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Summed Series Involving $_1F_2$ Hypergeometric Functions

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Abstract: Summation of infinite series has played a significant role in a broad range of problems in the physical sciences and is of interest in a purely mathematical context. In a prior paper, we found that the Fourier–Legendre series of a Bessel function of the first kind $J_N(kx)$ and modified Bessel functions of the first kind $I_N(kx)$ lead to an infinite set of series involving $_1F_2$ hypergeometric functions (extracted therefrom) that could be summed, having values that are inverse powers of the eight primes $1/\left(2^i3^j5^k7^l11^m13^n17^o19^p\right)$ multiplying powers of the coefficient k, for the first 22 terms in each series. The present paper shows how to generate additional, doubly infinite summed series involving $_1F_2$ hypergeometric functions from Chebyshev polynomial expansions of Bessel functions, and trebly infinite sets of summed series involving $_1F_2$ hypergeometric functions from Gegenbauer polynomial expansions of Bessel functions. That the parameters in these new cases can be varied at will significantly expands the landscape of applications for which they could provide a solution.

Keywords: Bessel functions; Fourier–Legendre series; Laplace series; Chebyshev polynomial expansions; Gegenbauer polynomial expansions; computational methods; Jacobi expansions; hypergeometric series summation

MSC: 33C10; 42C10; 41A10; 41A50; 33F10; 65D20; 68W30; 33C45

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1. Introduction

The summing of infinite series has played a key part in a broad range of problems in the physical sciences, from self-energy diagrams [1–3] to polarization [4]. See [5] for an excellent review of the summation of divergent asymptotic expansions. In particular, Mera et al. [6,7] and Pedersen et al. [8] use hypergeometric functions to sum series in perturbation theory.

In the present paper, we focus on summing trebly infinite sets of series involving $_1F_2$ hypergeometric functions. Historical antecedents of similar work include Chaundy [9], who expressed products of $_0F_1$ functions as infinite sums of $_2F_1$ functions, products of $_1F_1$ functions and products of $_2F_0$ functions as infinite sums of $_3F_2$ functions, and products of $_2F_1$ functions as infinite sums of $_4F_3$ functions. Additional combinations are found in Burchnall and Chaundy [10], Henrici [11], Gasper [12], and Jain and Verma [13].

Slater expressed generalized Whittaker functions [14] ($_pF_p$ functions having $p \geq 1$) as sums of other generalized Whittaker functions, and a few other $_pF_q$ functions. Additional forms of the latter are found in [15–21].

Of course, the Appell functions of the first through fourth kinds are defined [22] as infinite sums over $_2F_1$ functions, while some Meijer G-functions [23] and certain Kampé de Fériet's functions may be expressed [24] as infinite sums over $_pF_q$ functions.

The final stop in this historical sketch is the expression of $_pF_q$ functions as finite sums of other $_pF_q$ functions [25] that act as recurrence relations [26].

In a prior paper [27], I refined Keating's [28] derivation of the coefficient set of the Fourier–Legendre series for the Bessel function $J_N(kx)$ to be

$$J_N(kx) = \sum_{L=0}^{\infty} a_{LN}(k) P_L(x) \tag{1}$$

where

$$\begin{aligned} a_{LN}(k) &= \sqrt{\pi}(2L+1)2^{-L-1}i^{L-\mathrm{N}}\sum_{M=0}^{\infty}\frac{\left(\left(-\frac{1}{4}\right)^M k^{L+2M}\right)}{2^{L+2M+1}\left(M!\Gamma\left(L+M+\frac{3}{2}\right)\right)} \\ &\times \left(1+(-1)^{L+2M+\mathrm{N}}\right)\binom{L+2M}{\frac{1}{2}(L+2M-\mathrm{N})} \\ &= \frac{\sqrt{\pi}2^{-2L-2}(2L+1)k^L i^{L-N}}{\Gamma\left(\frac{1}{2}(2L+3)\right)}\left(1+(-1)^{L+N}\right)\binom{L}{\frac{L-N}{2}} \\ &\times {}_2F_3\left(\frac{L}{2}+\frac{1}{2},\frac{L}{2}+1;L+\frac{3}{2},\frac{L}{2}-\frac{N}{2}+1,\frac{L}{2}+\frac{N}{2}+1;-\frac{k^2}{4}\right) \\ &= \sqrt{\pi}\,2^{-2L-2}(2L+1)k^L i^{L-N}\left(1+(-1)^{L+N}\right)\Gamma(L+1) \\ &\times {}_2\tilde{F}_3\left(\frac{L}{2}+\frac{1}{2},\frac{L}{2}+1;L+\frac{3}{2},\frac{L}{2}-\frac{N}{2}+1,\frac{L}{2}+\frac{N}{2}+1;-\frac{k^2}{4}\right) \end{aligned}, \quad (2)$$

of which the final two steps were new in the prior work [27]. I included the final form using regularized hypergeometric functions [29]

$${}_2F_3(a_1,a_2;b_1,b_2,b_3;z) = \Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\,{}_2\tilde{F}_3(a_1,a_2;b_1,b_2,b_3;z) \quad (3)$$

and canceled the $\Gamma(b_i)$ with gamma functions in the denominators of the prefactors. This cancellation allows one to avoid infinities that arise whenever $N > 1$ is an integer larger than L, and of the same parity, which would otherwise result in indeterminacies in a computation when one tries to use the conventional form of the hypergeometric function.

After a further review of the literature, I found that Keating's result (the first line above) and my prior work (the second line above) are implicitly subsumed within Jet Wimp's 1962 Jacobi expansion [30] of the Anger–Weber function (his equations (2.10) and (2.11)) since Legendre polynomials are a subset of Jacobi polynomials, and the Bessel function $J_N(kx)$ is a special case of the Anger–Weber function $\mathbf{J}_\nu(kx)$ when ν is an integer. Wimp does not mention the calculational difficulties that were resolved through the third form above.

For the special cases of $N = 0$, 1, the order of the hypergeometric functions is reduced since the parameters are $a_2 = b_3$ and $a_1 = b_2$, respectively, giving

$$\begin{aligned} a_{L0}(k) &= \frac{\sqrt{\pi}i^L 2^{-2L-2}(2L+1)k^L}{\Gamma\left(\frac{1}{2}(2L+3)\right)}\left(1+(-1)^L\right)\binom{L}{\frac{L}{2}} \\ &\times {}_1F_2\left(\frac{L}{2}+\frac{1}{2};\frac{L}{2}+1,L+\frac{3}{2};-\frac{k^2}{4}\right) \\ &= \sqrt{\pi}i^L 2^{-2L-2}(2L+1)k^L\Gamma\left(\frac{L}{2}+1\right)\left(1+(-1)^L\right)\binom{L}{\frac{L}{2}} \\ &\times {}_1\tilde{F}_2\left(\frac{L}{2}+\frac{1}{2};\frac{L}{2}+1,L+\frac{3}{2};-\frac{k^2}{4}\right) \end{aligned}, \quad (4)$$

and

$$\begin{aligned} a_{L1}(k) &= \frac{\sqrt{\pi}i^{L-1}2^{-2L-2}(2L+1)k^L}{\Gamma\left(\frac{1}{2}(2L+3)\right)}\left(1+(-1)^{L+1}\right)\binom{L}{\frac{L-1}{2}} \\ &\times {}_1F_2\left(\frac{L}{2}+1;\frac{L}{2}+\frac{3}{2},L+\frac{3}{2};-\frac{k^2}{4}\right) \\ &= i^{L-1}2^{-L-2}(2L+1)k^L\Gamma\left(\frac{L}{2}+1\right)\left(1+(-1)^{L+1}\right) \\ &\times {}_1\tilde{F}_2\left(\frac{L}{2}+1;\frac{L}{2}+\frac{3}{2},L+\frac{3}{2};-\frac{k^2}{4}\right) \end{aligned}. \quad (5)$$

In each special case, the first form involving a hypergeometric function has no numerical indeterminacies, but I include the regularized hypergeometric function version for completeness.

The first 22 terms in the Fourier–Legendre series for $J_0(kx)$ (1) are given in Appendix A, with $k = 1$, as is an updated polynomial approximation created by expanding the Legendre polynomials into their constituent terms and gathering like powers. Since each Legendre polynomial in (A1) contributes to the constant term in both (A2) and (A3), their sum is

$$
\begin{array}{rcr}
0.91973041008976023931442119408062\times & (1) & \\
-0.15794205862585188757371396714436\times & \frac{1}{2}(3x^2-1)_{x\to 0} & \\
+0.00343840094460110923299688787207292\times & \frac{1}{8}(35x^4-30x^2+3)_{x\to 0} & \\
-0.0000291972184882872969366059098612566\times & \frac{1}{16}(231x^6-315x^4+105x^2-5)_{x\to 0} & \\
+\cdots & = & 1
\end{array} \quad (6)
$$

rather than some other number close to 1. This may formalized in a theorem for these summed series:

Theorem 1. *For integer h and for any values of k,*

$$
\sum_{L=0}^{\infty} \frac{\sqrt{\pi} i^L 2^{-2L-2}(1+(-1)^L)(2L+1)\binom{L}{\frac{L}{2}}}{\Gamma\left(\frac{1}{2}(2L+3)\right)} {}_1F_2\left(\frac{L}{2}+\frac{1}{2};\frac{L}{2}+1,L+\frac{3}{2};-\frac{k^2}{4}\right)k^L
$$
$$
\times \left\{ \frac{i^L 2^{-L/2}(L-1)!!\left(\frac{L}{2}+\frac{1}{2}\right)_h\left(-\frac{L}{2}\right)_h}{h!\frac{L}{2}!\left(\frac{1}{2}\right)_h} \right\} = \frac{(-1)^h 2^{-2h}}{h!\Gamma(h+1)}k^{2h}, \quad (7)
$$

within which $h = 0$ gives (6).

A researcher seeking to sum a series like this is likely to have the various factors expressed in alternative ways. For instance, the expression $\left(1+(-1)^L\right)$ in the first factor of this equation restricts the sum to even values of L, which is sometimes indicated instead as $\sum_{L=0}^{\infty}{}^{(2)}\ldots$. This restriction also means that the double factorial in the next line can be alternatively expressed as $(L-1)!! = \frac{2^{L/2}\Gamma\left(\frac{L}{2}+\frac{1}{2}\right)}{\sqrt{\pi}}$. The binomial $\binom{L}{\frac{L}{2}}$ can alternatively be expressed as a ratio of gamma functions, $\frac{\Gamma(L+1)}{\Gamma\left(\frac{L}{2}+1\right)^2}$, as can the Pochhammer symbols $(a)_h = \frac{\Gamma(a+h)}{\Gamma(a)}$. In Equation (26) of the previous paper [27], the term in curly brackets was given as

$$
\left\{ \frac{2^{-L}\binom{2L}{L}\left(\frac{1}{2}-\frac{L}{2}\right)_{\frac{L}{2}-h}\left(-\frac{L}{2}\right)_{\frac{L}{2}-h}}{\left(\frac{L}{2}-h\right)!\left(\frac{1}{2}-L\right)_{\frac{L}{2}-h}} \right\} \quad (8)
$$

because it used an alternative conversion of Legendre polynomials into ${}_2F_1$ hypergeometric functions [31] (p. 1044 No. 8911.1) [32] (p. 468 No. 7.3.1.206).

This was proved in the prior work for general h by extracting specific powers of x from the Legendre polynomials, most easily by converting them into ${}_2F_1$ hypergeometric functions [31] (p. 1044 No. 8911.2) [32] (p. 466 No. 7.3.1.182) and thence into a finite sum over ratios of Pochhammer symbols. For $h = 1$, the $P_2(x)$ through $P_{42}(x)$ terms add to give $-1/4$, the coefficient of x^2 term in both (A2) and (A3) if $k = 1$.

Including $k \neq 1$ poses no problem in (7) despite its appearance as the argument of the ${}_1F_2\left(\frac{L}{2}+\frac{1}{2};\frac{L}{2}+1,L+\frac{3}{2};-\frac{k^2}{4}\right)$ function, as well as the existence of a k^L factor in the argument of the sum. It ends up contributing a very clean factor of k^{2h} to the right-hand side of (7). I, thus, summed an infinite set of infinite sums of ${}_1F_2$ hypergeometric functions, though I numerically verified only those with $0 \leq h \leq 42$ (I had to take the upper limit on the number of terms in the series $\geq h+74$ in order to obtain a percent difference between left- and right-hand sides that was $\leq 10^{-33}$, because the first h terms in the series do not contribute. For $h = 0$, an upper limit on the number of terms in the series $\geq h+44$ was sufficient).

I likewise summed an infinite set of infinite sums of ${}_1F_2$ hypergeometric functions derived from the Fourier–Legendre series for $J_1(kx)$ (1):

Theorem 2. *For integer h and for any values of k,*

$$\sum_{L=1}^{\infty} \frac{\sqrt{\pi} i^{L-1} \left(1+(-1)^{L+1}\right)(2L+1) 2^{-3L-2} \binom{L}{\frac{L-1}{2}} \binom{2L}{L} (-1)^{-h+\frac{L}{2}-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(-h+\frac{L}{2}-\frac{1}{2}\right)!} k^L$$
$$\times \quad {}_1F_2\left(\frac{L}{2}+1; \frac{L}{2}+\frac{3}{2}, L+\frac{3}{2}; -\frac{k^2}{4}\right) \left[\frac{\left(\frac{1}{2}-\frac{L}{2}\right)_{\frac{L-1}{2}-h} \left(-\frac{L}{2}\right)_{\frac{L-1}{2}-h}}{\left(\frac{1}{2}-L\right)_{\frac{L-1}{2}-h}}\right]$$
$$= \frac{(-1)^h 2^{-2h-1}}{h!\Gamma(h+2)} k^{2h+1} . \tag{9}$$

The question naturally arises as to whether one can derive such a summed infinite series based on other polynomial expansions. In the following, one may answer in the affirmative for both Chebyshev and Gegenbauer polynomial expansions of Bessel functions.

2. Summed Series Involving ${}_1F_2$ Hypergeometric Functions from Chebyshev Polynomial Expansions of Bessel Functions

We wish to prove the following theorem for the summed series derived from Chebyshev polynomial expansions of the $J_0(kx)$ Bessel function:

Theorem 3. *For integer h and for any values of k,*

$$\sum_{L=0}^{\infty} \frac{(-1)^L 2^{-2L} \left(\frac{1}{2}-L\right)_{L-h} (-L)_{L-h}}{L!\Gamma(L+1)(L-h)!(1-2L)_{L-h}} \, {}_1F_2\left(L+\frac{1}{2}; L+1, 2L+1; -\frac{k^2}{4}\right) k^{2L}$$
$$= \frac{(-1)^h 2^{-2h}}{h!\Gamma(h+1)} k^{2h} . \tag{10}$$

Proof of Theorem 3. Wimp also applied his Jacobi expansion [30] to find Chebyshev polynomial expansions of Bessel functions, since [31] (p. 1060 No. 8.962.3)

$$P_{2n}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(z) = \frac{\left(\frac{1}{2}\right)_{2n}}{(2n)!} T_{2n}(z) . \tag{11}$$

Unlike the section above, the following expansion (his Equations. (3.6) and (3.7)) applies to non-integer indices as well:

$$J_\nu(kx) = (kx)^\nu \sum_{L=0}^{\infty} C_{L\nu}(k) T_{2L}(x) . \quad -1 \le x \le 1 \tag{12}$$

Since what one is expanding in Chebyshev polynomials is the function $J_\nu(kx)(kx)^{-\nu}$, the coefficients can only be given by the orthogonality of the Chebyshev polynomials if we include the full function in the defining integral,

$$C_{L\nu}(k) = \frac{(2-\delta_{L0})}{2} \frac{2}{\pi} \int_{-1}^{1} \left(J_\nu(kx)(kx)^{-\nu}\right) \frac{1}{\sqrt{1-x^2}} T_{2L}(x) \, dx , \tag{13}$$

which Wimp finds to be

$$C_{L\nu}(k) = \frac{(-1)^L k^{2L} 2^{-4L-\nu} (2-\delta_{L0})}{L!\Gamma(L+\nu+1)} \, {}_1F_2\left(L+\tfrac{1}{2}; L+\nu+1, 2L+1; -\tfrac{k^2}{4}\right) . \tag{14}$$

The first 22 terms in the Chebyshev polynomial expansion of $J_0(kx)$ (12), with $k = 1$ and 8, are given in Appendix A.2.

Since the constant term of every Chebyshev polynomial has magnitude one, and alternating sign, the sum of these times the coefficients—a_{2r} in Clenshaw's convention in which sums having a single prime indicate that the term with suffix zero is to be halved—is simply

$$\sum{}'(-1)^L a_{2r} = 1\,. \tag{15}$$

Clenshaw has, thus, given the first of the summation rules we wish to derive.

The general-h proof follows that of the prior paper. In order to sum the infinite set of infinite sums of ${}_1F_2$ hypergeometric functions derived from the Fourier–Legendre series expansion of Bessel functions (7) and (9), we extracted specific powers of x by converting Legendre polynomials into ${}_2F_1$ hypergeometric functions [32] (p. 468 No. 7.3.1.206). The equivalent conversion for Chebyshev polynomials is

$$T_{2L}(x) = 2^{2L-1}x^{2L}\,{}_2F_1\left(\frac{1}{2}-L,-L;1-2L;\frac{1}{x^2}\right)[1+\delta_{L0}]\,. \tag{16}$$

Note that in the above, I have augmented [32] (p. 468 No. 7.3.1.207) and [33] with the factor $[1+\delta_{L0}]$ that allows the conversion to be extended downward from their restriction: $L > 0$. When multiplied by the equivalent factor in Wimp's Chebyshev expansion (14), one obtains $(2-\delta_{L0})[1+\delta_{L0}] \equiv 2$ for all L. This is a strong argument for using the "sums should simply be sums" convention over Clenshaw's for the present analytical work.

The final step in the proof is to convert each ${}_2F_1$ hypergeometric function into a finite sum over ratios of Pochhammer symbols times inverse powers of x. (Let us use m for the summation index). One finds that, of the finite sum in (16) for a given L, the only term that contributes a power x^{2h} is

$$2^{2L-1}x^{2L}\sum_{m=L-h}^{L-h}\frac{x^{-2m}\left(\frac{1}{2}-L\right)_m(-L)_m}{m!(1-2L)_m}[1+\delta_{L0}]\,, \tag{17}$$

which may be more compactly written as

$$2^{2L-1}x^{2L-2(L-h)}\frac{\left(\frac{1}{2}-L\right)_{L-h}(-L)_{L-h}}{(L-h)!(1-2L)_{L-h}}[1+\delta_{L0}]\,. \tag{18}$$

Noting that multiplying the factor $(2-\delta_{L0})$from (14) by the above $[1+\delta_{L0}]$ gives another factor of 2 for all L, which completes the proof of (10). □

To numerically verify the lowest 43 summed series for $k \to 8$, one has to take the upper limit on the number of terms in the series $\geq h+18$ in order to obtain a percent difference between left- and right-hand sides that is $\leq 10^{-33}$, because the first h terms in the series do not contribute. For $k \to 5$, this reduces somewhat to $\geq h+15$. For $h = 0$, one needs 24 terms and 20 terms, respectively.

The first 22 terms in the Chebyshev polynomial expansion of $J_1(kx)$ (12), with $k = 1$ and 8, are given in Appendix A.2. The consequent summed series associated with the power x^{2h+1} in the Chebyshev expansion (12) is given in the following theorem:

Theorem 4. *For integer h and for any values of k,*

$$\sum_{L=1}^{\infty}\frac{(-1)^L 2^{-2L-1}\left(\frac{1}{2}-L\right)_{L-h}(-L)_{L-h}}{L!\Gamma(L+2)(L-h)!(1-2L)_{L-h}}k^{2L+1}\,{}_1F_2\left(L+\frac{1}{2};L+2,2L+1;-\frac{k^2}{4}\right)$$
$$= \frac{(-1)^h 2^{-2h-1}}{h!\Gamma(h+2)}k^{2h+1}\,. \tag{19}$$

Proof of Theorem 4. What changes in the proof as we move from $\nu = 0 \to 1$ is contained in the three factors

$$\frac{2^{-4L-\nu}}{\Gamma(L+\nu+1)}\,{}_1F_2\left(L+\frac{1}{2};L+\nu+1,2L+1;-\frac{k^2}{4}\right)\bigg|_{\nu\to 1} \tag{20}$$

in the coefficient $C_{L\nu}(k)$ of the defining series (12), while nothing does in the Chebyshev polynomial that multiplies it. Thus, nothing changes in the transformations (16)–(18) except that we now associate (17) and (18) with a power x^{2h} multiplied by $(kx)^{\nu}$. Indeed, we have not only proved Theorem 4, but also its extension to a series associated with the power $x^{2h+\nu}$ in the general-ν case: □

Theorem 5. *For integer h and for any values of k and ν,*

$$\sum_{L=1}^{\infty}\frac{(-1)^L 2^{-2L-\nu}\left(\frac{1}{2}-L\right)_{L-h}(-L)_{L-h}}{L!(L-h)!(1-2L)_{L-h}\Gamma(L+\nu+1)}k^{2L+\nu}\,{}_1F_2\left(L+\frac{1}{2};2L+1,L+\nu+1;-\frac{k^2}{4}\right)$$
$$= \frac{(-1)^h 2^{-2h-1}}{h!\Gamma(h+2)}k^{2h+1}\,. \tag{21}$$

To verify the lowest 43 summed series for $k \to 8$, one generally has to take the upper limit on the number of terms in the series $\geq h+20$ in order to obtain a percent difference between left- and right-hand sides that is $\leq 10^{-33}$, because the first h terms in the series do not contribute. For $k \to 5$, this reduces somewhat to $\geq h+16$. For $h = 0$, one needs 23 terms and 20 terms, respectively.

For large indices, such as $\nu = 17$, for instance, with $h = 5$, $k = 8$, the two sides of (21) sides diverge after 45 digits: $-1.335586213327781269795862205505422996145960793 \times 10^{-7}$. For small values of ν, however, neither side gives an accuracy beyond the 13th post-decimal place in the computer algebra program Mathematica 7 despite a command to do so, giving -68.7857424612620 for $h = 5$, $k = 8$, and $\nu = 0.17$. Complex values of ν likewise gave only 14 decimal placers in Mathematica, such as $1.15092097688009 + 1.83846320788943i$ for $h = 5$, $k = 8$, and $\nu = 17 + 30.3i$. Mathematica 13 likewise gives this more limited, but still excellent, accuracy.

In the prior paper [27], we noted that because the modified Bessel functions of the first kind $I_N(kx)$ are related to the ordinary Bessel functions by the relation [31] (p. 961 No. 8.406.3),

$$I_n(z) = i^{-n}J_n(iz)\,, \tag{22}$$

one merely needs to multiply by i^{-n} and set $k = i$ in (2) to obtain the $I_0(x)$ Fourier–Legendre series. Furthermore, one sees that I_0 expressed in powers of x is simply the J_0 version with all of the negative signs reversed. This is not true of (1) because the arguments of the Legendre polynomials do not undergo $x \to ix$ since they derive from the definition of the Fourier–Legendre series (1). The k-dependence is entirely within the coefficients $a_{LN}(k)$.

Therefore, the I_0 Legendre series expansion leads to no new set of summed series since these would simply be (7) with $k = i\kappa$. This is also the case for a Chebyshev expansion. Clenshaw [34] confirms this for $h = 0$ on pp. 34–35.

3. Summed Series Involving ${}_1F_2$ Hypergeometric Functions from Gegenbauer Polynomial Expansions of Bessel Functions

We wish to prove the following theorem for summed series derived from Gegenbauer polynomial expansions of the $J_0(kx)$ Bessel function:

Theorem 6. *For integer h and for any values of k and λ,*

$$\sum_{L=0}^{\infty} \frac{(-2)^{2L}(-L)_h\left(\lambda+\frac{1}{2}\right)_{2L}(L+\lambda)_h}{\sqrt{\pi}h!\left(\frac{1}{2}\right)_h\left(L+\frac{1}{2}\right)_{\frac{1}{2}}(L+\lambda)(2\lambda)_{2L}(2L+2\lambda)_{2L}B(\lambda,L+1)} k^{2L}\,{}_1F_2\left(L+\frac{1}{2};L+1,2L+\lambda+1;-\frac{k^2}{4}\right)$$
$$= \frac{(-1)^h 2^{-2h} k^{2h}}{h!\Gamma(h+1)} \quad . \tag{23}$$

Proof of Theorem 6. Although he does not explicitly do so, one may use Wimp's Jacobi expansion [30] to find Gegenbauer polynomial expansions of Bessel functions, since [31] (p. 1061 No. 8.962.4)

$$P_{2n}^{\left(\lambda-\frac{1}{2},\lambda-\frac{1}{2}\right)}(z) = \frac{\left(\lambda+\frac{1}{2}\right)_{2n} C_{2n}^{\lambda}(z)}{(2\lambda)_{2n}} \,. \tag{24}$$

Like those in the second section, the following expansion applies to both integer and non-integer indices:

$$J_\nu(kx) = (kx)^\nu \sum_{L=0}^{\infty} b_{L\nu}(k) C_{2L}^{\lambda}(x)\,, \quad -1 \le x \le 1 \tag{25}$$

where the coefficients are given by the orthogonality of the Chebyshev polynomials,

$$b_{L\nu}(k) = \frac{2^{2\lambda-1}(2L)!(2L+\lambda)\Gamma(\lambda)^2}{\pi\Gamma(2L+2\lambda)} \int_{-1}^{1} \left(J_\nu(kx)(kx)^{-\nu}\right)\left(1-x^2\right)^{-\frac{1}{2}+\lambda} C_{2L}^{\lambda}(x)\,dx\,, \tag{26}$$

as

$$b_{L\nu}(k) = \frac{(-1)^L a^{2L} 2^{2L-\nu}\left(\lambda+\frac{1}{2}\right)_{2L}}{\sqrt{\pi}(2\lambda)_{2L}(2L+2\lambda)_{2L}\left(L+\frac{1}{2}\right)_{\nu+\frac{1}{2}}} \quad {}_1F_2\left(L+\tfrac{1}{2};2L+\lambda+1,L+\nu+1;-\tfrac{a^2}{4}\right) \quad . \tag{27}$$

The first 22 terms in the Gegenbauer polynomial expansion of $J_0(kx)$ (25) are given in Appendix A.3, with $k=1$ and arbitrarily taking $\lambda=\frac{1}{4}$. One could, of course, test the technique using any value of λ, but since $T_\nu(z)=\frac{1}{2}\nu C_\nu^0(z)$ and $P_\nu(z)=C_\nu^{\frac{1}{2}}(z)$, the choice $\lambda=\frac{1}{4}$ seemed like the next most interesting value.

To extract the powers, we use the conversion for Gegenbauer polynomials that is equivalent to (16), which is [31] (GR5 p. 1051 No. 8.932.2)

$$C_{2L}^{\lambda}(x) = \frac{(-1)^L}{(L+\lambda)B(\lambda,L+1)}\,{}_2F_1\left(-L,L+\lambda;\frac{1}{2};x^2\right). \tag{28}$$

The final step in the proof is to convert each ${}_2F_1$ hypergeometric function into a finite sum over ratios of Pochhammer symbols times powers of x. (Let us use m for the summation index). We find that, of the finite sum in (28) for a given L, the only term that contributes a power x^{2h} is

$$\frac{(-1)^L}{(L+\lambda)B(\lambda,L+1)} \sum_{m=h}^{h} \frac{x^{2m}(-L)_m(L+\lambda)_m}{m!\left(\frac{1}{2}\right)_m}\,, \tag{29}$$

which may be more compactly written as

$$\frac{(-1)^L}{(L+\lambda)B(\lambda,L+1)} \frac{x^{2h}(-L)_h(L+\lambda)_h}{h!\left(\frac{1}{2}\right)_h}\,, \tag{30}$$

which completes the proof of (23). □

To verify the lowest 43 summed series for $k \to 8$, one generally has to take the upper limit on the number of terms in the series $\geq h + 20$ in order to obtain a percent difference between left- and right-hand sides that is $\leq 10^{-33}$, because the first h terms in the series do not contribute. For $k \to 5$, this reduces somewhat to $\geq h + 16$. For $h = 0$, one needs 23 terms and 20 terms, respectively.

This theorem has an identical right-hand side as for the Legendre (7) and Chebyshev (10) versions, and it holds for every value of λ. That is, we have just summed an infinite set of infinite sets of infinite series involving ${}_1F_2$ hypergeometric functions. To see how this plays out in practice, consider two extreme values, $\lambda = 2^{\pm 20}$. For $h = 1$ (associated with x^2) and $\lambda = 2^{-20}$, the first eight terms sum as

$$\begin{gathered} 0 \\ -0.23477602708172067919886133823697 8 \\ -0.014985695386016861195100449470218 2 \\ -0.000236617512932378657466894715711978 \\ -1.6535169502942828583471873729083 06 \times 10^{-6} \\ -6.4860878109272636032198430867196 85 \times 10^{-9} \\ -1.6263640871589308166157648744469 7 \times 10^{-11} \\ -2.8298673436017316204620773637913 3 \times 10^{-14} - \cdots \\ = -0.24999999999999996 \end{gathered} . \tag{31}$$

For $\lambda = 2^{20}$, the second term is almost sufficient by itself:

$$\begin{gathered} 0 \\ -0.24999995529664875664569456808574 6 \\ -4.4703346802500940786844771090904 18 \times 10^{-8} \\ -4.4408530558363249112241707668393 2 \times 10^{-14} \\ -3.0880872800619174625208926928076 0 \times 10^{-22} \\ -1.6565601438450955179762694653062 1 \times 10^{-29} \\ -7.2407328541556294150052694082045 3 \times 10^{-37} \\ -2.6716479542266127597304068054887 0 \times 10^{-44} - \cdots \\ = 0.25000000000000000000000000000000 \end{gathered} . \tag{32}$$

The final theorem we wish to prove is for series associated with the power $x^{2h+\nu}$ derived from the Gegenbauer polynomial expansion of $J_\nu(kx)$, the general ν case, which may be written as follows:

Theorem 7. *For integer h and for any values of k, λ, and ν,*

$$\begin{aligned} \sum_{L=0}^{\infty} & \frac{(-1)^{2L} 2^{2L-\nu} (-L)_h \left(\lambda + \frac{1}{2}\right)_{2L} (L+\lambda)_h}{\sqrt{\pi} h! \left(\frac{1}{2}\right)_h (L+\lambda)(2\lambda)_{2L} (2L+2\lambda)_{2L} \left(L + \frac{1}{2}\right)_{\nu + \frac{1}{2}} B(\lambda, L+1)} k^{2L+\nu} \\ \times & \ {}_1F_2\left(L + \frac{1}{2}; 2L + \lambda + 1, L + \nu + 1; -\frac{k^2}{4}\right) \\ = & \frac{(-1)^h 2^{-2h-\nu} k^{2h+\nu}}{h! \Gamma(h+\nu+1)} . \end{aligned} \tag{33}$$

Proof of Theorem 7. One may build the general case by examining what must be done to sum the series associated with the power x^{2h+1} derived from the Gegenbauer polynomial expansion of $J_1(kx)$, whose 22-term Gegenbauer polynomial expansion we display in Appendix A.3, with $k = 1$ and again arbitrarily taking $\lambda = \frac{1}{4}$.

What changes in the proof as we move from $\nu = 0 \to 1$ is contained in the three factors

$$\frac{2^{2L-\nu}}{\left(L+\frac{1}{2}\right)_{\nu+\frac{1}{2}}}\ {}_1F_2\left(L+\frac{1}{2};2L+\lambda+1,L+\nu+1;-\frac{k^2}{4}\right)\Bigg|_{\nu\to 1} \tag{34}$$

in the coefficient $b_{L\nu}(k)$ of the defining series (25), while nothing does in the Gegenbauer polynomial that multiplies it. Thus, nothing changes in the transformations (28)–(30) except that we now associate (29) and (30) with a power x^{2h} multiplied by $(kx)^{\nu}$. This completes the proof of the summed series associated with the power $x^{2h+\nu}$ in the general-ν case. □

To verify the lowest 43 summed series for $k \to 8$, one generally has to take the upper limit on the number of terms in the series $\geq h + 20$ in order to obtain a percent difference between left- and right-hand sides that is $\leq 10^{-33}$, because the first h terms in the series do not contribute. For $k \to 5$, this reduces somewhat to $\geq h + 16$. For $h = 0$, one needs 23 terms and 20 terms, respectively.

The summed series derived from the Gegenbauer polynomial expansions of $J_\nu(x)$ may be found for any value of ν, not just integer values, given that it is derived from Wimp's Jacobi expansion [30]. Thus, we have just summed an infinite set of infinite sets of doubly infinite series involving ${}_1F_2$ hypergeometric functions since the expression holds for every value of λ and holds for every value of ν.

Since $T_\nu(z) = \frac{1}{2}\nu C_\nu^0(z)$, by setting $\lambda = 0$ in (33) one may obtain a modest variation on the form given in (21), since we here use a ${}_2F_1$ hypergeometric function whose argument is x^2 in (33) and used a ${}_2F_1$ hypergeometric function whose argument is x^{-2} to prove (21).

An extension of the Legendre sets (7) and (9) to larger integer values of ν is not obvious, but one can obtain such a form directly from (33) since $P_\nu(z) = C_\nu^{\frac{1}{2}}(z)$, which applies even for non-integer values of ν.

4. Conclusions

I have shown how to sum doubly infinite sets of infinite series involving ${}_1F_2$ hypergeometric functions, derived from Chebyshev polynomial expansions of Bessel functions of the first kind $J_\nu(kx)$, and the trebly infinite sets of infinite series involving ${}_1F_2$ hypergeometric functions from the Gegenbauer polynomial expansions of $J_\nu(kx)$. The utility of any one of these summed series for future researchers is, of course, not guaranteed, but given the relative paucity of infinite series whose values are known (e.g., 24 pages in Gradshteyn and Ryzhik compared to their 900 pages of known integrals), one hopes that adding such multiply-infinite sets of infinite series of ${}_1F_2$ functions whose values are now known will be of use to some.

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Appendix A

In this Appendix, we display series expansions that could be used to provide computer programs with 33-digit accuracy, matching the IEEE extended precision in some compilers. These high-accuracy results are given here because they could needlessly distract the reader from the main point of the paper: summing infinite series. They are here to satisfy the interest of readers wishing to verify some detail in the main body of the paper and also because of their intrinsic interest.

Because of this focus, we will note only a few papers of interest in the vast field of computational research. Khajah and Ortiz [35] provide tables of somewhat higher accuracy (10^{-89}) for some elementary functions like the exponential function. Vujevic and Modrić [36] give highly accurate algorithms for the computation of complex-valued Bessel functions, as do Georgieva and Hofreither [37]. Yuste and Abad [38] give an iterative method to obtain approximations to Bessel functions $J_p(x)$ while Karatsuba [39] focuses on both increased accuracy and large arguments. Kislenkov [40] gives grid-oriented algorithms for computing modified Bessel functions, as does Takekawa [41] for parallel processing. For other sorts of functions, one finds a thorough review of recent work in approximation theory in Rao, Ayman-Mursaleen, and Aslan [42].

Appendix A.1. Legendre Series for $J_0(kx)$

The first 22 terms in the Fourier–Legendre series for $J_0(kx)$ (1), with $k = 1$, were given in a prior paper [27]

$$\begin{aligned}
J_0(x) \cong\ & 0.91973041008976023931442119408062000P_0(x) \\
- & 0.15794205862585188757371396714436370P_2(x) \\
+ & 0.0034384009446011092329968878720729150P_4(x) \\
- & 0.000029197218488287296936605909861256630P_6(x) \\
+ & 1.3173569524477809776556165631432800 \times 10^{-7}\ P_8(x) \\
- & 3.6845008442082030271737710960588660 \times 10^{-10}\ P_{10}(x) \\
+ & 7.0118300329938459282088033282114570 \times 10^{-13}\ P_{12}(x) \\
- & 9.6659643698589122636719953727533460 \times 10^{-16}\ P_{14}(x) \\
+ & 1.0096362768245464465253421709249360 \times 10^{-18}\ P_{16}(x) \\
- & 8.2666569559276378589919725841741170 \times 10^{-22}\ P_{18}(x) \\
+ & 5.4482448677627587258900828378394300 \times 10^{-25}\ P_{20}(x) \\
- & 2.9525271821373547516757746066634000 \times 10^{-28}\ P_{22}(x) \\
+ & 1.3388561588585344690808986700962000 \times 10^{-31}\ P_{24}(x) \\
- & 5.1549131860885129261932348378165820 \times 10^{-35}\ P_{26}(x) \\
+ & 1.7062315770385034501385640284676340 \times 10^{-38}\ P_{28}(x) \\
- & 4.9068935564277968574730979795682890 \times 10^{-42}\ P_{30}(x) \\
+ & 1.2374892007174793830205395762212930 \times 10^{-45}\ P_{32}(x) \\
- & 2.7590562375378718686045556885483640 \times 10^{-49}\ P_{34}(x) \\
+ & 5.4773822071727126291997146483964090 \times 10^{-53}\ P_{36}(x) \\
- & 9.7442003455788525506889460570506740 \times 10^{-57}\ P_{38}(x) \\
+ & 1.5622807116595044898280251489957700 \times 10^{-60}\ P_{40}(x) \\
- & 2.2690562838273943688360574705945990 \times 10^{-64}\ P_{42}(x)\ .
\end{aligned} \tag{A1}$$

In Equation (15) of the prior work [27], the last line above mistakenly had the wrong power, $-2.2690562838273943688360574705945990 \times 10^{-60}\ P_{42}(x)$, though the Fortran code in the appendix was correct: -2.2690562838273943688360574705945990 e-64 P(42 , x). Since all calculations in the prior work used the correct power—with the error only appearing after the editor requested a formatting change in the equation—this correction had no impact on any results in that paper.

That paper also expanded the Legendre polynomials into their constituent terms and gathered like powers in (A1) to give an updated polynomial approximation,

$$
\begin{aligned}
J_0(x) \cong\; & 1.000000000000000000000000000000000000\, x^0 \\
-\; & 0.250000000000000000000000000000000000\, x^2 \\
+\; & 0.015625000000000000000000000000000000\, x^4 \\
-\; & 0.000434027777777777777777777777777778\, x^6 \\
+\; & 6.781684027777777777777777777777778 \times 10^{-6}\; x^8 \\
-\; & 6.781684027777777777777777777777778 \times 10^{-8}\; x^{10} \\
+\; & 4.709502797067901234567901234567901235 \times 10^{-10}\; x^{12} \\
-\; & 2.402807549524439405391786344167296548 \times 10^{-12}\; x^{14} \\
+\; & 9.385966990329841427311665406903502142 \times 10^{-15}\; x^{16} \\
-\; & 2.896903392077111551639402903365278439 \times 10^{-17}\; x^{18} \\
+\; & 7.242258480192778879098507258413196097 \times 10^{-20}\; x^{20} \\
-\; & 1.496334396734045222954237036862230599 \times 10^{-22}\; x^{22} \\
+\; & 2.597802772107717400962217077885817011 \times 10^{-25}\; x^{24} \\
-\; & 3.842903509035084912666001594505646466 \times 10^{-28}\; x^{26} \\
+\; & 4.901662639075363409012757135849038860 \times 10^{-31}\; x^{28} \\
-\; & 5.446291821194848232223639681761004317 8 \times 10^{-34}\; x^{30} \\
+\; & 5.318644356635593976793356267197307791 \times 10^{-37}\; x^{32} \\
-\; & 4.600903422695150498956190542558224733 \times 10^{-40}\; x^{34} \\
+\; & 3.550079801462307483762492702591222788 \times 10^{-43}\; x^{36} \\
-\; & 2.458504017633176927813360597362342651 \times 10^{-46}\; x^{38} \\
+\; & 1.5365650110207355798833503733514641567 \times 10^{-49}\; x^{40} \\
-\; & 8.7106860035189091830121903251216788929 \times 10^{-53}\; x^{42} \\
\cong\; & 1 - \frac{x^2}{2^2} + \frac{x^4}{2^6} - \frac{x^6}{2^8 3^2} + \frac{x^8}{2^{14} 3^2} - \frac{x^{10}}{2^{16} 3^2 5^2} + \frac{x^{12}}{2^{20} 3^4 5^2} - \frac{x^{14}}{2^{22} 3^4 5^2 7^2} + \frac{x^{16}}{2^{30} 3^4 5^2 7^2} \\
-\; & \frac{x^{18}}{2^{32} 3^8 5^2 7^2} + \frac{x^{20}}{2^{36} 3^8 5^4 7^2} - \frac{x^{22}}{2^{38} 3^8 5^4 7^2 11^2} + \frac{x^{24}}{2^{44} 3^{10} 5^4 7^2 11^2} - \frac{x^{26}}{2^{46} 3^{10} 5^4 7^2 11^2 13^2} \\
+\; & \frac{x^{28}}{2^{50} 3^{10} 5^4 7^4 11^2 13^2} - \frac{x^{30}}{2^{52} 3^{12} 5^6 7^4 11^2 13^2} + \frac{x^{32}}{2^{62} 3^{12} 5^6 7^4 11^2 13^2} \\
-\; & \frac{x^{34}}{2^{64} 3^{12} 5^6 7^4 11^2 13^2 17^2} + \frac{x^{36}}{2^{68} 3^{16} 5^6 7^4 11^2 13^2 17^2} - \frac{x^{38}}{2^{70} 3^{16} 5^6 7^4 11^2 13^2 17^2 19^2} \\
+\; & \frac{x^{40}}{2^{76} 3^{16} 5^8 7^4 11^2 13^2 17^2 19^2} - \frac{x^{42}}{2^{78} 3^{18} 5^8 7^6 11^2 13^2 17^2 19^2}\,. \qquad \text{(A2)}
\end{aligned}
$$

The latter form is simply an inverse prime version of the first 22 terms of the well-known series representation [31] (p. 970 No. 8.440)

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+\nu}}{k!\Gamma(k+\nu+1)}. \tag{A3}$$

(There is a fascinating analog to this result arising from studies of the Bessel difference equation [43]).

Appendix A.2. Chebyshev Series for $J_0(kx)$ and $J_1(kx)$

The first 22 terms in the Chebyshev polynomial expansions of $J_0(kx)$ (12), with $k = 1$, are

$$\begin{aligned}
J_0(x) \cong\ & 0.88072557910260852856667169074495 94\, T_0(x) \\
-\ & 0.11738801116832431940624546392555 72\, T_2(x) \\
+\ & 0.00187321250371919483787087820392 9524\, T_4(x) \\
-\ & 0.00001314542297029262107182993119503582\, T_6(x) \\
+\ & 5.1672429668014370531710323599516 00 \times 10^{-8}\ T_8(x) \\
-\ & 1.2972182348547039630939753347598 65 \times 10^{-10}\ T_{10}(x) \\
+\ & 2.2588402346070019303202272439840 34 \times 10^{-13}\ T_{12}(x) \\
-\ & 2.8876213527680577644640584815978 16 \times 10^{-16}\ T_{14}(x) \\
+\ & 2.8248482562513800236212335360512 11 \times 10^{-19}\ T_{16}(x) \\
-\ & 2.1826990613090885138257260482900 21 \times 10^{-22}\ T_{18}(x) \\
+\ & 1.3657391838233660788193782973172 02 \times 10^{-25}\ T_{20}(x) \\
-\ & 7.0611257016995201808960516613482 97 \times 10^{-29}\ T_{22}(x) \\
+\ & 3.0671827272481380517401884837036 13 \times 10^{-32}\ T_{24}(x) \\
-\ & 1.1350928337149875004149669325259 64 \times 10^{-35}\ T_{26}(x) \\
+\ & 3.6217122517694898732484770933279 96 \times 10^{-39}\ T_{28}(x) \\
-\ & 1.0065554809142169137051345245121 48 \times 10^{-42}\ T_{30}(x) \\
+\ & 2.4585407871851352079070011229522 13 \times 10^{-46}\ T_{32}(x) \\
-\ & 5.3190864717767324194234250794886 87 \times 10^{-50}\ T_{34}(x) \\
+\ & 1.0264335330661426499433391909164 24 \times 10^{-53}\ T_{36}(x) \\
-\ & 1.7776511587214069163875858520769 82 \times 10^{-57}\ T_{38}(x) \\
+\ & 2.7784068926670943521736430130962 89 \times 10^{-61}\ T_{40}(x) \\
-\ & 3.9387172216790096541810927471029 98 \times 10^{-65}\ T_{42}(x) \quad -1 \le x \le 1\,.
\end{aligned} \tag{A4}$$

At the upper limit of applicability, $x = 1$, this gives 33-digit accuracy, $J_0(1) = 0.765197686557966551449717526102663$ (Even at $x = 8$, this gives a result accurate to 14 digits, $J_0(8) = 0.171650807137554$).

If one follows Clenshaw's [34] (p. 30) lead and instead takes $k = 8$, one obtains

$$
\begin{aligned}
J_0(x) \cong\ & \mathbf{0.3154559429497802391 27}5502330199159/2\, T_0(x/8) \\
-\ & \mathbf{0.00872344235285222129}0793322469895429\, T_2(x/8) \\
+\ & \mathbf{0.26517861320333680986 7}0778235911043\, T_4(x/8) \\
-\ & \mathbf{0.37009499387264977903}34193036836753\, T_6(x/8) \\
+\ & \mathbf{0.15806710233209726127}77155496720475\, T_8(x/8) \\
-\ & \mathbf{0.03489376941140888516}317328987958171\, T_{10}(x/8) \\
+\ & \mathbf{0.00481918006946760449}6778380314312767\, T_{12}(x/8) \\
-\ & \mathbf{0.00046062616620627504 7}5036418408154116\, T_{14}(x/8) \\
+\ & \mathbf{0.00003246032882100508}080625560924485746\, T_{16}(x/8) \\
-\ & \mathbf{1.76194690776215}07494597659664 07618 \times 10^{-6}\, T_{18}(x/8) \\
+\ & \mathbf{7.60816359241 87}818669737862 30699492 \times 10^{-8}\, T_{20}(x/8) \\
-\ & \mathbf{2.67925353055 57}672898335371633826306 \times 10^{-9}\, T_{22}(x/8) \\
+\ & \mathbf{7.84869631 4}4794644165295039051 01749 \times 10^{-11}\, T_{24}(x/8) \\
-\ & \mathbf{1.94383468 6}737016570620688424557753 \times 10^{-12}\, T_{26}(x/8) \\
+\ & \mathbf{4.1253205}9563437393261261841275 7652 \times 10^{-14}\, T_{28}(x/8) \\
-\ & \mathbf{7.5885}0812544754633761986081932 9317 \times 10^{-16}\, T_{30}(x/8) \\
+\ & \mathbf{1.2218}5158739614110344186197720 1729 \times 10^{-17}\, T_{32}(x/8) \\
-\ & \mathbf{1.7}367896077002367682937302427 13393 \times 10^{-19}\, T_{34}(x/8) \\
+\ & 2.1957932033195603536794938976 98779 \times 10^{-21}\, T_{36}(x/8) \\
-\ & 2.4855664193642922665379471752 58836 \times 10^{-23}\, T_{38}(x/8) \\
+\ & 2.5340246068189726911025857690 70259 \times 10^{-25}\, T_{40}(x/8) \\
-\ & 2.3390856270557447067120230520 59754 \times 10^{-28}\, T_{42}(x/8) \quad -8 \le x \le 8\,.
\end{aligned}
\tag{A5}
$$

where the bolding indicates the digits he displays (I have included an extra digit in some places to allow for appropriate rounding to his displayed digit). Clenshaw follows the usual convention (noted on his p. 7) for sums having a single prime to indicate that the term with suffix zero is to be halved (and if the prime is doubled, the highest term in the sum is also halved), as indicated in the first line of (A5). This factor-of-two difference arises from the normalization of the orthogonality relation for Chebyshev polynomials [31] (p. 1057 No. 8.949.9):

$$
\int_{-1}^{1} T_n(x) T_m(x) \frac{1}{\sqrt{1-x^2}} = \begin{cases} 0 & m \neq n \\ \pi/2 & m = n \neq 0 \\ \pi & m = n = 0 \end{cases} . \tag{A6}
$$

Since I am comparing Chebyshev expansions with both Legendre and Gegenbauer expansions, whose orthogonality relations [31] (p. 1043 No. 8.904 and p. 1054 No. 8.939.8, respectively) have no such third branch, all derivations are made much more straightforward if one adopts the perhaps iconoclastic notion that sums should simply be sums and displays the first line of (A5) as 0.15772797147489011956377511650995580 $T_0(x/8)$. Tumakov [44] also follows this convention.

At the upper limit of applicability, $x = 8$, (A5) gives 27-digit accuracy, $J_0(8) = 0.171650807137553906090869408$.

The first 22 terms in the $J_1(x)$ Chebyshev expansion (12) with $k = 1$ are

$$
\begin{aligned}
J_1(x) \cong\ & 0.46970979234338534413489721135386690 x T_0(x) \\
-\ & 0.02997305358809894507094444118401190 x T_2(x) \\
+\ & 0.00031549534017613301983071130328043 28 x T_4(x) \\
-\ & 1.65352859182766501038992113950921 1 \times 10^{-6}\ x T_6(x) \\
+\ & 5.18888911011410679295459957305875 0 \times 10^{-9}\ x T_8(x) \\
-\ & 1.08424512051533751907843246994385 7 \times 10^{-11}\ x T_{10}(x) \\
+\ & 1.61706952909405786982340192877847 6 \times 10^{-14}\ x T_{12}(x) \\
-\ & 1.80790397659252472339283152019513 1 \times 10^{-17}\ x T_{14}(x) \\
+\ & 1.57154394552172352917908369881577 1 \times 10^{-20}\ x T_{16}(x) \\
-\ & 1.09259164150827524205712235555384 0 \times 10^{-23}\ x T_{18}(x) \\
+\ & 6.21379179799224560944046955745357 5 \times 10^{-27}\ x T_{20}(x) \\
-\ & 2.94449582379001619717700078224763 4 \times 10^{-30}\ x T_{22}(x) \\
+\ & 1.18049666785025194409546707378197 9 \times 10^{-33}\ x T_{24}(x) \\
-\ & 4.05631803667506419837892165418943 9 \times 10^{-37}\ x T_{26}(x) \\
+\ & 1.20786664943663901463954976056210 2 \times 10^{-40}\ x T_{28}(x) \\
-\ & 3.14693235540340627309662083499269 9 \times 10^{-44}\ x T_{30}(x) \\
+\ & 7.23395787181933883311475244068191 1 \times 10^{-48}\ x P_{32}(x) \\
-\ & 1.47806433206975659397613866152380 9 \times 10^{-51}\ x T_{34}(x) \\
+\ & 2.70202982742698894332577295914228 5 \times 10^{-55}\ x T_{36}(x) \\
-\ & 4.44545111780577302266041590103220 0 \times 10^{-59}\ x T_{38}(x) \\
+\ & 6.61704504304166424639852722600757 8 \times 10^{-63}\ x T_{40}(x) \\
-\ & 8.95384220591825870800781380459216 9 \times 10^{-67}\ x T_{42}(x) \quad -1 \le x \le 1\,. \qquad \text{(A7)}
\end{aligned}
$$

At the upper limit of applicability, $x = 1$, this gives 33-digit accuracy, $J_1(1) = 0.440050585744933515959682203718915$ (Even at $x = 8$, this gives a result accurate to 16 digits, $J_1(8) = 0.2346363468539146$).

If one follows Clenshaw's [34] (p. 31) lead and instead takes $k = 8$, one obtains

$$
\begin{aligned}
J_1(x) \quad \cong \quad & \mathbf{1.2967175412105298416 7}3374221959245/2\frac{x}{8}T_0(\frac{x}{8}) \\
- \quad & \mathbf{1.19180116054121687250 7}032741866674\frac{x}{8}T_2(\frac{x}{8}) \\
+ \quad & \mathbf{1.28799409885767762038}2580899489350\frac{x}{8}T_4(\frac{x}{8}) \\
- \quad & \mathbf{0.66144393413454325277}28770946844658\frac{x}{8}T_6(\frac{x}{8}) \\
+ \quad & \mathbf{0.17770911723972828328}23229884383241\frac{x}{8}T(\frac{x}{8}) \\
- \quad & \mathbf{0.02917552480615420766}201489599627591\frac{x}{8}T_{10}(\frac{x}{8}) \\
+ \quad & \mathbf{0.00324027018268385746}6456539040415511\frac{x}{8}T_{12}(\frac{x}{8}) \\
- \quad & \mathbf{0.00026044438934858068}13446141103993105\frac{x}{8}T_{14}(\frac{x}{8}) \\
+ \quad & \mathbf{0.00001588701923993213}393104615470762 96\frac{x}{8}T_{16}(\frac{x}{8}) \\
- \quad & \mathbf{7.61758780540 03}48945692364404508548 \times 10^{-7}\ \frac{x}{8}T_{18}(\frac{x}{8}) \\
+ \quad & \mathbf{2.94970700727 77}1859082610099611219 0 \times 10^{-8}\ \frac{x}{8}T_{20}(\frac{x}{8}) \\
- \quad & \mathbf{9.424212981 56}70787185781738090560 09 \times 10^{-10}\ \frac{x}{8}T_{22}(\frac{x}{8}) \\
+ \quad & \mathbf{2.52812366 4}2784026571921989032537 96 \times 10^{-11}\ \frac{x}{8}T_{24}(\frac{x}{8}) \\
- \quad & \mathbf{5.7774041 9}1721418742769122933910453 \times 10^{-13}\ \frac{x}{8}T_{26}(\frac{x}{8}) \\
+ \quad & \mathbf{1.138571 5}2028111538530395132871782 4 \times 10^{-14}\ \frac{x}{8}T_{28}(\frac{x}{8}) \\
- \quad & \mathbf{1.95535}783329523711145715604973983 4 \times 10^{-16}\ \frac{x}{8}T_{30}(\frac{x}{8}) \\
+ \quad & \mathbf{2.95}301463983434660972205818426254 5 \times 10^{-18}\ \frac{x}{8}P_{32}(\frac{x}{8}) \\
- \quad & \mathbf{3.9}52934614113459501768862170679755 \times 10^{-20}\ \frac{x}{8}T_{34}(\frac{x}{8}) \\
+ \quad & 4.723067439441036227167716497766825 \times 10^{-22}\ \frac{x}{8}T_{36}(\frac{x}{8}) \\
- \quad & 5.068481382508651457731548219527637 \times 10^{-24}\ \frac{x}{8}T_{38}(\frac{x}{8}) \\
+ \quad & 4.912426488809207456168647750374833 \times 10^{-26}\ \frac{x}{8}T_{40}(\frac{x}{8}) \\
- \quad & 4.321688707060755263766813871186111 \times 10^{-28}\ \frac{x}{8}T_{42}(\frac{x}{8}) \qquad -8 \leq x \leq 8\,. \qquad \text{(A8)}
\end{aligned}
$$

where the bolding indicates the digits he displays (I have included an extra digit in some places to allow for appropriate rounding to his displayed digit). If one takes the iconoclastic route of not following his convention (noted on his p. 7) for sums having a single prime to indicate that the term with suffix zero is to be halved, the first line above would be $0.648358770605264920836687110979622 7\,\frac{x}{8}T_0(\frac{x}{8})$.

At the upper limit of applicability, $x = 8$, (A8) gives 29-digit accuracy, $J_1(8) = 0.23463634685391462438127665159$.

Appendix A.3. Gegenbauer Series for $J_0(kx)$ and $J_1(kx)$

The first 22 terms in the Gegenbauer polynomial expansions of $J_0(kx)$ (1), with $k = 1$ and arbitrarily taking $\lambda = \frac{1}{4}$, are

$$
\begin{aligned}
J_0(x) \cong\ & 0.904078771191585521024227636544096\, C_0^{\frac{1}{4}}(x) \\
-\ & 0.377480902332903752477356198652003\, C_2^{\frac{1}{4}}(x) \\
+\ & 0.00985645918454006348253321451683292\, C_4^{\frac{1}{4}}(x) \\
-\ & 0.0000929144245327682841642709978007872\, C_6^{\frac{1}{4}}(x) \\
+\ & 4.51192238929050409752370668969243 \times 10^{-7}\ C_8^{\frac{1}{4}}(x) \\
-\ & 1.33557953986611692879627373122257 \times 10^{-9}\ C_{10}^{\frac{1}{4}}(x) \\
+\ & 2.66185765952711910618049951726347 \times 10^{-12}\ C_{12}^{\frac{1}{4}}(x) \\
-\ & 3.81525698311458688534308130184138 \times 10^{-15}\ C_{14}^{\frac{1}{4}}(x) \\
+\ & 4.12174698882181605290995488668659 \times 10^{-18}\ C_{16}^{\frac{1}{4}}(x) \\
-\ & 3.47649878544013257006577318271996 \times 10^{-21}\ C_{18}^{\frac{1}{4}}(x) \\
+\ & 2.35284611436757064520926642520417 \times 10^{-24}\ C_{20}^{\frac{1}{4}}(x) \\
-\ & 1.30601535036874068380434807654702 \times 10^{-27}\ C_{22}^{\frac{1}{4}}(x) \\
+\ & 6.05330821302322601332159315076677 \times 10^{-31}\ C_{24}^{\frac{1}{4}}(x) \\
-\ & 2.37803900637432785965238868426667 \times 10^{-34}\ C_{26}^{\frac{1}{4}}(x) \\
+\ & 8.01904904818064037541914772609834 \times 10^{-38}\ C_{28}^{\frac{1}{4}}(x) \\
-\ & 2.34648500711595153019299896447757 \times 10^{-41}\ C_{30}^{\frac{1}{4}}(x) \\
+\ & 6.01437661018790357782076353076573 \times 10^{-45}\ C_{32}^{\frac{1}{4}}(x) \\
-\ & 1.36150461807454631533129344677808 \times 10^{-48}\ C_{34}^{\frac{1}{4}}(x) \\
+\ & 2.74196263898788782515776484348033 \times 10^{-52}\ C_{36}^{\frac{1}{4}}(x) \\
-\ & 4.94454559738665023143625856030709 \times 10^{-56}\ C_{38}^{\frac{1}{4}}(x) \\
+\ & 8.03022232133135996468784524426669 \times 10^{-60}\ C_{40}^{\frac{1}{4}}(x) \\
-\ & 1.18066972928855334355199708640780 \times 10^{-63}\ C_{42}^{\frac{1}{4}}(x) \quad -1 \le x \le 1\,.
\end{aligned}
\tag{A9}
$$

At the upper limit of applicability, $x = 1$, this gives 33-digit accuracy, $J_0(1) = 0.765197686557966551449717526102663$ (Even at $x = 8$, this gives a result accurate to 15 digits, 0.171650807137554). The convergence is not any faster than for the Chebyshev version (A5), so there is no strong motivation for programmers to switch to this representation of Bessel functions from the well-established computer codes for Chebyshev expansions. One obtains a different representation that has similar accuracy with 22 terms if one takes $\lambda = 4$, but Figure A1 shows that the convergence is slower until about twelve terms are included.

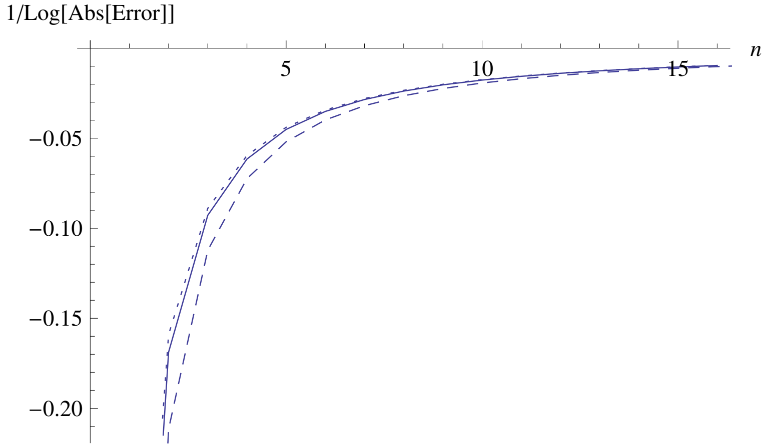

Figure A1. One divided by the logarithm of the absolute value of the error in the Gegenbauer expansion (A9) of $J_0(kx)$ with $k = 1$ and $\lambda = \frac{1}{4}$ (solid line) when successively more terms are added. One sees little difference in convergence from the Chebyshev version (A4) (dotted line). The Gegenbauer expansion (A5) of $J_0(kx)$ with $\lambda = 4$ (dashed line), on the other hand, displays somewhat slower convergence if one includes a dozen terms or fewer.

There is, however, interesting research into the utility of Gegenbauer expansions in an analytical context. To note just three examples, Bezubik, Dàbrowska, and Strasburger [45] derive an expansion of plane waves $e^{ir(\xi|\eta)}$ into an infinite series over $i^m(\alpha+m)J_{\alpha+m}(r)C_m^{\alpha}((\xi|\eta))$, and Elgindy and Smith-Miles [46] develop a numerical quadrature based on a truncated Gegenbauer expansion series. A third example is Jens Keiner's method [47] of converting from one expansion in $C_j^{\alpha}(x)$ to another expansion in $C_j^{b}(x)$. However, neither the relative numerical utility of Gegenbauer expansions, nor expansions in an analytical context will be explored further in this paper since it focuses instead on summing additional infinite series involving ${}_1F_2$ hypergeometric functions.

Turning now to Gegenbauer polynomial expansions of $J_1(kx)$ (1), with $k = 1$ and arbitrarily taking $\lambda = \frac{1}{4}$, the first 22 terms are

$$
\begin{aligned}
J_1(x) \quad \cong \quad & 0.4756834292754168073862242654710 41\, x\, C_0^{\frac{1}{4}}(x) \\
- \quad & 0.09622376780065818251326370185973 88\, x\, C_2^{\frac{1}{4}}(x) \\
+ \quad & 0.00165923280553475766418121861007493\, x\, C_{04}^{\frac{1}{4}}(x) \\
- \quad & 0.0000116849150281699572948996291216163\, x\, C_6^{\frac{1}{4}}(x) \\
+ \quad & 4.5303088506394388853845501270703 \times 10^{-8} x\, C_8^{\frac{1}{4}}(x) \\
- \quad & 1.1162341074884410545162588277692 8 \times 10^{-10} x\, C_{10}^{\frac{1}{4}}(x) \\
+ \quad & 1.9055029695700954979141872873389 9 \times 10^{-13} x\, C_{12}^{\frac{1}{4}}(x) \\
- \quad & 2.3886169220443579409283601933555 3 \times 10^{-16} x\, C_{14}^{\frac{1}{4}}(x) \\
+ \quad & 2.2929995378370815999190327978718 5 \times 10^{-19} x\, C_{16}^{\frac{1}{4}}(x) \\
- \quad & 1.7402020209449114207918662504749 4 \times 10^{-22} x\, C_{18}^{\frac{1}{4}}(x) \\
+ \quad & 1.0704776458798914169154263427097 0 \times 10^{-25} x\, C_{20}^{\frac{1}{4}}(x) \\
- \quad & 5.4460488520978026514672607761416 1 \times 10^{-29} x\, C_{22}^{\frac{1}{4}}(x) \\
+ \quad & 2.3297801734378369846444576516364 1 \times 10^{-32} x\, C_{24}^{\frac{1}{4}}(x) \\
- \quad & 8.4980095717422935738849721798933 5 \times 10^{-36} x\, C_{26}^{\frac{1}{4}}(x) \\
+ \quad & 2.6743976503584479086683701192287 0 \times 10^{-39} x\, C_{28}^{\frac{1}{4}}(x) \\
- \quad & 7.3361106801739760282462207462832 8 \times 10^{-43} x\, C_{30}^{\frac{1}{4}}(x) \\
+ \quad & 1.7696518802522535649775030550063 1 \times 10^{-46} x\, C_{32}^{\frac{1}{4}}(x) \\
- \quad & 3.7833304716828605938838986856828 5 \times 10^{-50} x\, C_{34}^{\frac{1}{4}}(x) \\
+ \quad & 7.2180499062674737114778866956416 8 \times 10^{-54} x\, C_{36}^{\frac{1}{4}}(x) \\
- \quad & 1.2365021083037866382708678883705 7 \times 10^{-57} x\, C_{38}^{\frac{1}{4}}(x) \\
+ \quad & 1.9124720630088783251263537724695 1 \times 10^{-61} x\, C_{40}^{\frac{1}{4}}(x) \\
- \quad & 2.6839995749795882850730754831304 1 \times 10^{-65} x\, C_{42}^{\frac{1}{4}}(x) \quad -1 \le x \le 1\,. \qquad \text{(A10)}
\end{aligned}
$$

At the upper limit of applicability, $x = 1$, this gives 33-digit accuracy, $J_1(1) = 0.440050585744933515959682203718915$ (Even at $x = 8$, this gives a result accurate to 16 digits, $J_1(8) = 0.2346363468539146$). One obtains a different representation with similar accuracy if one takes $\lambda = 4$ and 22 terms. The convergence is not any faster for either than for the Chebyshev version (A5), so there is no strong motivation for programmers to switch to this representation of Bessel functions.

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