Emergence of Space-Times from Gauge Theories in Gauge/Gravity Duality

Yuhma Asano

Department of Physics, Kyoto University

Ph. D. Thesis

January 2015

Abstract

Although gauge/gravity dualities have been getting understood with great effort, how the geometries emerge from and embed in gauge theories is still unclear. I show it in the case of the gauge/gravity duality between BMN model and the type IIA supergravity by investigating the exact partition function of the gauge theory. It is revealed that the saddle point equations on the gauge theory side are equivalent to the equations determining the dual geometry on the gravity side, and the ranges of eigenvalue distributions on the gauge theory side actually relates to the typical geometric scales. This thesis is based on [1, 2, 3].

Contents

1 Introduction

The discovery of a Higgs boson and the result of cross sections related to the Higgs bosons show that the standard model is almost completely correct below the energy scale that we can reach so far. However, the standard model does not include one of the fundamental interactions, the gravitation. Although the gravitation is negligible in experiments for particle physics due to the smallness of the coupling constant, it is not sufficient to describe nature without it. The problem is the non-renormalizable divergence that appears in gravitational theories. Therefore we need a new framework beyond ordinary quantum field theories, and the most successful one is superstring theory. We do not know theories consistently quantizing the gravitation other than superstring theory, and it is supposed as the ultimate theory for elementary particles.

A strong scenario connecting between the standard model and superstring theory is that the particles in the standard model and gravitons are described as modes of the string oscillation, and that our four-dimensional space-time is realized non-perturbatively as a background space-time. The perturbation theory in superstring theory shows that each oscillator mode has some quantum numbers and identified with a particle. This scenario is considered to be realized by the compactification or the brane configuration, which are somehow related by T-duality. To complete this scenario, one should show how the background space-time and the standard model realize from the string theory. Especially, the realization of the space-time is a quite important problem because it is assumed to exist at the beginning in quantum field theories.

In order to discuss how they realize, one needs a non-perturbative definition of string theory. The formulation presently understood as string theory is based on perturbation theory. After the discovery of the D-brane, non-perturbative phenomena was studied well, but D-branes themselves do not have an ability to describe non-perturbative phenomena. Hence, the non-perturbative formulation for string theory is necessary, and in fact, there are candidates of the non-perturbative formulation, such as string field theory, matrix models, and gauge/gravity dualities. Especially, both matrix models and gauge/gravity dualities state that a gauge theory somehow describes string theory, and it is believed that the eigenvalues of the matrices in the gauge theory construct the background geometry in string theory.

An old example of the realization of this belief is $c = 1$ matrix model (see for reviews [30, 31, 32]), which realizes non-perturbative formulation for two-dimensional bosonic string theory. The eigenvalue density of the model forms a spatial direction, and the model corresponds to two-dimensional gravity in the planar limit. If one takes an appropriate double scaling limit, it correctly corresponds to string theory. An extension of this idea is new matrix models. Banks-Fischler-Shenker-Susskind matrix model (BFSS model, also known as Matrix Theory) [11] is considered to describe M-theory in infinite momentum space non-perturbatively. This model is described by non-relativistic D0-branes because the fundamental degrees of freedom of M-theory are D0-branes in the limit. Hence, BFSS model is a one-dimensional matrix model. Ishibashi-Kawai-Kitazawa-Tsuchiya matrix model (IKKT model, also known as IIB matrix model) [12] is obtained by dimensional reduction of BFSS model to a point. This is considered to be the non-perturbative formulation of type IIB superstring theory. BFSS model and IKKT model can be interpreted as matrix regularizations of membranes and strings, respectively. The realization of gravity has been difficult for new matrix models, but gauge/gravity dualities provide clear view of classical gravity. The successful correspondence is AdS/CFT duality [27, 28, 29] between IIB classical supergravity on $AdS_5 \times S^5$ and $SU(N)$ four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory (SYM) in the large-N and strong 't Hooft coupling limit. Gauge/gravity dualities manifestly realize the holography principle, and considered to describe superstring theory with N and 't Hooft coupling finite.

Berenstein-Maldacena-Nastase matrix model (BMN model, also known as plane wave matrix model) [34], which is focused on in this paper, is one of the candidates of the non-perturbative formulation. This is conjectured as a non-perturbative definition of M-theory in pp-wave background geometry. This is a mass-deformation of BFSS model, and so it is naturally interpreted as a matrix regularization of membranes on pp-wave background. This model has many vacua and mass gap. They preserve $SU(2|4)$ symmetry and are given by fuzzy spheres, which are labeled by representations of the $SU(2)$ algebra. Therefore the vacua can be labeled by integers $(N_2^{(s)}, D_s)_{s=1,2,\cdots\Lambda}$ because a general form of the $SU(2)$ generators is a direct sum of irreducible representations, where s stands for a label of an irreducible representation, D_s stands for the dimension of the representation, $N_2^{(s)}$ for its multiplicity, and Λ for the number of kinds of irreducible representations.

The notable property of BMN model is that there is gauge/gravity dualities between vacua in BMN model and half-BPS gravity solutions of type IIA supergravity [33, 13]. As the vacua on the gauge theory side preserve $SU(2|4)$ symmetry, the isometry of the corresponding solutions is $R \times SO(3) \times SO(6)$, which is the bosonic part of $SU(2|4)$ symmetry. The most of the gravity solution is determined by the isometry, but two non-trivial directions are left. The part of the metric relative to those directions is determined by a boundary condition. The boundary condition is described by a fermionic droplet system, and is labeled by integers because the droplets are quantized. Thus the solutions are labeled by integers like the gauge theory side. Actually, the vacuum structure of BMN model corresponds to the structure of gravity solutions.

The aim of this paper is clarifying how the geometries embed in BMN model in the context of the gauge/gravity duality. Although this gauge/gravity correspondence does not produce our actual four-dimensional space-time, the mechanism of the emergence of geometries itself would be an essential clue for the emergence of the real space-time. The result is that a set of saddle point equations for the eigenvalue density of certain matrices in BMN matrix model is equivalent to that of the equations that determines the metric and fluxes of the corresponding geometry. This means that the eigenvalue density actually construct the corresponding geometry.

To obtain the corresponding geometries, the saddle point equations on the gauge theory side should be analyzed in the strong coupling region. Even though BMN model is a one-dimensional matrix model, it is difficult to compute correlators in the strong coupling region. The exact computation for BPS operators was achieved by the localization technique, which appeared in for example [8, 9, 4, 5, 6, 7]. The localization technique let us compute the vacuum expectation values of BPS operators exactly at one-loop order thanks to a fermionic symmetry, which is usually supersymmetry.

The emergence of geometries from other $SU(2|4)$ symmetric theories in the large-N limit can also be shown. This is because BMN model contains higher dimensional $SU(2|4)$ symmetric theories as special vacua in the limit [13, 15]. $\mathcal{N}=8$ SYM on $R\times S^2$ is realized from BMN model in the commutative limit of the fuzzy spheres. This is understood as "blowing-up" of D2-branes from D0-branes. $\mathcal{N}=4$ SYM on $R\times S^3/Z_k$ can be obtained by performing the Taylor's T-duality [17, 16] from the SYM on $R\times S^2$. S^3 of the SYM is naturally realized as a Hopf-fibration of S^2. This is interpreted as a theory on D3-branes wrapped around S^3/Z_k, obtained by T-duality of D2-branes. The successive application of these two procedures lets us obtain $\mathcal{N}=4$ SYM on $R\times S^3/Z_k$ from BMN model.

This paper is organized as follows. The gravity side for the gauge/gravity duality for $SU(2|4)$ is reviewed in section 2. The gauge theory side, BMN model and other $SU(2|4)$ symmetric theories, is summarized in section 3. Also, exact computation for BMN model is shown in that section. Then, the saddle point equations in the BPS sector are obtained in section 4. In the end of the section, the equation matching between gravity and gauge theory sides is shown. This means that the emergence of geometries is shown in the gauge theories.

2 Gauge/gravity duality for $SU(2|4)$ symmetric theories

In this section, I review the gauge/gravity duality for $SU(2|4)$ symmetric theories. The corresponding geometries were obtained as a special limit of half-BPS geometries in M-theory in [33], and later, the duality for $SU(2|4)$ symmetric theories was discussed by Lin and Maldacena [13]. They assumed the $SU(2|4)$ symmetric ansatz and then showed that finding the classical solutions is reduced to the problem of finding axisymmetric solutions to the three-dimensional Laplace equation with appropriate boundary conditions given by parallel charged conducting disks and a background potential.

In the simplest case of the supergravity solution, it has two interesting scaling limits in which the solution becomes the D2-brane solution or the NS5-brane solution constructed in [13]. In these limit, the equation determining the geometry is soluble and the solution can be explicitly obtained. We will also see this in this section.

2.1 Dual geometry of BMN model

First, let us see the Lin-Maldacena solution [13], which is the solution with $SU(2|4)$ symmetry in type IIA supergravity. It can be obtained by dimensional reduction of the eleven-dimensional supergravity solution with the symmetry. As the bosonic part of $SU(2|4)$ symmetry is $R\times S0(3)\times SO(6)$, there should be S^2 and S^5 in order for $SO(3)$ and $SO(6)$ to act on the geometry. By solving the Killing spinor equation with that ansatz in the condition that these spheres should shrink in a non-singular manner, one obtains the geometry and a four-form flux. The IIA solution is found by the toroidal compactification along a direction perpendicular to $R \times S^2 \times S^5$. In the string frame, it is given by

$$
\begin{aligned}
ds_{10}^2 &= \left(\frac{\ddot{V}-2\dot{V}}{-V''}\right)^{1/2}\left\{-4\frac{\ddot{V}}{\ddot{V}-2\dot{V}}dt^2 - 2\frac{V''}{\dot{V}}(dr^2+dz^2) + 4d\Omega_5^2 + 2\frac{V''\dot{V}}{\Delta}d\Omega_2^2\right\},\\
C_1 &= -\frac{(\dot{V}^2)'}{\ddot{V}-2\dot{V}}dt, \quad C_3 = -4\frac{\dot{V}^2V''}{\Delta}dt\wedge d\Omega_2,\\
B_2 &= \left(\frac{(\dot{V}^2)'}{\Delta}+2z\right)d\Omega_2, \quad e^{4\Phi} = \frac{4(\ddot{V}-2\dot{V})^3}{-V''\dot{V}^2\Delta^2},
\end{aligned}
\tag{2.1}
$$

where $\Delta = (\ddot{V}-2\dot{V})V'' - (\dot{V}')^2$ and the dots and primes denote $\frac{\partial}{\partial \log r}$ and $\frac{\partial}{\partial z}$, respectively. A remarkable feature of this solution is that it is written in terms of a single function $V(r,z)$.

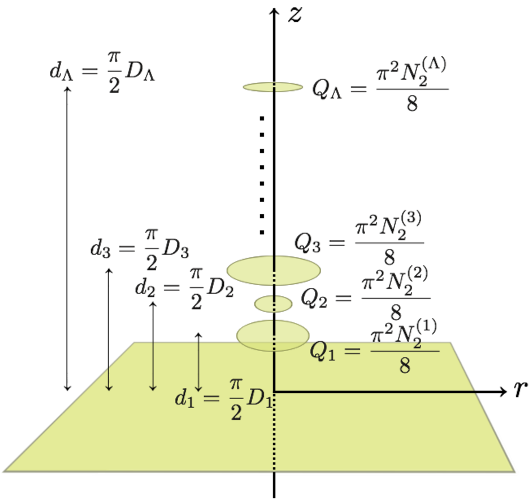

Figure 1: Electrostatic system corresponding to a general vacuum of BMN model. This figure is adapted from [3].

The Killing spinor equation in the supergravity imposes a condition that $V(r,z)$ satisfies the Laplace equation in a three-dimensional axisymmetric electrostatic system, where r and z represent coordinates for the transverse and the axial directions, respectively. The regularity of the metric requires that the electrostatic system must consist of some conducting disks with radii tuned such that the charge densities vanish at the edges. In addition, from the positivity of the metric, there must be a certain background potential. So, the potential $V(r,z)$ consists of these two contributions:

$$V(r,z) = V_{\text{b.g.}}(r,z) + \tilde{V}(r,z). \tag{2.2}$$

The electrostatic system is determined once a theory and its vacuum are specified. The electrostatic system relevant to BMN model consists of an infinite conducting plate at $z = 0$, some finite conducting disks in the region of $z \geq 0$ (Fig. 1) and the background potential of the form

$$V_{\text{b.g.}}(r,z) = V_0 \left(r^2 z - \frac{2}{3} z^3 \right), \tag{2.3}$$

where V_0 is a constant. The electrostatic system relevant to $\mathcal{N} = 8$ SYM on $R \times S^2$ consists of some finite conducting disks in the region $-\infty \leq z \leq \infty$ (Fig. 2 (left)) and the background potential of the form

$$V_{\text{b.g.}}(r,z) = W_0(r^2 - 2z^2), \tag{2.4}$$

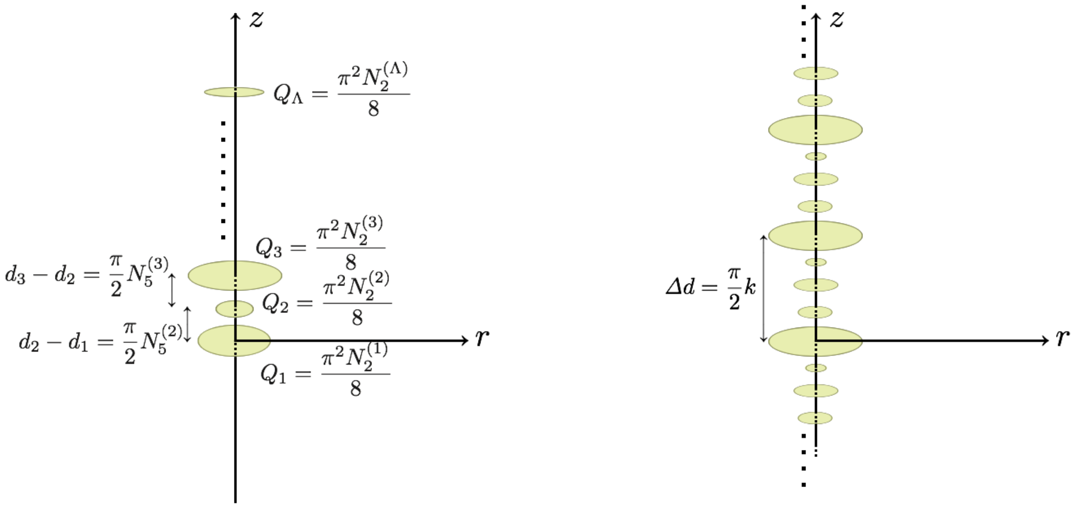

Figure 2: Electrostatic systems corresponding to a general vacuum of SYM on $R \times S^2$ (left) and SYM on $R \times S^3/Z_k$ (right). The conducting disks in the right figure are arranged periodically. This figure is adapted from [3].

where W_0 is a constant. The electrostatic system relevant to $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ consists of an infinite number of finite conducting disks arranged periodically along the z-axis (Fig. 2 (right)) and the background potential (2.4)[1].

The condition that the charge densities vanish at the edges of the disks relates the radii of the disks and the charges. So the independent parameters of this solution are the total charges and the z-coordinates of the disks, as well as V_0 in (2.3) or W_0 in (2.4).

While V_0 and W_0 are related to the magnitude of the dilaton, the other parameters turn out to be related to the brane charges. This geometry has an S^2 and an S^5 at each point on the r-z plane. One can show that, on the z-axis, the radius of S^5 becomes zero while, on the finite conducting disks, the radius of S^2 becomes zero. So one can consider a various non-contractible S^3's or S^6's which are given by fibering the S^2 or S^5 on the path (on the r-z plane) ending on different disks or on different segments of the z-axis, respectively. On each S^3 or S^6, one can measure the NSNS-flux or the RR-flux. When α' is set to 1, they are quantized as

$$N_5 = \frac{1}{(2\pi)^2} \int_{S^3} dB_2, \quad N_2 = \frac{1}{(2\pi)^5} \int_{S^6} (dC_3 - C_1 \wedge H_3), \tag{2.5}$$

where N_5 and N_2 are the integers which are a NS5-brane charge and D2-brane charge,

[1]Note that (2.4) is periodic up to terms of linear and constant in z, which do not contribute to the gravity solution (2.1).

respectively. This relates the parameters of the electrostatic system to the brane charges. As an example, let us consider the electrostatic system for BMN model with Λ finite plates (see Fig. 1). We denote the total charge, the radius and the z-coordinate of s-th disk by Q_s, R_s and d_s, respectively, where $s = 1, \cdots, \Lambda$. In this case, there are Λ independent non-contractible S^3's and the same number of S^6's in the geometry. d_s and Q_s are related to the NS5-brane charges $N_5^{(s)}$ and the D2-brane charges $N_2^{(s)}$ as

$$d_s - d_{s-1} = \frac{\pi N_5^{(s)}}{2}, \quad Q_s = \frac{\pi^2 N_2^{(s)}}{8}, \tag{2.6}$$

for $s = 1, \cdots, \Lambda$. Here $d_0 = 0$ denotes the position of the infinite plate. Hence, this system is labeled by a set of integers $\{N_2^{(s)}, N_5^{(s)}\}$.

Those integers correspond to the sizes of blocks of the matrix in the corresponding vacuum in BMN model. $N_2^{(s)}$ is identified with the multiplicity of a fuzzy sphere, and $N_5^{(s)}$ is proportional to a differences of the radii of the fuzzy spheres. In addition, in [35], the case of $\Lambda = 1$ was considered and V_0 was inferred to be related to the gauge coupling as

$$V_0 = \frac{hm^3}{8g^2}, \tag{2.7}$$

where h is a constant. In [2], the constant h was determined as

$$h = \frac{2}{\pi^2}. \tag{2.8}$$

Beyond the case of $\Lambda = 1$, the relation (2.7) with (2.8) also holds for the theory around the general vacuum. This can be seen also from the result of this paper (4.18).

2.2 Electrostatic problem for BMN model

In the following, we derive a Fredholm integral equation which determines $\tilde{V}(r,z)$ in (2.2). We consider the situation shown in Fig. 1 and denote the solution of $\tilde{V}(r,z)$ in the region $z \in [d_s, d_{s+1})$ by $V_s(r,z)$, where s runs from 0 to Λ and $(d_0, d_{\Lambda+1}) = (0, \infty)$ is assumed.

We first note that a general solution to the Laplace equation in an axially symmetric system is given by $J_0(ru)e^{\pm zu}$ where $J_0(z)$ is the Bessel function of the first kind of order zero and u is a positive real number. So we can write $V_s(r,z)$ as

$$V_s(r,z) = \int_0^\infty du (C_s(u)e^{zu} + D_s(u)e^{-zu}) J_0(ru). \tag{2.9}$$

Now we have the boundary condition that $\tilde{V}(r,z) \to 0$ as $z \to 0$ and $z \to \infty$. This means

$$C_0(u) = -D_0(u), \quad C_\Lambda(u) = 0. \tag{2.10}$$

We also have some continuation conditions for V_s's at $z = d_s$. First, at $z = d_s$, V_s should be equal to V_{s-1}. This is satisfied if

$$C_s(u)e^{d_s u} + D_s(u)e^{-d_s u} = C_{s-1}(u)e^{d_s u} + D_{s-1}(u)e^{-d_s u}. \tag{2.11}$$

Second, when $z = d_s$ and $r > R_s$, not only V_s but also $\frac{\partial V_s}{\partial z}$ should be continuous. So we have

$$\int_0^\infty duu \left(C_s(u)e^{d_s u} - D_s(u)e^{-d_s u} - C_{s-1}(u)e^{d_s u} + D_{s-1}(u)e^{-d_s u}\right) J_0(ru) = 0 \tag{2.12}$$

for $r > R_s$. Third, when $z = d_s$ and $r \leq R_s$ (i.e. on the conducting disk), the value of $V(r, z)$ should be constant: $V(r, d_s) = \Delta_s$. In terms of $C_s(u)$ and $D_s(u)$, this is written as

$$\int_0^\infty du \left(C_s(u)e^{d_s u} + D_s(u)e^{-d_s u}\right) J_0(ru) = \Delta_s - V_{\text{b.g.}}(r, d_s). \tag{2.13}$$

In order to solve the conditions (2.11), (2.12) and (2.13), we define

$$A_s(u) = u(C_s(u) - C_{s-1}(u))e^{d_s u} - u(D_s(u) - D_{s-1}(u))e^{-d_s u} \tag{2.14}$$

for $s = 1, 2, \cdots, \Lambda$. From (2.10) and (2.11), $C_s(u)$ and $D_s(u)$ can be written in terms of $A_s(u)$ as

$$\begin{aligned}
C_s(u) &= -\sum_{t=s+1}^{\Lambda} \frac{e^{-d_t u}}{2u} A_t(u), \\
D_s(u) &= \sum_{t=1}^{\Lambda} \frac{e^{-d_t u}}{2u} A_t(u) - \sum_{t=1}^{s} \frac{e^{d_t u}}{2u} A_t(u).
\end{aligned} \tag{2.15}$$

By substituting (2.14) and (2.15) to (2.12) and (2.13), we obtain

$$\begin{aligned}
&\int_0^\infty u^{-1} \sum_{t=1}^{\Lambda} (\delta_{st} + k_{st}(u)) A_t(u) J_0(ru) du = F_s(r), \quad (0 \leq r \leq R_s) \\
&\int_0^\infty A_s(u) J_0(ru) du = 0, \quad (R_s \leq r)
\end{aligned} \tag{2.16}$$

where $k_{st}(u)$ and $F_s(r)$ are given by

$$\begin{aligned}
k_{st}(u) &= -e^{-(d_s+d_t)u} + (1 - \delta_{st})e^{-|d_s - d_t|u}, \\
F_s(r) &= -2(\Delta_s - V_{\text{b.g.}}(r, d_s)).
\end{aligned} \tag{2.17}$$

As shown in appendix C, the equations (2.16) can be reduced to the integral equations, (C.28) and (C.29), for the functions $h_s(u)$ defined by (C.27)[2]. For our problem, it is more convenient to work with the variables

$$f_s(u) = -\frac{1}{4\sqrt{\pi}} u h_s(u). \tag{2.18}$$

Then, (C.29) is written for $\{f_s(x)\}$ as

$$\begin{aligned} f_s(x) + \frac{1}{\pi}\sum_{t=1}^{\Lambda}\int_{-R_t}^{R_t} du \left[-\frac{d_s+d_t}{(d_s+d_t)^2+(x-u)^2} + \frac{|d_s-d_t|}{(d_s-d_t)^2+(x-u)^2} \right] f_t(u) \\ = \frac{1}{\pi}\left(\Delta_s + \frac{2}{3}V_0 d_s^3 - 2V_0 d_s x^2 \right), \end{aligned} \tag{2.19}$$

and (C.28) shows that f_s is vanishing outside the region $[-R_s, R_s]$. Here we have defined $f_s(x)$ with negative x as $f_s(x) = f_s(-x)$ and extended the domain to the entire real line.

The function $f_s(x)$ can be interpreted as the charge density on the s-th conducting disk as follows. For $z = d_s$ and $r \leq R_s$, we have

$$\frac{\partial}{\partial z}V_s(r, d_s) - \frac{\partial}{\partial z}V_{s-1}(r, d_s) = K_{0,-\frac{1}{2}} h_s(r) = 4\int_r^{R_s} \frac{f_s'(u)}{\sqrt{u^2-r^2}} du. \tag{2.20}$$

On the other hand, this is equal to $-4\pi\sigma(r)$, where $\sigma(r)$ is the charge density for the r direction. Hence, the total charge on the disk can be computed as

$$Q_s = -2\int_0^{R_s} dr \int_r^{R_s} du \frac{r f_s'(u)}{\sqrt{u^2-r^2}} = \int_{-R_s}^{R_s} du f_s(u). \tag{2.21}$$

These relations show that $f_s(u)$ corresponds to the charge density on the s-th plate projected onto a diameter direction. These densities are fully determined by (2.19) and so is the potential which can be written in terms of $\{f_s(u)\}$ as

$$\tilde{V}(r, z) = \sum_{s=1}^{\Lambda}\int_{-R_s}^{R_s} dt \left[\frac{1}{\sqrt{(z-d_s+it)^2+r^2}} - \frac{1}{\sqrt{(z+d_s+it)^2+r^2}} \right] f_s(t). \tag{2.22}$$

Note that R_s and Δ_s are determined by $f_s(R_s) = 0$ and (2.21). The s-th disk radius, R_s, is related to the radius of s-th S^5 at the edge of the disk in the string frame, $R_{S^5}^{(s)}$, as [35]

$$R_s = \frac{R_{S^5}^{(s)\,2}}{4} \tag{2.23}$$

[2] n, I_{s1} and I_{s2} in appendix C corresponds to Λ, $[0, R_s)$ and $[R_s, \infty)$ in our problem, respectively.

in the string unit, $\alpha' = 1$. One can easily check this by using the Laplace equation to rewrite V'' and noting that $\dot{V} = 0$ on the disk.

Before closing this subsection, let us consider the simplest case, $\Lambda = 1$, which corresponds to the simplest vacuum in BMN model. It was studied in detail in [35]. In this case, some quantities are explicitly found. The electrostatic system associated with this solution consists of one infinite conducting plate at $z = 0$ and another finite conducting disk at $z = d > 0$ with radius R and charge Q. The background potential is given by $V_0(r^2 z - \frac{2}{3}z^3)$. Q and d are related to the brane charges as $N_5 = 2d/\pi$ and $N_2 = 8Q/\pi^2$.

In this case, it is convenient to rewrite $\tilde{V}$ as

$$\tilde{V}(r, z) = V_0 R^3 \phi_\kappa(r/R, z/R), \tag{2.24}$$

where $\kappa \equiv d/R$ and

$$\phi_\kappa(r, z) = \frac{\beta(\kappa)}{\pi} \int_{-1}^{1} dt \left(-\frac{1}{\sqrt{r^2 + (z + \kappa + it)^2}} + \frac{1}{\sqrt{r^2 + (z - \kappa + it)^2}} \right) g_\kappa(t). \tag{2.25}$$

Notice that $g_\kappa(t)$ is different from $f_1(t)$ by a factor. Here $\beta(\kappa)$ is a function of κ, which is determined later. $g_\kappa(t)$ is the solution to the Fredholm integral equation of the second kind,

$$g_\kappa(x) - \int_{-1}^{1} dy K_\kappa(x, y) g_\kappa(y) = 1 - \frac{2\kappa}{\beta(\kappa)} x^2 \tag{2.26}$$

with kernel

$$K_\kappa(x, y) = \frac{1}{\pi} \frac{2\kappa}{4\kappa^2 + (x - y)^2}. \tag{2.27}$$

Thus, if one defines $g_\kappa^{(n)}(t)$ by

$$g_\kappa^{(n)}(x) - \int_{-1}^{1} dy K_\kappa(x, y) g_\kappa^{(n)}(y) = x^n, \tag{2.28}$$

$\beta(\kappa)$ is given as

$$\beta(\kappa) \equiv 2\kappa \frac{g_\kappa^{(2)}(1)}{g_\kappa^{(0)}(1)}. \tag{2.29}$$

The equation (2.26) is solved by

$$g_\kappa(t) = g_\kappa^{(0)}(t) - \frac{2\kappa}{\beta(\kappa)} g_\kappa^{(2)}(t) = g_\kappa^{(0)}(t) - \frac{g_\kappa^{(0)}(1)}{g_\kappa^{(2)}(1)} g_\kappa^{(2)}(t). \tag{2.30}$$

The charge density $\sigma_\kappa(r)$ for the radial direction on the disk is related to $g_\kappa(t)$ as

$$\sigma_\kappa(r) = -\frac{\beta(\kappa)}{\pi^2}\int_r^1 dt \frac{g'_\kappa(t)}{\sqrt{t^2-r^2}}, \quad g_\kappa(t) = \frac{2\pi}{\beta(\kappa)}\int_{|t|}^1 dr \frac{r\sigma_\kappa(r)}{\sqrt{r^2-t^2}}. \tag{2.31}$$

The radius of the disk is related to the charge as

$$Q = q(\kappa)V_0 R^4, \quad q(\kappa) = \frac{\beta(\kappa)}{\pi}\int_{-1}^1 dt g_\kappa(t). \tag{2.32}$$

The parameters of the electrostatic problem were identified with the parameters in BMN model as [13, 35]

$$Q = \frac{\pi^2 N_2}{8}, \quad d = \frac{\pi}{2}N_5, \quad R = \left(\frac{\pi^2 g^2 N_2}{m^3 h q(\kappa)}\right)^{\frac{1}{4}}, \quad V_0 = \frac{hm^3}{8g^2}. \tag{2.33}$$

Here, h is a constant which does not depend on g^2/m^3, N_2 and N_5. In section 4, we determine the value of h from the gauge theory side.

2.3 Electrostatic problem for $\mathcal{N}=8$ SYM on $R\times S^2$

The electrostatic system associated with the gravity dual of $\mathcal{N}=8$ SYM on $R\times S^2$ is shown in Fig. 2 (left). The case where $\Lambda=2$ and $R_1=R_2$ was studied in [40]. Here, we generalize their result. It was shown in [35, 15] that the solution for this system can be obtained from the solution for BMN model by taking the D2-brane limit. After the redefinitions $d_s \to d + d_s \ (1\le s\le \Lambda)$, $z\to d+z$, D2-brane limit is written as

$$d\to\infty, \quad Q_s: \text{fixed}, \quad V_0 d = W_0: \text{fixed}. \tag{2.34}$$

Indeed, in this limit, Fig. 1 becomes Fig. 2 (left) and the background potential for BMN model (2.3) becomes

$$V_{\text{b.g.}}(r,z) \to -W_0\left(\frac{2d^2}{3}+2dz\right) + W_0(r^2-2z^2). \tag{2.35}$$

One can neglect the first term since it does not affect the gravity solution which depends only on $\dot V$, $\ddot V$, $\dot V'$ and V''. Thus, the background potential for BMN model (2.3) exactly reduces to that for SYM on $R\times S^2$ (2.4) in the limit (2.34).

By taking the D2-brane limit (2.34) of the integral equation (2.19) and the potential (2.25), we obtain

$$f_s(x) + \frac{1}{\pi}\sum_{t=1}^{\Lambda}\int_{-R_t}^{R_t} du \frac{|d_s-d_t|}{(d_s-d_t)^2+(x-u)^2} f_t(u) = \frac{1}{\pi}\left(\Delta'_s + 2W_0 d_s^2 - 2W_0 x^2\right), \tag{2.36}$$

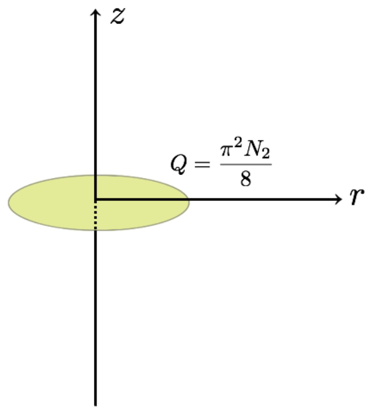

Figure 3: The electrostatic system for the D2-brane solution. This figure is adapted from [2].

and

$$\tilde{V}(r,z) = \sum_{s=1}^{\Lambda} \int_{-R_s}^{R_s} dt \frac{f_s(t)}{\sqrt{(z-d_s+it)^2+r^2}}, \tag{2.37}$$

respectively. Here Δ'_s is a constant potential on the s-th disk, $V(r,d_s) = \Delta'_s$ $(r < R_s)$. Note that R_s and Δ'_s are determined by $f_s(R_s) = 0$ and (2.21). The solution of (2.36) gives a general solution to the electrostatic problem for SYM on $R \times S^2$.

Let us consider the simplest case, $\Lambda = 1$. The limit is given as

$$d \to \infty, \quad Q: \text{ fixed}, \quad V_0 d = W_0: \text{ fixed}. \tag{2.38}$$

We can see that this limit corresponds to the large-κ limit.

By using the relation (2.33), one can rewrite this limit in terms of the parameters in BMN model as

$$N_5 \to \infty, \quad N_2: \text{ fixed}, \quad \frac{4\pi g^2}{m^2 N_5} \equiv g_{S^2}^2: \text{ fixed}. \tag{2.39}$$

The limit corresponds to the commutative limit of fuzzy spheres, where BMN model describes $U(N_2)$ $\mathcal{N} = 8$ SYM on $R \times S^2$. The radius of S^2 is given by $1/m = 1/2$. The fixed quantity g_{S^2} in (2.39) is the gauge coupling constant in this theory.

In this simplest case, the integral equation can be solved. The limit is $\kappa \to \infty$. The solution is

$$\beta(\kappa) \simeq 2\kappa,$$

$$
\begin{aligned}
q(\kappa) &\simeq \frac{8}{3\pi}\kappa, \\
f_\kappa(x) &\simeq 1 - x^2.
\end{aligned}
\tag{2.40}
$$

2.4 Electrostatic problem for $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$

The electrostatic system associated with the gravity dual of $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ is shown in Fig. 2 (right). The case for the trivial vacuum was studied in [40]. Here, we generalize their result. This can be obtained from that for SYM on $R \times S^2$ by compactifying the z direction to S^1 with the background potential intact [15].

We start from the solution for SYM on $R\times S^2$, (2.36) and (2.37), with disks periodically arranged. We change the labelling of the disks so that they are labelled by two integers (s, α), where $-\infty \leq s \leq \infty$ and $\alpha \in K \subset \{1, 2, \cdots, k\}$. s is a label of a single period and α is that of each disk in the period. So each period consists of $|K|$ conducting disks. We put the position of each disk to be

$$
d_{s,\alpha} = \frac{\pi}{2}\left(ks + \alpha - 1\right). \tag{2.41}
$$

The charge $Q_{s,\alpha}$ and the radius $R_{s,\alpha}$ of each disk is independent of s: $Q_{s,\alpha} = Q_\alpha$ and $R_{s,\alpha} = R_\alpha$. The charge density $f_{s,\alpha}(r)$ on each disk should also be independent of s:

$$
f_{s,\alpha}(r) = f_\alpha(r). \tag{2.42}
$$

Note that the naive substitutions of these conditions to (2.36) and (2.37) do not make sense because of the divergences coming from the periodicity. As remarked in [40], this divergence can be avoided by solving the electrostatic problem for the electric field rather than the potential. Hence, by differentiating (2.36) with respect to x and imposing the periodicity condition, one can obtain the integral equations for the charge densities $\{f_\alpha(r)\}$,

$$
f'_\alpha(x) + \sum_{\beta\in K}\int_{-R_\beta}^{R_\beta} du\, K_k\left(\frac{\alpha-\beta}{k}, x, u\right) f'_\beta(u) = -\frac{4}{\pi}W_0 x, \tag{2.43}
$$

where

$$
K_k(\nu, x, u) = \frac{1}{4\pi}\int_{-\infty}^{\infty} dp \frac{\cosh\left\{\frac{\pi k}{2}p\left(|\nu| - \frac{1}{2}\right)\right\}}{\sinh\frac{\pi k}{4}|p|}\left(e^{ip(x-u)} - e^{ip(x+u)}\right). \tag{2.44}
$$

The electric field generated by the conducting disks is obtained from (2.37) as

$$
E_r = \sum_{s=-\infty}^{\infty}\sum_{\alpha\in K}\int_{-R_\alpha}^{R_\alpha} dt \frac{r f_\alpha(t)}{((z - 2d_{s,\alpha} + it)^2 + r^2)^{\frac{3}{2}}},
$$

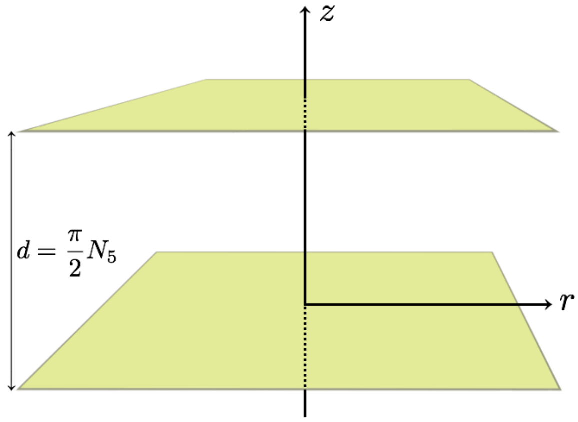

Figure 4: The electrostatic system for the NS5-brane solution. This figure is adapted from [2].

$$E_z = \sum_{s=-\infty}^{\infty} \sum_{\alpha \in K} \int_{-R_\alpha}^{R_\alpha} dt \frac{(z - 2d_{s,\alpha} + it) f_\alpha(t)}{((z - 2d_{s,\alpha} + it)^2 + r^2)^{\frac{3}{2}}}. \tag{2.45}$$

2.5 Electrostatic problem for little string theory on $R \times S^5$

Let us consider the NS5-brane solution in the simplest case, $\Lambda = 1$, although the general case can be also obtained straightforwardly. The NS5-brane solution is given by the form of (2.1), where the electrostatic system now consists of two infinite conducting plates separated by distance d as shown in Fig. 4. The electrostatic potential is given by

$$V_{NS5}(r, z) = \frac{1}{g_0} \sin\left(\frac{\pi z}{d}\right) I_0\left(\frac{\pi r}{d}\right), \tag{2.46}$$

where g_0 is a constant and I_n is the modified Bessel function of the first kind. For the explicit form of the geometry, see [13, 35].

Now let us consider the NS5-brane limit, in which the gravity dual solution written in terms of (2.2) with (2.3) is reduced to the NS5-brane solution constructed in [13]. The NS5-brane limit is given as a double scaling limit where both R and V_0 are sent to infinity in an appropriate way. Let us review the derivation of the precise form of the scaling limit [35]. We first make the Fourier expansion of the potential V with (2.3) in $r < R$ region as,

$$V(r, z) = V_0 R^3 \Delta(\kappa) \frac{z}{d} + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi z}{d}\right) I_0\left(\frac{n\pi r}{d}\right), \tag{2.47}$$

where $\kappa \equiv d/R$ and $\Delta(\kappa)$ is defined as

$$\Delta(\kappa) = \beta(\kappa) - \frac{2}{3}\kappa^3. \tag{2.48}$$

The restricted form of the expansion (2.47) follows from the conditions that V is regular at $r = 0$, constant ($V_0 R^3 \Delta(\kappa)$) at $z = d$ and zero at $z = 0$. Since the first term in (2.47) does not contribute to the geometry, the NS5-brane limit is a limit where

$$c_1 \to \frac{1}{g_0}, \quad c_n \to 0 \ \ (n > 1). \tag{2.49}$$

One can determine the coefficients c_n's by the inverse Fourier transformation at $r = R$ as

$$c_n = \left(I_0\left(\frac{n\pi}{\kappa}\right)\right)^{-1} 2V_0 R^3 p_n(\kappa), \tag{2.50}$$

where

$$p_n(\kappa) = \int_0^1 dy \left(\phi_\kappa(1, \kappa y) - \Delta(\kappa) y - \kappa y + \frac{2}{3}\kappa^3 y^3\right) \sin(n\pi y). \tag{2.51}$$

When $\kappa = d/R$ is small, $p_n(\kappa)$ behaves as

$$p_n(\kappa) \sim b_n \kappa^2, \tag{2.52}$$

where b_n are constants. Since $I_n(z) \sim e^z/\sqrt{2\pi z}$ for $z \gg 1$, we find for small κ that

$$c_n \sim 2b_n \sqrt{2\pi^2 n} e^{-\frac{n\pi R}{d}} V_0 (Rd)^{\frac{3}{2}}. \tag{2.53}$$

Then, the NS5-brane limit is given by

$$R \to \infty, \quad d: \text{ fixed}, \quad V_0 \to \frac{1}{g_0} \frac{1}{2b_1\sqrt{2\pi^2}} (Rd)^{-\frac{3}{2}} e^{\frac{\pi R}{d}}, \tag{2.54}$$

which realizes (2.49). Note that $\kappa = d/R$ goes to zero in this limit. The value of b_1 was computed numerically and found to be $b_1 = 0.040$ [35].

Using the relations (2.33), one can rewrite the limit (2.54) in the language of BMN model as

$$N_2 \to \infty, \quad \lambda \to \infty, \quad \frac{1}{N_2}\lambda^{\frac{5}{8}} e^{\frac{a}{N_5}\lambda^{\frac{1}{4}}} \equiv \tilde{g}_s: \text{ fixed}, \quad N_5: \text{ fixed}, \tag{2.55}$$

where $a = 2\pi^{\frac{1}{2}}/h^{\frac{1}{4}}$ and λ is the dimensionless 't Hooft coupling in BMN model,

$$\lambda = g^2 N_2 \left(\frac{2}{m}\right)^3. \tag{2.56}$$

The dual theory of the NS5-brane solution in IIA geometry is considered as a six-dimensional non-gravitational string theory called the IIA little string theory (LST)[36, 37, 38, 39]. The parameter $\tilde{g}_s$ is considered to be the string coupling constant of LST. The limit (2.55) predicts that the dynamics of BMN model near the NS5-brane limit is controlled by $\lambda^{1/4}$, and this can be confirmed by analyzing the gauge theory side [2].

Finally, let us see the solution in this $\kappa \sim 0$ case with $\Lambda = 1$. The solution can be obtained as

$$\begin{aligned} \beta(\kappa) &\simeq \kappa, \\ q(\kappa) &\simeq \frac{1}{8}, \\ f_\kappa(x) &\simeq \frac{1}{3\kappa}(1-x^2)^{\frac{3}{2}}. \end{aligned} \tag{2.57}$$

3 BMN model and other $SU(2|4)$ symmetric gauge theories

In this section, I review several properties of BMN model and other $SU(2|4)$ symmetric gauge theories. They all have many discrete vacua and the vacua have $SU(2|4)$ symmetry. These theories can be obtained by truncating $\mathcal{N} = 4$ super Yang-Mills theory on $R \times S^3$, which has superconformal symmetry forming $SU(2,2|4)$ group. $SU(2,2|4)$ group contains $SU(2)_L \times SU(2)_R \in SO(4)$ as a bosonic subgroup, and one obtains $SU(2|4)$ group by quotienting $SU(2,2|4)$ by a subgroup of $SU(2)_R$. Thus, the $SU(2|4)$ symmetric theories can be obtained by the consistent truncation which leaves only fields invariant under the subgroup of $SU(2)_R$. In the following, I start with $\mathcal{N} = 4$ SYM on $R \times S^3$, and obtain $SU(2|4)$ symmetric theories by a consistent truncation of it. Then, I show that $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ and $\mathcal{N} = 8$ SYM on $R \times S^2$ can be reproduced from BMN model in the large-N limit.

3.1 $\mathcal{N} = 4$ SYM on $R \times S^3$

$\mathcal{N} = 4$ SYM on $R \times S^3$ is obtained by the conformal mapping from that on R^4, which is obtained by dimensional reduction of $\mathcal{N} = 1$ SYM on $R^{9,1}$ along $5 + 1$ dimensions. In this paper, let us consider the theory in ten-dimensional Euclidean signature[3]. Its action

[3]We deal with gauge theories in the Euclidean signature so that the localization method would work. $\mathcal{N} = 4$ SYM in that original Lorentzian signature is obtained by the Wick-rotation in 10-direction.

is written as

$$S_{R\times S^3} = \frac{1}{g_{S^3}^2}\int d\tau d\Omega_3 \,\mathrm{Tr}\Big(\frac{1}{4}F_{MN}F^{MN} + \frac{m^2}{8}X_mX^m + \frac{i}{2}\Psi\Gamma^M D_M\Psi\Big), \qquad (3.1)$$

where

$$\begin{aligned}
&F_{1a} = \partial_1 A_a - \nabla_a A_1 - i[A_1, A_a], \quad F_{1m} = D_1 X_m = \partial_1 X_m - i[A_1, X_m],\\
&F_{ab} = \nabla_a A_b - \nabla_b A_a - i[A_a, A_b], \quad F_{am} = D_a X_m = \nabla_a X_m - i[A_a, X_m],\\
&F_{mn} = -i[X_m, X_n],\\
&D_1\Psi = \partial_1\Psi - i[A_1,\Psi], \quad D_a\Psi = \nabla_a\Psi - i[A_a,\Psi], \quad D_m\Psi = -i[X_m,\Psi]. \qquad (3.2)
\end{aligned}$$

Here, indices M, N run from 1 to 10, and indices m, n from 5 to 10. Index 1 corresponds to R direction, τ, and indices a, b run from 2 to 4, which indicate S^3 directions, with ∂_1 a derivative with respect to τ and ∇_a a covariant derivative on S^3 in the local Lorentz coordinates. A_1 and A_a is gauge fields, and X_m are scalar fields which transform as vector under $SO(6)$. Its gauge group is $U(N)$. Ψ consists of four Majorana spinor[4] fields, and is understood as a 16-component Majorana-Weyl fermion in the $(9+1)$-dimensional viewpoint. Due to the conformal mapping, the radius of the S^3 is determined as $2/m$, and the action has conformal mass terms and a three-point coupling of gauge fields.

For later convenience, I write down a detailed expression of the action:

$$\begin{aligned}
S_{R\times S^3} = \frac{1}{g_{S^3}^2}\int d\tau d\Omega_3 \,\mathrm{Tr}\Big[&\frac{1}{2}(\partial_1 A_b - \partial_b A_1 - i[A_1, A_b])^2\\
&+ \frac{1}{4}(2\varepsilon_{abc}A_c + \partial_a A_b - \partial_b A_a - i[A_a, A_b])^2\\
&+ \frac{1}{2}(D_1X_m)^2 + \frac{1}{2}(D_aX_m)^2 + \frac{1}{2}X_mX^m - \frac{1}{4}[X_m, X_n][X^m, X^n]\\
&+ \frac{i}{2}\Psi\Gamma^1\partial_1\Psi + \frac{i}{2}\Psi\Gamma^a\partial_a\Psi + \frac{3i}{8}\Psi\Gamma^{234}\Psi\\
&+ \frac{1}{2}\Psi\Gamma^1[A_1,\Psi] + \frac{1}{2}\Psi\Gamma^a[A_a,\Psi] + \frac{1}{2}\Psi\Gamma^m[X_m,\Psi]\Big], \qquad (3.3)
\end{aligned}$$

where ∂_a means a derivative transformed by the dreibein : $\partial_a = e_a^\mu\partial_\mu$. It is better to take the dreibein as right-invariant one because we will break $SU(2)_R$ isometry and dimensionally reduce the theory to lower dimensional ones. Though $SU(2)_L$ rotates indices a, b and c, they are contracted and so the action is invariant under $SU(2)_L$. This $SU(2)_L$ will remain after the dimensional reductions.

[4] "Majorana" appearing here and later means that the spinor satisfies the Majorana condition if it is Wick-rotated to the Lorentzian signature.

The parameters of this theory are conformal mass m, gauge coupling g_{S^3} and matrix size N_{S^3}. However, the independent parameters are $g_{S^3}^2/m^3$ and N_{S^3} because m has mass-dimension one and $g_{S^3}^2$ has mass-dimension three. Therefore, let us set m as 2 from now on.

The theory is invariant under the following fermionic transformations:

$$\begin{aligned}
&\delta_s A_1 = -i\Psi\Gamma_1\epsilon, \quad \delta_s A_a = -i\Psi\Gamma_a\epsilon, \quad \delta_s X_m = -i\Psi\Gamma_m\epsilon, \\
&\delta_s\Psi = \left(\frac{1}{2}F_{MN}\Gamma^{MN} - \frac{1}{2}X_m\tilde{\Gamma}^m(\Gamma^1\partial_1 + \Gamma^a\nabla_a)\right)\epsilon.
\end{aligned} \tag{3.4}$$

where ϵ is a Grassmann-even conformal Killing spinor satisfying

$$\partial_1\epsilon = \tilde{\Gamma}_1\tilde{\epsilon}, \quad \nabla_a\epsilon = \tilde{\Gamma}_a\tilde{\epsilon}, \tag{3.5}$$

so that δ_s is Grassmann odd. For $R\times S^3$, $\tilde{\epsilon}$ should satisfy

$$(\tilde{\Gamma}^1\partial_1 + \tilde{\Gamma}^a\nabla_a)\tilde{\epsilon} = -\frac{1}{2}\epsilon. \tag{3.6}$$

One can easily see this by noting that $(\Gamma^1\partial_1 + \Gamma^a\nabla_a)\epsilon = 4\tilde{\epsilon}$ and so

$$\Box\,\epsilon = \frac{1}{4}(\tilde{\Gamma}^1\partial_1 + \tilde{\Gamma}^a\nabla_a)(\tilde{\Gamma}^1\partial_1 + \Gamma^b\nabla_b)\epsilon = \frac{1}{4}\Box\,\epsilon - \frac{R_{R\times S^3}}{16}\epsilon, \tag{3.7}$$

where $R_{R\times S^3}$ is the scalar curvature of $R\times S^3$ of radius $2/m$, which is $3m^2/2 = 6$. When one considers the Wick-rotated theory, this transformation is interpreted as supersymmetry. Thus, let us also call this fermionic symmetry "supersymmetry" for simplicity from now on. Especially, if the direction of the Wick-rotation is 1-direction, it forms $SU(2|4)$ group. ϵ is solved with the ansatz $\tilde{\epsilon} = \pm\frac{1}{2}\Gamma^{190}\epsilon$ and the result is

$$\epsilon_+ = \begin{pmatrix} e^{\frac{\tau}{2}}\ \eta_1 \\ e^{\frac{\tau}{2}}\bar{g}\ \eta_2 \\ e^{-\frac{\tau}{2}}\ \eta_3 \\ e^{-\frac{\tau}{2}}\bar{g}\ \eta_4 \end{pmatrix} \quad \text{and} \quad \epsilon_- = \begin{pmatrix} e^{-\frac{\tau}{2}}g\ \eta_1 \\ e^{-\frac{\tau}{2}}\ \eta_2 \\ e^{\frac{\tau}{2}}g\ \eta_3 \\ e^{\frac{\tau}{2}}\ \eta_4 \end{pmatrix}, \tag{3.8}$$

for the upper and the lower sign in the ansatz, respectively. Here, $\eta_1,\cdots,\eta_4$ are four-component constant spinors, and g and $\bar{g}$ are 4×4 matrices defined by

$$\begin{aligned}
g &= e^{\frac{\varphi}{2}J_4}e^{\frac{\theta}{2}J_3}e^{\frac{\psi}{2}J_4}, \\
\bar{g} &= e^{-\frac{\varphi}{2}\bar{J}_4}e^{-\frac{\theta}{2}\bar{J}_3}e^{-\frac{\psi}{2}\bar{J}_4},
\end{aligned} \tag{3.9}$$

where θ,φ,ψ are the coordinates of S^3. Hence, there are 32 supersymmetries in the theory.

The vacuum of the theory is unique up to gauge transformation because the holonomy of the gauge connection on S^3 is trivial. As we see in the next subsection, $SU(2|4)$ symmetric theories have many discrete vacua in contrast. Their vacua possess 16 supersymmetries because the components in (3.8) that depend on the coordinates of S^3 should vanish in the theories.

3.2 $SU(2|4)$ symmetric theories

3.2.1 $\mathcal{N}=4$ SYM on $R \times S^3/Z_k$

One obtains this theory by quotienting S^3 by Z_k along its fiber direction S^1. Thus, it is the theory quantized by the fields invariant under Z_k in $\mathcal{N}=4$ SYM on $R \times S^3$. The Z_k action on the coordinate of S^3 is written as $(\theta, \varphi, \psi) \to (\theta, \varphi, \psi + 4\pi/k)$. The form of its action is the same as (3.1) and (3.3), but the fields are restricted to Z_k invariant ones.

Its vacuum is labeled by holonomies because the holonomy is not unique due to the homotopy of S^3/Z_k: $\pi_1(S^3/Z_k) = Z_k$. Thus, it can be found that $U^k = 1$. If one takes a gauge in which $A_1 = 0$ and A_4 is diagonal and constant in a patch, the gauge fields in the vacuum take values of

$$\begin{aligned} &\hat{A}_2 = \frac{1 \mp \cos\theta}{\sin\theta} \cos\varphi\, \chi, \quad \hat{A}_3 = \frac{1 \mp \cos\theta}{\sin\theta} \sin\varphi\, \chi, \quad \hat{A}_4 = \pm\chi, \\ &\chi = 2\ \mathrm{diag}(\frac{0}{2}\mathbf{1}_{M_1}, \frac{1}{2}\mathbf{1}_{M_2}, \frac{2}{2}\mathbf{1}_{M_3}, \cdots, \frac{k-1}{2}\mathbf{1}_{M_k}), \end{aligned} \tag{3.10}$$

up to gauge transformation, where $\mp$ correspond to patch N: $(0 \le \theta < \pi)$ and patch S: $(0 < \theta \le \pi)$, respectively, and $\{M_i\}_{i=1,\cdots,k}$ is a set of integers satisfying $\sum_{i=1}^{k} M_i = N_{S^3}$. Hence, the explicit form of the holonomy can be written as a diagonal form:

$$\begin{aligned} U &:= P\exp\left[i \oint_0^{2\pi/k} dx^4\, \hat{A}_4 \right]_{\theta=0} \\ &= \mathrm{diag}(\mathbf{1}_{M_1}, e^{2\pi i/k}\mathbf{1}_{M_2}, e^{4\pi i/k}\mathbf{1}_{M_3}, \cdots, e^{2\pi i(k-1)/k}\mathbf{1}_{M_k}). \end{aligned} \tag{3.11}$$

3.2.2 $\mathcal{N}=8$ SYM on $R \times S^2$

This is obtained simply by dimensional reduction of $\mathcal{N}=4$ SYM on $R \times S^3$ along the fiber direction of $S^3 \to S^2$. Note that, before the dimensional reduction, the derivative with respect to the local Lorentz coordinates in the right-invariant frame can be written as

$$\begin{pmatrix} \partial_2 \\ \partial_3 \\ \partial_4 \end{pmatrix} = 2\vec{e}_\varphi \partial_\theta - 2\vec{e}_\theta \frac{1}{\sin\theta}\partial_\varphi + \frac{2}{\sin\theta}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \partial_\psi$$

$$= 2\vec{e}_r \times \left(\vec{e}_\theta \partial_\theta + \vec{e}_\varphi \frac{1}{\sin\theta}\partial_\varphi\right) - \frac{2}{\sin\theta}\begin{pmatrix} -(1 \mp \cos\theta)\cos\varphi \\ -(1 \mp \cos\theta)\sin\varphi \\ \mp\sin\theta \end{pmatrix} \partial_y, \tag{3.12}$$

where $\vec{e}_r = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $\vec{e}_\theta = \partial_\theta \vec{e}_r$ and $\vec{e}_\varphi = \partial_\varphi \vec{e}_r / \sin\theta$, and $\mp$ correspond to patch N: $(0 \le \theta < \pi)$ and patch S: $(0 < \theta \le \pi)$, respectively. The fiber direction is y, and the coordinates of base space, S^2, are θ and φ. Let us decompose the gauge fields into the base space part and the part orthogonal to it as

$$\begin{pmatrix} A_2 \\ A_3 \\ A_4 \end{pmatrix} = 2\vec{e}_r \times \left(\vec{e}_\theta a_\theta + \vec{e}_\varphi \frac{a_\varphi}{\sin\theta}\right) + 2\vec{e}_r \Phi, \quad \begin{pmatrix} \Gamma^2 \\ \Gamma^3 \\ \Gamma^4 \end{pmatrix} = \frac{1}{2}\vec{e}_r \times \left(\vec{e}_\theta \Gamma^\theta + \vec{e}_\varphi \sin\theta\, \Gamma^\varphi\right) + \frac{1}{2}\vec{e}_r \Gamma^y, \tag{3.13}$$

and substitute the above expressions and $A_1 = a_1$ into (3.3) of SYM on $R \times S^3$. Then one obtains

$$\begin{aligned}
S_{R\times S^2} = \frac{1}{4g_{S^2}^2}\int d\tau d\Omega_2 \,\mathrm{Tr}\Big(& 2(f_{1\theta})^2 + 2(f_{1\varphi})^2 + 8\left(\frac{f_{\theta\varphi}}{\sin\theta} - \Phi\right)^2 \\
&+ \frac{1}{2}(2D_1\Phi)^2 + \frac{1}{2}(4D_\theta\Phi)^2 + \frac{1}{2}\left(\frac{4}{\sin\theta}D_\varphi\Phi\right)^2 + \frac{1}{2}(D_1X_m)^2 \\
&+ \frac{1}{2}(2D_\theta X_m)^2 + \frac{1}{2}\left(\frac{2}{\sin\theta}D_\varphi X_m\right)^2 - \frac{1}{2}[2\Phi, X_m]^2 + \frac{1}{2}X_m X^m \\
&- \frac{1}{4}[X_m, X_n][X^m, X^n] + \frac{i}{2}\Psi\Gamma^1\partial_1\Psi + \frac{i}{2}\Psi\Gamma^\theta\partial_\theta\Psi + \frac{i}{2}\Psi\Gamma^\varphi\partial_\varphi\Psi \\
&+ \frac{3i}{8}\Psi\Gamma^{234}\Psi + \frac{1}{2}\Psi\Gamma^1[a_1, \Psi] + \frac{1}{2}\Psi\Gamma^\theta[a_\theta, \Psi] + \frac{1}{2}\Psi\Gamma^\varphi[a_\varphi, \Psi] \\
&+ \frac{1}{2}\Psi\Gamma^y[\Phi, \Psi] + \frac{1}{2}\Psi\Gamma^m[X_m, \Psi]\Big),
\end{aligned} \tag{3.14}$$

by dropping off y-dependence in every field because y is the fiber direction. Here, the field strength is defined as $f_{\theta\varphi} = \partial_\theta a_\varphi - \partial_\theta a_\varphi - i[a_\theta, a_\varphi]$ and $f_{1\{\theta,\varphi\}} = \partial_1 a_{\{\theta,\varphi\}} - \partial_{\{\theta,\varphi\}} a_1 - i[a_1, a_{\{\theta,\varphi\}}]$. We denote their matrix size by N_{S^2}.

If one takes a gauge in which $a_1 = 0$ and Φ is diagonal, it is easily found that the vacuum is

$$\begin{aligned}
&\hat{a}_\theta = 0, \quad \hat{a}_\varphi = -(\cos\theta \mp 1)\,\hat{\Phi}, \\
&\hat{\Phi} = \mathrm{diag}(q_1 \mathbf{1}_{N_1}, q_2 \mathbf{1}_{N_2}, \cdots, q_\Lambda \mathbf{1}_{N_\Lambda}),
\end{aligned} \tag{3.15}$$

where $\mp$ correspond to patch N and S, respectively, and $\{N_s\}_{s=1,\cdots,\Lambda}$ is a set of integers satisfying $\sum_{s=1}^{\Lambda} N_s = N_{S^2}$. Here, each of $\{q_s\}_{s=1,\cdots,\Lambda}$ is quantized so that $2q_s$ should be an integer, and these are understood as Dirac monopole charges. Λ is the number of different values of monopole charges. This theory describes a gauge theory in the background produced by monopoles sitting at the center of S^2.

If one replaces ∂_y in (3.12) by $-i\hat{\Phi}$, it turns out to be the background covariant derivative around a vacuum plus $-i\hat{\Phi}$:

$$2\vec{e}_r \times \left\{ \vec{e}_\theta(\partial_\theta - i\hat{a}_\theta) + \vec{e}_\varphi \frac{1}{\sin\theta}(\partial_\varphi - i\hat{a}_\varphi) \right\} + 2\vec{e}_r(-i\hat{\Phi}). \tag{3.16}$$

It implies that Φ realizes the fiber direction of $S^3 \to S^2$, which we will see later.

3.2.3 BMN model

One obtains BMN model by dimensionally reducing the whole of S^3 of $\mathcal{N} = 4$ SYM on $R \times S^3$, that is, by breaking $SU(2)_R$ entirely. Thus, this is a one-dimensional quantum mechanics whose action is

$$S = \frac{1}{g^2} \int d\tau \, \mathrm{Tr}\Big(\frac{1}{4} F_{MN}F^{MN} + \frac{1}{2} X_m X^m + \frac{i}{2} \Psi \Gamma^M D_M \Psi \Big), \tag{3.17}$$

where

$$\begin{aligned}
&F_{1M} = D_1 X_M = \partial_1 X_M - i[X_1, X_M] \quad (M \neq 1), \\
&F_{ab} = 2\varepsilon_{abc}X_c - i[X_a, X_b], \quad F_{am} = D_a X_m = -i[X_a, X_m], \quad F_{mn} = -i[X_m, X_n], \\
&D_1\Psi = \partial_1 \Psi - i[X_1, \Psi], \quad D_a\Psi = \frac{1}{4}\varepsilon_{abc}\Gamma^{bc}\Psi - i[X_a, \Psi], \quad D_m\Psi = -i[X_m, \Psi].
\end{aligned} \tag{3.18}$$

Now the gauge field is only X_1. X_a and X_m are $SO(3)$ and $SO(6)$ scalar fields, respectively. Ψ consists of 16 spinor fields, and is understood again as a 16-component Majorana-Weyl fermion in the $(9+1)$-dimensional viewpoint. The matrix size is denoted by N.

If one takes the temporal gauge, $X_1 = 0$, the generic form of vacua is

$$\begin{aligned}
\hat{X}_a = -2L_a &= -2 \begin{pmatrix} \mathbf{1}_{N_2^{(1)}} \otimes L_a^{[D_1]} & & & & \\ & \ddots & & & \\ & & \mathbf{1}_{N_2^{(s)}} \otimes L_a^{[D_s]} & & \\ & & & \ddots & \\ & & & & \mathbf{1}_{N_2^{(\Lambda)}} \otimes L_a^{[D_\Lambda]} \end{pmatrix} \\
&= -2 \bigoplus_{s=1}^{\Lambda} \mathbf{1}_{N_2^{(s)}} \otimes L_a^{[D_s]},
\end{aligned} \tag{3.19}$$

where L_a's are the $SU(2)$ generators. $L_a^{[D_s]}$'s stand for $SU(2)$ generators in the D_s-dimensional representation, whose spin is $j_s = (D_s - 1)/2$, and Λ is the number of different representations. $N_2^{(s)}$ stands for the multiplicity of the D_s-dimensional representation. For the comparison with the gravity side, we define $N_5^{(s)}$ in BMN model by

$$N_5^{(s)} = D_s - D_{s-1}, \qquad (s = 1, \cdots, \Lambda) \tag{3.20}$$

where $D_0 := 0$. Then, a set of those integers, $\{N_2^{(s)}, N_5^{(s)}\}_{s=1,\cdots,\Lambda}$, is chosen from all of the partitions of N satisfying $\sum_{s=1}^{\Lambda} N_2^{(s)} D_s = N$. Generally, they are block diagonal matrices and labeled by $SU(2)$ representations. Each block can be understood as a fuzzy sphere whose radius is proportional to its spin. The fuzzy spheres are layered sharing their centers, and they are considered as concentric M2-branes. Hence, $N_2^{(s)}$'s are interpreted as M2-brane charges placed apart from each other. On the other hand, D_s's are interpreted as M5-brane charges[14].

3.3 Large-N equivalences

3.3.1 $\mathcal{N} = 8$ SYM on $R \times S^2$ from BMN model

Let us see the realization of $\mathcal{N} = 8$ SYM on $R \times S^2$ from BMN model in the large-N. This is achieved by the continuum limit of fuzzy spheres. To see this large-N equivalence, One needs to compare spherical harmonics and the action of covariant derivatives on them in BMN model and those in SYM on $R \times S^2$.

In BMN model, the fluctuation fields are expanded by fuzzy spherical harmonics. If one takes a vacuum (3.19), the fluctuation fields around it are composed of rectangular block matrices. The block in the s-th row and t-th column (let us call this (s,t)-block) is a $(N_2^{(s)} \times N_2^{(t)}) \otimes (D_s \times D_t)$ matrix. Let us write a (s,t)-block as $X^{(s,t)}$, and then it can be expanded by fuzzy spherical harmonics $\hat{Y}_{Jm(j_s,j_t)}$ as

$$X^{(s,t)}(\tau) = \sum_{J=|j_s-j_t|}^{j_s+j_t} \sum_{m=-J}^{J} X_{Jm}^{(PW)(s,t)}(\tau) \otimes \hat{Y}_{Jm(j_s j_t)}, \tag{3.21}$$

where $X^{(s,t)}(\tau)$ represents the block in the s-th row and t-th column of the fluctuation field. Fuzzy spherical harmonics satisfy the following properties:

$$\begin{aligned}
(L_a \circ)^2 \hat{Y}_{Jm(j_s j_t)} =& J(J+1) \hat{Y}_{Jm(j_s,j_t)}, \\
L_\pm \circ \hat{Y}_{Jm(j_s j_t)} =& \sqrt{(J \mp m)(J \pm m + 1)} \hat{Y}_{Jm\pm 1(j_s,j_t)}, \\
L_4 \circ \hat{Y}_{Jm(j_s j_t)} =& m \hat{Y}_{Jm(j_s,j_t)},
\end{aligned} \tag{3.22}$$

where $L_\pm = L_2 \pm iL_3$ and L_a's act on a $D_s \times D_t$ rectangular matrix $M^{(s,t)}$ as

$$L_a \circ M^{(s,t)} = L_a^{[D_s]} M^{(s,t)} - M^{(s,t)} L_a^{[D_t]}. \tag{3.23}$$

More explicitly, it realizes by taking a set of fuzzy spherical harmonics as

$$\hat{Y}_{Jm(j_s j_t)} = (D_s D_t)^{\frac{1}{4}} \sum_{m_s, m_t} (-)^{-j_s+m_t} C^{Jm}_{j_s m_s j_t m_t} |j_s m_s\rangle\langle j_t m_t|, \tag{3.24}$$

where $C^{Jm}_{j_s m_s j_t m_t}$'s are the Clebsch-Gordan coefficients. They satisfy the following equations: hermitian conjugates

$$(\hat{Y}_{Jm(j_s j_t)})^\dagger = (-)^{m-(j_s-j_t)} \hat{Y}_{J-m(j_t j_s)}, \tag{3.25}$$

and the orthogonality relation

$$\frac{1}{\sqrt{D_s D_t}} \mathrm{tr} \left\{ (\hat{Y}_{Jm(j_s j_t)})^\dagger \hat{Y}_{J'm'(j_s j_t)} \right\} = \delta_{J,J'} \delta_{m,m'}. \tag{3.26}$$

tr is taken as the trace of $D_t \times D_t$ matrices.

Around a vacuum in SYM on $R \times S^2$, there are Dirac monopoles. The fluctuation fields consist of rectangular block matrices and each of them is expanded by monopole spherical harmonics. The (s,t)-block is $N_s \times N_t$ matrix and function on $\tau \in R$ and $(\theta, \varphi) \in S^2$. In the case in which a monopole background exists, it can be expanded by a basis of the space of functions on S^2 as

$$X^{(s,t)}(\tau, \theta, \varphi) = \sum_{J=|q_s-q_t|}^{\infty} \sum_{m=-J}^{J} X^{(S^2)(s,t)}_{Jm}(\tau) Y_{Jm(q_s-q_t)}(\theta, \varphi). \tag{3.27}$$

Here, $X^{(s,t)}(\tau, \theta, \varphi)$ represents the block in the s-th row and t-th column of the fluctuation field in SYM on $R \times S^2$. This basis $Y_{Jmq}(\theta, \varphi)$ is called monopole spherical harmonics, which satisfy

$$\begin{aligned} (L_a^{(q)})^2 Y_{Jmq} &= J(J+1) Y_{Jmq}, \\ L_\pm^{(q)} Y_{Jmq} &= \sqrt{(J \mp m)(J \pm m + 1)} Y_{Jm\pm1q}, \\ L_4^{(q)} Y_{Jmq} &= m Y_{Jmq}. \end{aligned} \tag{3.28}$$

$L_a^{(q)}$'s are angular momentum operators with a monopole background, defined to satisfy

$$\begin{pmatrix} L_2^{(q)} \\ L_3^{(q)} \\ L_4^{(q)} \end{pmatrix} = -i\vec{e}_r \times \left(\vec{e}_\theta \partial_\theta + \vec{e}_\varphi \frac{1}{\sin\theta} \partial_\varphi \right) + \frac{q}{\sin\theta} \begin{pmatrix} -(1 \mp \cos\theta)\cos\varphi \\ -(1 \mp \cos\theta)\sin\varphi \\ \mp \sin\theta \end{pmatrix} \tag{3.29}$$

where $\mp$ correspond to patch N and S, respectively. They also satisfy the equations for complex conjugates

$$(Y_{Jmq})^* = (-1)^{m-q} Y_{J-m-q}, \tag{3.30}$$

and the orthogonality relation

$$\int \frac{d\Omega_2}{4\pi} (Y_{Jmq})^* Y_{J'm'q} = \delta_{JJ'}\delta_{mm'}. \tag{3.31}$$

From the above, it is found that the both theories have the same spectra of the fluctuations and the same action of background covariant derivatives around the vacuum on them if one takes the limit in which

$$2j_s + 1 = n + 2q_s, \quad n \to \infty \quad \text{with} \quad \frac{g^2}{n} = \frac{g_{S^2}^2}{4\pi} = \text{fixed} \tag{3.32}$$

and makes the identification that consists of $N_2^{(s)} = N_s$, $X_{Jm}^{(PW)(s,t)} \to X_{Jm}^{(S^2)(s,t)}$, $L_a \circ \to L_a^{(q_s - q_t)}$, $\hat{Y}_{Jm(j_s j_t)} \to Y_{Jm(q_s-q_t)}(\theta,\varphi)$, and $\mathrm{tr}\,/n \to \int d\Omega_2/(4\pi)$. This makes $|j_s - j_t| = |q_s - q_t|$ and $j_s + j_t \to \infty$, and so the spectra on the both sides get equivalent. It can be also proved that the interaction terms in BMN model and those in SYM on $R \times S^2$ are also equivalent to each other. Therefore, if one writes the both actions by the coefficients of spherical harmonics expansion, the action of BMN model around a vacuum, (3.19), in the large-N limit of (3.32) is the same as that of SYM on $R \times S^2$ around a vacuum, (3.15). It means that BMN model in that large-n limit realizes SYM on $R \times S^2$ around any vacua with N_{S^2} finite. This is true because these are massive theories and so there is no flat direction, which spontaneously breaks symmetry like $U(1)^d$.

3.3.2 $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ from $\mathcal{N} = 8$ SYM on $R \times S^2$ or BMN model

There are two ways to reproduce SYM on $R \times S^3/Z_k$ from lower dimensional theories. They are Taylor's T-duality[17, 16] and the large-N reduction[21, 20]. While the large-N reduction realizes SYM on $R \times S^3/Z_k$ in large-N_{S^3}, Taylor's T-duality realizes it with N_{S^3} finite. In this sense, Taylor's duality has an advantage than the large-N reduction. However, Taylor's T-duality demands the matrix size of the lower dimensional theories infinity from the beginning while the large-N reduction does not.

Taylor's T-duality

Let us see the realization of SYM on $R \times S^3/Z_k$ from SYM on $R \times S^2$ first. In this scheme, Λ in SYM on $R \times S^2$ should be infinite at the beginning, and so we make the label for blocks, s, run from $-\infty$ to ∞. To reproduce nontrivial vacua of SYM on $R \times S^3/Z_k$, one should relabel blocks from s to s and α with α running from 1 to k; for monopole charges, q_s should be replaced to $q_s^{(\alpha)}$ and N_s to $N_s^{(\alpha)}$.

In order to obtain SYM on $R\times S^3/Z_k$, let us set the vacuum in SYM on $R\times S^2$ to be

$$q_s^{(\alpha)}=\frac{ks}{2}+\frac{\alpha-1}{2},\quad N_s^{(\alpha)}=M_\alpha \qquad \text{for}\quad -\infty\leq s\leq\infty,\quad 1\leq\alpha\leq k. \tag{3.33}$$

Thus, $\hat{\Phi}$ is written as

$$\hat{\Phi}=\mathrm{diag}(\cdots,\frac{ks}{2}\mathbf{1}_{M_1},\cdots,\frac{ks+k-1}{2}\mathbf{1}_{M_k},\frac{k(s+1)}{2}\mathbf{1}_{M_1},\cdots,\frac{k(s+1)+k-1}{2}\mathbf{1}_{M_k},\cdots). \tag{3.34}$$

Let us make some blocks in the fluctuation fields around the vacuum are identified with each other as

$$X^{(s\,\alpha,t\,\beta)}(\tau,\theta,\varphi)=X^{(s+1\,\alpha,t+1\,\beta)}(\tau,\theta,\varphi)=:X_{\alpha\beta}^{(s-t)}(\tau,\theta,\varphi)\quad\text{for}\ -\infty<{}^\forall s,{}^\forall t<\infty, \tag{3.35}$$

It is a $M_\alpha\times M_\beta$ block matrix, and we denote the $(\sum_{\alpha=1}^k M_\alpha)\times(\sum_{\alpha=1}^k M_\alpha)$ matrix whose elements are those block matrices by $X^{(s-t)}(\tau,\theta,\varphi)$. Fluctuation field $X(\tau,\theta,\varphi)$ can be expanded as

$$X(\tau,\theta,\varphi)=\sum_{w=-\infty}^{\infty}X^{(S^2)(w)}(\tau,\theta,\varphi)\otimes U^w, \tag{3.36}$$

where U is defined as the infinite dimensional matrix satisfying $U_{st}=\delta_{s-t,1}$. Condition (3.35) is called orbifolding condition. If we denote

$$\sum_{w=-\infty}^{\infty}X_{\alpha\beta}^{(S^2)(w)}(\tau,\theta,\varphi)\otimes U^w \tag{3.37}$$

by $X^{(\alpha,\beta)}(\tau,\theta,\varphi)$, the (α,β)-block of the commutator of $\hat{\Phi}$ and a fluctuation field turns out to be

$$-i[\hat{\Phi},X]^{(\alpha,\beta)}=-i\sum_{w=-\infty}^{\infty}\frac{kw+\alpha-\beta}{2}X_{\alpha\beta}^{(S^2)(w)}(\tau,\theta,\varphi)\otimes U^w. \tag{3.38}$$

Then, let us see the fluctuation fields in SYM on $R\times S^3/Z_k$ around the vacuum (3.10). For each patch of S^2, the expression of the (α,β)-block of a fluctuation in a Fourier series expansion along the fiber direction, y, is generally written as

$$X(\tau,\theta,\varphi,y)^{(\alpha,\beta)}=\sum_{w=-\infty}^{\infty}X_{\alpha\beta}^{(S^3)(w)}(\tau,\theta,\varphi)e^{i\frac{kw}{2}y}, \tag{3.39}$$

because all fields are assumed to be periodic under $y \to y + 4\pi/k$. The action of the background covariant derivative along y around the vacuum for the w-th component of the Fourier series of the (α, β)-block is

$$\Big[\partial_y X - i[\frac{\chi}{2}, X]\Big]^{(\alpha,\beta;w)} = \frac{kw + \alpha - \beta}{2} X^{(S^3)(w)}_{\alpha\beta}(\tau, \theta, \varphi). \tag{3.40}$$

Now one can see that the spectra of the fluctuations on the both sides are the same if one makes the identification that consists of the relation (3.33), $X^{(S^2)(w)}_{\alpha\beta} \to X^{(S^3)(w)}_{\alpha\beta}$, $L_a^{(\frac{kw+\alpha-\beta}{2})} \to [\partial_a - i\hat{A}_a]^{(\alpha,\beta;w)}$, $U \to e^{i\frac{kw}{2}y}$. Also, the trace for U should be identified with $(k/4\pi)\int dy$, and

$$\frac{2\pi g^2_{S^2}}{k} = g^2_{S^3}. \tag{3.41}$$

There is no divergence of $\mathrm{tr}\,\mathbf{1}$ because it should be divided due to the orbifolding condition. Then, it appears in comparing the both free energies: $F_{S^2}/\,\mathrm{tr}\,\mathbf{1} = F_{S^3}$. By the above identification, the actions on the both sides, (3.14) and (3.3), become the same, and so SYM on $R \times S^2$ around the vacuum (3.15) with (3.33) and the orbifolding condition is equivalent to SYM on $R \times S^3/Z_k$ around the vacuum (3.10) because there is no flat direction.

One can obtain SYM on $R\times S^3/Z_k$ from BMN model by taking the procedure explained in section 3.3.1 and Taylor's T-duality successively. The relations for parameters are

$$\begin{aligned} &D^{(\alpha)}_s = n + ks + \alpha - 1, \quad n \to \infty \quad \text{with} \quad \frac{8\pi^2 g^2}{kn} = g^2_{S^3}, \\ &N^{(s\,\alpha)}_2 = M_\alpha. \end{aligned} \tag{3.42}$$

Large-N reduction

SYM on $R \times S^3/Z_k$ can be reproduced also by the usual large-N reduction. In order to obtain it, the vacuum in SYM on $R \times S^2$ should be set as

$$q^{(\alpha)}_s = \frac{ks}{2} + \frac{(\alpha-1)s}{2}, \qquad N^{(\alpha)}_s = M_\alpha \qquad \text{for} \quad -\frac{\Lambda}{2} \le s \le \frac{\Lambda}{2}. \tag{3.43}$$

Then, the double scaling limit which realizes SYM on $R \times S^3$ is

$$\Lambda \to \infty, \quad M_\alpha \to \infty, \quad \text{with} \quad \frac{2\pi g^2_{S^2} M_\alpha}{k} = g^2_{S^3} M_\alpha = \text{fixed} \quad \text{for} \;\; 1 \le \alpha \le k. \tag{3.44}$$

The contribution to the correlation functions in SYM on $R\times S^2$ or in SYM on $R\times S^3/Z_k$ is only from the planar diagrams in this limit. By the perturbative analysis, one can find that the planar diagrams in SYM on $R\times S^2$ and those in SYM on $R\times S^3/Z_k$ provide the same results. For the non-perturbative equivalence, we can see it in a BPS sector [1] by using the result of the localization method.

The realization of SYM on $R\times S^3$ from BMN model also requires the procedure in section 3.3.1 in addition to the large-N reduction. The double scaling limit is

$$\begin{aligned}
&D_s^{(\alpha)}=n+ks+\alpha-1, \quad n\to\infty,\\
&N_2^{(s\,\alpha)}=M_\alpha\to\infty \quad \text{with} \quad \frac{8\pi^2 g^2 M_\alpha}{kn}=g_{S^3}^2 M_\alpha=\text{fixed} \quad \text{for each } \alpha,
\end{aligned} \tag{3.45}$$

and then $\Lambda\to\infty$ should be taken.

3.4 Localization applied to BMN model

In this section, localization technique is applied to a BPS sector in BMN model. The BPS sector should be suitable for observing the correspondence between the vacua in BMN model and bubbling geometries.

The localization technique works if the following conditions hold:

1. There is a fermionic symmetry, say Q: $QS=0$, where S is the action.
2. There is a Q-exact quantity which can be added to the action and the bosonic part of which contains only positive-definite terms. If it is denoted by QV, it satisfies $Q^2V=0$.
3. The path-integral measure is closed[5].

Note that there is no need for nilpotency of Q. This Q is usually supersymmetry or the combination of supersymmetry and BRS symmetry. In order to compute a Q-closed operator, say $\mathcal{O}$, let us define a modified partition function $Z(t)$ and the vacuum expectation value of $\mathcal{O}$ in terms of $Z(t)$ as

$$Z(t):=\int[dX]e^{-S[X]-tQV}, \quad \langle\mathcal{O}\rangle=\int[dX]\mathcal{O}e^{-S[X]-tQV}/Z(t). \tag{3.46}$$

Then,

$$\frac{d}{dt}(\langle\mathcal{O}\rangle Z(t))=\int[dX]\mathcal{O}(QV)e^{-S[X]-tQV}=\int[dX](Q\,\mathcal{O}Ve^{-S[X]-tQV})$$

[5]This condition is imposed because we assume that the deformation term in (3.46) is QV. For example, if it is not satisfied and there is a functional U such that the path-integral measure times U is Q-closed, then the condition 2 should be modified to the statement that the bosonic terms of $U\times QV$ are positive definite.

$$= Q(\int [dX]\mathcal{O}Ve^{-S[X]-tQV}) = 0. \tag{3.47}$$

Taking $\mathcal{O}$ as 1, it is found that $Z(t)$ does not depend on t, nor does $\langle\mathcal{O}\rangle$ for any Q-closed $\mathcal{O}$. The original value of $\langle\mathcal{O}\rangle$ is computed by $Z(0)$, but its value computed by $Z(t)$ is the same for any t since it is independent of t. Therefore, $\langle\mathcal{O}\rangle$ can be computed also by $Z(t)$ at $t \to \infty$, which means that the fields in the Q-closed sector are localized around those satisfying $QV = 0$. Thus, the computation reduces to gaussian integrals around the localizing locus, and so it becomes exact at one-loop.

In general, localization technique is usually applied to theories on a compact space, but the theory we are analyzing is non-compact — real axis τ. As this non-compact space, the real axis, does not have finite volume, the partition function of BMN model on the real axis seems to have an infinite value. In order to make the effective action finite, it is necessary to impose on the fields that they are closer to vacua at $\tau \to \pm\infty$. By imposing it, one obtains a finite result because the fields in the BPS sector localize around the origin of the real axis, thanks to the τ-dependence of the Killing spinor of the BPS sector.

3.4.1 Determination of the BPS sector

We need to determine the appropriate BPS sector because it is emergence of geometries that we are interested in, in this paper.

A supersymmetry of BMN model is obtained by the simple dimensional reduction of (3.4) with (3.8).

$$\begin{aligned}
\delta_s X_M &= -i\Psi\Gamma_M\epsilon,\\
\delta_s \Psi &= \frac{1}{2}F_{MN}\Gamma^{MN}\epsilon - 2X_m\tilde{\Gamma}^m\tilde{\epsilon}.
\end{aligned} \tag{3.48}$$

The Killing spinor equation is now written as

$$\nabla_1\epsilon = \tilde{\Gamma}_1\tilde{\epsilon}, \quad \frac{1}{4}\varepsilon_{abc}\Gamma^{bc}\epsilon = \tilde{\Gamma}_a\tilde{\epsilon}. \tag{3.49}$$

Its solution is also obtained by the dimensional reduction of (3.8). Thus, $\eta_2 = \eta_4 = 0$ for ϵ_+ and $\eta_1 = \eta_3 = 0$ for ϵ_-. From now on, we concentrate on ϵ_+ and omit the subscript $+$. Then, let us set $\eta_3 = -J_4\eta_1$ so that $\delta_s(X_4 - iX_{10}) = 0$ would hold at $\tau = 0$. Now the Killing spinor is

$$\epsilon = e^{\frac{\tau}{2}\Gamma^{09}}e^{-\frac{\pi}{4}\Gamma^{49}}\begin{pmatrix}\eta_1\\0\\0\\0\end{pmatrix}. \tag{3.50}$$

We take $\eta_1 = (1,0,0,0)^T$ for simplicity. The Killing vector $v^M := \epsilon\Gamma^M\epsilon$ has the following components

$$v^{10} = 2i\cosh\tau, \quad v^4 = -2, \quad v^9 = 2\sinh\tau, \tag{3.51}$$

and the other elements are 0. There is a supersymmetric invariant operator defined by

$$\phi(\tau) := v^M X_M(\tau). \tag{3.52}$$

In the BPS sector determined by (3.50), ϕ becomes

$$\phi(\tau) = -2\left(X_4(\tau) - \sinh\tau X_9(\tau) - i\cosh\tau X_{10}(\tau)\right). \tag{3.53}$$

The condition that $\delta_s(X_4(0) - iX_{10}(0)) = 0$ or $\delta_s\phi(\tau) = 0$ can be understood as follows. In order to observe emergence of geometries in the context of the gauge/gravity duality, the classical configuration of the operators in BMN model corresponding to r- and z-directions should be determined by the equation of motion in the limit where the supergravity approximation is valid. $\phi(\tau)$ corresponds to the linear combination of r- and z-directions because X_4 is naively the direction perpendicular to S^2 as well as S^5 and so is $\sinh\tau X_9 + i\cosh\tau X_{10}$. Then, the configuration of $\phi(\tau)$ in that limit is considered to describe or have the information of how the metric of the corresponding geometry depends on r and z. This is why we are interested in the BPS sector satisfying this condition, since the equation of motion of ϕ is determined by the localization method in the sector.

The localization method requires off-shell supersymmetry. Following Berkovits' construction [23], the off-shell supersymmetry of BMN model is found that

$$\begin{aligned}
&\delta_s X_M = -i\Psi\Gamma_M\epsilon, \\
&\delta_s\Psi = \frac{1}{2}F_{MN}\Gamma^{MN}\epsilon - X_m\tilde{\Gamma}^m\Gamma^{190}\epsilon + K^i\nu_i, \\
&\delta_s K_i = i\nu_i\Gamma^M D_M\Psi,
\end{aligned} \tag{3.54}$$

with auxiliary fields K_i and additional bosonic spinors ν_i, which satisfy

$$\begin{aligned}
&\epsilon\Gamma^M\nu_i = 0, \\
&\frac{1}{2}(\epsilon\Gamma_N\epsilon)\tilde{\Gamma}^N_{\alpha\beta} = \nu^i_\alpha\nu^i_\beta + \epsilon_\alpha\epsilon_\beta, \\
&\nu_i\Gamma^M\nu_j = \delta_{ij}\epsilon\Gamma^M\epsilon.
\end{aligned} \tag{3.55}$$

Here, α, β are spinor indices, and i runs from 1 to 7 so as to equate the number of bosonic fields with a gauge fixing and that of fermionic fields. The action of BMN model is invariant under this symmetry if one adds to the action

$$-\frac{1}{g^2}\int d\tau\frac{1}{2}\mathrm{Tr}K_iK_i. \tag{3.56}$$

If ϵ satisfies (3.50), then ν_i is in the form of

$$\nu_i = \sqrt{2} e^{\frac{\pi}{2}\Gamma^{09}} e^{-\frac{\pi}{4}\Gamma^{49}} \Gamma^{i8} \begin{pmatrix} \eta_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (i = 1, 2, \cdots, 7) \tag{3.57}$$

Let us see another form of the supersymmetry by expanding fermionic fields as

$$\Psi = \Psi_{M'} \Gamma^{M'} \epsilon + \Upsilon_i \nu^i, \tag{3.58}$$

by the complete basis $\{\Gamma^{M'}\epsilon, \nu^i | M' = 1, \cdots, 9,\ i = 1, \cdots, 7\}$. Then the supersymmetry is written as

$$\begin{aligned} &\delta_s X_{M'} = -i(\epsilon\epsilon)\Psi_{M'}, \quad (\epsilon\epsilon)\delta_s \Psi_{M'} = (\delta_\phi + \delta_{U(1)})X_{M'}, \\ &(\epsilon\epsilon)\delta_s \Upsilon_i =: H_i, \qquad\quad \delta_s H_i = -i(\epsilon\epsilon)(\delta_\phi + \delta_{U(1)})\Upsilon_i, \quad \delta_s \phi = 0, \end{aligned} \tag{3.59}$$

where H_i is defined as

$$H_i = (\epsilon\epsilon) K_i + 2i\nu_i \tilde{\epsilon} X_{10} + s_i, \tag{3.60}$$

$$s_i := \nu_i \left(\frac{1}{2} \sum_{P',Q'=1}^{9} F_{P'Q'} \Gamma^{P'Q'} \epsilon - 2 \sum_{m=5}^{9} X_m \Gamma^m \tilde{\epsilon} \right). \tag{3.61}$$

δ_ϕ is the gauge transformation whose parameter is ϕ: $\delta_\phi X_M = D_M \phi$, $\delta_\phi \Upsilon = i[\phi, \Upsilon]$. $\delta_{U(1)}$ is $U(1)$ transformation which is a diagonal subgroup of $SO(3) \times SO(6)$:

$$\begin{aligned} &\delta_{U(1)} X_a = -2\varepsilon_{ab4} v^4 X^b, \\ &\delta_{U(1)} X_m = 2(-\delta_m^5 X_8 + \delta_m^8 X_5 - \delta_m^7 X_6 + \delta_m^6 X_7), \\ &\delta_{U(1)} \Upsilon_i = 2(\delta_{i1}\Upsilon_4 + \delta_{i2}\Upsilon_3 - \delta_{i3}\Upsilon_2 - \delta_{i4}\Upsilon_1 + \delta_{i6}\Upsilon_7 - \delta_{i7}\Upsilon_6). \end{aligned} \tag{3.62}$$

This $U(1)$ transformation corresponds to the linear combination of the Lie derivative along v^a on S^3 and $SU(2)$ subgroup of R-symmetry in the viewpoint of $\mathcal{N} = 4$ SYM. The explicit form of s_i is

$$\begin{aligned} s_1 &= 2c(F_{18} + F_{27} - F_{36}) + 2s(X_8 - F_{45}) + 2F_{59}, \\ s_2 &= 2c(-F_{17} + F_{28} + F_{35}) - 2s(X_7 + F_{46}) + 2F_{69}, \\ s_3 &= 2c(F_{16} - F_{25} + F_{38}) + 2s(X_6 - F_{47}) + 2F_{79}, \\ s_4 &= 2c(F_{15} + F_{26} + F_{37}) + 2s(X_5 + F_{48}) - 2F_{89}, \\ s_5 &= 2c(F_{23} + F_{58} - F_{67}) - 2sF_{14} - 2F_{19}, \end{aligned}$$

$$s_6 = 2c(-F_{13} + F_{57} + F_{68}) - 2sF_{24} - 2F_{29},$$
$$s_7 = 2c(F_{12} + F_{78} - F_{56}) - 2sF_{34} - 2F_{39}. \tag{3.63}$$

Now it is easy to see that δ_s^2 is the linear combination of gauge symmetry and $U(1)$ symmetry as

$$\delta_s^2 = -i(\delta_\phi + \delta_{U(1)}). \tag{3.64}$$

One can write the supersymmetry in a compact form by the following notation:

$$X := \begin{pmatrix} X_{M'} \\ (\epsilon\epsilon)\Upsilon_i \end{pmatrix}, \quad X' := \begin{pmatrix} -i(\epsilon\epsilon)\Psi_{M'} \\ H_i \end{pmatrix}.$$

Then, (X, X') forms doublets, and ϕ is a singlet. They transform as

$$\delta_s X = X', \quad \delta_s X' = -i(\delta_\phi + \delta_{U(1)})X, \quad \delta_s \phi = 0. \tag{3.65}$$

3.4.2 BRS symmetry and combined symmetry Q

It is convenient to fix a gauge by the BRS-quantization method so as to perform the localization technique. Thus, let us introduce ghost fields transforming under the BRS symmetry as

$$\begin{aligned}
&\delta_B X = -[C, X]_\mp, \quad \delta_B X' = -[C, X']_\mp, \\
&\delta_B C = a_0 - C^2, \quad \delta_B \phi = -[C, \phi], \\
&\delta_B \tilde{C} = b, \qquad \delta_B b = -[a_0, \tilde{C}], \\
&\delta_B \tilde{a}_0 = i\tilde{C}_0, \qquad \delta_B \tilde{C}_0 = i[a_0, \tilde{a}_0], \\
&\delta_B b_0 = iC_0, \qquad \delta_B C_0 = i[a_0, b_0], \quad \delta_B a_0 = 0,
\end{aligned} \tag{3.66}$$

where $[C,\ ]_\mp$ stands for the anti-commutator if the both fields are fermionic. Otherwise, it stands for $-iD_M C$ for X_M and the commutator, $[C, \Upsilon]$, for Υ. $C, \tilde{C}$ are the usual ghost and anti-ghost fields, and b is the Nakanishi-Lautrup field. Since C and $\tilde{C}$ have zero modes, a_0 is introduced as a ghost of ghost C, and other bosonic fields $b_0, \tilde{a}_0$ and fermionic fields $C_0, \tilde{C}_0$ are also introduced. The fields with subscript 0 are constant and zero modes for fuzzy spherical harmonics. δ_B^2 is the gauge transformation parameterized by the constant field a_0:

$$\delta_B^2 = -[a_0,\ ]. \tag{3.67}$$

We use the combined symmetry which consists of the supersymmetry and the BRS symmetry. The supersymmetry on the ghost fields and the fields accompanying them is defined as

$$\delta_s C = \phi, \quad \delta_s(\text{the other ghosts}) = 0. \tag{3.68}$$

Then, the combined symmetry $Q := \delta_s + \delta_B$ is

$$\begin{aligned}
&QX = X' - [C, X]_{\mp}, \quad QX' = -i(\delta_\phi + \delta_{U(1)})X - [C, X']_{\mp}, \\
&QC = \phi + a_0 - C^2, \quad Q\phi = -[C, \phi], \\
&Q\tilde{C} = b, \qquad\qquad\quad Qb = -[a_0, \tilde{C}], \\
&Q\tilde{a}_0 = i\tilde{C}_0, \qquad\qquad Q\tilde{C}_0 = i[a_0, \tilde{a}_0], \\
&Qb_0 = iC_0, \qquad\qquad QC_0 = i[a_0, b_0], \quad Qa_0 = 0.
\end{aligned} \tag{3.69}$$

One can find that Q^2 is $U(1)$ transformation plus the gauge transformation by a_0:

$$Q^2 = \mathcal{R}, \quad \mathcal{R} := -i\delta_{U(1)} - [a_0, \ ]. \tag{3.70}$$

One can write Q-symmetry in a compact form again. After redefining

$$\tilde{X}' := X' - [C, X]_{\mp}, \quad \tilde{\phi} := \phi + a_0 - C^2, \tag{3.71}$$

let us define

$$\begin{aligned}
&Z_0 = (X_{M'}, \tilde{a}_0, b_0), \quad Z_1 = (\Upsilon_i, C, \tilde{C}), \\
&Z_0' = (\tilde{\Psi}_{M'}, \tilde{C}_0, C_0), \quad Z_1' = (\tilde{H}_i, \tilde{\phi}, b).
\end{aligned} \tag{3.72}$$

Note that Z_0 and Z_1' are bosonic and Z_1 and Z_0' are fermionic. Then, (Z_i, Z_i') forms doublets, and Q-symmetry is written as

$$QZ_i = Z_i', \quad QZ_i' = \mathcal{R}Z_i, \tag{3.73}$$

where $i = 0, 1$.

3.4.3 Positive definite deformation

To computing BPS operators in BMN model, we use δ_s-symmetry for the localization technique. Then, one needs a positive definite deformation term for the action which is δ_s-exact. Let us define V_{matt} by

$$V_{matt} = \mathrm{Tr}\left[\Psi \overline{\delta_s \Psi}\right], \tag{3.74}$$

where

$$\overline{\delta_s \Psi} = \frac{1}{2} F_{MN} \tilde{\Gamma}^{MN} \epsilon + \frac{1}{2} X_m \tilde{\Gamma}^{am} \nabla_a \epsilon - K^i \nu_i. \tag{3.75}$$

Here, the bar makes the operator hermitian conjugate with K_i treated as anti-hermitian and others including X_{10} as hermitian. Actually, K_i should be anti-hermitian for the convergence of the path-integral. In the following, let us use $K_i^{(E)} = -iK_i$. The explicit form of V_{matt} is

$$V_{matt} = \mathrm{Tr}\left[\left(D_{M'} \bar{\phi} + \delta_{U(1)} X_{M'} \right) \Psi^{M'} + \bar{H}^i \Upsilon_i \right]. \tag{3.76}$$

The δ_s-exact term of this is the same as QV_{matt}. The bosonic part of it is calculated as

$$\begin{aligned}
QV_{matt}|_{bos} &= \delta_s V_{matt}|_{bos} \\
&= \mathrm{Tr}\Bigg[e^{\tau}(D_1 X_{10} + X_{10} - e^{-\tau} K_5^{(E)})^2 + e^{-\tau}(D_1 X_{10} - X_{10} + e^{\tau} K_5^{(E)})^2 \\
&\quad + 2c \sum_{a=2}^{4} (D_a X_{10})^2 + 2c \sum_{i \neq 5} (K^{(E)i})^2 + 2c(D_4 X_9)^2 - 2c[X_{10}, X_9]^2 \\
&\quad - 2c \sum_{m'=5}^{8} [X_{10}, X_{m'}]^2 + \mathcal{S} + 4 \sum_{a'=1}^{3} \Bigg[e^{-\tau} \left\{ F^+_{a'4} - \frac{1}{2} D_{a'}(e^{\tau} X_9) + F^+_{a'+4,8} \right\}^2 \\
&\quad + e^{\tau} \left\{ F^-_{a'4} + \frac{1}{2} D_{a'}(e^{-\tau} X_9) - F^-_{a'+4,8} \right\}^2 \Bigg] \Bigg],
\end{aligned} \tag{3.77}$$

where c stands for $\cosh \tau$, and we will use s for $\sinh \tau$ later. We define $\mathcal{S}$ by

$$\begin{aligned}
\mathcal{S} = & \, e^{\tau}(X_5 + D_1 X_5 + D_2 X_6 + D_3 X_7 + D_4 X_8 + e^{-\tau} F_{98})^2 \\
& + e^{-\tau}(X_5 - D_1 X_5 - D_2 X_6 - D_3 X_7 + D_4 X_8 - e^{\tau} F_{98})^2 \\
& + e^{\tau}(X_6 + D_1 X_6 - D_2 X_5 + D_3 X_8 - D_4 X_7 - e^{-\tau} F_{97})^2 \\
& + e^{-\tau}(X_6 - D_1 X_6 + D_2 X_5 - D_3 X_8 - D_4 X_7 + e^{\tau} F_{97})^2 \\
& + e^{\tau}(X_7 + D_1 X_7 - D_2 X_8 - D_3 X_5 + D_4 X_6 + e^{-\tau} F_{96})^2 \\
& + e^{-\tau}(X_7 - D_1 X_7 + D_2 X_8 + D_3 X_5 + D_4 X_6 - e^{\tau} F_{96})^2 \\
& + e^{\tau}(X_8 + D_1 X_8 + D_2 X_7 - D_3 X_6 - D_4 X_5 - e^{-\tau} F_{95})^2 \\
& + e^{-\tau}(X_8 - D_1 X_8 - D_2 X_7 + D_3 X_6 - D_4 X_5 + e^{\tau} F_{95})^2,
\end{aligned} \tag{3.78}$$

and $F^{\pm}$ by the selfdual and anti-selfdual parts of the field strength:

$$F^{\pm}_{a'b'} = \frac{1}{2}(F_{a'b'} \pm \frac{1}{2} \varepsilon_{a'b'c'd'} F^{c'd'}), \quad F^{\pm}_{m'n'} = \frac{1}{2}(F_{m'n'} \pm \frac{1}{2} \varepsilon_{m'n'p'q'} F^{p'q'}), \tag{3.79}$$

where $a', b', \cdots$ run from 1 to 4 and $m', n', \cdots$ from 5 to 8. These bosonic terms are positive definite if one treats $K_i^{(E)}$ as hermitian.

The localizing locus is obtained as the solution of $QV_{matt} = 0$. It is equivalent to the equations where each term of QV_{matt} is zero. In general, it is not so easy to solve it because there should be instanton configurations [24, 25] which extremize the Q-exact term, due to $e^{\pm\tau}$ factors in (3.77) and (3.78). The equations for obtaining instantons can be understood as mass-deformed Nahm equations [26]. The instanton effect is suppressed in the large-N limit, and so it is not considered in this paper. Then, the localizing configuration should become vacua at $\tau \to \pm\infty$, and furthermore, these both edges of the configuration should be the same. In this condition, the localizing locus in the temporal gauge turns out to be

$$\hat{X}_a = -2L_a \ (a = 2, 3, 4), \quad \hat{X}_{10} = \frac{M}{c}, \quad \hat{K}_5^{(E)} = \frac{M}{c^2}, \tag{3.80}$$

where M is a constant matrix which satisfies $[L_a, M] = 0$. These equations, of course, satisfy the condition mentioned above. Note that $\hat{\phi} = 4L_4 + 2iM$, which we will use later. M is decomposed, in the same manner as L_a in (3.19), as

$$M = \bigoplus_{s=1}^{\Lambda} M_s \otimes \mathbf{1}_{D_s}, \tag{3.81}$$

where M_s is an $N_2^{(s)} \times N_2^{(s)}$ constant matrix. One can take the gauge where M is diagonal. Let us denote the i-th element of the diagonalized M_s by m_{si} for $i = 1, \cdots, N_2^{(s)}$.

To perform the ordinary Faddeev-Popov gauge-fixing for the fluctuation fields, we define the ghost action S_{gh} by

$$\begin{aligned}
V_{gh} &= \mathrm{Tr}\left[\tilde{C}\left(iF + \frac{\xi_1}{2}b + ib_0\right) + C\left(\tilde{a}_0 - \frac{\xi_2}{2}a_0\right)\right], \\
S_{gh} &= t\int d\tau\, QV_{gh} = t\int d\tau \mathrm{Tr}\left[b\left(iF + \frac{\xi_1}{2}b + ib_0\right) - \tilde{C}\left(iQF + \frac{\xi_1}{2}[\tilde{C}, a_0] - C_0\right)\right. \\
&\qquad \left. +(\phi + a_0 - C^2)\left(\tilde{a}_0 - \frac{\xi_2}{2}a_0\right) - iC\tilde{C}_0\right],
\end{aligned} \tag{3.82}$$

where F is a gauge-fixing condition. This is not BRS-exact but Q-exact. We choose it because the action should be Q-closed, and it is indeed equivalent to the original gauge-fixing, which is performed by a BRS-exact action. F is chosen to be the following in this paper.

$$F = \sum_{a'=1}^{4} \hat{D}_{a'}\left(\frac{1}{\cosh\tau}X^{a'}\right), \tag{3.83}$$

where $\hat{D}_{a'}$ is defined as the background covariant derivative:

$$\hat{D}_1 X := \partial_1 X, \quad \hat{D}_a X := -i[\hat{X}_a, X]. \tag{3.84}$$

$\hat{X}_a$ is the vacuum configuration $-2L_a$, defined in (3.19). Now, by choosing the gauge in which $\xi_1 = \xi_2 = 0$, the ghost action is written as

$$\begin{aligned} S_{gh} = t \int d\tau \mathrm{Tr} \Bigg[& b\,(iF + ib_0) + \sum_{a'=1}^{4} \tilde{C}\hat{D}_{a'} \left(\frac{1}{\cosh\tau} D^{a'} C \right) + \tilde{C} C_0 \\ & + (\phi + a_0 - C^2)\tilde{a}_0 - iC\tilde{C}_0 - \sum_{a'=1}^{4} \tilde{C}\hat{D}_{a'} \left(\frac{1}{\cosh\tau} \Psi\Gamma^{a'}\epsilon \right) \Bigg]. \end{aligned} \tag{3.85}$$

One can note that, as is expected, zero modes of C, $\tilde{C}$ and b are equivalent to zero by integrating out $\tilde{C}_0$, C_0 and b_0, respectively. One also finds that a_0 is identified with ϕ by integrating $\tilde{a}_0$. The last term does not contribute to the path-integral because there is no diagram containing this term. If there exists such a diagram, it should have C insertions to terminate $\{C, \tilde{C}\}$ propagators and is not a physical diagram.

3.4.4 One-loop determinant

Now we are ready to compute the one-loop determinant around the localizing locus. After redefining the fields (3.72) so that $Z_i \to \hat{Z}_i + Z_i/\sqrt{t}$ and $Z'_i \to \hat{Z}'_i + Z'_i/\sqrt{t}$, let us take the limit $t \to \infty$, which makes all terms without the quadratic terms in the action vanish. Then, the action becomes

$$S_{cl} + Q(V_{matt} + V_{gh}) \tag{3.86}$$

where the first term is the classical action:

$$S_{cl} = \frac{1}{g^2} \int_{-\infty}^{\infty} d\tau \, \mathrm{Tr} \left(\frac{1}{2}(\partial_1 \hat{X}_{10})^2 + \frac{1}{2}\hat{X}_{10}^2 + \frac{1}{2}\hat{K}_5^{(E)\,2} \right) = \frac{2}{g^2} \mathrm{Tr}\, M^2, \tag{3.87}$$

and the second term is the quadratic ones in terms of fluctuation fields. Here, $V_{matt} + V_{gh}$ in the second term is in the form of

$$V_{matt} + V_{gh} = (Z'_0, Z_1) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} Z_0 \\ Z'_1 \end{pmatrix}, \tag{3.88}$$

where D_{ij}'s are linear differential operators. Then the Q-exact term is written as

$$Q(V_{matt} + V_{gh}) = (\mathcal{R}Z_0, Z'_1) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} Z_0 \\ Z'_1 \end{pmatrix} + (Z'_0, Z_1) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} Z'_0 \\ \mathcal{R}Z_1 \end{pmatrix}. \tag{3.89}$$

Let us define spaces V_{Z_0} and V_{Z_1} by the functional spaces of fluctuation fields Z_0 and Z_1, respectively, which are restricted to the ones that approach to vacua at $\tau \to \pm\infty$. By using a natural linear map by operator D_{10}, which maps V_{Z_0} to V_{Z_1}, one can see that the one-loop determinant becomes

$$Z_{1-\text{loop}} = \left(\frac{\det_{\text{coker}D_{10}}\mathcal{R}}{\det_{\text{ker}D_{10}}\mathcal{R}}\right)^{\frac{1}{2}}, \tag{3.90}$$

since the determinants in the numerator and denominator of the original one-loop determinant are partially canceled out. Here $\det_V$ stands for a determinant of the endomorphism $V \to V$. Then the kernel and cokernel are decomposed into the direct sum of the eigenspaces of $\mathcal{R}$ since D_{10} and $\mathcal{R}$ commute. If one writes the decomposition as

$$\text{ker}D_{10} = \bigoplus_i V_{r_i}, \quad \text{coker}D_{10} = \bigoplus_i V'_{r_i}, \tag{3.91}$$

where V_{r_i} and V'_{r_i} are the eigenspaces of r_i, an eigenvalue of $\mathcal{R}$, then the one-loop determinant can be written as

$$Z_{1-\text{loop}} = \prod_i r_i^{(\text{dim}V'_{r_i} - \text{dim}V_{r_i})/2}. \tag{3.92}$$

Hence, computing the one-loop determinant reduces to finding the index of D_{10} for every eigenspace of $\mathcal{R}$. The index is well-defined because BMN model is one-dimensional quantum mechanics and so D_{10} is a Fredholm operator.

In (3.88), $Z_1 D_{10} Z_0$ part is written as

$$\begin{aligned}\int d\tau\, \text{Tr}\Bigg[& 2s_i\Upsilon_i + i\tilde{C}(F + b_0) + C\tilde{a}_0 \\ & - \frac{i}{\epsilon\epsilon}\sum_{a'=1}^{4}\left(\delta_{U(1)}X_{a'} + 2\hat{D}_{a'}(v^4X_4 + v^9X_9) - i[X_{a'}, -2iM + v^4\hat{X}_4]\right)\hat{D}^{a'}C\Bigg].\end{aligned} \tag{3.93}$$

To compute the indices, one needs to find out from (3.93) the kernel and cokernel of D_{10} and eigenvalues of $\mathcal{R}$ acting on them. Let $f(\tau)$ be an n-dimensional vector which is an element of the kernel or cokernel of D_{10}. As they are subspaces of V_{Z_0} and V_{Z_1}, $f(\tau)$ should be 0 in the limit of $\tau \to \pm\infty$. Then, it satisfies an equation in the form of $Df(\tau) = 0$, where

$$Df_i(\tau) := \frac{\partial f_i}{\partial \tau}(\tau) + A_{ij}(\tau)f_j(\tau). \tag{3.94}$$

i, j are vector indices and running from 1 to n. Here, A is an $n \times n$ matrix which generally depends on τ, and goes to finite values in the limit of $\tau \to \pm\infty$. Essentially, the problem reduces to finding such $f(\tau)$'s. We use the following technique to find them.

The equation $Df(\tau) = 0$ can be diagonalized by the following gauge transformation:

$$U^{-1}(\tau)(\partial_1 + A(\tau))U(\tau) = A_d(\tau), \tag{3.95}$$

where A_d is a diagonal matrix. Then, the formal solution is

$$f(\tau) = U(\tau) \exp\left(- \int_0^\tau A_d(\tau')d\tau' \right) f_0, \tag{3.96}$$

where f_0 is a constant vector. If one takes $U(\tau)$ as a matrix that gets constant at $\tau \to \pm\infty$, $A_d(\pm\infty)$ is obtained by $U^{-1}(\pm\infty)A(\pm\infty)U(\pm\infty) = A_d(\pm\infty)$. This asymptotic values $A_d(\pm\infty)$ are the same as those of $\tilde{A}_d(\tau)$ in

$$V^{-1}(\tau)A(\tau)V(\tau) = \tilde{A}_d(\tau). \tag{3.97}$$

Let us denote the eigenvalues by $\lambda_1(\tau), \cdots, \lambda_n(\tau)$, and so $A_d(\tau) = \mathrm{diag}(\lambda_1(\tau), \cdots, \lambda_n(\tau))$. The equation (3.96) tells us that, the i-th component of f_0 can have a non-zero value if the following condition holds:

$$\lim_{\tau\to\infty} \mathrm{Re}\lambda_i(\tau) > 0 \quad \text{and} \quad \lim_{\tau\to-\infty} \mathrm{Re}\lambda_i(\tau) < 0. \tag{3.98}$$

This is because the exponential factor in $f_i(\tau)$ goes to 0 at $\tau \to \pm\infty$ under this condition. One finds that, if there are only k eigenvalues that satisfy the condition (3.98), the dimension of the kernel of D is

$$\dim(\ker D) = k. \tag{3.99}$$

When D is D_{10}, k is the dimension of $\ker D_{10}$; when D is the adjoint of D_{10}, it is the dimension of $\mathrm{coker}\, D_{10}$.

In the following, I will redefine the fields in order for easy view of $A_{ij}(\tau)$ and show explicit forms of $A_{ij}(\tau)$ for some sets of fields. the fields are expanded in terms of fuzzy spherical harmonics as (3.21).

Index for $X_5, \cdots, X_8$

These fields are interpreted as bosonic fields in hypermultiplet in the viewpoint of $\mathcal{N} = 4$ SYM. If one defines W_1 and W_2 as

$$W_1 = X_5 + iX_8, \quad W_2 = X_6 + iX_7, \tag{3.100}$$

the equations for $\ker D_{10}$ become

$$\partial_1 W_1 + 2i[L_-, W_2] + \frac{s}{c}(W_1 + 2[L_4, W_1]) = 0,$$
$$\partial_1 W_2 - 2i[L_+, W_1] + \frac{s}{c}(W_2 - 2[L_4, W_2]) = 0, \tag{3.101}$$

where $L_\pm = L_2 \pm iL_3$. Then, matrix A for $f = (W^{(s,t)}_{1JJ}, W^{(s,t)}_{1Jm}, W^{(s,t)}_{2Jm+1}, W^{(s,t)}_{2J-J})^T$ for $m = -J, -J+1, \cdots, J-1$ is written as

$$A = \begin{pmatrix} \frac{s}{c}(2J+1) & 0 & 0 & 0 \\ 0 & \frac{s}{c}(2m+1) & 2i\delta_- & 0 \\ 0 & -2i\delta_- & -\frac{s}{c}(2m+1) & 0 \\ 0 & 0 & 0 & \frac{s}{c}(2J+1) \end{pmatrix}, \tag{3.102}$$

where $\delta_\pm = \sqrt{(J \pm m)(J \mp m + 1)}$. Only the eigenvalues for $W^{(s,t)}_{1JJ}$ and $W^{(s,t)}_{2J-J}$ satisfy the condition (3.98). As a_0 is identified with $-\phi$, the action of $\mathcal{R}$ on these fields is

$$\mathcal{R}W^{(s,t)}_{1JJ} = 2\left\{(1+2J)W^{(s,t)}_{1JJ} + i(M_s W^{(s,t)}_{1JJ} - W^{(s,t)}_{1JJ} M_t)\right\},$$
$$\mathcal{R}W^{(s,t)}_{2J-J} = 2\left\{-(1+2J)W^{(s,t)}_{2J-J} + i(M_s W^{(s,t)}_{2J-J} - W^{(s,t)}_{2J-J} M_t)\right\}. \tag{3.103}$$

Note that $\mathcal{R}$ is the operator at the localizing locus (3.80) to compute the one-loop determinant. Thus, they contribute to the one-loop determinant as the following factor up to an overall constant:

$$\prod_{s,t=1}^{\Lambda} \prod_{J=|j_s-j_t|}^{j_s+j_t} \prod_{i=1}^{N_2^{(s)}} \prod_{j=1}^{N_2^{(t)}} \frac{1}{(2J+1)^2 + (m_{si} - m_{tj})^2}. \tag{3.104}$$

Index for $\Upsilon_1, \cdots, \Upsilon_4$

These fields are interpreted as fermionic fields in hypermultiplet in the view point of $\mathcal{N} = 4$ SYM. Defining

$$\xi_1 = \Upsilon_1 + i\Upsilon_4, \quad \xi_2 = \Upsilon_3 + i\Upsilon_2, \tag{3.105}$$

one finds that the equation for $\operatorname{coker} D_{10}$ tells us

$$A = \begin{pmatrix} -\frac{2s}{c}J & 0 & 0 & 0 \\ 0 & \frac{2s}{c}m & 2\delta_+ & 0 \\ 0 & 2\delta_+ & -\frac{2s}{c}(m-1) & 0 \\ 0 & 0 & 0 & -\frac{2s}{c}J \end{pmatrix}, \tag{3.106}$$

for $f = (\xi^{(s,t)}_{1J-J}, \xi^{(s,t)}_{1Jm}, \xi^{(s,t)}_{2Jm-1}, \xi^{(s,t)}_{2JJ})^T$ for $m = -J+1, -J+2, \cdots, J$. This does not contain any fields satisfying (3.98), and so it does not contribute to the one-loop determinant.

Index for $X_1, \cdots, X_4, X_9$ and $b_0, \tilde{a}_0$

These fields are interpreted as bosonic fields in vector multiplet and the bosonic fields associated with ghosts in the view point of $\mathcal{N}=4$ SYM. The equations for $\ker D_{10}$ are

$$F + b_0 = 0, \tag{3.107}$$

$$\tilde{a}_0 + \sum_{a'=1}^{4} 2\hat{D}^{a'}\left(\frac{i}{2c}\hat{D}_{a'}(v^4 X_4 + v^9 X_9)\right) + \left[\frac{F}{2}, -2iM + v^4 \hat{X}_4\right] = 0, \tag{3.108}$$

$$c(2X_4 - i[\hat{X}_2, X_3] + i[\hat{X}_3, X_2]) - s(\partial_1 X_4 + i[\hat{X}_4, X_1]) - \partial_1 X_9 = 0, \tag{3.109}$$

$$c(\partial_1 X_3 + i[\hat{X}_3, X_1]) - s(2X_3 + i[\hat{X}_2, X_4] - i[\hat{X}_4, X_2]) - i[\hat{X}_2, X_9] = 0, \tag{3.110}$$

$$c(\partial_1 X_2 + i[\hat{X}_2, X_1]) - s(2X_2 - i[\hat{X}_3, X_4] + i[\hat{X}_4, X_3]) + i[\hat{X}_3, X_9] = 0. \tag{3.111}$$

Some fields do not contribute to the one-loop determinant. (3.107) says that $b_0 = 0$ because $F \to 0$ at $\tau \to \pm\infty$ while b_0 is constant. Then, $F = 0$ holds. For $\tilde{a}_0$, the second and third terms in (3.108) goes to zero in that limit, and so one finds $\tilde{a}_0 = 0$. From (3.108) with $F = \tilde{a}_0 = 0$, the following equation holds:

$$-\partial_1\left(\frac{1}{c}\partial_1(X_4 - sX_9)\right) + \frac{4}{c}[L^a, [L_a, X_4 - sX_9]] = 0. \tag{3.112}$$

Let $g_{Jm}^{(s,t)}$ be the coefficient of $Y_{Jm(j_s,j_t)}$ in $X_4 - sX_9$. Then, one can see that, by using (3.112), $g_{Jm}^{(s,t)}$ satisfies

$$0 = \int_{-\infty}^{\infty} d\tau \partial_1\left(\frac{1}{c^2} g_{Jm}^{(s,t)} \partial_1 g_{Jm}^{(s,t)}\right) = \int d\tau \left[\left(\frac{\partial_1 g_{Jm}^{(s,t)}}{c}\right)^2 + \left(\frac{4J(J+1)-1}{c^2} + \frac{3}{2c^4}\right) g_{Jm}^{(s,t)\,2}\right], \tag{3.113}$$

since $g_{Jm}^{(s,t)}/c \to 0$ at $\tau \to \pm\infty$. Therefore, it is found that $g_{Jm}^{(s,t)}$ is zero when $J \neq 0$ because the integrand of the above equation is positive definite. For $J = 0$, the equation (3.112) is equivalent to $\partial_1(\partial_1 g_{00}^{(s,t)}/c) = 0$. Its solution is $g_{00}^{(s,t)} = \text{const.} \times s$. However, as $g_{00}^{(s,t)}$ should be finite or zero when $\tau \to \pm\infty$, the constant factor has to be zero, and $g_{00}^{(s,t)}$ is also zero. Thus, one has seen that $g_{Jm}^{(s,t)} = 0$ for any J, that is, $X_4 = sX_9$.

Now we have $F = 0$, (3.109), (3.110) and (3.111) with $X_4 = sX_9$, and they can be written as

$$\begin{aligned}
&- i\partial_1 X_1 + i\frac{s}{c}X_1 + [L_+, X_-] + [L_-, X_+] + 2s[L_4, X_9] = 0,\\
&- [L_+, X_-] + [L_-, X_+] + sX_9 - c\partial_1 X_9 + 2i\frac{s}{c}[L_4, X_1] = 0,
\end{aligned}$$

$$
\begin{aligned}
&c(\partial_1 X_+ - 2i[L_+, X_1]) - s(2X_+ - 2[L_4, X_+]) - 2c^2[L_+, X_9] = 0,\\
&c(\partial_1 X_- - 2i[L_-, X_1]) - s(2X_- + 2[L_4, X_-]) + 2c^2[L_-, X_9] = 0.
\end{aligned} \tag{3.114}
$$

Then, for $f = (X^{+(s,t)}_{Jm+1}/\sqrt{2}, X^{-(s,t)}_{Jm-1}/\sqrt{2}, X^{1(s,t)}_{Jm}, -icX^{9(s,t)}_{Jm})^T$ for $m = -J+1, -J+2, \cdots, J-1$ with $J \geq 1$, matrix A is read off from (3.114) as

$$
A = \begin{pmatrix} \frac{2ms}{c} & 0 & -\sqrt{2}i\delta_- & -\sqrt{2}i\delta_- \\ 0 & -\frac{2ms}{c} & -\sqrt{2}i\delta_+ & \sqrt{2}i\delta_+ \\ \sqrt{2}i\delta_- & \sqrt{2}i\delta_+ & -\frac{s}{c} & -\frac{2ms}{c} \\ \sqrt{2}i\delta_- & -\sqrt{2}i\delta_+ & -\frac{2ms}{c} & -\frac{2s}{c} \end{pmatrix}. \tag{3.115}
$$

There also remain $f = (X^{-(s,t)}_{JJ-1}/\sqrt{2}, X^{1(s,t)}_{JJ}, -icX^{9(s,t)}_{JJ})^T$ and $f = (X^{+(s,t)}_{J-J+1}/\sqrt{2}, X^{1(s,t)}_{J-J}, -icX^{9(s,t)}_{J-J})^T$. Matrices A for them are

$$
A = \begin{pmatrix} -\frac{2Js}{c} & -2i\sqrt{J} & 2i\sqrt{J} \\ 2i\sqrt{J} & -\frac{s}{c} & -\frac{2Js}{c} \\ -2i\sqrt{J} & -\frac{2Js}{c} & -\frac{2s}{c} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -\frac{2Js}{c} & -2i\sqrt{J} & -2i\sqrt{J} \\ 2i\sqrt{J} & -\frac{s}{c} & \frac{2Js}{c} \\ 2i\sqrt{J} & \frac{2Js}{c} & -\frac{2s}{c} \end{pmatrix}, \tag{3.116}
$$

respectively. One can see that these three do not satisfy the condition (3.98) by confirming that the determinant of each A for given J, m cannot be zero at any τ. Therefore, these bosonic fields do not contribute to the one-loop determinant.

Index for $C, \tilde{C}$ and $\Upsilon_5, \Upsilon_6, \Upsilon_7$

These fields are interpreted as fermionic fields in vector multiplet and the ghost and anti-ghost fields in the view point of $\mathcal{N} = 4$ SYM. The elements of $\operatorname{coker} D_{10}$ satisfy the following equations:

$$
\begin{aligned}
&-\frac{1}{c}\partial_1\tilde{C} - \frac{i}{c}[iM - 2L_4, \partial_1 C] - 8s[L_4, \Upsilon_5] - 8c[L_3, \Upsilon_6] + 8c[L_2, \Upsilon_7] = 0,\\
&\frac{1}{c}[L_2, \tilde{C}] + \frac{i}{c}[L_2, [iM - 2L_4, C]] + 4ic[L_3, \Upsilon_5] - 4is[L_4, \Upsilon_6] - 2c\partial_1\Upsilon_7 - 6s\Upsilon_7 = 0,\\
&\frac{1}{c}[L_3, \tilde{C}] + \frac{i}{c}[L_3, [iM - 2L_4, C]] - 4ic[L_2, \Upsilon_5] - 4is[L_4, \Upsilon_7] + 2c\partial_1\Upsilon_6 + 6s\Upsilon_6 = 0,\\
&\frac{1}{c}[L_4, \tilde{C}] - \partial_1\left(\frac{i}{c}\partial_1 C\right) + \frac{4i}{c}[L_{a'}, [L_{a'}, C]] + \frac{i}{c}[L_4, [iM - 2L_4, C]] + 2s\partial_1\Upsilon_5 + 6c\Upsilon_5\\
&+ 4is[L_2, \Upsilon_6] + 4is[L_3, \Upsilon_7] = 0,\\
&is\partial_1\left(\frac{1}{c}\partial_1 C\right) - \frac{4is}{c}[L_{a'}, [L_{a'}, C]] + 2\partial_1\Upsilon_5 + 4i[L_2, \Upsilon_6] + 4i[L_3, \Upsilon_7] = 0.
\end{aligned} \tag{3.117}
$$

There also exists a condition that C and $\tilde{C}$ do not have zero modes. Let us redefine the fields as $\tilde{C}' = (\tilde{C} - [iM - 2L_4, C])/(2\sqrt{2}c)$, $C' = iC/c$, $\Upsilon'_5 = \sqrt{2}\Upsilon_5$ and $\Upsilon_\pm = \Upsilon_6 \pm \Upsilon_7$, and define a new field as $d = \partial_1 C'$ so as to make the equations linear differential ones. Then, equations (3.117) become

$$\begin{aligned}
&\partial_1 C' - d = 0,\\
&\partial_1 d + \frac{3s}{c} d + 2C' - 4[L_{a'}, [L_{a'}, C']] + \frac{2\sqrt{2}}{c^2}[L_4, \tilde{C}'] + \frac{3\sqrt{2}}{c^2}\Upsilon'_5 = 0,\\
&\partial_1 \Upsilon_+ - \sqrt{2}i[L_+, \tilde{C}'] - \sqrt{2}i[L_+, \Upsilon'_5] + \frac{3s}{c}\Upsilon_+ - \frac{2s}{c}[L_4, \Upsilon_+] = 0,\\
&\partial_1 \Upsilon_- + \sqrt{2}i[L_-, \tilde{C}'] - \sqrt{2}i[L_-, \Upsilon'_5] + \frac{3s}{c}\Upsilon_- + \frac{2s}{c}[L_4, \Upsilon_-] = 0,\\
&\partial_1 \tilde{C}' + \frac{2s}{c}\tilde{C}' + \frac{2s}{c}[L_4, \Upsilon'_5] - \sqrt{2}i([L_+, \Upsilon_-] - [L_-, \Upsilon_+]) = 0,\\
&\partial_1 \Upsilon'_5 + \frac{2s}{c}[L_4, \tilde{C}'] + \frac{3s}{c}\Upsilon'_5 + \sqrt{2}i([L_+, \Upsilon_-] + [L_-, \Upsilon_+]) = 0.
\end{aligned} \tag{3.118}$$

Then, one finds that matrix A for $f = (C'^{(s,t)}_{Jm}, d^{(s,t)}_{Jm}, \Upsilon^{+(s,t)}_{Jm+1}, \Upsilon^{-(s,t)}_{Jm-1}, \Upsilon'^{5(s,t)}_{Jm}, \tilde{C}'^{(s,t)}_{Jm})^T$ for $m = -J+1, -J+2, \cdots, J-1$ is

$$A = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0\\
\frac{3s}{c} & 2-4J(J+1) & 0 & 0 & \frac{3\sqrt{2}}{c^2} & \frac{3\sqrt{2}m}{c^2}\\
0 & 0 & \frac{s}{c}(1-2m) & 0 & -\sqrt{2}i\delta_- & -\sqrt{2}i\delta_-\\
0 & 0 & 0 & \frac{s}{c}(1+2m) & -\sqrt{2}i\delta_+ & \sqrt{2}i\delta_+\\
0 & 0 & \sqrt{2}i\delta_- & \sqrt{2}i\delta_+ & \frac{3s}{c} & \frac{2ms}{c}\\
0 & 0 & \sqrt{2}i\delta_- & -\sqrt{2}i\delta_+ & \frac{2ms}{c} & \frac{2s}{c}
\end{pmatrix}. \tag{3.119}$$

This A does not contain any eigenvalues that satisfy (3.98).
For $f = (C'^{(s,t)}_{JJ}, d^{(s,t)}_{JJ}, \Upsilon^{-(s,t)}_{JJ-1}, \Upsilon'^{5(s,t)}_{JJ}, \tilde{C}'^{(s,t)}_{JJ})^T$, matrix A is written as

$$A = \begin{pmatrix}
0 & -1 & 0 & 0 & 0\\
\frac{3s}{c} & 2-4J(J+1) & 0 & \frac{3\sqrt{2}}{c^2} & \frac{3\sqrt{2}J}{c^2}\\
0 & 0 & \frac{s}{c}(1+2J) & -2i\sqrt{J} & 2i\sqrt{J}\\
0 & 0 & 2i\sqrt{J} & \frac{3s}{c} & \frac{2Js}{c}\\
0 & 0 & -2i\sqrt{J} & \frac{2Js}{c} & \frac{2s}{c}
\end{pmatrix}, \tag{3.120}$$

except for $J = 0$ because f does not have zero modes.[6] There is one eigenvalue of this A which satisfies (3.98). The action of $\mathcal{R}$ on the field is

$$\mathcal{R}f = 2\{2Jf + i(M_s f - f M_t)\}. \tag{3.121}$$

[6]From the sixth equation in (3.118), Υ'^5 would seem to have a zero mode which is the cokernel of D_{10}. However, the second equation in (3.118) says that the zero mode vanishes because the other fields does not have zero modes.

Similarly, for $f = (C'^{(s,t)}_{J-J}, d^{(s,t)}_{J-J}, \Upsilon^{+(s,t)}_{J-J+1}, \Upsilon'^{5(s,t)}_{J-J}, \tilde{C}'^{(s,t)}_{J-J})^T$,

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ \frac{3s}{c} & 2-4J(J+1) & 0 & \frac{3\sqrt{2}}{c^2} & -\frac{3\sqrt{2}J}{c^2} \\ 0 & 0 & \frac{s}{c}(1+2J) & -2i\sqrt{J} & -2i\sqrt{J} \\ 0 & 0 & 2i\sqrt{J} & \frac{3s}{c} & -\frac{2Js}{c} \\ 0 & 0 & 2i\sqrt{J} & -\frac{2Js}{c} & \frac{2s}{c} \end{pmatrix}, \tag{3.122}$$

except for $J = 0$. There is also one eigenvalue which satisfies (3.98), and the action of $\mathcal{R}$ is

$$\mathcal{R}f = 2\{-2Jf + i(M_s f - f M_t)\}. \tag{3.123}$$

Now we have the final field, which is $f = (\Upsilon^{+(s,t)}_{J-J}, \Upsilon^{-(s,t)}_{JJ})^T$. Matrix A for this field is

$$A = \begin{pmatrix} \frac{s}{c}(3+2J) & 0 \\ 0 & \frac{s}{c}(3+2J) \end{pmatrix}. \tag{3.124}$$

These eigenvalues satisfy the condition (3.98). The action of $\mathcal{R}$ is

$$\begin{aligned} \mathcal{R}\Upsilon^{+(s,t)}_{J-J} &= 2\left\{-(2+2J)\Upsilon^{+(s,t)}_{J-J} + i(M_s\Upsilon^{+(s,t)}_{J-J} - \Upsilon^{+(s,t)}_{J-J}M_t)\right\}, \\ \mathcal{R}\Upsilon^{-(s,t)}_{JJ} &= 2\left\{(2+2J)\Upsilon^{-(s,t)}_{JJ} + i(M_s\Upsilon^{-(s,t)}_{JJ} - \Upsilon^{-(s,t)}_{JJ}M_t)\right\}. \end{aligned} \tag{3.125}$$

From (3.121), (3.123) and (3.125), the contribution to the one-loop determinant is obtained as

$$\begin{aligned} &\prod_{s,t=1}^{\Lambda} \prod_{\substack{J=|j_s-j_t| \\ J\neq 0}}^{j_s+j_t} \prod_{i=1}^{N_2^{(s)}} \prod_{j=1}^{N_2^{(t)}} \{(2J)^2 + (m_{si} - m_{tj})^2\}^{1/2} \\ \times &\prod_{s,t=1}^{\Lambda} \prod_{J=|j_s-j_t|}^{j_s+j_t} \prod_{i=1}^{N_2^{(s)}} \prod_{j=1}^{N_2^{(t)}} \{(2J+2)^2 + (m_{si} - m_{tj})^2\}^{1/2}. \end{aligned} \tag{3.126}$$

As we have collected the contribution to the index, the one-loop determinant can be obtained now. Since M_s's are diagonalized, there exists a Vandermonde determinant factor: $\prod_s \prod_{i>j}(m_{si} - m_{sj})^2$. Combining (3.104), (3.126) and the Vandermonde determinant, the one-loop determinant is

$$Z_{1-\text{loop}} = \prod_{s,t=1}^{\Lambda} \prod_{J=|j_s-j_t|}^{j_s+j_t} \prod_{i=1}^{N_2^{(s)}} \prod_{j=1}^{N_2^{(t)}}{}' \left[\frac{\{(2J+2)^2 + (m_{si} - m_{tj})^2\}\{(2J)^2 + (m_{si} - m_{tj})^2\}}{\{(2J+1)^2 + (m_{si} - m_{tj})^2\}^2}\right]^{\frac{1}{2}}, \tag{3.127}$$

where $\prod'$ means that the second factor in the numerator with $s = t$, $J = 0$ and $i = j$ is not included in the product. This is the one-loop determinant around the vacuum labeled by $\{N_2^{(s)}, N_5^{(s)}\}_{s=1,\cdots,\Lambda}$. The partition function around the vacuum[7] is obtained as

$$Z = \int \prod_{s=1}^{\Lambda} \prod_{i=1}^{N_2^{(s)}} dm_{si}\, Z_{1-\mathrm{loop}}\, e^{-\frac{2}{g^2}\sum_s \sum_i D_s m_{si}^2}. \tag{3.128}$$

The exponent of the last factor of (3.128) is (3.87). This result is an exact one up to the instanton factor.

As a consistency check of our computation, we can reproduce a one-loop result of BMN model around the trivial background [1]. Furthermore, we can show that BMN model around the background corresponding to $\mathcal{N} = 4$ SYM on $R \times S^3$ becomes a Gaussian matrix model [1]. This is consistent with the results in [5, 18, 19].

Now we have the partition function, BPS operators, which are invariant under δ_s, can be computed exactly. Let us consider a "Wilson" loop operator[8] $W = \frac{1}{N}\mathrm{Tr}\, e^{-\pi i \phi}$. Its value at the localizing locus where the vacuum is labeled by $\{N_2^{(s)}, N_5^{(s)}\}_{s=1,\cdots,\Lambda}$ is

$$\frac{1}{N}\sum_{s=1}^{\Lambda}\sum_{i=1}^{N_2^{(s)}} D_s\, e^{2\pi m_{si}} \tag{3.129}$$

because the eigenvalues of $2L_4$ are integers. Then, the expectation value at the vacuum is

$$\langle W \rangle = \frac{1}{ZN}\sum_{t=1}^{\Lambda} D_t \sum_{j=1}^{N_2^{(t)}} \int \prod_{s=1}^{\Lambda}\prod_{i=1}^{N_2^{(s)}} dm_{si}\, e^{2\pi m_{tj}}\, Z_{1-\mathrm{loop}}\, e^{-\frac{2}{g^2}\sum_s \sum_i D_s m_{si}^2}. \tag{3.130}$$

4 Emergent bubbling geometry

In this section, we investigate BMN model around a general vacuum as well as $\mathcal{N} = 8$ SYM on $R \times S^2$ and $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$. First, we obtain the saddle point equation of the matrix integral (3.128) with (3.127), which is a equation for eigenvalue densities of the matrix, in two different ways. Then, we show that the eigenvalue densities satisfy the

[7] The partition function is originally defined as the summation of the contribution from all vacua. In this case, however, a "partial" partition function around a vacuum can be defined in the large-N limit because there are no instanton effect in the limit. In order to define it well, the condition mentioned at the above of (3.80) should be imposed in addition to the large-N limit.

[8] From the viewpoint of $\mathcal{N} = 4$ SYM, it is interpreted as the BPS Wilson loop whose contour is a great circle of S^3. It is obtained by dimensional reduction of $\mathcal{N} = 4$ SYM on $R \times S^3$.

same integral equations as the charge densities of the corresponding electrostatic system. We also show that the same relation holds for the other gauge theories.

In order to compare with the gravity side, one should analyze the gauge theories in the parameter region where the dual supergravity description is valid. In order for the supergravity approximation to be valid, the brane charges, $N_2^{(s)}$ and $N_5^{(s)}$, should be very large and $N_2^{(s)}$ should be much larger than $g^2 N_2^{(s)}$ and $N_5^{(s)}$ to suppress the bulk string coupling. In addition, the condition to suppress the α' corrections is needed. Therefore, the parameter region corresponds in the gauge theory side to the 't Hooft limit

$$N_2^{(s)} \to \infty, \quad \lambda^{(s)} = g^2 N_2^{(s)} = \text{fixed}, \tag{4.1}$$

and

$$D_s - D_{s-1} \gg 1, \quad \lambda^{(s)} \gg D_s, \tag{4.2}$$

for arbitrary s. The conditions (4.2) sufficiently make the α' corrections negligible. Note that $D_s - D_{s-1} = \frac{\pi}{2} N_5^{(s)}$ is the square of the S^5 radius in the s-th NS5-brane throat [13]. Also, as shown in appendix F, the second condition of (4.2) means that the radius of S^5 near the tip of a disk in the electrostatic system is large. In these limits, (4.1) and (4.2), one can evaluate the vacuum expectation values of BPS operators by applying the saddle point approximation, which becomes exact in these limits.

The procedure to obtain the corresponding geometry is summarized as follows.

1. Identify the appropriate BPS sector.

2. Perform the localization technique and obtain the saddle point equations.

3. The saddle point equations should be the equations of motion determining the corresponding geometry in the parameter region where the supergravity approximation is valid.

In this case, the appropriate BPS sector is the one under which $\phi(\tau)$ is invariant. ϕ is considered to correspond to points on the non-trivial two-dimensional subspace whose coordinates are r and z on the gravity side. In fact, X_4 in ϕ may correspond to z-direction and $X_9 \sinh\tau + iX_{10}\cosh\tau$ in ϕ to r-direction. The BPS configuration (3.80) tells us that $\hat{X}_4$ is constructed by representations of $SU(2)$, whose dimensions correspond to the distances between disks in terms of the electrostatic system, and that $\hat{X}_{10}$ has M, which appears later to correspond to electric charges spreading along r-direction on disks. z-direction is also understood as a direction emerged by the large-N reduction along the fiber of S^3 discussed in section 3.3.2. The insertion of ϕ into a vacuum expectation value breaks the original symmetry $R \times SO(3) \times SO(6)$ to $SO(2) \times SO(5)$, which is a symmetry of $S^2 \times S^5$ around a fixed point on it. This suggests that the two-dimensional space of r and z should be fibered on the point on $R \times S^2 \times S^5$.

4.1 Saddle point equation

As we know that all the dual geometries correspond to vacua in BMN model, only the partition function is needed[9]. In our case, two ways are known to obtain the saddle point equations from the partition function – a naive saddle point analysis and the Fermi gas approach.

4.1.1 Naive way

Let us obtain the saddle point equations by the usual saddle point analysis. When $|D_s - D_{s-1}| \gg 1$, one can rewrite the measure factor in (3.127) as

$$\begin{aligned}&\prod_{J=0}^{D_s-1} \frac{\{(2J+2)^2+(m_{si}-m_{sj})^2\}\{(2J)^2+(m_{si}-m_{sj})^2\}}{\{(2J+1)^2+(m_{si}-m_{sj})^2\}^2} \\ &= \tanh^2 \frac{\pi(m_{si}-m_{sj})}{2} \exp\left\{ \frac{2D_s}{(2D_s)^2+(m_{si}-m_{sj})^2} - \cdots \right\}\end{aligned} \tag{4.3}$$

for $s = t$, and

$$\begin{aligned}&\prod_{J=|D_s-D_t|/2}^{(D_s+D_t)/2-1} \frac{\{(2J+2)^2+(m_{si}-m_{tj})^2\}\{(2J)^2+(m_{si}-m_{tj})^2\}}{\{(2J+1)^2+(m_{si}-m_{tj})^2\}^2} \\ &= \exp\left\{ \frac{D_s+D_t}{(D_s+D_t)^2+(m_{si}-m_{tj})^2} - \frac{|D_s-D_t|}{(D_s-D_t)^2+(m_{si}-m_{tj})^2} + \cdots \right\}\end{aligned} \tag{4.4}$$

for $s \neq t$, where "$\cdots$" stands for $1/(D_s \pm D_t)$ corrections. We introduce the eigenvalue densities defined for each s as

$$\rho^{(s)}(x) = \sum_{i=1}^{N_2^{(s)}} \delta(x - m_{si}). \tag{4.5}$$

In the large $N_2^{(s)}$ limit (4.2), $\rho^{(s)}(x)$'s become continuous functions. Then, we obtain the effective action for (3.128) with (3.127)

$$S_{eff} = \sum_{s=1}^{\Lambda} \frac{2D_s}{g^2} \int dx\ x^2 \rho^{(s)}(x) - \sum_{s=1}^{\Lambda} \frac{1}{2} \int dxdy\ \log\tanh^2 \frac{\pi(x-y)}{2} \rho^{(s)}(x)\rho^{(s)}(y)$$

[9] If dual geometries correspond to BPS excited states in a gauge theory, expectation values of BPS operators also should be computed.

$$
-\sum_{s,t=1}^{\Lambda}\frac{1}{2}\int dxdy\left[\frac{D_s+D_t}{(D_s+D_t)^2+(x-y)^2}-\frac{|D_s-D_t|}{(D_s-D_t)^2+(x-y)^2}\right]\rho^{(s)}(x)\rho^{(t)}(y)
$$
$$
-\sum_{s=1}^{\Lambda}\mu_s\left(\int dx\rho^{(s)}(x)-N_2^{(s)}\right), \tag{4.6}
$$

where μ_s's are the Lagrange multipliers for the normalization of $\rho^{(s)}(x)$'s.

We assume that $\rho^{(s)}(x)$ has its support on $[-x_m^{(s)}, x_m^{(s)}]$. As shown in appendix F, in the limit of (4.1) and (4.2), the extents of $\rho^{(s)}(x)$ become large; $x_m^{(s)} \gg 1$. Using the fact that $x_m \log\tanh^2\frac{\pi x_m y}{2}$ can be approximated to $-\pi\delta(y)$ as $x_m \to \infty$, we obtain the following saddle point equations

$$
\rho^{(s)}(x)+\frac{1}{\pi}\sum_{t=1}^{\Lambda}\int_{-x_m^{(t)}}^{x_m^{(t)}}du\left[-\frac{D_s+D_t}{(D_s+D_t)^2+(x-u)^2}+\frac{|D_s-D_t|}{(D_s-D_t)^2+(x-u)^2}\right]\rho^{(t)}(u)
$$
$$
=\frac{\mu_s}{\pi}-\frac{2D_s}{\pi g^2}x^2, \tag{4.7}
$$

where $x_m^{(s)}$ and μ_s are determined from

$$
\rho^{(s)}(x_m^{(s)})=0 \quad \text{and} \tag{4.8}
$$
$$
\int_{-x_m^{(s)}}^{x_m^{(s)}} dx\rho^{(s)}(x)=N_2^{(s)}. \tag{4.9}
$$

4.1.2 Fermi gas approach

Here we show that the matrix integral (3.128) can be mapped to a one-dimensional interacting Fermi gas system with $N_2^{(s)}$ particles. We follow the method proposed in [42].

When $D_s - D_{s-1} \gg 1$, the measure factors in (3.128) converge to the product of $\tanh^2\frac{\pi(m_{si}-m_{sj})}{2}$ up to an over all constant. By using the Cauchy identity, we rewrite those hyperbolic tangent parts as

$$
\prod_{i\neq j}^{N_2^{(s)}}\tanh\frac{\pi(m_{si}-m_{sj})}{2}=\sum_{\sigma\in S_{N_2^{(s)}}}(-)^{\epsilon(\sigma)}\prod_{i=1}^{N_2^{(s)}}\frac{1}{\cosh\frac{\pi(m_{si}-m_{s\sigma(i)})}{2}}, \tag{4.10}
$$

where $\epsilon(\sigma)$ stands for the sign of the permutation σ.

We introduce the operators, $\hat{p}$ and $\hat{q}$, that obey the canonical Heisenberg algebra $[\hat{p},\hat{q}]=-i$. Let $\mathcal{H}$ be the usual representation space of this algebra which is an infinite dimensional Hilbert space spanned by the eigenstates of $\hat{q}$. We denote the eigenstates by

$|m\rangle$, which satisfy $\hat{q}|m\rangle = m|m\rangle$. Then, we have

$$\frac{1}{\cosh\frac{\pi(m_i-m_j)}{2}} = \frac{1}{\pi}\int dp\,\frac{1}{\cosh p}e^{ip(m_i-m_j)} = \left\langle m_i\left|\frac{2}{\cosh\hat{p}}\right|m_j\right\rangle. \tag{4.11}$$

We also introduce the Hilbert spaces for $N_2^{(s)}$ fermions, where s is the label of the particle species. It is a subspace of $\bigotimes_{s=1}^{\Lambda}\mathcal{H}^{\otimes N_2^{(s)}}$ and spanned by the antisymmetric states,

$$|\{m_1\},\cdots,\{m_\Lambda\}\} := \bigotimes_{s=1}^{\Lambda}\left(\frac{1}{N_2^{(s)}!}\sum_{\sigma_s\in S_{N_2^{(s)}}}(-)^{\epsilon(\sigma_s)}\left|m_{\sigma_s(1)}\right\rangle\otimes\left|m_{\sigma_s(2)}\right\rangle\otimes\cdots\otimes\left|m_{\sigma_s(N_2^{(s)})}\right\rangle\right). \tag{4.12}$$

We denote by $\hat{p}_{si}$ and $\hat{q}_{si}$ the canonical pair on the i-th Hilbert space of s-th species. They obey the commutation relations, $[\hat{p}_{si},\hat{q}_{tj}] = -i\delta_{ij}\delta_{st}$. With these notations, we can rewrite the matrix integral (3.128) as the partition function of a Fermi gas system,

$$Z = \mathrm{Tr}\,\hat{\rho} \tag{4.13}$$

where the trace is taken over the states (4.12) as

$$\mathrm{Tr}\,\hat{\rho} = \int\prod_i dq_i\,\{\{m_1\},\cdots,\{m_\Lambda\}\,|\hat{\rho}|\,\{m_1\},\cdots,\{m_\Lambda\}\}, \tag{4.14}$$

and the density matrix is given by

$$\hat{\rho} = \prod_{s=1}^{\Lambda}\left[\prod_{i=1}^{N_2^{(s)}}e^{-T(\hat{p}_{si})}\prod_{i=1}^{N_2^{(s)}}e^{-U_s(\hat{q}_{si})}\right]\prod_{s,t=1}^{\Lambda}\left[\prod_i^{N_2^{(s)}}\prod_j^{N_2^{(t)}}e^{-\frac{1}{2}W_{st}^{+}(\hat{q}_{si}-\hat{q}_{tj})}\prod_i^{N_2^{(s)}}\prod_j^{N_2^{(t)}}e^{\frac{1}{2}W_{st}^{-}(\hat{q}_{si}-\hat{q}_{tj})}\right]. \tag{4.15}$$

The functions $T(x)$, $U_s(x)$ and $W_{st}^{\pm}(x)$ are defined as follows.

$$\begin{aligned}
T(x) &:= \log\cosh x,\\
U_s(x) &:= \frac{2D_s}{g^2}x^2,\\
W_{st}^{+}(x) &:= -\frac{D_s+D_t}{(D_s+D_t)^2+x^2},\\
W_{st}^{-}(x) &:= -\frac{|D_s-D_t|}{(D_s-D_t)^2+x^2}.
\end{aligned} \tag{4.16}$$

Here, we have kept only the first term of the exponent in (4.3), and the first and second terms of that in (4.4) because we are interested in the large-$N_5^{(s)}$ limit. Even if $N_5^{(s)}$

is large, the first term should be kept since it can become comparable to the Gaussian potential in some parameter regions. The model defined by (4.13) is an interacting one-dimensional Fermi gas system of Λ species of $N_2^{(s)}$ fermions, where the interaction is given by $W^{\pm}_{st}(m_{si}-m_{tj})$.

The semi-classical limit of this model is described by the many-body Hamiltonian,

$$\hat{H}=\sum_s\sum_i\{T(\hat{p}_{si})+U_s(\hat{q}_{si})\}+\frac{1}{2}\sum_{s,t}\sum_i\sum_j\left\{W^{+}_{st}(\hat{q}_{si}-\hat{q}_{tj})-W^{-}_{st}(\hat{q}_{si}-\hat{q}_{tj})\right\}. \quad (4.17)$$

When $N_2^{(s)}$'s are large, we can apply the Thomas-Fermi approximation at zero temperature (see in appendix D) to the system (4.17). In this approximation, the original many-body path integral can be evaluated at a saddle point characterized by the mean-field density $\rho^{(s)}(x)$. $\rho^{(s)}(x)$ is assumed to have a single support $[-x_m^{(s)},x_m^{(s)}]$ and it is normalized as (4.9). $\rho^{(s)}(x)$ is determined by (D.9) which follows from the Thomas-Fermi equation at zero temperature. In our case, the equation (D.9) is given by (4.7) where μ_s is the chemical potential for s-th species. Here, we have made an approximation that $T(p)=\log\cosh p\sim|p|$. This is valid when $N_2^{(s)}$ is large.

The semi-classical equation (4.7) is expected to be valid when $x_m^{(s)}\gg 1$ in the large-$N_2^{(s)}$ limit. One can see that the quantum corrections in the Fermi gas model are indeed negligible [2] when $x_m^{(s)}\gg 1$. At least for $\Lambda=1$, one can also see that the condition $x_m^{(1)}\gg 1$ is written as $g^2N_2^{(1)}\gg N_5^{(1)}$ in terms of the original parameters in BMN model [2]. This is a strong coupling region of BMN model and corresponds to the region in the gravity side where the α' corrections are negligible.

4.2 Mapping to the gravity side

Notice that the saddle point equations of the eigenvalue densities (4.7) take a very similar form as the integral equations for the charge densities (2.19). In fact, by using the relations (2.6) and (2.7), one can find that they are exactly the same equations. Thus, we arrive at the relations

$$g^2\rho^{(s)}(x)=\frac{1}{V_0}\left(\frac{2}{\pi}\right)^3 f_s\left(\frac{\pi}{2}x\right), \quad (4.18)$$

and

$$\frac{\pi}{2}x_m^{(s)}=R_s. \quad (4.19)$$

Namely, the eigenvalue density on the gauge theory side has exactly the same functional form and parameter dependence as the charge density on the gravity side, up to the trivial rescaling. Hence, they can naturally be identified with each other and this identification relates the degrees of freedom on the gauge theory side to the background geometry on the gravity side. By integrating both sides of (4.18) over $[-x_m^{(s)}, x_m^{(s)}]$ and using (2.21) and (4.8), we find that those relations are consistent with (2.7) and (2.8).

If one finds exact solutions of (2.19) and (4.7), one can check the relations, (4.18) and (4.19), more explicitly. Although we could not find general exact solutions, still we can solve those equations in particular parameter regions. If a conducting disk is isolated at a distance from the other disks, the term with an integration in the integral equation of the disk becomes negligible. Then, the solution is simply given by a quadratic function. This is effectively the same situation as the D2-brane limit with $\Lambda = 1$ considered in [2]. In the same way, we can consider two isolated disks which effectively form the same system as the NS5-brane limit with $\Lambda = 1$ [2]. In these cases, one can find the exact solutions and check the relations, (4.19) and (4.18), more directly.

4.3 Higher dimensional $SU(2|4)$ symmetric gauge theories

The equivalence between the charge density and the eigenvalue density also holds for the other gauge theories with $SU(2|4)$ symmetry.

In the D2-brane limit (3.32), the partition function (3.128) reduces to the matrix integral for SYM on $R \times S^2$ given by

$$
\begin{aligned}
Z^{R\times S^2}_{\{(q_s,N_2^{(s)})\}} = \int \prod_{s=1}^{\Lambda}\prod_{i=1}^{N_2^{(s)}} dm_{si} \prod_{s=1}^{\Lambda}\Delta(m_s)^2 \prod_{s=1}^{\Lambda}\prod_{i,j=1}^{N_2^{(s)}} \left[\frac{1+\left(\frac{m_{si}-m_{sj}}{2}\right)^2}{\{1+(m_{si}-m_{sj})^2\}^2}\right]^{\frac{1}{2}} \\
\times \prod_{s,t=1}^{\Lambda}\prod_{\substack{J=|q_s-q_t|\\ J\neq 0}}^{\infty}\prod_{i=1}^{N_2^{(s)}}\prod_{j=1}^{N_2^{(t)}} \left[\frac{\left\{1+\left(\frac{m_{si}-m_{tj}}{2J+2}\right)^2\right\}\left\{1+\left(\frac{m_{si}-m_{tj}}{2J}\right)^2\right\}}{\left\{1+\left(\frac{m_{si}-m_{tj}}{2J+1}\right)^2\right\}^2}\right]^{\frac{1}{2}} e^{-\frac{m\pi}{g_{S^2}^2}\sum_{s,i} m_{si}^2},
\end{aligned}
\tag{4.20}
$$

where $\Delta(m_s) = \prod_{i<j}(m_{si} - m_{sj})$ is the Vandermonde determinant. The $\Lambda = 1$ case can be obtained from this equation and compared with the gravity side analysis in the end of section 2.3.

In addition, by taking the T-duality with (3.33) and (3.41), we end up with the matrix integral for SYM on $R \times S^3/Z_k$ given by

$$
Z^{R\times S^3/Z_k}_{\{(\alpha,N_2^{(\alpha)})\}} = \int \prod_{\alpha}\prod_{i=1}^{N_2^{(\alpha)}} dm_{\alpha i} \prod_{\alpha}\Delta(m_\alpha)^2 \prod_{\alpha}\prod_{i,j=1}^{N_2^{(\alpha)}} \left[\frac{1+\left(\frac{m_{\alpha i}-m_{\alpha j}}{2}\right)^2}{\{1+(m_{\alpha i}-m_{\alpha j})^2\}^2}\right]^{\frac{1}{2}}
$$

$$\times \prod_{u=-\infty}^{\infty} \prod_{\alpha,\beta} \prod_{\substack{J=|ku/2+(\alpha-\beta)/2| \\ J\neq 0}}^{\infty} \prod_{i=1}^{N_2^{(\alpha)}} \prod_{j=1}^{N_2^{(\beta)}} \left[\frac{\left\{1+\left(\frac{m_{\alpha i}-m_{\beta j}}{2J+2}\right)^2\right\}\left\{1+\left(\frac{m_{\alpha i}-m_{\beta j}}{2J}\right)^2\right\}}{\left\{1+\left(\frac{m_{\alpha i}-m_{\beta j}}{2J+1}\right)^2\right\}^2} \right]^{\frac{1}{2}}$$

$$\times e^{-\frac{4\pi^2}{kg^2_{S^3/Z_k}} \sum_{\alpha,i} m_{\alpha i}^2}. \tag{4.21}$$

If we apply the corresponding limits discussed in section 2.3 and 2.4 to the integral equation (2.19) of the charge density, we obtain the integral equations (2.36) and (2.43) for SYM on $R \times S^2$ and SYM on $R \times S^3/Z_k$, respectively. In these cases, the integral equations (2.36) and (2.43) for the charge densities can also be identified with the saddle point equations for (4.20) and (4.21), respectively, where the same relations as (4.18) and (4.19) hold. From the gauge theory side, the limits to obtain these higher dimensional gauge theories are discussed in section 3.3.1 and 3.3.2, and they are consistent with the gauge/gravity duality.

Note that the NS5-brane limit (2.55) can also be applied to the partition function (3.128). This is considered to reduce to the matrix integral for IIA LST on $R \times S^5$. For the $\Lambda = 1$ case, we have the equation (2.26) and the solution in this limit is (2.57).

5 Summary and Discussion

We showed that the bubbling geometries in type IIA supergravity are realized in the gauge theories with $SU(2|4)$ symmetry. It was also found that the charge densities of the electrostatic systems in the gravity dual are equivalent to the eigenvalue densities of the matrix integrals which govern the 1/4-BPS sector of the gauge theories.

On the gravity side, the bubbling geometries are given in terms of the electrostatic potential of electrostatic systems with conducting disks. First, we have considered the electrostatic system corresponding to BMN model around a general vacuum. We have shown that the boundary conditions of the potential are given by a system of dual integral equations. Extending the method to analyze the dual integral equations written in [41], we have reduced the dual integral equations to the Fredholm integral equations of the second kind for the charge densities on the disks. By taking the D2-brane limit or performing the T-duality as well as the D2-brane limit, we have also obtained the same type of integral equations for the charge densities in the electrostatic system corresponding to SYM on $R \times S^2$ or SYM on $R \times S^3/Z_k$.

On the gauge theory side, we have investigated the matrix integrals that describe 1/4-BPS sectors of the gauge theories. First, we have considered the case for BMN model around a general vacuum in the regime where the supergravity approximation is valid. In this regime, we have derived the saddle point equations of the eigenvalue densities of the matrix integral, which are almost the same integral equations for the charge densities on the gravity side. Then we have found that under the identifications of (2.6) and (2.7) the integral equations of the eigenvalue densities are exactly equivalent to those for the charge densities. As the D2-brane limit and the T-duality of BMN model lead to the other gauge theories with $SU(2|4)$ symmetry, that is, SYM on $R \times S^2$ and SYM on $R \times S^3/Z_k$, we have also shown the equivalence of the charge densities and the eigenvalue densities. Thus, we have concluded that since the bubbling geometries are completely determined by the charge densities the geometries are constructed from the eigenvalue densities of the gauge theories with $SU(2|4)$ symmetry.

For $\Lambda = 1$, the saddle point equation in the D2-brane limit and NS5-brane limit are soluble. By solving the equation in these limits, we found the value of a in (2.55), which was not fixed solely by the argument in [35]. We can reproduce the radius of S^5's in the D2-brane and the NS5-brane geometries in terms of the solutions on the gauge theory side.[2] These results strongly support our identification between the eigenvalue density and the electric charge density. In particular, our result in the NS5-brane limit reproduces the known behavior of the five-brane radius proportional to $\lambda^{1/4}$[14] and it gives a strong evidence for the description of five-branes in BMN model proposed in [14].

From the above observations, the results shown in this paper have provided a new evidence of this gauge/gravity duality. The procedure used for obtaining the dual geometries is expected to be applied to other proposed gauge/gravity dualities, for example dualities concerning ABJM theory [44], $\mathcal{N} = 2$ superconformal field theories [45], and so forth.

Finally, let us comment on IIA LST on $R \times S^5$, which is another theory with $SU(2|4)$ symmetry. Like other $SU(2|4)$ symmetric theories, LST on $R \times S^5$ is thought to have many discrete vacua and for each vacuum there exists a gravity dual given by type IIA bubbling geometry [13]. The gravity dual of LST around the trivial vacuum was elaborated in [35] and shown to be obtained from a double scaling limit of the gravity dual of BMN model around a particular vacuum. Although it is also expected that the gravity dual of LST around a general vacuum can be obtained from the same kind of double scaling limit of the gravity dual of BMN model, some careful analysis seems to be needed.

Acknowledgment

First, I would like to thank my collaborators relating to this work, who are Goro Ishiki, Shinji Shimasaki and Takashi Okada. I had learned so many things and been stimulated very much through the discussion with them. I greatly appreciate support from my supervisor, Hikaru Kawai. Not only pedagogical and fruitful discussion with him, was I helped with various things including private ones. I also would like to thank Asato Tsuchiya for collaborating with me and Jun Nishimura for teaching computational analysis for matrix models related to this work. Finally, I would like to thank the members of Theoretical Particle Physics Group at Dept. of Physics, Kyoto University.

A Notations for Gamma matrices

Our gamma matrices are the same as those in [5].

Let us denote the ten-dimensional 32×32 gamma matrices by γ^M $(M = 1, \cdots, 9, 10)$. They obey

$$\gamma^{\{M}\gamma^{N\}} = \eta^{MN}, \tag{A.1}$$

where η_{MN} is the local Lorentz metric: $\eta_{MN} = \mathrm{diag}(1, \cdots, 1)$ for $M, N = 1, \cdots, 9, 10$. The associated representation of $Spin(10)$ can be decomposed into two irreducible representations by the chirality,

$$\gamma^{11} \equiv -i\gamma^1 \cdots \gamma^9\gamma^{10}. \tag{A.2}$$

Then, let us decompose the ten-dimensional Dirac spinor into left-handed and right-handed Weyl spinors and express the gamma matrices γ^M in the block form,

$$\gamma^M = \begin{pmatrix} 0 & \tilde{\Gamma}^M \\ \Gamma^M & 0 \end{pmatrix}. \tag{A.3}$$

Every γ^M turns to be symmetric by being multiplied with the charge conjugate matrix C, which is the invariant matrix satisfying $C^{-1}\gamma^M C = -\gamma^{M\,T}$, $\gamma^{11}C = -C\gamma^{11}$ and $C^T = -C$. If C is taken as

$$C = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}, \tag{A.4}$$

Γ^M and $\tilde{\Gamma}^M$ are symmetric:

$$(\Gamma^M)^T = \Gamma^M, \ (\tilde{\Gamma}^M)^T = \tilde{\Gamma}^M. \tag{A.5}$$

γ^{MN}, Γ^{MN} and $\tilde{\Gamma}^{MN}$ are defined as

$$\gamma^{MN} \equiv \gamma^{[M}\gamma^{N]} = \begin{pmatrix} \tilde{\Gamma}^{[M}\Gamma^{N]} & 0 \\ 0 & \Gamma^{[M}\tilde{\Gamma}^{N]} \end{pmatrix} \equiv \begin{pmatrix} \Gamma^{MN} & 0 \\ 0 & \tilde{\Gamma}^{MN} \end{pmatrix}. \tag{A.6}$$

Then, the following equations hold:

$$\tilde{\Gamma}^{\{M}\Gamma^{N\}} = \Gamma^{\{M}\tilde{\Gamma}^{N\}} = \eta^{MN}, \tag{A.7}$$

$$\Gamma^M\Gamma^{PQ} = 4g^{M[P}\Gamma^{Q]} + \tilde{\Gamma}^{PQ}\Gamma^M. \tag{A.8}$$

There are some useful identities:

$$(\Gamma_M)_{\alpha_1\{\alpha_2}(\Gamma^M)_{\alpha_3\alpha_4\}} = 0, \tag{A.9}$$

$$(\Gamma^M)_{\alpha\delta}(\Gamma_M)_{\gamma\beta} = -\frac{1}{2}(\Gamma^M)_{\alpha\beta}(\Gamma_M)_{\gamma\delta} + \frac{1}{24}(\Gamma^{MNP})_{\alpha\beta}(\Gamma_{MNP})_{\gamma\delta}, \tag{A.10}$$

$$(\tilde{\Gamma}^{MN})_\alpha{}^\beta(\tilde{\Gamma}_{MN})_\gamma{}^\delta = 4(\Gamma^M)_{\alpha\gamma}(\tilde{\Gamma}_M)^{\beta\delta} - 2\,\delta_\alpha^\beta\delta_\gamma^\delta - 8\,\delta_\alpha^\delta\delta_\gamma^\beta, \tag{A.11}$$

where $\alpha, \beta, \cdots$ are spinor indices. The first equality is so called "triality", and the last two are Fierz identities. Decomposing the indices $M = 0, \cdots, 9$ into $a = 1, 2, 3, 4$ and $m = 0, 5, \cdots, 9$, one obtains the following identities

$$\Gamma_{am}\tilde{\Gamma}^a = -4\tilde{\Gamma}_m, \tag{A.12}$$

$$\Gamma^a\Gamma_{bc}\tilde{\Gamma}_a = 0, \tag{A.13}$$

$$\Gamma^a\Gamma_{bm}\tilde{\Gamma}_a = 2\tilde{\Gamma}_{bm}, \tag{A.14}$$

$$\Gamma^a\Gamma_{mn}\tilde{\Gamma}_a = 4\tilde{\Gamma}_{mn}. \tag{A.15}$$

The explicit form of the gamma matrices Γ^M and $\tilde{\Gamma}^M$ is written below.

$$\Gamma^i = \tilde{\Gamma}^i = \begin{pmatrix} 0 & E_i^T \\ E_i & 0 \end{pmatrix}, \ (i = 1, \cdots, 8)$$

$$\Gamma^9 = \tilde{\Gamma}^9 = \begin{pmatrix} 1_{8\times8} & 0 \\ 0 & -1_{8\times8} \end{pmatrix}, \quad \Gamma^{10} = -\tilde{\Gamma}^{10} = i\begin{pmatrix} 1_{8\times8} & 0 \\ 0 & 1_{8\times8} \end{pmatrix}. \tag{A.16}$$

We also use $\Gamma^0 := -i\Gamma^{10} = -\Gamma_0$. The 8×8 matrices E_i $(i = 1, \cdots, 8)$ are given as follows:

$$E_{a'} = \begin{pmatrix} J_{a'} & 0 \\ 0 & \bar{J}_{a'} \end{pmatrix} \ (a' = 1, 2, 3, 4), \quad E_{m'} = \begin{pmatrix} 0 & -J_{m'}^T \\ J_{m'} & 0 \end{pmatrix} \ (m' = 5, 6, 7, 8). \tag{A.17}$$

The 4×4 matrices $J_{a'}, \bar{J}_{a'}$ are given by

$$J_1 = 1_{4\times4}, \ \bar{J}_1 = 1_{4\times4},$$

$$J_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ J_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \ J_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{A.18}$$

$$\bar{J}_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \ \bar{J}_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ \bar{J}_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and $J_{m'}$ by

$$J_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, J_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{A.19}$$

$$J_7 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \ J_8 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices J_a and $\bar{J}_a$ satisfy

$$J_a J_b = -\delta_{ab}\mathbf{1}_4 + \varepsilon_{abc} J_c, \quad \bar{J}_a \bar{J}_b = -\delta_{ab}\mathbf{1}_4 - \varepsilon_{abc}\bar{J}_c \ \ (a,\, b,\, c = 2, 3, 4). \tag{A.20}$$

Note that, in this representation, "partial" chiralities are written as

$$\Gamma^{1234} = \Gamma^1\Gamma^2\Gamma^3\Gamma^4 = \begin{pmatrix} 1_{4\times 4} & 0 & 0 & 0 \\ 0 & -1_{4\times 4} & 0 & 0 \\ 0 & 0 & -1_{4\times 4} & 0 \\ 0 & 0 & 0 & 1_{4\times 4} \end{pmatrix}, \tag{A.21}$$

$$\Gamma^{5678} = \Gamma^5\Gamma^6\Gamma^7\Gamma^8 = \begin{pmatrix} 1_{4\times 4} & 0 & 0 & 0 \\ 0 & -1_{4\times 4} & 0 & 0 \\ 0 & 0 & 1_{4\times 4} & 0 \\ 0 & 0 & 0 & -1_{4\times 4} \end{pmatrix}.$$

B Notations for S^3

In this appendix, we review the notation S^3 coordinates and their properties.

Firstly, let us remind that S^3 can be written by a quaternion:

$$q = x^1 + \boldsymbol{i}x^2 + \boldsymbol{j}x^3 + \boldsymbol{k}x^4 \in \mathbb{H}, \quad |q|^2 = r^2$$
$$\Longrightarrow \ |q|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = r^2.$$

r is the radius of S^3, and $r = 2/m = 1$ in this paper. By noting that the algebra of quaternions is the same as $\mathfrak{su}(2)$ algebra, one can show that S^3 and $SU(2)$ group manifold are diffeomorphic. The coordinate of S^3 is written as an element of $SU(2)$:

$$g = \frac{x^1}{r}\mathbf{1} + i\frac{x^2}{r}\sigma_1 + i\frac{x^3}{r}\sigma_2 + i\frac{x^4}{r}\sigma_3 \in SU(2). \quad (g^\dagger g = \mathbf{1})$$

Also, g can be written by Euler angles as

$$g = e^{-i\varphi\sigma_3/2} e^{-i\theta\sigma_2/2} e^{-i\psi\sigma_3/2} \in SU(2).$$

The relationship between the Cartesian coordinates and Euler angles is

$$\begin{aligned} x^1 &= r\sin\frac{\theta}{2}\sin\frac{\varphi-\psi}{2}, \\ x^2 &= -r\sin\frac{\theta}{2}\cos\frac{\varphi-\psi}{2}, \\ x^3 &= -r\cos\frac{\theta}{2}\sin\frac{\varphi+\psi}{2}, \\ x^4 &= r\cos\frac{\theta}{2}\cos\frac{\varphi+\psi}{2}. \end{aligned}$$

This is convenient for discussing the relationship between S^3 and S^2. Its matrix elements are explicitly written as

$$g = \begin{pmatrix} e^{-i\frac{\varphi+\psi}{2}}\cos\frac{\theta}{2} & -e^{-i\frac{\varphi-\psi}{2}}\sin\frac{\theta}{2} \\ e^{i\frac{\varphi-\psi}{2}}\sin\frac{\theta}{2} & e^{i\frac{\varphi+\psi}{2}}\cos\frac{\theta}{2} \end{pmatrix},$$

where the ranges of the coordinates are $0 \le \theta \le \pi$, $0 \le \varphi < 2\pi$, $0 \le \psi < 4\pi$.

The isometry of S^3 is $SO(4) \cong SU(2)_L \times SU(2)_R$. $g^{-1}dg$ is left-invariant ($SU(2)_L \ni \bar{h} : g \to \bar{h}g$), and dgg^{-1} is right-invariant ($SU(2)_R \ni h : g \to gh$). If one writes the metric as

$$ds^2 = -\frac{r^2}{2}\operatorname{tr}[(g^{-1}dg)^2] = -\frac{r^2}{2}\operatorname{tr}[(dgg^{-1})^2],$$

its isometry is manifest. By using Euler angles, it turns out to be

$$ds^2 = \frac{r^2}{4}[d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\phi)^2]. \tag{B.1}$$

Since $\operatorname{tr}[g^{-1}dg] = d(\det g) = 0$, $g^{-1}dg$ can be expanded by Pauli matrices σ_a as

$$g^{-1}dg = -\frac{i}{r}\bar{e}^a\sigma_{a-1}, \quad dgg^{-1} = -\frac{i}{r}e^a\sigma_{a-1},$$

and their coefficients $\bar{e}^a$, e^a are the left-invariant dreibein and the right-invariant dreibein, respectively. The index a is contracted running from 2 to 4. The left-invariant dreibein is

$$\bar{e}^2 = \frac{r}{2}(d\theta\sin\psi - d\varphi\sin\theta\cos\psi),$$

$$\bar{e}^3 = \frac{r}{2}(d\theta \cos\psi + d\varphi \sin\theta \sin\psi),$$
$$\bar{e}^4 = \frac{r}{2}(d\psi + d\varphi \cos\theta).$$

One obtains the right-invariant dreibein by flipping the sign of θ, ϕ, ψ, swapping $\phi \leftrightarrow \psi$ and flipping the overall signs of the dreibeins:

$$\begin{aligned} e^2 &= \frac{r}{2}(-d\theta \sin\varphi + d\psi \sin\theta \cos\varphi), \\ e^3 &= \frac{r}{2}(d\theta \cos\varphi + d\psi \sin\theta \sin\varphi), \\ e^4 &= \frac{r}{2}(d\varphi + d\psi \cos\theta). \end{aligned} \tag{B.2}$$

From now on, let us concentrate on the right-invariant dreibein. One sees that the right-invariant dreibein satisfies the Maurer-Cartan equations,

$$de^a = \frac{1}{r}\varepsilon^{abc} e^b e^c$$

by acting a differential on $g^{-1}dg$. If one defines the spin connection as $\nabla_a A_b = e_a^{\ \mu}\partial_\mu A_b - \omega_{abc}A^c$, it is written as $\omega^{abc} = -\frac{1}{r}\varepsilon^{abc}$.

The Hopf fibration $S^3 \to S^2$ (fiber: $U(1) \cong S^1$) is realized in the following manner. Let us set patch N and S of $S^2 \cong SU(2)/U(1)$ as $0 \le \theta < \pi$ and $0 < \theta \le \pi$, respectively, and decompose g into L and h:

$$\begin{aligned} g &= Lh, \\ L &= e^{-i\varphi\sigma_3/2} e^{-i\theta\sigma_2/2} e^{\pm i\varphi\sigma_3/2}, \\ h &= e^{-i(\psi\pm\varphi)\sigma_3/2} =: e^{-iy\sigma_3/2}, \qquad (0 \le y < 4\pi) \end{aligned}$$

where $\pm$ correspond to patch N and S, respectively. It realizes S^2 with radius $r/2$ as L and $U(1)$ as h. $SU(2)_L$ symmetry out of the isometry of S^3 remains as the isometry of S^2.

Let us see a concrete realization of the fibration. By using Hopf coordinates, S^3 is expressed as

$$|z_1|^2 + |z_2|^2 = 1,$$

where $z_1 = -e^{-\frac{i}{2}(\psi-\phi)} \sin\frac{\theta}{2}$ and $z_2 = e^{-\frac{i}{2}(\psi+\phi)} \cos\frac{\theta}{2}$. The local trivialization of the fibration, $\phi^{-1} : S^3 \to (S^2, U(1))$, is

$$\phi_N^{-1} : (z_1, z_2) \mapsto (-z_1/z_2,\ z_2/|z_2|) = (e^{i\phi}\tan\frac{\theta}{2},\ e^{-iy/2}),$$
$$\phi_S^{-1} : (z_1, z_2) \mapsto (z_2/z_1,\ -z_1/|z_1|) = (e^{i(\pi-\phi)}\cot\frac{\theta}{2},\ e^{-iy/2}).$$

The transition functions at $\theta = \pi/2$ are $t_{NS} = -z_2/z_1 = e^{-i\phi}$, and the coordinates of S^2 is represented by the stereographic coordinates.

C Dual integral equations

Let $A(u)$ be a function defined on $[0,\infty)$. We define two functions on $[0,\infty)$ using $A(u)$ as

$$\phi(x) = \int_0^\infty u^{-2\alpha}(1+k(u))A(u)J_\nu(xu)du, \tag{C.1}$$

$$\chi(x) = \int_0^\infty A(u)J_\nu(xu)du, \tag{C.2}$$

where α is a positive half integer, $k(u)$ is a given weight function on $[0,\infty)$ and $J_\nu(z)$ is the Bessel function of the first kind of order ν defined by

$$J_\nu(z) = \sum_{s=0}^{\infty} \frac{(-1)^s (z/2)^{\nu+2s}}{s!\Gamma(\nu+s+1)}. \tag{C.3}$$

In this appendix, we consider a problem of finding a solution for $A(u)$, which solves the following equations called the dual integral equations.

$$\phi(x) = F(x) \quad \text{for} \quad x \in I_1. \tag{C.4}$$

$$\chi(x) = G(x) \quad \text{for} \quad x \in I_2. \tag{C.5}$$

Here we have divided the positive real line $[0,\infty)$ to two segments denoted by I_1 and I_2, which are written as $[0,c)$ and $[c,\infty)$, respectively. $F(x)$ and $G(x)$ are assumed to be known functions defined on I_1 and I_2, respectively. We will see that the problem reduces to a problem of solving a single Fredholm integral equation of the second kind [41].

In the following, for any function $f(x)$ on $[0,\infty)$, we denote by $f_1(x)$ and $f_2(x)$ the restrictions of $f(x)$ to I_1 and I_2, respectively. We assume that $f_i(x) = 0$ unless $x \in I_i$ $(i = 1,2)$, so that the original function can be written as $f(x) = f_1(x) + f_2(x)$. For example, the equation (C.4) can be written in this notation as $\phi_1(x) = F(x)$ for $x \in I_1$.

We also introduce the modified Hankel transformation,

$$S_{\eta,\alpha} f(x) := \left(\frac{2}{x}\right)^\alpha \int_0^\infty t^{1-\alpha} f(t) J_{2\eta+\alpha}(xt)dt. \tag{C.6}$$

The inverse transformation is given by

$$S_{\eta,\alpha}^{-1} = S_{\eta+\alpha,-\alpha}. \tag{C.7}$$

It is easy to see that

$$\phi(x) = \left(\frac{x}{2}\right)^\alpha S_{\nu/2-\alpha,2\alpha}\{(1+k)\cdot\psi\}(x), \tag{C.8}$$

$$\chi(x) = S_{\nu/2,0}\psi(x), \tag{C.9}$$

where $\psi(u)$ is defined by

$$A(u) = u\psi(u), \tag{C.10}$$

and the dot in (C.8) denotes the product of functions defined as usual by $f \cdot g(x) = f(x)g(x)$.

We first put

$$\psi(u) = S_{\nu/2,-\alpha}h(u), \tag{C.11}$$

and substitute this to (C.9). Then, after performing the inverse transformation (C.7) twice, we obtain

$$h(x) = S_{\nu/2+\alpha,-\alpha}S_{\nu/2,0}\chi(x) = K_{\nu/2+\alpha,-\alpha}\chi(x), \tag{C.12}$$

where we have defined

$$K_{\eta,\alpha+\beta} = S_{\eta,\alpha}S_{\eta+\alpha,\beta}. \tag{C.13}$$

One can show that the transformation $K_{\eta,\alpha}$ can be written as

$$K_{\eta,\alpha}f(x) = \begin{cases} \frac{2x^{2\eta}}{\Gamma(\alpha)}\int_x^\infty (u^2-x^2)^{\alpha-1}u^{-2\alpha-2\eta+1}f(u)du & \text{for } 0<\alpha, \\ -\frac{x^{2\eta-1}}{\Gamma(1+\alpha)}\frac{d}{dx}\int_x^\infty u^{-2\alpha-2\eta+1}(u^2-x^2)^{\alpha}f(u)du & \text{for } -1<\alpha<0. \end{cases} \tag{C.14}$$

See [41] for the definition for $\alpha < -1$. From (C.12) and (C.14), we find that the solution for $h_2(x)$ is given by

$$h_2(x) = K_{\nu/2+\alpha,-\alpha}\chi_2(x) = K_{\nu/2+\alpha,-\alpha}G(x). \tag{C.15}$$

Similarly, by applying the same calculation to (C.8), one can obtain the following equation for $x \in I_1$.

$$h_1(x) + S_{\nu/2-\alpha,\alpha}\{k \cdot S_{\nu/2,-\alpha}h_1\}(x) = H(x). \tag{C.16}$$

The function $H(x)$ is defined by[10]

$$H(x) = \left(\frac{2}{x}\right)^{2\alpha} I_{\nu/2,-\alpha}F(x) - S_{\nu/2-\alpha,\alpha}k(x)S_{\nu/2,0}G(x), \tag{C.17}$$

[10]The last term in (C.17) is obtained by using (C.15) and the relation, $S_{\eta,\alpha}K_{\eta+\alpha,\beta} = S_{\eta,\alpha+\beta}$.

where $I_{\eta,\alpha+\beta}$ is defined by

$$I_{\eta,\alpha+\beta} = S_{\eta+\alpha,\beta} S_{\eta,\alpha}, \tag{C.18}$$

and it is written more explicitly as

$$I_{\eta,\alpha} f(x) = \begin{cases} \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^x u^{2\eta+1} (x^2 - u^2)^{\alpha-1} f(u) du & \text{for} \;\; 0 < \alpha, \\ \frac{x^{-2\alpha-2\eta-1}}{\Gamma(1+\alpha)} \frac{d}{dx} \int_0^x u^{2\eta+1} (x^2 - u^2)^{\alpha} f(u) du & \text{for} \;\; -1 < \alpha < 0. \end{cases} \tag{C.19}$$

Note that $H(x)$ depends only on the known functions $k(x)$, $F(x)$ and $G(x)$. For the second term in the left-hand side of (C.16), we interchange the order of the integration, so that

$$S_{\nu/2-\alpha,\alpha}\{k \cdot S_{\nu/2,-\alpha} h_1\}(x) = \int_0^1 K(x,u) h_1(u) du, \tag{C.20}$$

where the integral kernel $K(x,u)$ is defined by

$$K(x,u) = u \left(\frac{u}{x}\right)^{\alpha} \int_0^{\infty} t k(t) J_{\nu-\alpha}(xt) J_{\nu-\alpha}(ut) dt. \tag{C.21}$$

Hence, we conclude that h_1 is the solution of the Fredholm integral equation of the second kind,

$$h_1(x) + \int_0^1 K(x,u) h_1(u) du = H(x). \tag{C.22}$$

The two equations (C.15) and (C.22) fully determines $h(x) = h_1(x) + h_2(x)$.

Now, we consider a generalization of this result to vector-valued functions. This is easy since the above problem is linear in $A(u)$. Let us consider a set of functions $\{A_s(u) | s = 1, \cdots, n\}$ which are determined by the equations,

$$\phi_s(x) = F_s(x) \quad \text{for} \;\; x \in I_1^{(s)}, \tag{C.23}$$

$$\chi_s(x) = G_s(x) \quad \text{for} \;\; x \in I_2^{(s)}, \tag{C.24}$$

where $\phi_s(x)$ and $\chi_s(x)$ are now defined by

$$\phi_s(x) = \int_0^{\infty} u^{-2\alpha} \sum_{t=1}^{n} (\delta_{st} + k_{st}(u)) A_t(u) J_\nu(xu) du, \tag{C.25}$$

$$\chi_s(x) = \int_0^{\infty} A_s(u) J_\nu(xu) du. \tag{C.26}$$

For each s, $F_s(x)$ and $G_s(x)$ are assumed to be known functions and $I_{1,2}^{(s)}$ are the two connected intervals, the sum of which is equal to $[0, \infty)$. If we write for each s

$$A_s(u) = u S_{\nu/2,-\alpha} h_s(u), \tag{C.27}$$

it is easy to see that $h_s(u)$ are determined by the following equations.

$$h_{s2}(x) = K_{\nu/2+\alpha,-\alpha} G_s(x). \tag{C.28}$$

$$h_{s1}(x) + \sum_{t=1}^{n} \int_0^1 K_{st}(x,u) h_{t1}(u) du = H_s(x), \tag{C.29}$$

where $H_s(x)$ and $K_{st}(x,y)$ are defined by

$$H_s(x) = \left(\frac{2}{x}\right)^{2\alpha} I_{\nu/2,-\alpha} F_s(x) - \sum_{t=1}^{n} S_{\nu/2-\alpha,\alpha} k_{st}(x) S_{\nu/2,0} G_t(x), \tag{C.30}$$

and

$$K_{st}(x,u) = u \left(\frac{u}{x}\right)^{\alpha} \int_0^{\infty} t k_{st}(t) J_{\nu-\alpha}(xt) J_{\nu-\alpha}(ut) dt. \tag{C.31}$$

D Thomas-Fermi approximation

In this appendix, we review the Thomas-Fermi approximation, which is the semi-classical limit of the Hartree approximation. We consider a one-dimensional many-body system at finite temperature $1/\beta$ that has a one-body Hamiltonian of the form $h(q,p) = T(p) + U(q)$ and a two-body interaction potential $W(q,q')$. The Hartree approximation is just the saddle-point evaluation of the path integral of this system and becomes exact when the particle number N goes to infinity. In this approximation, the saddle point is characterized by the mean-field density $\rho(x)$ that satisfies the normalization

$$\int dq \rho(q) = N. \tag{D.1}$$

$\rho(x)$ is determined by the following Hartree equation.

$$\rho(q) = \left\langle q \left| \frac{1}{e^{\beta(H(\hat{p},\hat{q})-\mu)} + 1} \right| q \right\rangle, \tag{D.2}$$

where μ is the chemical potential and $H(p,q)$ is the effective one-body Hamiltonian defined by

$$H(p,q) = T(p) + U(q) + \int dq' W(q,q') \rho(q'). \tag{D.3}$$

If one obtains $\rho(x)$ by solving the equation (D.2), then from (D.1) one can also compute the first derivative of the grand potential as

$$\frac{\partial J}{\partial \mu} = \int dq \rho(q). \tag{D.4}$$

The free energy is given by

$$F = \log Z = J(\mu(N)) - \mu(N)N. \tag{D.5}$$

In the semi-classical limit, the Hartree equation (D.2) reduces to

$$\rho(q) = \int \frac{dp}{2\pi\hbar} \frac{1}{e^{\beta(H(p,q)-\mu)}+1}. \tag{D.6}$$

This equation is called the Thomas-Fermi equation at finite temperature. When the temperature goes to zero, the equation (D.6) is further simplified to

$$\rho(q) = \int \frac{dp}{2\pi\hbar} \theta(\mu - H(p,q)). \tag{D.7}$$

Let us assume that the Fermi surface $\{(p,q)|\mu = H(p,q)\}$ is simply connected and symmetric under $p \to -p$. Then (D.7) implies that $\rho(q)$ is given by

$$\rho(q) = \frac{p_F(q)}{\pi\hbar}, \tag{D.8}$$

where $p_F(q)$ is the Fermi momentum. From the definition of $p_F(q)$, we obtain the following integral equation that determines $\rho(q)$.

$$\mu = T(\pi\hbar\rho(q)) + U(q) + \int dq' W(q,q')\rho(q'). \tag{D.9}$$

This equation can be regarded as an extremization condition for the Thomas-Fermi functional,

$$E_{\mathrm{TF}}[\rho] = \int dq\, t_{\mathrm{TF}}(q) + \int dq \rho(q) U(q) + \frac{1}{2}\int dq dq' \rho(q) W(q,q') \rho(q') - \mu \left(\int dq \rho(q) - N \right) \tag{D.10}$$

where $t_{\mathrm{TF}}(q)$ is the kinetic energy functional

$$t_{\mathrm{TF}}(q) = \int \frac{dp}{2\pi\hbar} T(p)\theta(\mu - H(p,q)), \tag{D.11}$$

and μ is the Lagrange multiplier associated with the constraint (D.1), which can be identified with the chemical potential at the saddle point. The free energy is given by (D.10) with ρ satisfying (D.9);

$$F = -\min_\rho E_{\mathrm{TF}}[\rho]. \tag{D.12}$$

E Saddle-point method for the D2-brane limit

In this appendix, we solve our matrix integral for the D2-brane limit in the planar limit by applying the usual saddle-point method [43]. We assume the one-cut solution. The matrix integral in this limit is given by

$$Z = \int \prod_i dq_i \prod_{i<j} \tanh^2 \left(\frac{\pi(q_i - q_j)}{2} \right) e^{-\frac{2\pi}{g_{S^2}^2} \sum_i q_i^2}. \tag{E.1}$$

By changing the integral variables to $z_i := \exp(\pi q_i + g_{S^2}^2 \pi/4)$, the path integral is reduced to

$$Z = \int \prod_i dz_i \prod_{i>j} \left(\frac{z_i - z_j}{z_i + z_j} \right)^2 e^{-\frac{2}{g_{S^2}^2 \pi} \sum_i (\log z_i)^2}. \tag{E.2}$$

The saddle-point equation is given by

$$\frac{2}{g_{S^2}^2 \pi} \frac{\log z_i}{z_i} - \sum_{j(\neq i)} \left(\frac{1}{z_i - z_j} - \frac{1}{z_i + z_j} \right) = 0. \tag{E.3}$$

Note that this equation is symmetric under the inversion, $z_i \to 1/z_i$. Let $[a, b]$ be the support of the eigenvalue distribution of z_i. From the inversion symmetry, it follows that $b = 1/a$. We define the resolvent as

$$W(z) = 4 g_{S^2}^2 \pi \sum_i \left(\frac{1}{z - z_i} - \frac{1}{z + z_i} \right). \tag{E.4}$$

This function has two branch cuts at $[a, b]$ and $[-b, -a]$. The eigenvalue distribution,

$$\rho(z) = \frac{1}{N_2} \sum_i \delta(z - z_i), \tag{E.5}$$

can be expressed as the discontinuity of $W(z)$ as usual,

$$W(z + i0) - W(z - i0) = -8\pi^2 i g_{S^2}^2 N_2 \rho(z). \tag{E.6}$$

We introduce a new variable $y = z^2$. The resolvent is also a holomorphic function of y. So let us denote $W(z) = P(y)$, where $P(y)$ is holomorphic in y. $P(y)$ has a single cut at $[a^2, b^2]$ on the y-plane. Using (E.3), one can easily get

$$P(y + i0) + P(y - i0) = \frac{8 \log y}{\sqrt{y}}, \tag{E.7}$$

where $y \in [a^2, b^2]$. By defining a new function,

$$\hat{P}(y) = \frac{P(y)}{\sqrt{(y-a^2)(y-b^2)}}, \tag{E.8}$$

one can convert (E.7) to the discontinuity equation,

$$\hat{P}(y+i0) - \hat{P}(y-i0) = \frac{1}{\sqrt{(y-a^2)(y-b^2)}} \frac{8\log y}{\sqrt{y}}. \tag{E.9}$$

This equation determines $\hat{P}$ up to the regular part. Since $\hat{P}(y) \sim y^{-2}$ when $y \to \infty$, the regular part should be vanishing. Thus, we obtain

$$\hat{P}(y) = \int_{a^2}^{b^2} \frac{dp}{2\pi} \frac{8\log p}{(y-p)\sqrt{p}} \frac{1}{\sqrt{(p-a^2)(b^2-p)}}, \tag{E.10}$$

and then the resolvent is given by

$$W(z) = 32 \int_a^b \frac{dq}{2\pi} \frac{\log q}{z^2 - q^2} \sqrt{\frac{(z^2-b^2)(z^2-a^2)}{(b^2-q^2)(q^2-a^2)}}. \tag{E.11}$$

From (E.6), the eigenvalue distribution is given by

$$\rho(x) = \frac{4}{\pi^3 g_{S^2}^2 N_2} P \int_a^b dq \frac{\log q}{q^2 - x^2} \sqrt{\frac{(b^2-x^2)(x^2-a^2)}{(b^2-q^2)(q^2-a^2)}}, \tag{E.12}$$

where $x \in [a, b]$ and $P\int$ means the principal value. Note that it satisfies

$$x\rho(x) = \frac{1}{x}\rho(1/x). \tag{E.13}$$

When the 't Hooft coupling $g_{S^2}^2 N_2$ is large, the integral in (E.12) can be performed. This limit will turn out to correspond to the large-b limit. By changing the variables in (E.12) as

$$\log b = -\log a = \alpha, \quad \log x = v\alpha, \quad \log q = u\alpha, \tag{E.14}$$

one can obtain

$$x\rho(x) + \frac{1}{x}\rho(1/x) = \frac{4\alpha^2}{\pi^3 g_{S^2}^2 N_2} P \int_{-1}^{1} du\, u \,\mathrm{sign}(u-v) + \mathcal{O}(\alpha^1). \tag{E.15}$$

Then, the integral can be easily performed. By using (E.13), one obtains

$$\rho(x) = \frac{2}{\pi^3 g_{S^2}^2 N_2} \frac{(\log b)^2 - (\log x)^2}{x}, \tag{E.16}$$

in the leading order of α. Since $\int_a^b \rho(x)dx = 1$ by definition, b is determined as

$$b = 1/a = \exp\left[\pi\left(\frac{3g_{S^2}^2 N_2}{8}\right)^{\frac{1}{3}}\right]. \tag{E.17}$$

Thus, b is indeed large when the 't Hooft coupling is large.

Using (E.16), one can easily compute the free energy of the matrix integral. The result is given by

$$\log Z = \frac{9\pi}{10}\frac{N_2^2}{(3g_{S^2}^2 N_2)^{1/3}}. \tag{E.18}$$

F Condition for large S^5 radius

In this appendix, we show that $\lambda^{(s)} \gg D_s$ is a sufficient condition for the large S^5 radius at the tips of the disks in the electrostatic problem. At the tip of a disk, the disk radius R and the radius R_{S^5} of S^5 are related as (2.23). Then, under the identification (4.19), the S^5 radius is large if and only if $x_m^{(s)} \gg 1$. In the following, we show that $x_m^{(s)} \gg 1$ if $\lambda^{(s)} \gg D_s$. We assume that the index s labels the disks in the order of the z-coordinate, namely, $D_{s-1} < D_s$ $(s = 1, 2, \cdots, \Lambda)$.

First we divide the theory described by (4.6) into three parts. The first is the free part, the action of which is given by

$$S_1 = \sum_{s=1}^{\Lambda}\int dx\left(\frac{2D_s}{g^2}(x^2 - \mu_s)\rho^{(s)}(x) + \frac{\pi}{2}(\rho^{(s)}(x))^2\right). \tag{F.1}$$

The second is the self-interaction part given by

$$S_2 = -\frac{1}{2}\sum_{s=1}^{\Lambda}\int dxdy\frac{2D_s}{(2D_s)^2 + (x-y)^2}\rho^{(s)}(x)\rho^{(s)}(y). \tag{F.2}$$

The third is the interaction between different s and t, defined by

$$S_3 = -\frac{1}{2}\sum_{s\neq t}\int dxdy\left[\frac{D_s + D_t}{(D_s + D_t)^2 + (x-y)^2} - \frac{|D_s - D_t|}{(D_s - D_t)^2 + (x-y)^2}\right]\rho^{(s)}(x)\rho^{(t)}(y). \tag{F.3}$$

The total theory is described by the sum of these. But for the moment, let us consider more generally the theory defined by $S(\alpha, \beta) = S_1 + \alpha S_2 + \beta S_3$, where α and β are parameters. We start with the simplest free theory with $\alpha = \beta = 0$. In this case, the extents of the eigenvalues can be easily estimated as $x_m^{(s)} \sim (\lambda^{(s)}/D_s)^{1/3}$. This gives a typical length scale of the free theory. From (F.2) and (F.3), one can also read off the typical length scale of the interaction potentials. For the self-interaction, it is given by $\Delta x \sim D_s$, where Δx denotes the separation distance between two eigenvalues. For the interaction between different s and t, the scale (for a fixed s) is equal or greater than $D_s^{3/4}$, namely, $\Delta x \gtrsim D_s^{3/4}$. The lower bound is saturated by the interaction between s and $t = s \pm 1$. Then, let us consider turning on the interactions to recover the theory with $\alpha = \beta = 1$. The typical scale $(\lambda^{(s)}/D_s)^{1/3}$ of the free theory should be modified by the interactions, which have structures with the length scale equal or greater than $D_s^{3/4}$ (Note that we always assume that $D_s \gg 1$). The modified scale should be at least greater than $\min((\lambda^{(s)}/D_s)^{1/3}, D_s^{3/4})$, since there is nothing which provides a finer scale than these. If the modified scale is $x_m^{(s)} \sim D_s^{3/4}$, this is always large enough when $D_s \gg 1$. If the modified scale is $x_m^{(s)} \sim (\lambda^{(s)}/D_s)^{1/3}$, this is large if $\lambda^{(s)} \gg D_s$. Therefore, we conclude that if $\lambda^{(s)} \gg D_s$, the typical extents of the eigenvalues are always much greater than 1.

References

[1] Y. Asano, G. Ishiki, T. Okada and S. Shimasaki, JHEP **1302**, 148 (2013) [arXiv:1211.0364 [hep-th]].

[2] Y. Asano, G. Ishiki, T. Okada and S. Shimasaki, JHEP **1405**, 075 (2014) [arXiv:1401.5079 [hep-th]].

[3] Y. Asano, G. Ishiki and S. Shimasaki, JHEP **1409**, 137 (2014) [arXiv:1406.1337 [hep-th]].

[4] N. A. Nekrasov, Adv. Theor. Math. Phys. **7**, 831 (2004) [hep-th/0206161].

[5] V. Pestun, arXiv:0712.2824 [hep-th].

[6] A. Kapustin, B. Willett and I. Yaakov, JHEP **1003**, 089 (2010) [arXiv:0909.4559 [hep-th]].

[7] N. Drukker, M. Marino and P. Putrov, Commun. Math. Phys. **306**, 511 (2011) [arXiv:1007.3837 [hep-th]].

[8] G. W. Moore, N. Nekrasov and S. Shatashvili, Commun. Math. Phys. **209**, 77 (2000) [hep-th/9803265].

[9] V. A. Kazakov, I. K. Kostov and N. A. Nekrasov, Nucl. Phys. B **557**, 413 (1999) [hep-th/9810035].

[10] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, JHEP **0204**, 013 (2002) [arXiv:hep-th/0202021].

[11] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, Phys. Rev. D **55** (1997) 5112 [arXiv:hep-th/9610043];

[12] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. B **498** (1997) 467 [arXiv:hep-th/9612115];

[13] H. Lin and J. M. Maldacena, Phys. Rev. D **74**, 084014 (2006) [hep-th/0509235].

[14] J. M. Maldacena, M. M. Sheikh-Jabbari and M. Van Raamsdonk, JHEP **0301**, 038 (2003) [hep-th/0211139].

[15] G. Ishiki, S. Shimasaki, Y. Takayama and A. Tsuchiya, JHEP **0611** (2006) 089 [arXiv:hep-th/0610038].

[16] T. Ishii, G. Ishiki, S. Shimasaki and A. Tsuchiya, JHEP **0705** (2007) 014 [arXiv:hep-th/0703021].

[17] W. Taylor, Phys. Lett. B **394**, 283 (1997) [hep-th/9611042].

[18] J. K. Erickson, G. W. Semenoff and K. Zarembo, Nucl. Phys. B **582**, 155 (2000) [hep-th/0003055].

[19] N. Drukker and D. J. Gross, J. Math. Phys. **42**, 2896 (2001) [hep-th/0010274].

[20] T. Ishii, G. Ishiki, S. Shimasaki and A. Tsuchiya, Phys. Rev. D **78** (2008) 106001 [arXiv:0807.2352 [hep-th]].

[21] T. Eguchi and H. Kawai, Phys. Rev. Lett. **48**, 1063 (1982).

[22] G. Ishiki, S. Shimasaki and A. Tsuchiya, JHEP **1111**, 036 (2011) [arXiv:1106.5590 [hep-th]].

[23] N. Berkovits, Phys. Lett. B **318**, 104 (1993) [hep-th/9308128].

[24] J. -T. Yee and P. Yi, JHEP **0302**, 040 (2003) [hep-th/0301120].

[25] H. Lin, Phys. Rev. D **74**, 125013 (2006) [hep-th/0609186].

[26] C. Bachas, J. Hoppe and B. Pioline, JHEP **0107**, 041 (2001) [hep-th/0007067].

[27] J. M. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231 [arXiv:hep-th/9711200].

[28] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428** (1998) 105 [arXiv:hep-th/9802109].

[29] E. Witten, Adv. Theor. Math. Phys. **2** (1998) 253 [arXiv:hep-th/9802150].

[30] I. R. Klebanov, In *Trieste 1991, Proceedings, String theory and quantum gravity '91* 30-101 and Princeton Univ. - PUPT-1271 (91/07,rec.Oct.) 72 p [hep-th/9108019].

[31] P. H. Ginsparg and G. W. Moore, In *Boulder 1992, Proceedings, Recent directions in particle theory* 277-469. and Yale Univ. New Haven - YCTP-P23-92 (92,rec.Apr.93) 197 p. and Los Alamos Nat. Lab. - LA-UR-92-3479 (92,rec.Apr.93) 197 p [hep-th/9304011].

[32] S. Mukhi, hep-th/0310287.

[33] H. Lin, O. Lunin and J. M. Maldacena, JHEP **0410**, 025 (2004) [hep-th/0409174].

[34] D. Berenstein, J. M. Maldacena and H. Nastase, JHEP **0204** (2002) 013 [arXiv:hep-th/0202021].

[35] H. Ling, A. R. Mohazab, H. -H. Shieh, G. van Anders and M. Van Raamsdonk, JHEP **0610**, 018 (2006) [hep-th/0606014].

[36] M. Berkooz, M. Rozali and N. Seiberg, Phys. Lett. B **408**, 105 (1997) [hep-th/9704089].

[37] N. Seiberg, Phys. Lett. B **408**, 98 (1997) [hep-th/9705221].

[38] O. Aharony, Class. Quant. Grav. **17**, 929 (2000) [hep-th/9911147].

[39] D. Kutasov, "Introduction to little string theory," Superstrings and related matters. Proceedings, Spring School, Trieste, Italy, April 2-10, 2001

[40] H. Ling, H. -H. Shieh and G. van Anders, JHEP **0702**, 031 (2007) [hep-th/0611019].

[41] I. N. Sneddon, "Mixed boundary value problems in potential theory" Amsterdam: North-Holland, 1966.

[42] M. Marino and P. Putrov, arXiv:1206.6346 [hep-th].

[43] T. Suyama, Nucl. Phys. B **856**, 497 (2012) [arXiv:1106.3147 [hep-th]].

[44] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, JHEP **0810**, 091 (2008) [arXiv:0806.1218 [hep-th]].

[45] D. Gaiotto and J. Maldacena, JHEP **1210**, 189 (2012) [arXiv:0904.4466 [hep-th]].