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*Research Article*

# Bound-State Solutions of the Klein-Gordon Equation with $q$-Deformed Equal Scalar and Vector Eckart Potential Using a Newly Improved Approximation Scheme

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We present the analytical solutions of the Klein-Gordon equation for $q$-deformed equal vector and scalar Eckart potential for arbitrary $l$-state. We obtain the energy spectrum and the corresponding unnormalized wave function expressed in terms of the Jacobi polynomial. We also discussed the special cases of the potential.

## 1. Introduction

The study of exactly solvable potentials has attracted much attention since the early development of quantum mechanics. For example, the exact solutions of the Klein-Gordon equation for an hydrogen atom and for a harmonic oscillator in 3D represent two typical examples [1–3]. When a particle is in a strong potential field, the relativistic effect must be considered, leading to the relativistic quantum mechanical description of such particle [4–7]. In the relativistic limit, the particle motions are commonly described using either the Klein-Gordon or the Dirac equations [4, 6] depending on the spin character of the particle. The spin-zero particles like the mesons are described by the Klein-Gordon equation. On the other hand, the spin-half particles such as electrons are described satisfactorily by the Dirac equation. One of the interesting problems in nuclear and high energy physics is to obtain exact solution of the Klein-Gordon and the Dirac equation. In recent years, many studies have been carried out to explore the relativistic energy eigenvalues and the corresponding wave functions of the Klein-Gordon and the Dirac equation [8–11].

These relativistic equations contain two objects: the vector potential $V(r)$ and the scalar potential $S(r)$. The Klein-Gordon equation with the vector and scalar potentials can be written as follows:

$$\left[-\left(i\frac{\partial}{\partial t}-V(r)\right)^2-\nabla^2+(S(r)+M)^2\right]\psi(r,\theta,\varphi)=0, \tag{1.1}$$

where $M$ is the rest mass, $i(\partial/\partial t)=E$ is the energy eigenvalues, and $V(r)$ and $S(r)$ are the vector and scalar potentials, respectively.

Recently, some authors have assumed that the scalar potential is equal to the vector potential and obtained the bound state of the Klein-Gordon and the Dirac equations with some potentials of interest such as Woods-Saxon's potential [12], Hartman's potential [13], Coulomb-like potentials [14], ring-shape pseudoharmonic potential [15], Kratzer's potential [16, 17], and Pöschl-Teller and Rosen Morse potential [18]. Different methods such as the asymptotic iteration method (AIM) [19], supersymmetric quantum mechanics (SUSSY) [20],and Nikiforov-Uvarov (NU) method [12, 21] have been used to solve the differential equation arising from these considerations.

However, the analytical solutions of the Klein-Gordon equation are possible only in the $s$-wave case with the angular momentum $l=0$ for some well-known potential models [22, 23]. Conversely, when $l\neq 0$, one can only solve approximately the Klein-Gordon equation and the Dirac equation for some potentials using a suitable approximation scheme [24].

The purpose of this work is to solve approximately the arbitrary $l$-state Klein-Gordon equation with $q$-deformed equal scalar and vector Eckart potential. This paper is organized as follows. In Section 2, we present the review of the NU method and its parametric form. Section 3 is devoted to the factorization method for the Klein-Gordon equation. Solution to the radial equation is presented in Section 4. Discussion of the result is given in Section 5. Finally, a brief conclusion is presented in Section 6.

## 2. Review of the Nikiforov-Uvarov (NU) Method and Its Parametric Form

The NU method [25] is based on the solution of a generalized second-order linear differential equation into the equation of hypergeometric type. The Schrödinger equation

$$\psi''(x)+(E-V(x))\psi(x)=0 \tag{2.1}$$

can be solved by this method. This can be done by transforming this equation into equation of hypergeometric type with appropriate transformation, $s=s(x)$:

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0. \tag{2.2}$$

In order to find the exact solution to (2.2), we set the wave function as

$$\psi(s)=\phi(s)\chi(s), \tag{2.3}$$

and substituting (2.3) into (2.2) reduces (2.2) into hypergeometric-type equation:

$$\sigma(s)\chi''(s) + \tau(s)\chi'(s) + \lambda\chi(s) = 0, \tag{2.4}$$

where the wave function $\phi(s)$ is defined as the logarithmic derivative [25]:

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}, \tag{2.5}$$

where $\pi(s)$ is at most first-order polynomials.

Likewise, the hypergeometric type function $\chi(s)$ in (2.4) for a fixed $n$ is given by the Rodriques relation as

$$\chi_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right], \tag{2.6}$$

where $B_n$ is the normalization constant and the weight function $\rho(s)$ must satisfy the condition

$$\frac{d}{ds}\left(\sigma(s)\rho(s)\right) = \tau(s)\rho(s) \tag{2.7}$$

with

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s). \tag{2.8}$$

In order to accomplish the condition imposed on the weight function $\rho(s)$, it is necessary that the classical orthogonal polynomials $\tau(s)$ be equal to zero to some point of an interval $(a, b)$ and its derivative at this interval at $\sigma(s) > 0$ will be negative; that is,

$$\frac{d\tau(s)}{ds} < 0. \tag{2.9}$$

Therefore, the function $\pi(s)$ and the parameters $\lambda$ required for the NU method are defined as follows:

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}, \tag{2.10}$$

$$\lambda = k + \pi'(s). \tag{2.11}$$

The $k$-values in (2.10) are possible to evaluate if the expression under the square root must be square of polynomials. This is possible, if and only if its discriminant is zero. With this condition, the new eigenvalues' equation becomes

$$\lambda = \lambda_n = -\frac{nd\tau}{ds} - \frac{n(n-1)}{2}\frac{d^2\sigma}{ds^2}, \quad n = 0, 1, 2, \ldots. \tag{2.12}$$

On comparing (2.11) and (2.12), we obtain the energy eigenvalues.

The parametric generalization of the NU method is given by the generalized hypergeometric-type equation as [26]

$$\psi''(s) + \frac{(\alpha_1 - \alpha_2 s)}{s(1-\alpha_3 s)}\psi'(s) + \frac{1}{s^2(1-\alpha_3 s)^2}\left[-\xi_1 s^2 + \xi_2 s - \xi_3\right]\psi(s) = 0. \tag{2.13}$$

Comparing (2.13) with (2.2), the following polynomials are obtained:

$$\widetilde{\tau}(s) = (\alpha_1 - \alpha_2 s), \qquad \sigma(s) = s(1-\alpha_3 s), \qquad \widetilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3. \tag{2.14}$$

Now substituting (2.14) into (2.10), we find

$$\pi(s) = \alpha_4 + \alpha_5 s \pm \left[(\alpha_6 - \alpha_3 k_\pm)s^2 + (\alpha_7 + k_\pm)s + \alpha_8\right]^{1/2}, \tag{2.15}$$

where

$$\begin{gathered} \alpha_4 = \frac{1}{2}(1-\alpha_1), \qquad \alpha_5 = \frac{1}{2}(\alpha_2 - 2\alpha_3), \qquad \alpha_6 = \alpha_5^2 + \xi_1, \\ \alpha_7 = 2\alpha_4\alpha_5 - \xi_2, \qquad \alpha_8 = \alpha_4^2 + \xi_3. \end{gathered} \tag{2.16}$$

The resulting value of $k$ in (2.15) is obtained from the condition that the function under the square root must be square of a polynomials, and it yields

$$k_\pm = -(\alpha_7 + 2\alpha_3\alpha_8) \pm 2\sqrt{\alpha_8\alpha_9}, \tag{2.17}$$

where

$$\alpha_9 = \alpha_3\alpha_7 + \alpha_2^2\alpha_8 + \alpha_6. \tag{2.18}$$

The new $\pi(s)$ for each $k$ becomes

$$\pi(s) = \alpha_4 + \alpha_5 s - \left[\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right)s - \sqrt{\alpha_8}\right], \tag{2.19}$$

for the $k_-$ value

$$k_- = -(\ \alpha_7 + 2\alpha_3\alpha_8) - 2\sqrt{\alpha_8\alpha_9}. \tag{2.20}$$

Using (2.8), we obtain

$$\tau(s) = \alpha_1 + 2\alpha_4 - (\alpha_2 - 2\alpha_5)s - 2\left[\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right)s - \sqrt{\alpha_8}\right]. \tag{2.21}$$

The physical condition for the bound-state solution is $\tau' < 0$, and thus

$$\tau'(s) = -2\alpha_3 - 2\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right) < 0. \tag{2.22}$$

With the aid of (2.11) and (2.12), we derive the energy equation as

$$(\alpha_2 - \alpha_3)n + \alpha_3 n^2 - (2n+1)\alpha_5 + (2n+1)\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right) + \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0. \tag{2.23}$$

The weight function $\rho(s)$ is obtained from (2.7) as

$$\rho(s) = s^{\alpha_{10}-1}(1-\alpha_3 s)^{(\alpha_{11}/\alpha_3)-\alpha_{10}-1}, \tag{2.24}$$

and together with (2.6), we have

$$\chi_n(s) = P_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-2\alpha_3 s), \tag{2.25}$$

where

$$\begin{aligned} \alpha_{10} &= \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}, \\ \alpha_{11} &= \alpha_2 - 2\alpha_5 + 2\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right), \end{aligned} \tag{2.26}$$

and $P_n^{(\alpha,\beta)}(s)$ are the Jacobi polynomials. The second part of the wave function is obtained from (2.5) as

$$\phi(s) = s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-(\alpha_{13}/\alpha_3)}, \tag{2.27}$$

where

$$\alpha_{12} = \alpha_4 + \sqrt{\alpha_8}, \qquad \alpha_{13} = \alpha_5 - \left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right). \tag{2.28}$$

Thus, the total wave function becomes

$$\psi(s) = N_{nl} s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-(\alpha_{13}/\alpha_3)} P_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-2\alpha_3 s), \tag{2.29}$$

whose $N_{nl}$ is the normalization constant.

## 3. Factorization Method for the Klein-Gordon Equation

The three-dimensional Klein-Gordon equation with mixed vector and scalar potentials can be written as

$$\left[\nabla^2 + (V(r)-E)^2 - (S(r)+M)^2\right]\psi(r,\theta,\varphi) = 0, \tag{3.1}$$

where $M$ is the rest mass, $E$ is the relativistic energy, and $S(r)$ and $V(r)$ are the scalar and vector potentials, respectively. $\nabla^2$ is the Laplace operator, $c$ is the speed of light, and $\hbar$ is the reduced Planck's constant which have been set to unity. In spherical coordinates, the Klein-Gordon equation for a particle in the present of Eckart potential $V(r)$ becomes

$$\begin{aligned}&\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}-2(EV(r)+MS(r))\right.\\&\left.\quad+V^2(r)-S^2(r)+E^2-M^2\right]\psi(r,\theta,\varphi)=0.\end{aligned}\tag{3.2}$$

If one assigns the corresponding spherical total wave function as

$$\psi(r,\theta,\varphi)=\frac{R(r)}{r}Y_{lm}(\theta,\varphi),\tag{3.3}$$

where

$$Y_{lm}(\theta,\varphi)=\Theta(\theta)\Phi(\varphi),\tag{3.4}$$

then the wave equation in (3.2) is separated into variables and the following equations are obtained:

$$\frac{d^2R(r)}{dr^2}+\left[E^2-M^2-2(EV(r)+MS(r))+V^2(r)-S^2(r)-\frac{\lambda}{r^2}\right]R(r)=0,\tag{3.5}$$

$$\begin{aligned}&\frac{d^2\Theta(\theta)}{d\theta^2}+\cot\theta\frac{d\Theta(\theta)}{d\theta}\left(\lambda-\frac{m^2}{\sin^2\theta}\right)\Theta(\theta)=0,\\&\frac{d^2\Phi(\varphi)}{d\varphi^2}+m^2\Phi(\varphi)=0,\end{aligned}\tag{3.6}$$

where $m^2$ and $\lambda=l(l+1)$ are the separation constants.

Equation (3.6) are spherical harmonic functions whose solutions are well known [27].

## 4. Solution of the Radial Equation

The $q$-deformed Eckart potential is defined from [23, 28, 29] as

$$V(r)=-\frac{V_1e^{-r/b}}{(1-qe^{-r/b})}+\frac{V_2e^{-r/b}}{(1-qe^{-r/b})^2},\tag{4.1}$$

where $V_1$, $V_2$ are the potential depth, $q$ is the deformation parameter, $b = 1/2\alpha$ is the parameter, and $\alpha$ is the range of the potential. The radial part of the Klein-Gordon equation in (3.5) for special case $V(r) = S(r)$ is written as

$$\frac{d^2R(r)}{dr^2} + \left[E^2 - M^2 - 2(E+M)V(r) - \frac{\lambda}{r^2}\right]R(r) = 0. \tag{4.2}$$

Substituting (4.1) into (4.2), we obtain

$$\frac{d^2R(r)}{dr^2} + \left[E^2 - M^2 + 2(E+M)\frac{V_1 e^{-r/b}}{\left(1-qe^{-r/b}\right)} - 2(E+M)\frac{V_2 e^{-r/b}}{\left(1-qe^{-r/b}\right)^2} - \frac{l(l+1)}{r^2}\right]R(r) = 0. \tag{4.3}$$

Obviously, this equation cannot be solved analytically for $l \neq 0$ due to the centrifugal term. Therefore, (4.3) can be evaluated by using a newly improved approximation scheme [30]:

$$\frac{1}{r^2} \approx \frac{1}{b^2}\left(C_0 + \frac{C_1 e^{-r/b}}{\left(1-qe^{-r/b}\right)} + \frac{C_2 e^{-2r/b}}{\left(1-qe^{-r/b}\right)^2}\right), \tag{4.4}$$

where $C_0$, $C_1$, and $C_2$ are three adjustable parameters.

Substituting (4.4) into (4.3), we obtain

$$\begin{aligned}\frac{d^2R(r)}{dr^2} + \Bigg[E^2 - M^2 + 2(E+M)\frac{V_1 e^{-r/b}}{\left(1-qe^{-r/b}\right)} - 2(E+M)\frac{V_2 e^{-r/b}}{\left(1-qe^{-r/b}\right)^2}&\\ - \frac{l(l+1)}{b^2}\left(C_0 + \frac{C_1 e^{-r/b}}{\left(1-qe^{-r/b}\right)} + \frac{C_2 e^{-2r/b}}{\left(1-qe^{-r/b}\right)^2}\right)\Bigg]R(r) = 0.&\end{aligned} \tag{4.5}$$

Using a new variable $s = e^{-r/b}$ and substituting in (4.5), we have the following hypergeometric equation:

$$\frac{d^2R(r)}{ds^2} + \frac{(1-qs)}{s(1-qs)}\frac{dR(s)}{ds} + \frac{1}{s^2(1-qs)^2}\left[\left(\varepsilon^2 q^2 + A\right)s^2 + \left(-2\varepsilon^2 q + B\right)s + \left(\varepsilon^2 + C\right)\right]R(s) = 0, \tag{4.6}$$

where

$$\begin{gathered}\varepsilon^2 = b^2\left(E^2 - M^2\right),\\ A = -2(E+M)b^2V_1 q - l(l+1)C_0 q^2 + l(l+1)C_1 q - l(l+1)C_2,\\ B = 2(E+M)b^2V_1 - 2b^2(E+M)V_2 + 2l(l+1)C_0 q - l(l+1)C_1,\\ C = -l(l+1)C_0.\end{gathered} \tag{4.7}$$

Comparing (4.6) with (2.13), we obtain the parameter set

$$
\begin{gathered}
\xi_1 = -\varepsilon^2 q^2 - A, \\
\xi_2 = -2\varepsilon^2 q + B, \\
\xi_3 = -\varepsilon^2 - C, \\
\alpha_1 = 1, \qquad \alpha_2 = \alpha_3 = q, \qquad \alpha_4 = 0, \\
\alpha_5 = \frac{-q}{2}, \qquad \alpha_6 = \frac{q^2}{4} - q^2\varepsilon^2 - A, \\
\alpha_7 = 2\varepsilon^2 q - B, \qquad \alpha_8 = -\left(\varepsilon^2 + C\right), \\
\alpha_9 = \frac{q^2}{4} - A - qB - q^2 C, \\
\alpha_{10} = 1 + 2i\sqrt{(\varepsilon^2 + C)}, \\
\alpha_{11} = 2q + 2\left(\sqrt{\frac{q^2}{4} - A - qB - q^2 C} + iq\sqrt{(\varepsilon^2 + C)}\right), \\
\alpha_{12} = i\sqrt{(\varepsilon^2 + C)}, \\
\alpha_{13} = -\frac{q}{2} - \left(\sqrt{\frac{q^2}{4} - A - qB - q^2 C} + iq\sqrt{(\varepsilon^2 + C)}\right).
\end{gathered}
\tag{4.8}
$$

Substituting (4.8) into (2.15), we obtain the polynomials $\pi(s)$ as

$$
\pi(s) = \frac{-qs}{2} \pm \left[\left(\frac{q^2}{4} - \varepsilon^2 q^2 - A - qk_{\pm}\right)s^2 + \left(2\varepsilon^2 q - B + k_{\pm}\right)s - \left(\varepsilon^2 + C\right)\right]^{1/2}. \tag{4.9}
$$

Substituting (4.8) into (2.17), we obtain $k_{\pm}$ as

$$
k_{\pm} = B + 2qC \pm 2\sqrt{(\varepsilon^2 + C)\left(A + qB + q^2\left(C - \frac{1}{4}\right)\right)}. \tag{4.10}
$$

Using (2.19), (2.20), and (4.8), we can obtain $\pi(s)$ and $k_-$ suitable for the NU method as

$$
\begin{gathered}
\pi(s) = \frac{-qs}{2} - \left[\left(\sqrt{\frac{q^2}{4} - A - qB - q^2 C} + iq\sqrt{(\varepsilon^2 + C)}\right)s - i\sqrt{(\varepsilon^2 + C)}\right], \\
k_- = B + 2qC - 2\sqrt{(\varepsilon^2 + C)\left(A + qB + q^2\left(C - \frac{1}{4}\right)\right)}.
\end{gathered}
\tag{4.11}
$$

Substituting (4.8) into (2.22), we obtain

$$\tau'(s) = -2q - 2\left(\sqrt{\frac{q^2}{4} - A - qB - q^2C} + iq\sqrt{(\varepsilon^2 + C)}\right) < 0, \tag{4.12}$$

which is the essential condition for bound-state (real) solution.

Substituting (4.8) into (2.23), we obtain the energy eigenvalues of the $q$-deformed Eckart potential as

$$E^2 - M^2 = \frac{-1}{4b^2}\left[\frac{\gamma + \left(\left(n + \frac{1}{2}\right) + \beta\right)^2}{\left(\left(n + \frac{1}{2}\right) + \beta\right)}\right]^2 + \frac{l(l+1)C_0}{b^2}, \tag{4.13}$$

where $\gamma = (-2(E + M)b^2V_1/q) + (l(l + 1)C_1/q) - (l(l + 1)C_2/q^2)$ and $\beta = \sqrt{(1/4) + (l(l+1)C_2/q^2) + (2b^2(E + M)V_2/q)}$.

The weight function $\rho(s)$ in (2.24) is obtained as

$$\rho(s) = s^{\mu}(1 - qs)^{\vartheta}, \tag{4.14}$$

where $\mu = 2i\sqrt{(\varepsilon^2 + C)}$ and $\vartheta = 2\sqrt{(1/4) - (A/q^2) - (B/q) - C}$ which gives the first part of the wave function in (2.3) using (2.25) as

$$\chi_n(s) = P_n^{(\mu,\vartheta)}(1 - 2qs). \tag{4.15}$$

The other wave function $\phi(s)$ is obtained from (2.27) as

$$\phi(s) = s^{\mu/2}(1 - qs)^{(1+\vartheta)/2}. \tag{4.16}$$

The radial wave function $R(r)$ expressed in terms of the Jacobi polynomials is obtained from (2.29):

$$R(r) = N_{nl}\left(e^{-r/b}\right)^{\mu/2}\left(1 - qe^{-r/b}\right)^{(1+\vartheta)/2}P_n^{(\mu,\vartheta)}\left(1 - 2qe^{-r/b}\right), \tag{4.17}$$

where $N_{nl}$ is the normalization constant.

Hence, the total wave function $\psi(r,\theta,\varphi)$ for the $q$-deformed Eckart potential is obtained using (3.3) as

$$\psi(r,\theta,\varphi) = N_{nl}\frac{1}{r}\left(e^{-r/b}\right)^{\mu/2}\left(1 - qe^{-r/b}\right)^{(1+\vartheta)/2}P_n^{(\mu,\vartheta)}\left(1 - 2qe^{-r/b}\right)Y_{lm}(\theta,\varphi). \tag{4.18}$$

## 5. Discussion

By setting some potential parameters into (4.1), we obtain some well-known potentials.

### 5.1. Hulthen's Potential

If we set $V_2 = 0$, $V_1 = V_0$, and $q = 1$ in (4.1), we obtain the Hulthen potential [31]

$$V(r) = \frac{-V_0 e^{-r/b}}{1 - e^{-r/b}}. \tag{5.1}$$

Substituting these parameters into (4.13) and (4.18) we obtain the energy eigenvalues and the corresponding wave function as

$$E^2 - M^2 = \frac{-1}{4b^2}\left[\frac{\tilde{\gamma} + \left((n + (1/2)) + \sqrt{(1/4) + l(l+1)c_2}\right)^2}{\left((n + (1/2)) + \sqrt{(1/4) + l(l+1)c_2}\right)}\right]^2 + \frac{l(l+1)C_0}{b^2}, \tag{5.2}$$

where $\tilde{\gamma} = -2(E + M)b^2 V_0 + l(l+1)c_1 - l(l+1)c_2$ and

$$\psi(r) = N_{nl}\frac{1}{r}\left(e^{-r/b}\right)^{\mu/2}\left(1 - e^{-r/b}\right)^{(1+\tilde{\vartheta})/2} P_n^{(\mu,\tilde{\vartheta})}\left(1 - 2e^{-r/b}\right) Y_{lm}(\theta, \varphi), \tag{5.3}$$

where

$$\begin{gathered}
\tilde{\vartheta} = 2\sqrt{\frac{1}{4} - \tilde{A} - \tilde{B} - C}, \\
\tilde{A} = -2(E + M)b^2 V_0 - l(l+1)c_0 + l(l+1)c_1 - l(l+1)c_2, \\
\tilde{B} = 2(E + M)b^2 V_0 + 2l(l+1)c_0 - l(l+1)c_1.
\end{gathered} \tag{5.4}$$

### 5.2. Modified Pöschl-Teller Potential

Setting $V_1 = 0$, $q = -1$, and $V_2 = -4V_0$ in (4.1), we obtain the modified Poschl-Teller potential of the form [32–35]

$$V(r) = \frac{-V_0}{\cos h^2(r/2b)}. \tag{5.5}$$

Substituting these parameters into (4.13) and (4.18), we obtain the energy spectrum and the corresponding eigen function as

$$E^2 - M^2 = \frac{-1}{4b^2}\left[\frac{-l(l+1)(c_1+c_2) + \left((n+(1/2)) + \sqrt{(1/4)+l(l+1)c_2+8b^2(E+M)V_0}\right)^2}{\left((n+(1/2)) + \sqrt{(1/4)+l(l+1)c_2+8b^2(E+M)V_0}\right)}\right]^2 + \frac{l(l+1)C_0}{b^2},$$
$$\psi(r) = N_{nl}\frac{1}{r}\left(e^{-r/b}\right)^{\mu/2}\left(1+e^{-r/b}\right)^{(1+\vartheta^1)/2}P_n^{(\mu,\vartheta^1)}\left(1+2e^{-r/b}\right)Y_{lm}(\theta,\varphi), \tag{5.6}$$

where

$$\vartheta^1 = 2\sqrt{\frac{1}{4} - A^1 + B^1 - C},$$
$$A^1 = -l(l+1)(c_0+c_1+c_2), \tag{5.7}$$
$$B^1 = 8b^2(E+M)V_0 - l(l+1)(2c_0+c_1),$$

respectively.

### 5.3. Morse's Potential

If we set $q = 0$, $V_1 = 0$ and $V_2 = V_0$ into (4.1), we obtain the Morse potential of the form [36]

$$V(r) = V_0 e^{-r/b}. \tag{5.8}$$

Substituting these parameters into (4.13) and (4.18), we obtain energy eigenvalues and wave function as

$$E^2 - M^2 = \frac{-1}{4b^2}(n+1)^2 + \frac{l(l+1)C_0}{b^2},$$
$$\psi(r) = N_{nl}\frac{1}{r}\left(e^{-r/b}\right)^{\mu/2}P_n^{(\mu,\epsilon)}Y_{lm}(\theta,\varphi), \tag{5.9}$$

where

$$\epsilon = 2\sqrt{\frac{1}{4} - C}, \tag{5.10}$$

respectively.

## 6. Conclusion

In this paper, we have studied the Klein-Gordon equation subject to equal $q$-deformed scalar and vector Eckart potentials. The energy and wave functions for bound states have been obtained by parametric form of the Nikiforov-Uvarov method. We also discussed some special cases of the potential.

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