|_{U_{ij}} \cdot \varphi_{ij}\big) \cdot \varphi_j|_{U_{ij}} &= \varphi_j^*|_{U_{ij}} \varphi_{ij}^* g_i|_{U_{ij}} \\ &= \varphi_j^*|_{U_{ij}} g_j|_{U_{ij}} \\ &= g'_j|_{U_{ij}}.\end{aligned} \tag{3.1.20}$$

which imply the commutativity of the diagram, and hence one can also deduce the following relation:

$$\big(g_i|_{U_{ij}} \cdot \varphi_{ij}\big) \cdot \varphi_j|_{U_{ij}} = \big(g_i|_{U_{ij}} \cdot \varphi_i|_{U_{ij}}\big) \cdot \varphi'_{ij} \tag{3.1.21}$$

$$\Longleftrightarrow$$

$$g_i|_{U_{ij}} \cdot \big(\varphi_{ij} \circ \varphi_j|_{U_{ij}}\big) = g_i|_{U_{ij}} \cdot \big(\varphi_i|_{U_{ij}} \circ \varphi'_{ij}\big) \tag{3.1.22}$$

$$\Longleftrightarrow$$

$$\varphi'_{ij} = \varphi_i^{-1}|_{U_{ij}} \circ \varphi_{ij} \circ \varphi_j|_{U_{ij}} \tag{3.1.23}$$

Thus, a morphism in $holim_{Grpds}(\mathcal{E}(U_\bullet))$ from $\big(\{g_i\}, \{\varphi_{ij}\}\big)$ to $\big(\{g'_i\}, \{\varphi'_{ij}\}\big)$ is a family

$$\Big\{\varphi_i \in Diff(U_i) : g'_i = \varphi_i^* g_i \text{ and } \varphi'_{ij} = \varphi_i^{-1}|_{U_{ij}} \circ \varphi_{ij} \circ \varphi_j|_{U_{ij}} \text{ for all } i, j.\Big\} \tag{3.1.24}$$

Now, for a covering family $\{U_i \subseteq U\}$ of U, *the canonical morphism*

$$\Psi : \mathcal{E}(U) \longrightarrow holim_{Grpds}(\mathcal{E}(U_\bullet)) \tag{3.1.25}$$

is defined as a functor of groupoids where

- for each object g in $FMet(U)$,

$$g \xrightarrow{\Psi} \Big(\{g|_{U_i}\}, \{\varphi_{ij} = id\}\Big), \tag{3.1.26}$$

  together with the trivial cocyle condition.

- for each morphism $g \xrightarrow[(g,\varphi)]{\sim} \varphi^* g$ with $\varphi \in Diff(U)$,

$$\left(g \xrightarrow[(g,\varphi)]{\sim} \varphi^* g\right) \xrightarrow{\Psi} \left(\{\varphi_i := \varphi|_{U_i}\}\right) \tag{3.1.27}$$

  where $\varphi|_{U_i}$ trivially satisfies the desired relation in 3.1.24 for being a morphism in $holim_{Grpds}(\mathcal{E}(U_\bullet))$.

Claim: Ψ is a fully faithful and essentially surjective functor between groupoids (cf. Theorem 2.2.1).

Proof of claim:

(i) Given a family of objects $\{g_i\}$ with the family of transition functions $\{\varphi_{ij} \in Diff(U_{ij})\}$ such that

$$g_j|_{U_{ij}} = \varphi_{ij}^* g_i|_{U_{ij}}$$

along with the cocycle condition

$$On\ U_{ijk}, \ \ \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}, \tag{3.1.28}$$

we need to show that these are patched together to form a metric $g \in FMet(U)$. In fact, this follows from the analysis of geometric structures [11] on objects in $\mathcal{C}$ together with the approach employed in [6]. The key is the following lemma:

Lemma 3.1.1. *All cocycles are trivializable on manifolds diffeomorphic to* $\mathbb{R}^n$.

Therefore, by Lemma 3.1.1, $\{\varphi_{ij} = id\}$ for all i, j, and hence it follows from the property

$$\varphi'_{ij} = \varphi_i^{-1}|_{U_{ij}} \circ \varphi_{ij} \circ \varphi_j|_{U_{ij}} \ \ for\ all\ i, j, \tag{3.1.29}$$

that we have the following observation:

$$id = \varphi_{ij} = \varphi_i^{-1}|_{U_{ij}} \circ \varphi_j|_{U_{ij}} \Leftrightarrow there\ exits\ \varphi\ such\ that\ \varphi|_{U_i} = \varphi_i \tag{3.1.30}$$

As we have a trivial cocycle condition with $\varphi_{ij} = id$, g_i's are glued together by transition functions φ_{ij} along with the trivial cocycle condition to form $g \in FMet(U)$ so that $g|_{U_i} = g_i$ and $\varphi|_{U_i} = \varphi_i$ for all i. Therefore, Ψ is *essentially surjective.*

(ii) We need to show that the map

$$Hom_{\mathcal{E}(U)}(g, g') \longrightarrow Hom_{holim_{Grpds}(\mathcal{E}(U_\bullet))}(\Psi(g), \Psi(g')) \tag{3.1.31}$$

is a bijection of sets. Let $g \xrightarrow[(g,\varphi)]{\sim} \varphi^* g$ be a morphism in $\mathcal{E}(U)$. Then Ψ sends (g, φ) to a family of morphisms

$$\{\varphi_i := \varphi|_{U_i}\} \tag{3.1.32}$$

where $g_i' = \varphi_i^* g_i$ with the condition $\varphi'_{ij} = \varphi_i^{-1}|_{U_{ij}} \circ \varphi_{ij} \circ \varphi_j|_{U_{ij}}$ for all i, j. Notice that as $\varphi_i := \varphi|_{U_i} = id$ for all i implies that φ must be the identity mapping in the first place, and hence we conclude that Ψ is injective on morphisms. Surjectivity, on the other hand, follows from the fact that the functor

$$C^\infty(\cdot, U) : \mathcal{C}^{op} \longrightarrow Sets, \quad V \mapsto C^\infty(V, U) \tag{3.1.33}$$

is a sheaf (for the fpqc-topology [6, 25, 49]). Indeed, $\imath : U_i \hookrightarrow U$ is an open embedding and φ_i is a diffeomorphism of U_i, one has a morphism

$$\imath \circ \varphi_i : U_i \longrightarrow U \;\; in \;\; C^\infty(U_i, U) \tag{3.1.34}$$

together with the suitable compatibility condition as above, and since $C^\infty(\cdot, U)$ is a sheaf on $\mathcal{C}$ in the sense of [25], by local-to-global properties of $C^\infty(\cdot, U)$ there is a diffeomorphism φ such that $\varphi|_{U_i} = \varphi_i$. Therefore, Ψ is surjective on morphisms as well. This completes the proof of claim.

◊

With the claim in hand together with the Definition 2.2.4, the canonical morphism

$$\Psi : \mathcal{E}(U) \longrightarrow holim_{Grpds}(\mathcal{E}(U_\bullet)) \tag{3.1.35}$$

is a weak equivalence (cf. Theorem 2.2.1) in $Grpds$, and this completes the proof of Theorem 3.1.1

□

Definition 3.1.1. The stack in Theorem 3.1.1

$$\mathcal{E} : \mathcal{C}^{op} \longrightarrow Grpds \tag{3.1.36}$$

is then called *the moduli stack of flat Lorentzian structures (or the moduli stack of solutions to the vacuum Einstein field equations with $\Lambda = 0$, i.e. that of Ricci flat pseudo-Riemannian metrics) on M.*

Once the specializations (such as considering vacuum with vanishing cosmological constant, etc...) are clear from the context, we sometimes call it directly *the moduli stack of Einstein gravity*.

3.2 Stacky equivalence of 3D quantum gravity with a gauge theory

As we have already discussed in the introduction (cf. Definition 1.1.3), one can introduce the notion of *equivalence* between quantum gravity and a gauge theory if the corresponding moduli spaces are isomorphic. That is, one requires

$$\phi : \mathcal{E}(M) \xrightarrow{\sim} \mathcal{M}_{flat}. \tag{3.2.1}$$

In fact, with the help of Theorem C.4.1, one has such *an equivalence of quantum gravity with a gauge theory in the case of 2+1 dimensional vacuum Einstein gravity (with vanishing cosmological constant) on a Lorentzian 3-manifold M of the form $\Sigma \times (0, \infty)$ where Σ is a closed Riemann surface of genus $g > 1$.* Now, we would like to show that the isomorphism ϕ naturally induces an *isomorphism between the corresponding stacks.*

We shall first revisit [6] and introduce a particular stack similar to BG_{con} (Example 2.11 in [6]). This helps us to view the space $\mathcal{M}_{flat}$ as a certain stack. Of course we first need to introduce the "flat" counterpart of this classifying stack BG_{con}. Just for simplicity we use $\mathcal{M}$ for the flat case whose construction is the same as that of BG_{con}. Note that the main ingredients of this structure are encoded by the theory of principal G-bundles in the following sense: Let $P \rightarrow M$ be a principal G-bundle on a 3-manifold M as above, and $\sigma \in \Gamma(U, P)$ a local trivializing section. We then schematically have

$$
\begin{array}{ccc}
P & \xrightarrow{\triangleleft\, G} & P \\
 & & \sigma \big\uparrow \big\downarrow \pi \\
 & & M
\end{array}
\tag{3.2.2}
$$

where $\triangleleft\, G$ denotes the right G-action on the smooth manifold P via diffeomorphisms of P. Let ω be a Lie algebra-valued connection one-form on P and $A := \sigma^*\omega$ its local representative, i.e., the Lie algebra-valued connection 1-form on M. A is also called *a local Yang-Mills field*. Then the space of *fields* is defined to be the infinite-dimensional space $\mathcal{A}$ of all G-connections on a principal G-bundle over M, i.e. $\mathcal{A} = \Omega^1(M) \otimes \mathfrak{g}$. Furthermore, the Chern-Simons action funtional $CS : \mathcal{A} \longrightarrow S^1$ is given by

$$CS[A] = \int_M \langle A, \mathrm{d}A + \frac{2}{3} A \wedge A \rangle \tag{3.2.3}$$

where $\langle \cdot, \cdot \rangle$ is a certain bilinear form on its Lie algebra $\mathfrak{g}$. The gauge group is locally of the form $\mathcal{G} = Map(U, G)$ that acts on the space $\mathcal{A}$ as follows: For all $\rho \in \mathcal{G}$ and $A \in \mathcal{A}$, we set

$$A \bullet \rho := \rho^{-1} \cdot A \cdot \rho + \rho^{-1} \cdot \mathrm{d}\rho. \tag{3.2.4}$$

The corresponding E-L equation in this case turns out to be

$$F_A = 0, \tag{3.2.5}$$

where $F_A = \mathrm{d}A + A \wedge A$ is *the curvature two-form* on M associated to A. Furthermore, under the gauge transformation, the curvature 2-form F_A behaves as follows:

$$F_A \longmapsto F_A \bullet \rho := \rho^{-1} \cdot F_A \cdot \rho \ \ for\ all\ \rho \in \mathcal{G}. \tag{3.2.6}$$

Note that the moduli space $\mathcal{M}_{flat}$ of flat connections on M, i.e. $A \in \mathcal{A}$ with $F_A = 0$, modulo gauge transformations emerges in many other areas of mathematics, such as topological quantum field theory, low-dimensional quantum invariants [78] or (infinite dimensional) Morse theory [41, 43, 64]. Note that for the gravitational interpretation (in the case of vanishing cosmological constant), one requires to consider the case of $G = ISO(2, 1)$ [9].

Lemma 3.2.1. *Let $\mathcal{C}$ be the category in Proposition 2.1.1 such that M is a Lorentzian 3-manifold topologically of the form $\Sigma \times \mathbb{R}$ where Σ is a closed Riemann surface of genus $g > 1$. The following assignment*

$$\mathcal{M} : \mathcal{C}^{op} \longrightarrow Grpds \tag{3.2.7}$$

defines a stack corresponding to the space $\mathcal{M}_{flat}$ where

1. *For each object U of $\mathcal{C}$, $\mathcal{M}(U)$ is a groupoid of flat G-connections on U with objects being the elements of the set $\Omega^1(U, \mathfrak{g})_{flat}$ of Lie algebra-valued 1-forms on U such that $F_A = 0$, and morphisms form the set*

$$Hom_{\mathcal{M}(U)}(A, A') = \{\rho \in \mathcal{G} : A' = A \bullet \rho\} \tag{3.2.8}$$

where the action of the gauge group $\mathcal{G}$, which is locally of the form $C^\infty(U', G)$, on $\Omega^1(U, \mathfrak{g})_{flat}$ is defined as above: For all $\rho \in \mathcal{G}$ and $A \in \mathcal{A}$, we set

$$A \bullet \rho := \rho^{-1} \cdot A \cdot \rho + \rho^{-1} \cdot \mathrm{d}\rho. \tag{3.2.9}$$

Furthermore, we denote a morphism

$$A \xrightarrow{\sim} A' = A \bullet \rho \tag{3.2.10}$$

in $Hom_{\mathcal{M}(U)}(A, A')$ by (A, ρ).

2. *To each morphism $U \xrightarrow{f} V$ in $\mathcal{C}$, i.e. $f : U \hookrightarrow V$ with $U \subset V$, one assigns*

$$\mathcal{M}(V) \xrightarrow{\mathcal{M}(f)} \mathcal{M}(U). \tag{3.2.11}$$

Here $\mathcal{M}(f) \in Fun(\mathcal{M}(V), \mathcal{M}(U))$ is a functor of categories whose action on objects and morphisms of $\mathcal{M}(V)$ is given as follows.

   (a) *For any object $A \in Ob(\mathcal{M}(V)) = \Omega^1(V, \mathfrak{g})_{flat}$,*

$$A \xrightarrow{\mathcal{M}(f)} f^*A \; (= A|_U) \tag{3.2.12}$$

where $\mathcal{M}(f)(A) := f^*A \in \Omega^1(U, \mathfrak{g})_{flat}$. Here we use the fact that the pullback (indeed the restriction to an open subset U in our case) of a flat connection in the sense that $F_A = 0$ is also flat.

(b) For any morphism $(A,\rho) \in Hom_{\mathcal{M}(V)}(A, A')$ with $\rho \in \mathcal{G}$ such that $A' = A \bullet \rho$, it follows from the fact that

$$f^*(A \bullet \rho) = f^*A \bullet f^*\rho \tag{3.2.13}$$

where $f^*\rho = \rho \circ f \in C^\infty(U, G)$, we conclude that $f^*(A \bullet \rho)$ lies in the orbit space of f^*A, and hence we get

$$\left(A \xrightarrow[(A,\rho)]{\sim} A' = A \bullet \rho\right) \xrightarrow{\mathcal{M}(f)} \left(f^*A \xrightarrow[(f^*A,\rho\circ f)]{\sim} f^*(A \bullet \rho) = f^*A \bullet f^*\rho\right) \tag{3.2.14}$$

where $\mathcal{M}(f)(A,\rho) := (f^*A, f^*\rho)$ is a morphism in $\mathcal{M}(U)$. Note that the identity 3.2.13 we mentioned above can indeed be proven by just local computations of the pullback of a connection A together with the action $A \bullet \rho$.

Proof. This is similar to the proofs of Proposition 2.1.1 and Theorem 3.1.1 in the special case where $n = 3$ and M as above. For a complete treatment to the generic case (i.e. without flatness requirement), see Examples 2.10 and 2.11 in [6]. For the flat case, on the other hand, one has exactly the same proof with $\Omega^1(U, \mathfrak{g})_{flat}$ instead of $\Omega^1(U, \mathfrak{g})$ thanks to the fact that the pullback of a flat connection by a canonical inclusion $U \hookrightarrow V$ between open subsets is also flat. □

To sum up, we have the following observations so far:

1. Before the stacky constructions, by Theorem C.4.1, we already have an isomorphism of moduli spaces

$$\phi : \mathcal{E}(M) \xrightarrow{\sim} \mathcal{M}_{flat} \tag{3.2.15}$$

   in the case of vacuum Einstein general relativity with the cosmological constant $\Lambda = 0$ on a Lorentzian 3-manifold of the form $M = \Sigma \times (0, \infty)$ where Σ is a closed Riemann surface of genus $g > 1$.

2. Let $\mathcal{C}$ be the category in Proposition 2.1.1. In Theorem 3.1.1, we have constructed *the stack of flat Lorentzian structures* on a generic n-manifold M

$$\mathcal{E} : \mathcal{C}^{op} \longrightarrow Grpds. \tag{3.2.16}$$

3. In Lemma 3.2.1, we have introduced *the classifying stack of principal G-bundles with flat connections* on Σ

$$\mathcal{M} : \mathcal{C}^{op} \longrightarrow Grpds \tag{3.2.17}$$

where $\mathcal{C}$, in that case, involves particular choices of dimension ($n = 3$) and the form of a manifold $M := \Sigma \times (0, \infty)$.

Given a closed Riemann surface Σ of genus $g > 1$, we now intend to show that if $\mathcal{C}$ is the category in Proposition 2.1.1 with the special case $(n = 3)$ where M is a Lorentzian 3-manifold of the form $\Sigma \times (0, \infty)$, then there exists *an invertible natural transformation*

$$\mathcal{C}^{op} \underset{\mathcal{M}}{\overset{\mathcal{E}}{\rightrightarrows}} Grpds, \quad \Phi \Downarrow \tag{3.2.18}$$

between these two stacks $\mathcal{E}$ and $\mathcal{M}$. This eventually provides a stacky extension of the isomorphism between the underlying moduli spaces.

Theorem 3.2.1. *Let $\mathcal{E}$ and $\mathcal{M}$ be as above. Then there exists an invertible natural transformation*

$$\Phi : \mathcal{E} \Rightarrow \mathcal{M}. \tag{3.2.19}$$

Proof. Given a closed Riemann surface Σ of genus $g > 1$, let $\mathcal{C}$ be the category of open subsets of $\Sigma \times (0, \infty)$ with morphisms being the canonical inclusions whenever $U \subset V$. Recall that, as we explained in section 1.1, any solution of the vacuum Einstein field equations with vanishing cosmological constant is locally *flat*. This means that the metric is locally equivalent to the standard Minkowski metric $\eta_{\mu\nu}$. Then, we first notice that it follows from gauge theoretic realization of 2+1 gravity (with $\Lambda = 0$) in Cartan's formalism that any solution to the vacuum Einstein field equation with $\Lambda = 0$ on any open subset of M defines a flat $ISO(2, 1)$-connection, and thus for any object U in $\mathcal{C}$, we have a natural map

$$\Phi_U : \mathcal{E}(U) \longrightarrow \mathcal{M}(U) \tag{3.2.20}$$

which is indeed *a functor of groupoids* defined as follows:

1. To each $g \in FMet(U)$, a solution to the vacuum Einstein field equations (with $\Lambda = 0$) over U, one assigns the corresponding flat $ISO(2,1)$-connection A^g in $\Omega^1(U, \mathfrak{iso}(2,1))_{flat}$ described by the Cartan's formalism. That is,

$$g \xrightarrow{\Phi_U} A^g. \tag{3.2.21}$$

2. Note that, as we addressed in Section C.1, Cartan's formalism not only provides the equivalence between the Einstein-Hilbert action functional for such a 2+1 gravity and the one for Chern-Simons theory with the gauge group $ISO(2,1)$, but also encodes the symmetries of each theory in the sense that *the diffeomorphism invariance of 2+1 gravity theory does correspond to the gauge invariance behaviour of the associated Chern-Simons theory (and vice versa).* It means that the equivalence classes $[g]$ of flat pseudo-Riemannian metrics correspond to the gauge equivalence classes of the associated connections $[A^g]$. Therefore, for any $g' \in [g]$, i.e. $g' = \varphi^* g$ for some diffeomorphism φ, the corresponding connections

$$A^g \;\; and \;\; A^{\varphi^* g} \tag{3.2.22}$$

are also gauge equivalent, and hence lie in the same equivalence class. That is, there exist $\rho_\varphi \in \mathcal{G}$, a gauge transformation associated to φ, such that

$$A^{\varphi^* g} = A^g \bullet \rho_\varphi. \tag{3.2.23}$$

In other words, such a correspondence can also be expressed as the following commutative diagram:

$$\begin{array}{ccc} g & \xrightarrow{\varphi} & \varphi^* g \\ \downarrow & & \downarrow \\ A^g & \xrightarrow[\rho_\varphi]{} & A^{\varphi^* g} = A^g \bullet \rho_\varphi \end{array} \tag{3.2.24}$$

together with a group isomorphism

$$Diff(U) \longrightarrow C^\infty(U, G), \quad \varphi \mapsto \rho_\varphi, \tag{3.2.25}$$

where $Diff(U)$ is endowed with the usual composition, and the group operation on $C^\infty(U, G)$ is given by the pointwise multiplication.

3. To each morphism $(g, \varphi) : g \longrightarrow \varphi^* g$ in $Hom_{\mathcal{E}(U)}(g, g')$, Φ_U associates a morphism

$$A^g \xrightarrow[(A^g, \rho_\varphi)]{\sim} A^g \bullet \rho_\varphi \; (= A^{\varphi^* g}) \tag{3.2.26}$$

where $\rho_\varphi \in C^\infty(U, ISO(2,1))$ is a gauge transformation corresponding to φ in accordance with the diagram in 3.2.24. Therefore, for any morphism $f : U \hookrightarrow V$ in $\mathcal{C}$, using 3.2.25, one also has the following commutative diagram:

$$\begin{array}{ccc} Diff(V) & \xrightarrow{f^* \; (= \cdot|_U)} & Diff(U) \\ {\scriptstyle by\ 3.2.25} \downarrow & & \downarrow {\scriptstyle by\ 3.2.25} \\ C^\infty(V, ISO(2,1)) & \xrightarrow[f^* \; (= \cdot|_U)]{} & C^\infty(U, ISO(2,1)) \end{array} \tag{3.2.27}$$

4. *Functoriality*. Given a composition of morphisms $g \xrightarrow{(g,\varphi)} \varphi^* g \xrightarrow{(\varphi^* g, \psi)} \psi^* \varphi^* g$ in $\mathcal{E}(U)$, that is

$$g \xrightarrow[(g,\varphi)]{} \varphi^* g \xrightarrow[(g,\psi)]{} \psi^* \varphi^* g = (\varphi \circ \psi)^* g, \quad \text{with } g \xrightarrow{(g \cdot \varphi) \cdot \psi = g \cdot (\varphi \circ \psi)} \psi^* \varphi^* g \tag{3.2.28}$$

we have the following commutative diagram

$$\begin{array}{ccccc} g & \xrightarrow{\varphi} & \varphi^* g & \xrightarrow{\psi} & \psi^* \varphi^* g = (\varphi \circ \psi)^* g \\ \downarrow & & \downarrow & & \downarrow \\ A^g & \xrightarrow{\rho_\varphi} & A^{\varphi^* g} & \xrightarrow{\rho_\psi} & A^{(\varphi \circ \psi)^* g} \end{array} \tag{3.2.29}$$

where, using the commutativity,

$$A^g \bullet \rho_{\varphi \circ \psi} = A^{(\varphi \circ \psi)^* g} = A^{\varphi^* g} \bullet \rho_\psi = (A^g \bullet \rho_\varphi) \bullet \rho_\psi, \tag{3.2.30}$$

and hence, with the abuse of notation by using just φ in place of (g, φ) (similarly for ψ and $\varphi \circ \psi$), one has

$$\rho_\varphi \cdot \rho_\psi = \rho_{\varphi \circ \psi} \tag{3.2.31}$$

which gives the desired functoriality in the sense that

$$\Phi_U(g, \varphi \circ \psi) = \rho_{\varphi \circ \psi} = \rho_\varphi \cdot \rho_\psi = \Phi_U(g, \varphi) \cdot \Phi_U(g, \psi). \tag{3.2.32}$$

Now, we need to show that for each morphism $f : U \to V$ in $\mathcal{C}$, i.e. $f : U \hookrightarrow V$ with $U \subset V$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{E}(V) & \xrightarrow{\Phi_V} & \mathcal{M}(V) \\ {\scriptstyle \mathcal{E}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{M}(f)} \\ \mathcal{E}(U) & \xrightarrow[\Phi_U]{} & \mathcal{M}(V) \end{array} \tag{3.2.33}$$

In fact, the commutativity follows from the definition 3.2.20 of the morphism Φ_U: Let $g \in FMet(V)$, then we get, from the construction and the restriction functor $\cdot|_U$, the natural diagram

$$\begin{array}{ccc} g & \longrightarrow & A^g \\ \downarrow & & \downarrow \\ g|_U & \longrightarrow & A^{g|_U} = A^g|_U \end{array} \tag{3.2.34}$$

Hence, a direct computation yields

$$\begin{aligned} (\mathcal{M}(f) \circ \Phi_V)(g) &= f^* A^g \\ &= A^g|_U \\ &= A^{g|_U} \quad \text{from 3.2.34} \\ &= A^{f^* g} \\ &= \Phi_U(f^* g) \\ &= (\Phi_U \circ \mathcal{E}(f))(g), \end{aligned} \tag{3.2.35}$$

which gives an "objectwise" commutativity of the diagram. Similarly, for any morphism

$$(g, \varphi) : g \longrightarrow \varphi^* g \text{ in } Hom_{\mathcal{E}(V)}(g, g') \tag{3.2.36}$$

one has another natural diagram again from the definition and the restriction functor as above

$$\begin{array}{ccc} \varphi & \longrightarrow & \rho_\varphi \\ \downarrow & & \downarrow \\ \varphi|_U & \longrightarrow & \rho_{\varphi|_U} = \rho_\varphi|_U \end{array} \tag{3.2.37}$$

Therefore, we obtain

$$
\begin{aligned}
(\mathcal{M}(f) \circ \Phi_V)(g, \varphi) &= (f^* A^g, f^* \rho_\varphi) \\
&= (A^{f^*g}, \rho_\varphi|_U) \quad \text{from 3.2.34} \\
&= (A^{f^*g}, \rho_{\varphi|_U}) \quad \text{from 3.2.37} \\
&= (A^{f^*g}, \rho_{f^*\varphi}) \\
&= \Phi_U(f^*g, f^*\varphi) \\
&= (\Phi_U \circ \mathcal{E}(f))(g, \varphi),
\end{aligned} \tag{3.2.38}
$$

which implies the desired "morphismwise" commutativity.

Therefore, Φ defines a natural transformation between $\mathcal{E}$ and $\mathcal{M}$ along with the collection

$$\big\{\Phi_U : \mathcal{E}(U) \longrightarrow \mathcal{M}(U)\big\}_{U \in Ob(\mathcal{C})} \tag{3.2.39}$$

of natural maps defined by means of Cartan geometric formulation of Einstein gravity together with the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{E}(V) & \xrightarrow{\Phi_V} & \mathcal{M}(V) \\
{\scriptstyle \mathcal{E}(f)} \downarrow & & \downarrow {\scriptstyle \mathcal{M}(f)} \\
\mathcal{E}(U) & \xrightarrow[\Phi_U]{} & \mathcal{M}(V)
\end{array} \tag{3.2.40}
$$

for each morphism $f : U \to V$ in $\mathcal{C}$, i.e. $f : U \hookrightarrow V$ with $U \subset V$.

The inverse construction, on the other hand, essentially follows from Mess' result (cf. Theorem C.4.1) that for each object U in $\mathcal{C}$, the map Φ_U is indeed invertible and the inverse map

$$\Phi_U^{-1} : \mathcal{M}(U) \to \mathcal{E}(U) \tag{3.2.41}$$

is defined as follows: Once we choose a hyperbolic structure on a closed orientable surface Σ of genus $g > 1$ and view it as a Riemannian surface, then a flat connection A defines the holonomy representation of such a hyperbolic structure, and hence a Fuchsian representation (cf. section C.3 and corollary C.3.2). Thus, by Theorem C.4.1, there exist a suitable flat pseudo-Riemannian manifold M whose flat structure given by a flat pseudo-Riemannian metric denoted by g_A such that $M = \Sigma \times (0, \infty)$ whose surface group representation agrees with the former one. Therefore, we have a well-defined assignment on objects

$$\Phi_U^{-1} : \mathcal{M}(U) \to \mathcal{E}(U), \;\; A \mapsto \; g_A \tag{3.2.42}$$

such that due to the fact that surface group representation agrees with the former one (cf. Theorem C.4.1), the flat connection A^{g_A} associated to g_A is exactly the connection we started with, i.e.

$$\Phi_U \circ \Phi_U^{-1} : A \longmapsto g_A \longmapsto A^{g_A} = A. \tag{3.2.43}$$

Then, by using the similar analysis as above, it is rather straightforward to check that we have a well-defined assignment Φ_U^{-1} on both objects and morphisms together with appropriate commutative diagram analogous to the one in 3.2.33, and hence Φ_U^{-1} is functor of groupoids as well. Thus, by construction, Φ^{-1} is indeed a natural transformation that serves as an inverse of the natural transformation Φ between two stacks. Therefore, one has *an invertible natural transformation* Φ

$$\mathcal{C}^{op} \underset{\mathcal{M}}{\overset{\mathcal{E}}{\;\;\Phi \Downarrow\;\;}} Grpds. \tag{3.2.44}$$

This completes the proof of Theorem 3.2.1.

CHAPTER 4

FORMAL MODULI PROBLEMS AND 3D EINSTEIN GRAVITY

We first intend to summarize what we have done so far and provide a kind of a recipe to motivate the derived geometric interpretation of a classical field theory. We then concentrate on a particular 2+1 dimensional Einstein gravity.

i. Employing the above approaches, describing a classical field theory boils down to the study of the moduli space $\mathcal{EL}$ of solutions to the Euler-Lagrange equations (the critical locus of the action functional). This is in fact encoded by a certain moduli functor.

ii. As stressed in Section 2.1, a moduli functor, however, would not be representable in general due to the existence of degenerate critical points or non-freeness of the action of the symmetry group on the space of fields. In order to avoid these sorts of problems (and to capture the perturbative behavior at the same time), one may replace the naïve notion of a moduli problem by *a formal moduli problem* as addressed in Section B.3.2.

iii. A formal moduli problem $\mathcal{F}$, on the other hand, turns out to be unexpectedly tractable notion in the sense that understanding $\mathcal{F}$, at the end of the day, boils down to finding a suitable $\mathcal{L}_\infty$ algebra $\mathfrak{g}$ (a dgla, in fact) such that $\mathcal{F}$ can be represented by the Maurer-Cartan functor $\mathcal{B}\mathfrak{g}$ associated to $\mathfrak{g}$.

iv. Having obtained an appropriate $\mathcal{L}_\infty$ algebra $\mathfrak{g}$, we can analyze the structure of $\mathfrak{g}$ so as to encode the aspects of the theory.

We shall first briefly give two relatively tractable examples. For details, see [3].

Example 4.0.1. Consider a free scalar *massless* field theory on a Riemannian manifold M with the space of fields being $C^{\infty}(M)$ and the action functional being of the form

$$\mathcal{S}(\phi) := \int\limits_{M} \phi \Delta \phi. \tag{4.0.1}$$

The corresponding E-L equation in this case turns out to be

$$\Delta \phi = 0, \tag{4.0.2}$$

and hence the moduli space EL of solutions to the E-L equations is the moduli space of harmonic functions

$$\big\{ \phi \in C^{\infty}(M) : \Delta \phi = 0 \big\}. \tag{4.0.3}$$

Now, having employed the derived enrichment $\mathcal{EL}$ of EL as described above, we need to find a suitable $\mathcal{L}_{\infty}$ algebra $\mathcal{E}$ whose Maurer-Cartan fuctor $\mathcal{BE}$ represents the formal moduli problem $\mathcal{EL}$. The answer is as follows: We define $\mathcal{E}$ to be the two-term complex concentrated in degrees 0 and 1

$$\mathcal{E} : C^{\infty}(M) \xrightarrow{\Delta} C^{\infty}(M)[-1], \tag{4.0.4}$$

equipped with a sequence $\{\ell_n\}$ of multilinear maps where $\ell_1 := \Delta$ and $\ell_i = 0$ for all $i > 1$. The Maurer-Cartan equation, on the other hand, turns out to be

$$\Delta \phi = 0. \tag{4.0.5}$$

Hence, the set of 0-simplices of the simplicial set $\mathcal{BE}(A)$ for A ordinary Artinian algebra is given as

$$\big\{ \phi : \Delta \phi = 0 \big\}. \tag{4.0.6}$$

For further details and interpretation of other simplices, see chapter 2 of [4] or chapter 4 of [3].

Example 4.0.2. We shall revisit Chern-Simons gauge theory on a closed, orientable 3-manifold X with the gauge group H. As usual, Let $P \to X$ be a principal H-bundle on X, $\mathfrak{h}$ denote the Lie algebra of H. Suppose $A \in \mathcal{A} := \Omega^1(X) \otimes \mathfrak{h}$ is the Lie algebra-valued connection 1-form on X such that the Chern-Simons action functional $CS : \mathcal{A} \longrightarrow S^1$ is given by

$$CS[A] = \int\limits_{X} \langle A, \mathrm{d}_{dR} A + \frac{2}{3} A \wedge A \rangle \tag{4.0.7}$$

where $\langle \cdot, \cdot \rangle$ is a certain bilinear form on $\mathfrak{h}$. Here, the gauge group $\mathcal{G}$ is locally of the form $Map(U, H)$ with the usual action on the space $\mathcal{A}$. The corresponding E-L equation, in this case, turns out to be

$$F_A := \mathrm{d}_{dR} A + A \wedge A = 0, \tag{4.0.8}$$

where F_A is the curvature two-form on X associated to A. Hence, the critical locus of CS modulo gauge transformations is the set

$$\big\{ [A] \in \Omega^1(X) \otimes \mathfrak{h} : \mathrm{d}_{dR} A + A \wedge A = 0 \big\}. \tag{4.0.9}$$

As before, we define a suitable $\mathcal{L}_\infty$ algebra $\mathfrak{g}$ encoding the formal moduli problem as follows:

$$\mathfrak{g} := \Omega^*(X) \otimes \mathfrak{h}[1], \tag{4.0.10}$$

where the only *non-zero* multilinear maps are $\ell_1 := \mathrm{d}_{dR}$ and $\ell_2 := [\cdot, \cdot]$ given as in Example B.3.1. Notice that the Maurer-Cartan equation, in this case, becomes

$$\mathrm{d}_{dR} A + \frac{1}{2}[A, A] = 0, \tag{4.0.11}$$

and hence the corresponding Maurer-Cartan functor $\mathcal{B}\mathfrak{g}$ yields the desired result. We shall elaborate the construction below. A relatively complete treatment can be found in chapter 4 of [3], chapter 5 of [14], or [40]. Furthermore, as stressed in [14], the space of all fields associated to the theory, which is encoded by a particular $\mathcal{L}_\infty$ algebra $\mathfrak{g}$, can also be interpreted in the Batalin-Vilkovisky formalism as follows:

- The space of degree -1 fields, so-called *ghosts*, corresponds to the space

$$\Omega^0(X) \otimes \mathfrak{h} = Map(X, H). \tag{4.0.12}$$

- The space of degree 0 fields, so-called *fields*, corresponds to the space

$$\Omega^1(X) \otimes \mathfrak{h}. \tag{4.0.13}$$

- The space of degree 1 fields, so-called *anti-fields*, corresponds to the space

$$\Omega^2(X) \otimes \mathfrak{h}. \tag{4.0.14}$$

- The space of degree 2 fields, so-called *anti-ghosts*, corresponds to the space

$$\Omega^3(X) \otimes \mathfrak{h}. \tag{4.0.15}$$

4.1 Formal moduli problem of Chern-Simons theory

A formal moduli problem encoding *deformation theory* of the flat H-bundles $P \to M$ on a closed orientable 3-manifold M can be defined as follows [3]: Let ∇ be a flat connection on P. Define an $\mathcal{L}_\infty$ algebra $\mathfrak{g}$ to be

$$\mathfrak{g} := \Omega^*(M) \otimes \mathfrak{h}, \tag{4.1.1}$$

where the only *non-zero* multilinear maps are $\ell_1 := \mathrm{d}_\nabla$ and $\ell_2 := [\cdot, \cdot]$ given as in Example B.3.1. Here, d_∇ denotes *the covariant derivative* defined as a coupling of the de Rham differential d_{dR} with the connection ∇:

$$d_\nabla := \mathrm{d}_{dR} + [\nabla, \cdot\,]. \tag{4.1.2}$$

Remark 4.1.1. As ∇ is flat, the differential d_∇ squares to zero, and hence one can form the following complex

$$\cdots \longrightarrow \Omega^i(M, \mathfrak{h}) \xrightarrow{d_\nabla} \Omega^{i+1}(M, \mathfrak{h}) \to \cdots \tag{4.1.3}$$

where $\Omega^i(M, \mathfrak{h})$ is just the short-hand standard notation for $\Omega^i(M) \otimes \mathfrak{h}$ for all i, and will be used repeatedly throughout the discussion below.

Starting with the 0-simplicies, let $(R, \mathfrak{m}_R)$ be a dg Artinian algebra with the maximal ideal $\mathfrak{m}_R$. As the difference $\nabla' - \nabla$ is again an $\mathfrak{h}$-valued 1-form on P for any $\mathfrak{h}$-valued 1-form ∇', the space $\Omega^i(M, \mathfrak{h})$ is in fact affine. A deformation of ∇ is then given by an element

$$A \in \Omega^1(M, \mathfrak{h}) \otimes \mathfrak{m}_R^0. \tag{4.1.4}$$

Therefore, *the curvature $F_{\nabla'}$ (or just $F(A)$) of the deformed connection* $\nabla' := \nabla + A$ is given as

$$\begin{aligned} F(A) &= \mathrm{d}_{dR}(\nabla + A) + \frac{1}{2}[\nabla + A, \nabla + A] \\ &= \mathrm{d}_{dR}\nabla + \frac{1}{2}[\nabla, \nabla] + \mathrm{d}_{dR}A + \frac{1}{2}[A, A] + [\nabla, A] \\ &= d_\nabla A + \frac{1}{2}[A, A] \quad (from\ F_\nabla = 0) \end{aligned} \tag{4.1.5}$$

such that from the usual Bianchi identity, we have

$$
\begin{aligned}
0 &= \mathrm{d}_{\nabla'} F_{\nabla'} \\
&= \mathrm{d}_{dR} F_{\nabla'} + [\nabla', F_{\nabla'}] \\
&= \mathrm{d}_{dR} F_{\nabla'} + [\nabla, F_{\nabla'}] + [A, F_{\nabla'}] \\
&= \mathrm{d}_{\nabla} F_{\nabla'} + [A, F_{\nabla'}]
\end{aligned} \tag{4.1.6}
$$

Now we denote $F_{\nabla'}$ by $F(A)$ to emphasize the connection A deforming ∇, then the computation above gives

$$
d_{\nabla} F(A) + [A, F(A)] = 0. \tag{4.1.7}
$$

Note that even if it captures the information about the deforming connection A, the notation $F(A)$ could be misleading in the sense that while $F(A)$ stands for the curvature of the deformed connection $\nabla' := \nabla + A$ (deformed by A), F_A denotes the curvature 2-form for the connection A. Now, one can define the following formal moduli problem $\mathcal{B}\mathfrak{g} \in Moduli_k$ for the Chern-Simons theory.

Lemma 4.1.1. *Let $(R, \mathfrak{m}_R)$ be a dg Artinian algebra with the maximal ideal $\mathfrak{m}_R$ and $n \in \mathbb{Z}_{\geq 0}$. Then the set of n-simplicies of $\mathcal{B}\mathfrak{g}(R)$ is given by*

$$
\mathcal{B}\mathfrak{g}(R)_n = \Big\{ A \in \bigoplus_{p+q+r=1} \Omega^p(M, \mathfrak{h}) \otimes \mathfrak{m}_R^q \otimes \Omega^r(\Delta^n) : \ \mathrm{d}_{\nabla} A \mp \mathrm{d}_R A \mp \mathrm{d}_{dR} A + \frac{1}{2}[A, A] = 0. \Big\}
$$

where $\mathrm{d}_{\nabla}, \mathrm{d}_R$ and d_{dR,Δ^n} denote the differentials on dg algebras $\Omega^p(M, \mathfrak{h}), R$ and $\Omega^*(\Delta^n)$ respectively. Furthermore, the choice of each $\mp$ sign can be determined as instructed in Remark B.3.3.

Proof. For the construction and details, we refer to Costello and Gwilliam's book [3], ch. 4.3, pg.30. □

Remark 4.1.2. One can recover the usual moduli space of flat H-connections from Lemma 4.1.1 together with some extra structures being manifested by higher simplicies. This essentially relates the gauge equivalent solutions in the following manner:

1. Let $(R, \mathfrak{m}_R)$ be an ordinary Artinian algebra with the maximal ideal $\mathfrak{m}_R$, and $n = 0$. Note that $\Omega^*(\Delta^0 = pt) \cong k$ if $* = 0$, else it is 0. As R can be viewed

a dg algebra concentrated in degree 0 (i.e. $q = 0$), and $n = 0$ with $\mathrm{d}_R = 0 = \mathrm{d}_{dR,\Delta^n}$, the only possible scenario in which one can form a cohomological degree 1 element is when $p = 1$. The set of 0-simplicies, therefore, is given as

$$\mathcal{B}\mathfrak{g}(R)_0 = \big\{A \in \Omega^1(M,\mathfrak{h}) \otimes \mathfrak{m}_R :\ \mathrm{d}_\nabla A + \frac{1}{2}[A,A] = 0\big\} \tag{4.1.8}$$

which is the usual moduli space of flat H-connections where $F(A) := \mathrm{d}_\nabla A + \frac{1}{2}[A,A]$ is the curvature of deformed connection as above.

2. Assume $(R,\mathfrak{m}_R)$ is again an ordinary Artinian algebra (i.e. $q = 0$) with the maximal ideal $\mathfrak{m}_R$, and $n = 1$. Now, since $n = 1$, only $\Omega^0(\Delta^1)$ and $\Omega^1(\Delta^1)$ will survive (i.e. $r \in \{0,1\}$). Thus, one has two possible configurations to form cohomologicaly degree 1 element:

$$(p,q,r)\ \in \{(1,0,0),(0,0,1)\}. \tag{4.1.9}$$

Therefore, a generic degree 1 element A in $\mathcal{B}\mathfrak{g}(R)_1$ has a decomposition

$$\Big(\Omega^1(M,\mathfrak{h}) \otimes \mathfrak{m} \otimes \Omega^0(\Delta^1)\Big) \oplus \Big(\Omega^0(M,\mathfrak{h}) \otimes \mathfrak{m} \otimes \Omega^1(\Delta^1)\Big) \tag{4.1.10}$$

where $\Omega^0(\Delta^1) \cong C^\infty([0,1])$ and $\Omega^1(\Delta^1) = span_k\{\mathrm{d}t\}$ such that A can be expressed in a local chart (U,x) as

$$A = A_0(t) + A_1(t) \cdot \mathrm{d}t. \tag{4.1.11}$$

Herein $A_0(t) = a_{0i}(x,t)\mathrm{d}x^i$ with $a_{0i}(x,t)$ is $\mathfrak{h} \otimes \mathfrak{m}$-valued smooth function on $[0,1]$, and $A_1(t)$ is a smooth $\mathfrak{h} \otimes \mathfrak{m}$-valued function on M parametrized by t. It follows from the properties of triple complexes outlined in Remark B.3.3 and the definition of the Maurer-Cartan equation B.3.5 that one can obtain the following equations [3]

$$\mathrm{d}_\nabla A_0(t) + \frac{1}{2}[A_0(t), A_0(t)] = 0 \tag{4.1.12}$$

$$\frac{\mathrm{d}A_0(t)}{\mathrm{d}t} + [A_1(t), A_0(t)] = 0, \tag{4.1.13}$$

together with the commutative diagram

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \Omega^0(M,\mathfrak{h})\otimes\mathfrak{m}\otimes\Omega^1(\Delta^1) & \xrightarrow{d_\nabla\otimes id^{\otimes 2}} & \Omega^1(M,\mathfrak{h})\otimes\mathfrak{m}\otimes\Omega^1(\Delta^1) & \longrightarrow & \cdots \\
 & & {\scriptstyle id^{\otimes 2}\otimes d_{dR,\Delta^1}}\uparrow & & \uparrow{\scriptstyle id^{\otimes 2}\otimes d_{dR,\Delta^1}} & & \\
\cdots & \longrightarrow & \Omega^0(M,\mathfrak{h})\otimes\mathfrak{m}\otimes\Omega^0(\Delta^1) & \xrightarrow[d_\nabla\otimes id^{\otimes 2}]{} & \Omega^1(M,\mathfrak{h})\otimes\mathfrak{m}\otimes\Omega^0(\Delta^1) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}
\tag{4.1.14}
$$

Here, Equation 4.1.12 implies that $\{A_0(t)\}$ defines a flat family of connections while Equation 4.1.13 implies that the gauge equivalence classes of the family $\{A_0(t)\}$ are independent of t up to homotopy defined by $A_1(t)$.

3. Higher simplicies provide an enriched and refined structure through which one can capture further relations between equivalences, and relations between such relations etc... In other words, higher simplicial structures in derived stacks allow us to encodes "symmetries between symmetries" and "symmetries between symmetries between symmetries" type argument. Different layers of the simplicial structure encode different levels of symmetries/equivalences. Each set of simplicies of a derived stack records further relations, and hence it is able to encode the "higher symmetries" argument above. For more details on the interpretation of higher simplicial structures in the case of gauge or free field theories, see [3], ch.4.

4.2 Formal moduli problem of 3D Einstein gravity

As an immediate application to the formulation of gauge theories in the language of formal moduli problems [3], we have a natural formal moduli problem for the Cartan geometric formulation of a 2+1 dimensional Einstein gravity.

Corollary 4.2.1. *The construction in Lemma 4.1.1 defines a natural formal moduli problem for the 2+1 Cartan's geometric formulation of vacuum Einstein gravity theory with vanishing cosmological constant.*

Proof. As manifestly analyzed in Proposition 2.1.1 and Theorem 3.1.1, we have the following groupoid-valued functor

$$\mathcal{E} : \mathcal{C}^{op} \longrightarrow \ Grpds \tag{4.2.1}$$

which defines a *stack* where for each object U of $\mathcal{C}$, $\mathcal{E}(U)$ is a groupoid of (Ricci) flat pseudo-Riemannian metrics on U with objects being the elements of *set* $FMet(U)$

$$FMet(U) := \big\{ g \in \Gamma(U, Met_M) : Ric(g) = 0 \big\} \tag{4.2.2}$$

of (Ricci) flat pseudo-Riemannian metrics on U where Met_M denotes metric "bundle" on M and $Ric(g)$ is the Ricci-tensor. Now, thanks to the Cartan formalism, which is briefly explained in Section C.1, one can reformalize such a gravity theory in which the Einstein-Hilbert action is presented as

$$\mathcal{I}'_{EH}[e, \omega] = \int\limits_M e^a \wedge \left(\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c \right) \tag{4.2.3}$$

where $\omega \in \Omega^1(LM, \mathfrak{so}(2,1))$ and $e \in \Omega^1(LM, \mathbb{R}^{2+1})$ are $\mathfrak{so}(2,1)$-valued Ehresmann connection 1-form on the frame bundle LM on M, and $e \in \Omega^1(LM, \mathbb{R}^{2,1})$ is *the coframe field.* The variation of this action, on the other hand, with respect to ω and e independently yields

$$\delta\mathcal{I}'_{EH}[e, \omega] = \int\limits_M tr(\delta\omega \wedge \Omega[\omega] + \delta e \wedge d_\omega e), \tag{4.2.4}$$

and thus the corresponding field equations are of the form

$$0 = \Omega[\omega] = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega] \tag{4.2.5}$$

$$0 = d_\omega e = \mathrm{d}e^a + [\omega, e]. \tag{4.2.6}$$

Note that [19], solving 4.2.6 for ω as a function of e and rewriting 4.2.5 as an equation of $\omega[e]$ give rise to the usual vacuum Einstein field equation. Now, set the corresponding Cartan connection

$$A \in \Omega^1(LM, \mathfrak{iso}(2,1)) \tag{4.2.7}$$

which can be expressed uniquely as a decomposition

$$A = \omega + e \ \in \Omega^1(LM, \mathfrak{so}(2,1)) \oplus \Omega^1(LM, \mathbb{R}^{2,1}). \tag{4.2.8}$$

Let $\mathfrak{g}$ be an $\mathcal{L}_\infty$ algebra

$$\mathfrak{g} := \Omega^*(M) \otimes \mathfrak{iso}(2,1), \tag{4.2.9}$$

where the only *non-zero* multilinear maps are $\ell_1 := \mathrm{d}_\nabla$ and $\ell_2 := [\cdot,\cdot]$ given as in Example B.3.1. Here, d_∇ denotes *the covariant derivative* defined as a coupling of the de Rham differential d_{dR} with the fixed flat connection ∇ on LM:

$$d_\nabla := \mathrm{d}_{dR} + [\nabla, \cdot\,]. \tag{4.2.10}$$

Then, from Lemma 4.1.1, we have a formal moduli problem $\mathcal{B}\mathfrak{g} \in Moduli_k$

$$\mathcal{B}\mathfrak{g} : \ dgArt_k \longrightarrow \ sSets, \ \ (R, \mathfrak{m}_R) \longmapsto \mathcal{B}\mathfrak{g}(R) \tag{4.2.11}$$

such that for each n, the set of n-simplicies of $\mathcal{B}\mathfrak{g}(R)$ is given by

$$\mathcal{B}\mathfrak{g}(R)_n = \Big\{ A \in \bigoplus_{p+q+r=1} \Omega^p(M, \mathfrak{iso}(2,1)) \otimes \mathfrak{m}_R^q \otimes \Omega^r(\Delta^n) : \ d_{tot}A + \frac{1}{2}[A,A] = 0 \Big\}.$$

where $d_{tot}A := \mathrm{d}_\nabla A \mp \mathrm{d}_R A \mp \mathrm{d}_{dR} A$. Note that the standard moduli data can be recovered by considering the set of 0-simplicies of $\mathcal{B}\mathfrak{g}(R)$ in the case of $(R, \mathfrak{m}_R)$ being an ordinary Artinian algebra with the maximal ideal $\mathfrak{m}_R$. Indeed, we have

$$\mathcal{B}\mathfrak{g}(R)_0 = \big\{ A \in \Omega^1(M, \mathfrak{iso}(2,1)) \otimes \mathfrak{m}_R : \ \mathrm{d}_\nabla A + \frac{1}{2}[A,A] = 0 \big\}. \tag{4.2.12}$$

When A has the unique decomposition, one has a reductive splitting [66, 67]:

$$F_A = 0 \iff \Omega[\omega] = 0 \ \ and \ \ d_\omega e = 0, \tag{4.2.13}$$

These are the desired defining relations. □

Remark 4.2.1. We should point out that all formal moduli constructions for Einstein gravity work with non-zero cosmological constant as well. As we stressed before, one has a Chern-Simons theory with either $G = SL(2,\mathbb{R}) \times SL(2;\mathbb{R})$ for $\Lambda < 0$ or $G = SL(2,\mathbb{C})$ for $\Lambda > 0$. Therefore, we end up with exactly the same constructions with different gauge groups.

CHAPTER 5

THE STRUCTURE OF OBSERVABLES

5.1 A naïve discussion on factorization algebras

[4, 3] study *factorization algebras* to provide a generalization of the Kontsevich's deformation quantization approach to quantum mechanics. In other words, while deformation quantization essentially encodes the nature of observables in *one*-dimensional quantum field theories, *the factorization algebra formalism* provides an n-dimensional generalization of this approach. To be more precise, we first recall how to describe observables in classical mechanics and those in the corresponding quantum mechanical system. Let (M, ω) be a symplectic manifold (a *phase space*), then we define *the space* A^{cl} *of classical observables on* M to be the space $C^\infty(M)$ of smooth functions on M. Hence, A^{cl} forms a *Poisson algebra* with respect to the Poisson bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$ given by

$$\{f, g\} := -w(X_f, X_g) = X_f(g) \ \ for \ all \ f, g \in C^\infty(M), \tag{5.1.1}$$

where X_f is *the Hamiltonian vector field associated to* $f \in C^\infty(M)$. Here, X_f is defined implicitly by the equation

$$\imath_{X_f}\omega = df, \tag{5.1.2}$$

where $\imath_{X_f}\omega$ denotes the usual *contraction* operator. With the geometric quantization formalism [51, 72, 74], a quantization concept boils down to the study of representation theory of (a certain subalgebra $\mathcal{A}$ of) classical observables in the sense that one can construct *a quantum Hilbert space $\mathcal{H}$ and a Lie algebra homomorphism* [1]

$$\mathcal{Q} : \mathcal{A} \subset \big(C^\infty(M), \{\cdot,\cdot\}\big) \longrightarrow \big(End(\mathcal{H}), [\cdot,\cdot]\big) \tag{5.1.3}$$

together with *Dirac's quantum condition*: For all $f, g \in \mathcal{A}$ we have

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = -i\hbar\mathcal{Q}\big(\{f,g\}\big) \tag{5.1.4}$$

where $[\cdot,\cdot]$ denotes the usual commutator on $End(\mathcal{H})$.

In accordance with the above set-up, while classical observables form a *Poisson algebra*, the space A^q of quantum observables forms *an associative algebra* which is related to the classical one by the quantum condition 5.1.4. Deformation quantization, in fact, serves as a mathematical treatment that captures this correspondence. In other words, it essentially encodes the procedure of *deforming commutative structures to non-commutative ones* for general Poisson manifolds [53].

Factorization algebras, on the other hand, are algebro-geometric objects which are manifestly described sheaf theoretically as follows:

Definition 5.1.1. *A prefactorization algebra $\mathcal{F}$* on a manifold M consists of the following data:

- For each open subset $U \subseteq M$, a cochain complex $\mathcal{F}(U)$.
- For each open subsets $U \subseteq V$ of M, a cochain map $\imath_{U;V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$.
- For any finite collection $U_1, ..., U_n$ of pairwise disjoint open subsets of $V \subseteq M$, V open in M, there is a morphism

$$\imath_{U_1,...U_n;V} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V) \tag{5.1.5}$$

together with *certain compatibility conditions:*

i. Invariance under the action of the symmetric group S_n permuting the ordering of the collection $U_1, ..., U_n$ in the sense that

$$\imath_{U_1,...U_n;V} = \imath_{U_{\sigma(1)},...U_{\sigma(n)};V} \; for \; any \; \sigma \in S_n. \tag{5.1.6}$$

[1] *A Lie algebra homomorphism* $\beta : \mathfrak{g} \rightarrow \mathfrak{h}$ is a linear map of vector spaces such that $\beta([X,Y]_\mathfrak{g}) = [\beta(X), \beta(Y)]_\mathfrak{h}$. Keep in mind that, one can easily suppress the constant "-$i\hbar$" in 5.1.4 into the definition of $\mathcal{Q}$ such that the quantum condition 5.1.4 becomes the usual compatibility condition that a Lie algebra homomorphism satisfies.

That is, the morphism $\imath_{U_1,\ldots U_n;V}$ is independent of the ordering of open subsets $U_1, ..., U_n$, but it depends only on the family $\{U_i\}$.

ii. *Associativity* condition: if $U_{i1} \amalg \cdots \amalg U_{in_i} \subset V_i$ and $V_1 \amalg \cdots \amalg V_k \subset W$ where $U_{ij}(resp.\ V_i)$ are pairwise disjoint open subsets of V_i (resp. W) with W open in M, then the following diagram commutes.

$$\begin{array}{ccc} \bigotimes_{i=1}^{k} \bigotimes_{j=1}^{n_i} \mathcal{F}(U_{ij}) & \longrightarrow & \bigotimes_{i=1}^{k} \mathcal{F}(V_i) \\ & \searrow & \downarrow \\ & & \mathcal{F}(W) \end{array} \tag{5.1.7}$$

With this definition in hand, a prefactorization algebra behaves like a co-presheaf except the fact that we use tensor product instead of a direct sum of cochain complexes. Furthermore, we can define ***a factorization algebra*** once we impose certain local-to-global conditions on a prefactorization algebra analogous to the ones imposed on presheaves [44]. For a complete discussion, we refer to Ch. 3 of [4] or [44].

Factorization algebras, in fact, serve as n-dimensional counterparts to those objects which are realized in deformation quantization formalism. In particular, one recovers observables in classical/quantum mechanics when we consider the case of $n = 1$ [4]. For instance, in a particular gauge theory, holonomy observables, namely *Wilson line operators*, can be formalized in terms of such objects. These are, in fact, the ones that are related to Witten's Knot invariants. They actually arise from the analysis of certain partition functions in three-dimensional Chern-Simons theory [78]. In this approach to perturbative quantum field theories, quantum observables in these types of theories form a factorization algebra. It turns out that a factorization algebra of quantum observables is related to a (commutative) factorization algebra of associated classical observables in the following sense:

Theorem 5.1.1. *(Weak quantization Theorem [4]): For a classical field theory and a choice of BV quantization,*

1. *The space Obs^q of quantum observables forms a factorization algebra over the ring* $\mathbb{R}[[\hbar]]$.

2. *$Obs^{cl} \cong Obs^q \ mod \ \hbar$ as a homotopy equivalence where Obs^{cl} denotes the associated factorization algebra of classical observables.*

Note that the theorem above is just a part of the story, and it is indeed *weak* in the sense that it is *not* able to capture the data related to Poisson structures. To provide a correct n-dimensional analogue of deformation quantization approach, we need to refine the notion of a classical field theory in such a way that the richness of this new set-up becomes visible. This is where *derived algebraic geometry* comes into play.

As we discussed above, the space of classical observables forms a (commutative) factorization algebra. This allows us to employ certain *cohomological methods* encoding the structure of observables in the following sense [4]: Factorization algebra Obs^{cl} of observables can be realized as a particular assignment analogous to a co-sheaf of *cochain complexes* as mentioned above. That is, for each open subset $U \subset M$ of M, $Obs^{cl}(U)$ has a $\mathbb{Z}$-graded structure

$$Obs^{cl}(U) = \bigoplus_{i \in \mathbb{Z}} Obs^{cl}_i(U)$$

together with suitable connecting homomorphisms $\mathrm{d}_i : Obs^{cl}_i(U) \to Obs^{cl}_{i+1}(U)$ for each i. Each cohomology group $H^i(Obs^{cl}(U))$ encodes the structure of observables as follows:

- *"Physically meaningful"* observables are the closed ones with cohomological degree 0, i.e., $\mathcal{O} \in Obs^{cl}_0(U)$ with $\mathrm{d}_0\mathcal{O} = 0$. (and hence $[\mathcal{O}] \in H^0(Obs^{cl}(U))$.)
- $H^1(Obs^{cl}(U))$ contains *anomalies*, i.e., obstructions for classical observables to be lifted to the quantum level. In gauge theory, for instance, there exist certain classical observables respecting gauge symmetries such that they do *not* admit any lift to quantum observables respecting gauge symmetries. This behaviour is indeed encoded by a non-zero element in $H^1(Obs^{cl}(U))$
- $H^n(Obs^{cl}(U))$ with $n < 0$ can be interpreted as symmetries, higher symmetries of observables etc. via higher categorical arguments.
- $H^i(Obs^{cl}(U))$ with $n > 1$ has no clear physical interpretation.

5.2 Constructions of Chevalley-Eilenberg complexes

In this section, we shall only present a treatment for the constructions of Chevalley-Eilenberg complexes and the corresponding homology/cohomology modules in the case of ordinary Lie algebras $\mathfrak{g}$. The reason for this restriction is just to make the argument more tractable and avoid complicated expressions, which possibly arise from the internal gradings and higher structural relations as in B.3.15 or B.3.16. Generalizations to dglas or $\mathcal{L}_\infty$ algebras are relatively straightforward procedures. Hence, we refer to [3] (App. A), [42] (ch. 21-23) or [10] (ch. 2.2). Now, the current discussion is based on constructions presented in [59].

Definition 5.2.1. Let $\mathfrak{g}$ be a Lie algebra over a field k of characteristic zero. We define *the universal enveloping algebra* $U\mathfrak{g}$ of $\mathfrak{g}$ as

$$U\mathfrak{g} := Tens(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y]) \tag{5.2.1}$$

where $Tens(\mathfrak{g}) := \bigoplus_{n\in\mathbb{N}} \mathfrak{g}^{\otimes n} = k \oplus \mathfrak{g} \oplus \big(\mathfrak{g} \otimes \mathfrak{g}\big) \oplus \cdots \oplus \mathfrak{g}^{\otimes i} \oplus \cdots$.

Now, by using *the universal enveloping algebra* $U\mathfrak{g}$ of $\mathfrak{g}$ one can introduce an $U\mathfrak{g}$-module

$$V_i(\mathfrak{g}) := U\mathfrak{g} \otimes_k \bigwedge^i \mathfrak{g} \ \text{ for all } i, \tag{5.2.2}$$

along with the natural maps

$$V_0(\mathfrak{g}) \longrightarrow U\mathfrak{g}/(\mathfrak{g}) \cong k \tag{5.2.3}$$

$$V_1(\mathfrak{g}) \longrightarrow V_0(\mathfrak{g}), \ \ u \otimes x \mapsto ux. \tag{5.2.4}$$

where $V_0(\mathfrak{g}) = U\mathfrak{g}$ and $V_1(\mathfrak{g}) = U\mathfrak{g} \otimes_k \mathfrak{g}$. Hence, we have an exact sequence

$$V_1(\mathfrak{g}) \longrightarrow V_0(\mathfrak{g}) \longrightarrow k \longrightarrow 0. \tag{5.2.5}$$

Definition 5.2.2. For $k > 1$, we define a morphism $d : V_k(\mathfrak{g}) \longrightarrow V_{k-1}(\mathfrak{g})$ as

$$\begin{aligned} d(u \otimes x_1 \wedge \cdots \wedge x_k) := & \sum_{i=1}^{k} (-1)^{i+1} u x_i \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_k \\ & + \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_k. \end{aligned}$$

Lemma 5.2.1. *$d^2 = 0$ and $\big(V_*(\mathfrak{g}), d\big)$ is indeed a chain complex.*

Proof. Let $k = 2$ and $u \otimes x_1 \wedge x_2$ given. Then we have

$$\begin{aligned} d(d(u \otimes x_1 \wedge x_2)) &= d(ux_1 \otimes x_2 - ux_2 \otimes x_1 - u \otimes [x_1, x_2]) \\ &= ux_1x_2 - ux_2x_1 - u(x_1x_2 - x_2x_1) \\ &= 0 \end{aligned} \tag{5.2.6}$$

The rest follows from the immediate induction on k. See [59] for the explicit expressions. □

The chain complex $V_*(\mathfrak{g}) := U\mathfrak{g} \otimes_k \bigwedge^* \mathfrak{g}$ is sometimes called *the Chevalley-Eilenberg (C-E) complex* or *the standard complex*. It essentially serves as a suitable *projective resolution* $\mathcal{P}$ [2] for the base field k in order to define $Tor_*^{U\mathfrak{g}}(k, M)$ and $Ext^*_{U\mathfrak{g}}(k, M)$ for any $\mathfrak{g}$-module M. In that respect, we have the following observations from [59].

Theorem 5.2.1. *$V_*(\mathfrak{g}) \longrightarrow k$ is a projective resolution of the (trivial) $\mathfrak{g}$-module k.*

Corollary 5.2.1. *Let $V_*(\mathfrak{g})$ be as above.*

1. *If M is a right $\mathfrak{g}$-module, then the Lie algebra homology modules*

$$H_*^{Lie}(\mathfrak{g}, M) = Tor_*^{U\mathfrak{g}}(k, M) \tag{5.2.7}$$

are the homology of the chain complex

$$M \otimes_{U\mathfrak{g}} V_*(\mathfrak{g}) = M \otimes_{U\mathfrak{g}} U\mathfrak{g} \otimes_k \bigwedge^* \mathfrak{g} \cong M \otimes_k \bigwedge^* \mathfrak{g} \tag{5.2.8}$$

We denote this tensor product complex by

$$k \otimes^{\mathbb{L}}_{U\mathfrak{g}} M : \quad \cdots \longrightarrow M \otimes_k \bigwedge^2 \mathfrak{g} \longrightarrow M \otimes_k \mathfrak{g} \longrightarrow M. \tag{5.2.9}$$

2. *If M is a left $\mathfrak{g}$-module, then the Lie algebra cohomology modules*

$$H^*_{Lie}(\mathfrak{g}, M) = Ext^*_{U\mathfrak{g}}(k, M) \tag{5.2.10}$$

are the cohomology of the chain complex

$$Hom_{\mathfrak{g}}(V_*(\mathfrak{g}), M) = Hom_{\mathfrak{g}}(U\mathfrak{g} \otimes_k \bigwedge^* \mathfrak{g}, M) \cong Hom_k(\bigwedge^* \mathfrak{g}, M) \tag{5.2.11}$$

[2] A projective resolution $\mathcal{P}$ of a module N is a free resolution of N such that the functor $Hom_{Mod}(A, \cdot)$ is exact.

together with an isomorphism

$$\bigwedge^i \mathfrak{g}^* \otimes_k M \cong Hom_k(\bigwedge^i \mathfrak{g}, M). \tag{5.2.12}$$

where $\mathfrak{g}^* = Hom_k(\mathfrak{g}, k)$ *is the dual space of* $\mathfrak{g}$*. We denote this tensor product complex by*

$$\mathbb{R}Hom_{U\mathfrak{g}}(k, M): \quad M \longrightarrow \mathfrak{g}^* \otimes_k M \longrightarrow \bigwedge^2 \mathfrak{g}^* \otimes_k M \longrightarrow \cdots \tag{5.2.13}$$

In this complex, an n*-cohain is just* k*-multilinear map*

$$f : \bigwedge^n \mathfrak{g} \longrightarrow M \tag{5.2.14}$$

together with the coboundary maps $\delta : Hom_k(\bigwedge^n \mathfrak{g}, M) \to Hom_k(\bigwedge^{n+1} \mathfrak{g}, M)$ *defined as follows: For any* $x_1 \wedge \cdots \wedge x_{n+1}$ *and* $f \in Hom_k(\bigwedge^n \mathfrak{g}, M)$*,*

$$\begin{aligned}\delta f\ (x_1 \wedge \cdots \wedge x_{n+1}) &:= \sum (-1)^{i+1} x_i f(x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge x_{n+1}) \\ &+ \sum_{i<j} (-1)^{i+j} f([x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge \hat{x_j} \wedge \cdots \wedge x_{n+1}).\end{aligned}$$

Remark 5.2.1. In the context of derived algebraic geometry, we have the following correspondences:

1. The tensor product complex $k \otimes^{\mathbb{L}}_{U\mathfrak{g}} M$ introduced in 5.2.9 defines *the derived tensor product* of $\mathfrak{g}$-modules k and M over $U\mathfrak{g}$. Note that this in fact boils down to the construction of *left derived functor* associated to the right exact functor $\cdot \otimes_R A$ for any R-module A. For an accessible introduction to left/right derived functors, see for instance [49], ch. 23.
2. The other complex $\mathbb{R}Hom_{U\mathfrak{g}}(k, M)$ defined in 5.2.13, on the other hand, is in fact *right derived functor associated to the left exact functor* $Hom_R(A, \cdot)$ for any R-module A.

Remark 5.2.2. Let $M := k$ be the trivial $\mathfrak{g}$-module. Then we denote the resulting complexes in 5.2.9 and 5.2.13 respectively by

$$C_*(\mathfrak{g}) \cong \bigwedge^* \mathfrak{g} \quad \text{and} \quad C^*(\mathfrak{g}) = Hom_k(\bigwedge^* \mathfrak{g}, k) \cong Hom_k(C_*(\mathfrak{g}), k). \tag{5.2.15}$$

They are sometimes referred as *C-E complexes* as well. As every commutative algebra should be interpreted as an algebra of functions on a certain space [3], we have the following naïve observations capturing the geometric realizations and the roles of C-E complexes in the context of derived algebraic geometry:

1. $C^*(\mathfrak{g})$ can be viewed as *an algebra* $\mathcal{O}(B\mathfrak{g})$ *of functions* on the classifying space $B\mathfrak{g}$.
2. $C_*(\mathfrak{g})$ can be considered as the space of distributions on $B\mathfrak{g}$.

As outlined in [42], one can also make sense of these definitions and geometric interpretations of $C^*(\mathfrak{g})$ and $C_*(\mathfrak{g})$ in the case of $\mathfrak{g}$ being a differential graded Lie algebra (or even being an $\mathcal{L}_\infty$ algebra [3]). Therefore, one can provide almost the same constructions with some modifications according to the graded structure of $\mathfrak{g}$. Note that, all kinds of $\mathcal{L}_\infty$ algebras we shall be interested in are, in fact, differential graded Lie algebras.

5.3 Factorization algebra of observables

As addressed in [3], in the case of classical field theories, one can make a reasonable measurement only on those fields which are the solutions to the Euler-Lagrange equations. Observables, therefore, are defined as functions

$$\mathcal{O} : EL \longrightarrow k \tag{5.3.1}$$

on the moduli space EL of solutions to the Euler-Lagrange equations. Now, we intend to extend this idea to the derived setting and provide an appropriate treatment with the notion of observables on a derived moduli stack of solutions to the E-L equations.

Given a classical field theory, let $\mathcal{L}$ be the corresponding local $\mathcal{L}_\infty$ algebra on a manifold M. As outlined in Section B.3.2 one can define a sheaf $\mathcal{BL}$ of formal moduli problem

$$\mathcal{BL} : Opens_M^{op} \longrightarrow Moduli_k, \quad U \longmapsto \mathcal{BL}(U) \tag{5.3.2}$$

where $\mathcal{BL}(U)$ can be considered as *a derived space* of solutions to field equations of the theory. As noted in Remark 5.2.2, the C-E complex $C^*(\mathcal{L}(U))$ associated to $\mathcal{L}(U)$ can be interpreted as *an "algebra" of functions on a derived space of solutions to the field equations over* U. Then, we can define the space of observables over U in the following natural way [3]:

Definition 5.3.1. *The space $Obs^{cl}(U)$ of observables with support on an open subset U of M* is defined to be a commutative differential graded k-algebra

$$Obs^{cl}(U) := C^*(\mathcal{L}(U)). \tag{5.3.3}$$

Note that it follows directly from the properties of PDEs and the construction of $\mathcal{BL}$ that if $U_1, ..., U_n$ are pairwise disjoint open subsets of U, then restrictions of solutions over U to each U_i induce a natural map

$$\mathcal{BL}(U) \longrightarrow \mathcal{BL}(U_1) \times \cdots \times \mathcal{BL}(U_n), \tag{5.3.4}$$

such that each function f over $\mathcal{BL}(U_i)$ can be pulled-back via the natural map above, and hence one obtains a morphism

$$Obs^{cl}(U_1) \otimes \cdots \otimes Obs^{cl}(U_n) \longrightarrow Obs^{cl}(U). \tag{5.3.5}$$

Therefore, the assignment Obs^{cl} admits the structure of pre-factorization algebra. Furthermore, as $\mathcal{BL}$ is a sheaf, it induces a local-to-global property on Obs^{cl} in a natural way. Thus, this observation essentially gives a sketch of the proof of the following proposition.

Proposition 5.3.1. *The assignment*

$$Obs^{cl} : U \longmapsto Obs^{cl}(U) \tag{5.3.6}$$

is a factorization algebra of observables.

5.4 Factorization algebra of observables for 3D Einstein gravity

Let $\mathfrak{g}$ be an $\mathcal{L}_\infty$ algebra

$$\mathfrak{g} := \Omega^*(M) \otimes \mathfrak{iso}(2,1), \tag{5.4.1}$$

where the only *non-zero* multilinear maps are $\ell_1 := \mathrm{d}_\nabla$ and $\ell_2 := [\cdot,\cdot]$ given as in Example B.3.1. Here, d_∇ denotes *the covariant derivative* defined as a coupling of the de Rham differential d_{dR} with the fixed flat connection ∇ on LM:

$$d_\nabla := \mathrm{d}_{dR} + [\nabla, \cdot\,]. \tag{5.4.2}$$

Then, as we discussed before, from Lemma 4.1.1, we have a formal moduli problem $\mathcal{B}\mathfrak{g} \in Moduli_k$

$$\mathcal{B}\mathfrak{g}: \ dgArt_k \longrightarrow \ sSets, \ \ (R, \mathfrak{m}_R) \longmapsto \mathcal{B}\mathfrak{g}(R) \tag{5.4.3}$$

for the vacuum Einstein gravity with vanishing cosmological constant in 3D Cartan formalism. Then, *the space of functions over an open subset U of M* can be defined to be a C-E complex associated to dgla $\mathfrak{g}(U) = \Omega^*(U) \otimes \mathfrak{iso}(2,1)$. That is, a factorization algebra of observables for this 3D Einstein gravity is given by

$$Obs^{cl}_{GR} : U \longmapsto Obs^{cl}_{GR}(U) \tag{5.4.4}$$

where $Obs^{cl}_{GR}(U) = C^*\big(\Omega^*(U) \otimes \mathfrak{iso}(2,1)\big)$.

Remark 5.4.1. As we pointed out before, non-zero cosmological constants would yield different gauge groups, and hence different Lie algebras. Therefore, a factorization algebra of observables for 3D Einstein gravities with non-vanishing cosmological constant would involve, instead of $\mathfrak{iso}(2,1)$, either $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2;\mathbb{R})$ for $\Lambda < 0$ or $\mathfrak{sl}(2,\mathbb{C})$ for $\Lambda > 0$.

CHAPTER 6

EPILOGUE

6.1 Concluding remarks

This thesis is about reformulating (vacuum) Einstein gravity with vanishing cosmological constant and exploring its aspects in the language of higher spaces. This work provides an investigation of "$stacky$" formulations and derived geometric interpretations for Einstein gravities in various scenarios. Inspired by the constructions in [3, 6, 7], we have the following observations and results.

(i) We provide, in n-dimensional set-up, a reformulation of vacuum Einstein gravity with vanishing cosmological constant by introducing a suitable moduli stack of *Ricci-flat metrics on a Lorentzian manifold of dimension* n. We first introduce a prestack associated with such a gravity theory as in Proposition 2.1.1. Next, an appropriate site structure on the underlying source category is chosen as in Theorem 3.1.1. As a final step, we make use of the homotopy theoretical definition of a stack to manifest the local-to-global property in a rather functorial manner. This eventually leads to a construction of *the moduli stack of Einstein gravity.*

(ii) One can alternatively reformulate such an Einstein gravity, especially in dimension $2+1$, as a particular gauge theory, namely Chern-Simons theory with gauge group being the Poincaré group $ISO(2,1)$ (see Appendix A.3). Hence we can realize the classical physical phase space of Einstein gravity as that of Chern-Simons theory, namely *the moduli space* $\mathcal{M}_{flat}$ *of flat connections*. In the 2+1 dimensional case, on the other hand, the Weyl tensor is identically zero. Then the Riemann tensor can locally be expressed in terms of R and $R_{\mu\nu}$, and

so we locally have $R_{\mu\nu\sigma\rho} = 0$ as well. That is, any solution of the vacuum Einstein field equations in 3-dimensions with vanishing cosmological constant is locally *flat*. Furthermore, it follows from the fact that any flat metric indeed defines a corresponding flat gauge connection, one has a canonical map

$$\phi : \mathcal{E}(M) \longrightarrow \mathcal{M}_{flat}. \tag{6.1.1}$$

between moduli spaces *(not invertible in the first place)* where $\mathcal{E}(M)$ denotes the moduli space of solutions to Einstein field equations in this setting. In that respect, we say that the quantum gravity is equivalent to gauge theory in the sense of the canonical formalism if this canonical map is, in fact, an isomorphism. So, the other key observation is about the consequence of this isomorphism which is indeed known to exist in a particular set-up. For details, see Appendix C. The equivalence of quantum gravity with gauge theory in 2+1 dimensions naturally (when it exists) induces an equivalence (of the corresponding stacks) as well. That is, in Theorem 3.2.1, we concentrate on 3D theories in a particular scenario. We upgrade the equivalence of certain 2+1 quantum gravities with gauge theory to *an isomorphism between the corresponding stacks* in the case where the underlying Lorentzian spacetime is of the form $\Sigma \times (0, \infty)$ with Σ being a closed Riemann surface of genus $g > 1$. Namely, there exists *an invertible natural transformation*

$$\mathcal{C}^{op} \; \underset{\mathcal{M}}{\overset{\mathcal{E}}{\Phi \Downarrow}} \; Grpds, \tag{6.1.2}$$

between the corresponding stacks $\mathcal{E}$ and $\mathcal{M}$.

(iii) Inspired by what has been already done for Chern-Simons theory [3], we introduce derived geometric constructions for 3D Einstein gravity as a natural example of the formulations in [3, 4]. In that respect, Corollary 4.2.1 is an immediate observation that directly follows from [3]. It essentially provides an obvious formal moduli problem in the case of 3D Cartan theory of vacuum Einstein gravity with vanishing cosmological constant. It is indeed straightforward to observe that this is just a particular case of the construction given for

Chern-Simons theory [3, 4, 14, 40]. Therefore, this part of the thesis can also be viewed as a detailed survey on the construction of formal moduli problems in the case of various classical field theories, including a certain 2+1 Einstein gravity as a particular example.

(iv) Once we adopt derived geometric interpretation of a classical field theory, *the algebraic structure* of observables becomes transparent in some way. Indeed, it can be described naturally in terms of a certain factorization algebra on the formal moduli problem of interest. In that respect, Section 5.1 is devoted to providing a survey on factorization algebras of observables. As a natural example of the constructions in [3, 4], we also present the factorization algebra of observables in 3D Cartan theory of (vacuum) Einstein gravity with/without cosmological constant (see section 5.4).

6.2 Future directions

A possible future direction, on the other hand, would be to elaborate further higher geometric structures on the (derived) stack of Einstein gravity in the context of derived symplectic (or Poisson) geometry [16, 20, 22]. Or one can also use a similar analysis to provide "stacky" formulations for other classical field theories, which may lead to alternative frameworks with new algebraic and geometric tools for the existing or upcoming problems in the theory of interest.

Appendix A

ASPECTS OF CARTAN GEOMETRY

As outlined in Section C.1, Cartan's formalism provides a fruitful reformulation of Einstein gravity in such a way that main aspects of the underlying geometric structures are encoded by a so-called *Cartan connection* which essentially consists of two parts: *a coframe field* e and *a spin connection* ω. *Cartan geometry*, informally speaking, can be thought of as a non-homogeneous generalization of *Klein geometry*, in which one can investigate underlying geometric properties of homogeneous spaces G/H in terms of their symmetry groups where G is a Lie group and $H \subset G$ closed subgroup. In other words, one has the following diagram [66] summarizing the relation between certain geometries of interest:

$$\begin{array}{ccc} Euclidean\ Geometry & \xrightarrow[\text{allow curvature}]{} & Riemannian\ Geometry \\ \text{generalize symmetry group}\ \big\downarrow & & \big\downarrow\ \text{generalize tangent space} \\ Klein\ Geometry & \xrightarrow[\text{allow curvature}]{} & Cartan\ Geometry \end{array}$$

Furthermore, as stressed in [67], symmetry groups of gravitational interest are as follows:

$$G := \begin{cases} SO(n,1), & \Lambda > 0\ (de\ Sitter) \\ ISO(n-1,1), & \Lambda = 0\ (Minkowski) \\ SO(n-1,2), & \Lambda < 0\ (Anti\ de\ Siter) \end{cases} \tag{A.0.1}$$

where the stabilizer group in each case is the Lorentz group $H = SO(n-1,1)$. We shall try to elaborate those ideas above in a self-contained manner by following [66, 67].

A.1 Preliminary definitions: Ehresmann and Cartan connections

In this section, we shall revisit a number of notions which are relatively standard and naturally emerge in the context of differential geometry of fibre bundles. For a complete treatment of the subject, we refer to [46, 68, 70].

Definition A.1.1. A smooth connected **Klein geometry** is a pair (G, H) where G is a Lie group and $H \subset G$ is a closed subgroup such that the quotient G/H space is connected.

As indicated in [66], the underlying structure of Klein geometry can also be formalized in the language of principal H-bundles over G/H. Given a principal H-bundle over G/H

$$\begin{array}{ccc} G & \xrightarrow{\ \triangleleft\, H\ } & G \\ & & \Big\downarrow \pi \\ & & G/H, \end{array} \tag{A.1.1}$$

the closeness of H, in this formulation, can be interpreted as choosing a subgroup H that serves a stabilizer of a point in the homogeneous space G/H. Throughout the Appendix A, we assume (G, H) is a Klein pair, and we denote their Lie algebras by $\mathfrak{g}$ and $\mathfrak{h}$ respectively.

Definition A.1.2. An **Ehresmann connection** on a principal (right) H-bundle $P \xrightarrow{\pi} M$ is an $\mathfrak{h}$-valued 1-form ω on P

$$\omega \in \Omega^1(P, \mathfrak{h}) \tag{A.1.2}$$

such that it satisfies the following conditions:

1. (H-equivariance) For each $h \in H$, one has

$$R_h^* \omega = Ad(h^{-1})\omega, \tag{A.1.3}$$

where R denotes the right multiplication on H and R^* its pullback, and $Ad : H \to Aut(\mathfrak{h})$ is the standard adjoint representation of the Lie subgroup H on its Lie algebra $\mathfrak{h}$. In fact, it is just defined as the derivative ϕ_e^* of the adjoint representation at the point e:

$$\phi : H \to Aut(H), \;\; X \mapsto \big(\phi_X : h \mapsto XhX^{-1}\big). \tag{A.1.4}$$

2. ω restricts to *the Maurer-Cartan form* $\omega_H : TP_x \to \mathfrak{h}$ on each fiber of P where *the Maurer-Cartan form* $\omega_H \in \Omega^1(P, \mathfrak{h})$ is the canonical 1-form defined as the derivative of the left multiplication L_h on H.

Definition A.1.3. *The curvature* $\Omega[\omega]$ of an Ehresmann connection ω is defined to be

$$\Omega[\omega] := \mathrm{d}\omega + \frac{1}{2}[\omega, \omega] \;\in \Omega^2(P, \mathfrak{h}), \tag{A.1.5}$$

where d is the usual de Rham differential on $\Omega^*(P, \mathfrak{h})$.

Definition A.1.4. A Cartan geometry $(P \overset{\pi}{\to} M, A)$ modeled on a Klein geometry (G, H) consists of a principal (right) H-bundle $P \overset{\pi}{\to} M$ equipped with a $\mathfrak{g}$-valued 1-form A on P, **a Cartan connection**,

$$A \in \Omega^1(P, \mathfrak{g}) \tag{A.1.6}$$

such that

1. For each $p \in P$, there is a linear isomorphism

$$A_p : T_pP \overset{\sim}{\to} \mathfrak{g}. \tag{A.1.7}$$

2. For each $h \in H$, one has

$$R_h^* A = Ad(h^{-1})A, \tag{A.1.8}$$

where R and Ad are as above.

3. A takes values in the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ on vertical vectors, and it restricts to *the Maurer-Cartan form* $A_H : TP_x \to \mathfrak{h}$ on fibers of P.

Similarly, **the curvature** $F[A]$ of a Cartan connection A is defined as

$$F[A] := \mathrm{d}A + \frac{1}{2}[A, A] \;\in \Omega^2(P, \mathfrak{g}). \tag{A.1.9}$$

Note that besides the isomorphism induced by a Cartan connection $A_p : T_pP \xrightarrow{\sim} \mathfrak{g}$ for each p, A takes values in a larger Lie algebra $\mathfrak{g}$ in contrast to the Ehresmann connection ω. Now, we shall investigate the relation between Cartan and Ehresmann connections in more detail. In order to accomplish this task and then elaborate the role of this interpretation in formalizing Einstein gravity, one first requires to introduce particular notions such as *reductive* and *symmetric* Cartan geometries.

Definition A.1.5. Given a Cartan geometry with a Klein model (G, H), one can always have a decomposition (as a vector space)

$$\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}. \tag{A.1.10}$$

1. A Lie algebra $\mathfrak{g}$ is called *a reductive splitting* if if the quotient $\mathfrak{g}/\mathfrak{h}$ is an $Ad(H)$-invariant subspace of $\mathfrak{g}$. That is, the decomposition A.1.10 is $Ad(H)$-invariant splitting.
2. A Cartan geometry is called *reductive* if $\mathfrak{g}$ is a reductive splitting.
3. A Lie algebra $\mathfrak{g}$ is called *symmetric* if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ is a reductive splitting and $\mathfrak{g}$ admits $\mathbb{Z}_2$-grading in the sense that

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{g}/\mathfrak{h}] \subseteq \mathfrak{g}/\mathfrak{h}, \quad [\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h}] \subseteq \mathfrak{h}. \tag{A.1.11}$$

4. A reductive Cartan geometry is called *symmetric* if $\mathfrak{g}$ is a symmetric Lie algebra.

Remark A.1.1. Each standard homogenous solution to Einstein equations (cf. Section 1.1), namely de Sitter, Minkowski and anti-de Sitter spacetimes with $\Lambda > 0$, $\Lambda = 0$ and $\Lambda < 0$ respectively, admits a suitable symmetric Cartan geometry. Hence, as we shall discuss below, the interaction between Cartan and Ehresmann connections is rather transparent in a way that the richness of the underlying geometric structure allows us to introduce the curvature formulas and other identities in relatively simple and tractable forms.

A.2 Reductive Cartan Geometry

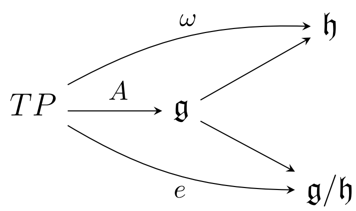
Given a Cartan geometry with a Klein model (G, H) and a reductive splitting

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}, \tag{A.2.1}$$

let A be a Cartan connection. Then A can be uniquely decomposed as

$$A = \omega + e, \tag{A.2.2}$$

where $\omega \in \Omega^1(P, \mathfrak{h})$ is the $\mathfrak{h}$-valued Ehresmann connection 1-form and $e \in \Omega^1(P, \mathfrak{g}/\mathfrak{h})$ is the $\mathfrak{g}/\mathfrak{h}$-valued 1-form, called *the coframe field.* We, in fact, have the following commutative diagram:



(A.2.3)

Now, in order to make the discussion more transparent and capture the gravitational interpretation, we shall assume that the symmetry group G is isomorphic to one of the groups indicated in A.0.1 with H being the Lorentz group $SO(n-1,1)$ such that $\mathbb{R}^{n-1,n} \cong G/SO(n-1,1)$. Then, in a local coordinate chart, one has the $\mathfrak{h}$-part of the Cartan connection A (i.e. the Lorentz/Ehresmann-part ω)

$$A^a_b = \omega^a_b \; for \; a = 0, 1, ..., n-1, \tag{A.2.4}$$

which is indeed the upper left block of matrix components of A together with the $\mathfrak{g}/\mathfrak{h}$-part of A being the last row and column of the matrix representation of A:

$$A^a_n = \frac{1}{\ell} e^a, \; A^n_a = -\frac{\epsilon}{\ell} e_a \;\; for \; a = 0, 1, ..., n-1. \tag{A.2.5}$$

where $\mathfrak{g}/\mathfrak{h} = \mathbb{R}^{n-1,n}$, $\epsilon := sign(\Lambda)$, Λ is the cosmological constant, and ℓ is a *length scaling constant* such that

$$\frac{\epsilon}{\ell^2} = \frac{2\Lambda}{(n-1)(n-2)}. \tag{A.2.6}$$

Therefore, the curvature 2-form $F[A]$ is given in local coordinates as follows:

$$F[A]^i_j = \mathrm{d}A^i_j + A^i_k \wedge A^k_j \tag{A.2.7}$$

where the $\mathfrak{so}(n-1,1)$-part can be expressed as

$$\begin{aligned} F[A]^a_b &= \mathrm{d}A^a_b + A^a_c \wedge A^c_b + A^a_n \wedge A^n_b && \text{(A.2.8)} \\ &= \mathrm{d}\omega^a_b + \omega^a_c \wedge \omega^c_b - \frac{\epsilon}{\ell^2} e^a \wedge e_b && \text{(A.2.9)} \\ &= \Omega[\omega]^a_b - \frac{\epsilon}{\ell^2} e^a \wedge e_b, && \text{(A.2.10)} \end{aligned}$$

and the $\mathbb{R}^{n-1,n}$-part, on the other hand, is given by

$$\begin{aligned} F[A]^a_n &= \mathrm{d}A^a_n + A^a_c \wedge A^c_n && \text{(A.2.11)} \\ &= \frac{1}{\ell}(\mathrm{d}e^a + \omega^a_c \wedge e^c) && \text{(A.2.12)} \\ &= \frac{1}{\ell} d_\omega e^a. && \text{(A.2.13)} \end{aligned}$$

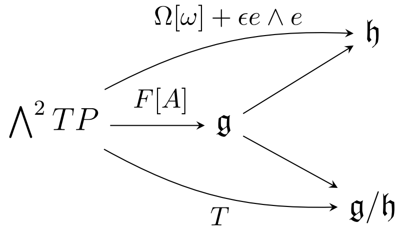
Here, d_ω denotes *the covariant derivative* defined as coupling of the de Rham differential d with the connection ω:

$$d_\omega e := \mathrm{d}e^a + [\omega, e]. \tag{A.2.14}$$

Letting $\ell := 1$ and $\epsilon \in \{-1, 0, 1\}$, the total curvature $F[A]$ of the Cartan connection A can be given as

$$F[A] = \Omega[\omega] + \frac{\epsilon}{2}[e, e] + d_\omega e \tag{A.2.15}$$

where $\Omega[\omega]$ is the $\mathfrak{h}$-valued Ehresmann curvature 2-form for ω (cf. Definition A.1.3) and $T := d_\omega e$ is the *torsion-part*. As before, we get the following commutative diagram:



(A.2.16)

Remark A.2.1. In *a generic reductive case*, the term $[e, e]$ could have both $\mathfrak{h}$- and $\mathfrak{g}/\mathfrak{h}$-parts

$$[e, e] = [e, e]_{\mathfrak{h}} + [e, e]_{\mathfrak{g}/\mathfrak{h}}, \tag{A.2.17}$$

and hence $F[A]$ splits as

$$F[A] = F[A]_{\mathfrak{h}} + F[A]_{\mathfrak{g}/\mathfrak{h}} \tag{A.2.18}$$

where $F[A]_{\mathfrak{h}}$ and $F[A]_{\mathfrak{g}/\mathfrak{h}}$ are called *the corrected curvature* and *torsion* respectively such that

$$F[A]_{\mathfrak{h}} = \Omega[\omega] + \frac{\epsilon}{2}[e,e]_{\mathfrak{h}} \tag{A.2.19}$$

$$F[A]_{\mathfrak{g}/\mathfrak{h}} = d_\omega e + \frac{\epsilon}{2}[e,e]_{\mathfrak{g}/\mathfrak{h}}. \tag{A.2.20}$$

But, in the case of *symmetric spaces*, from the $\mathbb{Z}_2$-graded structure of $\mathfrak{g}$ A.1.11, one has $[\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h}] \subseteq \mathfrak{h}$, and hence $[e,e]_{\mathfrak{g}/\mathfrak{h}} = 0$. Therefore, the $\mathfrak{h}$- and $\mathfrak{g}/\mathfrak{h}$-parts of the curvature are given as

$$F[A]_{\mathfrak{h}} = \Omega[\omega] + \frac{\epsilon}{2}[e,e] \ \in \Omega^2(P, \mathfrak{h}) \tag{A.2.21}$$

$$F[A]_{\mathfrak{g}/\mathfrak{h}} = d_\omega e \ \in \Omega^2(P, \mathfrak{g}/\mathfrak{h}). \tag{A.2.22}$$

Remark A.2.2. Furthermore, one can also make sense of the so-called *Bianchi identity* in any reductive Cartan geometrical framework. Given a Cartan connection A, as in the case of curvature 2-form, *the Bianchi identity*

$$0 = d_A F[A] = \mathrm{d}_{dR} F[A] + [A, F[A]] \tag{A.2.23}$$

for a Cartan curvature 2-form F_A also admits a reductive splitting:

$$d_A F[A] = 0 \iff d_\omega \Omega[\omega] = 0 \ \ and \ \ d_\omega^2 e + [e, \Omega[\omega]] = 0. \tag{A.2.24}$$

For local prescriptions, see [70], ch. 10.

A.3 Recasting 2+1 dimensional gravity

Having adopted the language with main ingredients ω and e as above, we shall revisit 2+1 dimensional gravity in the case of vanishing cosmological constant $\Lambda = 0$. As outlined in Section 1.1, the moduli space $\mathcal{E}(M)$ of solutions to the vacuum Einstein field equations (on M -for instance being of the form $\Sigma \times \mathbb{R}$ where Σ is a closed Riemann surface of genus g-)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{A.3.1}$$

becomes *the moduli space of Ricci-flat Lorentzian metrics on* M, or that of *flat geometric structures on* M, i.e. Lorentzian spacetimes that are locally modeled on $(ISO(2,1), \mathbb{R}^{2+1})$ where $\mathbb{R}^{2+1}$ denotes the usual Minkowski spacetime [11]. Minkowski spacetime, on the other hand, is indeed a symmetric spacetime as addressed in Remark A.1.1, and hence letting $\ell = 1$ and $n = 3$, the underlying Cartan geometry consists of the following data:

1. The underlying (G, H)-model is given as

$$ISO(2,1) = SO(2,1) \ltimes \mathbb{R}^{2,1} \tag{A.3.2}$$

   where $\mathbb{R}^{2,1} \cong ISO(2,1)/SO(2,1)$ together with the symmetric reductive splitting of the Lie algebra $\mathfrak{iso}(2,1)$

$$\mathfrak{iso}(2,1) = \mathfrak{so}(2,1) \oplus \mathbb{R}^{2,1}. \tag{A.3.3}$$

2. A principal H-bundle is defined as *the frame bundle* $LM \xrightarrow{\pi} M$ over M together with the corresponding Cartan connection

$$A \in \Omega^1(LM, \mathfrak{iso}(2,1)) \tag{A.3.4}$$

   which can be expressed uniquely as a decomposition

$$A = \omega + e \tag{A.3.5}$$

   where $\omega \in \Omega^1(LM, \mathfrak{so}(2,1))$ is the $\mathfrak{so}(2,1)$-valued Ehresmann connection 1-form on LM and $e \in \Omega^1(LM, \mathbb{R}^{2,1})$ is *the coframe field.*

3. As indicated in Remark A.2.1, the curvature of the Cartan connection A (with $\epsilon = 0$ as $\Lambda = 0$) is decomposed as

$$F[A] = \Omega[\omega] + d_\omega e \tag{A.3.6}$$

   where $\Omega[\omega]$ is the $\mathfrak{so}(2,1)$-valued curvature 2-form for ω and $d_\omega e$ is the *torsion-part.* Note that the usual notion of "flatness" for the Cartan connection can be encoded in this decomposition in the following sense:

   Definition A.3.1. A Cartan connection A is called **flat** if $F_A = 0$. Note that

$$F_A = 0 \iff \Omega[\omega] = 0 \ \ and \ \ d_\omega e = 0 \tag{A.3.7}$$

   where $\Omega[\omega] = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$ and $d_\omega e := \mathrm{d}e^a + [\omega, e]$.

4. Having introducing the Einstein-Hilbert action in the language of Cartan geometry as

$$\mathcal{I}'_{EH}[e,\omega] = \int\limits_M e^a \wedge \left(\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c \right) \tag{A.3.8}$$

where $\omega \in \Omega^1(LM, \mathfrak{so}(2,1))$ and $e \in \Omega^1(LM, \mathbb{R}^{2+1})$ are as above, the variation of this action w.t.r. ω and e independently yields

$$\delta\mathcal{I}'_{EH}[e,\omega] = \int\limits_M tr(\delta\omega \wedge \Omega[\omega] + \delta e \wedge d_\omega e), \tag{A.3.9}$$

and thus the corresponding field equations are of the form

$$0 = \Omega[\omega] = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega] \tag{A.3.10}$$

$$0 = d_\omega e = \mathrm{d}e^a + [\omega, e]. \tag{A.3.11}$$

Appendix B

DERIVED INTERPRETATION OF FIELD THEORIES

In this appendix, we would like to present a conceptional introduction to the interaction between derived geometry and physics based on the formalism that has been heavily studied by Costello and Gwilliam [3, 4]. Main motivations behind the current attempt are as follows:

(i) We would like to give a brief introduction to *derived algebraic geometry* [1, 54, 15], which can be, roughly speaking, thought of as a higher categorical refinement of ordinary algebraic geometry, and

(ii) to understand how certain *derived objects* naturally appear in mathematical physics and give rise to a formal mathematical treatment.

To make the first touch with physics and realize where derived geometry comes into play, we shall discuss certain notions and structures in a rather intuitive manner, such as *derived critical locus* and *shifted symplectic structures* [16, 23] on derived objects. Afterwards, we shall investigate a *derived* interpretation of a field theory: Together with Lagrangian formalism, one can realize, for instance, *a classical field theory on a smooth manifold* M as a sheaf of *derived stacks of solutions to the equations of motion* on M since it can be described as a *formal moduli problem* [3, 10] cut out by a system of certain PDEs, the so-called *Euler-Lagrange equations*.

B.1 Revisiting an underived set-up

We first recall how to define a naïve and algebro-geometric version of the definition of a classical field theory in Lagrangian formalism [24]:

Definition B.1.1. *A classical field theory* on a manifold M consists of the following data:

(i) the space $\mathbb{F}_M$ of *fields* of the theory defined to be the space $\Gamma(M, \mathcal{F})$ of sections of a particular *sheaf* $\mathcal{F}$ on M,

(ii) the action functional $\mathcal{S} : \mathbb{F}_M \longrightarrow k$ ($\mathbb{R}$ *or* $\mathbb{C}$).

Furthermore, if we want to describe a quantum system, as a third component we need to introduce (iii) the so-called path integral quantization formalism [24, 47, 51].

Remark B.1.1. In order to encode the dynamics of the system in a well-established manner, we need to study *the critical locus* $crit(\mathcal{S})$ of $\mathcal{S}$. One can determine $crit(\mathcal{S})$ by employing variational techniques for the functional $\mathcal{S}$ and that leads to define $crit(\mathcal{S})$ to be the space of solutions to the Euler-Lagrange equations modulo gauge equivalences. Therefore, a classical field theory can be thought of as *a study of the moduli space of solutions to the E-L equations.*

Definition B.1.2. A classical field theory on a manifold M is called

$$\begin{cases} scalar & \text{if } \ \mathbb{F}_M := C^\infty(M), \\ gauge & \text{if } \ \mathbb{F}_M := \mathcal{A}, \\ \sigma - model & \text{if } \ \mathbb{F}_M := Maps(M, N). \end{cases}$$

Here $\mathcal{A}$ is the space of all G-connections on a principal G-bundle over M, namely $\mathcal{A} = \Omega^1(M) \otimes \mathfrak{g}$, and $Maps(M, N)$ denotes the space of smooth maps from M to N for some fixed target manifold N.

Example B.1.1. [52]In accordance with the definitions above we consider the underlying theory (given as a σ-model) for a classical free particle of mass m moving in $\mathbb{R}^n$ together with a certain potential energy $V : \mathbb{R}^n \to \mathbb{R}$: Let $\mathbb{F}_M := Maps(M, \mathbb{R}^n)$ for $M := [0, 1]$ (in that case $\mathcal{F}$ is just the trivial bundle on M), and the action functional

$$\mathcal{S}(q) := \int_{[0,1]} \left(\frac{m||\dot{q}||^2}{2} - V(q) \right) \quad for \ all \ \ q : [0, 1] \longrightarrow \mathbb{R}^n.$$

Then the corresponding Euler-Lagrange equation becomes

$$m\ddot{q} = -grad\, V(q), \tag{B.1.1}$$

which is indeed *the Newton's equation of motion*.

Example B.1.2. Consider a classical free particle (of unit mass) moving in a Riemannian manifold N without any potential energy: Set $\mathbb{F}_M := Maps(M, N)$ with $M := [0, 1]$. Let $f \in Maps(M, N)$ be a smooth path in N, and the action functional given by

$$\mathcal{S}(f) := \frac{1}{2} \int_{[0,1]} ||\dot{f}||^2, \tag{B.1.2}$$

which is called *the energy functional* in Riemannian geometry (cf. [62] ch.5). Then the corresponding Euler-Lagrange equations in a local chart $x = (x^j)_{j=1,...,dimN}$ for N are given as

$$\ddot{f}^k + \Gamma^k_{ij} \dot{f}^i \dot{f}^j = 0 \;\; for \; k = 1, 2, ..., dimN, \tag{B.1.3}$$

where f^k denotes local component of f, i.e, $f^k := x^k \circ f$, and $\Gamma^k_{ij} := \Gamma^k_{ij}(f(t))$ are the Christoffel symbols for each i, j, k. These equations are indeed *geodesic equations in Riemannian geometry*.

Example B.1.3. [24] Consider the theory with free scalar massive fields. Let M be a Riemannian manifold and set $\mathbb{F}_M := C^\infty(M)$. Let $\phi \in \mathbb{F}_M$, then we define the action functional governing the theory as

$$\mathcal{S}(\phi) := \int_M \left(\frac{||d\phi||^2}{2} - \frac{m^2}{2} \phi^2 \right). \tag{B.1.4}$$

The corresponding E-L equation in this case reads as

$$(\Delta + m^2)\phi = 0. \tag{B.1.5}$$

Example B.1.4. Consider the $SU(2)$-Chern-Simons gauge theory ([78]) on a closed, orientable 3-manifold X, which can also be thought of as a non-trivial prototype example for a *3-TQFT* formalism in the sense of Atiyah [76]. Here, we may consider, in particular, an integral homology 3-sphere for some technical reasons [64]. For a complete mathematical treatment of the subject, see [24], [51]. Main ingredients of this structure are encoded by the theory of principal G-bundles in the following sense: Let $P \to X$ be a principal $SU(2)$-bundle on X, $\sigma \in \Gamma(U, P)$ a local trivializing section given schematically as

$$\begin{array}{ccc} P & \xrightarrow{\bullet SU(2)} & P \\ & & \sigma \uparrow \downarrow \pi \\ & & X \end{array} \tag{B.1.6}$$

Note that when $G = SU(2)$, P is a trivial principal bundle over X, i.e. $P \cong X \times SU(2)$ compatible with the bundle structure, and hence there exists a globally defined nowhere vanishing section $\sigma \in \Gamma(X, P)$. Assume ω is a Lie algebra-valued connection one-form on P. Let $A := \sigma^*\omega$ be its representative, i.e. the Lie algebra-valued connection 1-form on X, called *the Yang-Mills field*. Then the theory consists of the space $\mathbb{F}_X$ of *fields*, which is defined to be the infinite-dimensional space $\mathcal{A}$ of all $SU(2)$-connections on a principal $SU(2)$-bundle over X, i.e. $\mathcal{A} := \Omega^1(X) \otimes \mathfrak{g}$ (in that case $\mathcal{F}$ is the *"twisted"* cotangent bundle $T^*X \otimes \mathfrak{g}$), and the Chern-Simons action funtional $CS : \mathcal{A} \longrightarrow S^1$ given by

$$CS(A) := \frac{k}{4\pi} \int_X \mathrm{Tr}(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A), \quad k \in \mathbb{Z}, \tag{B.1.7}$$

together with the gauge group $\mathcal{G} = Map(X, SU(2))$ acting on the space $\mathcal{A}$ as follows: For all $g \in \mathcal{G}$ and $A \in \mathcal{A}$, we set

$$g \triangleleft A := g^{-1} \cdot A \cdot g + g^{-1} \cdot \mathrm{d}g. \tag{B.1.8}$$

The corresponding Euler-Lagrange equation in this case turns out to be

$$F_A = 0, \tag{B.1.9}$$

where $F_A = \mathrm{d}A + A \wedge A$ is the $\mathfrak{g}$-valued curvature two-form on X associated to $A \in \Omega^1(X) \otimes \mathfrak{g}$. Furthermore, under the gauge transformation, the curvature 2-form F_A behaves as follows: For all $g \in \mathcal{G}$

$$F_A \longmapsto g \triangleleft F_A := g^{-1} \cdot F_A \cdot g. \tag{B.1.10}$$

Note that the moduli space $\mathcal{M}_{flat}$ of flat connections modulo gauge transformations emerges in many other areas of mathematics, such as topological quantum field theory, low-dimensional quantum invariants for 3-manifolds and knots [78], and (infinite-dimensional) Morse theory [41, 43, 64].

B.2 Derived geometric formulations

Together with the *derived* interpretation of a classical field theory outlined in [3], one can employ a number of mathematical techniques and notions that naturally appear in derived algebraic geometry. For instance, we may consider a classical field theory as the study of *the derived critical locus* [16, 23] of the action functional since it can be considered as a formal moduli problem in the sense indicated above. Indeed, passing to the *derived* moduli space of solutions corresponds to Batalin-Vilkovisky formalism for a classical field theory which will be briefly discussed below. In derived algebraic geometry, any formal moduli problem arising as the derived critical locus admits a symplectic structure of cohomological degree -1 [23]. This observation is crucial and it ensures the existence of a symplectic structure on the space Obs^{cl} of classical observables. In the language of derived algebraic geometry, therefore, we have the following definition (see [3], ch. 3):

Definition B.2.1. *A (perturbative) classical field theory* is a formal elliptic moduli problem equipped with a symplectic form of cohomological degree -1.

Equivalently, one has the following definition (Appendix of [8] or chapter 3 of [3]):

Definition B.2.2. Let M denote the space of fields for some base manifold X, and $\mathcal{S} : M \to k$ a smooth action functional on M. ***A (perturbative) classical field theory*** is a sheaf of *derived stacks* (of *the derived critical locus* $dcrit(\mathcal{S})$ of the action functional $\mathcal{S}$) on M equipped with a symplectic form of degree -1.

We intend to *unpackage* Definition B.2.2 in an intuitive way. The remaining part of the current section will be devoted to that purpose.

B.2.1 Why does the term “derived" emerge?

We may first discuss naïve or underived realization of a classical field theory in the language of *intersection theory*. Let M denote the space of fields on a base manifold X as in Definition B.2.2. Assume M is a finite dimensional manifold. As indicated in the Remark B.1.1, a classical field theory can be considered as the study of the critical locus $crit(\mathcal{S}) \subset M$ of the action functional $\mathcal{S}$ on M. However, computing certain path integrals perturbatively around classical solutions to the E-L equations is usually problematic if the critical points are *degenerate*. To avoid such problems, one can employ a certain trick the so-called Batalin-Vilkovisky formalism which, roughly speaking, consists of adding certain fields, such as ghosts, anti-fieds etc..., to the functional [23]. This phenomena, on the other hand, can be formulated in the language of intersection theory as follows: We define $crit(\mathcal{S})$ to be the intersection of the graph $G(\mathrm{d}S) \subset T^*M$ of $\mathrm{d}S \in \Omega^1(M)$ with the zero-section of the cotangent bundle T^*M inside T^*M (cf. [3], ch. 5). That is,

$$crit(\mathcal{S}) := G(\mathrm{d}S) \cap M. \tag{B.2.1}$$

As in [16], by adopting algebro-geometric language (see [49], ch. 9), $crit(\mathcal{S})$ can be described in terms of *a fibered product* $M \times_{T^*M} M$ such that the following diagram commutes:

$$\begin{array}{ccc} crit(\mathcal{S}) := M \times_{T^*M} M & \longrightarrow & M \\ \big\downarrow & & \big\downarrow{\scriptstyle 0} \\ M & \xrightarrow{\quad \mathrm{d}\mathcal{S} \quad} & T^*M \end{array} \tag{B.2.2}$$

Even if M is a smooth manifold, for instance, the intersection $M \times_{T^*M} M$ would be highly problematic, and hence an object $\big(crit(\mathcal{S}), \mathcal{O}_{crit(\mathcal{S})}\big)$ generically fails to live in the same category, i.e. intersection would not define a manifold at all (e.g. non-transverse intersection of two submanifold is not a submanifold in general). In fact, problems arising from the degeneracy of critical points correspond to bad intersections in the above sense.

If we employ, however, derived set-up and introduce the derived geometric counterpart of a smooth manifold, namely *a derived manifold*, then we can circumvent the non-existence problem for a fibered product. It follows from the fact that the theory of derived schemes is equivalent to that of *dg-schemes* ([37, 38]) in the case of characteristic zero, one can also work with a dg-scheme $(X, \mathcal{O}_X)$ for which the structure sheaf $\mathcal{O}_X$ is a sheaf of commutative differential graded algebras. It should be noted that one has still the same base topological space, but the scheme structure is now more complicated and locally modeled on commutative graded algebras instead of the usual ones. Furthermore, one can show that the category (or the correct terminology would be the ∞-category) of derived manifolds admits the fibered product.

Therefore, this leads to the following motivation behind the use of *"derived"* formulation in Definition B.2.2:

> *One has to enlarge and re-design the notion of category together with new generalized objects in a way that the intersection of any two such objects always lives in the enlarged version of a category.*

This requires to re-organize *the local model* for the intersection of ringed spaces in the following sense: As in [3], instead of naïve intersection determined algebraically by

$$\mathcal{O}_{crit(\mathcal{S})} := \mathcal{O}_{G(dS)} \otimes_{\mathcal{O}_{T^*M}} \mathcal{O}_M, \tag{B.2.3}$$

we introduce *the derived version*:

$$\mathcal{O}_{dcrit(\mathcal{S})} := \mathcal{O}_{G(dS)} \otimes^{\mathbb{L}}_{\mathcal{O}_{T^*M}} \mathcal{O}_M. \tag{B.2.4}$$

where $\cdot \otimes^{\mathbb{L}}_{\mathcal{O}_{T^*M}} \cdot$ denotes *the derived tensor product.*

A digression on the definition of $\cdot \otimes^{\mathbb{L}}_{\mathcal{O}_{T^*M}} \cdot$**.** (cf. [55] ch. 0) Let R be a commutative ring, B a R-module. Then derived tensor product $\cdot \otimes^{\mathbb{L}}_R B$ arises from the construction of *the left-derived functor* associated to the right-exact functor (cf. [49], ch. 23)

$$\cdot \otimes_R B : Mod_R \to Mod_R. \tag{B.2.5}$$

Let A, B be two commutative algebras over R. Then the definition of $A \otimes^{\mathbb{L}}_R B$ naturally appears in the construction of the i^{th} Tor groups $Tor^R_i(A, B)$ given by the i^{th} homology of ***the tensor product complex*** $(P_\bullet \otimes_R B, \mathrm{d}')$:

$$\cdots \longrightarrow P_2 \otimes_R B \longrightarrow P_1 \otimes_R B \xrightarrow{\mathrm{d}'} P_0 \otimes_R B \longrightarrow 0 \tag{B.2.6}$$

where $P_\bullet$ is a projective resolution of A equipped with a differential d such that $(P_\bullet, \mathrm{d})$ becomes a commutative dg-algebra over R and $\mathrm{d}' = \mathrm{d} \otimes_R id_B$. Since B is a commutative R-algebra, the tensor product complex inherits the structure of a commutative dg-algebra over R as well, and we denote this tensor product complex by $A \otimes^{\mathbb{L}}_R B$. That is, we set

$$A \otimes^{\mathbb{L}}_R B := (P_\bullet \otimes_R B, \mathrm{d}'). \tag{B.2.7}$$

Remark B.2.1. The resulting commutative dg-algebra $A \otimes^{\mathbb{L}}_R B$ is independent of the choice of $(P_\bullet \otimes_R B, \mathrm{d}')$ up to a quasi-isomorphism. ***The end of a digression.***

If we go back to the local model discussion for the derived tensor products, then for each open subset $U \subseteq M$ we have

$$\mathcal{O}_{dcrit(\mathcal{S})}(U) := \mathcal{O}_{G(d\mathcal{S})}(U) \otimes^{\mathbb{L}}_{\mathcal{O}_{T^*M}(U)} \mathcal{O}_M(U) \tag{B.2.8}$$

where RHS corresponds to the tensor product complex of dg-algebra as above. Together with this local model, $\big(dcrit(\mathcal{S}), \mathcal{O}_{dcrit(\mathcal{S})}\big)$ becomes a dg-scheme with its structure sheaf $\mathcal{O}_{dcrit(\mathcal{S})}$ being the sheaf of commutative dg k-algebras such that $\mathcal{O}_{dcrit(\mathcal{S})}$ can be manifestly given as a Koszul resolution of $\mathcal{O}_M$ as a module over $\mathcal{O}_{T^*M}$:

$$\mathcal{O}_{dcrit(\mathcal{S})}: \quad \cdots \longrightarrow \Gamma(M, \wedge^2 TM) \longrightarrow \Gamma(M, TM) \xrightarrow{\imath_{\mathrm{d}\mathcal{S}}} \mathcal{O}_M \longrightarrow 0 \tag{B.2.9}$$

where $\Gamma(M, \wedge^i TM)$ is the space of ***polyvector fields of degree*** i *(or i-vector fields)* and $\imath_{\mathrm{d}\mathcal{S}}$ denotes the contraction with $\mathrm{d}\mathcal{S}$ in the sense that for any 1-vector field $X \in \Gamma(M, TM)$ we define

$$\imath_{\mathrm{d}\mathcal{S}}(X) := \mathrm{d}\mathcal{S}(X) = X\mathcal{S}. \tag{B.2.10}$$

Then, extending to i-vector fields by linearity, we set

$$\mathcal{O}_{dcrit(\mathcal{S})} := \left(\bigoplus_{i \in \mathbb{Z}_{\leq 0}} \Gamma(M, \wedge^i TM), \ \imath_{\mathrm{d}\mathcal{S}} \right). \tag{B.2.11}$$

Remark B.2.2. $\big(dcrit(\mathcal{S}), \mathcal{O}_{dcrit(\mathcal{S})}\big)$ admits a further derived structure; namely, a symplectic form of cohomological degree -1 (see [20], corollary 2.11). The description of this structure, however, is beyond the scope of the current discussion. For the construction, we refer to [20]. You may also see [21, 22] for an accessible presentation of PTVV's shifted symplectic geometry.

Remark B.2.3. The existence of such a derived geometric structure will be crucial when we discuss the notion of quantization for n-dimensional classical field theories. Indeed, this higher structure is really what we need, and it leads to an n-dimensional generalization of what we have already had in the case of quantization of classical mechanics. The language of derived algebraic geometry, in fact, allows us to discuss the concepts of symplectic and Poisson structures even in non-singular settings by introducing so-called *shifted symplectic and Poisson structures* on derived schemes. Once we have those higher structures, the concept of quantization, on the other hand, can also be formalized in terms of the tools from derived deformation theory. For details, see Section 5.1.

B.2.2 Why does “stacky" language come in?

Studying the critical locus of the action functional $\mathcal{S}$ is just one part of the story, and we have already observed that in order to avoid the degenerate critical points one requires to introduce the notions of a derived intersection and the derived critical locus, which is well-behaved than the usual one. For a more complete discussion, see [17, 25]. Another part of the story is related to the moduli nature of the problem. Indeed, one requires to quotient out by symmetries while studying the solution space of the E-L equations, but the quotient space might be highly problematic as well. For instance, the action of the gauge group $\mathcal{G}$ on a manifold X may not be free, and hence the resulting quotient $X/\mathcal{G}$ would not be a manifold, but it can be realized as an *orbifold* $[X//\mathcal{G}]$. It is indeed a particular *stack*, given by the *orbifold quotient* [6, 25].

A digression on moduli problems and stacks. *A moduli problem* is a problem of constructing a classifying space (or a moduli space $\mathcal{M}$) for certain geometric objects (such as manifolds, algebraic varieties, vector bundles etc...) up to their intrinsic symmetries. The wish-list for a "fine" moduli space $\mathcal{M}$ is as follows (see Appendix D for a rather complete treatment):

1. $\mathcal{M}$ is supposed to serve as a *parameter space* in a sense that there must be a one-to-one correspondence between the points of $\mathcal{M}$ and the *set* of isomorphism classes of objects to be classified:

$$\{points\ of\ \mathcal{M}\} \leftrightarrow \{isomorphism\ classes\} \tag{B.2.12}$$

2. The existence of universal classifying object.

In the language of category theory, a moduli problem can be formalized as a certain functor

$$\mathcal{F}:\ \mathcal{C}^{op} \longrightarrow\ Sets \tag{B.2.13}$$

which is called ***a moduli functor*** where $\mathcal{C}^{op}$ is the *opposite* category of the category $\mathcal{C}$ and $Sets$ is the category of sets. In order to make the argument more transparent, we take $\mathcal{C}$ to be the category Sch of k-schemes. Note that for each scheme $U \in Sch$,

$\mathcal{F}(U)$ is the *set* of isomorphism classes parametrized by U, and for each morphism $f : U \to V$ of schemes, we have a morphism $\mathcal{F}(f) : \mathcal{F}(V) \to \mathcal{F}(U)$ of sets. Together with the above formalism, the existence of a fine moduli space corresponds to the *representability* of the moduli functor $\mathcal{F}$ in the sense that

$$\mathcal{F} = Hom_{Sch}(\cdot, \mathcal{M}) \;\; for\ some\ \mathcal{M} \in Sch. \tag{B.2.14}$$

If this is the case, then we say that $\mathcal{F}$ *is represented by* $\mathcal{M}$.

In many cases, however, the moduli functor is *not* representable in the category Sch of schemes. This is essentially where the notion of *a stack* comes into play. The notion of *a stack* can be thought of as a first instance such that the ordinary notion of a category *no longer* suffices to define such an object. To make sense of this new object in a well-established manner and enjoy the richness of this *new structure*, we need to introduce a higher categorical notion, namely a *2-category* [25, 28]. The theory of stacks, therefore, employs higher categorical techniques and notions in a way that it provides a mathematical treatment for the representability problem by re-defining the moduli functor as a stack, *a particular groupoid-valued pseudo-functor with local-to-global properties*,

$$\mathcal{X} : C^{op} \longrightarrow \; Grpds \tag{B.2.15}$$

where $Grpds$ denotes the 2-category of groupoids with objects being categories $\mathcal{C}$ in which all morphisms are isomorphisms (these sorts of categories are called *groupoids*), 1-morphisms being functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between groupoids, and 2-morphisms being natural transformations $\psi : \mathcal{F} \Rightarrow \mathcal{F}'$ between two functors.

Remark B.2.4. In order to make sense of local-to-global (or "gluing") type arguments, one requires to introduce an appropriate notion of *topology on a category* $\mathcal{C}$. Such a structure is manifestly given in [25] and called *a Grothendieck topology* τ. Furthermore, a category $\mathcal{C}$ equipped with a Grothendieck topology τ is called a *site.* Note that if we have a site $\mathcal{C}$, then we can define *a sheaf on* $\mathcal{C}$ in a well-established manner as well. This essentially leads to *the functor of points*-type approach to define a scheme X in the following sense: Given a scheme X, one can define a sheaf (on the category Sch of schemes) by using the Yoneda functor $Hom_{Sch}(\cdot, X)$ as

$$\underline{X} : \; Sch^{op} \longrightarrow \; Sets \tag{B.2.16}$$

where $\underline{X} := Hom_{Sch}(\cdot, X)$. This is indeed a sheaf by the theorem of Grothendieck [25].

Remark B.2.5. Any 1-category (i.e. the usual category) can be realized as a 2-category in which there exists no non-trivial higher structures, i.e. 2-morphisms in a 1-category are just identities.

Remark B.2.6. By using a 2-categorical version of the Yoneda lemma, namely *2-Yoneda lemma* [25], one can show that the moduli functor $\mathcal{X}$ turns out to be representable in the 2-category $Stks$ of stacks. As in the case of derived intersections, we need to enlarge the category with certain non-trivial higher structures in a way that the moduli problem becomes representable in this enhanced version even if it was not in the first place. The price we have to pay is to adopt a higher categorical dictionary leading to the change in the level of abstraction in a way that objects under consideration become rather counter-intuitive. Indeed, stacks and 2-categories serve as motivating/prototype conceptual examples before introducing the notions like *∞-categories, derived schemes, higher stacks, and derived stacks* [17].

Definition B.2.3. Let $\mathcal{C}$ be a category in which all products exist. **A Grothendieck topology** τ on $\mathcal{C}$ consists of the following data.

1. For each object U in $\mathcal{C}$, *a collection* of families $\{U_i \xrightarrow{f_i} U\}$ of morphisms in $\mathcal{C}$, denoted by $\tau(U)$.
2. If $V \xrightarrow{f} U$ is an isomorphism, then $\{V \xrightarrow{f} U\} \in \tau(U)$.
3. If the family $\{U_i \xrightarrow{f_i} U\} \in \tau(U)$ and for each $i \in I$ one has a family $\{U_{ij} \xrightarrow{f_{ij}} U_i\}$ in $\tau(U_i)$, then
$$\{U_{ij} \xrightarrow{f_{ij} \circ f_i} U\} \ \in \tau(U). \tag{B.2.17}$$
4. Given a family $\{U_i \xrightarrow{f_i} U\}$ in $\tau(U)$ and a morphism $V \to U$ with the base change diagram
$$\begin{array}{ccc} V \times_U U_i & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ V & \xrightarrow{f} & U \end{array} \tag{B.2.18}$$
then $\{V \times_U U_i \to V\}$ is in $\tau(V)$.

Here, the families $\{U_i \to U\}$ in $\tau(U)$ are called **covering families for** U in the Grothendieck topology $\tau(U)$.

A motivating example. In the case of $\mathcal{C} = Top$, for any topological space X, the Grothendieck topology $\tau(X)$ corresponds the usual notion of open coverings $\{U_i \subseteq X\}_i$ of X with the maps φ_i being the usual inclusions (or open embeddings) such that

$$X \subseteq \bigcup_i U_i. \tag{B.2.19}$$

In that case, moreover, the fibered product $U_{ij} := U_i \times_X U_j$ in fact corresponds to the intersection of open subsets U_i and U_j in X.

Definition B.2.4. [49] Given two covariant functors between categories $\mathcal{F}, \mathcal{G} : \mathcal{A} \to \mathcal{B}$, **a natural transformation** is the data of morphisms

$$m_A : \mathcal{F}(A) \to \mathcal{G}(A) \ \text{ for all objects } A \in \mathcal{A} \tag{B.2.20}$$

such that for each morphism $A \xrightarrow{f} A'$ in $\mathcal{A}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(A') \\ {\scriptstyle m_A}\downarrow & & \downarrow{\scriptstyle m_{A'}} \\ \mathcal{G}(A) & \xrightarrow[\mathcal{G}(f)]{} & \mathcal{G}(A') \end{array} \tag{B.2.21}$$

Note that for a contravariant functor, one has the same definition with arrows $\mathcal{F}(f)$ and $\mathcal{G}(f)$ being reversed.

The end of a digression.

B.3 Towards the derived geometry of Einstein gravity

As outlined in Section 2.1, one can define a certain moduli functor $\mathcal{EL}$ corresponding to a given classical field theory as follows. Let $\mathcal{C}$ be the category, we set

$$\mathcal{EL} : \ \mathcal{C}^{op} \longrightarrow \ Sets, \ U \mapsto \mathcal{EL}(U), \tag{B.3.1}$$

where $\mathcal{EL}(U)$ is the *set* of isomorphism classes of solutions to the E-L equations over U. More precisely, $\mathcal{EL}(U)$ is the moduli space $EL(U)/\mathcal{G}$ of solutions to the E-L equations modulo gauge transformation. But, as we succinctly discussed in Section 2.1, the quotient space might be "bad" in the sense that it may fail to live in the same category. In other words, the moduli functor $\mathcal{EL}$, in general, is not representable in $\mathcal{C}$. In order to circumvent the problem, we introduce the "stacky" version of $\mathcal{EL}$ as the quotient moduli stack

$$[\mathcal{EL}/\mathcal{G}] : \mathcal{C}^{op} \longrightarrow Grpds, \;\; U \mapsto [\mathcal{EL}/\mathcal{G}](U), \tag{B.3.2}$$

where $[\mathcal{EL}/\mathcal{G}](U)$ is the *groupoid* of solutions to the E-L equations over U. Even if this explains the emergence of stacky language in Definition 1.1.2 in a rather intuitive way, the discussion above is just the tip of the iceberg and is still too naïve to capture the notion of *a derived stack*. We need further concepts in order to enjoy the richness of Definition 1.1.2, such as *the formal neighborhood of a point* in a derived stack, *a formal moduli problem* (in the sense of [10]), $\mathcal{L}_\infty$ algebras, the Maurer-Cartan equation for a $\mathcal{L}_\infty$ algebra $\mathfrak{g}$ and the associated Maurer-Cartan formal moduli problem $\mathcal{B}\mathfrak{g}$ etc... For an expository introduction to derived stacks, see [17]. The following material is mainly based on [3].

B.3.1 Formal moduli problems

In our setup, *formal moduli problems* are constructed to capture the formal geometries of moduli spaces of solutions to certain defining differential equations. The main motivation of the current digression on introducing the notion of *a formal moduli problem* can be outlined as follows. Consider a classical data (M, F_M, S, G) where F_M denotes the space of fields on some base manifold M, $\mathcal{S}$ is a smooth action functional on F_M, and G is a certain group encoding the symmetries. We define *a perturbative classical field theory on M* to be the sheaf $\mathcal{EL}$ of derived stacks of solutions to EOM on M: To each open subset U of M, one assigns

$$U \mapsto \mathcal{EL}(U) \in dStk \tag{B.3.3}$$

where $dStk$ denotes the ∞-category of derived stacks [17, 28] and $\mathcal{EL}(U)$ is given in the functor of points formalism as

$$\mathcal{EL}(U):\ cdga_k^{\leq 0} \longrightarrow\ sSets\ (or\ \infty - Grpds) \tag{B.3.4}$$

where $cdga_k^{\leq 0}$ and $sSets$ ($\infty-$*Grpds*) denote the category of commutative differential graded k-algebras and the ∞-category of simplical sets (∞-groupoids) respectively. Here $\mathcal{EL}(U)(R)$ is the *simplical set* of solutions to the defining relations (i.e. EL-equations) with values in R. In other words, the points of $\mathcal{EL}(U)$ form an ∞-groupoid. For more details on ∞-categories or related concepts, see [28].

As discussed above, in order to circumvent certain problems we work with the derived moduli space of solutions instead of the naïve one. Furthermore, we also intend to capture the *perturbative* behavior of the theory. Hence, this derived moduli space is defined as a *formal moduli problem*

$$\mathcal{EL}(U):\ dgArt_k \longrightarrow\ Ssets \tag{B.3.5}$$

where $dgArt_k$ is the (∞-)category of dg artinian algebras, where morphisms are simply maps of dg commutative algebras (cf. Appendix A of [3] or [28]).

Remark B.3.1. In order to remember the perturbative behavior around the solution $p \in \mathcal{EL}(U)$, we employ the notion of *a formal neighborhood* of a point (cf. [3], Appendix A). This concept essentially helps us to make the scheme structure sensitive enough to encode small thickenings of a point obtained by adding infinitesimal directions. To keep track such *infinitesimal directions* assigned to a point p, it is in fact more suited to use *dg Artinian algebras* as a local model for the scheme structure instead of the usual commutative k-algebras. That is, the scheme structure, informally speaking, is locally modeled on a kind of *nilpotent commutative dg-algebra* such that the structure consists of points with *infinitesimal directions* attached to them. Furthermore, every formal moduli functor can be manifested by using the language of $\mathcal{L}_\infty$ algebras in the sense of [10], which will be stressed below. Now, we intend to elaborate the content of Lurie's theorem.

Definition B.3.1. *A differential graded Artinian agebra* $(A, \mathfrak{m})$ is a commutative differential graded algebra

$$A = \bigoplus_{n \in \mathbb{Z}_{\leq 0}} A^n \tag{B.3.6}$$

over a field k concentrated in degrees ≤ 0 such that

1. Each graded component A^i is finite dimensional and $A^j = 0$ for $j << 0$,

2. A has an unique maximal ideal $\mathfrak{m}$ such that $A/\mathfrak{m} = \mathrm{k}$ and $\mathfrak{m}^N = 0$ for large N.

Definition B.3.2. *A formal moduli problem (or a particular derived stack)* is an ∞-functor (of ∞-categories)

$$\mathcal{F} :\ dgArt_k \longrightarrow\ Ssets \tag{B.3.7}$$

such that

1. $\mathcal{F}(\mathrm{k})$ is contractible.

2. $\mathcal{F}$ maps surjective morphisms of dg Artinian algebras to fibrations of simplicial sets.

3. Let A, B, C be dg Artinian algebras, and $B \to A$ and $C \to A$ surjective morphisms, then there exists a fibered product $B \times_A C$ such that

$$\mathcal{F}(B \times_A C) \to \mathcal{F}(B) \times_{\mathcal{F}(A)} \mathcal{F}(C) \tag{B.3.8}$$

is a weak equivalence.

Now, we shall be interested in constructing a particular kind of formal moduli problem which can be defined as the simplicial set of solutions to the so-called *Maurer-Cartan equations*. This concept, in fact, frequently emerges in the theory of $\mathcal{L}_\infty$-algebras. Therefore, we shall first provide a brief introduction to the theory of $\mathcal{L}_\infty$-algebras.

A digression on the theory of $\mathcal{L}_\infty$-algebras. Informally speaking, an $\mathcal{L}_\infty$ algebra $\mathfrak{g}$ can be considered as a certain dg Lie algebra endowed with a sequence $\{\ell_n\}$ of multilinear maps of (cohomological) degree $2-n$ as

$$\ell_n : \mathfrak{g}^{\otimes n} \longrightarrow \mathfrak{g}, \tag{B.3.9}$$

which are called *n-brackets* with $n = 1, 2, ...$ such that each bracket satisfies a certain graded anti-symmetry condition and n-Jacobi rule (for a complete treatment, see App.

A of [3] or [56]). In order to motivate the notion of a $\mathcal{L}_\infty$ algebra, we shall first investigate differential graded Lie algebras. Note that all kinds of $\mathcal{L}_\infty$ algebras we shall be interested in are indeed differential graded Lie algebras.

Definition B.3.3. *A differential graded Lie algebra* $\mathfrak{g}$ *over a ring* R is a dg R-module $(\mathfrak{g}, \mathrm{d})$ where

$$\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}^n \tag{B.3.10}$$

together with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \otimes_R \mathfrak{g} \to \mathfrak{g}$ such that for all $X, Y, Z \in \mathfrak{g}$, one has

1. *(Graded anti-symmetry)* $[X, Y] = -(-1)^{deg(X)deg(Y)}[Y, X]$.
2. *(Graded Leibniz rule)* $\mathrm{d}[X, Y] = [\mathrm{d}X, Y] + (-1)^{deg(X)}[X, \mathrm{d}Y]$.
3. *(Graded Jacobi rule)* $[X, [Y, Z]] = [[X, Y], Z] + (-1)^{degX degY}[Y, [X, Z]]$.

Example B.3.1. Let M be a smooth manifold and $\mathfrak{h}$ a Lie algebra. Then there exists a natural dgla structure (which will be central and will appear in the context of gauge theories) given as follows:

$$\mathfrak{g} := \Omega^*(M) \otimes \mathfrak{h}, \tag{B.3.11}$$

where the differential is the usual de Rham differential d_{dR} and the bracket $[\cdot,\cdot]$ is given by

$$[\alpha \otimes X, \beta \otimes Y] := \alpha \wedge \beta \otimes [X, Y]_{\mathfrak{h}}. \tag{B.3.12}$$

Definition B.3.4. An $\mathcal{L}_\infty$ algebra over R is a $\mathbb{Z}$-graded, projective R-module

$$\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}^n \tag{B.3.13}$$

equipped with a sequence

$$\{\ell_n : \mathfrak{g}^{\otimes n} \longrightarrow \mathfrak{g}\} \tag{B.3.14}$$

of multilinear maps of (cohomological) degree $2 - n$, which are called n*-brackets* with $n = 1, 2, ...$, such that each bracket satisfies the following conditions:

1. *Graded anti-symmetry:* For all n and for $i = 1, ..., n-1$ one has

$$\ell_n(a_1, a_2, ..., a_i, a_{i+1}, ..., a_n) = -(-1)^{deg(a_i)deg(a_{i+1})}\ell_n(a_1, a_2, ..., a_{i+1}, a_i, ..., a_n). \tag{B.3.15}$$

2. *The n-Jacobi rule:* For all n,

$$0 = \sum_{k=1}^{n} (-1)^k \Big(\sum_{i_1 < i_2 < \cdots < i_k < j_{k+1} < \cdots < j_n} (-1)^{sign(\sigma)} \ell_{n-k+1}\big(\ell_k(a_{i_1}, ..., a_{i_k}), a_{j_{k+1}}, ..., a_{j_n}\big)\Big) \tag{B.3.16}$$

where $\{i_1, i_2, ..., i_k, j_{k+1}, ..., j_n\} = \{1, 2, ..., n\}$ and $(-1)^{sign(\sigma)}$ denotes the sign of the permutation for assigning the element of the set $\{i_1, i_2, ..., i_k, j_{k+1}, ..., j_n\}$ to the element of $\{1, 2, ..., n\}$.

Remark B.3.2. From the definition of an $\mathcal{L}_\infty$ algebra, one can conclude that $\ell_1^2 = 0$, and ℓ_2 satisfies the conditions in the Definition B.3.3. Therefore, we also write $\ell_1 := \mathrm{d}$ and $\ell_2 := [\cdot, \cdot]$.

A first natural example of an $\mathcal{L}_\infty$ algebra. One can revisit Example B.3.1 and interpret $\mathfrak{g}$ as a $\mathcal{L}_\infty$ algebra in the following way:

$$\mathfrak{g} := \Omega^*(M) \otimes \mathfrak{h}, \tag{B.3.17}$$

where the only *non-zero* multilinear maps are $\ell_1 := \mathrm{d}_{dR}$ and $\ell_2 := [\cdot, \cdot]$ such that

$$[\alpha \otimes X, \beta \otimes Y] := \alpha \wedge \beta \otimes [X, Y]_{\mathfrak{h}}$$

for all $\alpha \otimes X, \beta \otimes Y \in \Omega^*(M) \otimes \mathfrak{h}$.

Definition B.3.5. For an $\mathcal{L}_\infty$ algebra $\mathfrak{g}$, *the Maurer-Cartan (MC) equation* is given as

$$\mathrm{d}\alpha + \sum_{n=2}^{\infty} \frac{1}{n!} \ell_n(\alpha^{\otimes n}) = 0 \tag{B.3.18}$$

where α is an element of degree 1.

Note that when we reconsider the case $\mathfrak{g} := \Omega^*(M) \otimes \mathfrak{h}$, the MC equation reduces to

$$\mathrm{d}_{dR}A + \frac{1}{2}[A, A] = 0 \text{ where } A \in \Omega^1(M) \otimes \mathfrak{h}. \tag{B.3.19}$$

The end of a digression.

We now like to present a construction for a particular kind of formal moduli problem, which can be defined as the simplicial set of solutions to *the Maurer-Cartan equations*.

Definition B.3.6. Let $\mathfrak{g}$ be an $\mathcal{L}_\infty$ algebra, $(A, \mathfrak{m})$ a dg Artinian algebra. We define the simplicial set $MC(\mathfrak{g} \otimes \mathfrak{m})$ of solutions to the Maurer-Cartan equation in $\mathfrak{g} \otimes \mathfrak{m}$ as follows:

$$MC(\mathfrak{g} \otimes \mathfrak{m}) \in Fun(\Delta, Sets) \tag{B.3.20}$$

where an n-simplex in the set $MC(\mathfrak{g} \otimes \mathfrak{m})_n$ of n-simplices is an element

$$\alpha \in \mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n) \tag{B.3.21}$$

of cohomological degree 1 that satisfies the Maurer-Cartan equation B.3.5, i.e.

$$\mathrm{d}\alpha + \sum_{n=2}^{\infty} \frac{1}{n!} \ell_n(\alpha^{\otimes n}) = 0. \tag{B.3.22}$$

Remark B.3.3. In Definition B.3.6, α is in fact an element of the tensor product complex $\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n)$ of dg algebras which is defined as

$$\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n) = \bigoplus_k (\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n))^k \tag{B.3.23}$$

where $\mathfrak{g} = \bigoplus_i \mathfrak{g}^i$ with the differential $\mathrm{d}_\mathfrak{g}$, $\mathfrak{m} = \bigoplus_i \mathfrak{m}^i$ with the differential d_A and $\Omega^*(\Delta^n)$ is the usual de Rham complex on the n-simplex Δ^n with the de Rham differential d_{dR}. Here Δ^n denotes an n-simplex in $\mathbb{R}^{n+1}$ given as a set

$$\Delta^n := \Big\{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1 \;\; and \;\; 0 \leq x_k \leq 1 \;\; for \; all \; k\Big\}.$$

Therefore, the degree k component of $\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n)$ is given by

$$(\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n))^k = \bigoplus_{p+q+r=k} \mathfrak{g}^p \otimes \mathfrak{m}^q \otimes \Omega^r(\Delta^n), \tag{B.3.24}$$

and hence we obtain *the total complex associated to the triple complex*

$$\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n) = \bigoplus_k \bigoplus_{p+q+r=k} \mathfrak{g}^p \otimes \mathfrak{m}^q \otimes \Omega^r(\Delta^n) \tag{B.3.25}$$

with *the total differential* $d_{tot}^k : (\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n))^k \to (\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n))^{k+1}$ defined by

$$d_{tot}^k = \sum_{p+q+r=k} d_1^{p,q,r} + (-1)^p d_2^{p,q,r} + (-1)^{p+q} d_3^{p,q,r} \tag{B.3.26}$$

where

$$d_1^{p,q,r} = \mathrm{d}_\mathfrak{g}^p \otimes id_A^q \otimes id_{\Omega^r}, \;\; \mathrm{d}_\mathfrak{g}^p : \mathfrak{g}^p \to \mathfrak{g}^{p+1} \tag{B.3.27}$$

$$d_2^{p,q,r} = id_\mathfrak{g}^p \otimes \mathrm{d}_A^q \otimes id_{\Omega^r}, \;\; \mathrm{d}_A^q : \mathfrak{m}^q \to \mathfrak{m}^{q+1} \tag{B.3.28}$$

$$d_3^{p,q,r} = id_\mathfrak{g}^p \otimes id_A^q \otimes \mathrm{d}_{dR}^r, \;\; \mathrm{d}_{dR}^r : \Omega^r \to \Omega^{r+1}. \tag{B.3.29}$$

For a more concrete treatment to the notions like double/triple complexes and their total complexes, see [28], Chapter 12. In order to illustrate the situation related to the triple complexes and motivate the structure of such "higher dimensional" cochain complexes, one can consider a rather simple setting in which A is assumed to be an ordinary k-algebra. Note that A can be viewed as a complex that is concentrated at degree 0, and all other components are trivial with differential being zero. Hence, in this situation, we can consider $\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n)$ as a double complex and write $\mathfrak{g} \otimes \Omega^*(\Delta^n)$ instead. Furthermore, we diagrammatically have

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathfrak{g}^p \otimes \Omega^{r+1}(\Delta^n) & \xrightarrow{d_1^{p,r+1}} & \mathfrak{g}^{p+1} \otimes \Omega^{r+1}(\Delta^n) & \longrightarrow & \cdots \\
 & & d_2^{p,r} \uparrow & & \uparrow d_2^{p+1,r} & & \\
\cdots & \longrightarrow & \mathfrak{g}^p \otimes \Omega^{r}(\Delta^n) & \xrightarrow[d_1^{p,r}]{} & \mathfrak{g}^{p+1} \otimes \Omega^{r}(\Delta^n) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}
\tag{B.3.30}
$$

where $d_1^{p,r} = \mathrm{d}_{\mathfrak{g}}^p \otimes id_{\Omega^r}$ and $d_2^{p,r} = id_{\mathfrak{g}}^p \otimes \mathrm{d}_{dR}^r$ for all p, r. Note that each square in the diagram is commutative, and hence different parts of the differential are compatible. For the precise structural relations, we again refer to [28], Ch.12.

Definition B.3.7. Given an $\mathcal{L}_\infty$ algebra $\mathfrak{g}$, we can define $\mathcal{B}\mathfrak{g} \in Fun(dgArt_k, sSets)$ associated to $\mathfrak{g}$ as follows:

$$
\mathcal{B}\mathfrak{g} : \ dgArt_k \longrightarrow \ sSets, \ \ (A, \mathfrak{m}) \longmapsto \mathcal{B}\mathfrak{g}[(A, \mathfrak{m})] := MC(\mathfrak{g} \otimes \mathfrak{m}) \tag{B.3.31}
$$

where the set of n-simplicies is defined as above (cf. Definition B.3.6):

$$
MC(\mathfrak{g} \otimes \mathfrak{m})_n = \Big\{ \alpha \in \bigoplus_{p+q+r=1} \mathfrak{g}^p \otimes \mathfrak{m}^q \otimes \Omega^r(\Delta^n) : \ \mathrm{d}\alpha + \sum_{n=2}^{\infty} \frac{1}{n!} \ell_n(\alpha^{\otimes n}) = 0. \Big\} \tag{B.3.32}
$$

Lemma B.3.1. *The functor $\mathcal{B}\mathfrak{g}$ is a formal moduli problem.*

Theorem B.3.1. *[10] Every formal moduli problem is represented by a Maurer-Cartan functor $\mathcal{B}\mathfrak{g}$ for some differential graded Lie algebra (or an $\mathcal{L}_\infty$ algebra) $\mathfrak{g}$ up* *to a weak equivalence. More precisely, there exists an equivalence of ∞-categories*

$$dgla_k \xrightarrow{\sim} Moduli_k \subset Fun(dgArt_k, Ssets) \tag{B.3.33}$$

where $dgla_k$ and $Moduli_k$ denote ∞-categories of differential graded Lie algebras over k and that of formal moduli problems over k respectively with k being a field of characteristic zero.

Remark B.3.4. Here, $dgla_k$ is in fact an ∞-category arising as the homotopy category with weak equivalences being (chains of) quasi-isomorphisms of the underlying dg k-modules. In that respect, two dgla's $\mathfrak{g}$ and $\mathfrak{g}'$ induce equivalent formal moduli problems provided that they are related to each other by a chain of quasi-isomorphisms. That is,

$$\mathcal{B}\mathfrak{g} \sim \mathcal{B}\mathfrak{g}' \Leftrightarrow \exists\, \phi = \{\phi_i\} : \mathfrak{g} \to \mathfrak{g}'. \tag{B.3.34}$$

where each ϕ_i is a degreewise quasi-isomorphism.

B.3.2 Sheaf of formal moduli problems

Having introduced the notion of a formal moduli problem compatible with the language of $\mathcal{L}_\infty$ algebras, it turns out that a formal moduli problem is an unexpectedly tractable notion -thanks to the Lurie's theorem B.3.1- in the sense that all kinds of formal moduli problems $\mathcal{F}$, up to weak equivalences, can be represented in a relatively simple form: $\mathcal{F} = \mathcal{B}\mathfrak{g}$ for some dgla $\mathfrak{g}$. Note that we are interested in particular formal moduli problems that define *derived moduli spaces of solutions to the Euler-Lagrange equations on an open subset U of M*. Therefore, we shall next seek for a well-defined notion of a "local" formal moduli problem with suitable local-to-global properties. The structure one requires is called *a local $\mathcal{L}_\infty$ algebra* [3, 82]. This will serve as a sheaf of $\mathcal{L}_\infty$ algebras associated to "local" formal moduli problems.

Definition B.3.8. Let M be a manifold. **A local $\mathcal{L}_\infty$ algebra on** M consists of the following data:

1. *A graded vector bundle $L \xrightarrow{\pi} M$ over M* where $L = \bigoplus_n L^n$ with the space of smooth sections being denoted by

$$\mathcal{L} := \Gamma(M, L). \tag{B.3.35}$$

Furthermore, we denote the space of local sections over an open subset U of M by

$$\mathcal{L}(U) := \Gamma(U, L). \tag{B.3.36}$$

2. *A differential operator* $\mathrm{d} : \mathcal{L} \to \mathcal{L}$ of cohomological degree 1 such that $\mathrm{d}^2 = 0$,
3. *A sequence* $\{\ell_n : \mathcal{L}^{\otimes n} \longrightarrow \mathcal{L}\}$ *of multilinear maps of (cohomological) degree* $2-n$ with $n \geq 2$ such that d along with the sequence $\{\ell_n\}$ endow $\mathcal{L}$ with the structure of an $\mathcal{L}_\infty$ algebra.

We have the following immediate and prototype example: Let $\mathfrak{h}$ be a Lie algebra, L the exterior algebra bundle over M

$$L := \bigwedge^* T^*M \otimes \mathfrak{h} \longrightarrow M \tag{B.3.37}$$

such that the corresponding sections are $\mathfrak{h}$-valued 1-forms where for all open subset U of M,

$$\mathcal{L}(U) = \Omega^*(U) \otimes \mathfrak{h}. \tag{B.3.38}$$

Note that one can revisit Example B.3.1 and interpret $\mathcal{L}(U)$ as an $\mathcal{L}_\infty$ algebra with the structure maps $\{\ell_n\}$ where the only *non-zero* multilinear maps on $\mathcal{L}$ are $\mathrm{d} := \mathrm{d}_{dR}$ and $\ell_2 := [\cdot, \cdot]$, which is given by

$$[\alpha \otimes X, \beta \otimes Y] := \alpha \wedge \beta \otimes [X, Y]_{\mathfrak{h}}. \tag{B.3.39}$$

Now we are in the place of introducing manifestly the following *sheaf* $\mathcal{BL}$ *of formal moduli problems associated to a given local* $\mathcal{L}_\infty$ *algebra* L. For the proof of being indeed a sheaf, we refer to [3]: Let M be a manifold and L a local $\mathcal{L}_\infty$ algebra on M. Then, we set

$$\mathcal{BL} : Opens_M^{op} \longrightarrow Moduli_k, \quad U \longmapsto \mathcal{BL}(U) \tag{B.3.40}$$

where $Opens_M$ is the category of open subsets of M with morphisms being canonical inclusions, and $Moduli_k$ is as in Theorem B.3.1. Here, for all open subset U of M $B\mathcal{L}(U)$ is *the* formal moduli problem

$$\mathcal{BL}(U) : \; dgArt_k \longrightarrow \; sSets, \;\; (A, \mathfrak{m}_A) \longmapsto \mathcal{BL}(U)[(A, \mathfrak{m}_A)] \tag{B.3.41}$$

where $\mathcal{BL}(U)[(A, \mathfrak{m}_A)] := MC(\mathcal{L}(U) \otimes \mathfrak{m}_A)$ and the set $MC(\mathcal{L}(U) \otimes \mathfrak{m}_A)[n]$ of n-simplicies is defined as in B.3.32 with the replacement of $\mathfrak{g}$ by $\mathcal{L}(U)$.

Appendix C

THE EQUIVALENCE OF 3D QUANTUM GRAVITY WITH GAUGE THEORY

Without referencing any stacky behavior, in this section, we shall try to elaborate the contribution of Mess and Goldman in constructing the equivalence addressed in 1.1.3 and provide a brief guideline to the existing literature.

C.1 Cartan's formalism and gauge theoretic interpretation of 2+1 gravity

In this section we shall revisit the aspects of Cartan's formalism in a rather succinct way and we refer to Appendix A for the detailed analysis of elegant structures arising in Cartan geometry. Cartan's formalism, roughly speaking, consists of the following data [5, 11, 12]:

1. A section e_i^a of the orthonormal frame bundle LM over M for each i. That is,

$$e_i^a \in \Gamma(M, LM) \tag{C.1.1}$$

   where i labels *the space indices* with respect to the local chart $\big(U_i, x\big)$ around the point $p \in M$ and a's are called *Lorentz indices* labeling vectors in the orthonormal basis $\{e_i^1, e_i^2, ..., e_i^{dimM^2}\}$ over U_i. Here, each fibre

$$LM_p = \Big\{ \big(e_i^1(p), ..., e_i^m(p)\big) : e_i^1(p), ..., e_i^m(p) \text{ forms a basis for } TM_p \Big\}$$

   of LM is isomorphic to $GL(n, \mathbb{R})$. Such e_i^a are called *vierbein.*

2. A $SO(2,1)$-connection (or *the spin connection*) one-form $\omega_{i\ b}^{a}$ on M. That is,

$$\omega_{i\ b}^{a} \in \Omega^1(M) \otimes \mathfrak{so}(2,1) \tag{C.1.2}$$

where ω_i is a Lie algebra-valued connection 1-form on LM such that $\omega_i^a := (e_i^a)^* \omega_i$.

3. Compatibility conditions on metric:

$$g_{ij} = e_i^a e_j^b \eta_{ab} \ \ and \ \ g^{ij} e_i^a e_j^b = \eta^{ab} \tag{C.1.3}$$

where η denotes the usual Minkowski metric.

The punchline is the following observation [5]: In 2+1 dimensional gravity, vierbein and spin connection can be considered as a pair (e_i^a, ω_i^a) such that they could be combined into a certain gauge field A with the gauge group $ISO(2,1)$ where ω_i^a in fact plays the role of so-called $SO(2,1)$-part of the connection A (or say *the Lorentz-part*), while e_i^a corresponds to *translation generators* of the Lie algebra $iso(2,1)$ of $ISO(2,1)$. For some technical reasons, vierbein is supposed to be invertible in order to avoid the non-degeneracy on the metric.

Now, the usual Einstein-Hilbert action

$$\mathcal{I}_{EH}[g] := \frac{1}{16\pi G} \int\limits_M \mathrm{dx}^3 \sqrt{-g} R \tag{C.1.4}$$

can be re-expressed by employing Cartan's formalism as follows [5]:

$$\mathcal{I}'_{EH}[e, \omega] = \int\limits_M e^a \wedge \left(\mathrm{d}\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \wedge \omega^c \right) \tag{C.1.5}$$

where $e^a = e_i^a \mathrm{d}x^i$ and $\omega^a = \frac{1}{2} \epsilon^{abc} \omega_{ibc} \mathrm{d}x^i$ together with an invariant non-degenerate, bilinear form $\langle \cdot, \cdot \rangle$ on the Lie algebra $iso(2,1)$ (with its generators J^a and P^a corresponding to Lorentz and translation generators resp.) defined as

$$\langle J_a, P_b \rangle = \delta_{ab} \ \ \langle J_a, J_b \rangle = \langle P_a, P_b \rangle = 0 \tag{C.1.6}$$

with the structure relations for the Lie algebra given as

$$[J_a, J_b] = \epsilon_{abc} J^c, \ \ [J^a, P^b] = \epsilon_{abc} P^c, \ \ [P^a, P^b] = 0. \tag{C.1.7}$$

Setting a gauge field as

$$A_i := P_a e_i^a + J_a \omega_i^a \tag{C.1.8}$$

where $A^g = A_i(x)dx^i$ in a local coordinate chart $x = (x^i)$ such that P_a and J_a correspond to translations and Lorentz generators resp. as above, one can define a Chern-Simons theory with gauge group $G = ISO(2,1)$ in accordance with the above bilinear form $\langle\cdot,\cdot\rangle$ so that the usual Chern-Simons action

$$CS[A] = \int\limits_M \langle A, \mathrm{d}A + \frac{2}{3}A \wedge A\rangle = \int\limits_M \langle A, \frac{1}{3}\mathrm{d}A + \frac{2}{3}F\rangle = \mathcal{I}'_{EH}[e,\omega] \qquad \text{(C.1.9)}$$

becomes exactly the same expression as $\mathcal{I}'_{EH}$. For computational details, see [5, 11, 12]. Note that obtaining the same action functional is just one part of the whole story, and one also requires to verify that the diffeomorphism invariance of 2+1 gravity must also be encoded in some way in (e,ω)-formalism. As stressed explicitly in [5, 11, 12], the notions of invariance in these two formalisms, i.e. the 2^{nd}-order (metric) formalism and 1^{st}-order (e,ω)- formalism, are related to each other in the sense that *the invariance under spacetimes diffeomorphisms in metric formalism corresponds to the invariance under the corresponding gauge transformations,* so-called *local Lorentz transformations* and *local translations* in (e,ω)- formalism. Due to the rather expository nature of this section, we cross our fingers and avoid the derivation of those correspondences to save some space and time! For a systematic treatment, we again refer to [5, 11, 12].

Having adopted Cartan's formalism with the above observations, one can manifestly associate to *2+1 vacuum Einstein gravity with* $\Lambda = 0$ *on* $M = \Sigma \times (0,\infty)$ a particular gauge theory, namely *Chern-Simons theory with the gauge group* $G = ISO(2,1)$. Herein Σ is a closed Riemann surface of genus $g > 1$. Furthermore, using the natural gauge theoretic approach, we have the corresponding field equations [19] in (e,ω)-language for $\mathcal{I}'_{EH}$:

$$d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \wedge \omega_c = 0 \qquad \text{(C.1.10)}$$

$$de^a + \epsilon^{abc}\omega_b \wedge e_c = 0 \qquad \text{(C.1.11)}$$

where the equation C.1.11 corresponds to the fact that the torsion $T^a = 0$ and ω_b serves as a so-called *soldering form* [5] through which one can make sense of the notion of *torsion*. The equation C.1.10, on the other hand, corresponds to the fact that ω is indeed a flat $SO(2,1)$-connection (or equivalently that the curvature of the metric $g_{ij} = e^a_i e^b_j \eta_{ab}$ vanishes). Furthermore, once we impose the field equations, we have the following observations:

1. e can be realized as a cotangent vector to the point ω in the space X of flat $SO(2,1)$-connections on Σ. As explained in [19], this follows naïvely from the following observation: Given a smooth curve $\omega(s)$ in X, then by imposing the EOMs in equation C.1.10, we get

$$d\omega^a(s) + \frac{1}{2}\epsilon^{abc}\omega_b(s) \wedge \omega_c(s) = 0, \tag{C.1.12}$$

   and taking derivatives would give

$$d\Big(\frac{d\omega^a}{ds}\Big) + \epsilon^{abc}\omega_b(s) \wedge \frac{d\omega^c}{ds} = 0. \tag{C.1.13}$$

   Then, from equation C.1.11 we have $\dfrac{d\omega^a}{ds} = e^a$. Therefore, if we consider the canonical/geometric quantization (in the sense of [72, 74]) of the cotangent bundle with coordinates $(q_1, ..., q_n, p_1, ..., p_n)$ where $p_i = \dot{q}_i$ for each i, then ω_i's in fact play the role of *coordinates* q_i's as e_i's are viewed as *momenta* p_i.

2. The observation above implies that a solution (e, ω) determines a point in the cotangent bundle T^*X, and hence one can realize the Poincaré group $ISO(2,1)$ as the cotangent bundle over the Lorentz group $SO(2,1)$. That is,

$$ISO(2,1) = T^*(SO(2,1)). \tag{C.1.14}$$

3. As we will discuss in more detail soon (cf. Section C.2), there is a one-to-one correspondence between the moduli space $\mathcal{M}_{flat}$ of flat G-connections on Σ and the moduli space $Hom(\pi_1(\Sigma), G)/G$ of representations of the surface group $\pi_1(\Sigma)$ in G [36] where G acts on $Hom(\pi_1(\Sigma), G)$ by conjugation.

4. Furthermore, we will have the following isomorphims from Mess [33] and Goldman [34]. These are based on Teichmüller theoretic treatment [35] of representations of the surface group $\pi_1(\Sigma)$ in the cases where $G = ISO(2,1)$

or $G = PSL(2,\mathbb{R}) \cong SO_0(2,1)$.

$$\begin{aligned}\mathcal{M}_{flat} &\cong Hom(\pi_1(\Sigma), ISO(2,1))/\sim \\ &\cong T^*\Big(Hom\big(\pi_1(\Sigma), PSL(2,\mathbb{R})\big)/\sim\Big) \\ &\cong T^*(Teich(\Sigma))\end{aligned} \tag{C.1.15}$$

where $Teich(\Sigma)$ denotes the Teichmüller space associated to the closed surface Σ of genus $g > 1$. Note that this observation will be the crucial if one requires the invertibility of the map ϕ (cf. Theorem C.4.1).

C.2 The holonomy representation of flat G-connections

As explicitly studied in [57], there is a one-to-one correspondence between the moduli space $\mathcal{M}_{flat}$ of flat G-connections on Σ and the moduli space $Hom(\pi_1(\Sigma), G)/G$ of (holonomy) representations of the surface group $\pi_1(\Sigma)$ in G [36] where G acts on $Hom(\pi_1(\Sigma), G)$ by conjugation. That is, we have

$$\mathcal{M}_{flat} \cong Hom(\pi_1(\Sigma), G)/G. \tag{C.2.1}$$

The correspondence is rather well-known and based on the techniques emerging in the theory of principal G-bundles [45, 46]. Sketch of the idea is as follows:

1. Let A be a G-connection on a principal G-bundle $P \xrightarrow{\pi} \Sigma$. Given a smooth path γ in Σ, for any $p \in \pi^{-1}(\gamma(0))$ there exists a unique horizontal path $\widetilde{\gamma}_p$ starting with $\widetilde{\gamma}_p(0) = p$, and hence we have the standard *parallel transport map*

$$T_\gamma : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(1)), \;\; p \mapsto \; \widetilde{\gamma}_p(1). \tag{C.2.2}$$

Note that if γ is a *loop*, then the corresponding lift $\widetilde{\gamma}_p$ lands in the same fiber, i.e. $\widetilde{\gamma}_p(1) \in \pi^{-1}(\gamma(0))$ (but *not* necessarily hits the same point, i.e. $\widetilde{\gamma}_p(1) \neq p$), and hence by definition of a principal G-bundle the parallel transport map becomes

$$T_\gamma : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(0)), \;\; p \mapsto \; \widetilde{\gamma}_p(1) = p \bullet g \;\text{ for some } g \in G$$

which yields the definition of *the holonomy group*

$$Hol_p(\gamma, A) = \{g_\gamma \in G : T_\gamma(p) = p \bullet g_\gamma\}. \tag{C.2.3}$$

As p varies along the fiber via the action of G, for any $h \in G$, one has

$$Hol_{p \bullet h}(\gamma, A) = h^{-1} \cdot Hol_p(\gamma, A) \cdot h. \tag{C.2.4}$$

2. Note that we haven't used the flatness of A in the above construction. The flatness of A will come into play in accordance with the following facts [57]:

   (a) The connection A is flat if and only if the holonomy group $Hol_p(\gamma, A)$ depends only on the homotopy class of γ in $\pi_1(\Sigma)$.

   (b) The holonomy of a flat A-connection for a contractible loop γ_0 is trivial:

   $$Hol_p(\gamma_0, A) = \{e\}. \tag{C.2.5}$$

   Now, one can define a *well-defined map*, compatible with the actions on both sides, as follows [57]:

   $$\mathcal{M}_{flat} \longrightarrow Hom(\pi_1(\Sigma), G)/G, \;\; [A] \mapsto \Big(\rho_{[A]} : [\gamma] \;\mapsto\; g_\gamma\Big).$$

3. Converse of the map requires a constructive argument (for more details, see [36], sec. 2.3) in the following sense: To a given representation $\rho : \pi_1(\Sigma) \to G$, one assigns a *flat* principal G-bundle $P_\rho \longrightarrow \Sigma$ as follows. First, consider the universal cover $\widetilde{\Sigma} \xrightarrow{q} \Sigma$. Notice that $\pi_1(\Sigma)$ acts on $\widetilde{\Sigma}$ via deck transformations because of the fact that for the universal cover $\widetilde{\Sigma} \xrightarrow{q} \Sigma$ one has

   $$Deck(\widetilde{\Sigma}) \cong \pi_1(\Sigma) \;\; and \;\; \widetilde{\Sigma} \longrightarrow \widetilde{\Sigma}/Deck(\widetilde{\Sigma}) \;\cong\; \Sigma, \tag{C.2.6}$$

   and hence $\widetilde{\Sigma} \xrightarrow{q} \Sigma$ in fact admits a principal $\pi_1(\Sigma)$-bundle structure. Now, given a representation $\rho : \pi_1(\Sigma) \to G$, consider the space $\widetilde{\Sigma} \times G$ and a right $\pi_1(\Sigma)$-action on $\widetilde{\Sigma} \times G$ as follows: for all $\gamma \in \pi_1(\Sigma)$ and $(x, g) \in \widetilde{\Sigma} \times G$, we define

   $$(x, g) \bullet \gamma := (x \cdot \gamma, \rho(\gamma^{-1}) \cdot g) \tag{C.2.7}$$

   where γ acts on $x \in \widetilde{\Sigma}$ via deck transformation as indicated above. Then, we can introduce an equivalence relation, and hence a quotient space as

   $$P_\rho := \widetilde{\Sigma} \times G/ \sim \tag{C.2.8}$$

   where $(x, g) \sim (y, h) \iff y = x \cdot \gamma$ and $h = \rho(\gamma^{-1}) \cdot g$ for some $\gamma \in \pi_1(\Sigma)$. Finally, from [36],

   $$P_\rho \xrightarrow{\pi} \Sigma, \;\; [(x, g)] \;\mapsto\; q(x), \tag{C.2.9}$$

indeed defines a *flat* principal G-bundle with a natural right G-action on P_ρ given by

$$[(x,g)]\cdot h := [(x,gh)] \text{ for any } h \in G. \tag{C.2.10}$$

The existence of the inverse of the map

$$\phi : \mathcal{E}(M) \longrightarrow \mathcal{M}_{flat}, \tag{C.2.11}$$

which leads to the equivalence of quantum gravity with a gauge theory in a sense discussed before, is essentially related to the analysis of topological components of the space $Hom(\pi_1(\Sigma), G)/G$ in the case of $G = PSL(2;\mathbb{R})$, which was studied in [34]. Now, our next task is, in a rather expository manner, to elaborate the role of a particular component of

$$Hom(\pi_1(\Sigma), PSL(2;\mathbb{R}))/PSL(2;\mathbb{R}), \tag{C.2.12}$$

namely *the Fuchsian representations*, in proving the existence of such ϕ^{-1}.

C.3 Fuchsian representations of the surface group $\pi_1(\Sigma)$ in $PSL(2,\mathbb{R})$

In [34], Goldman originally investigates the global topology of the space

$$Hom(\pi_1(\Sigma), G)/G \tag{C.3.1}$$

in the case of $G = PSL(2;\mathbb{R})$ where Σ is a closed orientable surface of genus $g > 1$ (*no a priori complex structure is assumed in the first place*). The results in [34], in fact, depend on the study of certain characteristic classes. For details, you may visit [36], sec. 4, as well. According to the previous observations, to a given representation $\rho \in Hom(\pi_1(\Sigma), G)/G$, we assign a *flat* principal G-bundle $P_\rho \longrightarrow \Sigma$ in a well-established manner. If $G = PSL(2,\mathbb{R})$, since $PSL(2,\mathbb{R})$ acts on $\mathbb{R}P^1 \backsimeq \mathbb{S}^1$ by orientation-preserving projective transformations, one can also define the associated $\mathbb{R}P^1$-bundle on Σ with the structure group $PSL(2,\mathbb{R})$ as

$$P_\rho \times \mathbb{R}P^1/PSL(2,\mathbb{R}) \xrightarrow{\pi'} \Sigma, \tag{C.3.2}$$

and hence we have the Euler number e associated to this $\mathbb{R}P^1$-bundle which induces the map

$$e : Hom(\pi_1(\Sigma), PSL(2;\mathbb{R}))/\sim \longrightarrow H^2(\Sigma,\mathbb{Z}) \cong \mathbb{Z}. \tag{C.3.3}$$

Goldman's results related to the connected components are as follows:

Theorem C.3.1. *The connected components of* $Hom(\pi_1(\Sigma), PSL(2;\mathbb{R}))/\sim$ *are the preimages* $e^{-1}(n)$ *of the map* e *where* $n \in \mathbb{Z}$ *such that*

$$|n| \leq |\chi(\Sigma)| = 2g - 2. \tag{C.3.4}$$

Also, it has precisely $4g-3$ *components and the maximal component* $e^{-1}(2g-2)$ *consists of discrete and faithful representations (which can be identified with* $Teich(\Sigma)$*).*

This motivates the following definition [36].

Definition C.3.1. A representation $\rho \in Hom(\pi_1(\Sigma), PSL(2;\mathbb{R}))/\sim$ is called **Fuchsian** if it is *discrete and faithful* , i.e. ρ is injective, its image $\rho(\pi_1(\Sigma))$ is a discrete subgroup of $PSL(2,\mathbb{R})$ such that the quotient space $PSL(2,\mathbb{R})/\rho(\pi_1(\Sigma)$ is compact.

With this definition in hand, from Goldman's theorem we have the following observation:

Corollary C.3.1. $\rho : \pi_1(\Sigma) \to PSL(2;\mathbb{R})$ *is Fuchsian if and only if* $e(\rho) = 2g - 2$.

Here, $e(\rho) = 2g - 2 = |e(T\Sigma)|$ where $e(T\Sigma)$ denotes the Euler number associated to the tangent bundle on Σ, and it is in fact equal to $-\chi(\Sigma)$. Note also that for a representation ρ with $e(\rho) = 2g - 2$, the corresponding $\mathbb{R}P^1$-bundle is isomorphic to the (unit) tangent bundle $T\Sigma$ over Σ for which one has $2g-2 = e(T\Sigma)$. Furthermore, we have the following corollary which can be taken as the definition of *a Fuchsian representation* as well. For more details, we refer to [33, 5, 36].

Corollary C.3.2. $\rho : \pi_1(\Sigma) \to PSL(2;\mathbb{R})$ *is Fuchsian* $\Longleftrightarrow$ *It arises from the holonomy of a hyperbolic structure on* Σ.

Remark C.3.1. As indicated at the beginning of the current section, we do *not* assume any a priori Riemannian surface structure on Σ. Now, if Σ is a closed Riemann surface of genus $g > 1$, then *by the Uniformization Theorem*, Σ admits a unique hyperbolic structure inherited from the one on the upper half plane $\mathbb{H}$. Therefore, Σ is locally modeled on $(Isom(\mathbb{H}), \mathbb{H})$. That is, Σ is locally isometric to $\mathbb{H}/\Gamma$ for some discrete subgroup $\Gamma \subset Isom(\mathbb{H}) \cong PGL(2,\mathbb{R})$. Therefore, *the choice of a hyperbolic structure (which is indeed parametrized by $Teich(\Sigma)$) defines automatically a Fuchsian representation ρ of surface group $\pi_1(\Sigma)$ in $PSL(2,\mathbb{R})$.*

C.4 The outline of Mess' Main Results

Now, we are in the place of discussing the results stated in Mess' paper [33] related to both *(i)* the existence of ϕ^{-1} (which leads to the desired equivalence of quantum gravity with gauge theory) and *(ii)* the construction of the quantum Hilbert space $\mathcal{H}_{\mathcal{E}(M)}$ associated to the classical phase space $\mathcal{E}(M)$ of 2+1 gravity with $\Lambda = 0$. The outline is as follows:

1. Mess establishes, by using Thurston theory (and Teicmuller theory), the following relation: For a closed Riemann surface Σ of genus $g > 1$,

   $$Hom\big(\pi_1(\Sigma), ISO(2,1)\big)\big/_{ISO(2,1)} \cong T^*\Big(Hom\big(\pi_1(\Sigma), PSL(2;\mathbb{R})\big)\big/_{PSL(2,\mathbb{R})}\Big).$$

   As stressed in Remark C.3.1, there is a one-to-one correspondence between hyperbolic structures on Σ and Fuchsian representation of $\pi_1(\Sigma)$, and hence we have

   $$Hom\big(\pi_1(\Sigma), ISO(2,1)\big)/\sim \;\cong\; T^*(Teich(\Sigma)). \tag{C.4.1}$$

2. One has the following theorem through which the equivalence of quantum gravity with gauge theory can be established in the sense of Definition 1.1.3.

   Theorem C.4.1. *([33], Prop. 2) Given a Fuchsian representation*

   $$\rho : \pi_1(\Sigma) \to PSL(2;\mathbb{R})$$

   with Σ a closed Riemann surface of genus $g > 1$, there exists a flat Lorentzian manifold M of the form $\Sigma \times (0,\infty)$ and holonomy $\psi : \pi_1(\Sigma) \to ISO(2,1)$ such that $\psi = \rho$.

3. As briefly stressed above, one can alternatively reformulate such an Einstein gravity, especially in dimension $2+1$, as a particular gauge theory, so-called *Chern-Simons theory* with the gauge group being the Poincaré group $ISO(2,1)$.

Hence, in the gauge theoretical interpretation, we can realize the classical physical phase space of Einstein gravity as that of Chern-Simons theory, namely *the moduli space* $\mathcal{M}_{flat}$ *of flat* $ISO(2,1)$*-connections* on M. Furthermore, it follows directly from the Cartan's geometric formulation of Einstein-Hilbert action and the analysis of the corresponding EOMs that any flat metric in fact defines a corresponding flat gauge connection. Thus, one has a canonical map of moduli spaces (*not* invertible in the first place)

$$\phi : \mathcal{E}(M) \longrightarrow \mathcal{M}_{flat}, \;\; g \longmapsto A^g. \tag{C.4.2}$$

In that respect, we say that the quantum gravity is equivalent to gauge theory in the sense of the canonical formalism if this canonical map is, in fact, an isomorphism. If $\mathcal{E}(M)$ denotes the moduli space of $2+1$ dimensional (vacuum) Einstein gravity with the vanishing cosmological constant on a Lorentzian 3-manifold $M = \Sigma \times (0, \infty)$ where Σ is a closed Riemann surface of genus $g > 1$, then from Theorem C.4.1, the map

$$\phi : \mathcal{E}(M) \xrightarrow{\sim} \mathcal{M}_{flat} \tag{C.4.3}$$

is *an isomorphism* where

$$\mathcal{M}_{flat} \;\cong\; Hom\big(\pi_1(\Sigma), ISO(2,1)\big)/\sim \;\cong\; T^*(Teich(\Sigma)). \tag{C.4.4}$$

Note that as indicated in Remark C.3.1, the choice of a hyperbolic structure on Σ gives rise to a certain Fuchsian representation ρ of surface group $\pi_1(\Sigma)$ in $PSL(2,\mathbb{R})$. Hence, Theorem C.4.1 applies once Σ is endowed with a Riemann surface structure.

4. These observations together with the equivalence mentioned above implies that quantization of 3D gravity in the case of $\Lambda = 0$ and $M = \Sigma \times (0, \infty)$ as above boils down to *the canonical quantization of the cotangent bundle* $T^*(Teich(\Sigma))$ for which the associated quantum Hilbert space $\mathcal{H}_{\mathcal{E}(M)}$ is defined as in section 3.1 of [5]:

$$\mathcal{H}_{\mathcal{E}(M)} = \mathcal{L}^2(Teich(\Sigma)). \tag{C.4.5}$$

Appendix D

INTRODUCTION TO MODULI THEORY AND STACKS

In this Appendix, we shall explore the main aspects of moduli theory along with some examples. Main motivations, ideas, and some definitions related to moduli theory have already appeared in Section 2.1 and Appendix B.2.2. In this section, we essentially try to elaborate on some loose ends. In that respect, one of the purposes of this current section is to understand how the introduction of stacks circumvents the non-representability problem of the corresponding moduli functor $\mathcal{F}$ by using the 2-category of stacks. To this end, we shall briefly revisit the basics of 2-category theory and present the 2-categorical Yoneda embedding lemma for the "refined" moduli functor $\mathcal{F}$. Indeed, it is *a particular groupoid-valued presheaf*

$$\mathcal{F} : \mathcal{C}^{op} \longrightarrow \ Grpds \qquad \text{(D.0.1)}$$

with local-to-global properties where $Grpds$ denotes the 2-category of groupoids with objects being categories $\mathcal{C}$ in which all morphisms are isomorphisms (these sorts of categories are called *groupoids*), 1-morphisms being functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between groupoids, and 2-morphisms being natural transformations $\psi : \mathcal{F} \Rightarrow \mathcal{F}'$ between two functors. For an accessible introduction to moduli theory and stacks, we refer to [18, 25]. For an extensive treatment to the case of moduli of curves, see [26, 27, 29]. Basics of 2-category theory and further discussions can be found in [28].

D.1 Functor of points, representable functors, and Yoneda's Lemma

Main aspects of a moduli problem of interest can be encoded by a certain functor, namely *a moduli functor* of the form

$$\mathcal{F} : \mathcal{C}^{op} \longrightarrow \ Sets \qquad \text{(D.1.1)}$$

where $\mathcal{C}^{op}$ is the *opposite* category of the category $\mathcal{C}$, and $Sets$ denotes the category of sets. Equivalently, it is just a *contravariant functor* from the category $\mathcal{C}$ to $Sets$. The existence of *a fine moduli space* corresponds to the representability of this moduli functor. More details are to be discussed below.

Definition D.1.1. Let $\mathcal{C}$ be a category. For any object U in $\mathcal{C}$ we define a functor

$$h_U : \mathcal{C}^{op} \longrightarrow Sets \tag{D.1.2}$$

as follows:

1. For each object $X \in Ob(\mathcal{C})$, $X \longmapsto h_U(X) := Mor_{\mathcal{C}}(X, U)$
2. For each morphism $X \xrightarrow{f} Y$,

$$\Big(X \xrightarrow{f} Y\Big) \longmapsto \Big(Mor_{\mathcal{C}}(Y, U) \xrightarrow{f^*} Mor_{\mathcal{C}}(X, U),\ g \mapsto g \circ f\Big).$$

This functor h_U is called "the Yoneda functor" or "the functor of points".

As we shall see below, this functor can be used to recover any object U of a category $\mathcal{C}$ by understanding morphisms into it via Yoneda's Lemma. Let $Fun(\mathcal{C}^{op}, Sets)$ denote the category of functors from $\mathcal{C}^{op}$ to $Sets$ with objects being functors $\mathcal{F} : \mathcal{C}^{op} \longrightarrow Sets$ and morphisms being natural transformations between functors, then using the definition of h_U, one can introduce the following functor as well:

$$h : \mathcal{C} \longrightarrow Fun(\mathcal{C}^{op}, Sets) \tag{D.1.3}$$

where

1. For each object $U \in Ob(\mathcal{C})$, $U \longmapsto h_U = Mor_{\mathcal{C}}(\cdot, U)$
2. To each morphism $U \xrightarrow{f} V$, h assigns a natural transformation

$$\mathcal{C}^{op} \ \overset{h_U}{\underset{h_V}{\rightrightarrows}}\ Sets. \qquad h_f \Downarrow \tag{D.1.4}$$

Here h_f is defined as follows:

(a) For each object X in $\mathcal{C}$, we set

$$h_f(X) : h_U(X) \to h_V(X), \ \ g \longmapsto f \circ g. \tag{D.1.5}$$

(b) Given a morphism $X \xrightarrow{\eta} Y$ in $\mathcal{C}$, from the associativity property of the composition map, the diagram

$$\begin{array}{ccc} h_U(Y) & \xrightarrow{f\circ} & h_V(Y) \\ {\scriptstyle \circ\eta}\downarrow & & \downarrow{\scriptstyle \circ\eta} \\ h_U(X) & \xrightarrow[f\circ]{} & h_V(X) \end{array} \tag{D.1.6}$$

commutes.

Definition D.1.2. [49] Let $\mathcal{C}, \mathcal{D}$ be two categories.

1. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is called *fully faithful* if for any objects $A, B \in \mathcal{C}$, the map

$$Hom_{\mathcal{C}}(A, B) \longrightarrow Hom_{\mathcal{D}}(\mathcal{F}(A), \mathcal{F}(B)) \tag{D.1.7}$$

is a bijection of sets.

2. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is called *essentially surjective* if for any objects $D \in \mathcal{D}$, there exists an object A in $\mathcal{C}$ such that one has an isomorphism of objects

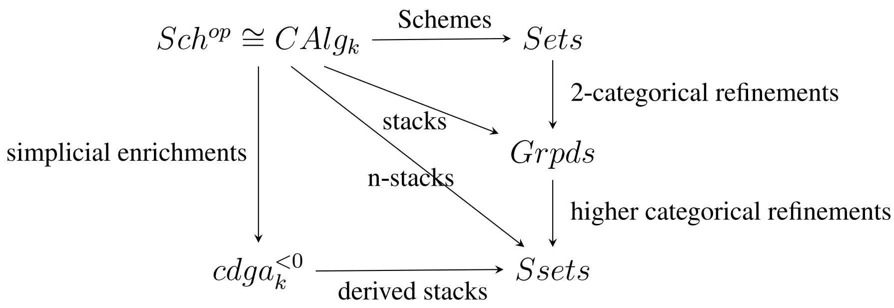
$$\mathcal{F}(A) \xrightarrow{\sim} D. \tag{D.1.8}$$

Lemma D.1.1. *(Yoneda's Lemma) The functor h above is fully faithful.*

Remark D.1.1.

1. Yoneda's lemma implies that the functor h serves as an embedding (sometimes it is also called *Yoneda's embedding*), and hence h_U determines U up to a unique isomorphism. Therefore, one can recover any object U in $\mathcal{C}$ by just knowing all possible morphisms into U. In the case of the category $Sch_{\mathbb{C}}$ of $\mathbb{C}$-schemes, for instance, it is enough to study the restriction of this functor to the full subcategory $Aff_{\mathbb{C}}$ of affine $\mathbb{C}$-schemes, in order to recover the scheme U.

2. Thanks to the Yoneda's embedding, one can also realize some algebro-geometric objects (like schemes, stacks, derived "spaces", etc...) as *a certain functor* in addition to standard ringed-space formulation. We have the following enlightening diagram [17] encoding such a functorial interpretation:



One way of interpreting this diagram is as follows: In the case of schemes (stacks resp.), for instance, such a functorial description implies that points of a scheme (a stack resp.) X form a *set* (a *groupoid* resp.). These kind of interpretations, in fact, suggest the name "functor of points".

3. The bad news is that not all functors $\mathcal{F} : \mathcal{C}^{op} \longrightarrow Sets$ are of the form h_U for some U in a general set-up. In other words, h is *not* essentially surjective in general. This in fact leads to the following definition:

Definition D.1.3. A functor $\mathcal{F} : \mathcal{C}^{op} \longrightarrow Sets$ is called *representable* if there exists $\mathcal{M} \in Ob(\mathcal{C})$ such that we have a natural isomorphism $\mathcal{F} \Leftrightarrow h_{\mathcal{M}}$. That is,

$$\mathcal{F} = Mor_{Sch_{\mathbb{C}}}(\cdot, \mathcal{M}) \text{ for some } \mathcal{M} \in Ob(\mathcal{C}). \tag{D.1.9}$$

If this is the case, then we say that $\mathcal{F}$ *is represented by* $\mathcal{M}$. In the case of moduli theory, $\mathcal{M}$ is then called a ***fine moduli space***. In the next section, we shall investigate the properties of $\mathcal{M}$.

D.2 Moduli theory in functorial perspective

D.2.1 Preliminaries

A moduli problem is a problem of constructing a classifying space (or *a moduli space* $\mathcal{M}$) for certain geometric objects (such as manifolds, algebraic varieties, vector bundles etc...) up to their intrinsic symmetries. In other words, a moduli space serves as a solution space of a given moduli problem of interest. In general, the set of isomorphism classes of objects that we would like to classify may not be able to provide sufficient information to encode geometric properties of the moduli space itself. Therefore, we expect a moduli space to behave well enough to capture the underlying geometry. Thus, this expectation leads to the following wish-list for $\mathcal{M}$ to be declared as a "fine" moduli space:

1. $\mathcal{M}$ is supposed to serve as a *parameter space* in a sense that there must be a one-to-one correspondence between the points of $\mathcal{M}$ and the *set* of isomorphism classes of objects to be classified:

$$\{points\ of\ \mathcal{M}\} \leftrightarrow \{isomorphism\ classes\ of\ objects\ in\ \mathcal{C}\} \qquad \text{(D.2.1)}$$

2. One ensures the existence of *universal classifying object*, say $\mathcal{T}$, through which all other objects parametrized by $\mathcal{M}$ can also be reconstructed. This, in fact, makes the moduli space $\mathcal{M}$ even more sensitive to the behavior of "families" of objects on any base object B. It is manifested by a certain representative morphism $B \to \mathcal{M}$. That is, for any family

$$\pi : X \longrightarrow B \qquad \text{(D.2.2)}$$

parametrized by some base scheme B where

$$X := \{X_b \in Ob(\mathcal{C}) : \pi^{-1}(b) = X_b,\ b \in B\},$$

there exits a unique morphism $f : B \to \mathcal{M}$ such that one has the following fibered product diagram:

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{T} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ B & \xrightarrow[f]{} & \mathcal{M} \end{array} \qquad \text{(D.2.3)}$$

where $X = B \times_{\mathcal{M}} \mathcal{T}$. That is, the family X can be uniquely obtained by pulling back the universal object $\mathcal{T}$ along the morphism f.

For an accessible overview, see [18]. Relatively complete treatments can be found in [30, 25].

Remark D.2.1. More formally, *a family over a base* B is a scheme X together with a morphism $\pi : X \to B$ of schemes where for each (closed point) $b \in B$ the fiber X_b is defined as fibered product

$$\begin{CD} X_b = \{b\} \times_B X @>>> X \\ @VVV @VV{\pi}V \\ \{b\} @>>{\imath}> B \end{CD} \tag{D.2.4}$$

where $\imath : \{b\} \hookrightarrow B$ is the usual inclusion map.

In the language of category theory, on the other hand, a moduli problem can be formalized as a certain functor

$$\mathcal{F} : \mathcal{C}^{op} \longrightarrow Sets, \tag{D.2.5}$$

which is called ***a moduli functor*** where $\mathcal{C}^{op}$ denotes the *opposite* category of the category $\mathcal{C}$, and $Sets$ is the category of sets. In other words, it is just a *contravariant functor* from the category $\mathcal{C}$ to $Sets$. In order to make the argument more transparent, we take $\mathcal{C}$ to be the category $Sch_{\mathbb{C}}$ of $\mathbb{C}$-schemes unless otherwise stated. Note that for each ($\mathbb{C}$-) scheme $U \in Sch$, $\mathcal{F}(U)$ is the *set* of isomorphism classes (of families) parametrized by U. For each morphism $f : U \to V$ of schemes, we have a morphism $\mathcal{F}(f) : \mathcal{F}(V) \to \mathcal{F}(U)$ of sets.

Example D.2.1. Given a scheme U, one can define $\mathcal{F}(U) := S(U)/\sim$ where $S(U)$ is *the set of families over the base scheme* U

$$S(U) := \Big\{ X \to U : X \text{ is a scheme over } U, \text{each fiber } X_u \text{ is } C_g \ \forall u \in U \Big\}$$

where C_g is a smooth, projective, algebraic curve of genus g. We say that two families $\pi : X \to U$ and $\pi' : Y \to U$ over U are *equivalent* if and only if there exists an isomorphism $f : X \xrightarrow{\sim} Y$ of schemes such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & U & \end{array} \tag{D.2.6}$$

On morphisms $\phi : U \to V$, on the other hand, we have

$$\mathcal{F}(\phi) : \mathcal{F}(V) \to \mathcal{F}(U), \;\; [X \to V] \longmapsto [U \times_V X \to U] \tag{D.2.7}$$

where $U \times_V X$ is the fibered product given by pulling back the family $X \to V$ along the morphism $\phi : U \to V$:

$$\begin{array}{ccc} U \times_V X & \longrightarrow & X \\ \downarrow & & \downarrow \\ U & \xrightarrow[f]{} & V \end{array} \tag{D.2.8}$$

With the above formalism in hand, the existence of a fine moduli space, therefore, corresponds to the *representability* of the moduli functor $\mathcal{F}$ in the sense that

$$\mathcal{F} = Mor_{Sch_{\mathbb{C}}}(\cdot, \mathcal{M}) \;\; for\ some\ \mathcal{M} \in Sch_{\mathbb{C}}. \tag{D.2.9}$$

If this is the case, then we say that $\mathcal{F}$ *is represented by* $\mathcal{M}$.

Remark D.2.2. Let $\mathcal{F} : \mathcal{C}^{op} \longrightarrow Sets$ be a moduli functor represented by an object M, then one can recast the desired properties of being a "fine" moduli space as follows:

1. Take $B := spec(\mathbb{C}) = \{*\}$, then from the representability we have

$$\mathcal{F}(\{*\}) \cong h_M(\{*\}) = Mor_{\mathcal{C}}(\{*\}, M). \tag{D.2.10}$$

   Note that the RHS is just the set of (closed) points of M, and LHS is the set of corresponding isomorphism classes.

2. When $B := M$, then we get an isomorphism

$$\mathcal{F}(M) \cong h_M(M) = Mor_{\mathcal{C}}(M, M), \tag{D.2.11}$$

   which allows us to define the universal object $\mathcal{T}$ to be the object corresponding to the identity morphism $id_M \in Mor_{\mathcal{C}}(M, M)$.

These observations yield the following corollary:

Corollary D.2.1. *If $\mathcal{F} : \mathcal{C}^{op} \longrightarrow Sets$ is a moduli functor represented by an object $\mathcal{M}$ in $\mathcal{C}$, then there exists an one-to-one correspondence between the set $\{X \to B\}$ of (equivalences classes of) families and $Mor_{\mathcal{C}}(B, \mathcal{M})$. That is,*

$$\{X \to B\}/\sim \longleftrightarrow Mor_{\mathcal{C}}(B, \mathcal{M}). \tag{D.2.12}$$

Furthermore, for a morphism $f : B \to \mathcal{M}$ corresponding to the equivalence class $\big[X \to B\big]$ of the family $X \to B$, we have

$$\big[X \to B\big] = \big[B \times_{\mathcal{M}} \mathcal{T} \to B\big]. \tag{D.2.13}$$

In many cases, however, a moduli functor is *not* representable in the category Sch of schemes. This is the place where the notion of a *stack* comes into play. In that situation, one can still make sense of the notion of a moduli space in a weaker sense. This version, namely *a coarse moduli space*, is still efficient enough to encode the isomorphism classes of points. That is, it has the correct points, and captures the geometry of moduli space. However, the sensitivity on the behavior of arbitrary families is no longer available. In other words, a coarse moduli space may not be able to distinguish two non-isomorphic families in many cases. Hence, the classification in this "family-wise" level is by no means possible. To elaborate the last statement, we first introduce the formal definition of a so-called *coarse moduli space*, and then we shall provide two important examples: (i) *the moduli problem of classifying vector bundles of fixed rank over an algebraic curve over a field* k [25], and (ii) *the moduli of elliptic curves* [18, 26].

Definition D.2.1. Let $\mathcal{C} := Sch_{\mathbb{C}}$ for the sake of simplicity. *A coarse moduli space* for a moduli functor $\mathcal{F} : \mathcal{C}^{op} \longrightarrow Sets$ consists of a pair (M, ψ) where M is an object in $\mathcal{C}$, and $\psi : \mathcal{F} \to h_M$ is a natural transformation such that

1. $\psi_{spec(\mathbb{C})} : \mathcal{F}(spec(\mathbb{C})) \to h_M(spec(\mathbb{C}))$ is a bijection of sets.

2. Such a pair (M, ψ) satisfies the following *universal property*: For any scheme N and any natural transformation $\phi : \mathcal{F} \to h_N$, there exists a unique morphism

$f : M \to N$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\psi} & h_M \\ & {\scriptstyle\phi}\searrow & \vdots\ \exists h_f \\ & & h_N \end{array} \tag{D.2.14}$$

Remark D.2.3. Here, $h_f : h_M \to h_N$ is the associated natural transformation of functors D.1.3. The second condition also implies that if it exists, a coarse moduli space M for a moduli functor $\mathcal{F}$ is unique up to a unique isomorphism.

Proposition D.2.1. *[30] Let (M, ψ) be a coarse moduli space for a moduli functor $\mathcal{F} : \mathcal{C}^{op} \longrightarrow Sets$ where M is a scheme and $\psi : \mathcal{F} \to h_M$ is the corresponding natural transformation. Then (M, ψ) is a fine moduli space if and only if the following conditions hold:*

1. *There exists a family $\mathcal{T} \to M$ such that $\psi_M(\mathcal{T}) = id_M \in Mor_{\mathcal{C}}(M, M)$.*

2. *For families $X \to B$ and $Y \to B$ on a base scheme B,*

$$[X \to B] = [Y \to B] \Longleftrightarrow \psi_B(X) = \psi_B(Y). \tag{D.2.15}$$

Proof. It follows directly from the definition of a fine moduli space. □

D.2.2 Moduli of vector bundles of fixed rank

We would like to investigate the moduli problem of classifying vector bundles of fixed rank over a smooth, projective algebraic curve X of genus g over a field k with char $k = 0$. We define the corresponding moduli functor

$$\mathcal{F}_X^n : Sch_{\mathbb{C}}^{op} \longrightarrow Sets \tag{D.2.16}$$

as follows: To each object U in $Sch_{\mathbb{C}}$, $\mathcal{F}_X^n$ assigns the set $\mathcal{F}_X^n(U)$ of isomorphism classes of families of vector bundles of rank n on X parametrized by U. That is,

$$\mathcal{F}_X^n(U) = \left\{ E \to U \times_{spec\mathbb{C}} X : E \text{ is a vector bundle of rank n } \right\} / \sim$$

Here, we say that two vector bundles $\pi : E \to U \times_{spec\mathbb{C}} X$ and $\pi' : E' \to U \times_{spec\mathbb{C}} X$ over $U \times_{spec\mathbb{C}} X$ are *equivalent* if and only if there exists an isomorphism $f : E \xrightarrow{\sim} E'$ of vector bundles such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & E' \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi'} & \\ & U \times_{spec\mathbb{C}} X & \end{array} \tag{D.2.17}$$

To each morphism $f : U \to V$ in $Sch_{\mathbb{C}}$, $\mathcal{F}_X^n$ assigns the map of vector bundles

$$\mathcal{F}_X^n(f) : \mathcal{F}_X^n(V) \to \mathcal{F}_X^n(U) \tag{D.2.18}$$

which is defined by pulling back of the vector bundle $E \to V \times_{spec\mathbb{C}} X$ along the morphism $f \times id_X$. Note that $U \times_{spec\mathbb{C}} X$ is just the usual direct product $U \times X$ with the projection maps $pr_1 : U \times X \to U$ and $pr_2 : U \times X \to X$ such that

$$\begin{array}{ccc} U \times_{spec\mathbb{C}} X & \xrightarrow{\; pr_2 \;} & X \\ {\scriptstyle pr_1} \downarrow & & \downarrow \\ U & \longrightarrow & spec\mathbb{C} \end{array} \tag{D.2.19}$$

Now, we would like to show that $\mathcal{F}_X^n$ can not be representable by some scheme M.

Claim: *$\mathcal{F}_X^n$ is not representable in $Sch_{\mathbb{C}}$.*

Proof. Assume that $\mathcal{F}_X^n$ is representable by a scheme M. That is, we have a natural isomorphism

$$\mathcal{F}_X^n \cong h_M. \tag{D.2.20}$$

Let U be a scheme and $E \in \mathcal{F}_X^n(U)$. Then, we have a vector bundle $E \to U \times X$ of rank n. Let L be a line bundle over U, then we can define *the induced bundle $pr_1^* L$ on $U \times X$* by pulling back L along the projection map pr_1. Therefore, we obtain a particular vector bundle

$$E' := E \otimes pr_1^* L \longrightarrow U \times X. \tag{D.2.21}$$

Indeed, E' is *a twisted bundle* where each fiber of E' is obtained by multiplying each fiber of E by a scalar. Hence, we produce a new vector bundle E' by "twisting" E

such that E' is *not* (globally) isomorphic to E. Let $\{U_i\}$ be a local trivializing cover of U such that $L|_{U_i}$ is trivial. Then it follows from the definition of E' that

$$E'|_{U_i \times X} \cong E|_{U_i \times X} \ \ \forall i. \tag{D.2.22}$$

As $\mathcal{F}^n_X$ is representable by a scheme M, there are morphisms $f_E : U \to M$ and $f_{E'} : U \to M$ in $Mor_{Sch_{\mathbb{C}}}(U, M)$ corresponding to E and E' respectively such that

$$f_E|_{U_i \times X} = f_{E'}|_{U_i \times X} \ \ \forall i. \tag{D.2.23}$$

Since h_M is a sheaf, it follows from the gluing axiom that all such morphisms are glued together nicely such that

$$f_E|_{U \times X} = f_{E'}|_{U \times X}. \tag{D.2.24}$$

But, from the representability of the functor $\mathcal{F}^n_X$, it implies that

$$E'|_{U \times X} \cong E|_{U \times X}. \tag{D.2.25}$$

This yields a contradiction to $E \ncong E'$. □

Remark D.2.4. The main reason behind the failure of the representability of $\mathcal{F}^n_X$ is that vector bundles have a number of non-trivial automorphisms, for instance, induced by a scalar multiplication as above. This example, in fact, may provide an important insight into why a generic moduli problem is destined to be non-representable in the category of schemes. In many cases, the main source of non-representability problems turns out to be *the existence of non-trivial automorphisms for the moduli problem of interest.*

D.2.3 Moduli of elliptic curves

In this section, we study the moduli space of elliptic curves and try to show how the existence of non-trivial automorphisms again leads to the non-representability of the corresponding moduli functor. The study of such a moduli problem is indeed a classical topic, and further discussion can be found elsewhere [18, 29, 26, 27].

Definition D.2.2. We first recall that one can define the notion of *an elliptic curve over* $\mathbb{C}$ in a number of equivalent ways. **An elliptic curve over** $\mathbb{C}$ is defined to be either of the following objects:

1. A Riemannian surface Σ of genus 1 with a choice of a point $p \in \Sigma$.

2. A quotient space $\mathbb{C}/\Lambda$ where $\Lambda = \omega_1 \cdot \mathbb{Z} \oplus \omega_2 \cdot \mathbb{Z}$ is a rank 2 lattice in $\mathbb{C}$ for each $\omega_i \in \mathbb{C}$.

3. A smooth algebraic curve of genus 1 and degree 3 in $\mathbb{P}^2_{\mathbb{C}}$.

We actually use the second characterization of an elliptic curve, namely the one given in terms of lattices. With this interpretation in hand, the study of the moduli of elliptic curves boils down to the study of integer lattices of full rank in $\mathbb{C}$.

$SL_2(\mathbb{Z})$-action on the upper half plane and the fundamental domain. Denote the upper half plane by $\mathbb{H} = \{z \in \mathbb{C} : im(z) > 0\}$. Then $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \frac{az+b}{cz+d} \quad \text{for all} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \text{ and } \forall z \in \mathbb{H}.$$

It is clear from the definition of the action that both $\pm I$ act on $\mathbb{H}$ in the same way, and hence we will concentrate on the action of $PSL_2(\mathbb{Z}) := SL_2(\mathbb{Z})/\{\pm I\}$ on $\mathbb{H}$. Then *the fundamental domain* $\Gamma := \mathbb{H}/PSL_2(\mathbb{Z})$ of this action turns out to be the set

$$\Gamma = \Big\{z : -\frac{1}{2} \leq Re(z) < \frac{1}{2},\ |z| > 1\Big\} \cup \Big\{z : -\frac{1}{2} \leq Re(z) \leq 0,\ \ |z| = 1\Big\}.$$

It is very well known that Γ is in fact a Riemann surface whose points correspond to the isomorphism classes of lattices of full rank in $\mathbb{C}$ up to homotheties. Note that any lattice $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ is isomorphic to a "normalized" lattice

$$\Lambda_\tau := 1 \cdot \mathbb{Z} \oplus \tau \cdot \mathbb{Z} \ \text{ for some } \tau \in \mathbb{H}. \tag{D.2.26}$$

We say that two lattices $\Lambda_{\tau_1} = 1 \cdot \mathbb{Z} \oplus \tau_1 \cdot \mathbb{Z}$ and $\Lambda_{\tau_2} = 1 \cdot \mathbb{Z} \oplus \tau_2 \cdot \mathbb{Z}$ with $\tau_i \in \mathbb{H}$ are *homothetic* if there exits $g \in PSL_2(\mathbb{Z})$ such that $\tau_2 = g \cdot \tau_1$. In other words, $\mathbb{H}/PSL_2(\mathbb{Z})$ serves as *a coarse moduli space* for isomorphism classes of elliptic curves $\mathbb{C}/\Lambda_\tau$ with $\tau \in \Gamma$. As we shall see soon, however, it turns out that the space Γ is not sensitive enough to parametrize certain families of elliptic curves. This amounts to say that not all families of elliptic curves over some base B correspond to morphisms $B \to \mathbb{H}/PSL_2(\mathbb{Z})$. Hence, Γ fails to become a fine moduli space.

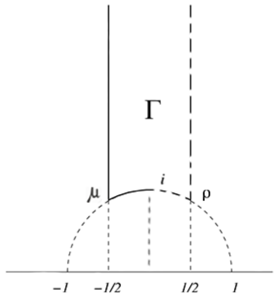


Figure D.1: The fundamental domain Γ where $\mu = e^{2\pi i/3}$ and $\rho = e^{2\pi i/6}$.

Remark D.2.5.

1. The fundamental domain Γ can also be represented as a free product of finite groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ as follows:

$$\begin{aligned}\Gamma =& \left\langle S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in PSL_2(\mathbb{Z}) \mid S^2 = (ST)^3 = I \right\rangle \\ \cong& \mathbb{Z}_2 \star \mathbb{Z}_3.\end{aligned}$$

2. The action of $PSL_2(\mathbb{Z})$ on $\mathbb{H}$ is not free. Indeed, it is routine to show that

$$Stab_{PSL_2(\mathbb{Z})}(\tau) \cong \begin{cases} \mathbb{Z}_2, & \tau = i \\ \mathbb{Z}_3, & \tau = \mu \text{ } or \text{ } \rho \\ \{e\}, & else. \end{cases} \tag{D.2.27}$$

3. It follows from the non-freeness of the action that one has special types of lattices, namely *the square lattice* Λ_i and the *hexagonal lattice* Λ_μ (or Λ_ρ) such that

$$Aut(\Lambda_\tau) = \begin{cases} \mathbb{Z}_4, & \tau = i \\ \mathbb{Z}_6, & \tau = \mu \text{ } or \text{ } \rho. \end{cases} \tag{D.2.28}$$

This gives rise to non-trivial automorphisms for the corresponding elliptic curves $\mathbb{C}/\Lambda_i$ and $\mathbb{C}/\Lambda_\mu$ by using, for instance, rotational symmetries of a square and that of a hexagon respectively. As before, *the existance of non-trivial automorphisms* allows us to produce some examples which eventually show that the corresponding moduli problem of elliptic curves can not be represented by the space $\mathbb{H}/PSL_2(\mathbb{Z})$. But, we should keep in mind that it becomes *the coarse moduli space.*

Moduli functor for the families of elliptic curves. We define the corresponding moduli functor

$$\mathcal{F}_{ell} : Sch_{\mathbb{C}}^{op} \longrightarrow Sets,\ U \mapsto \mathcal{F}_{ell}(U) \tag{D.2.29}$$

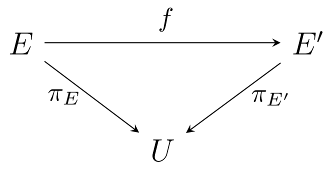
as follows: Given a scheme U, one can define $\mathcal{F}_{ell}(U) := S_{ell}(U)/\sim$ where $S_{ell}(U)$ is the set of (continuous) families of elliptic curves over the base scheme U:

$$S_{ell}(U) := \left\{ E \to U : \text{each fiber } E_u \text{ is } \mathbb{C}/\Lambda_{\tau(u)}\ \forall u \in U \right\} \tag{D.2.30}$$

where $\mathbb{C}/\Lambda_{\tau(u)}$ is an elliptic curve with $\tau : U \to \mathbb{H}/PSL_2(\mathbb{Z}),\ u \mapsto \tau(u)$, and $E = \bigsqcup_{u\in U} E_u$. We say that two families $\pi_E : E \to U$ and $\pi_{E'} : E' \to U$ over U are *equivalent* if and only if there exists an isomorphism $f : E \xrightarrow{\sim} E'$ of families such that on each fiber $E_u = \pi_E^{-1}(u)$, f restricts to an automorphism of elliptic curves

$$f|_{E_u} : E_u \xrightarrow{\sim} E'_u, \tag{D.2.31}$$

and the following diagram commutes:



(D.2.32)

From the previous discussion, it is not hard to observe that the space $\mathbb{H}/PSL_2(\mathbb{Z})$ in fact serves as the desired coarse moduli space for the moduli functor above. We now would like to show that the moduli functor $\mathcal{F}_{ell}$ is not representable by $\mathbb{H}/PSL_2(\mathbb{Z})$.

Claim: The moduli functor $\mathcal{F}_{ell}$ is not representable by $\mathbb{H}/PSL_2(\mathbb{Z})$.

Proof. Assume the contrary. Then, we have the following one-to-one correspondence between two sets:

$$Mor_{Sch_{\mathbb{C}}}(U, \mathbb{H}/PSL_2(\mathbb{Z})) \cong \mathcal{F}_{ell}(U). \tag{D.2.33}$$

We first consider a "constant" family E of elliptic curves on the interval $[0,1]$ where each fiber E_x is of the form

$$E_x := \mathbb{C}/\Lambda_i \ \text{ for all } x. \tag{D.2.34}$$

Recall that $Aut(\mathbb{C}/\Lambda_i) \cong \mathbb{Z}_4$. Let f be a non-trivial automorphism of $\mathbb{C}/\Lambda_i$ given as a multiplication by i,

$$f : \mathbb{C}/\Lambda_i \to \mathbb{C}/\Lambda_i, \; z \mapsto iz. \tag{D.2.35}$$

Then one can identify the fibers E_0 and E_1 along the morphism f so that a particular family $\mathcal{E}$ of elliptic curves over $\mathbb{S}^1$ can be obtained. Similarly, one can construct another family $\mathcal{E}'$ of elliptic curves over $\mathbb{S}^1$ by gluing the fibers E_0 and E_1 via the identity morphism. We then obtain two *non-isomorphic families* $\mathcal{E}$ and $\mathcal{E}'$ of elliptic curves over $\mathbb{S}^1$ with the generic fibers being all isomorphic. That is, $[\mathcal{E}] \neq [\mathcal{E}']$ such that

$$\mathcal{E}_x \cong \mathcal{E}'_x \cong E_x \text{ for all } x \in (0,1),$$

where $\mathcal{E}_x$ and $\mathcal{E}'_x$ denote the fibers of "twisted" and "trivial" families respectively. See Figure D.2.

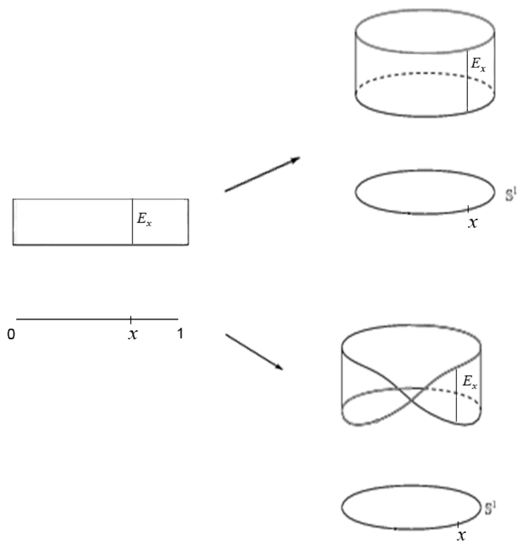


Figure D.2: The "trivial" and "twisted" families of elliptic curves over $\mathbb{S}^1$ with generically isomorphic fibers

As $\mathcal{F}_{ell}$ is representable by $\mathbb{H}/PSL_2(\mathbb{Z})$, there are corresponding morphisms $f_{\mathcal{E}}, \; f_{\mathcal{E}'}$: $\mathbb{S}^1 \to \Gamma$ for $\mathcal{E}$ and $\mathcal{E}'$ respectively such that

$$f_{\mathcal{E}}, \; f_{\mathcal{E}'} : \mathbb{S}^1 \longrightarrow \mathbb{H}/PSL_2(\mathbb{Z}), \; s \mapsto [i]. \tag{D.2.36}$$

That is, each representing morphism is just the constant map. Let $\{U_k\}$ be a local trivializing cover for $\mathbb{S}^1$ such that

$$\pi_{\mathcal{E}}^{-1}(U_k) \cong (\mathbb{C}/\Lambda_i) \times U_k \cong \pi_{\mathcal{E}'}^{-1}(U_k). \tag{D.2.37}$$

Then the representability condition implies that the representing morphisms are locally the same as well. That is,

$$f_{\mathcal{E}}|_{U_k} = f_{\mathcal{E}'}|_{U_k} \;\; \forall k. \tag{D.2.38}$$

Since $h_{\mathbb{H}/PSL_2(\mathbb{Z})} = Mor_{Sch_{\mathbb{C}}}(-, \mathbb{H}/PSL_2(\mathbb{Z}))$ is a sheaf, it follows from the gluing axiom that all such morphisms are glued together nicely. That is,

$$f_{\mathcal{E}} = f_{\mathcal{E}'} \;\; on \;\; \mathbb{S}^1. \tag{D.2.39}$$

But, it follows from the representability of the functor $\mathcal{F}_{ell}$ that we must have

$$[\mathcal{E}] = [\mathcal{E}'], \tag{D.2.40}$$

which is a contradiction.

$\square$

Remark D.2.6.

1. The construction above shows that the correspondence D.2.33 is not good enough to distinguish the "trivial" and "twisted" families of isomorphism classes of elliptic curves over $\mathbb{S}^1$ with generically isomorphic fibers. As before, the main source of this failure is due to the existence of non-trivial automorphism group for the fibers of the form $\mathbb{C}/\Lambda_i$. The existence of non-trivial automorphisms, on the other hand, is due to the fact that $PSL_2(\mathbb{Z})$ acts on $\mathbb{H}$ non-freely.

2. One way of circumventing this sort of problem is to change the way of organizing the moduli data. For instance, we can use the language of stacks, and redefine the moduli problem as a certain gruopoid-valued "functor"

$$\mathcal{F} : \mathcal{C}^{op} \longrightarrow \; Grpds \tag{D.2.41}$$

where $Grpds$ denotes *the 2-category of groupoids* with objects being categories $\mathcal{C}$ in which all morphisms are isomorphisms (these sorts of categories are called *groupoids*), 1-morphisms being functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between groupoids, and

2-morphisms being natural transformations $\psi : \mathcal{F} \Rightarrow \mathcal{F}'$ between two functors. Note that the groupoid structure, in fact, allows us to keep track of the non-trivial automorphisms as a part of the moduli data.

3. When we go back the example above, one can easily check that the space $\mathbb{H}/PSL_2(\mathbb{Z})$ can, in fact, be regarded as a *groupoid* where objects are the elements of $\mathbb{H}$, and the set of morphisms is the set $PSL_2(\mathbb{Z}) \times \mathbb{H}$. In fact,

$$\begin{aligned} Mor_{\mathbb{H}/PSL_2(\mathbb{Z})}(x,y) :=& \{g \in PSL_2(\mathbb{Z}) : y = g \cdot x\} \\ \cong& \, PSL_2(\mathbb{Z}) \times \mathbb{H}. \end{aligned} \tag{D.2.42}$$

Denote a morphism by (g, x): $x \mapsto g \cdot x$. Note that two morphisms (g, x) and (h, y) are composable if $x = h \cdot y$. Then we have

$$(g, h \cdot y) \circ (h, y) = (gh, y). \tag{D.2.43}$$

Furthermore, the inverse of the morphism (g, x) is $(g^{-1}, g \cdot x)$. Informally speaking, two non-isomorphic families $\mathcal{E}$ and $\mathcal{E}'$ above can be represented by points like $[i, f]$ and $[i, id]$ via suitable constant representing morphisms as above where the second slots in the parenthesis are to keep track of possible automorphisms distinguishing these families. The last statement will be elaborated in the next section. In literature, the space $\mathbb{H}/PSL_2(\mathbb{Z})$ is an example of an *orbifold*.

D.3 2-categories and Stacks

Stacks and 2-categories serve as motivating/prototype conceptual examples before introducing the notions like ∞*-categories, derived schemes, higher stacks and derived stacks* [17]. By using a 2-categorical version of the Yoneda lemma [25], one can show that the "refined" moduli functor

$$\mathcal{F} : \mathcal{C}^{op} \longrightarrow Grpds \tag{D.3.1}$$

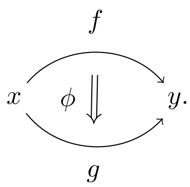
turns out to be representable in the 2-category $Stks$ of stacks. The price we have to pay is to adopt a higher categorical dictionary, which leads to the change in the level of abstraction in a way that objects under consideration may become rather counter-intuitive. We first briefly recall the basics of 2-category theory. For details, we refer to [28, 25].

D.3.1 A digression on 2-categories

Definition D.3.1. *A 2-category* $\mathcal{C}$ consists of the following data:

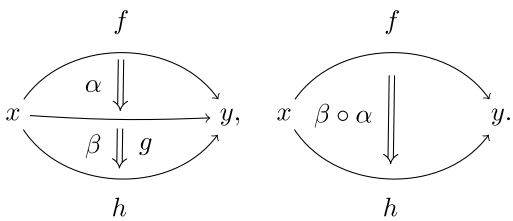
1. A collection of objects: $Ob(\mathcal{C})$.

2. For each pair x, y of objects, *a category* $Mor_{\mathcal{C}}(x, y)$. Here, objects of the category $Hom_{\mathcal{C}}(x, y)$ are called *1-morphisms* and are denoted either by $f : x \to y$ or $x \xrightarrow{f} y$. The morphisms of $Mor_{\mathcal{C}}(x, y)$ are called *2-morphisms* and are denoted either by $\phi : f \Rightarrow g$ or



(D.3.2)

The composition of two 2-morphisms $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$ in $Mor_{\mathcal{C}}(x, y)$ is called *a vertical composition*, and denoted by $\beta \circ \alpha : f \Rightarrow h$ or



(D.3.3)

A 2-morphism $\alpha : f \Rightarrow g$ is *invertible* if there exits a 2-morphism $\beta : g \Rightarrow f$ such that $\beta \circ \alpha = id_f$ and $\alpha \circ \beta = id_g$. Furthermore, an invertible 2-morphism $\alpha : f \Rightarrow g$ is called a *2-isomorphism*. It is sometimes denoted by $\alpha : f \Leftrightarrow g$.

3. For each triple x, y, z of objects in $\mathcal{C}$, there is *a composition functor*

$$\mu_{x,y,z} : Mor_{\mathcal{C}}(x, y) \times Mor_{\mathcal{C}}(y, z) \to Mor_{\mathcal{C}}(x, z) \qquad \text{(D.3.4)}$$

which is defined as follows:

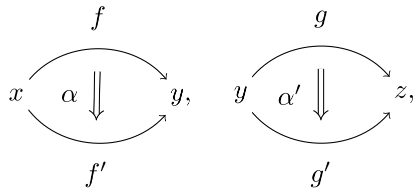
(a) On 1-morphisms, it acts as the usual composition of morphisms in $\mathcal{C}$:

$$\mu_{x,y,z} : \big(x \xrightarrow{f} y, y \xrightarrow{g} z\big) \longmapsto \big(y \xrightarrow{g \circ f} z\big)$$

(b) On 2-morphisms, it acts as a *horizontal composition*, denoted by $\star$:

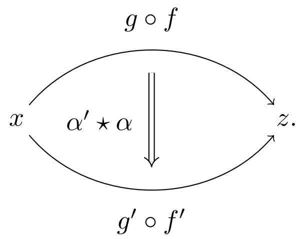
$$\mu_{x,y,z} : \big(f \overset{\alpha}{\Rightarrow} f', g \overset{\alpha'}{\Rightarrow} g'\big) \longmapsto \big(g \circ f \overset{\alpha' \star \alpha}{\Longrightarrow} g' \circ f'\big) \qquad \text{(D.3.5)}$$

That is, given two 2-morphisms



(D.3.6)

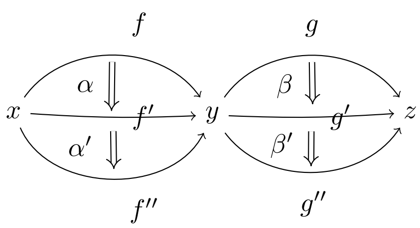
$\mu_{x,y,z}$ maps the pair $\big(f \overset{\alpha}{\Rightarrow} f', g \overset{\alpha'}{\Rightarrow} g'\big)$ of 2-morphisms to a 2-morphism



(D.3.7)

These data must satisfy the following conditions:

- For each object X of $\mathcal{C}$ and each 1-morphism $f : A \to B$, we have an identity 1-morphism $id_X : X \to X$ and an identity 2-morphism $id_f : f \Rightarrow f$ respectively.
- The composition of 1-morphisms (2-morphisms respectively) is associative.
- Horizontal and vertical compositions of 2-morphisms are "compatible" in the following sense. For a composition diagram



(D.3.8)

we have

$$(\beta' \circ \beta) \star (\alpha' \circ \alpha) = (\beta' \star \alpha') \circ (\beta \star \alpha). \qquad \text{(D.3.9)}$$

Example D.3.1. Every category can be realized as a 2-category. Indeed, let $\mathcal{C}$ be a category and $Mor_{\mathcal{C}}(A,B)$ denote the *set* of morphisms between two objects A, B. Then it is clear to observe that for any pair (A, B) of objects in $\mathcal{C}$ the set $Mor_{\mathcal{C}}(A,B)$ defines a category **Mor**$_{\mathcal{C}}$(A,B) whose objects (1-morphisms) are just morphisms $A \to B$ in $\mathcal{C}$, and morphisms (2-morphisms) are just identities. That is, there are no non-trivial higher morphisms in this realization. A category is sometimes called a *1-category*.

Remark D.3.1. Given a 2-category $\mathcal{C}$, one can obtain a category C_0 by defining $Ob(C_0) := Ob(\mathcal{C})$ and the "set" $Mor_{C_0}(A,B)$ of morphisms in C_0 to be

$$Mor_{C_0}(A,B) := Mor_{\mathcal{C}}(A,B)/\sim \tag{D.3.10}$$

where $f \sim g$ if there exits a 2-isomorphism $\alpha : f \Leftrightarrow g$. That is, $Mor_{C_0}(A,B)$ is just the set of isomorphism classes of 1-morphisms in $\mathcal{C}$.

Example D.3.2. A collection of categories forms a 2-category, namely the 2-category Cat of categories. Here, *objects* of Cat are just categories $\mathcal{C}$, *1-morphisms* in Cat are functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between two categories, and *2-morphisms* are natural transformations $\eta : \mathcal{F} \Rightarrow \mathcal{G}$ of functors. In this example, there are no non-trivial higher n-morphisms for $n > 2$. Once we allow such types of morphisms, we land in the territory of higher categories.

Definition D.3.2. Let $\mathcal{C}$ be a 2-category. Two objects X, Y in $\mathcal{C}$ are said to be *equivalent* if there exist a pair $(X \xrightarrow{f} Y, Y \xrightarrow{g} X)$ of 1-morphisms, and two 2-isomorphisms $\alpha : g \circ f \Leftrightarrow id_X$ and $\alpha' : f \circ g \Leftrightarrow id_Y$.

Definition D.3.3. *A pseudo-functor* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ *between two 2-categories* $\mathcal{C}, \mathcal{D}$ consists of the following data:

1. For each object A in $\mathcal{C}$, an object $\mathcal{F}(A)$ in $\mathcal{D}$,
2. For each 1-morphism $A \xrightarrow{f} B$ in $\mathcal{C}$, a 1-morphism $\mathcal{F}(A) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(B)$ in $\mathcal{D}$,
3. For each 2-morphism $\alpha : f \Rightarrow g$ in $\mathcal{C}$, a 2-morphism $\mathcal{F}(\alpha) : \mathcal{F}(f) \Rightarrow \mathcal{F}(g)$ in $\mathcal{D}$ satisfying the following conditions:
   (a) $\mathcal{F}$ respects 1- and 2-identities: $\mathcal{F}(id_A) = id_{\mathcal{F}(A)}$ and $\mathcal{F}(id_f) = id_{\mathcal{F}(f)}$ for all $A \in Ob(\mathcal{C})$ and $f \in Mor_{\mathcal{C}}(X,Y)$.

(b) $\mathcal{F}$ respects a composition of 1-morphisms up to a 2-isomorphism: Given a composition of 1-morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$, there is a 2-isomorphism

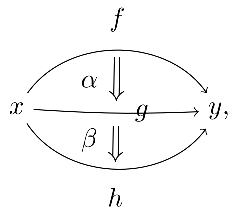
$$\phi^{\mathcal{F}}_{g,f} : \mathcal{F}(g \circ f) \Rightarrow \mathcal{F}(g) \circ \mathcal{F}(f) \tag{D.3.11}$$

such that the following diagram commutes *(encoding the associativity)*:

$$\begin{array}{ccc} \mathcal{F}(h \circ g \circ f) & \xRightarrow[\phi^{\mathcal{F}}_{h,g\circ f}]{} & \mathcal{F}(h) \circ \mathcal{F}(g \circ f) \\ \phi^{\mathcal{F}}_{h\circ g,f} \Big\Downarrow & & \Big\Downarrow id_{\mathcal{F}(h)} \star \phi^{\mathcal{F}}_{g,f} \\ \mathcal{F}(h \circ g) \circ \mathcal{F}(f) & \xRightarrow[\phi^{\mathcal{F}}_{h,g} \star id_{\mathcal{F}(f)}]{} & \mathcal{F}(h) \circ \mathcal{F}(g) \circ \mathcal{F}(f) \end{array} \tag{D.3.12}$$

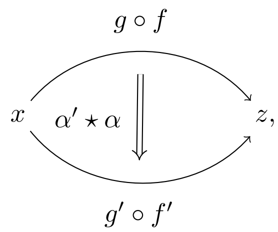
such that $\phi^{\mathcal{F}}_{f,id_A} = \phi^{\mathcal{F}}_{id_B,f} = id_{\mathcal{F}(f)}$.

(c) $\mathcal{F}$ respects both vertical and horizontal compositions: Given a vertical composition $\beta \circ \alpha : f \Rightarrow h$ for the diagram



(D.3.13)

we have $\mathcal{F}(\beta \circ \alpha) = \mathcal{F}(\beta) \circ \mathcal{F}(\alpha)$. Given a horizontal composition



(D.3.14)

with $\alpha : f \Rightarrow f'$ and $\alpha' : g \Rightarrow g'$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(g) \circ \mathcal{F}(f) & \xRightarrow[\mathcal{F}(\alpha') \star \mathcal{F}(\alpha)]{} & \mathcal{F}(g') \circ \mathcal{F}(f') \\ \phi^{\mathcal{F}}_{g,f} \Big\Downarrow & & \Big\Downarrow \phi^{\mathcal{F}}_{g',f'} \\ \mathcal{F}(g \circ f) & \xRightarrow[\mathcal{F}(\alpha' \star \alpha)]{} & \mathcal{F}(g' \circ f') \end{array} \tag{D.3.15}$$

Remark D.3.2. Notice that **a prestack** in Definition 2.1.1 is just a particular (contravariant) pseudo-functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ is an ordinary category and $\mathcal{D}$ is the 2-category $Grpds$ of groupoids. That is, it is a pseudo-functor $\mathcal{F} : C^{op} \to Grpds$ for some category C. Recall that, in the 2-category $Grpds$ of groupoids, objects are categories $\mathcal{C}$ in which all morphisms are isomorphisms (these sorts of categories are called *groupoids*), 1-morphisms are functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between groupoids, and 2-morphisms are natural transformations $\psi : \mathcal{F} \Rightarrow \mathcal{F}'$ between two functors. If $\mathcal{C}_\tau$ is a category $\mathcal{C}$ endowed with a Grothendieck topology τ, then **a stack** on $\mathcal{C}_\tau$ in Definition 2.2.4 is just a prestack with local-to-global properties w.r.t. τ. So, it can be considered as "a sheaf of groupoids" in a suitable sense.

D.3.2 2-category of Stacks and 2-Yoneda's Lemma

We like to present how stacks over a site $\mathcal{C}_\tau$ form a 2-category. To this end, we need to introduce the notions of 1- and 2-morphisms between two stacks.

Definition D.3.4. Let $\mathcal{C}$ be a category, and $\mathcal{X}, \mathcal{Y} : \mathcal{C}^{op} \longrightarrow Grpds$ be two prestacks. *A 1-morphism $F : \mathcal{X} \to \mathcal{Y}$ of prestacks* consists of the following data:

1. For each object A in $\mathcal{C}$, *a functor* $F_A : \mathcal{X}(A) \to \mathcal{Y}(A)$,
2. For each morphism $f : A \to B$ in $\mathcal{C}$, a 2-isomorphism

$$\mathcal{X}(B) \underset{\mathcal{Y}(f)\circ F_B}{\overset{F_A \circ \mathcal{X}(f)}{\Downarrow F_f}} \mathcal{Y}(A) \tag{D.3.16}$$

such that the following diagram commutes up to a 2-isomorphism

$$F_f : F_A \circ \mathcal{X}(f) \Longrightarrow \mathcal{Y}(f) \circ F_B$$

in $Grpds$:

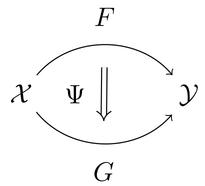
$$\begin{array}{ccc} \mathcal{X}(B) & \xrightarrow[F_B]{} & \mathcal{Y}(B) \\ {\scriptstyle \mathcal{X}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{Y}(f)} \\ \mathcal{X}(A) & \xrightarrow[F_A]{} & \mathcal{Y}(A) \end{array} \tag{D.3.17}$$

3. Given a composition of 1-morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$, the corresponding 2-isomorphisms F_f and F_g define $F_{g\circ f}$ in compatible with the natural 2-isomorphisms $\phi^{\mathcal{X}}_{g,f}$ and $\phi^{\mathcal{Y}}_{g,f}$. Indeed, using horizontal and vertical compositions of 2-morphisms, $F_{g\circ f} : F_A \circ \mathcal{X}(g \circ f) \Rightarrow \mathcal{Y}(g \circ f) \circ F_C$ is given by

$$F_{g\circ f} = \left(\phi^{\mathcal{Y}}_{g,f} \star id_{F_C}\right) \circ \left(id_{\mathcal{Y}(f)} \star F_g\right) \circ \left(F_f \star id_{\mathcal{X}(g)}\right) \circ \left(id_{F_A} \star \phi^{\mathcal{X}}_{g,f}\right).$$

Such a 1-morphism is just *a natural transformation* between two pseudo-functors of 2-categories.

Definition D.3.5. Let $\mathcal{C}$ be a category, and $F, G : \mathcal{X} \to \mathcal{Y}$ be a pair of 1-morphisms of prestacks. *A 2-morphism of between F and G*



(D.3.18)

is a collection $\left\{\Psi_A : F_A \Rightarrow G_A : A \in Ob(\mathcal{C})\right\}$ of invertible natural transformations of functors.

Remark D.3.3. A 1-morphism (2-morphism respectively) of stacks is defined as a 1-morphism (2-morphism resp.) of the underlying prestacks.

Proposition D.3.1. *[25]*

1. *Given a category $\mathcal{C}$, prestacks over $\mathcal{C}$ form a 2-category $PreStk_{\mathcal{C}}$ of prestacks where 1- and 2-morphisms are defined as above.*

2. *Stacks over a site $\mathcal{C}$ form a 2-category $Stk_{\mathcal{C}}$ of stacks with 1- and 2-morphisms being as above.*

Furthermore, it follows from the construction that we have the following observations [25].

Proposition D.3.2. *Let $\mathcal{C}$ be a category. Then the (1-) category $PreShv_{\mathcal{C}}$ of presheaves (of sets) over $\mathcal{C}$ is a full 2-subcategory of $PreStk_{\mathcal{C}}$. In addition, if $\mathcal{C}$ admits a site structure, then the (1-) category $Shv_{\mathcal{C}}$ of sheaves (of sets) over $\mathcal{C}$ is a full 2-subcategory of $Stk_{\mathcal{C}}$.*

Remark D.3.4. As we have already discussed in Remark D.1.1, Yoneda's lemma D.1.1 implies that any object X in a category $\mathcal{C}$ can be understood by studying all morphism into it. In other words, for any $X \in Ob(\mathcal{C})$ one has a particular *sheaf of sets*

$$h_X : \mathcal{C}^{op} \longrightarrow \; Sets, \quad Y \longmapsto h_X(Y) := Mor_{\mathcal{C}}(Y, X) \tag{D.3.19}$$

which uniquely determines X. If $\mathcal{C}$ admits a suitable site structure, then it follows from Proposition D.3.2 that the sheaf $Mor_{\mathcal{C}}(\cdot, X)$ can be considered as a stack with trivial 2-morphisms. We denote this stack by $\underline{X}$.

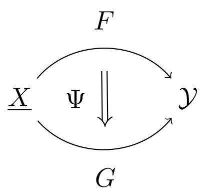
Lemma D.3.1. *(2-categorical Yoneda's Lemma for prestacks) Let $\mathcal{Y} : \mathcal{C}^{op} \longrightarrow Grpds$ be a prestack over a category $\mathcal{C}$. Then for each object X in $\mathcal{C}$, there exits an equivalence of categories*

$$\mathcal{Y}(X) \cong Mor_{PreStk_{\mathcal{C}}}(\underline{X}, \mathcal{Y}). \tag{D.3.20}$$

Proof. First, let us try to understand the objects of interests in the statement. On the left hand side, $\mathcal{Y}(X)$ is a groupoid, i.e. a category for which all morphisms are isomorphisms. On the right hand side, $Mor_{PreStk_{\mathcal{C}}}(\underline{X}, \mathcal{Y})$ is the category of morphisms in the 2-category $PreStk_{\mathcal{C}}$. 1-morphisms are a collection

$$\big\{ F_A : \underline{X}(A) \to \mathcal{Y}(A) : A \in Ob(\mathcal{C}) \big\}$$

of functors with some compatibility conditions as above. Such collection is denoted by $F : \underline{X} \to \mathcal{Y}$. On the other hand, 2-morphisms are of the form



(D.3.21)

where for each object Z, $\Phi_Z : F_Z \Rightarrow G_Z$ is an invertible natural transformation. Therefore, for any $f \in \underline{X}(Z)$, we have an isomorphism $\Psi_{Z,f} : F_Z(f) \xrightarrow{\sim} G_Z(f)$ in $\mathcal{Y}(Z)$. To show the desired equivalence, we introduce the following functors:

1. We define the functor $\Theta : Mor_{PreStk_{\mathcal{C}}}(\underline{X}, \mathcal{Y}) \longrightarrow \mathcal{Y}(X)$ as follows:

   (a) On objects (1-morphisms), $\big(F : \underline{X} \to \mathcal{Y}\big) \longmapsto F_X(id_X)$

(b) On morphisms (2-morphisms),

$$\big(\Psi : F \Rightarrow G\big) \longmapsto \big(\Psi_{X,id_X} : F_X(id_X) \xrightarrow{\sim} G_X(id_X)\big)$$

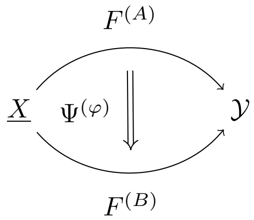
2. Define the functor $\eta : \mathcal{Y}(X) \longrightarrow Mor_{PreStk_{\mathcal{C}}}(\underline{X}, \mathcal{Y})$ as follows:

   (a) On objects A in $\mathcal{Y}(X)$, $A \longmapsto \big(F^{(A)} : \underline{X} \to \mathcal{Y}\big)$. Here $F^{(A)}$ is given as a collection $\{F_U^{(A)} : U \in Ob(\mathcal{C})\}$ of functors such that

$$F_U^{(A)} : \underline{X}(U) \to \mathcal{Y}(U),\ (U \xrightarrow{f} X) \longmapsto \mathcal{Y}(f)(A) \qquad \text{(D.3.22)}$$

   where $\mathcal{Y}(f) : \mathcal{Y}(X) \to \mathcal{Y}(U)$. Notice that $\underline{X}(U) = Mor_{\mathcal{C}}(U, X)$ is just a *set* in the first place, but, as we remarked before, it can be viewed as a category for which all morphisms are identities. Therefore, $F_U^{(A)}$ acts on morphisms of $\underline{X}(U)$ trivially. That is, it maps id_f to $id_{F_U^{(A)}(f)}$.

   (b) η sends morphisms $\varphi : A \xrightarrow{\sim} B$ in $\mathcal{Y}(X)$ to 2-morphisms



(D.3.23)

   where $\Psi^{(\varphi)}$ is a collection $\big\{\Psi_U^{(\varphi)} : F_U^{(A)} \Longrightarrow F_U^{(B)} : U \in Ob(\mathcal{C})\big\}$ of invertible natural transformations of functors where for each object $f : U \to X$ in $\underline{X}(U)$, there is an isomorphism

$$\Psi_U^{(\varphi)}(f) : F_U^{(A)}(f) \longrightarrow F_U^{(B)}(f),\ \Psi_U^{(\varphi)}(f) := \mathcal{Y}(f)(\varphi).$$

   Notice that $\mathcal{Y}(f) : \mathcal{Y}(X) \to \mathcal{Y}(U)$ is a functor of groupoids, and hence it maps φ to an isomorphism.

3. Now, we like to show that the compositions of η and Θ are identities up to 2-isomorphisms.

   (a) Let A be an object in $\mathcal{Y}(X)$. Then we have

$$\begin{aligned}
\big(\Theta \circ \eta\big)(A) &= \Theta\big(F^{(A)} : \underline{X} \to \mathcal{Y}\big) \\
&= F_X^{(A)}(id_X) \\
&= \mathcal{Y}(id_X)(A) \ \text{ by definition of } F_X^{(A)} \\
&= id_{\mathcal{Y}(X)}(A) = A \ \text{ since } \mathcal{Y}(id_X) \text{ is a functor of groupoids}
\end{aligned}$$

(b) Let $F : \underline{X} \to \mathcal{Y}$ be an object in $Mor_{PreStk_{\mathcal{C}}}(\underline{X}, \mathcal{Y})$. Then we get

$$\big(\eta \circ \Theta\big)\big(F\big) = \eta(F_X(id_X)) = F^{(F_X(id_X))} \tag{D.3.24}$$

where the 1-morphism $F^{(F_X(id_X))}$ is defined by, for all $U \in Ob(\mathcal{C})$,

$$F_U^{(F_X(id_X))} : \underline{X}(U) \to \mathcal{Y}(U),\ (U \xrightarrow{f} X) \longmapsto \mathcal{Y}(f)(F_X(id_X))$$

Claim. $\mathcal{Y}(f)(F_X(id_X)) \cong F_U(f)$.

Proof. It follows directly from the definition of a 2-morphism that for a morphism $U \xrightarrow{f} X$, there exists

$$F_f : F_U \circ \underline{X}(f) \Longrightarrow \mathcal{Y}(f) \circ F_X$$

such that the following diagram commutes up to a 2-isomorphism F_f:

$$\begin{array}{ccc} \underline{X}(X) & \xrightarrow[F_X]{} & \mathcal{Y}(X) \\ {\scriptstyle \underline{X}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{Y}(f)} \\ \underline{X}(U) & \xrightarrow[F_U]{} & \mathcal{Y}(U) \end{array} \tag{D.3.25}$$

where $\underline{X}(f) : \underline{X}(X) \to \underline{X}(U),\ g \mapsto g \circ f$. Therefore, we have

$$\big(\mathcal{Y}(f) \circ F_X\big)(id_X) \cong F_U(id_X \circ f) = F_U(f), \tag{D.3.26}$$

which proves the claim. □

Therefore, since the claim holds for all U, we conclude that

$$\big(\eta \circ \Theta\big)\big(F\big) = F \tag{D.3.27}$$

As a result, we get the desired equivalence of categories

$$\eta : \mathcal{Y}(X) \overset{\sim}{\leftrightarrows} Mor_{PreStk_{\mathcal{C}}}(\underline{X}, \mathcal{Y}) : \Theta. \tag{D.3.28}$$

□

Remark D.3.5. 2-categorical Yoneda's lemma implies that if $\mathcal{Y} : \mathcal{C}^{op} \longrightarrow Grpds$ is a moduli functor for some moduli problem, then *it is always representable by $\mathcal{Y}$ in the 2-category of (pre-)stacks.*

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CURRICULUM VITAE

PERSONAL INFORMATION

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Web page: http://users.metu.edu.tr/berktav/
Languages: Turkish, English (fluent) and French (intermediate)

EDUCATION

Degree	Institution	Year of Graduation
Ph.D. in Mathematics	METU	2021
B.S. in Mathematics with High Honor	METU	2013
Minor in Physics	METU	2013

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
2014 - 2021	METU	Research Assistant (including Teaching & Coordination)
2012 - 2014	METU	Student Assistant

RESEARCH INTERESTS

Differential/algebraic geometry, category theory, higher categorical structures in geometry and physics, moduli theory, and mathematical physics. For more details, please visit my web page.

PUBLICATIONS & PREPRINTS

1. K. İ. Berktav, *On Local Theory of Shifted Contact Structures on Derived Schemes (work in progress)*
2. K. İ. Berktav, *Stacky Formulations of Einstein Gravity (2019).* arXiv:1907.00665. (submitted)
3. K. İ. Berktav, *Notes on moduli theory, stacks and 2-Yoneda's Lemma (2020)*, available on my webpage
4. K. İ. Berktav, *A Note on Derived Geometric Interpretation of Classical Field Theories (2019)*. arXiv:1904.13331.
5. K. İ. Berktav, *An Introduction to Geometric Quantization and Witten's Quantum Invariant (2019).* arXiv:1902.10813.(Proceedings of the Workshop on Mathematical Topics in Quantization, Galatasaray University, 2018)

EVENTS AND ACTIVITIES

I. Talks given

Colloquium.

- Formal Moduli Problems and Theories of Physics,
  18th Workshop on Quantization, Dualities and Integrable Systems — Fall 2020
- Higher structures in physics, UCGEN Seminars — Fall 2020
- Symplectic structures on derived schemes,
  ODTU-Bilkent AG Seminars — Spring 2020
- Formal moduli problems and classical field theories,
  ODTU-Bilkent AG Seminars — Fall 2019
- Towards the Stacky Formulation of Einstein Gravity @ — Spring 2019
  1. Mathematical Physics Days 2019, Koç University, Istanbul, Turkey

2. Mathematical and Theoretical Physics Afternoons, Boğaziçi University, Istanbul, Turkey
3. ODTU-Bilkent AG Seminars, Ankara, Spring 2019

- Derived Geometric Interpretation of Classical Field Theories,
  METU, NCC. Spring 2019
- An Introduction to Geometric Quantization,
  Workshop at Galatasaray University, Istanbul, Turkey Fall 2018
- Derived Geometry and Physics,
  Department of Mathematics, METU. Spring 2017 - 2018

Graduate Seminars, Department of Mathematics,

- Stacky Formulations in Physics,
  Ohio State University Student Algebraic Geometry Seminar Fall 2020
- Towards the stacky formulations,
  METU Graduate Seminar Fall 2019
- Topological Aspects of Chern-Simons Gauge Theory Fall 2017 - 2018
- An Introduction to Classification of Four-Manifolds Fall 2016 - 2017
- Spacetime Singularities Spring 2015 - 2016
- Rapidly Expanding Universe Fall 2015 - 2016
- A Brief Survey of General Relativity Spring 2014 - 2015
- The Postulates of Quantum Mechanics Fall 2014 - 2015

Lecture Series

- Mini-lecture series on homotopy theory of stacks,
  METU Algebraic geometry student seminar. Fall 2019
- Lectures on Category-theoretical Formulations of Field Theories,
  IMBM, Boğaziçi University. Fall 2019

- Derived Geometry and Physics, Ankara-Istanbul AGNT - XXIV, Spring 2018
- Introduction to Floer's Instanton Homology,
  Department of Mathematics, METU Fall 2017 - 2018
- An Introduction to CFT, Department of Physics, METU Fall 2017 - 2018
- Geometric Quantization, Department of Physics, METU Spring 2016 - 2017

II. Some Events Participated - Visits

- Summer School: *New Perspectives in Gromov-Witten Theory*, Summer 2019
  IMJ-PRG, Sorbonne Université, Paris, France.
- Workshop: *Derived algebraic geometry and its applications*, Spring 2019
  Mathematical Sciences Research Institute (MSRI),
  Berkeley, CA., U.S.A.
- Séminaire de Mathématiques Supérieures (SMS) 2018:
  Derived Geometry and Higher Categorical Structures *Summer 2018*
  in Geometry and Physics,
  The Fields Institute, Toronto, Canada.
- Algebraic Geometry and Number Theory Meetings,
  Istanbul Center for Mathematical Sciences (IMBM), Turkey.
- Gökova Geometry/Topology Conferences, Turkey.

TEACHING EXPERIENCE

1. Calculus with Analytic Geometry (First-year course for engineers)
2. Calculus of Functions of Several Variables (First-year course for engineers)
3. Advanced Calculus 1 & 2 (Second-year courses for math students)
4. Complex Calculus (Third-year course for math students)
5. Introduction to Mathematical Analysis (Third-year course for math students)
6. Differential Geometry (Third-year course for math students)

HONORS AND SCHOLARSHIPS

- METU Graduate Performance Awards
  from METU Graduate School of Natural and Applied Sciences. 2014 - 2015
- Selected to be placed on *the High Honor Roll of METU* (7-times).
- Selected to be placed on *the Honor Roll of METU* (1-times).
- The Merit Scholarship from the Department of Mathematics, METU. 2011-2012

REFERENCES

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