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Electromagnetic finite-size effects beyond the point-like approximation

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Abstract. We present a model-independent and relativistic approach to analytically derive electromagnetic finite-size effects beyond the point-like approximation. The key element is the use of electromagnetic Ward identities to constrain vertex functions, and structure-dependence appears via physical form-factors and their derivatives. We apply our general method to study the leading finite-size structure-dependence in the pseudoscalar mass (at order $1/L^3$) as well as in the leptonic decay amplitudes of pions and kaons (at order $1/L^2$). Knowledge of the latter is essential for Standard Model precision tests in the flavour physics sector from lattice simulations.

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1 Introduction

Lattice quantum chromodynamics (QCD) allows for systematically improvable Standard Model (SM) precision tests from numerical simulations performed in a finite-volume (FV), discretised Euclidean spacetime. In order to reach (sub-)percent precision in lattice predictions, also strong and electromagnetic isospin breaking corrections have to be included. The latter are encoded via quantum electrodynamics (QED), but the inclusion of QED in a FV spacetime is complicated because of Gauss' law [1]. This problem is related to zero-momentum modes of photons and the absence of a QED mass-gap. Several prescriptions of how to include QED in a finite volume have been formulated and the one used here is $\mathrm{QED_L}$ where the spatial zero-modes are removed on each time-slice. The long-range nature of QED in addition enhances the FV effects (FVEs), which typically leads to power-law FVEs that are larger than the exponentially suppressed ones for single-particle matrix elements in QCD alone.

The FVEs for a QCD+QED process depend on properties of the involved particles, including masses and charges, but also structure-dependent quantities such as electromagnetic form-factors and their derivatives. In order to analytically capture the finite-volume scaling fully, one cannot neglect hadron structure, and in the following we develop a relativistic and model-independent method to go beyond the point-like approximation at order e^2 in $\mathrm{QED_L}$.

We consider a space-time with periodic spatial extents L but with infinite time-extent. To exemplify the method, we first consider the pseudoscalar mass in Sec. 2, and then proceed to leptonic decays in Sec. 3. The discussion is based on the results in Ref. [2], and the reader is referred there for further technical details.

2 Pseudoscalar Mass

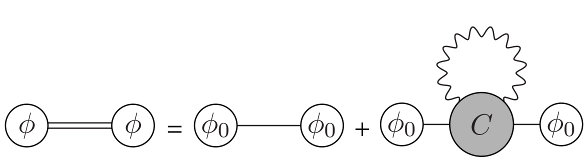
To study the finite-size scaling in the mass $m_P(L)$ of a charged hadronic spin-0 particle P, we first define the full QCD+QED infinite-volume two-point Euclidean correlation function

$$C_2^\infty(p) = \int d^4x\, \langle 0|\, \mathrm{T}[\phi(x)\phi^\dagger(0)]\, |0\rangle\, e^{-ipx}\,. \tag{1}$$

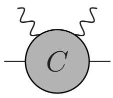
Here ϕ is an interpolating operator coupling to P, and $p = (p_0, \mathbf{p})$ is the momentum. We denote the finite-volume counterpart of this correlator $C_2^L(p)$, but for the moment only consider $C_2^\infty(p)$. This can be diagrammatically represented as

$C_2^\infty(p) =$  $= Z_P \cdot D(p) \cdot Z_P\,, \qquad D(p) = \dfrac{Z(p^2)}{p^2 + m_P^2}\,, \qquad Z_P = \langle 0|\, \phi(0)\, |P, \mathbf{p}\rangle\,,$ (2)

where the double-line represents the QCD+QED propagator $D(p)$, the ϕ-blob is the overlap between ϕ and P and $Z(p^2) = 1 + \mathcal{O}(p^2 + m_P^2)$ is the residue of the propagator. Expanding $C_2^\infty(p)$ in (2) around $e = 0$ yields

 $+\, \mathcal{O}(e^4)\,,$ (3)

where quantities with subscript 0 are evaluated in QCD alone. The grey blob is the Compton scattering kernel defined via

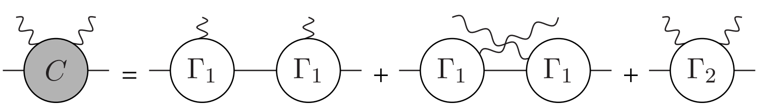


$$= C_{\mu\nu}(p,k,q) = \int d^4x\, d^4y\, d^4z\, e^{ipz+ikx+iqy} \frac{\langle 0|\,\mathrm{T}[\phi(0)J_\mu(x)J_\nu(y)\phi^\dagger(z)]\,|0\rangle}{Z_{P,0}^2\, D_0(p)D_0(p+k+q)}\,. \tag{4}$$

Here k and q are incoming photon momenta and $J_\mu(x)$ is the electromagnetic current. Note that the unphysical dependence on the arbitrary interpolating operator ϕ must cancel for any physical quantity, and when the external legs in $C_{\mu\nu}(p,k,q)$ go on-shell the kernel is nothing but the physical forward Compton scattering amplitude. Using (3) the electromagnetic mass-shift of the meson is readily obtained in terms of an integral over the photon loop-momentum k. One may follow an equivalent procedure for the finite-volume correlation function $C_2^L(p)$, where the integral over the spatial momentum $\mathbf{k}$ is replaced by a sum. The leading electromagnetic FVEs in the mass, $\Delta m_P^2(L)$, are thus given by the sum-integral difference

$$\Delta m_P^2(L) = -\frac{e^2}{2} \lim_{p_0^2\to -m_P^2} \left(\frac{1}{L^3}\sum_{\mathbf{k}}{}' - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{dk_0}{2\pi}\, \frac{C_{\mu\mu}(p,k,-k)}{k^2}\bigg|_{\mathbf{p}=0}\,, \tag{5}$$

where the rest-frame $\mathbf{p} = 0$ was chosen for convenience and the primed sum indicates the omission of the photon zero-mode $\mathbf{k} = 0$ in $\mathrm{QED_L}$. The analytical dependence on $1/L$ including structure-dependence can now be obtained from this formula through a soft-photon expansion of the integrand, i.e. an expansion order by order in $|\mathbf{k}|$ which is directly related to the expansion in $1/L$ via $|\mathbf{k}| = 2\pi|\mathbf{n}|/L$ where $\mathbf{n}$ is a vector of integers. The first step is to decompose $C_{\mu\nu}(p,k,q)$ into two irreducible electromagnetic vertex functions Γ_1 and Γ_2 according to



$$= \quad + \quad + \quad . \tag{6}$$

The vertex functions depend in general on the structure of the particle, as can be seen from e.g. the form-factor decomposition

$$\Gamma_1 = \Gamma_\mu(p,k) = (2p+k)_\mu\, F(k^2,(p+k)^2,p^2) + k_\mu\, G(k^2,(p+k)^2,p^2)\,, \tag{7}$$

where $F(k^2,(p+k)^2,p^2)$ and $G(k^2,(p+k)^2,p^2)$ are structure-dependent electromagnetic form-factors depending on three virtualities. This means that F and G contain off-shell effects, but we stress that these non-physical quantities always cancel in the FVEs. The cancellation occurs since the vertex functions $\Gamma_{1,2}$ are related to each other and the propagator $D_0(p)$ via Ward identities. An example of an off-shell relation is $F(0,p^2,-m_{P,0}^2) = Z_0(p^2)^{-1}$. The derivatives of $Z_0(p^2)$ are already known in the literature as $\delta D^{(n)}(0)$ [3] and z_n [4], but these could in principle be set to zero as they always cancel in the final results. The Ward identities further yield G as a function of F. The form-factor F also contains physical information, and for our purposes it suffices to know that $F^{(1,0,0)}(0,-m_{P,0}^2,-m_{P,0}^2) = F'(0) = -\langle r_P^2\rangle/6$, where $\langle r_P^2\rangle$ is the physical electromagnetic charge radius of P which is well-known experimentally [5].

Using our definitions of the vertex functions in $C_{\mu\nu}(p,k,q)$ in (5) we obtain the FVEs

$$\Delta m_P^2(L) = e^2 m_P^2 \left\{ \frac{c_2}{4\pi^2 m_P L} + \frac{c_1}{2\pi(m_P L)^2} + \frac{\langle r_P^2\rangle}{3m_P L^3} + \frac{C}{(m_P L)^3} + O\left[\frac{1}{(m_P L)^4}\right] \right\}\,, \tag{8}$$

where the c_j are finite-volume coefficients specific to $\mathrm{QED_L}$ arising from the sum-integral difference in (5). These are discussed in detail in Ref. [2]. Here we see the charge radius $\langle r_P^2 \rangle$ appearing at order $1/L^3$ and its coefficient agrees with that derived within non-relativistic scalar QED [6]. However, there is an additional structure-dependent term C related to the branch-cut of the forward, on-shell Compton amplitude. This contribution can be found, in other forms, also in Refs. [3, 7], and only arises because of the $\mathrm{QED_L}$ prescription with the subtracted zero-mode. Its value is currently unknown but one can show $C > 0$ [2], meaning that it cannot cancel the charge radius contribution. Note that all unphysical off-shell contributions from the form-factors F and G have vanished.

3 Leptonic Decays

Leptonic decay rates of light mesons are of the form $P^- \to \ell^- \bar{\nu}_\ell$, where P is a pion or kaon, ℓ a lepton and ν_ℓ the corresponding neutrino. These are important for the extraction of the Cabibbo-Kobayashi-Maskawa matrix elements $|V_{us}|$ and $|V_{ud}|$ [8, 9]. The leading virtual electromagnetic correction to this process yields an infrared (IR) divergent decay rate Γ_0. One must therefore add the real radiative decay rate $\Gamma_1(\Delta E)$ for $P^- \to \ell^- \bar{\nu}_\ell \gamma$, where the photon has energy below ΔE, to cancel the IR-divergence in Γ_0. The IR-finite inclusive decay rate is thus $\Gamma(P^- \to \ell^- \nu_\ell[\gamma])$, and following the lattice procedure first laid out in Ref. [8] we may write

$$\Gamma_0 + \Gamma_1(\Delta E_\gamma) = \lim_{L\to\infty} [\Gamma_0(L) - \Gamma_0^{\mathrm{uni}}(L)] + \lim_{L\to\infty} [\Gamma_0^{\mathrm{uni}}(L) + \Gamma_1(L, \Delta E_\gamma)] \,. \tag{9}$$

Here, Ref. [8] chose to add and subtract the universal finite-volume decay rate $\Gamma^{\mathrm{uni}}(L)$, calculated in point-like scalar QED in Ref. [4], to cancel separately the IR-divergences in Γ_0 and Γ_1. In the following we are interested in only the first term in brackets. The subtracted term $\Gamma_0^{\mathrm{uni}}(L)$ cancels the FVEs in $\Gamma_0(L)$ through order $1/L$, and hence $\Gamma_0(L) - \Gamma_0^{\mathrm{uni}}(L) \sim O(1/L^2)$. Structure-dependence enters at order $1/L^2$. With the goal of systematically improving the finite-volume scaling order by order including structure-dependence, we replace the universal contribution by

$$\Gamma_0^{\mathrm{uni}}(L) \longrightarrow \Gamma_0^{(n)}(L) = \Gamma_0^{\mathrm{uni}}(L) + \sum_{j=2}^{n} \Delta\Gamma_0^{(j)}(L) \,, \tag{10}$$

where $n \geq 2$ and $\Delta\Gamma_0^{(j)}(L)$ contains the FVEs at order $1/L^j$. This means that the finite-volume residual instead scales as $\Gamma_0(L) - \Gamma_0^{(n)}(L) \sim O(1/L^{n+1})$. We may parametrise $\Gamma_0^{(n)}(L)$ in terms of a finite-volume function $Y^{(n)}(L)$ according to

$$\Gamma_0^{(n)}(L) = \Gamma_0^{\mathrm{tree}} \left[1 + 2\frac{\alpha}{4\pi} Y^{(n)}(L)\right] + O\left(\frac{1}{L^{n+1}}\right) , \tag{11}$$

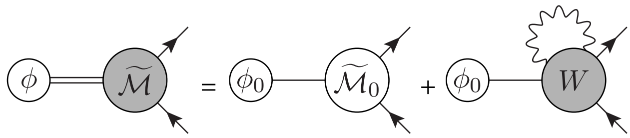
where Γ_0^{tree} is the tree-level decay rate.

Since we are interested in the leading structure-dependent contribution we consider $Y^{(2)}(L)$. In order to derive it, we define the QCD+QED correlation function

$$C_W^{rs}(p, p_\ell) = \int d^4z\, e^{ipz}\, \langle \ell^-, \mathbf{p}_\ell, r; \nu_\ell, \mathbf{p}_{\nu_\ell}, s|\, \mathrm{T}[O_W(0)\phi^\dagger(z)]\, |0\rangle \ , \tag{12}$$

where $p_\ell = (p_\ell^0, \mathbf{p}_\ell)$ is the momentum of the on-shell lepton of mass m_ℓ, $p_{\nu_\ell} = (p_{\nu_\ell}^0, \mathbf{p}_{\nu_\ell})$ is the momentum of the massless neutrino and $O_W(0)$ is the four-fermion operator of the decay in question. We may diagrammatically represent this in a similar way as for the mass according to

$C_W^{rs}(p, p_\ell) =$



. (13)

The grey blob containing W is of order e^2 and can be separated, just like the Compton amplitude, into several irreducible vertex functions. The exact definitions of these vertex functions are quite involved and can be found in Ref. [2], but several comments can be made. First of all, the vertex functions are related to various structure-dependent form-factors containing both on-shell and off-shell information. Again, the off-shellness must cancel. The vertex functions also contain physical structure-dependent information (similar to how Γ_1 depends on the charge radius) and for $Y^{(2)}(L)$ this is the axial-vector form-factor $F_A(-m_P^2) = F_A^P$ from the real radiative decay $P^- \to \ell^- \bar{\nu}_\ell \gamma$.

By performing the amputation on the external meson leg in (12) to obtain the matrix element needed for the decay rate in (11), one finds the finite-volume function $Y^{(2)}(L)$ to be

$$\begin{aligned} Y^{(2)}(L) \;=\; & \frac{3}{4} + 4\,\log\left(\frac{m_\ell}{m_W}\right) + 2\,\log\left(\frac{m_W L}{4\pi}\right) + \frac{c_3 - 2\left(c_3(\mathbf{v}_\ell) - B_1(\mathbf{v}_\ell)\right)}{2\pi} - \\ & - 2\,A_1(\mathbf{v}_\ell)\left[\log\left(\frac{m_P L}{2\pi}\right) + \log\left(\frac{m_\ell L}{2\pi}\right) - 1\right] - \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\,c_2 - 4\,r_\ell^2\,c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right] + \\ & + \frac{1}{(m_P L)^2}\left[-\frac{F_A^P}{f_P}\,\frac{4\pi\,m_P\left[(1+r_\ell^2)^2\,c_1 - 4\,r_\ell^2\,c_1(\mathbf{v}_\ell)\right]}{1-r_\ell^4} + \frac{8\pi\left[(1+r_\ell^2)\,c_1 - 2\,c_1(\mathbf{v}_\ell)\right]}{(1-r_\ell^4)}\right]. \end{aligned} \tag{14}$$

Here, $r_\ell = m_\ell/m_P$, $\mathbf{v}_\ell = \mathbf{p}_\ell/E_\ell$ the lepton velocity in terms of the energy E_ℓ, and m_W the W-boson mass. Also, c_k, $A_1(\mathbf{v}_\ell)$, $B_1(\mathbf{v}_\ell)$ and $c_j(\mathbf{v}_\ell)$ are finite-volume coefficients defined in Ref. [2]. Note that no unphysical quantities appear. At order $1/L^2$, there is one structure-dependent contribution proportional to F_A^P and the other term is purely point-like. This result is in perfect agreement with Ref. [4] for the universal terms up to $O(1/L)$, which we derived in a completely different approach. The numerical impact of the $1/L^2$-corrections is studied in Ref. [2].

4 Conclusions

We have presented a relativistic and model-independent method to derive electromagnetic FVEs beyond the point-like approximation. We are currently working to obtain the leading FVEs for semi-leptonic kaon decays, relevant for future precision tests in the SM flavour physics sector.

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