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In-medium heavy-quark interactions from lattice QCD

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Heavy quarks may be produced in the earliest stage of ultra-relativistic heavy-ion collisions, and may or may not bind into quarkonia. They probe the full evolution of the strongly-coupled medium created in these collisions.

The kinetic thermalization of unbound heavy quarks is understood in terms of the heavy-quark diffusion coefficient. Recently, first results with a non-trivial quark sea were made possible by the use of gradient flow. These results covered temperatures of 195 MeV < T < 352 MeV and included the dependence on the heavy-quark mass. As the light sea quarks become more important near the crossover, we extend these studies with almost physical quark sea to lower temperatures and with the same unphysical quark sea to higher temperatures, thus mapping out the quark sea dependence in the temperature window from around $T \simeq 150$ MeV to the GeV level. At such high temperatures appropriately resummed weak-coupling methods have become predictive, whereas the constraining power of the lattice approach has ceased.

In-medium quarkonia are subject to a dynamical melting process, which can be understood in terms of the static potential or of low-lying quarkonia levels. EFT calculations predict a complex potential, whose imaginary part corresponds to the thermal width of these levels. The real part is Debye screened only in certain hierarchies between thermal and non-relativistic scales. Yet these hierarchies apply in the large-time limit and thus are irrelevant for the dynamical melting. A recent calculation in (2+1)-flavor QCD at T < 352 MeV reveals a large imaginary part and provides no evidence for Debye screening. While another recent study in the quenched approximation is consistent with these results. Direct lattice calculations of bottomonia correlators support these conclusions, either of temporal correlators via lattice non-relativistic QCD or of spatial correlators via highly-improved staggered quarks.

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1. Introduction

Both the asymptotic freedom at UV scales, where the weak-coupling approach applies, as well as a wide range of strictly non-perturbative, mutually interrelated IR phenomena are well-established in experimental and theoretical studies of nuclear matter (see, e.g., Refs. [1–7] for recent reviews). Asymptotic freedom predicts that nuclear matter can be understood at UV scales in terms of its fundamental weakly-interacting partons [8], i.e. gluons and quarks. This state of hot nuclear matter is called quark-gluon plasma (QGP), whose properties are determined by the interplay of its thermal scales: the temperature T, the Debye mass $m_D \sim gT$, and the magnetic scale g^2T. These are hierarchically ordered as $g^2T \ll m_D \ll \pi T$ in the strict weak-coupling limit, which is only reached at asymptotically high T. The IR regime of nuclear matter, which is reached at sufficiently low temperatures and densities, can be described to a fair approximation as a gas of more or less non-interacting hadrons and resonances with their non-perturbative vacuum properties [9]. Thus, this low-temperature state of nuclear matter is a hadron gas. A description in terms of such a hadron gas leads to exponential growth in the density of states and an eventual breakdown at high T [10].

The study of heavy-quark (HQ) interaction in the QGP [11–13] is essential for understanding the general properties of strongly-interacting matter produced in a heavy-ion collision (HIC). Heavy flavors, i.e. charm or bottom, probe the entire non-equilibrium dynamics of the nuclear fireball as they are produced only in its earliest stages, while their flavor numbers is conserved throughout its evolution. Thus, the study of their in-medium properties is essential for understanding many features of QGP, such as its near-perfect fluid behavior [14–16]. In these proceedings, we review the current status of in-medium heavy flavors with lattice QCD These proceedings are organized as follows. First, we summarize the state of the art of HQ diffusion and HQ transport coefficient calculations via LQCD in Sec. 2. Next, we summarize the state of the art concerning in-medium quarkonia via LQCD in Sec. 3, i.e. the heavy-quark-antiquark ($Q\bar{Q}$) potential and bottomonia correlators. Finally, we provide a summary and an outlook in Sec. 4.

2. Heavy-quark transport

The interaction of isolated in-medium HQ can be effectively described by Langevin dynamics [17] for sufficiently large HQ masses m_h. Their motion is characterized in terms of a few so-called transport coefficients related by Einstein relations. Among these, the HQ momentum diffusion coefficient κ plays a central role. First, it is the least problematic transport coeffcient to obtain from lattice QCD. And second, the HQ momentum distribution has been precisely measured in HIC at RHIC [18, 19] and LHC [20]. κ quantifies the average momentum transferred between QGP and HQ by random momentum kicks, uncorrelated in time, with 3κ being the mean squared momentum transfer per unit of time. Physically, it describes how much a HQ loses momentum and how fast it thermalizes in the QGP before becoming part of the collective motion. For sufficiently large m_h, the relaxation time is expected to be $\frac{m_h}{T}$ times larger than that of light quarks [17, 21], where T is the medium temperature. Recent measurements have confirmed that this relaxation time is shorter than the mean QGP lifetime, suggesting fast relaxation of HQ and supporting the strongly-coupled nature of the nuclear fireball. Because of this importance accurate results for κ are a key target for experimental and theoretical research.

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Lattice field theory is a first-principles approach to the non-perturbative aspects of Quantum Chromodynamics (QCD), including κ extraction. The calculation requires a highly non-trivial spectral function (SF) reconstruction from lattice correlation functions. Like all transport coefficients, κ is directly encoded in the low-frequency part of the SF called the transport peak. This inversion problem is extremely challenging and dominates the uncertainties of the extracted SF and hence of κ. The transport coefficient κ is known at next-to-leading order (NLO) in hard thermal loop (HTL)-resummed perturbation theory [22], but this calculation is reliable only for asymptotically high T. In the present work we build on previous studies to extract the HQ transport coefficients via LQCD [25–35]. The HotQCD collaboration calculated κ in a $(2+1)$-flavor LQCD framework with a physical strange and rather unphysical light quarks with mass $m_l = \frac{m_s}{5}$ [33, 34] as line of constant physics (LCP), resulting in a continuum pion mass $M_\pi \simeq 320\,\mathrm{MeV}$, and for a window between $T = 195$ to $352\,\mathrm{MeV}$. While the effect of light-quark masses is expected to be negligible for sufficiently high T, it is important for $T \lesssim 250\,\mathrm{MeV}$, i.e. in the scaling window of the chiral transition.

In these proceedings we provide a synopsis of an extension of this previous calculation to an almost physical point, with $m_l = \frac{m_s}{20}$ and a continuum pion mass $M_\pi \simeq 160\,\mathrm{MeV}$, and for a wider window in T, from values close to the pseudocritical temperature $T_{pc} \simeq 156\,\mathrm{MeV}$ up to the GeV scale. We have generated new ensembles and extended many existing ones.Our study also corrects some flaws present in previous lattice determinations of κ. In particular, we use the correct form of the NLO perturbative prediction of the electric SF [37], which was shown to have a missing contribution [38, 39], and we also correct a mistake in the matching factor used in [34, 35] to convert the magnetic correlator to the $\overline{\mathrm{MS}}$ scheme. Furthermore, we skip an intermediate conversion of the magnetic UV SF to a physical scheme, and keep it in the $\overline{\mathrm{MS}}$ scheme throughout. The details of our calculation, its ensembles and technical improvements are covered in our upcoming publication [36].

The Green-Kubo relation characterizes the linear response of a thermalized QGP to perturbations induced by the motion of particles. The HQ momentum diffusion coefficient κ can be extracted from the SF encoded in the HQ vector-vector correlation function in thermodynamic equilibrium,

$$G^{ij}(\tau,\boldsymbol{p}) = \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}\boldsymbol{x}\boldsymbol{p}} \langle J^i_V(0,0) J^j_V(\tau,\boldsymbol{x})\rangle = \int_0^\infty \frac{d\omega}{2\pi} \rho^{ij}(\omega,\boldsymbol{p},T) K(\omega,\tau)\,, \tag{1}$$

with the HQ vector current $J^\mu_V = \bar{\psi}\gamma^\mu\psi$ and the integration Kernel

$$K(\omega,\tau) = \frac{\cosh(\omega\tau - \omega/2T)}{\sinh(\omega/2T)}\,. \tag{2}$$

κ is then defined as the zero-frequency/zero-momentum limit of the SF $\rho(\omega,\boldsymbol{p},T)$

$$\kappa(T) = 2T \lim_{\boldsymbol{p}\to 0} \lim_{\omega\to 0} \rho(\omega,\boldsymbol{p},T)/\omega\,. \tag{3}$$

A direct LQCD determination of κ from quarkonium correlation functions is difficult, because extremely fine lattices are needed to resolve HQ modes. Yet for HQ mass significantly larger than the typical QCD scale Λ_{QCD} or temperature T, we can use heavy-quark effective theory (HQET) to expand the QCD Lagrangian in powers of $\frac{1}{m_h}$ [22–24]. Then, κ can be expressed as a combination of chromo-electrical and chromo-magnetic transport coefficients,

$$\kappa = \kappa_E + \frac{2}{3}\langle v^2\rangle \kappa_B + O(m_h^{-2})\,, \tag{4}$$

where $\kappa_{E,B}$ are defined via Eq. (3) for the corresponding SF $\rho_{E,B}(\omega,\boldsymbol{p},T)$. The magnetic contribution κ_B controls the HQ mass dependence, which enters via the mean squared thermal velocity scaling as $\langle v^2\rangle \propto \frac{T}{m_h}$ at leading order. The pure electrical coefficient κ_E is called the static limit. Thermal chromo-fieldstrength correlation functions are defined as [22–24],

$$G_F(\tau,T) = -\sum_{i=1}^{3} \frac{\langle \mathrm{Re\,Tr}(U(\beta,\tau,\boldsymbol{x})F_i(\tau,\boldsymbol{x},\boldsymbol{x}')U(\tau,0,\boldsymbol{x}')F_i(0,\boldsymbol{x}',\boldsymbol{x}))\rangle}{3\langle \mathrm{Tr}\,P\rangle} . \quad (5)$$

$\beta = \frac{1}{T}$ is the inverse temperature, τ is Euclidean time separation, and $U(\tau_1,\tau_2,\boldsymbol{x})$ is a temporal Wilson line between τ_1 and τ_2 at the spatial site $\boldsymbol{x}$. The chromo-fieldstrengths $F_i = E_i$ or B_i are realized through extended operators as [22],

$$\begin{aligned} E_i^{(\pm)}(\tau,\boldsymbol{x},\boldsymbol{x}+\hat{i}) &= U_i(\tau,\boldsymbol{x})U_4(\tau,\boldsymbol{x}+\hat{i}) - U_4(\tau,\boldsymbol{x})U_i(\tau,\boldsymbol{x}+\hat{4}) , \\ B_i^{(\pm)}(\tau,\boldsymbol{x},\boldsymbol{x}+\hat{j}+\hat{k}) &= \frac{1}{2}\epsilon_{ijk}\left[U_j(\tau,\boldsymbol{x})U_k(\tau,\boldsymbol{x}+\hat{j}) - U_k(\tau,\boldsymbol{x})U_j(\tau,\boldsymbol{x}+\hat{k})\right] , \end{aligned} \quad (6)$$

where in each correlator the two fieldstrengths connect the same two points in opposite spatial directions on different time slices at 0 or τ, and the intermediate temporal Wilson line connects their shifted endpoints. The correlation functions are normalized by the mean trace of the Polyakov loop $\mathcal{P} = \langle \mathrm{Re\,Tr}\,U(\beta,0)\rangle$, and each expectation value includes an implicit volume average. Both gluonic correlators are noisy and require renormalization factors; these are finite for G_E [40] but logarithmically divergent for G_B [23, 41]. We tackle both issues using gradient flow [42–45], which has been shown to achieve renormalized results consistent for pure gauge backgrounds [29, 31], where multi-level updating [46] is applicable as an alternative. In particular, gradient flow has made LQCD calculation of κ with dynamical quarks feasible in the first place.

In the following we sketch how the $\overline{\mathrm{MS}}$ correlation function for the SF reconstruction is obtained. We combine ensembles of two different LCPs, namely $m_l = \frac{m_s}{20}$ and $m_l = \frac{m_s}{5}$, and demonstrate that the influence of m_l is negligible for $T \gtrsim 220\,\mathrm{MeV}$. We also demonstrate that our ensembles are sufficiently close in T such that T interpolations are reliable. We restrict the lattice correlators in τ and τ_F to $0.25 \le \tau T \le 0.50$ and $0.25 \le \frac{\sqrt{\tau_\mathrm{F}}}{\tau} \le 0.30$, respectively. The former prevents prohibitively large lattice artifacts at small τ. The latter makes sure to have a reasonably linear τ_F dependence due to $\frac{\sqrt{\tau_\mathrm{F}}}{a} \ge 1$ and $\sqrt{\tau_\mathrm{F}}T \ll 1$ [29, 31, 47]. We define improved correlators

$$G_F(\tau,T,\tau_\mathrm{F},a) \equiv G_F^{\mathrm{data}}(\tau,T,\tau_\mathrm{F},a) \times \frac{G^{\mathrm{norm}}(\tau,T)}{G_F^{\mathrm{latt}}(\tau T,a\tau_\mathrm{F},aT)} , \quad (7)$$

where $G^{\mathrm{norm}}(\tau,T)$ is the leading order (LO) continuum correlation function with the coupling removed [22], while $G_F^{\mathrm{latt}}(\tau T,a\tau_\mathrm{F},aT)$ is its counterpart at finite lattice spacing a and finite flow time τ_F [33]. We then first interpolate $G_F(\tau,T,\tau_\mathrm{F},a)$ in τT at fixed τ_F and a. Then we simultaneously extrapolate $G_F(\tau T,\tau_\mathrm{F},a)$ for all τT to the continuum limit at fixed $\frac{\sqrt{\tau_\mathrm{F}}}{\tau}$ using parameters $G_F^{\mathrm{cont}}(\tau T,T,\frac{\sqrt{\tau_\mathrm{F}}}{\tau})$ and m in

$$G_F\left(\tau T,T,\frac{\sqrt{\tau_\mathrm{F}}}{\tau},\frac{a}{r_1}\right) = G_F^{\mathrm{cont}}(\tau T,T,\tfrac{\sqrt{\tau_\mathrm{F}}}{\tau}) + \left(\tfrac{m}{\tau T}\right)^2 \times \left(\frac{a}{r_1}\right)^2 . \quad (8)$$

Before the flow extrapolation the magnetic correlator $G_M^{\mathrm{cont}}(\tau T,T,\frac{\sqrt{\tau_\mathrm{F}}}{\tau})$ must be converted to another regularization scheme. We choose $\overline{\mathrm{MS}}$, since there is no need for the correlator in a physical scheme [50] as long as logarithmic terms do not become too large. We use four different $\overline{\text{MS}}$ scales, $\mu = 2, 3, 4,$ or 5 GeV, and convert the gradient flow correlator at scale $\mu_\text{F} = \frac{1}{\sqrt{8\tau_\text{F}}}$ as

$$G_B^{\overline{\text{MS}}}(\tau T, T, \mu) = Z_\text{match}(\mu, \mu_\text{F}) G_B^\text{GF}(\tau T, T, \mu_\text{F}) \,, \quad \ln Z_\text{match}(\mu, \mu_\text{F}) = -\gamma_0 g^2_{\overline{\text{MS}}}(\mu) \left(\ln \frac{\mu^2}{4\mu_\text{F}^2} + \gamma_\text{E} \right) \,, \tag{9}$$

with $\gamma_0 = 3/(8\pi^2)$ being the leading anomalous dimension [41] and $\gamma_\text{E} = 0.5772156649$ the Euler-Mascheroni constant. The resummation of higher orders in the matching factor helps keeping logarithms in check. In previous analyses a mistake in the matching's implementation had led to slightly large results for κ_B. Yet due to the error budget being dominated by SF reconstruction, this mistake turned out as insignificant. For G_E such a conversion to $\overline{\text{MS}}$ scheme is not necessary [40]. For each τT and T both correlators are linearly extrapolated to zero flow time via

$$G_F(\tau T, \tau_\text{F}) = G_F^{\tau_\text{F} \to 0}(\tau T) + m_F \, \tau_\text{F} \,, \tag{10}$$

where the slopes $m_{E,B}$ are (independent) dependent of τT for the (electric) magnetic correlator.

The SF reconstruction is the most important and intricate aspect of the calculation. We estimate the SF through a variety of theoretically motivated Ansätze. For the IR part we use Eq. (3) and treat κ as a fit parameter. For the UV part we use the vacuum perturbative QCD (pQCD) SF at LO or NLO [22] behaving as $\rho(\omega, T) \propto \omega^3$ and supply prefactors $K_{E,B}$ that are fitted to account for non-perturbative effects. We use in each electric pQCD SF a theoretically motivated scale, i.e. for LO $\mu_T = 4\pi T \exp\left[-\gamma_E - \frac{N_c - 8N_f \ln 2}{22N_c - 4N_f}\right] \mu_T \simeq 9.1T$ (for $N_c = N_f = 3$) suggested by dimensionally reduced QCD [49], or for NLO the one that minimizes the logarithmic corrections,

$$\mu^\text{NLO}(\omega) \equiv \mu_\text{opt} = 2\omega \exp\left[\frac{(6\pi^2 - 149)N_c + 20N_f}{6(11N_c - 2N_f)}\right] \simeq 0.549\omega \text{ (for } N_c = N_f = 3) \,. \tag{11}$$

$K_E^\text{NLO} \simeq 1$ at high temperatures as a consequence of using the correct value of $\mu^\text{NLO}(\omega)$ (instead of $\mu^\text{NLO}(\omega) \simeq 14.7427\omega$ used in previous publications due to a mistake [37]). For the magnetic LO or NLO SF we simply choose $\mu = 2, 3, 4,$ or 5 GeV, which keeps logarithmic corrections smaller than 18% with a frequency cutoff of $\frac{\omega}{T} \leq 10^3$. Finally, we use a wide range of models for interpolating and switching between the two limiting behaviors, i.e.

$$\rho_\text{max}(\omega, T) = \max\{\rho^\text{IR}(\omega, T), \rho^\text{UV}(\omega)\} \,, \tag{12}$$

$$\rho_\text{smax}(\omega, T) = \sqrt{{\rho^\text{IR}}^2(\omega, T) + {\rho^\text{UV}}^2(\omega)} \,, \tag{13}$$

$$\rho_\text{sum}(\omega, T) = \rho^\text{IR}(\omega, T) + \rho^\text{UV}(\omega) \,. \,, \tag{14}$$

$$\rho_\text{plaw}(\omega, T) = \begin{cases} \rho^\text{IR}(\omega, T) & 10^{-6} \leq \omega/T \leq \omega_\text{IR} \\ c\omega^p & \omega_\text{IR} \leq \omega/T \leq \omega_\text{UV} \\ \rho^\text{UV}(\omega) & \omega_\text{UV} \leq \omega/T \leq 10^3 \end{cases} \,. \tag{15}$$

Most of these models have been considered before [33–35, 37, 48], and some have smoother or sharper transitions. We also use smooth polynomials interpolations with additional fit parameters c, p in an intermediate region with varied IR boundary $\omega_\text{IR} \propto T$ and UV boundary $\omega_\text{UV} = 2\pi T$.

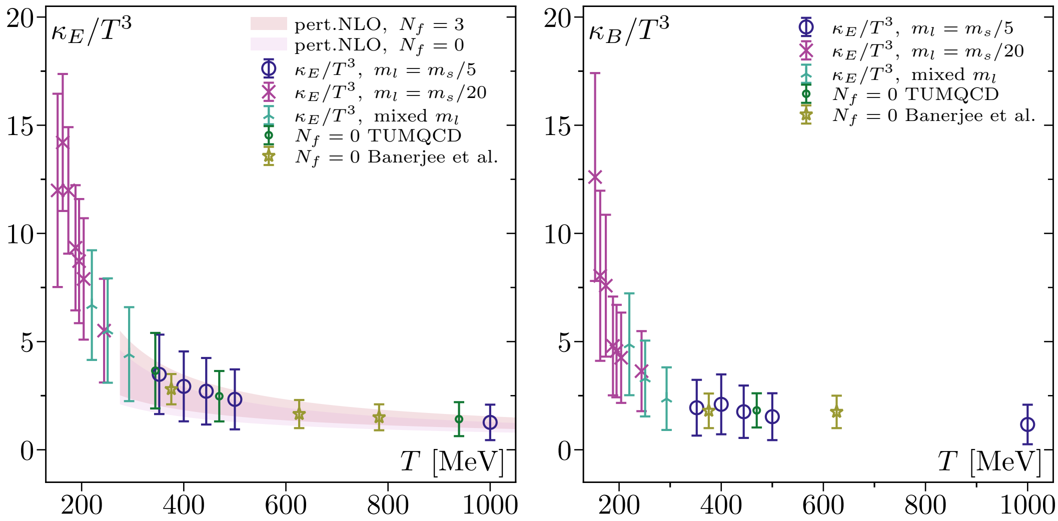

Figure 1: Comparison of κ_E or κ_B between $N_f = 3$ or $N_f = 0$. Above $T \simeq 300$ MeV, the LQCD results agree with the reorganized perturbative expansion. Figure from [36]

Then central value and symmetrized error of κ are obtained from the median and spread of the various models, while the statistical error turns out to be insignificant in comparison.

Next, we compare to the predictions for κ_E at $O(g^4)$, either using HTL resummation at LO [37, 51] or within the kinetic theory [51, 52]. The former predicts a physically non-sensical negative value for κ_E, which is then fixed at NLO [51], whereas the latter avoids this already at LO through the LO gluon self-energy, i.e. the Debye mass m_D. Once the perturbative series is reorganized and the first few formally subleading corrections due to the gluon self-energy are explicitly absorbed into a redefinition of κ_E [28], the convergence of the perturbative series is considerably improved. Above $T \gtrsim 300$ MeV there is convincing agreement between the weak-coupling and LQCD results and QCD yields only slightly larger $\kappa_{E,B}$ than its quenched approximation, see Fig. 1. Our calculation suggests that $G_{E,B}$, and thus $\kappa_{E,B}$ at higher temperatures can be predicted using the current lattice results. The HQ spatial diffusion coefficient D_s can be obtained via

$$D_s = \frac{2T^2}{\kappa} \frac{\langle p^2 \rangle}{3MT} \,. \tag{16}$$

We obtain the thermal expectation values $\langle v^2 \rangle$ in Eq. (4) and $\langle p^2 \rangle$ in Eq. (16) from a thermal quasiparticle (QP) model [53]. The HQ spatial diffusion coefficient D_s is much smaller than smaller than most phenomenological estimates and approaches the AdS/CFT bound [24] calculated assuming strong coupling from above, while it is consistent with the HTL resummed NLO result at $T > 300$ MeV. All technical details are covered in our upcoming publication [36].

3. Quarkonia

Multiple in-medium HQ at close distance can still exist in highly correlated QP states reminiscent of bound states in the vacuum, albeit with some form of medium modification. We have used this property in obtaining the HQ spatial diffusion coefficient D_s, see Sec. 2. The non-relativistic bound state problem of QCD is characterized by a non-relativistic hierarchy of scales, namely, the HQ mass m_h, the inverse distance $\frac{1}{r}$, the internal momenta $p \sim \frac{\alpha_s}{r}$, and the binding energies $E \sim \frac{\alpha_s^2}{r}$, where $\alpha_s \ll 1$. A Coulombic bound state has $r \sim \frac{1}{\alpha_s m_h}$. The notion of in-medium quarkonia gathered lots of interest due to the assumption that color screening in QGP would make the $Q\bar{Q}$ interaction short ranged [54]. Thus, quarkonia states could not be formed in QGP, which would lead to quarkonia suppression. The study of quarkonia production in HIC is a major part of the experimental HIC program; for a recent review see [55].

The idea of having a screened potential between a $Q\bar{Q}$ pair in QGP is closely related to the exponential screening of the free energy of an infinitely heavy $Q\bar{Q}$ pair in QGP, which is well established by LQCD calculations, see e.g. [56]. However, the free energy of $Q\bar{Q}$ describes the in-medium $Q\bar{Q}$ interaction at macroscopic time scales much larger than $\frac{1}{T}$, where the $Q\bar{Q}$ is fully thermalized. For in-medium quarkonia, however, the individual thermal or quarkonia hierarchies become intertwined and permit many different in-medium hierarchies. Only some of these give rise to Debye screening as in the free energy. In particular, for understanding the in-medium quarkonia's properties one needs to know if and how the $Q\bar{Q}$ interaction is modified at (time) scales comparable to quarkonia's internal ones. The effective field theory (EFT) approach provides a natural framework to address this problem when the weak-coupling approach is applicable [57–59], i.e. at high T. Depending on the separation of the quarkonia scales and the thermal scales the $Q\bar{Q}$ potential could be modified by QGP differently and should acquire an imaginary part that was missing in the seminal model [54]. However the real part of this potential does not necessarily have a screened form in this approach [58]. The underlying hierarchy that produces such a screened form [57] is not the one to which dynamical quarkonia melting can be attributed [59].

How to study the modification of $Q\bar{Q}$ interactions in QGP beyond weak coupling remains an unsolved problem. However, one could define a $Q\bar{Q}$ potential at $T > 0$ in analogy to the $T = 0$ case in terms of Wilson loops of size $\tau \times r$ [61], and write these in terms of the r-dependent SF

$$W(\tau, r, T) = \int_{-\infty}^{+\infty} d\omega e^{-\omega\tau} \rho_r(\omega, T) \,. \tag{17}$$

The distance r between the $Q\bar{Q}$ acts as the SF label, analogous to a quark mass. At $T = 0$, the SF's lowest delta function peak corresponds to the ground state potential. If there were a QP state for moderately high T, then the SF has include on top of some continuum a dominant Lorentzian peak that is regularized by non-potential effects; the peak position and width determine the real and imaginary parts of the in-medium $Q\bar{Q}$ potential, respectively [60]. Yet for very high T the SF might lack any well-defined peak such that a QP state and a potential cannot be defined at all. The existence of a well-defined peak in $\rho_r(\omega, T)$ is a necessary, yet insufficient condition for an in-medium potential picture of quarkonia at $T > 0$. This complex potential has been obtained from static correlation functions in recent studies using the quenched approximation [61–64] or various $(2 + 1)$-flavor LQCD frameworks [65–69].

Bottomonia levels in the vacuum may be studied directly using lattice NRQCD [70–77]. In-medium bottomonia have been studied as well by extending these approaches to $T > 0$ [78–88]. The effective masses of static or non-relativistic correlation functions show qualitatively consistent features, see Fig. 2. While there are distinct, operator dependent effects at very small ($\tau \simeq a$) or very large ($\tau \simeq \frac{1}{T}$) Euclidean times, the central region $a \ll \tau \ll \frac{1}{T}$ is operator independent for

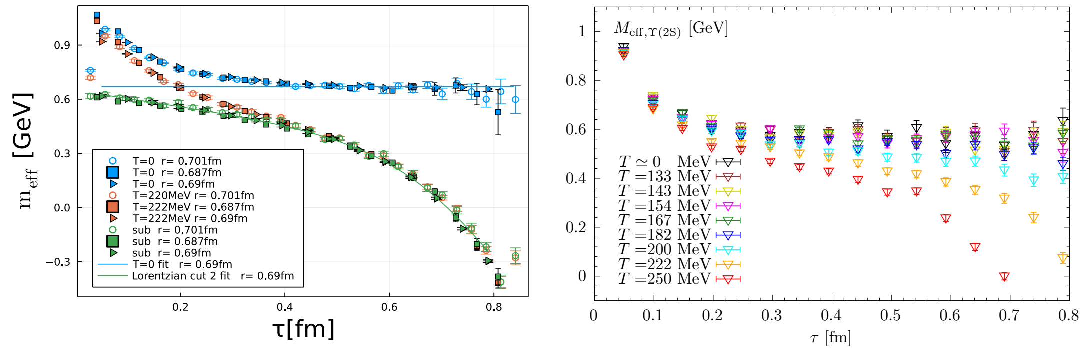

Figure 2: Effective masses of quarkonia correlation functions in the static limit (i.e., smeared gauge-fixed Wilson line correlators, left panel, from [69]) or lattice non-relativistic QCD (NRQCD) (i.e., $\Upsilon(2S)$ with wavefunction-optimized interpolating operator [88]). The features at $T = 222$ MeV are quite consistent.

static or non-relativistic quarkonia [68]. The small τ correlation function is dominated by the UV part of the SF, which turns out to be T independent within uncertainties. In particular, since this UV part has a very strong interpolating operator dependence, it is not a spectral property. Therefore, it could be fixed to its vacuum form, or removed altogether by subtraction, see [64, 68, 69, 85–88]. The intermediate τ part, which extends after UV subtraction consistently to $\tau \gtrsim a$ as well, can be understood as being due to a low-lying broadened peak corresponding to a QP groundstate and a well-separated, even lower-lying feature with much smaller spectral weight. The broadened peak is known to be some regularized Lorentzian [60], which is confirmed in HTL resummed perturbation theory for some large distance hierarchy [57]. The first two cumulants of this SF serve as model-independent proxies for the physical energy and width parameters of this Lorentzian; these suggest a largely T independent real part of the potential and a large imaginary part scaling with T and rT, independent of the details how the Lorentzian has been regularized, e.g. a Gaussian [64, 68, 69, 85–87], or a Lorentzian with hard [69, 88] or smooth cutoff [88]. The smooth cut Lorentzian yields a similar SF as a self-consistent T-matrix approach [94], with only minor differences for the thermal width, while the width of the hard cut Lorentzian is somewhat larger. The lower-lying feature gives rise to the considerable curvature seen at $\tau \gg \frac{T}{2}$, which is due to the interaction of backward propagating states of the medium with the forward propagating $Q\bar{Q}$ pair [68]; this effect is quantitatively irrelevant for low T or sufficiently compact $Q\bar{Q}$ systems. The existing lattice data do not permit sufficient resolution for understanding the third or higher cumulamts

As for HQ transport the in-medium quarkonia properties have to be extracted by means of a highly non-trivial SF reconstruction from lattice correlation functions. The obtained low-energy features change significantly if different assumptions are encoded in the framework to accomplish the SF reconstruction [64, 68]. In particular, the static correlation functions have been modeled with a different, HTL-inspired Ansatz [63, 64, 68], too. This Ansatz assumes that the potential's support is concentrated in the $\tau \simeq \frac{T}{2}$ region. This results in a complex potential with a screened real part qualitatively similar to the one in HTL-resummed perturbation theory [57]. This screened potential has a smaller imaginary part than the unscreened potentials obtained from models with support in the small $\tau \ll \frac{T}{2}$ region. Similar screened results are obtained for any reasonable spectral model assuming the potential's support is concentrated in the $\tau \simeq \frac{T}{2}$ region, i.e. the accummulated effect of the thermal width up to the $\tau \simeq \frac{T}{2}$ region is attributed to the screening of the potential. Whether or not support in the $\tau \ll \frac{T}{2}$ or the $\tau \simeq \frac{T}{2}$ region is physically adequate for obtaining the potential is a question that could not be answered through LQCD studies so far.

While a variety of model-agnostic Bayesian techniques have been introduced [89–91], their inherent assumptions are not necessarily met by noisy LQCD correlation functions [64, 68], and none appear to systematically outperform all others, see e.g. [92]. In particular, on the one hand, if the interpolating operator dependent UV part of the correlator is subtracted, the potential is controlled by the $\tau \ll \frac{T}{2}$ region and Bayesian methods recover an unscreened potential [64]. On the other hand, if the interpolating operator dependent UV part of the correlator is kept, the potential is dominated by the $\tau \simeq \frac{T}{2}$ region and Bayesian methods recover a screened potential [64]. Direct analytic continuation followed by a Padé interpolation [93] has been applied with inconclusive outcomes [64, 68]. Thus, similar to the HQ transport, direct fits of the LQCD correlation functions taking known limiting behavior of the SF into account appears to provide the most robust estimates so far [64, 68, 69, 85–88].

These static or NRQCD studies are complementary to calculation of spatial or screening correlation functions using relativistic quark formulations [97–107] or Wilson loops [108]. These spatial correlation functions avoid the non-trivial SF reconstruction, but still reveal the suppression of thermal behavior or spin-dependent interactions for larger quark masses consistent with EFT ideas [100, 108–110]. The non-perturbative contributions to the screening mass persist even at the electroweak scale [101, 107, 108], and provide further evidence for a consistency of the static or non-static description of the interactions with the thermal medium.

4. Summary

There has been significant progress in the lattice calculation of in-medium heavy-flavor observables. Gradient flow has been of crucial importance for bringing down the statistical noise on unquenched backgrounds. While both the HQ momentum diffusion coefficient κ and the complex $Q\bar{Q}$ potential are now accessible on realisitic ensembles in (2+1)-flavor QCD, the SF reconstruction remains the dominant uncertainty limiting the achievable precision for both quantities. Results for both observables support the picture of a strongly-coupled medium up to about $T \lesssim 300\,\mathrm{MeV}$.

On the one hand, the recent LQCD calculations suggest a HQ transport coefficient close to the AdS/CFT bound in the crossover region. Yet the same transport coefficient could be smoothly connected to the weak-coupling regime already at $T > 300\,\mathrm{MeV}$. On the other hand, the recent LQCD calculations suggest the survival of some low-lying quarkonia up to well over $T \simeq 300\,\mathrm{MeV}$. Thus, while some hadronic quasiparticle states may persist and deconfinement remains incomplete in this hot medium, hard probes may still be insensitive to those degrees of freedom.

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References

[1] Y. Akiba, A. Angerami, H. Caines, A. Frawley, U. Heinz, B. Jacak, J. Jia, T. Lappi, W. Li and A. Majumder, *et al.* [arXiv:1502.02730 [nucl-ex]].

[2] H. T. Ding, F. Karsch and S. Mukherjee, doi:10.1142/9789814663717_0001.

[3] G. Aarts, J. Aichelin, C. Allton, R. Arnaldi, S. A. Bass, C. Bedda, N. Brambilla, E. Bratkovskaya, P. Braun-Munzinger and G. E. Bruno, *et al.* Eur. Phys. J. A **53**, no.5, 93 (2017) [arXiv:1612.08032 [nucl-th]].

[4] A. Bazavov *et al.* [USQCD], Eur. Phys. J. A **55**, no.11, 194 (2019) [arXiv:1904.09951 [hep-lat]].

[5] J. Zhao, K. Zhou, S. Chen and P. Zhuang, Prog. Part. Nucl. Phys. **114**, 103801 (2020) [arXiv:2005.08277 [nucl-th]].

[6] A. Bazavov and J. H. Weber, Prog. Part. Nucl. Phys. **116**, 103823 (2021) [arXiv:2010.01873 [hep-lat]].

[7] J. H. Weber, PoS **LATTICE2021**, 014 (2022) [arXiv:2202.03054 [hep-lat]].

[8] D. J. Gross, R. D. Pisarski and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).

[9] F. Karsch, K. Redlich and A. Tawfik, Eur. Phys. J. C **29**, 549-556 (2003) [arXiv:hep-ph/0303108 [hep-ph]].

[10] R. Hagedorn, Nuovo Cim. Suppl. **3**, 147-186 (1965) CERN-TH-520.

[11] J. C. Collins and M. J. Perry, Phys. Rev. Lett. **34**, 1353 (1975).

[12] J. Adams *et al.* [STAR], Nucl. Phys. A **757**, 102-183 (2005) [arXiv:nucl-ex/0501009 [nucl-ex]].

[13] I. Arsene *et al.* [BRAHMS], Nucl. Phys. A **757**, 1-27 (2005) [arXiv:nucl-ex/0410020 [nucl-ex]].

[14] R. Rapp, P. B. Gossiaux, A. Andronic, R. Averbeck, S. Masciocchi, A. Beraudo, E. Bratkovskaya, P. Braun-Munzinger, S. Cao and A. Dainese, *et al.* Nucl. Phys. A **979**, 21-86 (2018) [arXiv:1803.03824 [nucl-th]].

[15] X. Dong, Y. J. Lee and R. Rapp, Ann. Rev. Nucl. Part. Sci. **69**, 417-445 (2019) [arXiv:1903.07709 [nucl-ex]].

[16] M. He, H. van Hees and R. Rapp, Prog. Part. Nucl. Phys. **130**, 104020 (2023) [arXiv:2204.09299 [hep-ph]].

[17] G. D. Moore and D. Teaney, Phys. Rev. C **71**, 064904 (2005) [arXiv:hep-ph/0412346 [hep-ph]].

[18] A. Adare *et al.* [PHENIX], Phys. Rev. Lett. **98**, 172301 (2007) [arXiv:nucl-ex/0611018 [nucl-ex]].

[19] B. I. Abelev *et al.* [STAR], Phys. Rev. Lett. **98**, 192301 (2007) [erratum: Phys. Rev. Lett. **106**, 159902 (2011)] [arXiv:nucl-ex/0607012 [nucl-ex]].

[20] B. Abelev *et al.* [ALICE], JHEP **09**, 112 (2012) [arXiv:1203.2160 [nucl-ex]].

[21] B. Svetitsky, Phys. Rev. D **37**, 2484-2491 (1988).

[22] S. Caron-Huot, M. Laine and G. D. Moore, JHEP **04**, 053 (2009) [arXiv:0901.1195 [hep-lat]].

[23] A. Bouttefeux and M. Laine, JHEP **12**, 150 (2020) [arXiv:2010.07316 [hep-ph]].

[24] J. Casalderrey-Solana and D. Teaney, Phys. Rev. D **74**, 085012 (2006) [arXiv:hep-ph/0605199 [hep-ph]].

[25] D. Banerjee, S. Datta, R. Gavai and P. Majumdar, Phys. Rev. D **85**, 014510 (2012) [arXiv:1109.5738 [hep-lat]].

[26] A. Francis, O. Kaczmarek, M. Laine, T. Neuhaus and H. Ohno, Phys. Rev. D **92**, no.11, 116003 (2015) [arXiv:1508.04543 [hep-lat]].

[27] N. Brambilla *et al.* [TUMQCD], PoS **LATTICE2019**, 108 (2019) [arXiv:1912.00689 [hep-lat]].

[28] N. Brambilla, V. Leino, P. Petreczky and A. Vairo, Phys. Rev. D **102**, no.7, 074503 (2020) [arXiv:2007.10078 [hep-lat]].

[29] L. Altenkort, A. M. Eller, O. Kaczmarek, L. Mazur, G. D. Moore and H. T. Shu, Phys. Rev. D **103**, no.1, 014511 (2021) [arXiv:2009.13553 [hep-lat]].

[30] D. Banerjee, S. Datta and M. Laine, JHEP **08**, 128 (2022) [arXiv:2204.14075 [hep-lat]].

[31] N. Brambilla *et al.* [TUMQCD], Phys. Rev. D **107**, no.5, 054508 (2023) [arXiv:2206.02861 [hep-lat]].

[32] D. Banerjee, S. Datta, R. V. Gavai and P. Majumdar, Nucl. Phys. A **1038**, 122721 (2023) doi:10.1016/j.nuclphysa.2023.122721 [arXiv:2206.15471 [hep-ph]].

[33] L. Altenkort *et al.* [HotQCD], Phys. Rev. Lett. **130**, no.23, 231902 (2023) [arXiv:2302.08501 [hep-lat]].

[34] L. Altenkort *et al.* [HotQCD], Phys. Rev. Lett. **132**, no.5, 051902 (2024) [arXiv:2311.01525 [hep-lat]].

[35] L. Altenkort, D. de la Cruz, O. Kaczmarek, G. D. Moore and H. T. Shu, Phys. Rev. D **109**, no.11, 11 (2024) [arXiv:2402.09337 [hep-lat]].

[36] L. Altenkort *et al.* [HotQCD], “Temperature Dependence of Heavy Quark Diffusion from 2+1-Flavor Lattice QCD.”

[37] Y. Burnier, M. Laine, J. Langelage and L. Mether, JHEP **08**, 094 (2010) [arXiv:1006.0867 [hep-ph]].

[38] D. de la Cruz, A. M. Eller and G. D. Moore, Phys. Rev. D **110**, no.9, 094057 (2024) [arXiv:2410.01578 [hep-ph]].

[39] B. Scheihing-Hitschfeld and X. Yao, Phys. Rev. D **108**, no.5, 054024 (2023) [erratum: Phys. Rev. D **109**, no.9, 099902 (2024)] [arXiv:2306.13127 [hep-ph]].

[40] C. Christensen and M. Laine, Phys. Lett. B **755**, 316-323 (2016) [arXiv:1601.01573 [hep-lat]].

[41] E. Eichten and B. R. Hill, Phys. Lett. B **243**, 427-431 (1990).

[42] R. Narayanan and H. Neuberger, JHEP **03**, 064 (2006) [arXiv:hep-th/0601210 [hep-th]].

[43] M. Luscher, Commun. Math. Phys. **293**, 899-919 (2010) [arXiv:0907.5491 [hep-lat]].

[44] M. Lüscher, JHEP **08**, 071 (2010) [erratum: JHEP **03**, 092 (2014)] [arXiv:1006.4518 [hep-lat]].

[45] M. Luscher and P. Weisz, JHEP **02**, 051 (2011) [arXiv:1101.0963 [hep-th]].

[46] M. Luscher and P. Weisz, JHEP **09**, 010 (2001) [arXiv:hep-lat/0108014 [hep-lat]].

[47] A. M. Eller and G. D. Moore, Phys. Rev. D **97**, no.11, 114507 (2018) [arXiv:1802.04562 [hep-lat]].

[48] S. Caron-Huot, Phys. Rev. D **79**, 125009 (2009) [arXiv:0903.3958 [hep-ph]].

[49] K. Kajantie, M. Laine, K. Rummukainen and M. E. Shaposhnikov, Nucl. Phys. B **503**, 357-384 (1997) [arXiv:hep-ph/9704416 [hep-ph]].

[50] M. Laine, JHEP **06**, 139 (2021) [arXiv:2103.14270 [hep-ph]].

[51] S. Caron-Huot and G. D. Moore, Phys. Rev. Lett. **100**, 052301 (2008) [arXiv:0708.4232 [hep-ph]].

[52] S. Caron-Huot and G. D. Moore, JHEP **02**, 081 (2008) [arXiv:0801.2173 [hep-ph]].

[53] P. Petreczky, Eur. Phys. J. C **62**, 85-93 (2009) [arXiv:0810.0258 [hep-lat]].

[54] T. Matsui and H. Satz, Phys. Lett. B **178**, 416-422 (1986).

[55] A. Rothkopf, Phys. Rept. **858**, 1-117 (2020) doi:10.1016/j.physrep.2020.02.006 [arXiv:1912.02253 [hep-ph]].

[56] A. Bazavov *et al.* [TUMQCD], Phys. Rev. D **98**, no.5, 054511 (2018) [arXiv:1804.10600 [hep-lat]].

[57] M. Laine, O. Philipsen, P. Romatschke and M. Tassler, JHEP **03**, 054 (2007) [arXiv:hep-ph/0611300 [hep-ph]].

[58] N. Brambilla, J. Ghiglieri, A. Vairo and P. Petreczky, Phys. Rev. D **78**, 014017 (2008) [arXiv:0804.0993 [hep-ph]].

PoS(QCHSC24)021

[59] M. Laine, Nucl. Phys. A **820**, 25C-32C (2009) [arXiv:0810.1112 [hep-ph]].

[60] Y. Burnier and A. Rothkopf, Phys. Rev. D **86**, 051503 (2012) [arXiv:1208.1899 [hep-ph]].

[61] A. Rothkopf, T. Hatsuda and S. Sasaki, Phys. Rev. Lett. **108**, 162001 (2012) [arXiv:1108.1579 [hep-lat]].

[62] Y. Burnier and A. Rothkopf, Phys. Rev. D **95**, no.5, 054511 (2017) [arXiv:1607.04049 [hep-lat]].

[63] D. Bala and S. Datta, Phys. Rev. D **101**, no.3, 034507 (2020) [arXiv:1909.10548 [hep-lat]].

[64] R. N. Larsen, G. Parkar, A. Rothkopf and J. H. Weber, Phys. Rev. D **110**, no.11, 114501 (2024) [arXiv:2402.10819 [hep-lat]].

[65] Y. Burnier, O. Kaczmarek and A. Rothkopf, JHEP **12**, 101 (2015) [arXiv:1509.07366 [hep-ph]].

[66] P. Petreczky *et al.* [TUMQCD], Nucl. Phys. A **967**, 592-595 (2017) [arXiv:1704.08573 [hep-lat]].

[67] P. Petreczky, A. Rothkopf and J. Weber, Nucl. Phys. A **982**, 735-738 (2019) [arXiv:1810.02230 [hep-lat]].

[68] D. Bala *et al.* [HotQCD], Phys. Rev. D **105**, no.5, 054513 (2022) [arXiv:2110.11659 [hep-lat]].

[69] A. Bazavov *et al.* [HotQCD], Phys. Rev. D **109**, no.7, 074504 (2024) [arXiv:2308.16587 [hep-lat]].

[70] B. A. Thacker and G. P. Lepage, Phys. Rev. D **43**, 196-208 (1991).

[71] G. P. Lepage, L. Magnea, C. Nakhleh, U. Magnea and K. Hornbostel, Phys. Rev. D **46**, 4052-4067 (1992) [arXiv:hep-lat/9205007 [hep-lat]].

[72] C. T. H. Davies, K. Hornbostel, A. Langnau, G. P. Lepage, A. Lidsey, J. Shigemitsu and J. H. Sloan, Phys. Rev. D **50**, 6963-6977 (1994) [arXiv:hep-lat/9406017 [hep-lat]].

[73] S. Meinel, Phys. Rev. D **79**, 094501 (2009) [arXiv:0903.3224 [hep-lat]].

[74] S. Meinel, Phys. Rev. D **82**, 114502 (2010) [arXiv:1007.3966 [hep-lat]].

[75] T. C. Hammant, A. G. Hart, G. M. von Hippel, R. R. Horgan and C. J. Monahan, Phys. Rev. Lett. **107**, 112002 (2011) [erratum: Phys. Rev. Lett. **115**, 039901 (2015)] [arXiv:1105.5309 [hep-lat]].

[76] R. J. Dowdall *et al.* [HPQCD], Phys. Rev. D **85**, 054509 (2012) [arXiv:1110.6887 [hep-lat]].

[77] J. O. Daldrop *et al.* [HPQCD], Phys. Rev. Lett. **108**, 102003 (2012) [arXiv:1112.2590 [hep-lat]].

[78] G. Aarts, S. Kim, M. P. Lombardo, M. B. Oktay, S. M. Ryan, D. K. Sinclair and J. I. Skullerud, Phys. Rev. Lett. **106**, 061602 (2011) [arXiv:1010.3725 [hep-lat]].

[79] G. Aarts, C. Allton, S. Kim, M. P. Lombardo, M. B. Oktay, S. M. Ryan, D. K. Sinclair and J. I. Skullerud, JHEP **11**, 103 (2011) [arXiv:1109.4496 [hep-lat]].

[80] G. Aarts, C. Allton, S. Kim, M. P. Lombardo, M. B. Oktay, S. M. Ryan, D. K. Sinclair and J. I. Skullerud, JHEP **03**, 084 (2013) [arXiv:1210.2903 [hep-lat]].

[81] G. Aarts, C. Allton, S. Kim, M. P. Lombardo, S. M. Ryan and J. I. Skullerud, JHEP **12**, 064 (2013) [arXiv:1310.5467 [hep-lat]].

[82] G. Aarts, C. Allton, T. Harris, S. Kim, M. P. Lombardo, S. M. Ryan and J. I. Skullerud, JHEP **07**, 097 (2014) [arXiv:1402.6210 [hep-lat]].

[83] S. Kim, P. Petreczky and A. Rothkopf, Phys. Rev. D **91**, 054511 (2015) [arXiv:1409.3630 [hep-lat]].

[84] S. Kim, P. Petreczky and A. Rothkopf, JHEP **11**, 088 (2018) [arXiv:1808.08781 [hep-lat]].

[85] R. Larsen, S. Meinel, S. Mukherjee and P. Petreczky, Phys. Rev. D **100**, no.7, 074506 (2019) [arXiv:1908.08437 [hep-lat]].

[86] R. Larsen, S. Meinel, S. Mukherjee and P. Petreczky, Phys. Lett. B **800**, 135119 (2020) [arXiv:1910.07374 [hep-lat]].

[87] R. Larsen, S. Meinel, S. Mukherjee and P. Petreczky, Phys. Rev. D **102**, 114508 (2020) [arXiv:2008.00100 [hep-lat]].

[88] H. T. Ding, W. P. Huang, R. Larsen, S. Meinel, S. Mukherjee, P. Petreczky and Z. Tang, [arXiv:2501.11257 [hep-lat]].

[89] M. Asakawa, T. Hatsuda and Y. Nakahara, Prog. Part. Nucl. Phys. **46**, 459-508 (2001) [arXiv:hep-lat/0011040 [hep-lat]].

[90] A. Rothkopf, J. Comput. Phys. **238**, 106-114 (2013) [arXiv:1110.6285 [physics.comp-ph]].

[91] Y. Burnier and A. Rothkopf, Phys. Rev. Lett. **111**, 182003 (2013) [arXiv:1307.6106 [hep-lat]].

[92] L. Del Debbio, A. Lupo, M. Panero and N. Tantalo, [arXiv:2410.09944 [hep-lat]].

[93] R. A. Tripolt, I. Haritan, J. Wambach and N. Moiseyev, Phys. Lett. B **774**, 411-416 (2017) [arXiv:1610.03252 [hep-ph]].

[94] Z. Tang, S. Mukherjee, P. Petreczky and R. Rapp, [arXiv:2411.09132 [nucl-th]].

[95] S. Shi, K. Zhou, J. Zhao, S. Mukherjee and P. Zhuang, Phys. Rev. D **105**, no.1, 014017 (2022) [arXiv:2105.07862 [hep-ph]].

[96] D. Bala *et al.* [HotQCD], [arXiv:2412.17570 [hep-lat]].

[97] F. Karsch, E. Laermann, S. Mukherjee and P. Petreczky, Phys. Rev. D **85**, 114501 (2012) [arXiv:1203.3770 [hep-lat]].

[98] A. Bazavov, F. Karsch, Y. Maezawa, S. Mukherjee and P. Petreczky, Phys. Rev. D **91**, no.5, 054503 (2015) [arXiv:1411.3018 [hep-lat]].

[99] A. Bazavov, S. Dentinger, H. T. Ding, P. Hegde, O. Kaczmarek, F. Karsch, E. Laermann, A. Lahiri, S. Mukherjee and H. Ohno, *et al.* Phys. Rev. D **100**, no.9, 094510 (2019) [arXiv:1908.09552 [hep-lat]].

[100] P. Petreczky, S. Sharma and J. H. Weber, Phys. Rev. D **104**, no.5, 054511 (2021) [arXiv:2107.11368 [hep-lat]].

[101] M. Dalla Brida, L. Giusti, T. Harris, D. Laudicina and M. Pepe, JHEP **04**, 034 (2022) [arXiv:2112.05427 [hep-lat]].

[102] R. Thakkar and P. Hegde, JHEP **07**, 171 (2023) [arXiv:2304.10884 [hep-lat]].

[103] L. Giusti, T. Harris, D. Laudicina, M. Pepe and P. Rescigno, Phys. Lett. B **855**, 138799 (2024) [arXiv:2405.04182 [hep-lat]].

[104] A. Krasniqi, M. Cè, R. J. Hudspith and H. B. Meyer, Phys. Rev. D **110**, no.11, 114506 (2024) [arXiv:2407.01657 [hep-lat]].

[105] H. T. Ding, J. B. Gu, S. T. Li and R. Thakkar, [arXiv:2501.11262 [hep-lat]].

[106] Y. Aoki, H. Fukaya, S. Hashimoto, I. Kanamori, Y. Nakamura, C. Rohrhofer, K. Suzuki and D. Ward, [arXiv:2501.12675 [hep-lat]].

[107] M. Cè, L. Giusti, D. Laudicina, M. Pepe and P. Rescigno, [arXiv:2502.07674 [hep-lat]].

[108] D. Bala, O. Kaczmarek, P. Petreczky, S. Sharma and S. Tah, [arXiv:2501.17943 [hep-lat]].

[109] E. Eichten and F. Feinberg, Phys. Rev. D **23**, 2724 (1981).

[110] V. Koch, E. V. Shuryak, G. E. Brown and A. D. Jackson, Phys. Rev. D **46**, 3169 (1992) [erratum: Phys. Rev. D **47**, 2157 (1993)] [arXiv:hep-ph/9204236 [hep-ph]].