Maxwell's equations, quantum physics and the quantum graviton

Alexander Gersten and Amnon Moalem
Department of Physics, Ben-Gurion University of the Negev, Beer-Sheva, Israel

Abstract. Quantum wave equations for massless particles and arbitrary spin are derived by factorizing the d'Alembertian operator. The procedure is extensively applied to the spin one photon equation which is related to Maxwell's equations via the proportionality of the photon wavefunction Ψ to the sum $\mathbf{E} + i\mathbf{B}$ of the electric and magnetic fields. Thus Maxwell's equations can be considered as the first quantized one-photon equation. The photon wave equation is written in two forms, one with additional explicit subsidiary conditions and second with the subsidiary conditions implicitly included in the main equation. The second equation was obtained by factorizing the d'Alembertian with 4×4 matrix representation of "relativistic quaternions". Furthermore, scalar Lagrangian formalism, consistent with quantization requirements is developed using derived conserved current of probability and normalization condition for the wavefunction. Lessons learned from the derivation of the photon equation are used in the derivation of the spin two quantum equation, which we call the quantum graviton. Quantum wave equation with implicit subsidiary conditions, which factorizes the d'Alembertian with 8×8 matrix representation of relativistic quaternions, is derived. Scalar Lagrangian is formulated and conserved probability current and wavefunction normalization are found, both consistent with the definitions of quantum operators and their expectation values. We are showing that the derived equations are the first quantized equations of the photon and the graviton.

Key words: Maxwell's equations, one photon quantum equation, quantum graviton

1. Introduction

In the past, relativistic wave equations were successfully obtained by factorizing the Klein-Gordon operator. The Dirac equation was derived from the relativistic condition on the Energy E, mass m, and momentum $\mathbf{p}$:

$$\left(E^2 - c^2\mathbf{p}^2 - m^2c^4\right) I^{(4)}\Psi = 0, \tag{1}$$

where $I^{(4)}$ is the 4×4 unit matrix and Ψ is a four component column (bispinor) wave function. Eq. (1) was factorized into

$$\left[EI^{(4)} + \begin{pmatrix} mc^2I^{(2)} & c\mathbf{p}\cdot\boldsymbol{\sigma} \\ c\mathbf{p}\cdot\boldsymbol{\sigma} & -mc^2I^{(2)} \end{pmatrix}\right]\left[EI^{(4)} - \begin{pmatrix} mc^2I^{(2)} & c\mathbf{p}\cdot\boldsymbol{\sigma} \\ c\mathbf{p}\cdot\boldsymbol{\sigma} & -mc^2I^{(2)} \end{pmatrix}\right]\Psi = 0, \tag{2}$$

where $I^{(2)}$ is the 2×2 unit matrix and $\boldsymbol{\sigma}$ is the Pauli spin one-half vector matrix with the components

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

For massless particles wave equations can be obtained by factorizing the zero mass wave operator. The two component (massless) neutrino equation can be derived from the factorization

$$\left(E^2 - c^2\mathbf{p}^2\right) I^{(2)}\psi = \left[EI^{(2)} - c\mathbf{p}\cdot\boldsymbol{\sigma}\right]\left[EI^{(2)} + c\mathbf{p}\cdot\boldsymbol{\sigma}\right]\psi = 0, \tag{4}$$

where ψ is a two component spinor wavefunction.

In the past there were many attempts to describe Maxwell's equation as the quantum photon equation[1]. In this work our presentation is more complete and contains new elements. We extend them to the derivation of the wave equation of the quantum graviton. Our starting point is the factorization of the d'Alembertian. For spin one the factorization leads to the photon wave function. We present different factorizations which lead to quantum photon equations. In all of them, the somewhat mysterious substitution of the wave function $\mathbf{\Psi} = \mathcal{N}\ (\mathbf{E} + i\mathbf{B})$, leads to Maxwell's equations. Above $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields respectively, and $\mathcal{N}$ is a normalization constant (normalizable only for functions that fall off sufficiently rapidly). Thus Maxwell's equations are the one-photon quantum equations. We construct Lagrangians for the photon equations and evaluate the energy-momentum tensors and probability currents, which appear to be consistent with the definitions of quantum operators and their expectation values. We extend our findings to derivations of other zero mass and arbitrary spin quantum equations. We worked out the details of the zero mass spin two quantum equation which is linked to the quantum graviton.

In Sec. 2 and Sec. 4 the photon wave equation is derived by factorizing the wave operator (the d'Alembertian). From the photon equation Maxwell's equations are derived in Sec. 3. In Sec. 2 and Sec. 4 two procedures will be presented: one with subsidiary conditions, in the second procedure the subsidiary conditions will be integrated into the usual form of Maxwell's equations. In Sec. 4 scalar Lagrangian was formulated and conserved probability current was found, both consistent with the definitions of quantum operators and their expectation values. In Sec. 5 the photon Lagrangian was converted into a scalar Lagrangian for Maxwell's equations. In Sec. 6 relativistic quaternions are introduced in order to work properly with the space-time metric. The factorization of the d'Alembertian is achieved with matrix representations of the relativistic quaternions. In Sections 7-8 a general method for deriving quantum equations for zero mass particles and arbitrary spins is presented. In Sec. 9 we use this method to derive again Maxwell's equations, this time with spherical tensor formulation. In Sec. 10 the quantum graviton (spin 2) wave equation (with explicit and implicit subsidiary conditions) are considered. The quantum wave equation with implicit subsidiary conditions, which factorizes the d'Alembertian with 8×8 matrix representation of relativistic quaternions, is derived. Scalar Lagrangian is formulated and conserved probability current was found, both consistent with the definitions of quantum operators and their expectation values. In Sec. 11 the covariance of the equations is explained. Sec. 12 contains our conclusions. In the following symbols with hat over them (like $\hat{E}$) will denote quantum operators.

2. The photon equation

We shall derive the photon equation from the following decomposition[2],

$$\left(\frac{\hat{E}^2}{c^2} - \hat{\mathbf{p}}^2\right) I^{(3)} = \left(\frac{\hat{E}}{c} I^{(3)} + \hat{\mathbf{p}}\cdot\mathbf{S}\right)\left(\frac{\hat{E}}{c} I^{(3)} - \hat{\mathbf{p}}\cdot\mathbf{S}\right) - \begin{pmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & \hat{p}_x\hat{p}_z \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & \hat{p}_y\hat{p}_z \\ \hat{p}_z\hat{p}_x & \hat{p}_z\hat{p}_y & \hat{p}_z^2 \end{pmatrix}, \tag{5}$$

where $I^{(3)}$ is a 3×3 unit matrix, and $\mathbf{S}$ is a spin one vector matrix with components,

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6}$$

and with the properties,

$$[S_x, S_y] = iS_z, \quad [S_z, S_x] = iS_y, \quad [S_y, S_z] = iS_x, \quad \mathbf{S}^2 = 2I^{(3)}. \tag{7}$$

The decomposition (5) can be verified directly by substitution. The matrix on the right hand side of Eq. (5) can be rewritten as:

$$\begin{pmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & \hat{p}_x\hat{p}_z \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & \hat{p}_y\hat{p}_z \\ \hat{p}_z\hat{p}_x & \hat{p}_z\hat{p}_y & \hat{p}_z^2 \end{pmatrix} = \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{pmatrix} \begin{pmatrix} \hat{p}_x & \hat{p}_y & \hat{p}_z \end{pmatrix}. \tag{8}$$

From Eqs.(5-6, and 8) , the photon equation can be obtained,

$$\left(\frac{\hat{E}^2}{c^2} - \hat{\mathbf{p}}^2\right)\mathbf{\Psi} = \left(\frac{\hat{E}}{c}I^{(3)} + \hat{\mathbf{p}}\cdot\mathbf{S}\right)\left(\frac{\hat{E}}{c}I^{(3)} - \hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi} - \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{pmatrix}(\hat{\mathbf{p}}\cdot\mathbf{\Psi}) = 0, \tag{9}$$

where $\mathbf{\Psi}$ is a 3 component (column) wave function. Eq.(9) will be satisfied if the two equations,

$$\left(\frac{\hat{E}}{c}I^{(3)} - \hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi}^{(+)} = 0, \tag{10}$$

$$\hat{\mathbf{p}}\cdot\mathbf{\Psi}^{(+)} = 0, \tag{11}$$

are simultaneously satisfied. A second possibility is to replace Eq. (9) with,

$$\left(\frac{\hat{E}^2}{c^2} - \hat{\mathbf{p}}^2\right)\mathbf{\Psi} = \left(\frac{\hat{E}}{c}I^{(3)} - \hat{\mathbf{p}}\cdot\mathbf{S}\right)\left(\frac{\hat{E}}{c}I^{(3)} + \hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi} - \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{pmatrix}(\hat{\mathbf{p}}\cdot\mathbf{\Psi}) = 0. \tag{12}$$

Eq.(12) will be satisfied if the two equations,

$$\left(\frac{\hat{E}}{c}I^{(3)} + \hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi}^{(-)} = 0, \tag{13}$$

$$\hat{\mathbf{p}}\cdot\mathbf{\Psi}^{(-)} = 0 \tag{14}$$

are simultaneously satisfied. Above $\mathbf{\Psi}^{(+)}$ and $\mathbf{\Psi}^{(-)}$ refer to forward and backward helicities.

3. Maxwell's equations

Maxwell's equations will be derived from Eqs. (10) and (11). We will show below that if in Eqs. (10) and (11) the quantum operator substitutions,

$$\hat{E} \Longrightarrow i\hbar\frac{\partial}{\partial t}, \quad \hat{\mathbf{p}} \Longrightarrow -i\hbar\boldsymbol{\nabla}, \tag{15}$$

and the wavefunction substitution

$$\mathbf{\Psi} = \mathcal{N}\left(\mathbf{E} + i\mathbf{B}\right), \tag{16}$$

are made, as a result the Maxwell equations will be obtained. In Eq. (16) $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields respectively, and $\mathcal{N}$ is a constant, with units of $\left[\left(\sqrt{energy}\right)^{-1}\right]$, which will be determined later on (see Eq. (66)). Indeed, one can easily check from Eqs. (6) and (15) that the following identity is satisfied,

$$(\hat{\mathbf{p}}\cdot\mathbf{S})\,\mathbf{\Psi}=\hbar\boldsymbol{\nabla}\times\mathbf{\Psi}. \tag{17}$$

From Eqs. (10) , (11) , (15) and 17) we obtain,

$$i\hbar\frac{\partial}{\partial t}\mathbf{\Psi}=c\hbar\boldsymbol{\nabla}\times\mathbf{\Psi}=c\,(\hat{\mathbf{p}}\cdot\mathbf{S})\,\mathbf{\Psi}=\mathcal{H}\mathbf{\Psi}, \tag{18}$$

$$-i\hbar\boldsymbol{\nabla}\cdot\mathbf{\Psi}=0, \tag{19}$$

where $\mathcal{H}$ can be defined as the Hamiltonian operator. Although the definition of the energy operator $\hat{E}$ in Eq. (15) is only formal, in practice it can be replaced by the Hamiltonian. The definition of the Hamiltonian does not depend on the subsidiary conditions Eq. (19).

The constants $\hbar$ and $\mathcal{N}$ can be cancelled out in Eqs. (18), (19), and after replacing $\mathbf{\Psi}$ by Eq. (16), the following equations are obtained,

$$\boldsymbol{\nabla}\times(\mathbf{E}+i\mathbf{B})=\frac{i}{c}\frac{\partial\,(\mathbf{E}+i\mathbf{B})}{\partial t}, \tag{20}$$

$$\boldsymbol{\nabla}\cdot(\mathbf{E}+i\mathbf{B})=0. \tag{21}$$

If in Eqs. (20) and (21) the electric and magnetic fields are real, the separation into the real and imaginary parts will lead to the Maxwell equations (without sources),

$$\boldsymbol{\nabla}\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \tag{22}$$

$$\boldsymbol{\nabla}\times\mathbf{B}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}, \tag{23}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E}=0, \tag{24}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B}=0. \tag{25}$$

One should note that the Plank constant $\hbar$ was cancelled out earlier, in Eqs. (18) and (19), which explains its absence in the Maxwell equations. Here to be noted that from Eq. (15) and Eq. (6),

$$\hat{E}^*=-\hat{E},\quad \hat{\mathbf{p}}^*=-\hat{\mathbf{p}},\quad \mathbf{S}^*=-\mathbf{S}, \tag{26}$$

and therefore, the complex conjugate of the photon equation Eq. (10) ,

$$\left[\left(\frac{\hat{E}}{c}I^{(3)}-\hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi}^{(+)}\right]^*=-\left(\frac{\hat{E}}{c}I^{(3)}+\hat{\mathbf{p}}\cdot\mathbf{S}\right)\left(\mathbf{\Psi}^{(+)}\right)^*=0, \tag{27}$$

where $\left(\mathbf{\Psi}^{(+)}\right)^*$ satisfies Eq. (13) for backward helicity.

4. The photon equation with an implicit subsidiary condition

Maxwell's equations without sources are[3],

$$\partial_\mu F^{\mu\nu} = 0, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0, \quad \nu = 0, 1, 2, 3, \tag{28}$$

where the antisymmetric tensor $F^{\mu\nu}$ and its dual $\tilde{F}^{\mu\nu}$ are defined via the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ respectively as,

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{29}$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}, \tag{30}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the totally antisymmetric tensor ($\epsilon^{0123} = 1, \quad \epsilon^{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta}$). The sum of the $F^{\mu\nu}$ and $i\tilde{F}^{\mu\nu}$ is the self-dual antisymmetric tensor, which depends only on the combination $\mathbf{\Psi} = \mathcal{N}\left(\mathbf{E}+i\mathbf{B}\right)$,

$$F^{\mu\nu} + i\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -E_x - iB_x & -E_y - iB_y & -E_z - iB_z \\ E_x + iB_x & 0 & iE_z - B_z & B_y - iE_y \\ E_y + iB_y & B_z - iE_z & 0 & iE_x - B_x \\ E_z + iB_z & iE_y - B_y & B_x - iE_x & 0 \end{pmatrix}, \tag{31}$$

$$= \frac{1}{\mathcal{N}} \begin{pmatrix} 0 & -\Psi_x & -\Psi_y & -\Psi_z \\ \Psi_x & 0 & i\Psi_z & -i\Psi_y \\ \Psi_y & -i\Psi_z & 0 & i\Psi_x \\ \Psi_z & i\Psi_y & -i\Psi_x & 0 \end{pmatrix} = \frac{1}{\mathcal{N}}\left(R_1\Psi_x + R_2\Psi_y + R_3\Psi_z\right), \tag{32}$$

with,

$$R_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The self-dual Maxwell equations take the form,

$$\mathcal{N}\partial_\mu\left(F^{\mu\nu} + i\tilde{F}^{\mu\nu}\right) = \partial_\mu\left(R_i^{\mu\nu}\Psi_i\right) = R_i^{\mu\nu}\partial_\mu\Psi_i = -\left(\alpha_\mu\right)^{\nu i}\partial_\mu\Psi_i = 0, \tag{33}$$

where,

$$\alpha_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{34}$$

$$\alpha_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \tag{35}$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \tag{36}$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{37}$$

and the Ψ_i is part of the the four component wave function,

$$\Phi = \begin{pmatrix} 0 \\ \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix} \tag{38}$$

The free field photon equation becomes

$$i\hbar\left(\alpha_\mu \partial^\mu\right)\Phi = \left(\frac{\hat{E}}{c}a_0 - \hat{\mathbf{p}}\cdot\boldsymbol{\alpha}\right)\Phi = 0. \tag{39}$$

Compare this expression with our previous result of Eqs. (10) and (11). Note that the α_i $(i = 1, 2, 3)$ matrices contain as submatrices the spin one matrices of Eq. (6). The subsidiary condition $\hat{\mathbf{p}}\cdot\boldsymbol{\Psi} = 0$ is added to the first row and first column of the α_i $(i = 1, 2, 3)$ matrices, i.e.,

$$\alpha_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & & & \\ 0 & & (S_x) & \\ 0 & & & \end{pmatrix}, \tag{40}$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & & & \\ -1 & & (S_y) & \\ 0 & & & \end{pmatrix}, \tag{41}$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & & & \\ 0 & & (S_z) & \\ -1 & & & \end{pmatrix}. \tag{42}$$

Thus the α_i $(i = 1, 2, 3)$ matrices can be constructed from the photon equation with spin one matrices and the subsidiary condition. In Eq. (39) only differential operators act on the wave function Φ, they eliminate any constant in spacetime coordinates, therefore Φ can be replaced by

$$\Phi = \begin{pmatrix} \lambda \\ \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix}, \tag{43}$$

where λ is an arbitrary constant, independent of the space-time coordinates. The Hermitian conjugate of Φ is,

$$\Phi^H = \begin{pmatrix} \lambda^* & \Psi_x^* & \Psi_y^* & \Psi_z^* \end{pmatrix}. \tag{44}$$

With these new definitions one can define the Lagrangian density L,

$$L = \frac{\Phi^H}{2}\left(\hat{E}a_0 - c\hat{\mathbf{p}}\cdot\boldsymbol{\alpha}\right)\Phi - \frac{\Phi^T}{2}\left(\hat{E}a_0 - c\hat{\mathbf{p}}\cdot\boldsymbol{\alpha}^*\right)\Phi^*, \tag{45}$$

where Φ^T is the transpose of Φ,

$$\Phi^T = \begin{pmatrix} \lambda & \Psi_x & \Psi_y & \Psi_z \end{pmatrix}. \tag{46}$$

Using the definitions of Eqs. (40) , (42) and (6) one can reduce Eq. (45) to,

$$\frac{L}{c} = \frac{\mathbf{\Psi}^H}{2}\left(\frac{\hat{E}}{c}I^{(3)} - \hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi} + \lambda^*\left(\hat{\mathbf{p}}\cdot\mathbf{\Psi}\right) - \frac{\mathbf{\Psi}^T}{2}\left(\frac{\hat{E}}{c}I^{(3)} + \hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi}^* - \lambda\left(\hat{\mathbf{p}}\cdot\mathbf{\Psi}^*\right), \tag{47}$$

where,

$$\mathbf{\Psi} = \begin{pmatrix} \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix}. \tag{48}$$

One can see in Eq. (47) that the constants λ and λ^* play the role of Lagrangian multipliers. By varying the Lagrangian L with respect to λ and λ^* the subsidiary condition Eq. (11) and its complex conjugate are recovered. By varying the Lagrangian L with respect to $\mathbf{\Psi}^*$ Eq. (10) is obtained. By varying L with respect to $\mathbf{\Psi}$, Eq. (27) is recovered. The probability current is evaluated in Appendix A, with the result,

$$\partial_t\left(\mathbf{\Psi}^H\mathbf{\Psi}\right) + c\boldsymbol{\nabla}\cdot\left(\mathbf{\Psi}^H\mathbf{S}\mathbf{\Psi}\right) = 0. \tag{49}$$

Thus $\mathbf{\Psi}^H\mathbf{\Psi}$ is interpreted as the density of probability which should be normalized to unity (and is a constant of motion),

$$\int\int\int dxdydz\left(\mathbf{\Psi}^H\mathbf{\Psi}\right) = 1, \tag{50}$$

under the condition that the wavefunction vanishes properly at infinity. This is a realistic condition if the wavefunction is a wave packet. In this case the wavefunctions form a base for an Hilbert space. Please note that the constant in Eq. (43) does not appear in the equations that determine the normalization, namely Eqs. (48),(49) and (50).

Having a Lagrangian, one can compute the corresponding energy-momentum tensor,

$$T_{\mu\nu} = \sum_j \frac{\partial L}{\partial\left(\frac{\partial u_j}{\partial x\mu}\right)} \frac{\partial u_j}{\partial x_\nu} - L\delta_{\mu\nu}, \tag{51}$$

where u_j stand for the components of $\mathbf{\Psi}$ and $\mathbf{\Psi}^*$. For T^{00} we obtain,

$$\frac{T^{00}}{c} = \frac{T_{00}}{c} = \frac{\mathbf{\Psi}^H}{2}\left(\hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi} + \frac{\mathbf{\Psi}^T}{2}\left(\hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi}^* = \frac{\mathbf{\Psi}^H}{2}\frac{\hat{E}}{c}\mathbf{\Psi} - \frac{\mathbf{\Psi}^T}{2}\frac{\hat{E}}{c}\mathbf{\Psi}^*, \tag{52}$$

and

$$\int\int\int dxdydzT^{00} = \frac{1}{2}\int\int\int dxdydz\left[\mathbf{\Psi}^H\hat{E}\mathbf{\Psi} - \mathbf{\Psi}^T\hat{E}\mathbf{\Psi}^*\right]. \tag{53}$$

Applying the energy operator on Eq. (50) one obtains,

$$\hat{E}\int\int\int dxdydz\left(\mathbf{\Psi}^H\mathbf{\Psi}\right) = \int\int\int dxdydz\left[\mathbf{\Psi}^H\hat{E}\mathbf{\Psi} + \mathbf{\Psi}^T\hat{E}\mathbf{\Psi}^*\right] = 0 \tag{54}$$

Hence

$$\int\int\int \mathbf{\Psi}^T\hat{E}\mathbf{\Psi}^* dxdydz = -\int\int\int \mathbf{\Psi}^H\hat{E}\mathbf{\Psi} dxdydz. \tag{55}$$

Substituting Eq. (55) into Eq. (53) we finally obtain,

$$\int\int\int dxdydzT^{00} = \int\int\int dxdydz\mathbf{\Psi}^H\hat{E}\mathbf{\Psi} =< \hat{E} >, \tag{56}$$

which is the expectation value of the energy operator. For the T^{0k} components we have,

$$\frac{T^{0k}}{c} = -\frac{T_{0k}}{c} = \mathbf{\Psi}^H\frac{\hat{p}_k}{2}\mathbf{\Psi} - \mathbf{\Psi}^T\frac{\hat{p}_k}{2}\mathbf{\Psi}^*$$

$$= \mathbf{\Psi}^H\frac{\hat{p}_k}{2}\mathbf{\Psi} + \left(\mathbf{\Psi}^H\frac{\hat{p}_k}{2}\mathbf{\Psi}\right)^* = \mathrm{Re}\left(\mathbf{\Psi}^H\hat{p}_k\mathbf{\Psi}\right), \tag{57}$$

and

$$\int\int\int dxdydz\frac{T^{0k}}{c} = \int\int\int dxdydz\mathbf{\Psi}^H\left(\hat{p}_k\right)\mathbf{\Psi} =< \hat{p}_k >, \quad k = 1, 2, 3, \tag{58}$$

which are the expectation values of real momenta.

Let us prove that $\hat{E}$ is a self adjoint operator. Indeed, from Eq. (15) $\hat{E}^* = -\hat{E}$, and using Eq. (55) we obtain,

$$\int\int\int dxdydz\left(\hat{E}\mathbf{\Psi}\right)^H\mathbf{\Psi} = -\int\int\int dxdydz\left(\hat{E}\mathbf{\Psi}^H\right)\mathbf{\Psi} \tag{59}$$

$$= -\int\int\int dxdydz\mathbf{\Psi}^T\hat{E}\mathbf{\Psi}^* = \int\int\int dxdydz\mathbf{\Psi}^H\hat{E}\mathbf{\Psi}. \tag{60}$$

In a similar way one can prove that the momentum operator $\hat{p}_k$ (for real momenta) is self adjoint,

$$\int\int\int dxdydz\left(\hat{p}_k\mathbf{\Psi}\right)^H\mathbf{\Psi} = -\int\int\int dxdydz\left(\hat{p}_k\mathbf{\Psi}^H\right)\mathbf{\Psi}$$

$$= -\int\int\int dxdydz \mathbf{\Psi}^T \hat{p}_k \mathbf{\Psi}^* = \int\int\int dxdydz \mathbf{\Psi}^H \hat{p}_k \mathbf{\Psi}. \tag{61}$$

Therefore Eq. (39), with the normalization condition Eq. (50), is the first quantized equation of the photon.

5. Lagrangian for Maxwell's equations without sources

One can write a Lagrangian for Maxwell's equations simply by substituting $\mathbf{\Psi} = \mathcal{N}(\mathbf{E} + i\mathbf{B})$ into the photon Lagrangian Eq. (47). One obtains,

$$\frac{L}{c\mathcal{N}} = -\mathbf{E}^T(\hat{\mathbf{p}}\cdot\mathbf{S})\mathbf{E} - \mathbf{B}^T(\hat{\mathbf{p}}\cdot\mathbf{S})\mathbf{B} + \mathbf{i}\mathbf{E}^T\frac{\hat{E}}{c}\mathbf{B} - i\mathbf{B}^T\frac{\hat{E}}{c}\mathbf{E} + \frac{\lambda^*}{2}\hat{\mathbf{p}}\cdot\mathbf{\Psi} - \frac{\lambda}{2}\hat{\mathbf{p}}\cdot\mathbf{\Psi}^* \tag{62}$$

$$= -\hbar\left(\mathbf{E}\cdot(\nabla\times\mathbf{E}) + \mathbf{B}\cdot(\nabla\times\mathbf{B}) + \frac{1}{c}\mathbf{E}\cdot(\partial_t\mathbf{B}) - \frac{1}{c}\mathbf{B}\cdot(\partial_t\mathbf{E})\right)$$

$$-\hbar\left(\mathrm{Im}(\lambda)\nabla\cdot\mathbf{E} - \mathrm{Re}(\lambda)\nabla\cdot\mathbf{B}\right). \tag{63}$$

The conserved probability current Eq. (49) takes the form,

$$\partial_t\left(\mathbf{\Psi}^H\mathbf{\Psi}\right) + c\boldsymbol{\nabla}\cdot\left(\mathbf{\Psi}^H\mathbf{S}\mathbf{\Psi}\right) = \mathcal{N}^2\left[\partial_t\left(\mathbf{E}^2 + \mathbf{B}^2\right) + 2c\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B})\right] = 0, \tag{64}$$

which coincides with the Poynting theorem. The normalization coefficient $\mathcal{N}$ can be found from the normalization requirement,

$$\int\int\int dxdydz \mathbf{\Psi}^H\mathbf{\Psi} = |\mathcal{N}|^2\int\int\int dxdydz\left(\mathbf{E}^2 + \mathbf{B}^2\right) = 1, \tag{65}$$

hence,

$$|\mathcal{N}|^{-2} = \int\int\int dxdydz\left(\mathbf{E}^2 + \mathbf{B}^2\right). \tag{66}$$

Thus the probability density for one photon is,

$$\mathbf{\Psi}^H\mathbf{\Psi} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{\int\int\int dxdydz\left(\mathbf{E}^2 + \mathbf{B}^2\right)}, \tag{67}$$

and the normalized photon wavefunction is,

$$\mathbf{\Psi} = \frac{\mathbf{E} + i\mathbf{B}}{\sqrt{\int\int\int dxdydz\left(\mathbf{E}^2 + \mathbf{B}^2\right)}}. \tag{68}$$

6. Relativistic quaternions

The α_μ matrices of the photon equation Eqs.(34-37) form a representation of what we will call "relativistic quaternions". They satisfy the following relations,

$$(\alpha_1)^2 = (\alpha_2)^2 = (\alpha_3)^2 = (\alpha_0)^2 = \alpha_0, \tag{69}$$

$$\alpha_1\alpha_2 = i\alpha_3 \quad \text{(and cyclic permutation)}, \tag{70}$$

$$\alpha_k\alpha_l = -\alpha_l\alpha_k, \quad \text{for} \quad k \neq l, \quad k, l = 1, 2, 3. \tag{71}$$

The α_μ matrices factorize the d'Alembertian, therefore also the relativistic quaternions will factorize it. The relativistic quaternions are a particular case of complex quaternions. Let $\mathbf{Q}$ be a quaternion, which traditionally is written as,

$$\mathbf{Q} = c_0\mathbf{1} + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}, \tag{72}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \quad \mathbf{ij} = -\mathbf{ji}, \quad \mathbf{ik} = -\mathbf{ki}, \quad \mathbf{kj} = -\mathbf{jk}, \quad \mathbf{ijk} = -\mathbf{1}, \tag{73}$$

where the c_μ are real coefficients. Let us rewrite Eq. (72) as,

$$\mathbf{Q} = c_0\mathbf{q}_0 + c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + c_3\mathbf{q}_3, \tag{74}$$

$$\mathbf{q}_0 \equiv \mathbf{1}, \quad \mathbf{q}_1 \equiv \mathbf{i}, \quad \mathbf{q}_2 \equiv \mathbf{j}, \quad \mathbf{q}_3 \equiv \mathbf{k}. \tag{75}$$

The conjugate quaternion is defined as,

$$\mathbf{Q}^C = c_0\mathbf{q}_0 - c_1\mathbf{q}_1 - c_2\mathbf{q}_2 - c_3\mathbf{q}_3. \tag{76}$$

Using Eqs. (72-76) we obtain,

$$\mathbf{Q}^C\mathbf{Q} = \mathbf{q}_0\left(c_0^2 + c_1^2 + c_2^2 + c_3^2\right). \tag{77}$$

The quaternions are not suitable for factorizing the d'Alembertian. The factorization can be done with complex quaternions, for which the coefficients c_μ in Eq. (72) or Eq. (74) can be complex. The relativistic quaternion with complex coefficients is defined as,

$$\mathbf{R} = c_0\mathbf{r}_0 + c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3, \tag{78}$$

$$\mathbf{r}_0 = \mathbf{q}_0 = \mathbf{1}, \quad \mathbf{r}_1 = i\mathbf{q}_1, \quad \mathbf{r}_2 = i\mathbf{q}_2, \quad \mathbf{r}_3 = i\mathbf{q}_3, \tag{79}$$

and its conjugate as,

$$\mathbf{R}^C = c_0\mathbf{r}_0 - c_1\mathbf{r}_1 - c_2\mathbf{r}_2 - c_3\mathbf{r}_3, \tag{80}$$

The multiplication table of the $\mathbf{r}_\mu$ base (and its representations) is,

$\times$	$\mathbf{r}_0$	$\mathbf{r}_1$	$\mathbf{r}_2$	$\mathbf{r}_3$
$\mathbf{r}_0$	$\mathbf{r}_0$	$\mathbf{r}_1$	$\mathbf{r}_2$	$\mathbf{r}_3$
$\mathbf{r}_1$	$\mathbf{r}_1$	$\mathbf{r}_0$	$i\mathbf{r}_3$	$-i\mathbf{r}_2$
$\mathbf{r}_2$	$\mathbf{r}_2$	$-i\mathbf{r}_3$	$\mathbf{r}_0$	$i\mathbf{r}_1$
$\mathbf{r}_3$	$\mathbf{r}_3$	$i\mathbf{r}_2$	$-i\mathbf{r}_1$	$\mathbf{r}_0$

. (81)

and,

$$\mathbf{R}^C\mathbf{R} = \mathbf{R}\mathbf{R}^C = \mathbf{r}_0\left(c_0^2 - c_1^2 - c_2^2 - c_3^2\right), \tag{82}$$

which allows factorization of the d'Alembertian.

Another important advantage of the relativistic quaternions is that they have a representation in terms of Pauli matrices Eq. (3),

$$\mathbf{R}_{\boldsymbol{\sigma}} = c_0\boldsymbol{\sigma}_0 + c_1\boldsymbol{\sigma}_x + c_2\boldsymbol{\sigma}_y + c_3\boldsymbol{\sigma}_z, \tag{83}$$

i.e. $\mathbf{R}_{\boldsymbol{\sigma}}$ is a second rank mixed spinor. In this way we have shown the connection between the relativistic quaternions and spinors. Now we can rewrite the photon (and Maxwell's) equations Eq. (39) as spinor equation,

$$\left(\frac{\hat{E}}{c}\boldsymbol{\sigma}_0 - \hat{p}_x\boldsymbol{\sigma}_x - \hat{p}_y\boldsymbol{\sigma}_y - \hat{p}_z\boldsymbol{\sigma}_z\right)(\lambda\boldsymbol{\sigma}_0 + \boldsymbol{\Psi}_x\boldsymbol{\sigma}_x + \boldsymbol{\Psi}_y\boldsymbol{\sigma}_y + \boldsymbol{\Psi}_z\boldsymbol{\sigma}_z) = 0. \tag{84}$$

In the section devoted to the quantum photon (spin 1) other representation of the relativistic quaternion in terms of 4×4 matrices is given in Eqs. (124) and (125). Similarly, in the section devoted to the quantum graviton (spin 2) other representation of the relativistic quaternion in terms of 8×8 matrices is given in Eqs. (145) and (147).

The 4×4 matrices Eqs. (34), (37), (124) and (125), which are representations of the relativistic quaternions, have eigenvalues -1 and 1 (twice). This is consistent with the condition of having helicity in the forward and backward directions only. The above matrices contain implicitly the subsidiary conditions, which is consistent with their role in suppressing the helicity components in other than forward and backward directions. Other properties are:

$$\det(\alpha_i) = \det(\beta_i) = 1, \quad i = x, y, z, \tag{85}$$

$$\mathrm{tr}(\alpha_i) = \mathrm{tr}(\beta_i) = 0. \tag{86}$$

The Pauli matrices σ_i have eigenvalues -1 and 1 and,

$$\det(\sigma_i) = -1, \quad \mathrm{tr}(\sigma_i) = 0. \tag{87}$$

Taking into account that the α_i and β_i have the same eigenvalues as the σ_i and the validity of Eqs. (85-87), we can conclude that α_i and β_i are equivalent to the matrices

$$\Sigma_i^{(4)} = \begin{pmatrix} \sigma_i & \\ & \sigma_i \end{pmatrix}, \quad i = 0, x, y, z, \tag{88}$$

In a similar way one can show that the 8×8 γ_i matrices of Eqs. (145) and (147) are equivalent to the 8×8 matrices

$$\Sigma_i^{(8)} = \begin{pmatrix} \sigma_i & & & \\ & \sigma_i & & \\ & & \sigma_i & \\ & & & \sigma_i \end{pmatrix}. \tag{89}$$

The above results can be generalized. Thus the relativistic quaternions have representations in terms of 4s×4s matrices

$$\Sigma_i^{(4s)} = \begin{pmatrix} \sigma_i & & & \\ & \sigma_i & & \\ & & \ddots & \\ & & & \sigma_i \end{pmatrix}. \tag{90}$$

An example of the above representation can be obtained by replacing Eq. (84) with,

$$\left(\frac{\hat{E}}{c}\Sigma_0^{(4)} + \hat{p}_x\Sigma_x^{(4)} + \hat{p}_y\Sigma_y^{(4)} + \hat{p}_z\Sigma_z^{(4)}\right)\begin{pmatrix} \lambda + \Psi_z \\ \Psi_x + i\Psi_y \\ \Psi_x - i\Psi_y \\ \lambda - \Psi_z \end{pmatrix} = 0, \tag{91}$$

or to use,

$$\begin{pmatrix} \lambda+\Psi_z \\ \Psi_x+i\Psi_y \\ \Psi_x-i\Psi_y \\ \lambda-\Psi_z \end{pmatrix} = \begin{pmatrix} \lambda+\Psi_0 \\ -\sqrt{2}\Psi_1 \\ \sqrt{2}\Psi_{-1} \\ \lambda-\Psi_0 \end{pmatrix}. \tag{92}$$

7. Dirac's equations for massless particles of any spin

Dirac has derived equations for massless particles with spin s, which in the ordinary vector notation[4] are,

$$\{s\hat{p}_0 + S_x\hat{p}_x + S_y\hat{p}_y + S_z\hat{p}_z\}\,\psi = 0, \tag{93}$$

$$\{s\hat{p}_x + S_x\hat{p}_0 - iS_y\hat{p}_z + iS_z\hat{p}_y\}\,\psi = 0, \tag{94}$$

$$\{s\hat{p}_y + S_y\hat{p}_0 - iS_z\hat{p}_x + iS_x\hat{p}_z\}\,\psi = 0, \tag{95}$$

$$\{s\hat{p}_z + S_z\hat{p}_0 - iS_x\hat{p}_y + iS_y\hat{p}_x\}\,\psi = 0, \tag{96}$$

where ψ a $(2s+1)$ component wave function and S_n are the spin $(2s+1)\times(2s+1)$ matrices which satisfy,

$$[S_x, S_y] = iS_z, \quad [S_z, S_x] = iS_y, \quad [S_y, S_z] = iS_x, \quad S_x^2 + S_y^2 + S_z^2 = s(s+1)I^{(s)}. \tag{97}$$

Above the $\hat{p}_n$ are the momenta, $\hat{p}_0 = \hat{E}/c$, $\hat{E}$ the energy, and $I^{(s)}$ is a $(2s+1)\times(2s+1)$ unit matrix. As we shall see below, for the case $s = 1$, Eq. (93) will lead to the Faraday and Ampere-Maxwell laws. The Gauss laws can be derived from Eqs. (93-96) in a way which will be described below. Eqs. (93-96) were analyzed extensively by Bacry[5], who derived them using Wigner's condition [6] on the Pauli-Lubanski vector W^μ for massless fields,

$$W^\mu = s\hat{p}^\mu, \quad \mu = x_0, x, y, z. \tag{98}$$

In the next section Eqs. (93-96) will be retrieved by factorizing the d'Alembertian.

8. Wave equations for massless particles of any spin

Using the metric convention $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$, we will employ a special form of the Pauli-Lubanski vector operator W^μ suggested by H. Bacry[5] for massless particles,

$$W^\mu = \frac{-i}{2}\varepsilon^{\mu\nu\rho\lambda}S_{\rho\lambda}\hat{p}_\nu, \quad S_{\rho\lambda} = \begin{pmatrix} 0 & S_x & S_y & S_z \\ -S_x & 0 & -iS_z & iS_y \\ -S_y & iS_z & 0 & -iS_x \\ -S_z & -iS_y & iS_x & 0 \end{pmatrix} \equiv (\mathbf{S}, i\mathbf{S}), \tag{99}$$

where $S_{\rho\lambda}$ is the antysymmetric spin tensor, $\mathbf{S}$ is the spin vector matrix operator, s is the spin of the particle with zero mass and the components of W^μ are $(2s+1)\times(2s+1)$ matrices. We shall use the following relation[7],

$$W_\mu W^\mu = -s\,(s+1)\,\hat{p}_\mu\hat{p}^\mu I^{(s)}, \tag{100}$$

where $I^{(s)}$ is the $(2s+1)\times(2s+1)$ unit matrix. One can easily see that,

$$\left(W_\mu - s\hat{p}_\mu\right)\left(W^\mu + s\hat{p}^\mu\right) = -s\left(2s+1\right)\hat{p}_\mu\hat{p}^\mu I^{(s)}, \tag{101}$$

or

$$\left(W_\mu + s\hat{p}_\mu\right)\left(W^\mu - s\hat{p}^\mu\right) = -s\left(2s+1\right)\hat{p}_\mu\hat{p}^\mu I^{(s)}. \tag{102}$$

Eq. (102) or Eq. (101) will be the basis for all our derivations. From Eq. (102) one can see that if ψ, the wavefunction, which is a $(2s+1)$ component spherical tensor operator of rank s satisfies,

$$\left(W^\mu + s\hat{p}^\mu\right)\psi = 0, \quad \mu = 0,1,2,3, \tag{103}$$

also the basic one-particle requirement,

$$\hat{p}_\mu\hat{p}^\mu I^{(s)}\psi = 0, \tag{104}$$

will be satisfied. Explicitly we have,

$$W^0 = W_0 = -S_x\hat{p}_x - S_y\hat{p}_y - S_z\hat{p}_z, \tag{105}$$

$$-W^1 = W_x = -S_x\hat{p}_0 - iS_y\hat{p}_z + iS_z\hat{p}_y, \tag{106}$$

$$-W^2 = W_y = -S_y\hat{p}_0 - iS_z\hat{p}_x + iS_x\hat{p}_z, \tag{107}$$

$$-W^3 = W_z = -S_z\hat{p}_0 - iS_x\hat{p}_y + iS_y\hat{p}_x. \tag{108}$$

Eqs. (105) and (108) can be presented in a nonmanifestly covariant form[5] as,

$$\left(\hat{\mathbf{p}}\cdot\mathbf{S} - s\hat{p}^0\right)\psi = 0, \tag{109}$$

$$\left(\mathbf{S}\hat{p}^0 + i\mathbf{S}\times\hat{\mathbf{p}} - s\hat{\mathbf{p}}\right)\psi = 0, \tag{110}$$

where $\hat{\mathbf{p}}$ the particle momentum and $\mathbf{S}$ is the spin vector matrix operator with the properties given in Eq. (97). Dirac[4] has suggested to use Eq. (109) as the basic one and to substitute from it $s\hat{p}^0 = -\hat{\mathbf{p}}\cdot\mathbf{S}$ into Eqs. (110). In this way one obtains,

$$\left(\frac{1}{s}\mathbf{S}\left(\hat{\mathbf{p}}\cdot\mathbf{S}\right) + i\mathbf{S}\times\hat{\mathbf{p}} - s\hat{\mathbf{p}}\right)\psi = 0, \tag{111}$$

which are subsidiary conditions on Eq. (109). But the three Eqs. (111) are not independent. If one multiplies Eqs. (111) by $\mathbf{S}$ one finds,

$$\mathbf{S}\cdot\left(\frac{1}{s}\mathbf{S}\left(\hat{\mathbf{p}}\cdot\mathbf{S}\right) + i\mathbf{S}\times\hat{\mathbf{p}} - s\hat{\mathbf{p}}\right) = 0, \tag{112}$$

and only one of the three Eqs. (111) is independent. According to our experience the simplest one is the z-component equation,

$$\left(\frac{1}{s}S_z\left(\hat{\mathbf{p}}\cdot\mathbf{S}\right) + i\left(S_x\hat{p}_y - S_y\hat{p}_x\right) - s\hat{p}_z\right)\psi = 0. \tag{113}$$

Now we can collect our results and make the quantum transition substitutions,

$$\hat{\mathbf{p}} \rightarrow -i\hbar\boldsymbol{\nabla}, \quad \hat{p}^0 \rightarrow i\hbar\frac{\partial}{c\partial t}, \tag{114}$$

and obtain the final result,

$$\left(si\hbar\frac{\partial}{c\partial t} + i\hbar\mathbf{S}\cdot\boldsymbol{\nabla}\right)\psi = 0, \tag{115}$$

with the subsidiary conditions,

$$-i\hbar\left[\frac{1}{s}S_z\left(\mathbf{S}\cdot\boldsymbol{\nabla}\right) + i\left(S_x\frac{\partial}{\partial y} - S_y\frac{\partial}{\partial x}\right) - s\frac{\partial}{\partial z}\right]\psi = 0. \tag{116}$$

In the equations we derived above, the wavefunctions are $(2s+1)$ component spherical tensor operators of rank s. In order to be consistent with the Cartesian representation, we will have to work in terms of spherical tensors. A consistent formalism is worked out in Appendix D.

9. The photon equation revisited

We will present the photon equation in the angular momentum representation using the formalism given by Eqs.(99-D.12) of Appendix D. The wavefunction and the spin matrices are,

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_0 \\ \Psi_{-1} \end{pmatrix}, \tag{117}$$

$$\tilde{S}_x = \frac{-1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \tilde{S}_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ \tilde{S}_z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{118}$$

$$I^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

We have 4 equations for $s = 1$. The main equation,

$$\left(\hat{\mathbf{p}}\cdot\tilde{\mathbf{S}} - \frac{\hat{E}}{c}I^{(3)}\right)\Psi = 0, \tag{119}$$

and the subsidiary equations,

$$\left(\tilde{\mathbf{S}}\left(\hat{\mathbf{p}}\cdot\tilde{\mathbf{S}}\right) + i\tilde{\mathbf{S}}\times\hat{\mathbf{p}} - \hat{\mathbf{p}}I^{(3)}\right)\Psi = 0. \tag{120}$$

The z- subsidiary condition is,

$$\left[\tilde{S}_z\left(\tilde{S}_x\hat{p}_x + \tilde{S}_y\hat{p}_y + \tilde{S}_z\hat{p}_z\right) + i\tilde{S}_x\hat{p}_y - i\tilde{S}_y\hat{p}_x - \hat{p}_z I^{(3)}\right]\Psi$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2}\sqrt{2}\hat{p}_x - \frac{1}{2}i\sqrt{2}\hat{p}_y & -\hat{p}_z & -\frac{1}{2}\sqrt{2}\hat{p}_x - \frac{1}{2}i\sqrt{2}\hat{p}_y \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_0 \\ \Psi_{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ \hat{p}_{-1} & -\hat{p}_0 & \hat{p}_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_0 \\ \Psi_{-1} \end{pmatrix} = 0, \tag{121}$$

which yields,

$$\hat{p}_{-1}\Psi_1 - \hat{p}_0\Psi_0 + \hat{p}_1\Psi_{-1} = -\hat{p}_x\Psi_x - \hat{p}_y\Psi_y - \hat{p}_z\Psi_z = -\hat{\mathbf{p}} \cdot \mathbf{\Psi} = \mathbf{0},$$

so that the subsidiary condition becomes,

$$\hat{\mathbf{p}} \cdot \mathbf{\Psi} = \mathbf{0}, \tag{122}$$

as before in Eq. (11). The main equation and the subsidiary condition form the photon equation in the angular momentum representation. In order to get an equation which include the subsidiary condition implicitly, we proceed in a similar way to the derivation of Eqs.(40-43). One introduces the wavefunction,

$$\Phi = \begin{pmatrix} \lambda \\ \Psi_1 \\ \Psi_0 \\ \Psi_{-1} \end{pmatrix}, \tag{123}$$

where λ is an arbitrary constant, and complements the $\tilde{S}_x, \tilde{S}_y, \tilde{S}_z$ matrices to 4×4 matrices $\beta_x, \beta_y, \beta_z$,which include implicitly the subsidiary condition. The required matrices are,

$$\beta_x = \frac{-1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}; \ \beta_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}; \tag{124}$$

$$\beta_z = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \ \beta_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{125}$$

One can check that these matrices provide a representation of the relativistic quaternions Eqs. (69) - (79), and therefore factorize the d'Alembertian,

$$\left(\beta_0\partial_0 + \beta_x\partial_x + \beta_y\partial_y + \beta_z\partial_z\right)\left(\beta_0\partial_0 - \beta_x\partial_x - \beta_y\partial_y - \beta_z\partial_z\right)$$

$$= \left(\partial_0^2 - \partial_x^2 - \partial_y^2 - \partial_z^2\right)\beta_0. \tag{126}$$

The (free) photon equation takes the form,

$$\left(\hat{E}\beta_0 - c\hat{\mathbf{p}}\cdot\boldsymbol{\beta}\right)\Phi = 0, \tag{127}$$

where Φ is given in Eq.(123). In a similar way to the derivation of the photon Lagrangian Eq.(45) one can construct the Lagrangian L,

$$L = \frac{\Phi^H}{2}\left(\hat{E}\beta_0 - c\hat{\mathbf{p}}\cdot\boldsymbol{\beta}\right)\Phi - \frac{\Phi^T}{2}\left(\hat{E}\beta_0 - c\hat{\mathbf{p}}\cdot\boldsymbol{\beta}^*\right)\Phi^*, \tag{128}$$

where Φ^T is the transpose of Φ,

$$\Phi^T = \begin{pmatrix} \lambda & \Psi_1 & \Psi_0 & \Psi_{-1} \end{pmatrix}, \tag{129}$$

and Φ^H is the Hermitian conjugate of Φ which, according to Appendix B, is,

$$\Phi^H = \begin{pmatrix} \lambda^* & -\Psi^*_{-1} & \Psi^*_0 & -\Psi^*_1 \end{pmatrix}. \tag{130}$$

By varying the Lagrangian L with respect to the components of Φ^*, Eq.(127) is obtained. By varying L with respect to the components of Φ, the equation with the opposite helicity,

$$\left(\hat{E}\beta_0 - c\hat{\mathbf{p}}\cdot\boldsymbol{\beta}^*\right)\Phi^* = 0, \tag{131}$$

is obtained.

10. The quantum graviton

We now turn to a derivation of the graviton (spin 2) equation in the angular momentum representation following the formalism of Eqs.(99-D.10). The wavefunction $\mathbf{\Psi}$ is,

$$< 2, m|\mathbf{\Psi} >= \begin{pmatrix} \Psi_2 \\ \Psi_1 \\ \Psi_0 \\ \Psi_{-1} \\ \Psi_{-2} \end{pmatrix}, \tag{132}$$

and the spin matrices are,

$$\tilde{S}_x = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & -\sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & -\sqrt{\frac{3}{2}} & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{133}$$

$$\tilde{S}_y = i\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{134}$$

$$\tilde{S}_z = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \; I^{(5)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{135}$$

We have 4 equations for $s = 2$, the main equation,

$$\left(\hat{\mathbf{p}}\cdot\tilde{\mathbf{S}} - 2\frac{\hat{E}}{c}I^{(5)}\right)\Psi = 0, \tag{136}$$

and the subsidiary conditions,

$$\left(\frac{1}{2}\tilde{\mathbf{S}}\left(\hat{\mathbf{p}}\cdot\tilde{\mathbf{S}}\right)+i\tilde{\mathbf{S}}\times\hat{\mathbf{p}}-2\hat{\mathbf{p}}I^{(5)}\right)\Psi=0. \tag{137}$$

The z- subsidiary condition is,

$$\left[S_z\left(S_x\hat{p}_x+S_y\hat{p}_y+S_z\hat{p}_z\right)/2+iS_x\hat{p}_y-iS_y\hat{p}_x-2\hat{p}_zI^{(5)}\right]\Psi=K\Psi=0, \tag{138}$$

where,

$$K=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{3}{\sqrt{2}}\hat{p}_{-1} & -\frac{3}{2}\hat{p}_0 & \sqrt{3}\hat{p}_1 & 0 & 0 \\ 0 & \sqrt{3}\hat{p}_{-1} & -2\hat{p}_0 & \sqrt{3}\hat{p}_1 & 0 \\ 0 & 0 & \sqrt{3}\hat{p}_{-1} & -\frac{3}{2}\hat{p}_0 & \frac{3}{\sqrt{2}}\hat{p}_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{139}$$

The 3 subsidiary conditions are each valid up to a multiplicative constant. We will choose the multiplicative constants so that the following decomposition is obtained,

$$-4\hat{p}_\mu\hat{p}^\mu I^{(s)}\Psi=\left(W_0-2\frac{\hat{E}}{c}I^{(5)}\right)\left(W_0+2\frac{\hat{E}}{c}I^{(5)}\right)\Psi+P^HP\Psi, \tag{140}$$

where P^H is the Hermitian conjugate of P. The normalized subsidiary condition will fulfil the equation,

$$P\Psi=0, \tag{141}$$

where,

$$P=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{6}\hat{p}_{-1} & -\sqrt{3}\hat{p}_0 & \hat{p}_1 & 0 & 0 \\ 0 & \sqrt{3}\hat{p}_{-1} & -2\hat{p}_0 & \sqrt{3}\hat{p}_1 & 0 \\ 0 & 0 & \hat{p}_{-1} & -\sqrt{3}\hat{p}_0 & \sqrt{6}\hat{p}_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{142}$$

In order to get an equation with the subsidiary conditions included in it implicitly, one has to introduce an eight-component wavefunction,

$$\Phi=\begin{pmatrix} \lambda_1 \\ \lambda_0 \\ \lambda_{-1} \\ \Psi_2 \\ \Psi_1 \\ \Psi_0 \\ \Psi_{-1} \\ \Psi_{-2} \end{pmatrix}, \tag{143}$$

where $\lambda_1,\lambda_0,\lambda_{-1}$ are arbitrary constants, and to complement the 5×5 $\tilde{S}_x,\tilde{S}_y,\tilde{S}_z$ matrices into 8×8 matrices $\gamma_x,\gamma_y,\gamma_z$, so that the first 3 raws and 3 columns generate the subsidiary conditions. In addition we require that these matrices factorize the d'Alembertian. The required matrices are:

$$\gamma_x = \frac{1}{2}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \sqrt{3} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & \sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\sqrt{3} \\ \sqrt{3} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & -1 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & -\sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & -\sqrt{\frac{3}{2}} & 0 & 0 & 0 & -\sqrt{\frac{3}{2}} & 0 & -1 \\ 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{144}$$

$$\gamma_y = \frac{i}{2}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & -\sqrt{3} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & -\sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\sqrt{3} \\ \sqrt{3} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & 1 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & \sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 & -1 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{145}$$

$$\gamma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \tag{146}$$

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{147}$$

One can check that the $\gamma_0\gamma_x, \gamma_y, \gamma_z$, form a representation of the relativistic quaternion, and therefore factorize the d'Alembertian, i.e.,

$$\begin{aligned} & \left(\gamma_0\partial_0 + \gamma_x\partial_x + \gamma_y\partial_y + \gamma_z\partial_z\right)\left(\gamma_0\partial_0 - \gamma_x\partial_x - \gamma_y\partial_y - \gamma_z\partial_z\right) \\ = \; & \left(\partial_0^2 - \partial_x^2 - \partial_y^2 - \partial_z^2\right)\gamma_0. \end{aligned} \tag{148}$$

With this in mind, the (free) quantum graviton equation takes the form,

$$\left(2\hat{E}\gamma_0 - c\hat{\mathbf{p}}\cdot\boldsymbol{\gamma}\right)\Phi = 0, \tag{149}$$

where Φ is given in Eq. (143). More details about the properties of the $\gamma_x, \gamma_y, \gamma_z$ matrices can be found in Appendix C. In a similar way to the derivation of the photon Lagrangian, Eq. (128), one can construct also a Lagrangian for the quantum graviton,

$$L = \frac{\Phi^H}{2}\left(2\hat{E}\gamma_0 - c\hat{\mathbf{p}}\cdot\boldsymbol{\gamma}\right)\Phi - \frac{\Phi^T}{2}\left(2\hat{E}\gamma_0 - c\hat{\mathbf{p}}\cdot\boldsymbol{\gamma}^*\right)\Phi^*, \tag{150}$$

where Φ^T is the transpose of Φ, and Φ^H is the Hermitian conjugate of Φ which, according to Appendix B, is,

$$\Phi^H = \left(\begin{array}{cccccccc} \lambda_{-1}^* & \lambda_0^* & \lambda_1^* & \Psi_{-2}^* & -\Psi_{-1}^* & \Psi_0^* & -\Psi_1^* & \Psi_2^* \end{array}\right). \tag{151}$$

By varying the Lagrangian L with respect to the components of Φ^* Eq. (149) is obtained. By varying L with respect to the components of $\mathbf{\Phi}$, the equation with the opposite helicity,

$$\left(2\hat{E}\gamma_0 - c\hat{\mathbf{p}}\cdot\boldsymbol{\gamma}^*\right)\Phi^* = 0, \tag{152}$$

is obtained. The probability current, evaluated in a similar way as in Appendix A, is,

$$\partial_t\left(\mathbf{\Psi}^H\mathbf{\Psi}\right) + \frac{c}{2}\boldsymbol{\nabla}\cdot\left(\mathbf{\Psi}^H\tilde{\mathbf{S}}\mathbf{\Psi}\right) = 0. \tag{153}$$

Here $\mathbf{\Psi}^H\mathbf{\Psi}$ is interpreted as the density of probability which should be normalized to unity,

$$\int\int\int dxdydz\mathbf{\Psi}^H\mathbf{\Psi} = 1, \tag{154}$$

under the condition that the wavefunction (as a wave packet) vanishes properly at infinity. The constants in Eq. (143) do not appear in the equations that determine the normalization, namely Eqs. (132),(153) and (154).Using the Lagrangian Eq. (150), the expectation values of the energy and momenta are,

$$\int\int\int dxdydzT^{00} = \int\int\int dxdydz\mathbf{\Psi}^H\hat{E}\mathbf{\Psi} =< \hat{E} >, \tag{155}$$

$$\int\int\int dxdydzT^{0k} = \int\int\int dxdydz\mathbf{\Psi}^H\left(c\hat{p}_k\right)\mathbf{\Psi} =< c\hat{p}_k >, \quad k = 1, 2, 3, \tag{156}$$

in agreement with the definitions of quantum operators. Therefore Eq. (149) with the normalization condition Eq. (154) is the first quantized equation of the graviton.

11. Covariance of the equations

The quantum equations for massless particles of spin s were derived from the covariant equations Eqs. (100-104) with the result

$$\left(s\hat{E}\eta_0 - c\hat{\mathbf{p}}\cdot\boldsymbol{\eta}\right)\Phi = 0, \tag{157}$$

where $\eta_0, \eta_1, \eta_2, \eta_3$ are $4s \times 4s$ matrices representing the relativistic quaternions and Φ is a $4s$ component wave function with $2s - 1$ zeros. Equations similar to Eq. (157) were analyzed by Lomont[[8]] (including the problem of the zeros in the wave functions) and were shown to be covariant. Moreover, as $\left(s\hat{E}\eta_0 - c\hat{\mathbf{p}}\cdot\boldsymbol{\eta}\right)$ factorizes the d'Alembertian, each component of the wave function satisfies the relativistic wave equation.

As an example let us take Eq. (39) for $s = 1$, which was derived by combining two covariant Maxwell equations Eq. (28), therefore it should be also covariant. The wave function Eq. (38) has a zero component which results from the sum of two covariant equations, therefore it will remain zero in any frame. The representation under which the wave function Eq. (38) transforms is $D^{(1/2,1/2)}$ which has basis of spin 0 (one component) and spin 1 (3 components). As the photon is of spin 1, the spin zero component has to be eliminated. This is the reason of the zero component in Eq. (38) and it has to stay this way in all frames. The spin one 3 component wave function Eq. (18), Eq. (19) or Eq. (48) (proportional to $\mathbf{E} + \mathbf{iH}$) is also covariant as was shown by Laporte and Uhlenbeck[[9]], the 3 components belong to a second rank symmetric spinor.

Similar situation appears in handling the "graviton". The 8 component wave function transforms under $D^{(3/2,1/2)}$ which has basis of spin 1 (3 components) and spin 2 (5 components). In order to eliminate the spin one contribution the wave function has to have 3 zeros.

12. Summary and conclusions

We have shown above, how the quantum photon equation and the resulting Maxwell's equations can be derived from first principles, similar to those which have been used to derive the Dirac relativistic electron equation. We have worked out a general method of deriving quantum equations for massless particles of any spin, based on the factorization of the d'Alembertian. We have applied this procedure for the spin one (photon) and the spin two (graviton) in two ways, in order to write free particle wave equations with explicit and implicit subsidiary conditions. To this aim the factorization was achieved by using "relativistic quaternions" and their matrix representations (See Sec.6).

The unexplained substitution of the photon wavefunction Ψ in terms of the electric $\mathbf{E}$ and the magnetic field $\mathbf{B}$,

$$\mathbf{\Psi}=\mathcal{N}\left(\mathbf{E}+i\mathbf{B}\right), \tag{158}$$

leads to Maxwell's equations. Thus Maxwell's equations can be considered as the first quantized one-photon quantum equation. This fact was not realized for a long time, because Maxwell's equations do not contain the Planck constant $\hbar$. The Planck constant $\hbar$ and the normalization constant $\mathcal{N}$ are cancelled out in Eqs. (18) and (19).

Maxwell's equations were derived using different representations of relativistic quaternions, i.e., in terms of α and β matrices, Eqs. (39) and (127), respectively. It was demonstrated that these equations can be obtained from scalar Lagrangians and each one of them leads to Maxwell's equations. In addition a conserved probability current Eq. (49) was derived, which allowed definition of the probability density and wavefunction normalization. The energy-momentum tensors were calculated and found to be consistent with the quantum expectation values of the energy and momentum operators.

The quantum graviton equation was derived first with explicit subsidiary conditions, Eqs. (136) and (139). Rather than using the z subsidiary condition, Eqs. (137), we defined normalized subsidiary conditions, Eqs. (141, 142), which allowed construction of a single quantum graviton equation Eq. (149) in terms of 8×8 γ matrices Eqs. (144-147). The 8×8 γ matrices form a representation of the relativistic quaternions and in this way allow the factorization of the d'Alembertian. Moreover the subsidiary conditions are implicitly included in the new equation. We have included in the 8-component wavefunction Eq. (143) 3 constant components $\lambda_1, \lambda_0, \lambda_{-1}$ which allowed us to construct a scalar Lagrangian from which all the equations can be derived. A conserved probability current Eq. (153) was derived, which allowed to define the probability density and wavefunction normalization. The energy-momentum tensor was found to be consistent with the quantum expectation values of the energy and momentum operators.

In this way we have shown that the quantum graviton equation Eq. (149) satisfies the basic requirements of a quantum theory.

In summary, we have developed a formalism of quantum equations for massless particles of arbitrary spin and applied it to spin one (photon) and spin two (graviton). The formalism allowed us to formulate scalar Lagrangians, derive probability currents and wavefunction normalization and to understand Maxwell's equations as first quantized photon equation. The same procedures were applied to the quantum graviton. It remains to understand the connection between the quantum graviton and gravitation.

Appendix A. Conserved currents of probability

The photon equation Eq.(10) and Eq.(27), satisfies,

$$\left(\frac{\hat{E}}{c}I^{(3)} - \hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi} = 0, \tag{A.1}$$

$$\left(\frac{\hat{E}}{c}I^{(3)} + \hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi}^* = 0. \tag{A.2}$$

From Eqs.(A.1-A.2) we have,

$$\mathbf{\Psi}^H\left(\frac{\hat{E}}{c}I^{(3)} - \hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi} + \mathbf{\Psi}^T\left(\frac{\hat{E}}{c}I^{(3)} + \hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi}^*$$

$$= \frac{\hat{E}}{c}\left(\mathbf{\Psi}^H\mathbf{\Psi}\right) - \mathbf{\Psi}^H\left(\hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi} + \mathbf{\Psi}^T\left(\hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi}^*. \tag{A.3}$$

Let us show that,

$$\mathbf{\Psi}^H\left(\hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi} - \mathbf{\Psi}^T\left(\hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi}^* = \hat{\mathbf{p}}\cdot\left(\mathbf{\Psi}^H\mathbf{S}\mathbf{\Psi}\right). \tag{A.4}$$

Indeed,

$$\mathbf{\Psi}^H\left(\hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi} - \mathbf{\Psi}^T\left(\hat{\mathbf{p}}\cdot\mathbf{S}\right)\mathbf{\Psi}^* =$$

$$\left[\Psi_y^*\left(-i\hat{p}_x\Psi_z\right) + \Psi_z^*\left(i\hat{p}_x\Psi_y\right) + \Psi_y\left(i\hat{p}_x\Psi_z^*\right) + \Psi_z\left(-i\hat{p}_x\Psi_y^*\right)\right]$$

$$+\left[\Psi_x^*\left(i\hat{p}_y\Psi_z\right) + \Psi_z^*\left(-i\hat{p}_y\Psi_x\right) + \Psi_x\left(-i\hat{p}_y\Psi_z^*\right) + \Psi_z\left(i\hat{p}_y\Psi_x^*\right)\right]$$

$$+\left[\Psi_x^*\left(-i\hat{p}_z\Psi_y\right) + \Psi_y^*\left(i\hat{p}_z\Psi_x\right) + \Psi_x\left(i\hat{p}_z\Psi_y^*\right) + \Psi_y\left(-i\hat{p}_z\Psi_x^* + i\hat{p}_x\Psi_z^*\right)\right]$$

$$= \hat{p}_x\left[i\left(\Psi_z^*\Psi_y\right) - i\left(\Psi_y^*\Psi_z\right)\right] + \hat{p}_y\left[i\Psi_x^*\Psi_z - i\Psi_z^*\Psi_x\right] + \hat{p}_z\left[i\Psi_y^*\Psi_x - i\Psi_x^*\Psi_y\right]$$

$$= \hat{p}_x\left(\mathbf{\Psi}^H\mathbf{S}_x\mathbf{\Psi}\right) + \hat{p}_y\left(\mathbf{\Psi}^H\mathbf{S}_y\mathbf{\Psi}\right) + \hat{p}_z\left(\mathbf{\Psi}^H\mathbf{S}_z\mathbf{\Psi}\right) = \hat{\mathbf{p}}\cdot\left(\mathbf{\Psi}^H\mathbf{S}\mathbf{\Psi}\right).$$

Finally from Eq.(A.3) and Eq.(A.4) one obtains,

$$\frac{\hat{E}}{c}\left(\mathbf{\Psi}^H\mathbf{\Psi}\right) - \hat{\mathbf{p}}\cdot\left(\mathbf{\Psi}^H\mathbf{S}\mathbf{\Psi}\right) = 0, \tag{A.5}$$

or

$$\partial_t\left(\mathbf{\Psi}^H\mathbf{\Psi}\right) + c\boldsymbol{\nabla}\cdot\left(\mathbf{\Psi}^H\mathbf{S}\mathbf{\Psi}\right) = 0. \tag{A.6}$$

Eq.(A.6) is the conserved current of probability and $\mathbf{\Psi}^H\mathbf{\Psi}$ is interpreted as the density of probability which should be normalized to unity.

Appendix B. Complex conjugation of spherical tensors

In this appendix we relate spherical tensors to Cartesian ones and evaluate the outcome of complex conjugation of spherical tensors. For spherical vectors $\mathbf{V}^{(1)}$ (tensors of rank 1) we have,

$$\mathbf{V}_m^{(1)} \equiv \begin{pmatrix} V_1^{(1)} \\ V_0^{(1)} \\ V_{-1}^{(1)} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}}\left(V_x + iV_y\right) \\ V_z \\ \frac{1}{\sqrt{2}}\left(V_x - iV_y\right) \end{pmatrix}, \tag{B.1}$$

where V_x, V_y, V_z are the Cartesian components of the vector $\mathbf{V}$,

$$\mathbf{V}_i \equiv \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}}\left(V_1^{(1)} - V_{-1}^{(1)}\right) \\ \frac{i}{\sqrt{2}}\left(V_1^{(1)} + V_{-1}^{(1)}\right) \\ V_z \end{pmatrix}. \tag{B.2}$$

Let us construct the spherical tensor $\mathbf{V}_m^{C(1)}$ out of the complex conjugates of $\mathbf{V}_i$,

$$\mathbf{V}_m^{C(1)} \equiv \begin{pmatrix} -\frac{1}{\sqrt{2}}\left(V_x^* + iV_y^*\right) \\ V_z^* \\ \frac{1}{\sqrt{2}}\left(V_x^* - iV_y^*\right) \end{pmatrix}. \tag{B.3}$$

Taking the complex conjugate of Eq.(B.1) and the definition Eq.(B.3) one finds,

$$\left(V_m^{(1)}\right)^* = (-1)^m V_{-m}^{C(1)}, \quad m = 1, 0, -1. \tag{B.4}$$

The Hermitian conjugate of $\mathbf{V}^{(1)}$ can be found from Eqs.(B.1-B.4),

$$\left(\mathbf{V}_m^{(1)}\right)^H \equiv \begin{pmatrix} V_1^{(1)} \\ V_0^{(1)} \\ V_{-1}^{(1)} \end{pmatrix}^H = \begin{pmatrix} \left(V_1^{(1)}\right)^* & \left(V_0^{(1)}\right)^* & \left(V_{-1}^{(1)}\right)^* \end{pmatrix} \tag{B.5}$$

$$= \begin{pmatrix} -V_{-1}^{C(1)} & V_0^{C(1)} & -V_1^{C(1)} \end{pmatrix}.$$

We also find that,

$$\mathrm{Re}\left(V_m^{(1)}\right) = \frac{1}{2}\left(V_m^{(1)} + (-1)^m V_{-m}^{C(1)}\right), \tag{B.6}$$

$$\mathrm{Im}\left(V_m^{(1)}\right) = \frac{-i}{2}\left(V_m^{(1)} - (-1)^m V_{-m}^{C(1)}\right), \quad m = 1, 0, -1. \tag{B.7}$$

For tensors of rank 2 $\mathbf{T}_m^{(2)}$ the procedure is similar. Their relation to a traceless symmetric Cartesian tensors ($S_{11} + S_{22} + S_{33} = 0$) is,

$$\mathbf{T}_m^{(2)} \equiv \begin{pmatrix} T_2^{(2)} \\ T_1^{(2)} \\ T_0^{(2)} \\ T_{-1}^{(2)} \\ T_{-2}^{(2)} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{3}{2}}\left(S_{11} - S_{22} + 2iS_{12}\right) \\ -\sqrt{6}\left(S_{13} + iS_{23}\right) \\ 2S_{33} - S_{11} - S_{22} = -3\left(S_{11} + S_{22}\right) \\ \sqrt{6}\left(S_{13} - iS_{23}\right) \\ \sqrt{\frac{3}{2}}\left(S_{11} - S_{22} - 2iS_{12}\right) \end{pmatrix}, \tag{B.8}$$

where S_{ij} are the components of the corresponding traceless Cartesian tensor. The inverse relations are,

$$S_{11} = \frac{1}{\sqrt{24}}\left(T_2^{(2)} + T_{-2}^{(2)}\right) - \frac{1}{6}T_0^{(2)},$$

$$S_{22} = -\frac{1}{\sqrt{24}}\left(T_2^{(2)} + T_{-2}^{(2)}\right) - \frac{1}{6}T_0^{(2)},$$

$$S_{12} = S_{21} = -i\frac{1}{\sqrt{24}}\left(T_2^{(2)} - T_{-2}^{(2)}\right),$$

$$S_{13} = S_{31} = -\frac{1}{\sqrt{24}}\left(T_1^{(2)} - T_{-1}^{(2)}\right),$$

$$S_{23} = S_{32} = i\frac{1}{\sqrt{24}}\left(T_1^{(2)} + T_{-1}^{(2)}\right),$$

$$S_{33} = -S_{11} - S_{22} = \frac{1}{3}T_0^{(2)}.$$

Let us construct the spherical tensor $\mathbf{T}_m^{C(2)}$ out of the complex conjugates of $\mathbf{S}_{ij}$,

$$\mathbf{T}_m^{C(2)} \equiv \begin{pmatrix} T_2^{C(2)} \\ T_1^{C(2)} \\ T_0^{C(2)} \\ T_{-1}^{C(2)} \\ T_{-2}^{C(2)} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{3}{2}}\left(S_{11}^* - S_{22}^* + 2iS_{12}^*\right) \\ -\sqrt{6}\left(S_{13}^* + iS_{23}^*\right) \\ 2S_{33}^* - S_{11}^* - S_{22}^* \\ \sqrt{6}\left(S_{13}^* - iS_{23}^*\right) \\ \sqrt{\frac{3}{2}}\left(S_{11}^* - S_{22}^* - 2iS_{12}^*\right) \end{pmatrix}, \tag{B.9}$$

Taking the complex conjugate of Eq.(B.8) and the definition Eq.(B.9) one finds,

$$\left(T_m^{(2)}\right)^* = (-1)^m\, T_{-m}^{C(2)}, \quad m = 2, 1, 0, -1, -2. \tag{B.10}$$

and,

$$\mathrm{Re}\left(T_m^{(2)}\right) = \frac{1}{2}\left(T_m^{(2)} + (-1)^m\, T_{-m}^{C(2)}\right), \tag{B.11}$$

$$\mathrm{Im}\left(T_m^{(2)}\right) = \frac{-i}{2}\left(T_m^{(2)} - (-1)^m\, T_{-m}^{C(2)}\right), \quad m = 2, 1, 0, -1, -2. \tag{B.12}$$

The Hermitian conjugate of $\mathbf{T}_m^{(2)}$ can be found from Eqs.(B.8-B.10),

$$\begin{aligned} \begin{pmatrix} T_2^{(2)} \\ T_1^{(2)} \\ T_0^{(2)} \\ T_{-1}^{(2)} \\ T_{-2}^{(2)} \end{pmatrix}^H &= \begin{pmatrix} \left(T_2^{(2)}\right)^* & \left(T_1^{(2)}\right)^* & \left(T_0^{(2)}\right)^* & \left(T_{-1}^{(2)}\right)^* & \left(T_{-2}^{(2)}\right)^* \end{pmatrix} \\ &= \begin{pmatrix} T_{-2}^{C\,(2)} & -T_{-1}^{C\,(2)} & T_0^{C(2)} & -T_1^{C(2)} & T_2^{C(2)} \end{pmatrix}. \end{aligned} \tag{B.13}$$

Appendix C. Properties of the γ matrices

The matrix P Eq. (142) can be presented as a scalar product of the matrices Π_x, Π_y, Π_z, with the vector $\hat{p}_x, \hat{p}_y, \hat{p}_z$,

$$P = \hat{\mathbf{p}} \cdot \mathbf{\Pi}, \tag{C.1}$$

where,

$$\Pi_x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{C.2}$$

$$\Pi_y = i \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{C.3}$$

$$\Pi_z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.4}$$

The $\gamma_x, \gamma_y, \gamma_z$,matrices are constructed according to,

$$\gamma_k = \frac{1}{2} \begin{pmatrix} \begin{pmatrix} -\tilde{S}_k^{(1)} \end{pmatrix} & \begin{pmatrix} \Pi_k \end{pmatrix} \\ \begin{pmatrix} \Pi_k^H \end{pmatrix} & \begin{pmatrix} \tilde{S}_k^{(2)} \end{pmatrix} \end{pmatrix}, \quad k = x, y, z \tag{C.5}$$

where $\tilde{S}_k^{(2)}$ are spin 2 matrices Eqs. (133), 135) and $\tilde{S}_k^{(1)}$ are spin 1 matrices Eqs. (118).

Appendix D. Spherical tensor formalism

We follow the definition of the scalar product for irreducible spherical tensors of rank J. Let $A_m \equiv \langle A|Jm)$, and $B_m \equiv \langle B|Jm)$, $-J \leq m \leq J$, be such tensors, the scalar product is defined[10] according to,

$$\langle A|B\rangle \equiv (A \cdot B) = \sum_{m=-J}^{J} \langle A|Jm\rangle\langle Jm|B\rangle \tag{D1}$$

$$= \sum_{m=-J}^{J} (-1)^m \langle A|Jm\rangle\langle B|-m, J\rangle = \sum_{m=-J}^{J} (-1)^m A_m B_{-m}, \tag{D2}$$

from which we infer that,

$$\sum (-1)^m \left|J, m\right.\!>\!< J, -m| = \sum (-1)^m \left|J, -m\right.\!>\!< J, m| = I^{(J)}, \tag{D.1}$$

is the identity operator in the subspace of given J. Let us denote,

$$(-1)^m \left|J, -m\right.\!> = \left|J, -\tilde{m}\right.\!>, \tag{D.2}$$

hence,

$$\sum \left|J, -\tilde{m}\right.\!>\!< J, m| = I^{(J)}. \tag{D.3}$$

Within this formalism,

$$< J, -\tilde{m}_2|(W^\mu + s\hat{p}^\mu I^{(s)})|\Psi > \tag{D.4}$$

$$= \sum_{m_1} < J, -\tilde{m}_2|(W^\mu + s\hat{p}^\mu I^{(s)})|J, -\tilde{m}_1 >< J, m_1|\Psi >= 0,$$

where $< J, m_1|\Psi >$ is a column vector. The matrix elements of the spin operators satisfy,

$$\tilde{S}_{m_2,m_1} =< J, -\tilde{m}_2|S_x|J, -\tilde{m}_1 >= - < J, m_2|S_x|J, m_1 >, \ i.e. \ \tilde{S}_x = -S_x. \tag{D.5}$$

In a similar way we obtain,

$$\tilde{S}_y = S_y, \quad and \quad \tilde{S}_z = -S_z. \tag{D.6}$$

For practical calculations we introduce,

$$S_+ = S_x + iS_y, \ S_- = S_x - iS_y, S_x = \frac{1}{2} \ (S_+ + S_-) \, , \ S_y = \frac{-i}{2} \ (S_+ - S_-) \, , \tag{D.7}$$

with their matrix elements given by,

$$\langle s, m+1| \, S_+ \left|sm'\right\rangle = \sqrt{(s-m)\,(s+m+1)}\delta_{mm'}, \tag{D.8}$$

$$\langle s, m-1| \, S_- \left|sm'\right\rangle = \sqrt{(s+m)\,(s-m+1)}\delta_{mm'}, \tag{D.9}$$

$$\langle sm| \, S_z \left|sm'\right\rangle = m\delta_{mm'}, \quad -s \leq m \leq s. \tag{D.10}$$

The momentum vector $\hat{\mathbf{p}}$ in this representation is,

$$< 1, m|\hat{\mathbf{p}} >= \begin{pmatrix} \hat{p}_1 \\ \hat{p}_0 \\ \hat{p}_{-1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}}\,(\hat{p}_x + i\hat{p}_y) \\ \hat{p}_z \\ \frac{1}{\sqrt{2}}\,(\hat{p}_x - i\hat{p}_y) \end{pmatrix}. \tag{D.11}$$

The inverse relation is,

$$\begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}}\hat{p}_1 + \frac{1}{\sqrt{2}}\hat{p}_{-1} \\ \frac{i}{\sqrt{2}}\hat{p}_1 + \frac{i}{\sqrt{2}}\hat{p}_{-1} \\ \hat{p}_0 \end{pmatrix}. \tag{D.12}$$

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