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Branes changing signature

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Abstract. We explore the possibility of having a good description of classical signature change in the brane-world or shell cosmology scenarios.

1. Introduction

The aim of this work is to show, in simple terms, that a natural scenario for the classical change of signature in the physical spacetime is provided by the brane-world models [1, 2, 3] (see also [4] for an exhaustive list of references), the more general shell cosmologies [5] or, in general, by every higher-dimensional theory [6] which may contain domain walls and/or branes. The main idea was put forward in [7], and the issue has been retaken in order to go deeper in the study of the possible behaviours in different settings.

1.1. Why signature change?

Although the occurrence of a change of signature in our spacetime is debatable, this possibilty has been considered several times in the literature during the last decade [8]. The motivations behind the signature change at a classical level are grounded on the possibility it offers to escape singularities in General Relativity [10], as for instance, replacing the big-bang by a Euclidean region prior to the birth of time. On similar terms, the signature change has provided a classical description of both the quantum tunneling [11] and the no-boundary proposal [12] approaches in the prescription of the Universes's wave function in quantum cosmology. Alternatively, one could also simply argue that the Einstein's equations do not fix the signature 'a priori', and that a spacetime may, in principle at least, undergo a change of signature.

1.2. General classical signature change

The general classical signature change is described by a 4-dimensional manifold endowed with a 2-covariant tensor, say $(\mathcal{V}, g)$, such that $\mathcal{V}$ is the closure of the union of two open disjoint subsets $\mathcal{V}_L$ and $\mathcal{V}_E$, so that $(\mathcal{V}_L, g|_L)$ and $(\mathcal{V}_E, g|_E)$ are Riemannian manifolds with Lorentzian and Euclidean signature respectively. Also, we will assume that $\mathcal{V}_L$ and $\mathcal{V}_E$ are not empty, and that there is a hypersurface $S \equiv \overline{\mathcal{V}}_L \cap \overline{\mathcal{V}}_E$ which is not empty either. Thence $g|_S$ degenerates, this is, S is the set of signature change.

Given this basic scheme, there is still much freedom to devise and study a change of signature in a manifold. Regarding the metric, strong discussions emerged on whether the change of signature had to be produced by a discontinuous g or should be better described by a continous g (see [8],[9]).

As for the structure of the set of signature change S, it has been normally assumed in the literature that S is a Euclidean (spacelike) hypersurface everywhere, see Figure 1. But still,

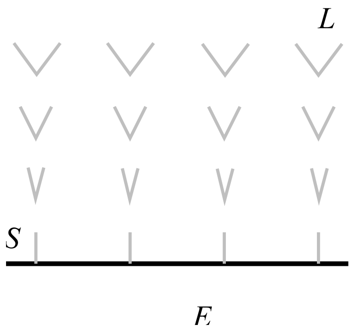

Figure 1. An schematic figure for a signature change S set being spacelike everywhere. Here and in the following figures we indicate the Lorentzian and Euclidean regions by L and E respectively, and the lightcones (only present in L) in grey. Note that in this example the lightcones collapse onto a single direction at points on S, which corresponds to a single degeneration direction of $g|_S$.

there are many more possibilities which have been excluded by hand, with no justification (see also Figure 2):

(i) S may not be a submanifold.

(ii) $g|_S$ may have many directions of degeneration, or even become zero. As an example, one can consider the manifold with metric

$$ds^2 = -tdt^2 + t^2\left(dx^2 + dy^2 + dz^2\right)$$

(iii) $g|_S$ may have only one degeneration direction, but tangent to S. S can even be generic (different character at different points).

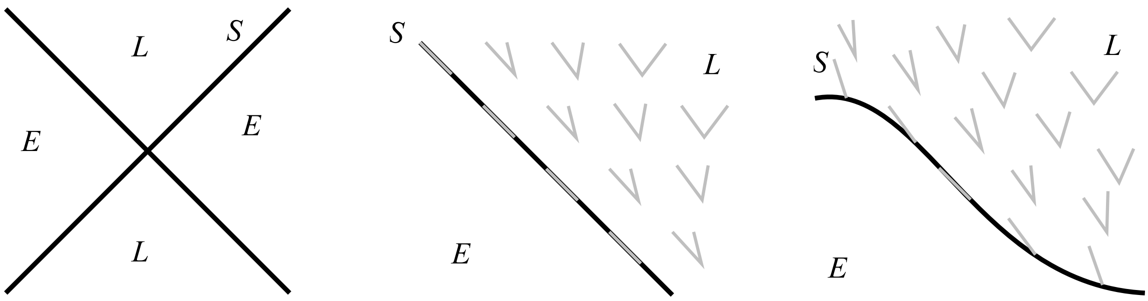

Figure 2. Figures for the possibilities (i) and the two in (iii) (degeneration tangent to S and S generic), respectively

1.3. Why on the brane?

So far, signature change had received ad hoc descriptions.

The main idea behind our proposal in [7] was simply that higher-dimensional theories, and in particular brane-world (or shell cosmology) settings, provide natural scenarios for the change of signature in the physical spacetime. In geometrical terms, branes (as well as shells, domain walls, etc...) are submanifolds of a higher dimensional spacetime (the bulk) with a metric which is differentiable everywhere except on the brane where it is only continuous. The jump in the derivatives of the metric is related to the distributional part of the energy- momentum tensor via the so-called Israel formula.

Generically, branes are taken to be timelike submanifolds so that the induced geometry is Lorentzian and the brane can describe the four-dimensional spacetime where we live. However, smooth submanifolds can have varying causal character from point to point (see Figure 3), this is, the induced metric may change character from point to point. This means that, a priori, nothing prevents the possibility of having perfectly regular branes which change its character from (say) spacelike to timelike, or which are partly null, or even more complicated possibilities. The first case corresponds to a signature-changing brane. The interesting property is that both the bulk and the brane can be regular everywhere even though the change of signature may appear as a dramatical event when seen from within the brane, see the example in Figure 3. Notice that the signature in the bulk is left unchanged, so that our work differs significantly from other studies [13].

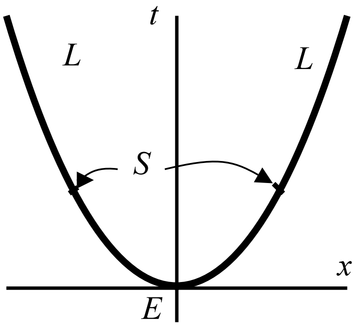

Figure 3. Simple example of a submanifold that changes character: $t = x^2$ in M_2. The curve is timelike, hence Lorentzian, for $x^2 > 1$, and spacelike, hence Euclidean, for $x^2 < 1$. It is null at $x^2 = 1$, the character changing set S.

In our proposal, the study of change of signature becomes the simple geometrical analysis of embedded submanifolds in the bulk: a well-posed mathematical problem without pathologies. It is remarkable that some of the traditional ad hoc assumptions concerning signature change are shown to become pure necessary conditions in the brane case.

For the study of general n-dimensional submanifolds $\mathcal{V}$ embedded in a Lorentzian $(n+1)$-dimensional space $(\mathcal{M}, g|_{\mathcal{M}})$ we will assume that

Assumption 1.1 *$\mathcal{V}_E$, $\mathcal{V}_L$ and S are non-empty, and S has no interior.*

This means that, in the particular case of a 4-dimensional $\mathcal{V}$, we will be considering a change of signature from an Euclidean to a Lorentzian region of $\mathcal{V}$ across a connected (3-dimensional) hypersurface S. We do not consider cases in which $\mathcal{V}$ is composed by two Lorentzian regions separated by a set S where $\mathcal{V}$ becomes null. With only this assumption one can then prove the following result, in the form of an automatic restriction on S:

Result 1.1 *$g|_S$ has only one degeneration direction.*

The structure of S in this scenario is therefore restricted, because we have that possibility (ii) above is not possible. In this regard, though, we must stress here that the result presented in [7] stating that S must be spacelike is not correct. We are grateful to Aguirre-Dabán and Lafuente-López, who let us know a counter-example to that statement [14].

2. Usual 5-dim brane-world scenario

The usual 5-dimensional brane-world scenario is based on a 5-dimensional Lorentzian manifold $(\mathcal{M}, g_{\mathcal{M}})$ composed of two connected C^3 manifolds with boundary $(\mathcal{M}^{\pm}, g_{\mathcal{M}^{\pm}})$ so that $\mathcal{M} =$

$\overline{\mathcal{M}^+ \cup \mathcal{M}^-}$, with continuous but not differentiable metric $g_{\mathcal{M}}$, and such that the intersection $\mathcal{V} \equiv \overline{\mathcal{M}^+} \cap \overline{\mathcal{M}^-}$ is a timelike submanifold in $\mathcal{M}$, the brane.

The energy-momentum tensor on the brane $\tau_{\mu\nu}$ is defined as the delta part of the energy-momentum tensor in $\mathcal{M}$ except for a 4-dimensional "cosmological" constant on $\mathcal{V}$. The energy-momentum tensor of the bulk, in distributional terms, reads

$$T^{\mathcal{M}}_{\mu\nu} = \theta^+ T^+_{\mu\nu} + (1-\theta^+) T^-_{\mu\nu} + S_{\mu\nu} \delta|_{\mathcal{V}}, \tag{1}$$

where θ^+ stands for the Heaviside distribution which is one at points on $\mathcal{M}^+$, $T^{\pm}_{\mu\nu}$ correspond to the energy-momentum tensors on $\mathcal{M}^{\pm}$, and $S_{\mu\nu}$ is a tensor with support on $\mathcal{V}$ (as well as $\delta|_{\mathcal{V}}$), which is finally decomposed as

$$S_{\mu\nu} = -\Lambda_4 g_{\mu\nu} + \tau_{\mu\nu} \qquad \text{where} \qquad g_{\mu\nu} = g_{\mathcal{M}\mu\nu} - n_\mu n_\nu,$$

n_μ being the unit normal to $\mathcal{V}$. The tensor $S_{\mu\nu}$ relates to the jumps of the derivatives of $g_{\mathcal{M}}$ on $\mathcal{V}$ by the Israel formula

$$[K_{\mu\nu}] = -\kappa_5^2 \left(S_{\mu\nu} - \frac{1}{3} g_{\mu\nu} S \right),$$

where $[f] \equiv f^+ - f^-$ stands for the difference of any object as computed at either side, $K^{\pm}_{\mu\nu} \equiv g^\alpha_\mu g^\beta_\nu \nabla^{\pm}_\alpha n_\beta$ are the extrinsic curvatures of $\mathcal{V}$ as seen from $\mathcal{V}^{\pm}$, and κ_5 corresponds to the constant of proportionality between the Einstein tensor and the energy-momentum tensor of the 5-dimensional Einstein equations in the bulk.

2.1. Proper "brane-world"

The first brane-world models with one brane, the Randall-Sundrum II model [3], correspond to the above construction for an AdS_5 bulk with, say Λ_5, and a Z_2 mirror symmetry, so that $\mathcal{V}$ is the set of fixed points (the mirror). In that case we have

$$T^{\pm}_{\mu\nu} = -\Lambda_5 g^{\pm}_{\mathcal{M}\mu\nu}, \qquad \text{so that} \qquad T^{\mathcal{M}}_{\mu\nu} = -\Lambda_5 g_{\mathcal{M}\mu\nu} + S_{\mu\nu}\delta|_{\mathcal{V}},$$

and

$$K^+_{\mu\nu} = -K^-_{\mu\nu} = -\frac{1}{2}\kappa_5^2 \left(S_{\mu\nu} - \frac{1}{3} g_{\mu\nu} S \right).$$

2.2. Vacuum branes

Another common construction corresponds to vacuum branes, where only a 4-dimensional cosmological constant (the tension of the brane) describes the brane. In this case one has

$$S_{\mu\nu} = -\Lambda_4 g_{\mu\nu} \quad \Longleftrightarrow \quad [K_{\mu\nu}] = -\kappa_5^2 \frac{\Lambda_4}{3} g_{\mu\nu}.$$

3. Generic (signature changing) submanifolds

The description of a generic submanifold starts by considering a C^3 $(\mathcal{M}, g_{\mathcal{M}})$ $(n+1)$-dimensional Lorentzian manifold, a C^3 abstract V n-dimensional manifold, and a C^3 embedding $\Phi : V \to \mathcal{M}$. We define then the submanifold in $\mathcal{M}$ by $\mathcal{V} \equiv \Phi(V)$.

The inherited metric on V is given by $g = \Phi^*(g_{\mathcal{M}})$. $\mathcal{V}$ readily determines $\boldsymbol{N}$ a normal one-form to $\mathcal{V}$: a nowhere-zero one-form defined at points on $\mathcal{V}$ satisfying $\Phi^*(\boldsymbol{N}) = 0$. The corresponding normal vector $\vec{N}$ is then defined by $N^\mu \equiv g^{\mu\nu}_{\mathcal{M}} N_\nu$.

With these ingredients at hand one has that $\vec{N}$ is tangent to $\mathcal{V}$ at p if and only if $\det(g)|_p = 0$, this is, g is degenerate at p.

With this description of a generic manifold one can now proceed to the construction of a generic brane (or shell) by joining two such $\mathcal{M}^+$ and $\mathcal{M}^-$ across given respective generic submanifolds $\mathcal{V}^+$ and $\mathcal{V}^-$, taken as boundaries, whose points are identified as $\mathcal{V}$ by using a common V, so that $\Phi^+(V) = \Phi^-(V) \equiv \mathcal{V}$.

To obtain the matching conditions (and thus the generalised Israel formula) one also needs to identify the tangent spaces of $\mathcal{M}^\pm$ at every $p \in \mathcal{V}$. A necessary condition is that the inherited metric at either side of $\mathcal{V}$ is the same, this is

$$[g] = 0.$$

These are the so-called *preliminary junction conditions*, and are sufficient to identify the part of the tangent spaces at points on $\mathcal{V}$ which is tangent to $\mathcal{V}$, this is $T\mathcal{V}^+ = T\mathcal{V}^- \equiv T\mathcal{V}$ at every $p \in \mathcal{V}$. But $T\mathcal{V}$ is a n-dimensional plane, so we need a further direction transverse to $\mathcal{V}$. A choice of transverse vector, highly non-unique, is the so-called rigging vector $\vec{\ell}$, and clearly satisfies $\ell^\mu N_\mu \neq 0$. Given $\vec{\ell}$, the tangent space $T_p\mathcal{M}$ for any $p \in \mathcal{V}$ and thence any vector there can be decomposed as

$$\begin{aligned} T_p\mathcal{M} &= <\vec{\ell}>_p \oplus T_p\mathcal{V} \\ \vec{v} &= v^\ell \vec{\ell} + \vec{v}^{||} \end{aligned}$$

Performing this decomposition on either side, $\mathcal{M}^+$ and $\mathcal{M}^-$, so that $\vec{\ell}^-$ points outwards from $\mathcal{M}^-$ and $\vec{\ell}^+$ points inwards to $\mathcal{M}^+$, suitable conditions exist on $\vec{\ell}^+$ and $\vec{\ell}^-$ that ensure their identification (as $\vec{\ell}$) all over $\mathcal{V}$, and thus the identification of $T\mathcal{M}^+$ and $T\mathcal{M}^-$ [15].

Having these identifications at hand, one can prove that there is a coordinate chart defined on $(\mathcal{M}, g_\mathcal{M})$ in which the metric is continuous [16] (see also [15]). This allows us to define the Riemann and hence the Einstein tensor in a distributional sense all over $(\mathcal{M}, g_\mathcal{M})$, which leads to the generalised Israel formula.

3.1. Generalised Israel formula

Making use of the projector $T\mathcal{M} \to T\mathcal{V}$,

$$P^\mu{}_\nu = \delta^\mu_\nu - \ell^\mu n_\nu, \qquad \text{where} \qquad n_\nu \equiv \frac{1}{\ell^\alpha N_\alpha} N_\nu,$$

we define

$$\mathcal{H}_{\mu\nu} \equiv P^\alpha{}_\mu P^\beta{}_\nu \nabla_\alpha \ell_\beta,$$

where $\vec{\ell}$ denotes, obviously, the identified rigging at both sides. This is the so-called rigged second fundamental form, and plays the role of $K_{\mu\nu}$ for timelike and spacelike $\mathcal{V}$. Indeed, if $\mathcal{V}$ is everywhere timelike or spacelike one can choose $\vec{\ell}$ appropiately (the unit normal) and $\mathcal{H}_{\mu\nu}$ reduces to $K_{\mu\nu}$. Then,

$$T^\mathcal{M}_{\mu\nu} = \theta^+ T^+_{\mu\nu} + (1-\theta^+) T^-_{\mu\nu} + S_{\mu\nu}\delta|_\mathcal{V}$$

with

$$\begin{aligned} \kappa_5^2 S_{\mu\nu} &= n^\alpha[\mathcal{H}_{\alpha\mu}]n_\nu + n^\alpha[\mathcal{H}_{\alpha\nu}]n_\mu - n^\alpha n_\alpha[\mathcal{H}_{\mu\nu}] - n_\mu n_\nu[\mathcal{H}^\alpha{}_\alpha] \\ &\quad - g_{\mathcal{M}\mu\nu}\left(n^\alpha n^\beta[\mathcal{H}_{\alpha\beta}] - n^\alpha n_\alpha[\mathcal{H}^\beta{}_\beta]\right)\Big|_\mathcal{V}. \end{aligned} \tag{2}$$

The two most relevant features of $S_{\mu\nu}$ is first, that it is defined in $\mathcal{V}$, because $n^\mu S_{\mu\nu} = 0$. Secondly, it can be shown that $S_{\mu\nu}$ does not depend on the choice of $\vec{\ell}$.

3.2. Some results

Given the above construction one can prove the following results [7, 16]:

Result 3.1 *It is impossible to join two identical bulks with signature changing boundary $\mathcal{V}$ to produce a bulk $\mathcal{M}$ with continuous $g_{\mathcal{M}}$.*

In short, this means that Z_2 *mirror symmetry is incompatible with signature change.* Therefore, the brane in the "proper" brane-world scenario (2.1) cannot undergo a signature change. Looking at this result from the reverse point of view, if, for any reason, the Z_2 symmetry is a consequence of a given theory, then that theory prevents the existence of a change of signature.

Result 3.2 *The previous result provides a counter-example of a well known theorem by Clarke and Dray [17, 15] stating that the necessary and sufficient conditions for a junction with continuous metric is that $[g] = 0$ (the preliminary junction conditions).*

Indeed, on top of $[g] = 0$, to ensure a continuous metric we also need the existence of two rigging vectors $\vec{\ell}^{+}$ and $\vec{\ell}^{-}$ satisfying the convenient conditions that allow their identification. The error in [17] was in proving that $[g] = 0$ is enough for the existence of such $\vec{\ell}^{+}$ and $\vec{\ell}^{-}$.

Result 3.3 $K_{\mu\nu} \propto g_{\mu\nu}$ *everywhere* $\Longrightarrow$ $\mathcal{V}$ *cannot change signature.*

Result 3.4 *Moreover, if two spacetimes $(\mathcal{M}^{+}, g^{\pm}_{\mathcal{M}})$ are joined across $\mathcal{V}$ satirfying $[K_{\mu\nu}] \propto g_{\mu\nu}$ everywhere, then $\mathcal{V}$ cannot change signature.*

In other words, $S_{\mu\nu} \propto g_{\mu\nu}$ is incompatible with signature change. Then, in particular, vacuum branes (see Section 2.2) cannot change signature. This can be interpreted as the fact that *some fields must become excited to produce a signature change* (or, at least, near there).

4. Signature changing branes (shells) in AdS_5

From the previous results, in order to construct a signature changing brane, or better called shell, we have to consider the matching of two different anti-de Sitter bulks, AdS_5 and $\widetilde{AdS}_5$. These are defined by corresponding cosmological constants Λ_5 and $\widetilde{\Lambda}_5$. Defining $\Lambda_5 = -6\lambda^2$, their line-elements can be cast in the form

$$\begin{aligned} ds^2 &= -(k+\lambda^2 r^2)dt^2 + (k+\lambda^2 r^2)^{-1}dr^2 + r^2 d\Omega^2_{\Upsilon^3}, \\ ds^2 &= -(k+\tilde{\lambda}^2 \tilde{r}^2)d\tilde{t}^{\,2} + (k+\tilde{\lambda}^2 \tilde{r}^2)^{-1}d\tilde{r}^2 + \tilde{r}^2 \widetilde{d\Omega^2_{\Upsilon^3}}, \end{aligned}$$

where $d\Omega^2_{\Upsilon^3} = d\chi^2 + f_k^2(\chi)(d\theta^2 + \sin^2\theta)d\phi^2$, for $f_1(\chi) = \sin\chi$, $f_0(\chi) = \chi$, $f_{-1}(\chi) = \sinh\chi$. The construction chosen is performed preserving the symmetries of the homogeneous spaces $d\Omega^2_{\Upsilon^3}$. The symmetry preserving $\mathcal{V}$ is given in parametric form by

$$\begin{aligned} &\Psi_{+}(\xi,\varphi^m) : \{t = t(\xi), r = r(\xi), \phi^m = \varphi^m\} \quad (\mathcal{V}^{+}), \\ &\Psi_{-}(\xi,\varphi^m) : \{\tilde{t} = \tilde{t}\,(\xi), \tilde{r} = \tilde{r}(\xi), \tilde{\phi}^m = \varphi^m\} \quad (\mathcal{V}^{-}), \end{aligned}$$

where $\varphi^m = \{\chi, \theta, \phi\}$. Denoting the derivative with respect to ξ by a dot, the $[g] = 0$ conditions read

$$\begin{aligned} &r(\xi) = \tilde{r}(\xi) \equiv a(\xi) \\ &\dot{t} = \frac{\sigma a}{k+\lambda^2 a^2}\sqrt{\frac{\dot{a}^2}{a^2} - N\left(\frac{k}{a^2}+\lambda^2\right)}, \quad \dot{\tilde{t}} = \frac{\sigma a}{k+\tilde{\lambda}^2 a^2}\sqrt{\frac{\dot{a}^2}{a^2} - N\left(\frac{k}{a^2}+\tilde{\lambda}^2\right)}, \end{aligned}$$

for some function $N(\xi)$ and $\sigma^2 = 1$. The induced metric on $\mathcal{V}$ reads then

$$ds^2_{\mathcal{V}} = N(\xi)d\xi^2 + a^2(\xi)\ d\Omega^2_{\Upsilon^3_{\varphi}}. \tag{3}$$

Clearly, if there is a region where $N < 0$, $\mathcal{V}$ is a 4-dimensional FLRW spacetime, and in fact, any FLRW model can be recovered. We will discuss this further later.

4.1. Energy-momentum tensor on the brane $\mathcal{V}$

The $S_{\mu\nu}$ tensor can be now computed using (2). The result is a tensor of a perfect-fluid type, with two eigenvalues $\hat{\varrho}$ (along the ξ direction) and $\hat{p}$ (along the rest), whose expressions read

$$\kappa_5^2\,\hat{\varrho} = \frac{3\sigma}{(\ell_+^\alpha N_\alpha^+)}\left[\epsilon\sqrt{\frac{\dot{a}^2}{a^2} - N\left(\frac{k}{a^2}+\tilde{\lambda}^2\right)} - \sqrt{\frac{\dot{a}^2}{a^2} - N\left(\frac{k}{a^2}+\lambda^2\right)}\right],$$

$$\kappa_5^2(\ell_+^\alpha N_\alpha^+)\left(\hat{p}+\frac{2}{3}\hat{\varrho}\right) = \epsilon\sigma\frac{\left(\tilde{\lambda}^2 N + \frac{\dot{N}}{2N}\frac{\dot{a}}{a} - \frac{\ddot{a}}{a}\right)}{\sqrt{\frac{\dot{a}^2}{a^2} - N\left(\frac{k}{a^2}+\tilde{\lambda}^2\right)}} - \sigma\frac{\left(\lambda^2 N + \frac{\dot{N}}{2N}\frac{\dot{a}}{a} - \frac{\ddot{a}}{a}\right)}{\sqrt{\frac{\dot{a}^2}{a^2} - N\left(\frac{k}{a^2}+\lambda^2\right)}},$$

where

$$(\ell_+^\alpha N_\alpha^+) = \epsilon_1\left(2\frac{\dot{a}^2}{k+a^2\lambda^2} - N\right), \quad \epsilon_1^2 = 1, \quad \epsilon^2 = 1.$$

The different signs correspond to different regions within $\mathcal{V}$ (determined by σ) or the four different choices of matchings that depend on the relative orientation of the rigging vectors, that correspond to the combinations of the signs of ϵ and ϵ_1. All this is shown in Figure 4.

In fact, it can be shown that $\epsilon = -1$ is incompatible with a change of signature.

4.2. Cosmology on the Lorentzian part of the brane $\mathcal{V}_L$

At points where $N \neq 0$ (at the Lorentzian part we have $N < 0$) we can define

$$\varrho \equiv \hat{\varrho}\ |\ell_+^\alpha N_\alpha^+|/\sqrt{|N|}, \quad p \equiv \hat{p}\ |\ell_+^\alpha N_\alpha^+|/\sqrt{|N|},$$

so that the energy-conservation law is obtained in standard form

$$\dot{\varrho} + 3\frac{\dot{a}}{a}(\varrho + p) = 0.$$

This choice of normalisation may seem ad-hoc, but it corresponds precisely to the choice of the unit normal as the rigging vector, and one therefore recovers the usual extrinsic curvature K_{ab}, the Israel formula and hence, the usual energy-momentum tensor on the brane. Indeed, when $N \neq 0$ we choose

$$\vec{\ell}_+ = -\frac{\epsilon_1\ \mathrm{sign}(N)}{\sqrt{|N|}}\vec{N}^+, \quad \vec{\ell}_- = -\frac{\epsilon\ \epsilon_1\ \mathrm{sign}(N)}{\sqrt{|N|}}\vec{N}^-,$$

so that all the relations above between $\vec{\ell}_\pm$ and $\boldsymbol{N}^\pm$ hold together with

$$(\ell_+^\alpha N_\alpha^+) = \epsilon(\ell_-^\alpha N_\alpha^-) = \epsilon_1\sqrt{|N|} \quad \longrightarrow \quad \frac{|\ell_+^\alpha N_\alpha^+|}{\sqrt{|N|}} = \frac{|\ell_-^\alpha N_\alpha^-|}{\sqrt{|N|}} = 1,$$

noting that $\mathrm{sign}(N) = -1$ for the Lorentzian part of $\mathcal{V}$.

Therefore, *ϱ and p are the energy density and pressure measured within $\mathcal{V}_L$*. Changing to cosmic time T, such that $\dot{T}(\xi) = \sqrt{-N(\xi)}$, the line-element (3)simply becomes

$$ds^2_{\mathcal{V}_L} = -dT^2 + a^2 d\Omega^2_{\Upsilon^3_\varphi}.$$

The equations definig ϱ and p read then

$$\varrho' + 3\frac{a'}{a}(\varrho + p) = 0, \tag{4}$$

$$\frac{\kappa_5^2}{3}\varrho = \sigma\epsilon_1\left[\epsilon\sqrt{\frac{a'^2}{a^2} + \frac{k}{a^2} + \tilde{\lambda}^2} - \sqrt{\frac{a'^2}{a^2} + \frac{k}{a^2} + \lambda^2}\right], \tag{5}$$

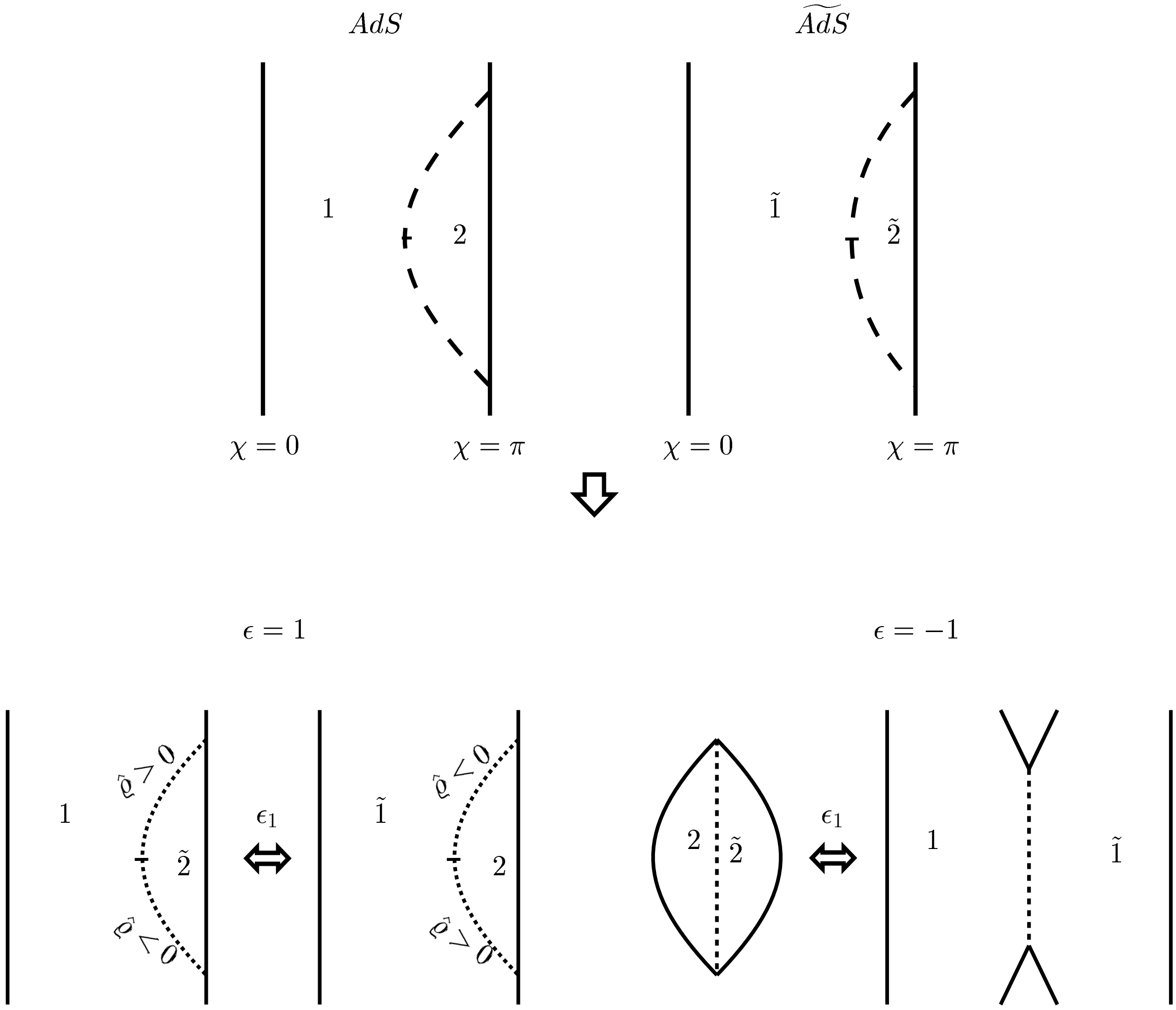

Figure 4. The four different possible matchings between AdS and $\widetilde{AdS}$ in the case $k = 1$. The anti de-Sitter diagrams for $k = 1$ are drawn at the top. The slashed curves represent $\mathcal{V}^+$ and $\mathcal{V}^-$, that divide AdS and $\widetilde{AdS}$ into two parts, respectively. The pairs of choices of regions (halves) to be taken and joined depend on the relative signs of the rigging vectors, ϵ, and whether $\vec{\ell}$ points inwards or outwards from/to one of the halves, which is determined by ϵ_1. For $\epsilon = 1$ we obtain one of two possbilities on the bottom left, which differ on the sign of ϵ_1, and for $\epsilon = -1$, the two on the bottom right.

which imply

$$\frac{a'^2}{a^2} + \frac{k}{a^2} = \frac{1}{4\rho^2}\left[\left(\tilde{\lambda}^2 + \lambda^2 - \rho^2\right)^2 - 4\tilde{\lambda}^2\lambda^2\right]. \tag{6}$$

This corresponds to the Friedmann equation in these "asymmetric" brane cosmologies.

Note that with the case $\epsilon = -1$, $\lambda = \tilde{\lambda}$ the usual "proper" brane of section 2.1 is recovered. The case with $\epsilon = 1$ corresponds to the "shell cosmologies" in [5].

Recalling that we need $\epsilon = 1$ to have a change of signature in addition to the fact that $a|_S \neq 0$ (this is where $N = 0$), ϱ attains a finite value at the change of signature.

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