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Black Holes in Three Dimensional Higher Spin Gravity

by

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Summary

Inside a theory of gravity coupled to higher spin fields in three dimensions, the problem on the existence of black hole solutions carrying higher spin charges is studied. A consistent thermodynamic description for static and circularly symmetric higher spin fields living on the solid torus is given by purely topological considerations.
As the higher spin symmetries are bigger than the diffeomorphisms group, in this theory, the usual geometrical notions to define regular black holes solutions, i.e., curvature, causal structure, etc., are ruled out. This rise the necessity of find them by considering other, more direct, methods which are worked out in this thesis based mainly on topological considerations and on coordinates matters. Finally, a simple ansatz to build regular higher spin black holes and higher spin fields is given.

Chapter 1

Introduction

During the lasts 80 years, higher spin field theory, which originally begins as a very unpopular topic pushed only by a very few enthusiast, through the years has evolved to become into a very interesting, rich and active field of research. The principal motivation for a physicist to study them is due to that higher spin fields, of non-fundamental nature, has been observed as Hadronic resonances (composed quark bound states) in nature, where higher spin theory has proven to be very useful in their description. On other side, string theory naturally predicts the existence of an infinite tower of massive higher spin fields as forming part of the full string spectrum of excited states.This is another reason to study them if one seriously takes the string theory picture as a fundamental theory of nature. Moreover, without appealing to string theory, an old group theoretical analysis to build physical theories, based on the representations of the Lorentz group, reveals that besides the allowed usual lower spins, which are truly found to be part in the standard model description of nature, higher spin fields are also allowed to exist without any consistency problems but freely. However, in the physical world we are interested in study interacting systems, but interacting higher spins has been shown to be a very difficult topic of research. At quantum field theory level, there exists several no-go theorems concerning their interactions (Weinberg, Weinberg-Witten, Coleman-Mandula) that, at present energy scales, rules them out where conventional perturbative field theory methods are used. If higher spins really do exist in nature, they must interact with gravity as this is known to be universal. The field of research dedicated to its study is called 'Higher Spin Gravity', which has been advocated in the hope to find consistent UV completions for a quantum theory of gravity, which by itself, is well known that is plagued with infinities. The prospects of the higher spin gravity theory is that infinities may be eliminated by the addition of higher spin fields, in an analogous fashion as the addition of local super-symmetry to gravity leads to super-gravity which possess an improved UV behaviour than just ordinary gravity.

In this thesis, we work in a three dimensional theory of higher spin fields which are non-minimally coupled to gravity. In particular this theory describes a highly coupled system of a tower of fundamental higher spin fields, from spin 2 up to a spin N, in which each spin appears only once. This theory, besides from already being complicated enough on its technical issues, it presents several challenging theoretical concepts to any researcher in the field, because at the present moment there is an enormous lack of knowledge about the geometric concepts regarding higher spin fields. However, the main objective of this work has been precisely to find out if this theory admits regular black hole solutions (or some sensible possible generalizations of them), carrying fundamental higher spin charges as additional 'hairs' to use in their description. Also, a thermodynamic study of these solutions has been carried out and also given for all N, providing a successful and consistent description through a generalized higher spin euclidean partition function, in a saddle point approximation, which naturally arises as the exponentiation of the on-shell action of the theory.

In order to facilitate the exposure, reading and understanding of the background theory, and thus, of the subsequently original contribution of this work, the content of this thesis has been arranged in 8 separate chapters. The contents treated from chapter (2) to chapter (7) are basically a bibliographic revision of the literature, made with a very personal touch on its explanation and straightforward exposition possessing a focus on the relevant concepts for this work. Chapter (8) is a collaboration, which has been published in [50]. Finally, chapter (9) is a personal work, which has not been published yet. However, some parts of its contents will be hopefully shown in a forthcoming publication.

The content of this thesis is organized as follows: Chapter (2) is a review, where mainly from a group theoretic point of view, higher spin nature of the fields is explained by its definition as irreducible unitary representations of the Poincare group. This chapter intends to be purely motivational for the reader and, in this way, a brief list about fundamental higher spin issues, inside the usual field theory context, is also treated. In chapter (3), we review the free theory of higher spin fields in a usual Lagrangian formulation. This material can be understood as complement of chapter (2), however it is not less important, because here an explicit connection is made with usual field theories. Chapter (4) is a very brief review where the inconsistency problem of minimally coupled higher spin interactions is introduced. A brief exposition about the issues one faces when one naively tries to minimally couple higher spin fields $s \geq 2$ to themselves or, also, to lower spin fields $s < 2$ is shown. This chapter is roughly focussed on the main general idea of the inconsistency problem, and on its exceptional solution in three dimensions. Until chapter (4) the content is introductory and it was made in order for the reader to get a feeling of the higher spin scenario, before moving to the more complicated fully interacting higher spin scenario, which is the core of the theoretical framework of this thesis. Therefore, if the reader is unfamiliar with these higher spin basic concepts, these chapters should not be skipped from the reading.
Chapter (5) is one of the most important chapters regarding the theoretical framework we use in this thesis. Here, a consistent theory of interacting higher spins fields with gravity in three dimensions is developed and its fundamentals are shown in an extended frame formalism thanks to the help of the three-dimensional Chern-Simons theory. Thus, its reading turns out to be indispensable. In chapter (6) we roughly discuss the main ideas of AdS/CFT holography, which are also used on this thesis. To this end a pedagogical example, in the simplest case of the scalar field on a fixed AdS background, is given. However, if the reader is unfamiliar with AdS/CFT holography, its reading turns out to be indispensable in order to understand the main concepts used on this work. Chapter (7), it is devoted to fulfil a gap in the link between bosonic holography which is done in the metric like formalism and treated in chapter (6), and holography done in the Chern Simons formalism, which is the main formalism used in this work. Chapter (8) is about the interpretation of the euclidean AdS/CFT partition function in saddle point approximation over static and circularly symmetric solutions defined on the torus, solutions which must gives rise to black hole solutions in the bulk, as the thermodynamic partition function which describes the thermodynamic properties of these prospective black hole solutions. Finally in chapter (9),

we explicitly show the construction of regular black hole solutions carrying higher spin charges. We first work the case of a single spin 3 coupled to gravity, to then pass to the case of one spin 3 and one spin 4 coupled to gravity.

Chapter 2

Introduction to Higher Spins

2.1 Introduction

The scope of this chapter is to give the reader a context for higher spin theory. This chapter begins introducing Wigner's classification of particles as unitary irreducible representations of the Poincar group, labelling these irreps by mass and spin. First we make use of the quadratic Casimir invariant operator of the Poincare group, to separate the different class of momentum that may exist according to the rest mass and energy of the free particles. Then inside a given class, we make a classification of its irreducible pieces by making an irreducible classification of the Wigner little group. This allows us to see single particles as irreducible representations of the Poincare group. In order to see these particles as forming part of some kind of fields, as is usually seen from the fields excitation paradigm, we then move into a spin classification of the Lorentz group, reducible under the rotation subgroup, which allows us to represent massless higher spin fields ($s \geq 2$) as traceless and completely symmetric tensors (for an excellent review see [1]). Then we review how, from the Bargman and Wigner study on relativistic wave equations, one can interpret the removing of the lower spin gauge degrees of freedom a spin s possess, as the necessary conditions that lead to an energy bounded from below. In section two, in the context of quantum field theory over flat space-time, we enumerate and roughly explain some old no-go theorems concerning higher spin interactions, and some possible ways to surpass them (for excellent reviews see [2],[3]).

2.2 Wigner classification of fundamental particles by mass and spin

In an old but seminal work, using the method of induced representations ([4]), Wigner has shown how to classify fundamental particles as unitary representations of the Poincaré group in $D = 4$. He studied one-particle states in QFT [1], and showed that these states, under general Lorentz transformations, only transform non-trivially under the subgroup called the Wigner little group. To achieve this, he first used the quadratic Casimir [2] of $D = 4$ Poincaré algebra:

[1]One particle states of course are given as the eigenstates of the Hamiltonian related to a free theory. These states are labeled by the particle's momentum and possibly some quantum numbers denoting the particle's species such as spin.

[2]An usual invariant operator of a Lie algebra

$C_2 = P_\mu P^\mu = -M^2$ acting on the space of one particle states, whose eigenvalues are essentially labeled by the mass of the particle. Then, in order to classify the different classes of one particle states, i.e. non-related by a proper orthochronous [3] Lorentz Transformations [4], he used the Casimir related mass value: M, and the energy value: p_0 of its four momentum vector. The different classes of momentum which arise according this classification are:

$$p^2 = -M^2 < 0 \text{ and } p_0 < 0: \text{ Massive and Negative energy} \tag{2.1}$$
$$p^2 = -M^2 < 0 \text{ and } p_0 > 0: \text{ Massive and Positive energy} \tag{2.2}$$
$$p^2 = -M^2 = 0 \text{ and } p_0 < 0: \text{ Massless and Negative energy} \tag{2.3}$$
$$p^2 = -M^2 = 0 \text{ and } p_0 > 0: \text{ Massless and Positive energy} \tag{2.4}$$
$$p^2 = -M^2 > 0: \text{ Tachyonic imaginary mass} \tag{2.5}$$
$$p_\mu = 0$$

In a free theory, from the above classes, only (2.2) and (2.4) are considered physical [5]. Sitting in the class (2.2), corresponding to massive and positive energy particles, one can study the subgroup $\bar{\Lambda}^+$ of the proper orthochronous Lorentz transformations Λ^+, which leaves invariant a particular four-momentum representative vector $\kappa^\mu \sim p^\mu$ of this class, i.e: $\kappa^\mu = (\bar{\Lambda}^+)^\mu{}_\nu \kappa^\nu$. Without loosing generality the representative for this class of massive and positive energy particles can be chosen in the rest frame to be: $\kappa_\mu = (1, 0, 0, 0)$. The group given by $\bar{\Lambda}^+$ in this way, it is called the Wigner Little Group for massive particles and it is direct to see that it correspond to the rotation group $\bar{\Lambda}^+ = SO(3)$. This group has different $(2s+1) \times (2s+1)$-dimensional spin s representations, in which a given spin s representation posses $2s + 1$ states. Thus in $D = 4$, massive particles posses $2s + 1$ physical degrees of freedom.

For the class of massless and positive energy particles (2.4), which is a light-like class of 4-momentum, we cannot choose a rest frame, because it does not exist one in which we are able to be at rest with the particle [6], so, without loosing generality, we are forced to choose in the simplest case as some representative the vector: $\kappa_\mu = (1, 1, 0, 0)$. Then, it is direct to see that the group given by $\bar{\Lambda}^+$ in this case should contain $SO(2)$, but it is not so straight forward to see that it also should contain translations in R_2 (see [4]). Thus, in the class of massless particles, Wigner Little Group is the isometry group of R_2, usually called $ISO(2)$, which is non-compact. However, there exist a known theorem, that says that finite dimensional unitary representations of a non-compact group are not faithful, but only infinite dimensional ones are. In the case of $ISO(2)$, this would corresponds to continuous spin representations, given by the non-compact part of translations in R_2, and because we haven't observed this kind of spin in nature, we exclude these representations with continuous spin. [7]. This is as if we are left with

[3] Which preserves the direction of time

[4] An element of $SO(3,1)^+$, i.e., identity connected and orthochronous ($\Lambda^0{}_0 = 1$) Lorentz transformation which excludes parity and temporal Inversion operations, which can be seen as a pair of discrete elements belonging to the full Lorentz Group $O(3,1)$. Also, the use of $SO(3,1)^+$ instead of $O(3,1)^+$ is well founded as Wigner himself proved that any symmetry transformation, which is continuously connected with the identity, acting on the space of one particle states must be represented as a linear unitary operator acting on this space [4]

[5] up to 2.1 and 2.3 in the case of anti-particles, which are interpreted as to move backward in time, the analysis in this case is essentially the same, but it is restricted to the negative branch of frequencies ($p_0 < 0$). So we will restrict here only to the classes with positive energy.

[6] Due to its massless nature, the particle propagates with the speed of light.

[7] which basically is due to exclude the translation operators from the group which acts unfaithful over finite

only $SO(2)$ as the Wigner Little group for massless particles, and $SO(2)$ algebra has only one element, but including parity, we have two elements which leads to the well known result that massless particles in $D = 4$ only carry two degrees of freedom, i.e. the helicities $\pm h$. Their different spin representations can be given by considering topological arguments but they are beyond the scope of this section.

Also, as $SO(2)$ for massless particles, and as $SO(3)$ for massive particles, all of its different spin representations can be exhausted by traceless and completely symmetric tensors, i.e., different single row of arbitrary length Young tableaux [8], each corresponding to different spin representations.

The same analysis can be made, for the D dimensional Poincare group, case in which we have $SO(D-1)$ for massive particles, and $SO(D-2)$ for massless particles. However for $D > 4$ subtleties arise as we just cannot use only single row Young tableaux for $SO(N)$ with $N > 4$. i.e. for all the different spin representations that may exist.

With the previous analysis, we can see beforehand that spin for massive particles are not the same as spin for massless particles, which is reflected in the fact that the number of physical degrees of freedom of some given spin s field, in these two different cases does not match.

Thus, the Wigner classification of single particles as unitary irreducible representations of the Poincare group, can be reduced to the problem of classify the irreducible representations of the Wigner little group for some given class of momentum of the particle. In order to see these single particles inside the usual context of field theory, i.e., as the excitations coming from a field, we will now move onto a classification of the higher spin fields by constructing different spin representations of the full Lorentz group, but which are reducible under its rotation subgroup.

Building different spin representations of the Lorentz Group:

The homogeneous Poincare algebra or Lorentz algebra $SO(3,1)$ is given by:

$$[M_{\mu\nu}, M_{\rho\sigma}] = M_{\rho\nu}\eta_{\sigma\mu} + M_{\mu\rho}\eta_{\nu\sigma} - M_{\sigma\nu}\eta_{\rho\mu} - M_{\mu\sigma}\eta_{\nu\rho} \tag{2.6}$$

From the above generators, the Boost K_i, and rotations J_i generators can be written as:

$$K_i = M_{0\,i} \quad J_i = i\epsilon_{i\,j\,k} M_{j\,k} \tag{2.7}$$

and the Lorentz Algebra in terms of the new generators is left as:

$$[J_i, J_j] = i\epsilon_{i\,j\,k} J_k \tag{2.8}$$
$$[K_i, K_j] = -i\epsilon_{i\,j\,k} K_k \tag{2.9}$$
$$[J_i, K_j] = i\epsilon_{i\,j\,k} K_k \tag{2.10}$$

dimensional unitary representations.

[8]This is a classification of tensors under its symmetry properties, given by how do they transform under the permutation group acting on their indices.

by choosing the combinations:

$$N_i = \frac{1}{2}(J_i + iK_i) \;\; N_i^\dagger = \frac{1}{2}(J_i - iK_i) \tag{2.11}$$

the Lorentz algebra $SO(3,1)$ splits out as two hermitian conjugated copys of $SU(2)$ algebra, that do not see each other:

$$[N_i, N_j] = i\epsilon_{ijk}N_k \tag{2.12}$$
$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger \tag{2.13}$$
$$[N_i, N_j^\dagger] = 0 \tag{2.14}$$

So, we have the following isomorphism at algebra level $SO(3,1) \sim SU(2) \oplus SU(2)$.

In order to build different spin representations of $SO(3,1)$, we can use the different spin representations of $SU(2)$. The representations of $SU(2)$ are well known, and we can use a Cartan basis to work with it. This traduces in choosing some combinations of the generators of $SU(2)$ as follows: we will have one Cartan operator which is diagonal in this basis with weight s, and two lader operators which upper and lower the weights of the basis. We choose the highest weight basis such that the upper operator acting on a highest weight state annihilates it.
Then for some given highest weight s, $s = 0, 1/2, 1, \ldots$, we can find a $(2s+1)\times(2s+1)$ dimensional representation of $SU(2)$, in which the Cartan operator has the eigenvalues: $-s, -s+1, \ldots, +s$, because it is represented in a spin s basis composed by $2s+1$ states. In this way we can exhaust all the different spin s representations of $SU(2)$.

We do the same for the other copy of $SU(2)$, and use the pair (s_1, s_2) to denote the highest weights of the two different $SU(2)$ representations, and also to denote a possible $SO(3,1)$ representation. Given that the rotation operator is given by $J_i = N_i + N_i^\dagger$, we can use the standard rules of adding angular momentum to build the different possible spin representations of $SO(3,1) \sim SU(2) \oplus SU(2)$ algebra. Also, given that at the Lie Algebra level $SO(3,1)$ is the direct sum of the two $SU(2)$, the possible highest weight basis of $SO(3,1)$ will be given by the sum of the highest weights of $SU(2)$, i.e. $s_1 + s_2$, which has allowed values for the total angular momentum given by $j = \mid s_1 - s_2 \mid, \mid s_1 - s_2 \mid + 1, \ldots, s_1 + s_2$.

One cannot use arbitrary values of (s_1, s_2) for labelling possible representations of $SO(3,1)$. We can see this by considering that the space-time inversion operator V has to be included on the full Lorentz Group. This operator leaves invariant even-rank tensors, but changes the sign when act on odd-rank tensors. It is the inclusion of this discrete operator that, in order for this operation exist, leads to the constraint on the spectrum of the two Cartan operators for each copy of $SU(2)$ to be equal. This constraint, in terms of the highest weights values of the two $SU(2)$ representation, it is traduced in the two allowed cases for the representations of the Lorentz Algebra:

- 1) $(\frac{s}{2}, \frac{s}{2})$ representation gives a rise to a spin s bosonic representations of $SO(3,1)$ in which the allowed values of total angular momentum are $j = 0, 1, 2, 3 \ldots, s$ and possible values of J_3 component are $m = -j, \ldots, j$

- 2) $(\frac{s}{2}, \frac{s+1}{2}) \oplus (\frac{s+1}{2}, \frac{s}{2})$ representation gives a rise to a spin $\frac{s}{2}$ fermionic representations of $SO(3,1)$ with the allowed values of total angular momentum $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{s}{2}$, so possible values of J_3 component are $m = -j, \ldots, j$

With this at hand, in general we see that a spin s bosonic, ($\frac{s}{2}$ fermionic) field contains invariant subspaces containing all integer (half-integer) lower spins up to and including s ($\frac{s}{2}$), thus these representations are reducible under the rotation subgroup. From now on, and along all the text, when we refer to the phrase 'invariant subspaces' we will refer that they are invariant with respect to the rotation subgroup.

Now, we will see that the representations $(\frac{s}{2}, \frac{s}{2})$ are rank-s completely symmetric space-time tensors and traceless in any pair of space-time index. The vector representation $(\frac{1}{2}, \frac{1}{2})$ of $SO(3,1)$, can be constructed by using the fundamental representation of $SU(2)$ given by the Pauli matrices. The elements in which the fundamental representation of $SU(2) \times SU(2)$ acts are two-index spinor tensor fields $\Psi^{\alpha\dot{\alpha}}$, which are related to the vector representation via the Pauli matrices as: $\phi^{\mu} = \sigma^{\mu}_{\alpha\dot{\alpha}}\phi^{\alpha\dot{\alpha}}$. Also, the representations of $(\frac{s}{2}, \frac{s}{2})$ acts on $2n$-index spinor tensor fields $\phi^{\alpha_1 \ldots \alpha_n, \dot{\alpha}_1 \ldots \dot{\alpha}_n}$ for all $n \leq s$. They are symmetric in all the α_i, and symmetric in all the $\dot{\alpha}_i$ spinor indices. By using Pauli matrices $\sigma^{\mu}_{\alpha_n\dot{\alpha}_n}$ to contract each pair of α and $\dot{\alpha}$ indices, and taking the direct sum of all the spinor representations with $n \leq s$, this results in a rank-s space-time symmetric tensor field representation as:

$$\phi^{\mu_1 \ldots \mu_s} = \sigma^{\mu_1}_{\alpha_1\dot{\alpha}_1} \cdots \sigma^{\mu_s}_{\alpha_s\dot{\alpha}_s}\phi^{\alpha_1\dot{\alpha}_1 \ldots \alpha_s\dot{\alpha}_s} \tag{2.15}$$

Using the standard properties of Pauli matrices and the rules for spinor calculus, it can be shown they are also traceless in each pair of space-time indices ([5]).

In an analogous fashion, one can show that the half integer spin representations of $SO(3,1)$ are spinor tensor fields, symmetric in the space-time indices, and of rank $s - \frac{1}{2}$, which in particular for the spin s representation are given by: $\Psi^{\alpha}_{\mu_1 \ldots \mu_{s-\frac{1}{2}}}$, and satisfy a gamma-traceless condition: $\gamma^{\nu}_{\alpha\dot{\beta}}\Psi^{\alpha}_{\nu\,\mu_1 \ldots \mu_{s-\frac{3}{2}}} = 0$

Then, since the work of Bargman and Wigner ([6]) on the classification of relativistic wave equations, it becomes clear that in order to eliminate the lower spin invariant subspaces that a bosonic rank-s tensor field has, one has to impose the divergence/ transversality condition on this field:

$$\partial^{\nu}\phi_{\nu\mu_1 \ldots \mu_{s-1}} = 0 \tag{2.16}$$

And an analogous condition for the half integer spin fermionic fields, given by:

$$\partial^{\nu}\psi^{\alpha}_{\nu\mu_1 \ldots \mu_{s-1}} = 0 \tag{2.17}$$

Such constraints imposes naturally that the lower spin $s-1$ degrees of freedom given by this divergence be eliminated, which when analyzed in the Lagrangian context for a spin s field, as Fierz and Pauli ([5]) shown before Bargman and Wigner in this context, they traduces in a necessary condition for the positivity of the energy of the fields.

Lastly, in ([6]) was shown that having gauged away the lower spin gauge degrees of freedom that can exist inside a massless spin s field, and using the quadratic Casimir operator of the Poincare algebra applied on it, one have that the free massless field has to satisfy the on mass shell conditions:

$$\Box\phi_{\mu_1\ldots\mu_s} = 0 \tag{2.18}$$

and similarly for fermions, which traduces in :

$$\gamma^{\nu}_{\alpha\dot{\beta}}\partial_\nu \Psi^{\alpha}_{\mu_1\ldots\mu_{s-\frac{1}{2}}} = 0 \tag{2.19}$$

Together the divergence (transversality) condition [9], with the on mass shell conditions, are the so called 'physicality conditions' for the single traceless (gamma-traceless) bosonic (fermionic) fields.

2.3 Obstructions to higher spin interactions in flat QFT (No-Go theorems)

As we saw above, mainly from a group theoretical point of view, free massive and massless fields of arbitrary high spin, are allowed to exist in which their physical part are given by unitary irreducible representation of the Poincare group. In fact, as we will see in chapter (3) in the context of a Lagrangian derived theory, the free theory for massive and massless higher spin field is well posed and constructed. Even if higher spin fields has not been observed in nature at the present scales of energies we manage in the laboratory, this does not means that they cannot exist. The problem with higher spin begins when one try to introduce interactions of higher spin fields with any kind of other field (for an excellent review see [2, 3]). In this subsection we will roughly review the main problems which arise with a higher spin interacting theory in the context of usual quantum field theory over flat space-time.

2.3.1 Trivial scattering matrix obstruction (Coleman-Mandula no-go theorem)

The Coleman-Mandula theorem [7] on all the possible symmetries of the S-Matrix, and the Haag-Lopuszanski-Sohnius extension of the above [8] for super-symmetric theories, under its restricted but usual assumptions, put several constraints on the kind of symmetries that an interacting theory defined in flat space-time can have, in order for its S-Matrix to be non-trivial. Coleman-Mandula theorem essentially states that if one try to combine the Poincare symmetries with another group of internal symmetries, in order for the theory be non-trivial, i.e., the scattering matrix be different from one, then the only allowed form for the extended symmetry group, is given by the direct product of external (Poincare) and internal (Bosonic) symmetries. Turning this analisys into graded Lie Algebras, Haag-Lopuszansk-Sohnius have shown that one

[9] Here we are working with massless fields, and as such, these are only some possible gauge conditions, but for a massive theory, these conditions should be understood as the natural conditions, i.e., that follows from the equations of motion for the massive field. As an example, Lorentz condition follows directly from the Proca equations of motion in the case of the massive spin 1 field.

can also have super-symmetry as another ingredient of the puzzle, now using the super-Poincare algebra extension of the Poincare group, and in which the bosonic internal symmetry algebra is commuting with the elements of the Poincare Algebra but not with the extended elements of the super-algebra.

In the massless bosonic higher spin context, by definition, there exist some bosonic higher spin symmetries, thus the higher spin bosonic generators of that symmetries must exist, but these generators turns out to be non-commuting with the elements of the Poincare algebra. Therefore in general we see, that at difference with respect to the lower spin scenario, the symmetry of the higher spin theory will not be one which is the direct product of bosonic higher spin generators time Poincare group nor even a super-symmetric extension of it, and in this way higher spins are automatically ruled out by Coleman-Mandula or by its Haag-Lopuszanski-Sohnius extension (see [2]) .
However, this theorem can be circumvented in other background space-times such as Anti de Sitter space, in which there is no notion of asymptotic states where the scattering matrix can make some sense. In that kind of spaces-times however there exist other types of observables which can be defined making use of conformal field theories which lives on the boundary of the AdS space.

2.3.2 Highly constrained values of coupling and/or momenta due to conservation laws (Weinberg low energy no-go theorem)

Weinberg low energy theorem [9] basically puts restrictions on the kind of bosonic massless particles that can interact at low energies, with each other and with itself. This is a no-go theorem only for low energy interactions. For higher spin particles it basically says that no higher spin particle with $s > 2$ can interact at low energy.
Weinberg low energy theorem goes as follows, consider we have a non trivial scattering process which involves N external particles with ingoing momenta p_j and spin s_j $(j = 1, \ldots, N)$, in which we have that an additional bosonic particle of spin s is absorbed with an arbitrary but soft momentum q at the ith external leg (see figure 2.1).

The scattering matrix element in which this process occurs (figure 2.1), posses a part that controls the absorption of the spin s massless particle given by a vertex of type $s - s_j - s_j$ with coupling constant g_j, that at low energy can be factored from the rest of the process as show in figure 2.2

The spin s polarization $\varepsilon_{\mu_1 \ldots \mu_s}(q)$ that appears in 2.2 is not Lorentz covariant. Under a Lorentz transformation it transforms as:

$$\varepsilon_{\mu_1 \ldots \mu_s}(q) \rightarrow \varepsilon_{\mu_1 \ldots \mu_s}(q) + q_{(\mu 1}\chi_{\mu_2 \ldots \mu_s)}$$

where $\chi_{\mu_1 \ldots \mu_{s-1}}$ is a symmetric tensor of rank $(s - 1)$.

Then, to eliminate the contribution coming from the spurious lower spin gauge degrees of freedom with spin: $0, \ldots s - 1$ pertaining to the spin s emitted particle, we must demand that the S-matrix be Lorentz invariant. This is accomplished with the following condition, which

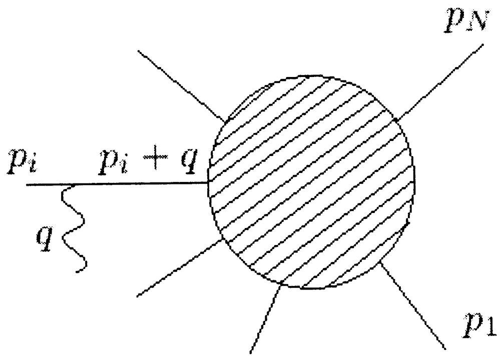

Figure 2.1: Scattering matrix element corresponding to the absortion of a soft spin s particle with momentum q

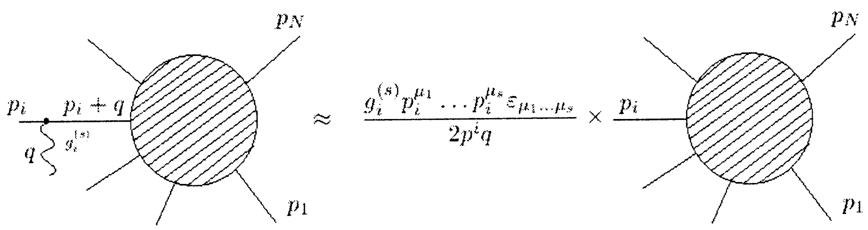

Figure 2.2: Factorization of the process at low energy

involves the couplings and incoming momentum as:

$$\sum_{i=1}^{N} g_i^{(s)} p_i^{\mu_1} \dots p_i^{\mu_{s-1}} = 0, \qquad \forall p_i \tag{2.21}$$

The above relation for spin $s = 1$ reduces just to the charge conservation: $\sum_{i=1}^{N} g_i^{(1)} = 0$, as is well known in quantum electrodynamics. For a spin $s = 2$, it reduces to the relation $\sum_{i=1}^{N} g_i^{(2)} p_i^{\mu_1} = 0, \ \forall p_i$, and considering the momentum conservation law for the momentum of the incoming particles, given by $\sum_{i=1}^{N} p_i^{\mu_1} = 0$, it reduces just to $g_i^{(2)} = g^{(2)}$, which states that the coupling of any particle with the graviton field must be the same. As Weinberg states it, this is the counterpart of the equivalence principle in quantum field theory.

For spin $s \geq 3$, the equation (2.21) has no solution for the arbitrary incoming momenta, which automatically leads to $g_i^{(s)} = 0, \ s > 2$. Thus, this relation states that no higher spin particle can interact at low energy. i.e. mediate long range interactions.

In particular this theorem do not rules out higher spin interactions unless we demand that the same Lagrangian should describes the physics of both IR and UV sectors.

2.3.3 No higher spin Lorentz covariant conserved currents and nor gauge invariant (Weinberg-Witten no-go theorem)

Weinberg-Witten theorem [10] as they states, says:

- Theorem 1). A theory that allows the construction of a Lorentz-covariant conserved four-vector current J_ν cannot contain massless particles of spin $j > 1/2$ with non-vanishing values of the conserved charge $\int J_0 d^3x$

- Theorem 2). A theory that allows the construction of a conserved Lorentz covariant energy-momentum tensor $T_{\mu\nu}$ for which $\int T_{0\nu} d^3x$ is the energy-momentum four-vector cannot contain massless particles of spin $j > 1$.

This theorem put constraints on the kind of allowed vertex interactions that some kind of theories can have. It is usually applied to abelian gauge theories which makes use of minimal coupling prescription between some kind of massless spin s_1 field, and the spin s_1 conserved Noether currents coming from some massless spin s_2 field.

For example, in the case of electrodynamics and gravitation, theorem 2) states that effectively a Lorentz covariant and gauge invariant and conserved energy momentum tensor (spin 2 Noether current) can be constructed for the electromagnetic field (spin 1 field), and thus the energy of the field can be localized, which allows the corresponding minimal coupling to the graviton (spin 2 field).

Another example, is in the case of the massless graviton, where 2) already states that a Lorentz covariant and gauge invariant, at the same time, energy momentum tensor (spin 2 current) for the graviton itself, cannot be constructed, and thus the energy of the field cannot be localized, forbidding minimal coupling to itself. But as we know, this does not forbids to the graviton interact with itself in a non-minimally way, the example of this is the existence of general relativity. [10]

This theorem in particular for the graviton $s = 2$ and also for higher spin fields $s > 2$ rules out the possibility of construct a gauge invariant and Lorentz covariant energy momentum tensor for these fields, which rules out the minimal coupling to gravitation as a valid prescription for these fields.

[10]Another way to see this is that the covariantization of the Fierz-Pauli Lagrangian, which must be see as the minimal coupling prescription for a spin 2, does not give us back general relativity as a result.

Chapter 3

Review of Free bosonic massless higher spin fields over a fixed maximally symmetric background

3.1 Introduction

In this chapter we will focus on free bosonic higher spin fields, because this thesis directly works with them, so from here on, fermionic fields will be left aside. More-over we will be interested on free massless bosonic higher spin fields. This is because the free massive higher spin field theory in D dimensions can be directly derived from the free massless higher spin field theory in $D+1$ dimensions, via Kaluza-Klein reduction, i.e., compactifying the extra dimension, as a Wigner little group analysis indirectly suggest. In this chapter we will expose the most straightforward approach to get into free massless higher spins which is principally due to Fronsdal [11]. However, for the sake of historical completeness (and also brevity), we will give only the strange historical context on how Fronsdal get its theory for free massless higher spin fields, by taking the massless limit of an interacting theory of massive higher spin fields with the electromagnetic field. It was Fierz who first work on Higher Spin in ([12]), then Fierz and Pauli [5] try to attack the problem of the coupling of a massive spin s field with the spin 1 electromagnetic field at the level of the equations of motions and of the physicality conditions, by directly replacing partial derivatives with spin 1 covariant ones, spotting that several inconsistencies arise because that is not a proper modification to account for interactions, instead this leads to algebraically inconsistent equations. Then Singh and Hagen ([13]) were successful in attack this problem by getting the equations of motion, and physicality conditions, directly from a more general Lagrangian. In order to do this for a spin s field, they introduced physicality conditions via suitable Lagrange multipliers, and also introduce some auxiliary lower spin fields with spins from $0 \ldots s-1$ in the game, to then impose that these lower spin fields be turned off when interactions are absent. To this end, they fixed the value of the Lagrange multipliers a posteriori, such that at on shell level, the equations of motions gives the correct physicality conditions, which reduces to the Bargman and Wigner ones, mean at the same time, the lower auxiliary fields are turned off when interactions are turned off, all in a consistent way. Finally comes Fronsdal [11], who take the massless limit of the Singh and Hagen Lagrangian to realize that all the auxiliary lower spin fields decouples, except the auxiliary field of spin $s-2$, which he combined with the single traceless spin s field into only one, symmetric, but now double traceless rank s tensor field. Then due to the massless nature of the fields, the free higher spin theory was settled as a gauge theory, thus Fronsdal payed the price of having to tackled its gauge symmetry problem. Fortunately, after Fronsdal work, the theory took a simpler form than before, and its grounds were then well posed. For excellent reviews see the works [14], [15]. Finally, Fronsdal equations of motion can be written in a very algorithmic way for any spin, with the only requirement they satisfy some symmetry preserving criteria (gauge invariance).

3.2 Free bosonic Fronsdal's higher spin fields in flat space

3.2.1 Fronsdal's equations of motion

For a spin s Bosonic field (rank s symmetric tensor field), Fronsdal equations of motion in D-dimensional Minkowski space-time are:

$$\mathcal{F}_{\mu_1\ldots\mu_s} \equiv \Box\Phi_{\mu_1\ldots\mu_s} - s\partial_{(\mu_1}\partial_\nu\Phi^\nu{}_{\mu_2\ldots\mu_s)} + \frac{s(s-1)}{2}\partial_{(\mu_1}\partial_{\mu_2}\Phi^\nu{}_{\nu\,\mu_3\ldots\mu_s)} = 0 \tag{3.1}$$

Where the left hand side of (3.1) defines the Fronsdal tensor. From here on, a pair of parentheses will denote a complete symmetrization of all the non-contracted indices it enclose, divided by the total number of terms used for the complete symmetrization.

These equations are a natural generalization of the equations of motion for lower spin massless bosonic fields:

$$s = 0,\ \text{Scalar},\ \Box\Phi = 0,\ \text{No gauge symmetries} \tag{3.2}$$

$$s = 1,\ \text{Vector},\ \Box\Phi_\mu - \partial_\mu\partial_\nu\Phi^\nu = 0,\ \text{Gauge symm.:}\ \delta\Phi_\mu = \partial_\mu\epsilon \tag{3.3}$$

$$s = 2,\ \text{Tensor},\ \Box\Phi_{\mu\nu} - 2\partial_\alpha\partial_{(\mu}\Phi^\alpha{}_{\nu)} + \partial_\mu\partial_\nu\Phi^\alpha{}_\alpha = 0,\ \text{Gauge symm.:}\ \delta\Phi_{\mu\nu} = 2\partial_{(\mu}\epsilon_{\nu)} \tag{3.4}$$

The equation 3.1 is left invariant under the gauge transformation:

$$\delta\Phi_{\mu_1\ldots\mu_s} = s\partial_{(\mu_1}\varepsilon_{\mu_2\ldots\mu_s)} \tag{3.5}$$

with a rank $s-1$ symmetric tensor gauge parameter $\varepsilon_{\mu_1\ldots\mu_{s-1}}$, which is traceless in any pair of indices. i. e.:

$$\varepsilon^\nu{}_{\nu\mu_1\ldots\mu_{s-3}} = 0 \tag{3.6}$$

as can be seen by a direct calculation:

$$\delta\mathcal{F}_{\mu_1\ldots\mu_s} = \frac{1}{2}s(s-1)(s-2)\partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\varepsilon^\nu{}_{\mu_4\ldots\mu_s)\nu} = 0 \tag{3.7}$$

Clearly this condition on the gauge parameter becomes relevant for $s \geq 3$.

If we want to be able to build a local gauge invariant [1] action under the gauge transformation 3.5, which of course give us the equations of motions (3.1) when arbitrary varied, we need one more restriction on this system. This restriction consist in that the spin s field has to be double-traceless in any pair of indices [16], but of course this has to be imposed "off mass shell":

[1] up to a total derivative or boundary term

$$\Phi^{\nu_1\nu_2}{}_{\nu_1\nu_2\mu_1\ldots\mu_{s-4}} = 0 \tag{3.8}$$

This condition, obviously becomes relevant for $s \geq 4$, and is directly related to the fact that, due that the imposed gauge invariance of the theory, at the action level i.e. off shell, demands that exist some generalized Bianchi identities that has to be identically satisfied, as we will see below.

The gauge invariant action under the gauge transformation 3.5 is defined as:

$$S = \int d^d x \Phi^{\mu_1\ldots\mu_s}\mathcal{G}_{\mu_1\ldots\mu_s} \tag{3.9}$$

Where:

$$\mathcal{G}_{\mu_1\ldots\mu_s} \equiv \mathcal{F}_{\mu_1\ldots\mu_s} - \frac{1}{4}s(s-1)\eta_{(\mu_1\mu_2}\mathcal{F}^{\gamma}_{\mu_3\ldots\mu_s)\gamma} \tag{3.10}$$

Using the above definition one can see that the following is satisfied:

$$\partial^\gamma \mathcal{G}_{\gamma\mu_2\ldots\mu_s} \propto \partial_{(\mu_2}\partial_{\mu_3}\partial_{\mu_4}\Phi^{\nu_1\nu_2}{}_{\nu_1\nu_2\mu_5\ldots\mu_s)} \tag{3.11}$$

In which it can be see, that imposing 3.8 we have the following Bianchi identities:

$$\partial^\gamma \mathcal{G}_{\gamma\mu_2\ldots\mu_s} = 0 \tag{3.12}$$

With these at hand, the gauge invariance of the action under 3.5 up to a boundary term follows as [2]:

$$\begin{aligned} \delta S &= \int d^d x \delta\Phi^{\mu_1\ldots\mu_s}\mathcal{G}_{\mu_1\ldots\mu_s} + \Phi^{\mu_1\ldots\mu_s}\delta\mathcal{G}_{\mu_1\ldots\mu_s} && (3.13)\\ &= \int d^d x s\partial^{(\mu_1}\epsilon^{\mu_2\ldots\mu_s)}\mathcal{G}_{\mu_1\ldots\mu_s} + 0 && (3.14)\\ &= b.t. + \int d^d x \epsilon^{\mu_2\ldots\mu_s}\partial^\gamma\mathcal{G}_{\gamma\mu_2\ldots\mu_s} = b.t. + 0 && (3.15)\\ & && (3.16) \end{aligned}$$

The equations of motion follows from varying with respect to ϕ as:

$$\mathcal{G}_{\mu_1\ldots\mu_s} \equiv \mathcal{F}_{\mu_1\ldots\mu_s} - \frac{1}{4}s(s-1)\eta_{(\mu_1\mu_2}\mathcal{F}^{\gamma}_{\mu_3\ldots\mu_s)\gamma} = 0 \tag{3.17}$$

and considering 3.8 the off shell double trace of the Fronsdal tensor vanish, i.e, $\mathcal{F}^{\alpha\beta}{}_{\alpha\beta\mu_5\ldots\mu_s} = 0$. Thus, taking the single-trace of the e.o.m. 3.17, and using the vanishing double trace identity of the Fronsdal tensor, we are left with the on shell equation:

$$\mathcal{G}_{\mu_3\ldots\mu_s\gamma}{}^\gamma = (1 - \frac{d}{4}s(s-1))\mathcal{F}_{\mu_3\ldots\mu_s\gamma}{}^\gamma = 0 \tag{3.18}$$

[2] where from the first to second line, we have made use explicit of the gauge invariance of 3.10 tensor trough the gauge invariance of the Fronsdal tensor under 3.5, and in the third line we have made an integral by parts, and then used 3.12.

Which for arbitrary d and s, (except for the special case $d = 2, s = 2$), has as solution the vanishing of the on shell single trace of the Fronsdal tensor: $\mathcal{F}_{\mu_3 \dots \mu_s \gamma}{}^{\gamma} = 0$, which replacing into 3.17, leads to the equivalent equation of motion:

$$\mathcal{G}_{\mu_1 \dots \mu_s} = \mathcal{F}_{\mu_1 \dots \mu_s} = 0 \tag{3.19}$$

For the interested reader, there exist a formulation due to Francia and Sagnotti ([17],[18]), which makes no use of a constrained gauge parameter as 3.6, neither makes use of a constrained Fronsdal field as 3.8, motivated by the contact between String Theory and Higher Spin Fields, in which these restrictions seems rather unnatural, but this formulation presents other highly complex issues as now the action principle should be non-local and higher derivative and it is beyond the scope of this thesis, so we won't insist on this.

3.2.2 Gauge degrees of freedom and the de Donder-gauge fixing

In D-dimensions, a symmetric rank-s tensor field has $C(D-1+s, s)$ [3] number of independent components. The double trace of a rank s tensor field, is a rank $s-4$ tensor field, imposing this double trace to be zero, we have $C(D-1+s-4, s-4)$ conditions. So in principle a Fronsdal massless spin s field has a total of $C(D-1+s, s) - C(D-1+s-4, s-4)$ independent components.

But we have gauge symmetries, so not all this components are physical, we have redundant mathematical unphysical information we want to gauge away. For this we can perform a gauge transformation with a symmetric and single-traceless rank $s-1$ tensor gauge parameter, which carries a number of $C(D-1+s-1, s-1) - C(D-1+s-3, s-3)$ independent components, and in this way fix this same number of components inside the spin s field by choosing some gauge condition.

With respect to the gauge condition, a natural generalization of the Lorentz covariant gauge fixing for $s = 1$, and of the de Donder covariant gauge fixing for $s = 2$, it is the generalized de Donder covariant gauge condition for arbitrary spin:

$$H_{\mu_2 \dots \mu_s} \equiv \partial^{\gamma} \Phi_{\gamma \mu_2 \dots \mu_s} - \frac{1}{2}(s-1) \partial_{(\mu_2} \Phi^{\gamma}_{\mu_3 \dots \mu_s)\gamma} = 0 \tag{3.20}$$

This reduces to the Lorentz gauge for $s = 1$, and to de Donder gauge for $s = 2$.

With this gauge fixing condition, the equations of motion are left as a wave equation for the spin s field:

$$\Box \Phi_{\mu_1 \dots \mu_s} = 0 \tag{3.21}$$

And, in fact we see it describes a massless field.

However, this gauge fixing condition does not fix the gauge completely, because we can still perform a gauge transformation whose gauge parameters satisfy:

$$\Box \varepsilon_{\mu_1 \dots \mu_{s-1}} = 0 \tag{3.22}$$

[3]This is the binomial coefficient $C(n, k) = \binom{n}{k}$

In which clearly we have another set of $C(D-1+s-1,s-1)-C(D-1+s-3,s-3)$ components that we can gauge away to completely fix the gauge.

So the total number of propagating degrees of freedom is:

$$\begin{aligned} DoF(s,D) &= C(D-1+s,s)-C(D-1+s-4,s-4) \\ &\quad -2\times(C(D-1+s-1,s-1)-C(D-1+s-3,s-3)) \\ &= C(D-5+s,s)+2C(D-5+s,s-1) \end{aligned} \tag{3.23}$$

In particular we see that in $D=4$ for any $s>0$. i.e. only gauge fields [4] we have $DoF(s,4)=2$, which is the well known result that massless gauge fields carry only two physical propagating degrees of freedom. i.e. the helicities $\pm h$.
Another important particular case that follows is that in $D=3$, for any $s>0$ we have $DoF(s,3)=0$, which is related to, the well known fact, that in three dimensions the little group of massless particles $ISO(1)$ is trivial, but including parity we have as the little group $\{1,-1\}\times\Re$, and excluding continuous spin degrees of freedom ($\Re$), we are only allowed to distinguish between bosons and fermions with $\{1,-1\}$ part. These cases are well known results in field theory.

Thus, the off shell vanishing doubly-trace condition on the spin s field, together with the generalized de Donder gauge fixing, are necessary to eliminated all the lower spin $s-1$ invariant subspaces a spin s field may have.

By using a further gauge condition due to ([19]), a considerably simplification with respect to on-shell fields can be done. This preserves the generalized de Donde gauge condition and one recovers the Bargman-Wigner form of the fields, this is the vanishing single trace condition on the spin s field, in which the generalized de Donder gauge reads simply as the transversality condition:

$$H_{\mu_2\ldots\mu_s}\equiv\partial^\gamma\Phi_{\gamma\mu_2\ldots\mu_s}=0 \tag{3.24}$$

And the equations of motion are still 3.21, but now considering the spin s field is single traceless, i.e: $\Phi^\gamma{}_{\gamma\mu_3\ldots\mu_s}=0$.

3.3 Free bosonic Fronsdal's higher spin fields over fixed (A)dS space

In the last section Fronsdal theory was presented in a flat Minkowski background. In this section we will move into the other two possible classes of a maximally symmetric backgrounds. i.e. de Sitter and Anti de Sitter space. The reason for choose this class of backgrounds, resides in the fact their Weyl tensor are zero, and furthermore, due that the tracefull parts of the Riemann tensor can be completely expressed in terms of the metric and the cosmological constant. This requirement as a necessity, is easily explained as follows, if we move into a arbitrary but fixed (non-dynamical) curved background to put the free theory to live in, in principle we just have to covariantize all off the expressions with respect to the background. We are not adding any

[4] we are excluding the spin $s=0$ scalar boson, which has no gauge symmetries and thus it is considered as a matter field

interactions, we are just changing the background. This demands to promote partial derivatives to covariant ones, but covariant derivatives no longer commute. This is a problem because when we take gauge variations of the covariantized spin s action, terms proportional to the full Riemann tensor will appear, which will eventually spoil the gauge invariance of the theory, in such a way that it wont be possible to be recovered unless we have chosen some background which has vanishing Weyl tensor, which in the case of a maximally symmetric background, Riemann tensor is completely determined in terms of the background metric and the cosmological constant, which guaranties we will be able to modify the action (and thus the equations of motions derived from it), by adding some suitable terms which keep alive the gauge invariance of the free covariantized theory. One part of this same problem shows up when we try to add interactions but we will cover this in the next chapter.

More explicitly, due that covariant derivatives no longer commute, when we take the gauge variation of the (A)dS covariantized Fronsdal tensor, i.e the tensor (3.1) but with all its partial derivatives changed to (A)dS covariant derivatives defined in the sequel by $\bar{\nabla}$, under the (AdS) covariantized gauge transformation:

$$\delta\Phi_{\mu_1\ldots\mu_s} = s\bar{\nabla}_{(\mu_1}\varepsilon_{\mu_2\ldots\mu_s)} \tag{3.25}$$

the covariantized Fronsdal tensor is no longer gauge invariant:

$$\delta\mathcal{F}_{\mu_1\ldots\mu_s} \neq 0 \tag{3.26}$$

In order to recover gauge invariance of the covariant Fronsdal tensor, we have to modify it by adding to it some terms which ensures its gauge invariance, giving rise to a modified covariant Fronsdal tensor:

$$\hat{\mathcal{F}}_{\mu_1\ldots\mu_s} = \mathcal{F}_{\mu_1\ldots\mu_s} + \Lambda((s^2+(D-6)s-2(D-3))\Phi_{\mu_1\ldots\mu_s} + s(s-1)\bar{g}_{(\mu_1\mu_2}\Phi_{\mu_3\ldots\mu_s)\gamma}{}^{\gamma}) \tag{3.27}$$

where $g_{\bar{\mu}\nu}$ is the (A)dS background metric and where $\Lambda = -\frac{1}{l^2}$ for AdS, and $\Lambda = \frac{1}{l^2}$ for dS space-times.

The modified Fronsdal tensor, now results to be gauge invariant under 3.25, i.e:

$$\delta\hat{\mathcal{F}}_{\mu_1\ldots\mu_s} = 0 \tag{3.28}$$

And the gauge invariant action under the gauge transformation 3.25 is defined as:

$$S = \int d^dx\sqrt{-\bar{g}}\,\Phi^{\mu_1\ldots\mu_s}\hat{\mathcal{G}}_{\mu_1\ldots\mu_s} \tag{3.29}$$

Where:

$$\hat{\mathcal{G}}_{\mu_1\ldots\mu_s} \equiv \hat{\mathcal{F}}_{\mu_1\ldots\mu_s} - \frac{1}{4}s(s-1)\eta_{(\mu_1\mu_2}\hat{\mathcal{F}}^{\gamma}_{\mu_3\ldots\mu_s)\gamma} \tag{3.30}$$

This gauge invariant action results to be uniquely defined up to a boundary term, and under some ordering convention for the covariant derivatives.
The equations of motions are now:

$$\hat{\mathcal{G}}_{\mu_1\ldots\mu_s} = 0 \tag{3.31}$$

Which, analogously to the last section, they are simplified to:

$$\hat{\mathcal{F}}_{\mu_1\ldots\mu_s} = 0 \tag{3.32}$$

The counting of propagating degrees of freedom remains exactly the same as in the case of flat backgrounds, as it should be. But now the generalized covariant de Donder gauge fixing condition is:

$$H_{\mu_2\ldots\mu_s} \equiv \bar{\nabla}^\gamma \Phi_{\gamma\mu_2\ldots\mu_s} - \frac{1}{2}(s-1)\bar{\nabla}_{(\mu_2}\Phi^\gamma_{\mu_3\ldots\mu_s)\gamma} = 0 \tag{3.33}$$

Which leads to the gauge fixed equations of motions as:

$$\Box\Phi_{\mu_1\ldots\mu_s} + \Lambda((s^2+(d-6)s-2(d-3))\Phi_{\mu_1\ldots\mu_s} + s(s-1)\bar{g}_{(\mu_1\mu_2}\Phi_{\mu_3\ldots\mu_s)\gamma}{}^\gamma) = 0 \tag{3.34}$$

In which we can see that the coefficients $m_1 \equiv \Lambda((s^2+(d-6)s-2(d-3))$ and $m_2 \equiv \Lambda s(s-1)$ plays the role of something like mass terms [20] for the spin s gauge field in (A)dS. To interpret this, we can see that the deformed covariant Fronsdal tensor (3.27) has acquired a part proportional to the cosmological constant, which is linear in the spin s field. This will reflects itself like a "mass terms" in the Lagrangian when we have moved from flat into (A)dS background. However, gauge symmetries has not been lost in the process, they are still there. Thus in (A)dS backgrounds we can still talk about massless or gauge fields even having present in the Lagrangian, or in the equations of motions, something like a "mass term" for them if its form is given by m_1 and m_2. These are called Fronsdal masses.

As in the flat case, staying on the generalized de Donder gauge condition, for on shell fields, one can further gauge away the trace of the spin s field, and the generalized de Donder gauge reduces to transversality condition in (A)dS:

$$H_{\mu_2\ldots\mu_s} \equiv \bar{\nabla}^\gamma \Phi_{\gamma\mu_2\ldots\mu_s} = 0 \tag{3.35}$$

The equations of motions are reduced to:

$$\Box\Phi_{\mu_1\ldots\mu_s} + \Lambda((s^2+(d-6)s-2(d-3))\Phi_{\mu_1\ldots\mu_s} = 0 \tag{3.36}$$

Thus, the single-traceless, transversality and 3.36 are the physicality conditions for on shell fields in (A)dS.

The mass-like term $m_1 \equiv \Lambda((s^2+(d-6)s-2(d-3))$ in (3.36) , also appears in the action (3.29) trough (3.27) and (3.30), and is responsible for what is called the Breitenlohner-Freedman bound of the energy from below [21].

Chapter 4

Review of Higher spins and inconsistency in a minimally coupled system

4.1 Massless higher spins interacting minimally with gravity

Attempts to introduce interactions of fields with spin $s \geq 2$ with the gravitational field has been shown to be plagued with inconsistencies. The gauge symmetries of the original non-interacting sectors, when minimally coupled, are spoiled in such a way that it is not possible to recover gauge invariance, neither in a deformed way. Aragone-Deser spots this fact for a spin 2 minimally coupled to gravity in ([22]), and in the hyper-gravity context ([23],[24]) for a spin $\frac{5}{2}$. In the special case of super-gravity, i.e. for a spin $\frac{3}{2}$ Rarita-Schwinger field, the coupling to gravity pass the test, due to some very special properties as the Fierz rearrangement identities of the gamma matrices which allows to build a gauge invariant action. However, in general, for fields with $s \geq 2$ a gravitational minimal coupling prescription is condemned to fail.

4.1.1 Aragone-Deser obstruction to preserve higher spin symmetries in the minimal coupling prescription in $D > 3$

Consider we are in D-dimensions, If we try to couple a spin s field to gravity minimally. i.e. using the usual prescription that follows:

- 1) Take the free spin s field action defined over flat background. Then covariantize the free action over what will be the arbitrary, but now dynamical gravitational field, i.e. promoting partial derivatives to covariant ones, and promoting the integral over flat coordinates to arbitrary ones, i.e. with its respective Jacobian in terms of the metric.

- 2) Add this resulting spin s Lagrangian to the gravitational one, i.e. Einstein-Hilbert Lagrangian.

then we will see that, in general for $s \geq 2$, several inconsistencies arises, given rise to a non-consistent interacting theory. This means that the interacting theory do not posses gauge symmetries: nor the original ones, neither a somehow deformed symmetries from the original ones which comes from the free system, as one naively would expect. Thus we have introduced the interactions inconsistently, because in this process we have lost the gauge invariance of the theory, which, in final instance, leads to a wrong number of propagating degrees of freedom of the interacting theory, i.e., these number will differ from the number of propagating D.o.F that the free theory originally has which could lead to several contradictions.

To illustrate this problem, which is absent in the $s = 1$ case, but present for the cases with $s \geq 2$, lets take the simplest example: consider we have a spin 2 field, and we want to couple it to gravitation using the minimal coupling prescription, thus covariantizing the Frondal tensor, and choosing some usefull conventional ordering for the covariant derivatives [25]:

$$\mathcal{F}_{\mu_1\mu_2} = \Box\Phi_{\mu_1\mu_2} - \nabla^\lambda\nabla_{\mu_1}\Phi_{\mu_2\lambda} + \nabla_{\mu_1}\nabla^\lambda\Phi_{\mu_2\lambda} + \nabla_{\mu_1}\nabla_{\mu_2}\Phi_\lambda{}^\lambda \tag{4.1}$$

and considering to deform the gauge symmetries, as is given by the covariantization with respect to the dynamical gravitational field:

$$\delta\Phi_{\mu_1\mu_2} = 2\nabla_{(\mu_1}\varepsilon_{\mu_2)} \tag{4.2}$$

Then the tensor 4.1 is not gauge invariant under 4.2, and in fact it transform as:

$$\delta F_{\mu_1\mu_2} = 2R_{\alpha\mu_1\mu_2\beta}\nabla^\alpha\varepsilon^\beta - 2\varepsilon^\alpha\nabla_\alpha R_{\mu_1\mu_2} + R_{(\mu_1\alpha}\nabla^\alpha\varepsilon_{\mu_2)} - 3R_{(\mu_1\alpha}\nabla_{\mu_2)}\varepsilon^\alpha \tag{4.3}$$

From 4.3, we see there are parts in which the Ricci scalar, and Ricci tensor appears and these parts are considered innocuous because they can be gauged away by a suitable deformation of the gauge transformations that acts on the metric (deformed diffeomorphisms). But the parts in which the full Riemann tensor appears [1] are considered dangerous, because they cannot be compensated by a deformed gauge transformation done on the metric. This means that for $s \geq 2$ we cannot modify the action by adding some 'suitable' terms to it, with the hope that when gauge transformed the metric with now some 'suitable' deformed gauge transformation, its transformations will cancel the unwanted Riemann terms that spoil the gauge invariance of the Fronsdal tensor part of the action.

4.1.2 Surpassing Aragone-Deser obstruction in three dimensions

In three dimension, for any metric the Weyl tensor vanish, thus the Riemann tensor can be fully expressed in terms of its tracefull parts, i.e, Ricci tensor, ricci scalar, and in terms of the metric. This allows us to build terms to add to the action, that when gauge transformed the metric in a suitably deformed way, it compensates the non-invariant terms coming from the Fronsdal tensor. Thus in three dimensions Aragone-Deser obstruction is not an obstruction to build an interacting action starting with minimal coupling as prescription, but in the process of adding terms to the Lagrangian, in order to recover the gauge invariance under some deformed gauge

[1] for $s \geq 2$, due to the spacetime index structure, the full Riemann tensor is always present.

transformation of the metric, of course the action is not minimally coupled anymore, but it has non-minimal terms that (omitting the space-time indices) schematically goes as:

$$\mathcal{L}_{\mathcal{NM}} = \Phi \ldots R \ldots \Phi \ldots + \ldots \tag{4.4}$$

Where Φ denotes the spin s field and R denotes the Riemann tensor and its tracefull parts and the dots represent possible contractions.
In fact, in ([25]) it has been constructed an action for a spin s field, non-minimally coupled to gravity in which for $s \geq 2$ terms as 4.4 are always present.

4.2 Some comments about a precursor: Vasiliev's interacting theory

As respect to theories of interacting higher spin fields, besides the theory we will directly use in this thesis, and thus deserves the full next chapter, probably the only other known existing theory of fully interacting higher spin fields, is due to Vasiliev in his pioneer work ([26]). Vasiliev construct his theory in a very geometric way, in a parallel way with the first order formulation of the equations of motion for gravity. Furthermore, in this formalism, Vasiliev theory is an extension of the last, and that is why the name Higher Spin Gravity has been given to it. This theory is defined over maximally symmetric backgrounds. It is important to recall that Vasiliev theory is a classical theory, i.e. the only objects which are known from this theory are its equations of motion, sadly an action principle is still lacking. However, Vasiliev equations of motions enjoy gauge symmetries under deformed gauge transformations of the free Fronsdal theory for all spins, and the last is contained in its weak field expansions, which gives rise, to its lowest orders on the fields, to its weak interacting and free limits.
The theory itself is extremely complicated because in order to account for the interactions and gauge symmetries consistently, in a similar way as the Singh and Hagen work, it requires the introduction of auxiliary fields of all spins, and also requires that an infinite tower of non-auxiliary fields of all spin from $s = 0 \ldots \infty$ in order to exist consistently. This principally is due that Vasiliev used an infinite higher spin gauge algebra, which is given by an infinite dimensional extension of the Lorentz sub-algebra contained in the isometry algebra of the maximally symmetric background, and its is realized in terms of fermionic oscilators and star products. Thus, there posses generalized geometric objects, as generalized spin connections which gives rise to the auxiliary fields, and generalized vielbeins which gives rise to the higher spin fields. We wont turn into this theory in this thesis, but for the sake of completeness, and as a precursor of all interacting higher spin theories, we cannot leave it without mention it. We can also say that this theory is extremely complicated because due to the infinite oscilator realization of the higher spin algebra, its information is quite encoded, and thus extremely large expressions can appear when one try to extract something known in the usual field theory language, g.e., just to obtain the (A)dS background of the theory, one can use two full page of calculations, or, g.e., to get the free scalar field equation in the non interacting theory one can use four full pages. Fortunately, we don't work with this theory in this thesis, instead we work with a simpler theory, which will be reviewed in the next chapter, but which has somehow its roots in Vasiliev's work.

Chapter 5

Review of Three dimensional Higher Spin Gravity

5.1 Introduction

In the last chapters we saw that the fields described by Fronsdal equations (3.1) in three dimensions for $s > 1$ does not propagate local degrees of freedom. However, we will keep talking about them as higher spin fields, because even if the bulk dynamics is trivial, when studied on AdS backgrounds they can lead to a non-trivial dynamics living on the AdS boundary. This fact is what motivates the study of this chapter about a fully non-linear interacting higher spin field theory in three dimensions. The interacting higher spin gauge theory which will be presented in this chapter, worked out in [27], is a fully interacting theory of spins from $s = 2, \ldots, N$ in which each spin is present only once (almost simultaneously it was also worked out by [28] and its super-symmetric generalization were worked out in [29]). This theory is consistent because it posses some non-linearly deformed higher spin gauge symmetries, which allows to have the same number of physical D.o.F. that the free theory has, i.e., zero in the bulk but which can be non-zero at the AdS boundary. Furthermore, in the linearized limit, this theory falls into Fronsdal theory for free massless higher spin fields. The reason for work in three dimensions is simply that, as many examples has shown [30, 31, 32, 33], in three dimensions life is so much easier than in higher dimensions, and also very interesting properties can be found. This theory, besides higher spins $s > 2$, it posses a spin 2, thus, being fully non-linear, it also contains gravity, but not as an isolated sector of the theory, but as a fully mixed one, on its non-linear interactions, with the other spins $s > 2$. As such, one can expect to find very interesting things, as all the interesting things that pure AdS gravity in three dimensions posses, i.e.: black holes solutions [31, 32, 33], solutions with asymptotic conformal symmetries [34, 35], etc.

5.2 Review of pure AdS gravity: Its action as the difference of two Chern Simons actions

After the works ([36],[30]), it is a well known fact that gravity with negative cosmological constant in three dimensions can be formulated, in its frame formalism, with the help of two Chern Simons (CS) actions were both gauge fields are valued over the $SL(2, \Re)$ algebra. It is the purpose of this section to show this construction, because this will be the starting point to then incorporate higher spin interactions in the theory.

The Cherns Simons functional in three dimentions is given by:

$$I_{CS}[A] = \frac{k}{4\pi}\int tr(\, A\wedge dA + \frac{2}{3}A\wedge A\wedge A\,) \tag{5.1}$$

where k is the called CS level, A is a one form $A = A^a_\mu T_a dx^\mu$ defined over a semi-simple Lie Algebra $\mathcal{A}$ expanded by the T_a generators, and where tr stands for a symmetric and non-degenerated bilinear form defined on $\mathcal{A}$, i.e. its Killing metric.
The CS theory has a lot of interesting properties:

- 1) It is written purely in terms of differential forms, thus it is diffeomorphism invariant.
- 2) It is a topological action, i.e., it does not need a metric to be defined, and its integral over a compact manifold is a number.
- 3) It is gauge invariant on manifold without boundaries.
- 4) Defined on manifolds with boundary, under gauge transformations, it change as the WZW action at the boundary.
- 5) The equations of motion are flat connections, i.e., of vanishing curvature $F[A] = 0$, so it does not posses local degrees of freedom.
- 6) On topologically non-trivial manifolds, i.e., non simply-connected [1], it can have non-trivial solutions.

It is well known that three dimensional gravity with negative cosmological constant can be written by the use of one CS action valued over the $O(2,2)$ algebra (which is isomorphic to AdS_3 isometry algebra), using the gauge connection:

$$A = e^a P_a + \omega^{ab} M_{ab} \tag{5.2}$$

Where P_a, M_{ab} are the generators of $O(2,2)$, which satisfies:

$$[M_{ab}, M_{cd}] = \eta_{ac}M_{db} - \eta_{ad}M_{cb} - \eta_{bc}M_{da} + \eta_{bd}M_{ca} \tag{5.3}$$

$$[M_{ab}, P_c] = \eta_{cb}P^a - \eta_{ca}P^b \tag{5.4}$$

$$[P_a, P_b] = \frac{1}{l^2}M_{ab} \tag{5.5}$$

[1] i.e. with a hole, such that there can exist some class of curves on it which cannot be contracted into a point.

and using the killing form for this algebra given by:

$$tr(M_{ab}M_{cd}) = 0 \quad (5.6)$$
$$tr(P_a M_{bc}) = \epsilon_{abc} \quad (5.7)$$
$$tr(P_a P_b) = 0 \quad (5.8)$$
$$(5.9)$$

With $k = \frac{1}{4G_3}$, where G_3 is the Newton constant in three dimensions, the CS action is left as:

$$I_{CS} = \frac{1}{16\pi G}\int \epsilon_{a\,b\,c}\,(e^a \wedge R^{b\,c} + \frac{1}{3l^2}\, e^a \wedge e^b \wedge e^c) \quad (5.10)$$

The equations of motion are $F[A] \equiv dA + A \wedge A = 0$ (i.e., the solution is a flat A gauge connection), and reads explicitly as:

$$R^{ab} - \frac{1}{2l^2} e^a \wedge e^b \;=\; 0 \;\; \text{AdS curvature} \quad (5.11)$$
$$T^a \;=\; 0 \;\; \text{zero torsion condition} \quad (5.12)$$

Where as usual, the Riemann curvature two-form is defined as $R^{ab} \equiv d\,\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$, and the torsion two-form is defined as $T^a \equiv d\,e^a + \omega^a{}_b \wedge e^b$.

In three dimensions, a two-index antisymmetric tensor can be dualized into a vector, thus in this case we can define the Lorentz generators $M_{ab} \equiv \epsilon_{abc} M^c$ to simplify the writing of the $SO(2,2)$ algebra as:

$$[M_a, M_b] = \epsilon_{abc} M^c$$
$$[M_a, P_b] = \epsilon_{abc} P^c$$
$$[P_a, P_b] = \frac{1}{l^2}\epsilon_{abc} M^c$$

By the same reason, this induces the rewriting of the spin connection as $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}$, which is left with the same index structure that the dreibein e^a has. [2]

If we now further decomposes the algebra elements as:

$$M_a = J_a + \bar{J}_a \quad (5.13)$$
$$P_a = \frac{1}{l}(J_a - \bar{J}_a) \quad (5.14)$$

the (5.13) algebra, splits out as two $SL(2,\Re)$, which does not see each other:

$$[J_a, J_b] = \epsilon_{abc} J^c \quad (5.15)$$
$$[\bar{J}_a, \bar{J}_b] = \epsilon_{abc} \bar{J}^c \quad (5.16)$$
$$[J_a, \bar{J}_b] = 0 \quad (5.17)$$

which means that $SO(2,2) \equiv SL(2,\Re) \oplus SL(2,\Re)$.

[2] Consequently, we will also have the dualization of the curvature: $R^a = \frac{1}{2}\epsilon^{abc} R_{bc}$.

The Killing form of a $g \oplus g$ algebra, where g Algebra is expanded by some generators J_a, can be generically decompose as:

$$tr(J_a\bar{J}_b) = 0, \;\; tr(J_aJ_b) = \eta_{ab}, \;\; tr(\bar{J}_a\bar{J}_b) = \eta_{ab} \tag{5.18}$$

Where η_{ab} is the Killing form of the g algebra.
This suggest to use the simplified action, given by:

$$I_{grav} = I_{CS}[A] - I_{CS}[\bar{A}] \tag{5.19}$$

Where now, we use two gauge connections, in where each are valued on the same single copy of the $SL(2,\Re)$ algebra:

$$A = A^a_\mu J_a dx^\mu \tag{5.20}$$

$$\bar{A} = \bar{A}^a_\mu J_a dx^\mu \tag{5.21}$$

In order to recover the frame formalism objects, starting with the two new connections in the new action (5.19), and due that now, dreibein and spin connection shares the same index structure [3], we can take the linear combinations to form the frame fields:

$$\begin{aligned} e &= \frac{l}{2}(A - \bar{A}) \\ \omega &= \frac{1}{2}(A + \bar{A}) \end{aligned}$$

where l denotes the AdS radius.

Then is straight-forward to see that making use of the above dictionary, replacing in (5.19), one recovers the action for the frame formalism:

$$I_{CS}[A] - I_{CS}[\bar{A}] = \frac{1}{16\pi G}\int tr(e \wedge R + \frac{1}{3l^2} e \wedge e \wedge e) \tag{5.22}$$

But now (5.22) requires that the two CS levels k be equal, and equal to $k = \frac{l}{4G_3}$, and where tr is taken with the killing metric of one single $SL(2,\Re)$ copy. This action, in its explicit $SL(2,\Re)$ Lie algebra index structure, reads as:

$$I_{CS}[A] - I_{CS}[\bar{A}] = \frac{1}{8\pi G}\int (e^a \wedge R_a + \epsilon_{abc}\frac{1}{6l^2}\, e^a \wedge e^b \wedge e^c) \tag{5.23}$$

The equations of motion, now are two flat connections $F[A] = 0, \bar{F}[\bar{A}] = 0$, which using the map (5.22) are reduced to ((5.11),(5.12)). These equations, now due to the dualization of spin connection, reads as:

$$0 = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \wedge \omega_c + \frac{1}{2l^2}\epsilon^{abc}e_b \wedge e_c \tag{5.24}$$

$$0 = de^a + \epsilon^{abc}\omega_b \wedge e_c \tag{5.25}$$

[3]From here on, and along all the text, we will only use the dualized expressions for the frame indices that spin connection one form and Riemann curvature two form posses, i.e., with only one frame index.

The contact with the metric formulation is given by using:

$$g_{\mu\nu} \equiv tr(e_{(\mu}e_{\nu)}) = \eta_{ab}e^a_\mu e^b_\nu \tag{5.26}$$

and, as usual, imposing the dreibein postulate, i.e., vanishing covariant derivative of the dreiben with respect to all its indices (frame and space-time indices):

$$\partial_\mu e^a_\nu + \epsilon^a{}_{bc}\omega^b_\mu e^c_\nu - \Gamma^\beta_{\mu\nu}e^a_\beta = 0 \tag{5.27}$$

together combined with the vanishing torsion condition (5.25), we can solve for the spin connection in terms of the dreibein as:

$$\omega_d \equiv \frac{1}{2}\epsilon_{dab}\omega^{ab}_\mu = \frac{1}{2}\epsilon_{dab}(2e^{\nu[a}\partial_{[\mu}e^{b]}_{\nu]} - e_{\mu c}e^{\nu[a}e^{b]}_\alpha\partial_\nu e^{c\alpha}) \tag{5.28}$$

Thus, instead of working with one $SO(2,2)$ gauge connection as in (5.10), we have reduced the problem to work with two $SL(2,\Re)$ connections as in (5.19). This point, will be the starting point for a generalization to include higher spins by extending the $SL(2,\Re)$ gauge algebra to $SL(N,\Re)$ gauge algebra.

5.3 Review of full interacting Higher Spin Gravity Action in Chern Simons formulation

In [37], Blencowe has generalized the CS construction for pure AdS Gravity in order to incorporate higher spin fields, falling this way into Vasiliev equations of motions for an infinite tower of higher spin fields but in three dimensions. However his construction, rather complicated just as Vasiliev one, makes use of an infinite extension of each copy of the $SL(2,\Re)$ algebras in such a way that also an infinite number of auxiliary fields has to be added for the consistency of the theory. In [27], Campoleoni et a.l. have made significant simplifications to Blencowe's work, by showing that in three dimensions, choosing the extended the gauge algebra of the CS theory in a proper way, it is not necessary that the introduced algebra be an infinite one, in order to account for higher spin fields and their interactions. Thus, in this way they showed we can construct an interacting theory with a finite spectrum of fields with spins from $s = 2, \ldots, N$ in which each spin s is present only once. The purpose of this section to show this construction (for excellent review see also [38]).

The idea behind the work ([27][37]), is that instead of introduce non-linear interactions in the free Fronsdal action, which hopefully respect some deformed non-linear gauge symmetries in the metric-like formalism,i.e., constructing a non -linear gauge invariant action under them, the idea is to introduce the interactions, simply, in the frame formalism. In order to achieve this it is easier to work, more precisely, with the Chern Simons actions by enhancing the algebra of the gauge connections. This construction guaranties gauge invariance a priori. Particularly in ([27]) this algebra has been chosen in such a way that in the linearized limit this gauge algebra accounts properly for the Fronsdal Fields and the gauge symmetries they posses. Thus in this way CS formulation allows us to construct a gauge invariant action for the first order formalism in a very easy way, to a posteriori interpret the metric-like field counterpart, analogously as one does in the case of pure AdS Gravity given in the last section.

For this consider an action which is the difference of two CS actions as:

$$S = I_{CS}[A] - I_{CS}[\bar{A}] \tag{5.29}$$

with $k = \bar{k} = \frac{l}{4G}$, but now each is valued over the $SL(3,\Re)$ algebra [4], and split the $SL(3,\Re)$ generators as:

$$[J_a, J_b] = \epsilon_{abc} J^c \tag{5.30}$$

$$[J_a, T_{bc}] = \epsilon^m{}_{a(b} T_{c)m} \tag{5.31}$$

$$[T_{ab}, T_{cd}] = -(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}) J^m \tag{5.32}$$

where eq. (5.30) is the $SL(2,\Re)$ subalgebra contained in $SL(3,\Re)$, and the generators T_{ab} in eq. (5.31) are symmetric and traceless tensors that accounts as the complementary higher spin generators [5], which according to (5.31) transform as irreducible $SL(2,\Re) \sim SO(2,1)$ Lorentz tensors. Finally (5.32) close the $SL(3,\Re)$ algebra.

In the fundamental representation of $SL(3,\Re)$ ((3×3) dimensional matrix representation), the higher spin generators T_{ab} can be constructed from the 3×3 dimensional adjoint representation of $SL(2,\Re)$ generators, by using symmetrized and traceless products of these generators as:

$$T_{ab} \equiv J_{(a} J_{b)} - \frac{2}{3} \eta_{ab} J_c J^c \tag{5.33}$$

where $\eta_{ab} \equiv tr(J_a J_b)$ is the Killing metric of the $SL(2,\Re)$ part. Note that the matrix trace is $Tr(T_{ab}) = 0$, thus they fulfill traceless properties of a Lie Algebra generators. Also note that $\eta^{ab} T_{ab} = T^a{}_a = 0$.

Then construct the gauge connections as:

$$A = (A^a_\mu J_a + A^{ab}_\mu T_{ab}) dx^\mu \tag{5.34}$$

$$\bar{A} = (\bar{A}^a_\mu J_a + \bar{A}^{ab}_\mu T_{ab}) dx^\mu \tag{5.35}$$

And from them, lets construct the generalized dreibein and spin connection:

$$e = \frac{l}{2}(A - \bar{A}) \tag{5.36}$$

$$\omega = \frac{1}{2}(A + \bar{A}) \tag{5.37}$$

[4] In order to extend the gravitational theory, in principle it is not necessary to choose this algebra, it is only necessary that the chosen algebra posses a non-degenerate symmetric bilinear form, and contains $SL(2,\Re)$ as a sub-algebra. The first is in order to be able to use CS actions to define it, and the last is in order to be able to identify what would be the gravitational part of the theory when we set the rest of the field content equal to zero. However, one thus should consider that different spectrum of fields, i.e. with different spin, should arise as the field content of the theory, depending on the chosen algebra. By the way, this spectrum, may or may not contain higher spins, and respectively they may or may not fall in the free Fronsdal theory at linearized limit. Furthermore, a particular chosen gauge algebra can have different possibles spectrum depending on the choice made to embed the $SL(2,\Re)$ algebra into the chosen bigger one.

[5] As these generators has two Lorentz index, they are spin 2 generators, however they will represent the higher spin part of the components of the gauge field which also carry an additional spin 1 because they posses one space-time index

These objects can be expanded as:

$$e = (e^a_\mu J_a + e^{ab}_\mu T_{ab})dx^\mu \quad (5.38)$$
$$\omega = (\omega^a_\mu J_a + \omega^{ab}_\mu T_{ab})dx^\mu \quad (5.39)$$

where e^a_μ and ω^a_μ accounts for the usual dreibein and spin connections, i.e. spin 2 fields, and e^{ab}_μ and ω^{ab}_μ accounts for the spin 3 fields, which being contracted with higher spin generators T_{ab} are completely symmetric and traceless in its frame indices, which means they are higher spin irreducible representations of the Lorentz group in the frame indices.

After the map given by ((5.36)), the action is left as:

$$S = I_{CS}[A] - I_{CS}[\bar{A}] = \frac{1}{16\pi G}\int tr(e \wedge R + \frac{1}{3l^2} e \wedge e \wedge e) \quad (5.40)$$

where $R = d\omega + \omega \wedge \omega$ and the tr is taken over only one single $SL(3,\Re)$ copy, which in its index structure it explicitly reads as:

$$S = \frac{1}{8\pi G}\int [e^a \wedge (d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c + 2\epsilon_{abc}\omega^{bd} \wedge \omega^c{}_d) \quad (5.41)$$
$$+2e^{ab} \wedge (d\omega_{ab} + \epsilon_{cd(a}\omega^c \wedge \omega_{b)}{}^d) + \frac{1}{6l^2}\epsilon_{abc}(e^a \wedge e^b \wedge e^c + 12e^a \wedge e^{bd} \wedge e^c{}_d)] \quad (5.42)$$

Meanwhile in the CS formulation the equations of motion are those of flats connections $F[A] = 0$, $F[\bar{A}] = 0$, in the frame formalism the equations of motions for the spin 2 fields: e^a, w^a, reads as:

$$de^a + \epsilon^{abc}\omega_b \wedge e_c + 4\epsilon^{abc}e_{bd} \wedge \omega_c{}^d = 0 \quad (5.43)$$
$$d\omega^a + \frac{1}{2}\epsilon^{abc}(\omega_b \wedge \omega_c + \frac{1}{l^2}e_b \wedge e_c) + 2\epsilon^{abc}(\omega_{bd} \wedge \omega_c{}^d + \frac{1}{l^2}e_{bd} \wedge e_c{}^d) = 0 \quad (5.44)$$

Note that ((5.43),(5.44)) can be directly compared with the equations of motion of the $SL(2,\Re)$ theory ((5.24),(5.25)), to directly see the extra contributions coming from the spin 3 fields e^{ab}, w^{ab} in the $SL(3,\Re)$ case.
The equations of motion for the spin 3 fields e^{ab}, w^{ab}, reads:

$$de^{ab} + \epsilon^{cd(a}\omega_c \wedge e_d{}^{b)} + \epsilon^{cd(a}e_c \wedge \omega_d{}^{b)} = 0 \quad (5.45)$$
$$d\omega^{ab} + \epsilon^{cd(a}\omega_c \wedge \omega_d{}^{b)} + \frac{1}{l^2}\epsilon^{cd(a}e_c \wedge e_d{}^{b)} = 0 \quad (5.46)$$
$$(5.47)$$

5.4 Contact with the metric-like formulation

The action (5.29) is invariant under the infinitesimal gauge transformations:

$$\delta A = D\lambda = d\lambda + [A,\lambda] \quad (5.48)$$
$$\bar{\delta}\bar{A} = \bar{D}\bar{\lambda} = d\bar{\lambda} + [\bar{A},\bar{\lambda}] \quad (5.49)$$

which with the help of the map:

$$\Lambda = \lambda + \bar{\lambda}, \varepsilon = \lambda - \bar{\lambda} \tag{5.50}$$

can be mapped into the frame formulation as the transformations:

$$\begin{aligned} \delta e &= d\varepsilon + [\omega, \varepsilon] + [e, \Lambda] \\ \delta \omega &= d\Lambda + [\omega, \Lambda] + \frac{1}{l^2}[e, \varepsilon] \end{aligned} \tag{5.51}$$

which leaves the frame action (5.40) invariant.
The parameters Λ and ε of the map ((5.50)), can be explicitly written as:

$$\Lambda = \Lambda^a J_a + \Lambda^{ab} T_{ab} \tag{5.52}$$
$$\varepsilon = \varepsilon^a J_a + \varepsilon^{ab} T_{ab} \tag{5.53}$$

where the Lie algebra components Λ^a gives rise to local Lorentz transformations, and the components ε^a gives rise to diffeomorphisms, while the components Λ^{ab} gives rise to a generalized spin 3 local Lorentz-like transformation, and ε^{ab} gives rise to a generalized spin 3 diffeomorphism-like transformation.

Note that, from ((5.51)), besides the usual action of the spin 2 parameters (Λ^a , ε^a) on the spin 2 fields (e^a , ω^a) (which in the following expression is omitted), also the spin 3 parameters (Λ^{ab} , ε^{ab}) of the transformation acts non trivially on the spin 2 fields as:

$$\delta e^a = 4\epsilon^{abc}\omega_{bd}\varepsilon_c{}^d + 4\epsilon^{abc}e_{bd}\Lambda_c{}^d \tag{5.54}$$
$$\delta \omega^a = 4\epsilon^{abc}\omega_{bd}\Lambda_c{}^d + \frac{4}{l^2}\epsilon^{abc}e_{bd}\varepsilon_c{}^d \tag{5.55}$$

Also, from ((5.51)), the spin 2 and spin 3 parameters acts on the spin 3 field as:

$$\delta e^{ab} = d\,\varepsilon^{ab} + \epsilon^{cd(a}\omega_c\,\varepsilon_d{}^{b)} + \epsilon^{cd(a}e_c\,\Lambda_d{}^{b)} + \epsilon^{cd(a}\omega^{b)}{}_c\,\varepsilon_d + \epsilon^{cd(a}e^{b)}{}_c\,\Lambda_d \tag{5.56}$$
$$\delta \omega^{ab} = d\,\Lambda^{ab} + \epsilon^{cd(a}\omega_c\,\Lambda_d{}^{b)} + \frac{1}{l^2}\epsilon^{cd(a}e_c\,\varepsilon_d{}^{b)} + \epsilon^{cd(a}\omega^{b)}{}_c\,\Lambda_d + \frac{1}{l^2}\epsilon^{cd(a}e^{b)}{}_c\,\varepsilon_d \tag{5.57}$$
$$\tag{5.58}$$

To identify what would be the higher spin generalization of the metric-like fields, which in the pure gravity $SL(2, \Re)$ case, the metric (spin 2 field) it is invariant under local Lorentz transformations, and it is given as:

$$g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu = tr_{\text{SL2}}(e_{(\mu}e_{\nu)}) \tag{5.59}$$

,

in the higher spin $SL(N,\Re)$ theory one ask for which combinations of the fields are also invariant under the generalized local lorentz-like fields with Λ parameters. An analysis reveal that for the $SL(3,\Re)$ case, the combinations given by:

$$g_{\mu\nu} = tr_{\mathrm{SL3}}(e_{(\mu}e_{\nu)}) \tag{5.60}$$
$$g_{\mu\nu\gamma} = tr_{\mathrm{SL3}}(e_{(\mu}e_{\nu}e_{\gamma)}) \tag{5.61}$$
$$\tag{5.62}$$

fulfills this invariance under a gauge transformation with full Λ (Λ^a spin 2, and Λ^{ab} spin 3 local Lorentz-like parameters).

However, the spin 2 and spin 3 diffeomorphism related parameters ε, through (5.54) acts on the fields (5.60) in an unexpected and very mixed way. This has as consequence that meaningful quantities in General Relativity as good coordinates invariant, in the higher spin setting are not so meaningful because they are no longer invariant under higher spin transformations, e.g., one can change the causal structure of a space-time in which we have a higher spin by simply making a higher spin transformation. One possible way to understand this, as we saw in chapter (2), is by the fact that in general, higher spin transformations are made with higher spin gauge parameters, which always must somehow carry lower spins invariant subspaces on its gauge parameters, which in last instance, in the interacting system, these lower spin invariant subspaces must acts on the lower spin fields. This spots the necessity of posses a enhanced setting of higher spin geometry with higher spin geomteric concepts, which is at the moment unknown.

Now, trough (5.60), we have only a partial contact with a metric-like formulation of this theory. A full metric-like formulation of this theory (action and equations of motion) it is not known yet, because we don't have a full dictionary to go from the frame formulation to the metric-like one. The reason of this, is that, at difference as in the case of pure AdS gravity ($SL(2,\Re)$ case), we don't know a way on how to generically invert the generalized dreibein in order to solve the equations of null torsion (5.43) and null generalized higher spin torsion (5.45), and metricity, for the generalized spin connection in terms of the generalized dreibein.

5.5 Recovering the free higher spin Fronsdal equations in the linearized limit

From the above definitions for the full metric-like fields, one can recover the free Fronsdal metric-like fields by making a weak field expansion, at linearized order, around some AdS background values, i.e., the exact AdS metric which is left invariant by the six killing vectors of AdS_3, and a zero spin 3 field which is left invariant also by the ten traceless killing tensors of AdS_3. This configuration, as a background demand, is the configuration with the maximal amount of symmetry.

Consider for this to have, $\bar{e}^a_\mu$ as the AdS background dreibein and $\bar{\omega}^a_\mu$ as the AdS background spin connection solutions, both in the principal embedding (see section (4.7)), and $\bar{e}^{ab}_\mu = 0$ as the AdS higher spin background dreibeins and $\bar{\omega}^{ab}_\mu = 0$ as the AdS higher spin background connections, thus consider to have the fields as linearized around this background configurations

[6]:

$$e^a = \bar{e}^a + h^a \quad (5.63)$$
$$e^{ab} = h^{ab} \quad (5.64)$$

and from (5.60), the linearized Fronsdal fields follows as:

$$\Phi_{\mu_1 \dots \mu_s} = \bar{e}^{a_1}{}_{(\mu_1} h_{\mu_2) a_1} \quad (5.65)$$

$$\Phi_{\mu_1 \dots \mu_s} = \bar{e}^{a_1}{}_{(\mu_1} \bar{e}^{a_2}{}_{\mu_2} h_{\mu_3) a_1 a_2} \quad (5.66)$$

It was shown in [27] that these linearized fields, satisfy the free Fronsdal's equations, and that the linearized gauge transformations coming from the linearization of (5.51), given by the diffeomorphism-like related gauge parameter ε, reduces to the free Fronsdal's gauge transformations on fixed AdS that we saw in chapter 2.

5.6 Extension to arbitrary N: the $SL(N, \Re)$ theory

The above construction can be done for $SL(N, \Re) \times SL(N, \Re)$ CS theories.

In this case we split the $SL(N, \Re)$ algebra as:

$$[J_a, J_b] = \epsilon_{abc} J^c \quad (5.67)$$

$$[J_a, T_{b_1 \dots b_s}] = \epsilon^m{}_{a(b_1} T_{b_2 \dots b_{s-1})m} \; , \quad s = 2, \dots, N-1 \quad (5.68)$$

where J_a expand a N dimensional representation of the $SL(2, \Re)$ algebra and where the generators in (5.68) are symmetric and single traceless in all its indices. This line shows they transform as irreducible higher spin representations of the Lorentz group $SO(2,1) \sim SL(2, \Re)$.

One can construct the gauge fields of the CS theory considering, besides the $SL(2, \Re)$ part, also the Lie algebra components corresponding to the higher spin generators:

$$A = (A^a_\mu J_a + A^{a_1 a_2}_\mu T_{a_1 a_2} + \dots + A^{a_1 \dots a_{s-1}}_\mu T_{a \dots a_{s-1}}) dx^\mu \quad (5.69)$$
$$\bar{A} = (\bar{A}^a_\mu J_a + \bar{A}^{a_1 a_2}_\mu T_{a_1 a_2} + \dots + \bar{A}^{a_1 \dots a_{s-1}}_\mu T_{a \dots a_{s-1}}) dx^\mu \quad (5.70)$$

As before, we construct the frame fields as in (5.36), and we will have:

$$e = (e^a_\mu J_a + e^{a_1 a_2}_\mu T_{a_1 a_2} + \dots + e^{a_1 \dots a_{s-1}}_\mu T_{a \dots a_{s-1}}) dx^\mu \quad (5.71)$$
$$\omega = (\omega^a_\mu J_a + \omega^{a_1 a_2}_\mu T_{a_1 a_2} + \dots + \omega^{a_1 \dots a_{s-1}}_\mu T_{a \dots a_{s-1}}) dx^\mu \quad (5.72)$$

Then we can construct the $N-1$ 'metric-like fields' as:

[6]Note we are linearizing around trivial spin 3 background values, i.e., we are considering the spin 3 field e^{ab} as being its own fluctuations as the solution with maximal symmetry, i.e., the background demands it be.

$$g_{\mu_1\mu_2} = tr(e_{(\mu_1}e_{\mu_2)}) \tag{5.73}$$
$$g_{\mu_1\mu_2\mu_3} = tr(e_{(\mu_1}e_{\mu_2}e_{\mu_3)}) \tag{5.74}$$
$$\vdots \tag{5.75}$$
$$g_{\mu_1\mu_2\mu_3\dots\mu_N} = tr(e_{(\mu_1}e_{\mu_2}e_{\mu_3}\dots e_{\mu_N)}) \tag{5.76}$$

This is allows us to have, $N-1$ independent fields. This is because the $SL(N,\Re)$ algebra posses $N-1$ Casimir invariants corresponding to the trace of the powers $p=2,\dots,N$ of an arbitrary algebra element. Thus the above fields, omitting space-time index (and omitting the Lie agebra indices) can be seen as constructed by taking the trace of these different $N-1$ powers of the dreibein one form e.

The free Fronsdal higher spin fields now follows by linearizing around AdS background values, which are chosen to be given by only non-null components of the spin 2 generators J_a that corresponds to the principal embedding (see below):

$$\Phi_{\mu_1\dots\mu_s} = \bar{e}^{a_1}{}_{(\mu_1}\bar{e}^{a_2}{}_{\mu_2}\dots\bar{e}^{a_{s-1}}{}_{\mu_{s-1}}e_{\mu_s)a_1a_2\dots a_{s-1}} \tag{5.77}$$

5.7 Different $SL(2,\Re)$ embeddings into the $SL(N,\Re)$ theory

Now we will see that having the $SL(N,\Re)$ theory, we can describe different theories, i.e., with different field content, depending on the different embeddings of the $SL(2,\Re)$ algebra, into the $SL(N,\Re)$ algebra, that we chose to describe the pure gravitational part[7] (see the works [39],[40]).

The generators of $SL(N,\Re)$ algebra can be arranged by choosing several different sets composed by three generators, which together form an $SL(2,\Re)$ algebra. A particular choice of the possible $SL(2,\Re)$ sets, describes what is called an $SL(2,\Re)$ embedding into $SL(N,\Re)$ algebra. Depending on the chosen $SL(2,\Re)$ set, we will have that the rest of the generators of the complementary Lie algebra space to fill the whole $SL(N,\Re)$ algebra, will transform according to some definite rules, in each case, under the chosen $SL(2,\Re)$ set, which can be analyzed on the same footing for all the possible $SL(2,\Re)$ sets, by studying the adjoint representation of the $SL(N,\Re)$ algebra. In the adjoint representation of $SL(N,\Re)$ we will have its N^2-1 generators represented by $(N^2-1)\times(N^2-1)$ dimensional matrices to expand the whole algebra. Then we can bring each $(N^2-1)\times(N^2-1)$ dimensional generator into a Jordan block-diagonal form, accommodating in this way all the generators in different sets, each sets corresponding to the different blocks of some definite $n\times n$ dimensional size categorized by it spin s and given by $n=2s+1$. We will call this idea of spin s as conformal spin, but it is essentially the same concept of spin that arise when one study finite dimensional representations of e.g. $SU(2)$. Thus, each set of generators of some fixed conformal spin s will transform under a representations of the $SL(2,\Re)$ algebra of the same conformal spin s. In this way, by choosing different embeddings, we will have different branchings for the generators of the $SL(N,\Re)$ algebra according on the spin of the $SL(2,\Re)$ part in which they transform.

[7]Please understand the pure gravitational part, as the part corresponding to a chosen $SL(2,\Re)$ algebra which remains when we turn off the components along the other generators that completes $SL(N,\Re)$

For example in the case of $SL(3,\Re)$ algebra, we will have two branchings. One is given by choosing the so called 'principal embedding' of the $SL(2,\Re)$ into $SL(3,\Re)$, and is characterized by the fact that the $SL(3,\Re)$ algebra in the adjoint representation splits (in this case) in two sets of generators: one is a set of conformal spin 1 (3×3 dimensional) and the other set is of conformal spin 2 (5×5) dimensional, and of course due to the dimensions, each set occurs once, giving the branching $\bar{8} = \bar{3} \oplus \bar{5}$. Also, note that this is the branching corresponding to choose the J_a $SL(2,\Re)$ generators in the adjoint of $SL(2,\Re)$ (3 dimensional representation, i.e. spin 1 representation) and the T_{ab} generators of eq. (5.31), which in the 8 dimensional adjoint of $SL(3,\Re)$ can be seen as represented by a (5×5) block matrix, and thus as transforming under a spin 2 representation of the $SL(2,\Re)$ algebra, as the equation (5.31) shows.
The other existing branching in $SL(3,\Re)$ is given by the so called 'diagonal embedding' of $SL(2,\Re)$ into $SL(3,\Re)$. In this case the generators splits in four sets of generators, each set is of different spin related size, and they are given by the branching $\bar{8} = \bar{3} \oplus 2 \times \bar{2} \oplus \bar{1}$. This say that the first set transform under a conformal spin 1 representation of $SL(2,\Re)$, the second set appears twice and, each copy, transform under a spin $\frac{1}{2}$ representation of $SL(2,\Re)$, and the third set transform as a conformal spin 0.

Now, concerning the Lie algebra components of the gauge field, to see the real field content of the theory, the analysis is as follows: Consider to take the conformal spin s of each set of generators, then consider the fact that the $SL(3,\Re)$ gauge fields carry also 'spin 1' vector space-time index. Thus the final field content that a theory with definite $SL(2,\Re)$ embedding into $SL(3,\Re)$ posses, will be given by adding 'one' to the conformal spin s that each set of a given branching posses. This means that the theory defined in the principal embedding will posses, as field content, one spin 2 field and one spin 3 field. On the other side the theory defined on the diagonal embedding will posses as field content, one spin 2, two spin $\frac{3}{2}$ bosonic [8] Rarita Schwinger field, and one spin 1 field. Thus, we learn that different embeddings of the $SL(2,\Re)$ 'gravitational ' part, leads to different theories with the same $SL(N,\Re)$ algebra.

In particular, note that by construction the free Fronsdal Higher spin theory would be only recovered when we choose the principal embedding as the gravitational part, thus if we are trying to describe a theory of interacting higher spin, i.e., which falls into the free Fronsdal equation in the linearized limit, we better work with the principal embedded $SL(2,\Re)$ part as the gravitational part. This last point, at least in my concern, has not been made explicit in the literature before. In fact it is very natural to wonder about if the diagonal embedding posses a possible higher spin interpretation (may be as composed states of lower spin particles) due that its spectrum does not posses fundamental spins higher than 2.

5.8 Solutions of the $SL(3,\Re)$ theory with asymptotic W_3 symmetries a la Brown-Henneaux

In [27] it has been also shown that this formulation of a fully non-linear interacting spin 2 and spin 3 fields, posses a solution with enhanced conformal asymptotic symmetries given by the W_3 Zamolodchikov algebras ([41]) with a non-trivial central charge, which is the same as in the

[8]Bosonic, because the algebra is realized with commutators, not with anti-commutators as in super-gravity in which one uses a super-algebra

case of pure AdS gravity, a la Brown-Henneaux [34] which was done in the metric formalism. To achieve this, in [27], it has been worked in the CS gauge connections (directly linked with the frame formalism), where it was chosen the right AdS boundary conditions on the on shell bulk gauge fields. For this it was used the asymptotic AdS gauge connection with leading behaviour at $\mathcal{O}(r^2)$ pertaining to the principal embedding of $SL(2,\Re)$ into $SL(3,\Re)$, and then it has been looked for a $SL(3,\Re)$ solution which departs softly from the background, i.e., only departs asymptotically from the asymptotic AdS solution in the asymptotics sub-leading terms $\mathcal{O}(1)$.

$$\begin{aligned}(A - A_{AdS})|_{\text{Boundary}} &= \mathcal{O}(1) \\ (\bar{A} - \bar{A}_{AdS})|_{\text{Boundary}} &= \mathcal{O}(1)\end{aligned} \tag{5.78}$$

This choice of boundary conditions, when seen at the boundary, constitutes an asymptotic gauge fixing that allows to the authors to isolate the 'would be physical' degrees of freedom of the full interacting system, which propagates at the boundary with the correct spin ($s = 2, 3$). This result should be considered somehow expected because of the AdS/CFT conjecture we will see in the next chapter (6).

In order to find this solutions, the authors considered the following flat solutions, working in coordinates r and light-cone coordinates $x_{\pm} = \frac{t}{l} \pm \varphi$ coordinates:

$$A_r = b^{-1}\partial_r b \ , \ A_+ = b^{-1}a(x_+)b \ , \ A_- = 0 \tag{5.79}$$

$$\bar{A}_r = b\partial_r b^{-1} \ , \ \bar{A}_+ = 0 \ , \ \bar{A}_- = b\bar{a}(x_-)b^{-1} \tag{5.80}$$

where, in order to obtain them, it was chosen a partial (not complete) gauge fixing at the boundary, trough the boundary conditions $A_- = 0$ (and $\bar{A}_+ = 0$ for the other copy), and trough the choice for the radial group element at the boundary, which was chosen to be $b(r) = e^{ln(r)L_0}$. Also, in (5.79,5.80) the fields $a(x_+)$ and $\bar{a}(x_-)$ respectively, are allowed to carry remaining unfixed pure gauge degrees of freedom, by taking values in the whole $SL(3,\Re)$ algebra [9]. This partial gauge fixing at the boundary, trough the equations of motions, extend to the whole interior of the bulk manifold leading to chiral fields as shown in (5.79,5.80). And it is also the gauge fixing condition which allows to keep the CS theory invariant in a manifold with boundary[10] under 'gauge fixing preserving' gauge transformations. And it is well known that in the CS theory with a boundary the CS solutions, i.e., for one copy, e.g., lets say A in (5.79), leads to a Kac-Moody algebra in the Poisson bracket structure of the phase space for the remaining degrees of freedom (inside $a(x_+)$), and analogously for the other copy $\bar{A}$ (inside $\bar{a}(x_-)$). However, these solutions are not asymptotically AdS, thus it is necessary to further fix some of the remaining gauge freedom by using the criteria given by (5.78), which is the same to demand the desired AdS boundary conditions. As a final result, this further gauge fixing procedure gives gauge connections with a minimum amount of remaining pure gauge degrees of freedom, which can be interpreted as physical at the boundary:

$$a(x_+) \equiv L_1 + \frac{2\pi}{k}\mathcal{L}(x_+)L_{-1} - \frac{\pi}{2k}\mathcal{W}(x_+)W_{-2} \ , \ \bar{a}(x_-) \equiv L_{-1} + \frac{2\pi}{k}\bar{\mathcal{L}}(x_-)L_1 - \frac{\pi}{2k}\bar{\mathcal{W}}(x_-)W_2 \tag{5.81}$$

[9]For the convention over the generators of $SL(3,\Re)$ algebra used here, see appendix

[10]For a discussion of the CS theory with boundary see appendix.

Understanding the symmetry in the connections A and $\bar{A}$, the discussion that follows will be given for only one copy A because for the other copy $\bar{A}$ can be done analogously.

Having fixed the gauge for A as in (5.79) with $a(x_+)$ as in (5.81), they look for the allowed gauge transformations which preserves the gauge choice for the gauge fixed components A_r, A_- as in (5.79), i.e., that leaves them invariant. But they also required that the remaining gauge freedom inside $a(x_+)$ as in (5.81), be allowed to change under this gauge transformation, but in such a way that leaves invariant the generator structure of (5.81), i.e., only allowing to change the finally remaining pure gauge degrees of freedom $\mathcal{L}(x_+), \mathcal{W}(x_+))$, finding that the gauge transformations with parameter $\Lambda = b^{-1}(r)\lambda(x_+)b(r)$ fulfill this condition, where:

$$\lambda(x_+) = \sum_{i=-1}^{1} \varepsilon^i(x_+)L_i + \sum_{m=-2}^{2} \chi^m(x_+)W_m \tag{5.82}$$

and where the parameters inside (5.82) are conveniently expressed with the help of the redefinitions $\varepsilon \equiv \varepsilon^1$ and $\chi \equiv \chi^2$ as:

$$\begin{aligned}
\varepsilon^0 &= -\varepsilon' &(5.83)\\
\varepsilon^{-1} &= \frac{1}{2}\varepsilon'' + \frac{2\pi}{k}\varepsilon\mathcal{L} + \frac{4\pi}{k}\chi\mathcal{W} &(5.84)\\
\chi^1 &= -\chi' &(5.85)\\
\chi^0 &= \frac{1}{2}\chi'' + \frac{4\pi}{k}\chi\mathcal{L} &(5.86)\\
\chi^{-1} &= -\frac{1}{6}\chi''' - \frac{10\pi}{3k}\chi'\mathcal{L} - \frac{4\pi}{3k}\chi\mathcal{L}' &(5.87)\\
\chi^{-2} &= \frac{1}{24}\chi'''' - \frac{4\pi}{3k}\chi''\mathcal{L} + \frac{7\pi}{6k}\chi'\mathcal{L}' + \frac{\pi}{3k}\chi\mathcal{L}'' + \frac{4\pi^2}{k^2}\chi\mathcal{L}^2 - \frac{\pi}{2k}\varepsilon\mathcal{W} &(5.88)
\end{aligned}$$

Considering (5.83), it was found that under this transformation the fields $\mathcal{L}(x_+)$ and $\mathcal{W}(x_+)$ transform as:

$$\begin{aligned}
\delta_\varepsilon\mathcal{L} &= \varepsilon\mathcal{L}' + 2\varepsilon'\mathcal{L} + \frac{k}{4\pi}\varepsilon''' &(5.89)\\
\delta_\varepsilon\mathcal{W} &= \varepsilon\mathcal{W}' + 3\varepsilon'\mathcal{W} &(5.90)\\
\delta_\chi\mathcal{L} &= 2\chi\mathcal{W}' + 3\chi'\mathcal{W} &(5.91)\\
\delta_\chi\mathcal{W} &= -\frac{1}{3}\{2\chi\mathcal{L}''' + 9\chi'\mathcal{L}'' + 15\chi''\mathcal{L}' + 10\chi'''\mathcal{L} + \frac{k}{4\pi}\chi^{(5)} + \frac{64\pi}{k}(\chi\mathcal{L}\mathcal{L}' + \chi'\mathcal{L}^2)\} &(5.92)
\end{aligned}$$

where a prime denotes derivative with respect to x_+.

Note that eq. (5.89) says that the field $\mathcal{L}(x_+)$ is a non-primary field of conformal weight 2, i.e., energy momentum tensor of what would be the conformal boundary theory, and (5.90) says that $\mathcal{W}(x_+)$ is primary field of conformal weight 3 (see appendix).

Furthermore, just as it was done by Brown-Henneaux in [34], for the spin 2 case, where W_2 Virasoro conformal algebra was found, in [27] it was shown that this asymptotic W_3 conformal symmetry algebra can be realized canonically, trough the Dirac Brackets, worked out directly from the constrained CS theory with the circle as a boundary.

The Poisson Bracket algebra of the reduced phase space of the theory, was calculated in [27] and leads to the W_3 algebra:

$$\{\mathcal{L}(\varphi),\ \mathcal{L}(\varphi')\} = -(\delta(\varphi-\varphi')\mathcal{L}'(\varphi) + 2\delta'(\varphi-\varphi')\mathcal{L}(\varphi) + \frac{k}{4\pi}\delta'''(\varphi-\varphi')) \tag{5.93}$$

$$\{\mathcal{L}(\varphi),\ \mathcal{W}(\varphi')\} = -(2\delta(\varphi-\varphi')\mathcal{W}'(\varphi) + 3\delta'(\varphi-\varphi')\mathcal{W}(\varphi)) \tag{5.94}$$

$$\begin{aligned}\{\mathcal{W}(\varphi),\ \mathcal{W}(\varphi')\} = & -\frac{1}{3}(2\delta(\varphi-\varphi')\mathcal{L}'''(\varphi) + 9\delta'(\varphi-\varphi')\mathcal{L}''(\varphi) + 15\delta''(\varphi-\varphi')\mathcal{L}'(\varphi) \\ & +10\delta'''(\varphi-\varphi')\mathcal{L}(\varphi) + \frac{k}{4\pi}\delta^{(5)}(\varphi-\varphi') \\ & +\frac{64\pi}{k}(\delta(\varphi-\varphi')\mathcal{L}(\varphi)\mathcal{L}'(\varphi) + \delta'(\varphi-\varphi')\mathcal{L}^2(\varphi)))\end{aligned} \tag{5.95}$$

where the equal time dependence of the fields in (5.93) has been omitted.

Chapter 6

Review of AdS/CFT Holography for bosonic lower spins

6.1 Introduction

The AdS/CFT conjecture was firstly spotted in the work of Maldacena [42] inside the context of string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills in four dimensions. After that, it was further clarified and generalized by Witten in [43], where it was also stated that the holographic phenomena will be present in any theory which posses a conformal symmetry. Subsequently, the conjecture was quickly developed even further, leading to highly extended works, such as the one by Aharony et al. [44], and being applied to many systems such as strongly coupled QCD, condensed matter systems, etc. At today, AdS/CFT is a highly developed tool to study quantum systems at regimes in which perturbative methods fails, and although not fully understood, the application of AdS/CFT to systems possessing higher spins fields is not an exception. Actually, there is a lot of research works that are being carried out on this line. Regarding the importance of this review chapter to the original work of this thesis, we can say that it lies in the fact that in the theoretical framework we have a two dimensional CFT with W_N symmetries defined on the boundary, and we would like to understand, in a sensible AdS/CFT picture, the three dimensional AdS gravitational solutions associated to these two dimensional CFT.

6.2 AdS/CFT conjecture at level of symmetries

AdS/CFT is a tool for build and/or study theories based on the duality given by equivalence of having a theory with the conformal group as the group of symmetries living in flat d dimensional compactification of Minkowsky space, and a gravitational theory with (global or asymptotic) AdS isometries in $d + 1$ dimensions. This is due to the isomorphism of these two groups which is $SO(d, 2)$.

In $d > 2$ dimensions [1] a theory which posses conformal symmetries (a scale invariant theory), can be understood as a theory with the symmetry of the conformal group expanded by the

[1] In two dimensions the conformal group is infinite dimensional, this fact can be seen directly by considering that in two dimensions we can work in the complex plane, thus considering that any holomorphic function defined on it can give rise to a conformal transformation, by expanding it in a Laurent series and then by linear independence of the terms in the series, we will see that each term in the series will be accompanied by a generator of the conformal group in two dimensions. Of course this result can be also obtained if we do not work in the complex

generators:

$$[M_{\mu\nu}, M_{\rho\sigma}] = M_{\rho\nu}\eta_{\sigma\mu} + M_{\mu\rho}\eta_{\nu\sigma} - M_{\sigma\nu}\eta_{\rho\mu} - M_{\mu\sigma}\eta_{\nu\rho} \quad (6.1)$$
$$[P_\rho, M_{\mu\nu}] = (\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \quad (6.2)$$
$$[K_\rho, M_{\mu\nu}] = (\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \quad (6.3)$$
$$[K_\mu, P_\nu] = 2(\eta_{\mu\nu}D - J_{\mu\nu}) \quad (6.4)$$
$$[D, P_\mu] = P_\mu \quad (6.5)$$
$$[D, K_\mu] = -K_\mu \quad (6.6)$$
$$(6.7)$$

Where $M_{\mu\nu}$ Lorentz transformations generators, P_μ are translations generators, D are dilatations generators, and K_μ are generators related to special conformal transformations. These generators acts on the field configurations that the theory posses, as they usually does in field theory, depending on the nature of the field, by means of finite or infinite dimensional representations.

Also, this symmetry group is isomorphic to the isometry group of AdS_{d+1} space, i.e. the group of transformations in a $d+2$ dimensional space that leaves invariant the quadric:

$$\sum_{i=1}^{d} x_i^2 - \sum_{n=1}^{2} y_n^2 = -R^2 \quad (6.8)$$

The group action in the fundamental representation is on a vector defined in $\Re^{d,2}$. The induced metric on this surface is the AdS_{d+1} metric, and the isometry group of this surface can be realized infinitesimally trough its Killing vectors as generators, and the Lie bracket as operation (with Lie derivatives), and the action on field configurations which lives on AdS is realized trough the Lie derivative along the AdS Killing vectors. The algebra of Killing vectors is isomorphic to the $SO(d,2)$ algebra.

Furthermore the conformal boundary of a $d+1$ dimensional AdS space is the compactification of d dimensional Minkowsky space by adding some points at infinity [2]. Thus, the isometry group of AdS_{d+1}, at the boundary, can be seen as acting as the d dimensional conformal group does on the field configurations at the boundary.

6.3 AdS/CFT correspondence at level of quantum theories

The AdS/CFT correspondence originally ([43]) states that:

$$e^{W[\bar{\varphi}]} = \int [D\Phi] e^{-I_{grav}[\Phi]} \Big|_{\partial\text{AdS}\Phi\sim\bar{\varphi}} \quad (6.9)$$

Where $W[\bar{\varphi}]$ is the effective action of d-dimensional theory, which classically posses conformal symmetries (which may or may not survive at quantum level), and the right hand side is

plane by a straight forward analysis done by studying the properties of conformal transformations done in a general two dimensional metric. The conformal group in the two dimensional case will contain a sub-algebra which is isomorphic to the $SO(2,2)$ algebra and is isomorphic to the algebra of killing vectors in exact AdS_3.

[2]Analogously as the Riemann sphere is the compactification of $\Re^2$

the partition function of a $d+1$-dimensional theory, which may or may not include gravity and in which, classically, its gravitational background (dynamical or not, repectively) asymptotically enjoy the symmetries of AdS_{d+1} space-time. In the right hand side, the path integral over all the possible bulk fields configurations, it is constrained to be only over bulk fields which at the boundary of AdS space its leading behavior, carries as coefficient the value of the boundary field $\bar{\varphi}$.

On the left hand side of (6.9), for some d dimensional QFT, we have that the effective action of connected green functions $W(\bar{\varphi})$ is given by:

$$e^{W[\bar{\varphi}]} = \int [D\mathcal{O}] e^{-S_{\text{CFT}}[\mathcal{O}_i] + \int d^d x \bar{\varphi}_i \mathcal{O}_i} = \langle e^{\int d^d x \bar{\varphi}_i \widehat{\mathcal{O}}_i} \rangle_{\text{CFT}} \tag{6.10}$$

Note we added a subscript 'CFT' to recall that this discussion, which is valid in general, in this case will be related to the CFT side of the correspondence. In the notation used in (6.10), $\bar{\varphi}_i$ denote the sources which couples to $\mathcal{O}_i$ CFT fields, and $\widehat{\mathcal{O}}_i$ denote its quantum counterpart operators of the quantum 'CFT'. As usual, this allows us to construct correlations functions for the operators by taking functional derivatives of W with respect to the sources, and then setting the sources to zero, e.g.,:

$$\langle \widehat{\mathcal{O}}_1 \dots \widehat{\mathcal{O}}_n \rangle = \frac{\delta^n W}{\delta \bar{\varphi}_1 \dots \delta \bar{\varphi}_n} |_{\bar{\varphi}=0} \tag{6.11}$$

Of course higher-lower order correlators can be computed.
On the right hand side of (6.9), i.e. the AdS_{d+1} bulk side, the partition function for the $d+1$ dimensional gravitational theory, in a saddle point (semi classical) approximation, i.e., allowing only the contribution which comes from the classical configuration (for bulk on-shell fields given by φ), is defined as:

$$Z_{grav}(\varphi) = e^{I_{grav0}(\varphi)} \tag{6.12}$$

where φ are on-shell bulk fields, and $I_{\text{grav } 0}(\varphi)$ is the on shell [3] gravitational action.

In saddle point approximation of the quantum bulk theory, the AdS/CFT conjecture states that:

$$W(\bar{\varphi}) = I_{\text{grav } 0}(\varphi) \tag{6.13}$$

subject that the on-shell bulk fields φ posses boundary values given by $\varphi|_{\partial\text{AdS}} \sim \bar{\varphi}$, which of course now, due to the on-shell condition, they has to be given by boundary conditions compatible with the on shell condition for bulk fields in an AdS background, or in other background (a dynamical one if includes gravity) which asymptotically goes as AdS does. Note that, in the saddle point approximation, the bulk on-shell action means that we are dealing with a classical theory in $d+1$ dimensions, and on the other hand, the effective action of the d dimensional conformal theory means we are dealing with a quantum theory with conformal symmetries in the d dimensional boundary of the $d+1$ dimensional AdS space-time.

[3] To be the on-shell action, besides from being the action of the theory valued at the on-shell field configurations, it has to be such that its functional derivatives are well defined in order for this action truly posses an extrema. If this is not the case it has to be re-defined by adding suitable boundary terms to it.

6.4 Holographic renormalization and the radial coordinate paradigm

In AdS/CFT correspondence, specifically in the AdS side, one usually choose coordinates of the base manifold, such that one of them is normal to the *AdS* boundary. This coordinate runs from the boundary, along the bulk, up to the 'interior point' of the bulk manifold. The other coordinates are the ones that successful describes the boundary, and thus they are used to sitting the CFT fields to be defined on them. One usually uses some symmetric configuration of the bulk manifold, e.g., in euclidean AdS/CFT one can think in the sphere, in which the picture can be visualized as:

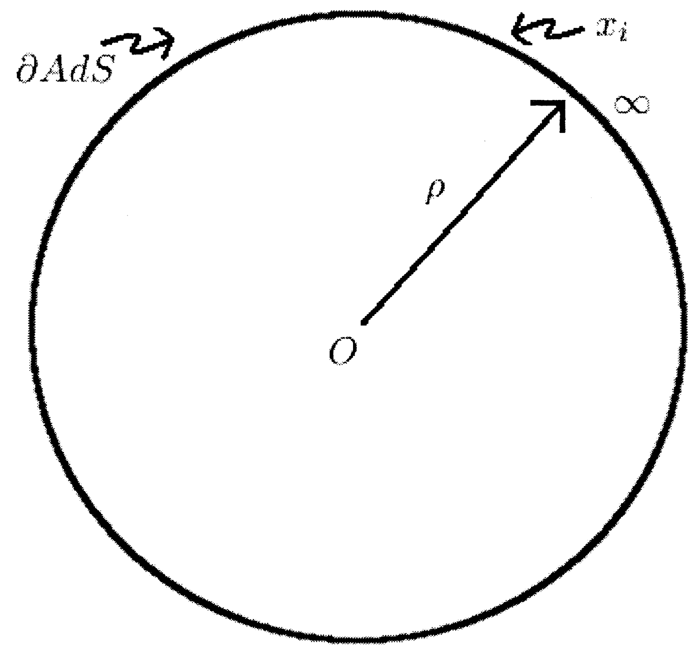

Figure 6.1: AdS_3 euclidean ball

Once we have properly chosen the coordinates to perform AdS_{d+1}/CFT_d computations, it may happen (and it usually does) that the 'gravitational' action I_{grav} valued on the on shell bulk fields configuration, turns out to be infinite at the d dimensional boundary. These infinities are divergences coming from the radial coordinate at the 'point' in which the boundary is located. On the other hand, looking the other side of the correspondence in (6.13), this phenomena manifest itself by infinites in the effective action W. Infinites in the quantum theory spots the necessity of carry out a renormalization process of W to get ride of them. This is the first clue that the radial coordinate is somehow related to an energy scale at which we are looking the quantum d dimensional theory. In order to renormalize the d dimensional boundary theory we can work on the gravitational side, and introduce a regulator, i.e., a cutoff in the radial coordinate to be able to identify the divergent terms, and then we can construct local boundary counter-terms to eliminate them, giving rise this way to a renormalized on shell gravitational action which is finite [45]. The existence of local counter terms, in general has been proven to be guarantied in the work [46]. Also, it could happen that in order for this renormalized (finite) on shell action posses a well defined Dirichlet problem, we could need to add some suitable finite local terms to it. The final result of this process is a renormalized effective action W_{ren} for the d dimensional theory which, after the renormalization process, could be or could not be conformal at some points of the 'energy scale' related to the radial coordinate of the bulk side. In fact, most of the systems, e.g. the dual d dimensional theory of AdS_{d+1} gravity (is only one of them) has shown to posses an anomaly called conformal anomaly (see [47]) at the boundary, such that if the classical boundary theory which gave rise to W_{ren} is conformal, the consistent quantization of this theory kills this symmetry at the quantum level, at the same energy scales at which the classical theory do posses it. However, it could happen that the conformal invariance at quantum level can be recovered at some other energy scales, i.e., leading to a null beta function.

Note that in the gravitational side of the correspondence, the AdS boundary is located at large distances, so from this side of the theory, the boundary divergences correspond to IR divergences of the 'would be' the quantum $d+1$ dimensional bulk theory, according to right hand side of the full correspondence (6.9). Furthermore, by a hand waving argument one can say that large distances from the centre at the bulk side (near the boundary, from the interior of the bulk point of view distance is large), corresponds to short distances from the boundary at the CFT side (near the boundary, from the boundary point of view distance is small), and just as this suggest it has been shown that the AdS/CFT conjecture use to relates, i.e., spots a duality relation [42] between weakly (AdS) / strongly (CFT) coupled theories at large bulk distances (IR), and between strongly (AdS) / weakly (CFT) coupled theories at short bulk distances (UV).

6.5 Abused and over-simplified scalar field example in exact AdS

One of the most abused and simplest examples of the AdS/CFT correspondence is the case of a massless scalar field living on fixed AdS background. The scalar field, massive or massless does not posses gauge symmetries, and here resides its simplicity: no gauge symmetries means no gauge fixing is needed at the boundary to isolate some 'would be' physical boundary degrees of freedom, thus it is hard to imagine how this can shed some light in the massless higher spin problem. However, it results to be the best example to explain the correspondence, because many of the general ideas of the correspondence are present, and now we will show it in detail. For this consider, we are in fixed AdS background with is metric given in Poincare coordinates:

$$ds^2 = \frac{1}{z^2}(dz^2 + dx_i dx^i), \text{where}\, i \,\text{runs from}\, i = 1, \ldots, d \tag{6.14}$$

where the boundary of AdS space is located at $z = 0$, and the interior is located at $z = \infty$.

The massive scalar field action on fixed AdS background is given by:

$$I_{AdS} = \frac{1}{2} \int d^{d+1}x\sqrt{g}(\nabla_\mu \phi \nabla^\mu \phi + m^2 \phi^2) \tag{6.15}$$

Its equations of motion are:

$$\Box\phi - m^2\phi = 0 \tag{6.16}$$

to get some knowledge about the asymptotic (near $z = 0$) boundary behavior of the solutions, consider first to solve for spherically symmetric solutions:

$$((1-d)z\partial_z + z^2\partial_z^2)\phi - m^2\phi = 0 \tag{6.17}$$

whose solutions are:

$$\phi(z) = \phi_0 z^{d-\Delta} + \phi_1 z^{\Delta} \tag{6.18}$$

where the exponent Δ, called conformal dimension [4], is one of the two solutions of the mass parameterization:

$$m^2 = \Delta(\Delta - d) \tag{6.19}$$

If we now we allow for a dependence on the boundary coordinates x_i, we can solve:

$$((1-d)z\partial_z + z^2\partial_z^2 + z^2\partial_i\partial^i)\phi - m^2\phi = 0 \tag{6.20}$$

which has an asymptotic solution (near $z = 0$) in a series expansion that schematically goes as:

$$\phi(z, x_i) = z^{d-\Delta}(\phi_0(x_i) + \mathcal{O}(z)) + z^{\Delta}(\phi_1(x_i) + \mathcal{O}(z)) \tag{6.21}$$

For the discussion that follows, lets assume that in in (6.21) we have chosen the value of Δ such that $d - \Delta < \Delta$, thus ϕ_0 will be the coefficient of the leading power near $z = 0$.

The asymptotic expansion (6.21) is useful to identify what would be the sources and what would be the vacuum expectation values (Vev's) in the boundary theory. To see this consider the variation of the action 6.15:

$$\delta I_{\text{AdS}} = -\int_{AdS} d^{d+1}x\sqrt{g}(\Box\phi - m^2\phi)\delta\phi + \int_{\partial AdS} d^d x\sqrt{g}\nabla^z\phi\delta\phi \tag{6.22}$$

Then plugging the on shell solution (6.21) on the variation 6.22, the bulk part is zero, and we only get contributions from the boundary (near $z = 0$):

$$\begin{aligned}
\delta I_{\text{AdS o.s}} &= 0_{\text{bulk}} + \int_{\partial\text{AdS}} d^d x \frac{1}{z^{d+1}} z^2\partial_z(z^{d-\Delta}\phi_0 + \ldots + z^{\Delta}\phi_1 + \ldots)(z^{d-\Delta}\delta\phi_0 + \ldots + z^{\Delta}\delta\phi_1 + \ldots) \\
&= \int_{\partial\text{AdS}} d^d x(\{(d-\Delta)z^{d-2\Delta}\phi_0\delta\phi_0 + \ldots\} + \{\Delta\phi_1\delta\phi_0 + (d-\Delta)\phi_0\delta\phi_1\} \\
&\quad + \{\Delta z^{2\Delta-d}\phi_1\delta\phi_1 + \ldots\})
\end{aligned} \tag{6.23}$$

The boundary part has been divided into three parts distinguished by curly brackets: the first part is divergent, the second part is finite and the third part is vanishing, as $z \to 0$.

The vanishing part does not provide information. On the other hand, it was shown in [46] that in general terms, i.e., for any theory, the divergent part can always be written as a total variation, thus it can always be suppressed from the original on shell action action $I_{\text{AdS on shell}}$ by

[4] It is the conformal dimension of the operator of the CFT, this can be seen by the fact that the full scalar field has to posses conformal dimension equal to zero, thus from (6.18) it follows that ϕ_1...

adding suitable local counter-terms to it, to build in this way a new renormalized on shell action in the holographic renormalization process. Also, in order to get a well posed Dirichlet problem, further terms corresponding to a total variation of the zero mode (finite part) are needed to be added to the renormalized on shell action. The final outcome of the application of this process to the variation of the original on shell action, is directly seen from (6.23) to be:

$$\delta I_{\text{ren}} = (2\Delta - d)\phi_1 \delta\phi_0(x) \tag{6.24}$$

This expression says that the final renormalized on shell action I_{ren} will be asymptotically well behaved (finite), and possessing an extrema when ϕ_0 is held fixed at the boundary. Leading to the relation:

$$\frac{\delta I_{\text{ren}}}{\delta\phi_0(x)} = (2\Delta - d)\phi_1(x) \tag{6.25}$$

where, according the correspondence, this means we will have a relation between the sub-leading component $\phi_1(x)$ of the bulk field at the boundary, and the Vev's in presence of sources, given by:

$$\langle \mathcal{O}(x) \rangle_{\text{sources}} = (2\Delta - d)\phi_1(x) \tag{6.26}$$

Also, having carefully followed the holographic renormalization process discussed above, but being applied directly to the original on-shell action (not to its variation) one gets the renormalized action I_{ren} as:

$$I_{ren} = \frac{1}{2}(2\Delta - d)\int d^d x \phi_0(x)\phi_1(x) \tag{6.27}$$

which using the correspondence means that the leading component ϕ_0 of the bulk field at the boundary, sources an operator $\mathcal{O}$ at the boundary:

$$W_{ren} = \frac{1}{2}\int d^d x \phi_0(x)\mathcal{O}(x) \tag{6.28}$$

Also, as (6.24) shows explicitly, the renormalized on shell action I_{ren} has to be considered as a function only on the sources $\phi_0(x)$. In fact note that, if we consider $\phi_0(x)$ and $\phi_1(x)$ as independent, from (6.27) we will have:

$$\frac{\delta I_{\text{ren}}}{\delta\phi_0(\vec{w})} = \frac{1}{2}(2\Delta - d)\phi_1(\vec{w}) \tag{6.29}$$

which gives a wrong answer because (6.29) does not coincide with (6.25) by a factor of one half. Thus, in order to get a well posed prescription and the correspondence makes sense, we have to consider the sources: $\phi_0(x)$ and Vevs: $\phi_1(x)$ as to be related in some way. Now we will see that they has to be related satisfying some regularity conditions at the interior $z = \infty$ of the bulk manifold.

To illustrate the process lets take the Fourier modes of the field $\phi_p(z) = \int d^d p \phi(z, x) e^{i x_i p^i}$ then making some further simplifications, eq. (6.20) can be reduced into a modified Bessel equation for the Fourier modes, whose solutions are in terms of modified Bessel functions:

$$\phi_p(z) = C_1 z^{\frac{d}{2}} I_{-\Delta+\frac{d}{2}}(pz) + C_2 z^{\frac{d}{2}} K_{\Delta-\frac{d}{2}}(pz) \tag{6.30}$$

This solution posses two integration constants, which has to be fixed some way to achieve regularity at the interior. As the modified Bessel function $I_a(pz)$ has $\lim_{z\to\infty} I_a(pz) = \infty,\ \forall a$, regularity at the interior demands $C_1 = 0$. However, in this Fourier transformed form of the solutions, it is difficult to see directly the relation that the non-transformed asymptotic leading $\phi_0(x)$ and sub-leading $\phi_1(x)$ components has to satisfy.

A possible way to see the relation directly, it is considering the already well behaved (finite) exact solution of (6.20) obtained by the method of green functions [43], which is regular at the interior ($z = \infty$) of the bulk manifold:

$$\phi_\Delta(z,\vec{x}) = \int d^dy K_\Delta(z,\vec{x};\vec{y})\phi_0(\vec{y}) \tag{6.31}$$

where:

$$K_\Delta(z,\vec{x};\vec{y}) = C_\Delta \left(\frac{z}{z^2+(\vec{x}-\vec{y})^2}\right)^\Delta \tag{6.32}$$

is called the boundary to bulk propagator, whose role as (6.31) show, is to build bulk fields starting from boundary fields, and C_Δ is an integration constant unimportant for this discussion.

Being regular, the solution (6.31) already encodes the regularity conditions that the components $\phi_0(x)$, $\phi_1(x)$ has to satisfy. In order to see it explicitly, we expand the form of $K_\Delta(z,\vec{x};\vec{y})$ near the boundary $z = 0$, considering two cases:

when $\vec{x} \neq \vec{y}$ in which, near $z = 0$, we have:

$$K_\Delta(z,\vec{x};\vec{y}) \approx C_\Delta \frac{z^\Delta}{(\vec{x}-\vec{y})^{2\Delta}} + \dots \tag{6.33}$$

where ... means terms of higher orders than z^Δ.

The other case is when $\vec{y}$ is near $\vec{x}$, case in which under the integral sign the function given by $z^{-d}K_\Delta(z,\vec{x};\vec{y})$ near $z = 0$ behaves as a multiple of the delta function (witten):

$$\int d^dy z^{-d}K_\Delta(z,\vec{x};\vec{y})f(y) \underset{z\to 0, x\to y}{\approx} z^{-\Delta}f(x) \tag{6.34}$$

using these results we have near $z = 0$:

$$\phi_\Delta(z,\vec{x}) = z^{d-\Delta}\phi_0(\vec{x}) + C_\Delta z^\Delta \int d^dy \frac{\phi_0(\vec{y})}{(\vec{x}-\vec{y})^{2\Delta}} + \underset{\text{regular terms}}{\dots} \tag{6.35}$$

Comparing with (6.21) at order z^Δ, we can identify the regularity condition on the asymptotic components, being:

$$\phi_1(\vec{x}) = C_\Delta \int d^dy \frac{\phi_0(\vec{y})}{(\vec{x}-\vec{y})^{2\Delta}} \tag{6.36}$$

Plugging this result in (6.27) we have:

$$I_{ren} = \frac{1}{2}(2\Delta - d)C_\Delta \int d^dx \int d^dy \frac{\phi_0(x)\phi_0(y)}{(\vec{x}-\vec{y})^{2\Delta}} + \dots \tag{6.37}$$

In which it can be see that taking functional derivatives with respect to the source $\phi(\vec{w})$ we have the right answer:

$$\frac{\delta I_{ren}}{\delta\phi_0(\vec{w})} = (2\Delta - d)C_\Delta \int d^dy \frac{1}{(\vec{w}-\vec{y})^{2\Delta}}\phi_0(\vec{y}) = (2\Delta - d)\phi_1(\vec{w}) \tag{6.38}$$

which correctly coincides with (6.25). Furthermore this expression becomes null when we turn off the source, which is an expected result.

Finally, taking one further derivative with respect to the source we have the two point function (propagator):

$$\frac{\delta^2 I_{ren}}{\delta\phi_0(\vec{x})\delta\phi_0(\vec{y})} = \frac{(2\Delta - d)C_\Delta}{(\vec{x} - \vec{y})^{2\Delta}} \equiv \langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle \tag{6.39}$$

which has the correct distance dependence for an operator $\mathcal{O}(\vec{x})$ of conformal dimension Δ.

6.6 Stressing the importance of bulk regular solution at the interior of the bulk manifold in AdS/CFT correspondence

Summarizing, with the above example, we saw that in principle, when we study boundary conditions for the bulk fields, the sources can be free asymptotically, i.e. they are not related with the Vev's at the boundary, but if we then go into the interior of the bulk manifold, regularity of the solution, which is needed for the consistency of the AdS/CFT correspondence, demands they be related at the boundary. In other words, for the consistency of the correspondence, it is not sufficient to have an on shell bulk solution, it is rather necessary to have a regular bulk solution which has to be accomplish as such, by some regularity conditions between sources and Vev's, which in final instance will lead to some integrability conditions necessary for the consistency of the correspondence. Consequently, this means that in order to compute correlation functions by taking functional derivatives of the renormalized on shell gravitational action, in this process we have to consider the on shell action as a function only in the sources. Once we have taken the functional derivative, and only then, we can set the sources equal to zero to get the correct correlation functions between operators. It is important to recall that this scheme repeats itself in other examples of the AdS/CFT correspondence, and in particular we will use it when we tackle the problem of higher spin black holes in chapter (8).

6.7 Further comments about the correspondence

Having shown the most simple example of the AdS/CFT correspondence, some comments are in order about the use of AdS/CFT:

- If we have a theory with other type of metric-like field content, e.g, vector fields, symmetric tensor fields, etc, Lorentz invariance of the boundary theory demands that we have to consider that the bulk fields defines, at the boundary, sources of the same nature in the CFT, which must respectively couple (sources) to its operator counterpart in the CFT. This means: as we saw in the explicit example, a scalar bulk field defines a scalar source at the boundary, this scalar source at the boundary it is coupled to a scalar operator of the CFT. Similarly, a vector bulk field defines a vector source at the boundary, this vector source at the boundary it is coupled to a vector operator of the CFT, etc. If we have AdS gravity as the bulk theory, the metric field at the boundary defines a boundary metric, which acts as a source in the boundary CFT. The corresponding CFT operator, sourced by this boundary metric, is the Energy-momentum tensor of the boundary CFT. For Higher spins ,this scheme must be in this same way (see [19]).

- The AdS/CFT correspondence can be applied by starting with an explicit theory in the bulk (AdS side), and then, after some work, one can compute the correlators between operators of the boundary theory, without explicitly make some realization of the boundary theory, i.e., without having knowledge of what is the explicit CFT Lagrangian. Something like this is what was done in the scalar field case. When working with gauge theories, in this case one usually can compute the symmetries that the dual CFT posses.
- Similarly, if we only know the symmetries of a CFT, without explicitly know which is the CFT Lagrangian, we can build the bulk theory starting from the boundary data, i.e., starting from sources and Vevs (see [45]).
- In the process of the holographic renormalization (see also [45], [19]), started with a given bulk Lagrangian (as in the above scalar field example), we could have build explicitly if we want, a new bulk Lagrangian which accounts for introducing interactions (or self-interactions) along the bulk, which were absent in the original bulk Lagrangian. These extra pieces would correspond to the needed deformation of the bulk theory in order for its quantization to be finite. This phenomena in AdS/CFT language it is usually called by the phrase 'Adding interactions along the bulk'.

Chapter 7

Review of Holography in Chern Simons formulation

7.1 Introduction

In this section we will see how from a CS action in a manifold with boundary, but without a metric structure, and only linear in first order derivatives of the fields, people use to do holography.

7.2 Chern-Simons holography from the gauge connections without a metric structure

Recall that the Chern Simons action is a special case which cannot be directly connected with the AdS/CFT picture as was done for the scalar case in chapter (6), because its equations of motion are only first order in derivatives, and furthermore it lacks of a metric structure on all its expressions. Thus, in a first attempt is not directly to see how to carry out an holographic study for a system which is describe by the CS action, as one would do it for e.g. the scalar field case.

7.2.1 Global W_3 symmetries from Chern-Simons theory

In chapter (5) we saw how from a globally defined (radial independent) gauge fixing in the classical CS theory with boundary, one can obtain a gauge fixed CS gauge connection with remnant classical W_3 symmetry, in which the remaining degrees of freedom, i.e., the fields $\mathcal{L}(x_+), \mathcal{W}(x_+)$ can be interpreted as the 'physical' degrees of freedom of a theory which lives on the two dimensional boundary. Now we will see how this result can be connected with the AdS/CFT picture, in which holography allows to interpret these fields as vacuum expectation values for their analogous operators $\widehat{\mathcal{L}}(x_+), \widehat{\mathcal{W}}(x_+)$ of a quantum theory with W_3 conformal symmetry at the boundary of AdS space-time.

7.2.2 Adding sources: CFT Ward identities from a deformation of the CFT action

Now, in the higher spin context, we will see the problem of how the symmetries of the classical system manifest itself in the quantum theory as some existent relation between operators correlation functions, leading, in this way to some identities between correlators called Ward identities. The main idea of this section, in the higher spin context, has been developed in [48], which uses a slightly different line of though than the earlier work [49] in which Ward identities computations were developed in light cone gauge coordinates for holomorphic fields. It is worth to mention that the work [48] has been partially inspired by the work [50] presented in this thesis in which the particular case for constant fields was firstly developed to subsequently be strongly supported with some of the ideas presented in [51] from a Hamiltonian point of view. Even if the work [50] chronologically appears earlier than [48], for an easier exposure of the ideas the work [48] will be shown firstly because it makes an easier contact with what is pretend to be shown here.

To start, lets focus on one side of the AdS/CFT correspondence and compute the effective action for the two dimensional conformal theory that lives on the boundary in the usual way, with the partition function: $Z[\bar{\varphi}] = e^{W(\bar{\varphi})} = \langle e^{\int d^2x \bar{\varphi}\mathcal{O}} \rangle_{CFT}$, where $\bar{\varphi}$ are the sources and $\widehat{\mathcal{O}}$ are the CFT fields or the symmetry currents of the CFT, etc. In the following lets think that we arrive to this expression starting with Hamiltonian path integral method as if we were functionally integrating over the Hamiltonian phase space. In this way, all the expectation values $\langle \ldots \rangle$ can be thought as taken with the Hamiltonian path integral of the CFT. Thus a possible insertion of the exponential operator $e^{\int d^2x \bar{\varphi}\mathcal{O}}$, inside correlations functions, can be understood as a Hamiltonian deformation of the CFT action. At the moment, it is unknown to us the explicit form of the CFT action (Lagrangian or Hamiltonian), but we know through the analysis done in chapter 4, that it posses a conformal symmetry which is realizable canonically through the Poisson bracket of the charges 5.93, which is well defined at the Boundary 'point'. Thus this symmetry has to be a symmetry of the unknown CFT action, an due to it is canonically realizable it must be a symmetry of the CFT hamiltonian. Thus the deformation we will introduce on the CFT, it will be a symmetry of the Hamiltonian CFT system.

Now, specifically in our case, we will have:

$$Z[\varepsilon, \mu] = e^{W(\varepsilon,\mu)} = \langle e^{\int d^2x(\varepsilon\mathcal{L}+\mu\mathcal{W})} \rangle_{CFT} \tag{7.1}$$

This expression does not tell us anything about the method used to compute the effective action, and certainly it should not depend on the method (Lagrangian or Hamiltonian). But as we are interested in compute the Ward identities for correlators in presence of sources, we need to use it without turning off the sources.

In the following we will make emphasis in the quantum operators description of the system by denoting explicitly the CFT fields inside the original path integral, as quantum operators inside brackets:

$$Z[\varepsilon, \mu] = \langle e^{\int d^2x(\varepsilon\widehat{\mathcal{L}}+\mu\widehat{\mathcal{W}})} \rangle_{CFT} \tag{7.2}$$

where temporal ordering at the left side has been assumed.

Now, consider the expectation values for the operators in presence of sources, by functionally differentiating (7.2) with respect to one of the sources, g.e., lets say ε, and showing explicitly the temporal ordering denoted by the operator '$\mathcal{T}$', we get:

$$\langle\widehat{\mathcal{L}}\rangle_{\varepsilon,\mu} = \frac{\delta Z}{\delta\varepsilon} = \langle\mathcal{T}(e^{\int d^2x(\varepsilon\widehat{\mathcal{L}}+\mu\widehat{\mathcal{W}})}\widehat{\mathcal{L}})\rangle \tag{7.3}$$

and by taking temporal derivatives at both sides of (7.3) we get:

$$\partial_{t'}\langle\widehat{\mathcal{L}}(t',\varphi')\rangle_{\varepsilon,\mu} = \partial_{t'}\langle\int dtd\varphi\varepsilon(t,\varphi)\mathcal{T}(\widehat{\mathcal{L}}(t,\varphi)\widehat{\mathcal{L}}(t',\varphi')) + \mu(t,\varphi)\mathcal{T}(\widehat{\mathcal{W}}(t,\varphi)\widehat{\mathcal{L}}(t',\varphi'))\rangle_{\varepsilon\mu} \tag{7.4}$$

Now considering an expression of the form:

$$f(t',\varphi') = \int dtd\varphi\ \alpha(t,\varphi)\langle\mathcal{T}(\widehat{\mathcal{X}}(t',\varphi')\widehat{\mathcal{Y}}(t,\varphi))\rangle \tag{7.5}$$

Dividing into two parts the integral interval in (7.5),i.e. first for $-\infty \leq t \leq t'$ and then for $t' \leq t \leq \infty$, and considering explicitly the temporal ordering we get:

$$f(t',\varphi') = \int_{t_1}^{t'} dtd\varphi\ \alpha(t,\varphi)\langle\widehat{\mathcal{X}}(t',\varphi')\widehat{\mathcal{Y}}(t,\varphi)\rangle - \int_{t_2}^{t'} dtd\varphi\ \alpha(t,\varphi)\langle\widehat{\mathcal{Y}}(t,\varphi)\widehat{\mathcal{X}}(t',\varphi')\rangle \tag{7.6}$$

then from 7.6, it follows exactly the identity:

$$\partial_{t'}f(t',\varphi') = \int d\varphi\ \alpha(t',\varphi)\langle[\widehat{\mathcal{X}}(t',\varphi')\,,\,\widehat{\mathcal{Y}}(t',\varphi)]\rangle \tag{7.7}$$

Then using (7.7) applied on (7.4) we get:

$$\partial_t\langle\widehat{\mathcal{L}}(t,\varphi')\rangle_{\varepsilon,\mu} = \langle\int d\varphi\varepsilon(t,\varphi)[\widehat{\mathcal{L}}(t,\varphi)\,,\,\widehat{\mathcal{L}}(t,\varphi')] + \mu(t,\varphi)[\widehat{\mathcal{W}}(t,\varphi)\,,\,\widehat{\mathcal{L}}(t,\varphi')]\rangle_{\varepsilon,\mu} \tag{7.8}$$

An analogous computation can be done for the Vev of the operator $\widehat{\mathcal{W}}$ in presence of sources:

$$\partial_t\langle\widehat{\mathcal{W}}(t,\varphi')\rangle_{\varepsilon,\mu} = \langle\int d\varphi\varepsilon(t,\varphi)[\widehat{\mathcal{L}}(t,\varphi)\,,\,\widehat{\mathcal{W}}(t,\varphi')] + \mu(t,\varphi)[\widehat{\mathcal{W}}(t,\varphi)\,,\,\widehat{\mathcal{W}}(t,\varphi')]\rangle_{\varepsilon,\mu} \tag{7.9}$$

Now is when the symmetry comes into play. Lets consider the equal-time Poisson bracket of the W_3 algebra of the reduced phase space 5.93 at boundary. This algebra is satisfied by the global charges which lives on the boundary where it also lives the conformal theory as we saw in chapter 4. Now promote the fields to operators and classical bracket to quantum commutators:

$$\begin{aligned}
[\widehat{\mathcal{L}}(\varphi),\ \widehat{\mathcal{L}}(\varphi')] &= -(\delta(\varphi-\varphi')\widehat{\mathcal{L}}'(\varphi) + 2\delta'(\varphi-\varphi')\widehat{\mathcal{L}}(\varphi) + \frac{k}{4\pi}\delta'''(\varphi-\varphi'))\\
[\widehat{\mathcal{L}}(\varphi),\ \widehat{\mathcal{W}}(\varphi')] &= -(2\delta(\varphi-\varphi')\widehat{\mathcal{W}}'(\varphi) + 3\delta'(\varphi-\varphi')\widehat{\mathcal{W}}(\varphi))\\
[\widehat{\mathcal{W}}(\varphi),\ \widehat{\mathcal{W}}(\varphi')] &= -\frac{1}{12}(2\delta(\varphi-\varphi')\widehat{\mathcal{L}}'''(\varphi) + 9\delta'(\varphi-\varphi')\widehat{\mathcal{L}}''(\varphi) + 15\delta''(\varphi-\varphi')\widehat{\mathcal{L}}'(\varphi)\\
&\quad +10\delta'''(\varphi-\varphi')\widehat{\mathcal{L}}(\varphi) + \frac{k}{4\pi}\delta^{(5)}(\varphi-\varphi')\\
&\quad +\frac{64\pi}{k}(\delta(\varphi-\varphi')\widehat{\mathcal{L}}(\varphi)\widehat{\mathcal{L}}'(\varphi) + \delta'(\varphi-\varphi')\widehat{\mathcal{L}}^2(\varphi)))
\end{aligned} \tag{7.10}$$

Then we plug the expressions (7.10) into (7.8) (and also into (7.9)), and denoting the expectation values of the operators in presence of sources as:

$$\mathcal{L} = \langle\widehat{\mathcal{L}}\rangle_{\varepsilon,\mu} \tag{7.11}$$
$$\mathcal{W} = \langle\widehat{\mathcal{W}}\rangle_{\varepsilon,\mu} \tag{7.12}$$

after taking the angular integral, making use of several Dirac delta identities, we get:

$$\begin{aligned}\dot{\mathcal{L}} &= \frac{k}{4\pi}\varepsilon''' + 2\mathcal{L}\varepsilon' + \mathcal{L}'\varepsilon + 2\mathcal{W}'\chi + 3\mathcal{W}\chi' \\ \dot{\mathcal{W}} &= \mathcal{W}'\varepsilon + 3\mathcal{W}\varepsilon' - \frac{5}{4}\mathcal{L}'\chi'' - \frac{5}{6}\mathcal{L}\chi''' - \frac{k}{48\pi}\chi^{(5)} - \frac{3}{4}\mathcal{L}''\chi' - \frac{1}{6}\mathcal{L}'''\chi - \frac{16\pi}{3k}\mathcal{L}(\mathcal{L}'\chi + \mathcal{L}\chi')\end{aligned} \tag{7.13}$$

Expressions (7.13) are the time-evolution equations for the charges. From these expressions, considering the time evolution that each parameter produces by itself, in each one of the fields, we get:

$$\begin{aligned}\delta_\varepsilon\mathcal{L} &= \frac{k}{4\pi}\varepsilon''' + 2\mathcal{L}\varepsilon' + \mathcal{L}'\varepsilon \\ \delta_\chi\mathcal{L} &= 2\mathcal{W}'\chi + 3\mathcal{W}\chi' \\ \delta_\varepsilon\mathcal{W} &= \mathcal{W}'\varepsilon + 3\mathcal{W}\varepsilon' \\ \delta_\chi\mathcal{W} &= -\frac{5}{4}\mathcal{L}'\chi'' - \frac{5}{6}\mathcal{L}\chi''' - \frac{k}{48\pi}\chi^{(5)} - \frac{3}{4}\mathcal{L}''\chi' - \frac{1}{6}\mathcal{L}'''\chi - \frac{16\pi}{3k}\mathcal{L}(\mathcal{L}'\chi + \mathcal{L}\chi')\end{aligned} \tag{7.14}$$

These expressions are the Ward identities associated to W_3 algebra.

7.2.3 Recovering CFT Ward identities holographically from the classical CS bulk constraint

Now we will carry out the computations of the CFT Ward identities, but holographically, i.e., deriving them from the classical CS bulk constraint. For this we step into the radial gauge given by $A_\rho = 0$ and consider we have a gauge fixed A_ϕ connection, which satisfy the equations of motion $F_{\rho\phi} = 0$ given by:

$$A_\phi = L_1 + \frac{4\pi}{k}\mathcal{L}(t,\phi)L_{-1} + \frac{8\pi}{k}\mathcal{W}(t,\phi)W_{-2} \tag{7.15}$$

Then we consider a general A_t given along all the $SL(3,\Re)$ components:

$$\begin{aligned}A_t &= \varepsilon(t,\phi)L_1 + \varepsilon_0(t,\phi)L_0 + \varepsilon_{-1}(t,\phi)L_{-1} \\ &\quad +\chi(t,\phi)W_2 + \chi_1(t,\phi)W_1 + \chi_0(t,\phi)W_0 + \chi_{-1}(t,\phi)W_{-1} + \chi_{-2}(t,\phi)W_{-2}\end{aligned} \tag{7.16}$$

After that, for arbitrary charges $\mathcal{L}, \mathcal{W}$, we solve the CS equations of motions $F_{t\phi} = 0$. This system fix 6 of the 8 parameters which enters in the components of the $SL(3,\Re)$ Lie algebra along a_t, where the χ, ε components of the a_t field, are left undetermined. However we have 8 equations, thus in order to completely fulfil the equations of motion, we need that two extra conditions be imposed on the system. These condition gives the time evolution of the fields $\mathcal{L}, \mathcal{W}$.

In details, the values of the six parameters inside (7.16) which partially solves the equations of motion $F_{t\phi} = 0$ are:

$$\begin{aligned}
\varepsilon_0 &= -\varepsilon' \\
\varepsilon_{-1} &= \varepsilon'' + \frac{4\pi}{k}\mathcal{L}\varepsilon + \frac{8\pi}{k}\mathcal{W}\chi \\
\chi_1 &= -\frac{1}{2}\chi' \\
\chi_0 &= \frac{1}{3}\left(\frac{1}{2}\chi'' + \frac{4\pi}{k}\mathcal{L}\chi\right) \\
\chi_{-1} &= -\frac{1}{3}\left(\frac{1}{2}\chi''' + \frac{4\pi}{k}\mathcal{L}\chi + \frac{10\pi}{k}\mathcal{L}\chi'\right) \\
\chi_{-2} &= -\frac{1}{6}\chi'''' - \frac{16\pi}{3k}\mathcal{L}\chi'' - \frac{14\pi}{3k}\mathcal{L}'\chi' - \frac{4\pi}{3k}\mathcal{L}''\chi - \frac{16\pi^2}{k^4}\mathcal{L}^2\chi + \frac{8\pi}{k}W\varepsilon
\end{aligned} \tag{7.17}$$

where a 'prime' denotes derivative with respect to ϕ.

The system is completely solved if also the charges follows the temporal evolution given by:

$$\dot{\mathcal{L}} = \frac{k}{4\pi}\varepsilon''' + 2\mathcal{L}\varepsilon' + \mathcal{L}'\varepsilon + 2\mathcal{W}'\chi + 3\mathcal{W}\chi' \tag{7.18}$$

$$\dot{\mathcal{W}} = \mathcal{W}'\varepsilon + 3\mathcal{W}\varepsilon' - \frac{5}{4}\mathcal{L}'\chi'' - \frac{5}{6}\mathcal{L}\chi''' - \frac{k}{48\pi}\chi^{(5)} - \frac{3}{4}\mathcal{L}''\chi' - \frac{1}{6}\mathcal{L}'''\chi - \frac{16\pi}{3k}\mathcal{L}(\mathcal{L}'\chi + \mathcal{L}\chi') \tag{7.19}$$

where a dot denotes temporal derivative.

From the above expressions (7.18),(7.19), we consider the time evolution that each parameter ε, χ produces by itself in each of one the fields $\mathcal{L}, \mathcal{W}$:

$$\begin{aligned}
\delta_\varepsilon\mathcal{L} &= \frac{k}{4\pi}\varepsilon''' + 2\mathcal{L}\varepsilon' + \mathcal{L}'\varepsilon \\
\delta_\chi\mathcal{L} &= 2\mathcal{W}'\chi + 3\mathcal{W}\chi' \\
\delta_\varepsilon\mathcal{W} &= \mathcal{W}'\varepsilon + 3\mathcal{W}\varepsilon' \\
\delta_\chi\mathcal{W} &= -\frac{5}{4}\mathcal{L}'\chi'' - \frac{5}{6}\mathcal{L}\chi''' - \frac{k}{48\pi}\chi^{(5)} - \frac{3}{4}\mathcal{L}''\chi' - \frac{1}{6}\mathcal{L}'''\chi - \frac{16\pi}{3k}\mathcal{L}(\mathcal{L}'\chi + \mathcal{L}\chi')
\end{aligned} \tag{7.20}$$

These expressions are exactly the same as (7.14), but to confirm that they are the Ward identities, they must come from variations of the action with respect to the sources.

If we compute the on shell variation of the CS action, the bulk part is zero because the equations of motion holds, and we are left with only a boundary term, where the explicit parameters of the solutions appears as:

$$\begin{aligned}\delta I^0_{CS} &= 0_{\text{bulk}} + \frac{k}{4\pi}\int dtd\phi(A_\phi\delta A_t - A_t\delta A_\phi) \\ &= -\int dtd\phi\left(\frac{k}{2\pi}\delta\varepsilon'' + 4\mathcal{L}\delta\varepsilon + 2\chi\delta\mathcal{W} + 6\mathcal{W}\delta\chi\right) \end{aligned} \tag{7.21}$$

then, we can sum to the action a local term B_{tot} as:

$$B_{tot} = \int dtd\phi\left(\frac{k}{2\pi}\varepsilon'' + 2\chi\mathcal{W}\right) \tag{7.22}$$

defining a new action $I_{new} = I^0_{CS} + B_{tot}$ whose variations gives:

$$\delta I_{new} = -4\int dtd\phi\,(\mathcal{L}\delta\varepsilon + \mathcal{W}\delta\chi) \tag{7.23}$$

From the variation of this action we read,

$$\begin{aligned}\frac{\delta I_{new}}{\delta\varepsilon} &= -4\mathcal{L} \\ \frac{\delta I_{new}}{\delta\chi} &= -4\mathcal{W}\end{aligned}$$

which confirms that the expressions given in (7.20), are the Ward identities for correlation functions derived from I_{new}.

Chapter 8

The Action for Higher Spin (N) Black Holes in three dimensions

8.1 Introduction

In this chapter we will see that the well defined, i.e., with an allowed extrema, euclidean on shell CS action, defined on the non-trivial topology of the solid torus, leads to a consistent thermodynamic picture for higher spin black holes. This result has been motived by the fact that in [49] it has been conjectured the existence of a partition function trough some integrability conditions which arises as consequence of some holonomy conditions that must be imposed on the fields as they are defined on the torus. The integrability conditions demands that a well defined functional exist from which, by functional differentiation, one can extract Vev's of some physical quantities of interest. In this chapter we will show that the above functional is in fact the CS action, but properly modified by adding some properly defined boundary terms, and subsequently valued on the solutions of the equations of motion, which after that, is usually called the on shell CS action. The result of this chapter has been published in the work [50].

8.2 Classifying solutions by holonomies

Euclidean black holes lives on the non-trivial solid torus topology and as we will be interested in evaluating the action for this type of solutions, in what follows we will work on this topology (see figure 8.1), but fixing the rank of coordinates a priori as:

$$0 \leq t \leq 1 \ , \ 0 \leq \phi \leq 2\pi \ , \ 0 \leq r \leq \infty \tag{8.1}$$

It is important to recall that one usually works with a free periodicity β on the temporal coordinate:

$$0 \leq t \leq \beta \ , \ 0 \leq \phi \leq 2\pi \ , \ 0 \leq r \leq \infty \tag{8.2}$$

which is usually related to the inverse temperature of the solution. But with the choice (8.1) of a priori fixed-rank topology, if one want to keep the freedom of the periodicity β of the fields on the topological manifold (8.2), one should recover this freedom by an explicit apparition of

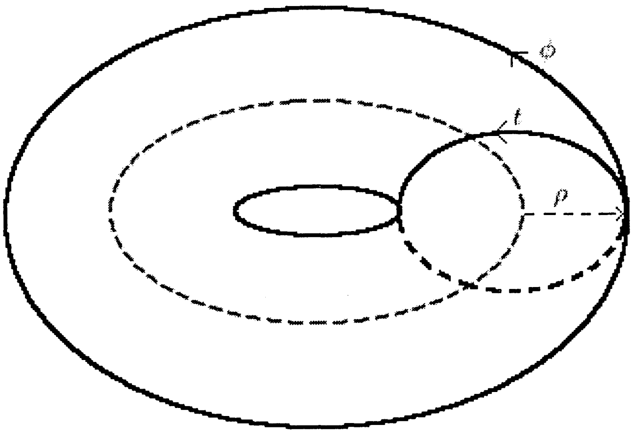

Figure 8.1: Solid torus topology

this periodicity-parameter, but now inside the fields, as one straight-forward change of coordinates in the integral sign would demand.

The solutions to CS equation of motion $F = 0$ are flat connections, which at first naive sight could lead one to think there is no interesting solutions. In fact, one can argue that a particular solution, being a flat connection, can be obtained by gauge transforming the trivial solution $A = 0$ with a particular group element, and thus one can think that all the solutions are gauge equivalent to the trivial one. This is only true when we have defined our gauge fields on simply connected manifolds with no holes inside, in which we can use any particular well defined gauge transformation to build any particular solution starting from the trivial one. Over this trivial class of topological manifolds the converse is also true, supposing we have a solution with $A \neq 0$ we can bring this solution to the zero solution by performing a well defined gauge transformation. However, as we are working on the non-trivial topology of the solid torus, the above statement is not longer true and we must classify solutions by its holonomies around contractile and non-contractile cycles.

The whole point of holonomies is the following: suppose you have a gauge connection over a non-trivial manifold and that you want to bring it to zero by a gauge transformation, the answer to the question if this is really possible, depends on whether the space-time components of the gauge connection are along contractile or non-contractile cycles. As the manifold is non-trivial, i.e., with some holes inside, one can define different class of curves on it. If there is only one hole, one class will be the class of contractile curves into a point, which are defined as going around a contractile cycle, and the other possible class will be the non-contractile ones which goes along the non-contractile cycle that surrounds the hole. Then, coming back to the question, the answer is that space-time components of the gauge field which goes as a contractile curve does, can be made zero by a well defined gauge transformation, meanwhile space-time components which goes as a non-contractile curve does cannot be made zero by a well defined gauge transformation, but only by means a multivalued gauge transformation which leads, at final instance, to a gauge field which is singular at the points in which the multivalued gauge group element changes to another branch.

In order to build black holes, we need that the metric-like fields be static and circularly symmetric, and we can guaranty this by considering the set of static and circularly symmetric solutions, which has the form:

$$\begin{aligned} A_r &= g_1^{-1}\partial_r g_1 \\ A_\phi &= g_1^{-1}a_\phi g_1 \\ A_t &= g_1^{-1}a_t g_1 \end{aligned} \tag{8.3}$$

where $g_1 = g_1(r)$ is a $SL(N,\Re)$ group element which depend only on the radial coordinate, and a_t, a_ϕ are constant $SL(N,\Re)$ algebra matrices which satisfy $[a_t, a_\phi] = 0$. Analogous type of solutions (static and circularly symmetric) are use for the other copy of the gauge fields:

$$\begin{aligned} B_r &= g_2^{-1}\partial_r g_2 \\ B_\phi &= g_2^{-1}b_\phi g_2 \\ B_t &= g_2^{-1}b_t g_2 \end{aligned} \tag{8.4}$$

Due that we are working from a theory whose action can be written as the difference of two CS actions, and we will be mainly working with this total action, we will focus all of the following discussion only for one copy of the gauge field (and one CS action), because for the other copy this discussion is analogous and straight forwardly constructed.

For the small case connection, which fulfils the equations of motion $[a_t, a_\phi] = 0$, it is obvious that $a_t = f(a_\phi)$. Thus, we can write the general solution as a power series in a_ϕ to then remove the trace in order that a_t be an algebra element. But due to the Cayley-Hamilton theorem [1] we can use the matrix valued characteristic polynomial to rewrite the power series solution as a polynomial with a finite amount of terms. In the case of the $SL(N,\Re)$ algebra, the resulting polynomial is of order $N-1$ and, as such, a number of $N-1$ arbitrary parameters are introduced. Finally, after removing the trace, the most general form of a_t is left as:

$$a_t = \sigma_2 a_\phi + \sigma_3\left(a_\phi^2 - \frac{I_{N\times N}}{N}\mathrm{Tr}(a_\phi^2)\right) + \cdots + \sigma_N\left(a_\phi^{N-1} - \frac{I_{N\times N}}{N}\mathrm{Tr}(a_\phi^{N-1})\right) \tag{8.5}$$

where $\sigma_2, \sigma_3, \ldots, \sigma_N$ are the $N-1$ arbitrary parameters, which will turn to be crucial for the description of the thermodynamic behavior of the solution because they will play the role of chemical potentials.

Also, it will be very convenient to use the definition of the i-th power-like invariant Casimir Operator- related parameters:

$$Q_2 = \frac{1}{2}\mathrm{Tr}(a_\phi^2), \qquad Q_3 = \frac{1}{3}\mathrm{Tr}(a_\phi^3), \quad \ldots \quad , Q_N = \frac{1}{N}\mathrm{Tr}(a_\phi^N). \tag{8.6}$$

This set of parameters will also be used for the thermodynamic description, playing the roles of charges. Furthermore, in the sequel we will see that the pairs: σ_n, Q_n, for $n = 2, \ldots, N$ play the role of canonically conjugated thermodynamics variables.

[1] It is a trivial theorem that says that any square matrix satisfy its own characteristic polynomial.

At this point of the discussion it is important to stress that for a given solution of the equations of motion fulfills the holonomy conditions discussed above, the charges Q_n must be fixed in terms of chemical potentials σ_n or vice-versa, more of this will be discussed in the next sections.

8.3 The on shell action for higher spin black holes

Lets consider to have the CS action as:

$$I_{CS}[A] = \frac{k}{4\pi}\int tr(\, A\wedge dA + \frac{2}{3}A\wedge A\wedge A\,) \tag{8.7}$$

and take its variations to get:

$$\delta I_{CS} \;=\; \frac{k}{4\pi}\int_{\mathbb{R}\times T_2} \mathrm{Tr}(F\wedge\delta A) + \frac{k}{4\pi}\int_{T_2}\mathrm{Tr}(A\wedge\delta A) \tag{8.8}$$

where, due that on shell $F=0$, the on shell bulk part is zero and we are left with only a boundary term:

$$\delta I_{CS}|_0 \;=\; 0_{\mathrm{bulk}} \;+\; \frac{k}{4\pi}\int_{r\to\infty} dt d\phi\,\mathrm{Tr}(A_\phi\delta A_t - A_t\delta A_\phi)\,. \tag{8.9}$$

Then, considering a solution of the form (8.3), is very easy to see that the radial dependent group element plays no role in this analysis, in fact plugging (8.3) inside (8.9) we have:

$$\delta I_{CS}|_0 \;=\; \frac{k}{4\pi}\int dt d\phi\,\mathrm{Tr}(a_\phi\delta a_t - a_t\delta a_\phi + 2[a_t, a_\phi]g\delta g^{-1})\,. \tag{8.10}$$

where the last term vanish for static and circularly symmetric solutions, and even if the solutions are not of this sort, the last term under the integral sign vanish as consequence of the periodicity of the fields living on the solid torus, giving us:

$$\delta I_{CS}|_0 \;=\; \frac{k}{4\pi}\int dt d\phi\,\mathrm{Tr}(a_\phi\delta a_t - a_t\delta a_\phi) \tag{8.11}$$

Then, considering the explicit form for the small case connections that a solution must satisfy, given by (8.5) which inserted in (8.11) gives:

$$
\begin{aligned}
\delta I_{CS}|_0 &= \frac{k}{2}\sum_{n=2}^{N} \mathrm{Tr}\left(a_\phi \delta(\sigma_n a_\phi^{n-1}) - \sigma_n a_\phi^{n-1}\delta a_\phi\right) \\
&= \frac{k}{2}\sum_{n=2}^{N} \mathrm{Tr}\left(a_\phi^n \delta\sigma_n + \frac{n-2}{n}\sigma_n \delta(a_\phi^n)\right) \\
&= \frac{k}{2}\sum_{n=2}^{N} (nQ_n\delta\sigma_n + (n-2)\sigma_n\delta Q_n) \\
&= k\sum_{n=2}^{N} Q_n\delta\sigma_n + \delta\left(\frac{k}{2}\sum_{n=2}^{N}(n-2)\sigma_n Q_n\right)
\end{aligned}
\tag{8.12}
$$

where in the third equality of (8.12), we have used (8.6), and where in the last line, the last term is a total variation which we can pass it into the other side to define a new action I_{os} given by:

$$
I_{\text{os}} \equiv I_{CS}|_0 - \sum_{n=2}^{N} \frac{k}{2}(n-2)\sigma_n Q_n \tag{8.13}
$$

such that 8.12 guaranties that the new action I_{os} it is a function only on the chemical potentials σ_n as:

$$
\delta I_{\text{os}} = k\sum_{n=2}^{N} Q_n\delta\sigma_n \tag{8.14}
$$

This expression tell us that the action (8.13) posses an extrema when chemical potentials σ_n are fixed, and as we must have $I_{\text{os}}(\sigma_n)$ this action is adequate for a '(grand) canonical' description of the system, where the charges Q_n can fluctuate. Similarly, by performing a Legendre transformation acting on (8.13), one can define a new action $I_{\text{os}}^{\text{New}}$ which will depends only on the charges Q_n as:

$$
I_{\text{os}}^{\text{New}} = I_{\text{os}} - k\sum_{n=2}^{N} Q_n\sigma_n \tag{8.15}
$$

such that varying (8.15) and considering (8.14) we get:

$$
\delta I_{\text{os}}^{\text{New}} = -k\sum_{n=2}^{N} \sigma_n\delta Q_n \tag{8.16}
$$

In this case $I_{\text{os}}^{\text{New}}(Q_n)$ will posses an extrema when the charges are kept fixed. The action $I_{\text{os}}^{\text{New}}(Q_n)$ is adequate for a 'micro-canonical' description as the chemical potentials are allowed to fluctuate.

Now, we still need the value of the CS action valued on shell: $I_{CS}|_0$. To find it we can pick the angle ϕ as a special coordinate to do a foliation of the solid torus. This foliation is regular everywhere as is made by using planes (regular surfaces) which do not intersect at any point. Using a 2+1 decomposition using the angle ϕ as the direction in which we do the foliation, we get:

$$I_{CS} = \frac{k}{4\pi}\int d\phi \int d^2x \mathrm{Tr}\epsilon^{\alpha\beta}(-A_\alpha\partial_\phi A_\beta + A_\phi F_{\alpha\beta}) - \frac{k}{4\pi}\int_{r\to\infty} dt d\phi\, \mathrm{Tr}(A_t A_\phi)\,. \tag{8.17}$$

In the bulk part, this foliation is covariant in the two dimensional planes at some constant ϕ, thus we can use a well defined set of coordinates to value it. Also, as respect to the on shell value of (8.17), the bulk part is zero as consequence of the circular symmetry of the fields (i.e.: $\partial_\phi A_\beta = 0$) and also due to the fact that it is a solution (i.e.: $F_{\alpha\beta} = 0$), thus we get:

$$I_{CS}|_0 = -\frac{k}{4\pi}\int_{r\to\infty} dt d\phi\, \mathrm{Tr}(A_t A_\phi)\,. \tag{8.18}$$

But, considering (8.3), the radial dependence is again factored out giving:

$$I_{CS}|_0 = -\frac{k}{4\pi}\int_{r\to\infty} dt d\phi\, \mathrm{Tr}(a_t a_\phi)\,. \tag{8.19}$$

and using (8.5) and (8.6) in (8.19) we have:

$$I_{CS}|_0 = -\frac{k}{2}\sum_{n=2}^{N} n\,\sigma_n\, Q_n. \tag{8.20}$$

Finally, using (8.20) in (8.13) the new action I_{os} is:

$$I_{\mathrm{os}} = -k\sum_{n=2}^{N}(n-1)\,\sigma_n\, Q_n \tag{8.21}$$

Note that if we vary the explicit final form of (8.21), then (8.14) is not automatically satisfied. In fact, as we said before, (8.14) tell us that I_{os} must be only a function of the chemical potentials, and as such, in order to vary it consistently, i.e.: fulfil consistency with (8.14), we have to consider some conditions which allows to express the charges Q_n in terms of the chemical potentials σ_n before take the variation.

The converse is also true for the micro canonical on-shell action, which with the value (8.20) in (8.15) we get:

$$I_{\mathrm{os}}^{\mathrm{New}} = -k\sum_{n=2}^{N} n\,\sigma_n\, Q_n \tag{8.22}$$

In the next section we will see that the right conditions for the consistency of the on shell variational problem are precisely given by the trivial holonomy conditions along the temporal (contractile) cycle.

Lastly, note that being the full action of the theory, given by the difference of two CS actions with two independent gauge fields: $I_{\mathrm{total}} = I_{\mathrm{CS}}[A] - I_{\mathrm{CS}}[\bar{A}]$, then the obvious result of this section is that the full on-shell CS action for the higher spin black holes will be given by the straight forward extension of the above expressions which consider both copies as. For example for the total on-shell CS action appropriate for higher spin black holes in a '(grand) canonical' description we have:

$$I_{\text{os}}(\sigma_n, \bar{\sigma}_n) = -k\sum_{n=2}^{N}(n-1)\,\sigma_n\,Q_n + k\sum_{n=2}^{N}(n-1)\,\bar{\sigma}_n\,\bar{Q}_n \tag{8.23}$$

such that:

$$\delta I_{\text{os}} = k\sum_{n=2}^{N} Q_n\delta\sigma_n - k\sum_{n=2}^{N}\bar{Q}_n\delta\bar{\sigma}_n \tag{8.24}$$

meanwhile in a 'micro-canonical' description we have:

$$I_{\text{os}}^{\text{New}}(Q_n, \bar{Q}_n) = -k\sum_{n=2}^{N} n\,\sigma_n\,Q_n + k\sum_{n=2}^{N} n\,\bar{\sigma}_n\,\bar{Q}_n \tag{8.25}$$

such that:

$$\delta I_{\text{os}}^{\text{New}} = -k\sum_{n=2}^{N}\sigma_n\delta Q_n + k\sum_{n=2}^{N}\bar{\sigma}_n\delta\bar{Q}_n \tag{8.26}$$

8.4 Holonomy conditions and Consistency: Spin 2, Spin 3 and Spin 4 examples

In this section we will see explicitly how the above mechanism works for different spins $N = 2, 3, 4$ examples. As explained at the beginning of this chapter, interesting gauge fields solutions defined on the solid torus must fulfils some holonomy conditions, and as we have two class of curves in the solid torus, i.e., contractile and non-contractile, we must have:

The non-trivial holonomy restriction along the non-contractile angular cycle is:

$$\mathcal{P}e^{\oint a_\varphi d\varphi} \neq I_N \tag{8.27}$$

The trivial holonomy condition along the contractile temporal cycle is:

$$\mathcal{P}e^{\oint a_t dt} = I_N \tag{8.28}$$

where I_N denotes the $N \times N$ dimensional identity matrix.

We also have similar independent holonomy conditions for the other copy. As the formulation of the full CS action is symmetric in treatment in both copies, we will restrict here to prove the consistency for only one copy, because for the other copy is done analogously in a straight forward manner.

From the above expressions, the first one (8.27) is a restriction, which gives us no condition, but rather suggest we must keep free the charges which are defined trough (8.6) such that a_ϕ cannot be brought to zero via a regular gauge transformation. The second one (8.28) is a condition which states that a_t can be brought to zero by a regular gauge transformation. In order to impose this condition one must solve the chemical potential as functions of the charges as we will see below. In the $SL(N,\Re)$ theory, the trivial holonomy conditions (8.28), for constant gauge fields, traduces into $N-1$ conditions on the eigenvalues of a_t, or equivalently into conditions on the $N-1$ power-like Casimirs of a_t. These $N-1$ conditions allows us to solve for the $N-1$ chemical potentials σ_n in terms of the $N-1$ charges Q_n.

8.4.1 N=2 example

In the simplest example, i.e., the $SL(2,\Re)$ theory, we can make the choice of parametrizing the a_ϕ gauge fields explicitly in terms of its Casimir as:

$$a_\phi = \begin{bmatrix} 0 & Q_2 \\ 1 & 0 \end{bmatrix} \tag{8.29}$$

by the way, it is important to recall that the above result is independent of the choice made for the matrix parameterisation of the gauge fields.

The trivial holonomy condition reads as:

$$Tr(a_t^2) = q^2 \tag{8.30}$$

with q fixed as $q=\sqrt{2}\pi$. Using a_t as given in (8.5) (with $N=2$), this condition explicitly reads as:

$$2\,{\sigma_2}^2 Q_2 - q^2 = 0 \tag{8.31}$$

where, considering σ_2 as a function of Q_2, differentiating one gets:

$$\frac{\partial Q_2}{\partial \sigma_2} = -2\,\frac{Q_2}{\sigma_2} \tag{8.32}$$

Using explicitly 8.21, for $N=2$ we have:

$$I_{\text{os}} = -\,k\sigma_2\,Q_2 \tag{8.33}$$

using 8.33 and 8.32 one gets:

$$\frac{\partial I_{\text{os}}}{\partial \sigma_2} = kQ_2 \tag{8.34}$$

which coincides with 8.14 at $N=2$.

8.4.2 N=3 example

At $N=3$, i.e., for $SL(3,\Re)$ theory, we can choose as a possible parameterization in terms of a_ϕ Cassimirs as:

$$a_\phi = \begin{bmatrix} 0 & \frac{1}{2}Q_2 & Q_3 \\ 1 & 0 & \frac{1}{2}Q_2 \\ 0 & 1 & 0 \end{bmatrix} \tag{8.35}$$

The trivial holonomy condition now reads as:

$$
\begin{aligned}
Tr(a_t^2) &= q^2 &(8.36)\\
Tr(a_t^3) &= 0 &(8.37)\\
& &(8.38)
\end{aligned}
$$

with q fixed as $q = 2\sqrt{2}\pi$. Using (8.5), with $N = 3$, these conditions are written as:

$$
\begin{aligned}
&\frac{2}{3}\sigma_3{}^2 Q_2{}^2 + 2\,\sigma_2{}^2 Q_2 + 6\,\sigma_2\,\sigma_3\,Q_3 - q^2 = 0\\
&-\frac{2}{9}\sigma_3{}^3 Q_2{}^3 + 2\,\sigma_3\,Q_2{}^2\sigma_2{}^2 + 3\,\sigma_3{}^2 Q_2\,\sigma_2\,Q_3 + 3\,\sigma_2{}^3 Q_3 + 3\,\sigma_3{}^3 Q_3{}^2 = 0
\end{aligned}
\tag{8.39}
$$

where, considering chemical potentials σ_2, σ_3 as functions of the charges Q_2, Q_3, differentiating the above expressions one gets:

$$
\begin{aligned}
\frac{\partial Q_2}{\partial \sigma_2} &= \frac{-6}{N_3}\left(3\,\sigma_3\,Q_3 - \sigma_2\,Q_2\right) &(8.40)\\
\frac{\partial Q_2}{\partial \sigma_3} &= \frac{1}{N_3}\left(-4\,\sigma_3\,Q_2{}^2 + 9\,\sigma_2\,Q_3\right) &(8.41)\\
\frac{\partial Q_3}{\partial \sigma_2} &= \frac{1}{N_3}\left(-4\,\sigma_3\,Q_2{}^2 + 9\,\sigma_2\,Q_3\right) &(8.42)\\
\frac{\partial Q_3}{\partial \sigma_3} &= \frac{-2\,Q_2}{N_3}\left(3\,\sigma_3\,Q_3 - \sigma_2\,Q_2\right) &(8.43)
\end{aligned}
$$

where $N_3 = 4\,\sigma_3{}^2 Q_2 - 3\,\sigma_2{}^2$.

Due that at $N = 3$ we have more than one pair of canonically conjugated variables, this is the first time we observe integrability conditions in the space of chemical potentials, which is absent in the $N = 2$ case. In fact, comparing (8.41) and (8.42), we directly see:

$$
\frac{\partial Q_2}{\partial \sigma_3} = \frac{\partial Q_3}{\partial \sigma_2} \tag{8.44}
$$

For $N = 3$ in 8.21, we explicitly have:

$$
I_{\rm os} = -k\left(\sigma_2\,Q_2 + 2\sigma_3\,Q_3\right) \tag{8.45}
$$

differentiating 8.45 and using (8.40),(8.41),(8.42),(8.43) one directly gets:

$$
\begin{aligned}
\frac{\partial I_{\rm os}}{\partial \sigma_2} &= kQ_2 &(8.46)\\
\frac{\partial I_{\rm os}}{\partial \sigma_3} &= kQ_3 &(8.47)
\end{aligned}
$$

which coincides with 8.14 at $N = 3$.

8.4.3 N=4 example

At $N = 4$, i.e., for the $SL(4,\Re)$ theory, a convenient parameterization of a_ϕ in term of its Casiimirs is given by:

$$a_\phi = \begin{bmatrix} 0 & \frac{1}{3}Q_2 & \frac{1}{2}Q_3 & Q_4 - \frac{7}{18}{Q_2}^2 \\ 1 & 0 & \frac{1}{3}Q_2 & \frac{1}{2}Q_3 \\ 0 & 1 & 0 & \frac{1}{3}Q_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{8.48}$$

or, due that this analysis is independent on the explicit matrix parameterization of the gauge connection, provided we write it in terms of the a_ϕ Casimirs, we could equivalently have used:

$$a_\phi = \begin{bmatrix} 0 & \frac{1}{10}Q_2\sqrt{3} & \frac{1}{12}Q_3\sqrt{3} & \frac{1}{6}Q_4 - \frac{41}{600}{Q_2}^2 \\ \sqrt{3} & 0 & \frac{1}{5}Q_2 & \frac{1}{12}Q_3\sqrt{3} \\ 0 & 2 & 0 & \frac{1}{10}Q_2\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \tag{8.49}$$

which is a more appropriate parameterization for the next chapter when we will try to build explicit black hole metrics.

The trivial holonomy condition now reads as:

$$Tr(a_t^2) = 20\,q^2 \tag{8.50}$$
$$Tr(a_t^3) = 0 \tag{8.51}$$
$$Tr(a_t^4) = 164\,q^4 \tag{8.52}$$
$$\tag{8.53}$$

but now with q fixed as $q = \pi$. Using (8.5) with $N = 4$, this conditions explicitly reads as:

$$\begin{aligned} 0 \;=\;& 7\,\sigma_3\,Q_2\,\sigma_4\,Q_3 + 6\,\sigma_3\,Q_3\,\sigma_2 + 8\,\sigma_4\,Q_4\,\sigma_2 + 6\,{\sigma_4}^2 Q_4\,Q_2 - {\sigma_3}^2{Q_2}^2 + \frac{3}{4}\,{\sigma_4}^2{Q_3}^2 \\ & + 2\,{\sigma_2}^2 Q_2 - {\sigma_4}^2{Q_2}^3 + 4\,{\sigma_3}^2 Q_4 - 20\,q^2 \end{aligned} \tag{8.54}$$

$$\begin{aligned} 0 \;=\;& 24\,\sigma_4\,Q_4\,\sigma_2\,\sigma_3\,Q_2 + \frac{9}{2}\,{\sigma_4}^3 Q_4\,Q_2\,Q_3 + 12\,{\sigma_3}^2 Q_4\,\sigma_4\,Q_3 + \frac{15}{4}\,\sigma_3\,Q_2\,{\sigma_4}^2{Q_3}^2 \\ & + \frac{21}{2}\,{\sigma_2}^2 Q_2\,\sigma_4\,Q_3 - 6\,\sigma_2\,{Q_2}^3\sigma_4\,\sigma_3 + \frac{21}{2}\,\sigma_2\,{Q_2}^2{\sigma_4}^2 Q_3 + 6\,{\sigma_3}^2 Q_3\,\sigma_2\,Q_2 \\ & + \frac{9}{2}\,\sigma_3\,{Q_3}^2\sigma_2\,\sigma_4 + 3\,{\sigma_4}^2 Q_4\,\sigma_2\,Q_3 + 3\,{\sigma_4}^2 Q_4\,{Q_2}^2\sigma_3 + 12\,\sigma_3\,Q_4\,{\sigma_2}^2 + 12\,\sigma_3\,{\sigma_4}^2{Q_4}^2 \\ & - 3\,{\sigma_2}^2{Q_2}^2\sigma_3 - \frac{3}{2}\,{\sigma_4}^2{Q_2}^4\sigma_3 + \frac{9}{4}\,{\sigma_4}^3{Q_2}^3 Q_3 - \frac{3}{8}\,{\sigma_4}^3{Q_3}^3 + 3\,{\sigma_2}^3 Q_3 + 3\,{\sigma_3}^3{Q_3}^2 \end{aligned} \tag{8.55}$$

$$\begin{aligned} 0 \;=\;& \frac{3}{2}\,{\sigma_3}^2{Q_2}^5{\sigma_4}^2 + 3\,\sigma_4\,{Q_3}^3{\sigma_3}^3 - 2\,{\sigma_3}^4{Q_2}^2 Q_4 - 6\,{\sigma_2}^2{Q_2}^4{\sigma_4}^2 + 12\,{\sigma_4}^4{Q_4}^2{Q_2}^2 \\ & + 18\,{\sigma_3}^2{Q_3}^2{\sigma_2}^2 - 3\,{\sigma_3}^2{Q_2}^3{\sigma_2}^2 + \frac{21}{8}\,{\sigma_4}^4{Q_3}^2{Q_2}^3 - 2\,\sigma_2\,{Q_2}^5{\sigma_4}^3 + 3\,{\sigma_2}^3{Q_3}^2\sigma_4 \\ & - 3\,{\sigma_4}^4{Q_2}^4 Q_4 + 2\,{\sigma_3}^4 Q_2\,{Q_3}^2 - 4\,{\sigma_2}^3{Q_2}^3\sigma_4 + 24\,{\sigma_4}^2{Q_4}^2{\sigma_2}^2 + 12\,{\sigma_2}^2 Q_2\,{\sigma_3}^2 Q_4 \\ & + 28\,{\sigma_3}^3 Q_3\,\sigma_2\,Q_4 + 40\,{\sigma_4}^3{Q_4}^2\sigma_2\,Q_2 + 48\,\sigma_4\,{Q_4}^2\sigma_2\,{\sigma_3}^2 + 36\,{\sigma_4}^2{Q_4}^2 Q_2\,{\sigma_3}^2 \\ & + 8\,{\sigma_4}^3 Q_3\,\sigma_3\,{Q_4}^2 - \frac{15}{2}\,{\sigma_3}^3{Q_2}^3\sigma_4\,Q_3 + \frac{159}{8}\,{\sigma_3}^2{Q_2}^2{\sigma_4}^2{Q_3}^2 - 7\,{\sigma_3}^3{Q_2}^2 Q_3\,\sigma_2 \\ & - 15\,{\sigma_3}^2{Q_2}^3{\sigma_4}^2 Q_4 + \frac{25}{8}\,\sigma_3\,Q_2\,{\sigma_4}^3{Q_3}^3 - \frac{13}{2}\,\sigma_3\,{Q_2}^4{\sigma_4}^3 Q_3 + \frac{39}{4}\,{\sigma_4}^2{Q_3}^2{\sigma_2}^2 Q_2 \\ & + \frac{17}{2}\,{\sigma_4}^3{Q_3}^2\sigma_2\,{Q_2}^2 + \frac{9}{4}\,{\sigma_4}^2{Q_3}^3\sigma_3\,\sigma_2 + 4\,{\sigma_4}^3{Q_3}^2 Q_4\,\sigma_2 + \frac{9}{4}\,{\sigma_4}^4{Q_3}^2 Q_4\,Q_2 \\ & + \frac{21}{2}\,{\sigma_4}^2{Q_3}^2{\sigma_3}^2 Q_4 + 14\,{\sigma_2}^3 Q_2\,\sigma_3\,Q_3 + 24\,{\sigma_2}^3 Q_2\,\sigma_4\,Q_4 + 24\,{\sigma_2}^2{Q_2}^2{\sigma_4}^2 Q_4 \\ & + \frac{1}{4}\,{\sigma_3}^4{Q_2}^4 + \frac{21}{64}\,{\sigma_4}^4{Q_3}^4 + 4\,{\sigma_3}^4{Q_4}^2 + 4\,{\sigma_4}^4{Q_4}^3 + 4\,{\sigma_2}^4 Q_4 - 12\,{\sigma_3}^2{Q_2}^2\sigma_4\,Q_4\,\sigma_2 \\ & + 21\,\sigma_3\,{Q_2}^2\sigma_4\,Q_3\,{\sigma_2}^2 - 3\,\sigma_3\,{Q_2}^3{\sigma_4}^2 Q_3\,\sigma_2 + 42\,{\sigma_3}^2 Q_2\,\sigma_4\,{Q_3}^2\sigma_2 + 43\,\sigma_3\,{Q_2}^2{\sigma_4}^3 Q_3\,Q_4 \\ & + 30\,{\sigma_3}^3 Q_2\,\sigma_4\,Q_3\,Q_4 + 48\,\sigma_3\,Q_3\,{\sigma_2}^2\sigma_4\,Q_4 + 102\,\sigma_3\,Q_2\,{\sigma_4}^2 Q_3\,Q_4\,\sigma_2 - 164\,q^4 \end{aligned} \tag{8.56}$$

where, considering Q_2, Q_3, Q_4 as a function of $\sigma_2, \sigma_3, \sigma_4$, differentiating one gets:

$$\frac{\partial Q_2}{\partial \sigma_2} = \frac{1}{N_4}(-24\,\sigma_3\,Q_3\,\sigma_2 + 8\,{\sigma_2}^2 Q_2 + 64\,{\sigma_3}^2 Q_4 - 48\,\sigma_4\,Q_4\,\sigma_2 + 27\,{\sigma_4}^2{Q_3}^2 \\ -12\,\sigma_3\,Q_2\,\sigma_4\,Q_3 - 32\,{\sigma_3}^2{Q_2}^2 + 36\,\sigma_2\,{Q_2}^2\sigma_4 - 72\,{\sigma_4}^2 Q_4\,Q_2 + 36\,{\sigma_4}^2{Q_2}^3) \tag{8.57}$$

$$\frac{\partial Q_2}{\partial \sigma_3} = \frac{1}{N_4}(12\,{\sigma_2}^2 Q_3 - 32\,\sigma_3\,Q_4\,\sigma_2 - 18\,\sigma_3\,{Q_3}^2\sigma_4 + 8\,{\sigma_3}^2 Q_3\,Q_2 + 12\,\sigma_2\,Q_2\,\sigma_4\,Q_3 \\ +8\,\sigma_2\,{Q_2}^2\sigma_3 + 36\,{\sigma_4}^2 Q_4\,Q_3 - 18\,{Q_2}^2{\sigma_4}^2 Q_3) \tag{8.58}$$

$$\frac{\partial Q_2}{\partial \sigma_4} = \frac{1}{N_4}(18\,{\sigma_4}^2{Q_2}^4 + 12\,\sigma_2\,{Q_2}^3\sigma_4 - 16\,{\sigma_3}^2{Q_2}^3 - 36\,{\sigma_4}^2 Q_4\,{Q_2}^2 + 18\,Q_2\,{\sigma_4}^2{Q_3}^2 \\ -28\,\sigma_3\,Q_3\,\sigma_2\,Q_2 + 32\,{\sigma_3}^2 Q_2\,Q_4 - 24\,\sigma_3\,Q_4\,\sigma_4\,Q_3 - 9\,{Q_3}^2\sigma_2\,\sigma_4 + 12\,{\sigma_3}^2{Q_3}^2 \\ +16\,Q_4\,{\sigma_2}^2) \tag{8.59}$$

$$\frac{\partial Q_3}{\partial \sigma_2} = \frac{1}{N_4}(12\,{\sigma_2}^2 Q_3 - 32\,\sigma_3\,Q_4\,\sigma_2 - 18\,\sigma_3\,{Q_3}^2\sigma_4 + 8\,{\sigma_3}^2 Q_3\,Q_2 + 12\,\sigma_2\,Q_2\,\sigma_4\,Q_3 \\ +8\,\sigma_2\,{Q_2}^2\sigma_3 + 36\,{\sigma_4}^2 Q_4\,Q_3 - 18\,{Q_2}^2{\sigma_4}^2 Q_3) \tag{8.60}$$

$$\frac{\partial Q_3}{\partial \sigma_3} = \frac{1}{N_4}(16\,Q_4\,{\sigma_2}^2 + 12\,{\sigma_3}^2{Q_3}^2 - 36\,{Q_3}^2\sigma_2\,\sigma_4 - 16\,\sigma_3\,Q_3\,\sigma_2\,Q_2 - 4\,{\sigma_2}^2{Q_2}^2 \\ +48\,\sigma_3\,Q_4\,\sigma_4\,Q_3 + 48\,\sigma_4\,Q_4\,\sigma_2\,Q_2 - 144\,{\sigma_4}^2{Q_4}^2 + 18\,Q_2\,{\sigma_4}^2{Q_3}^2 \\ -36\,\sigma_3\,{Q_2}^2\sigma_4\,Q_3 - 12\,\sigma_2\,{Q_2}^3\sigma_4 + 108\,{\sigma_4}^2 Q_4\,{Q_2}^2 - 18\,{\sigma_4}^2{Q_2}^4) \tag{8.61}$$

$$\frac{\partial Q_3}{\partial \sigma_4} = \frac{1}{N_4}(12\,\sigma_4\,{Q_2}^4\sigma_3 + 27\,{\sigma_4}^2{Q_2}^3 Q_3 + 8\,\sigma_2\,{Q_2}^3\sigma_3 - 72\,\sigma_4\,Q_4\,{Q_2}^2\sigma_3 + 42\,\sigma_2\,{Q_2}^2\sigma_4\,Q_3 \\ -54\,{\sigma_4}^2 Q_4\,Q_2\,Q_3 - 30\,\sigma_3\,Q_2\,\sigma_4\,{Q_3}^2 + 14\,{\sigma_2}^2 Q_2\,Q_3 - 32\,Q_4\,\sigma_2\,\sigma_3\,Q_2 \\ +27\,{\sigma_4}^2{Q_3}^3 + 96\,\sigma_3\,\sigma_4\,{Q_4}^2 - 48\,\sigma_4\,Q_4\,\sigma_2\,Q_3 - 6\,\sigma_3\,{Q_3}^2\sigma_2 + 16\,{\sigma_3}^2 Q_4\,Q_3) \tag{8.62}$$

$$\frac{\partial Q_4}{\partial \sigma_2} = \frac{1}{N_4}(18\,{\sigma_4}^2{Q_2}^4 + 12\,\sigma_2\,{Q_2}^3\sigma_4 - 16\,{\sigma_3}^2{Q_2}^3 - 36\,{\sigma_4}^2 Q_4\,{Q_2}^2 + 18\,Q_2\,{\sigma_4}^2{Q_3}^2 \\ -28\,\sigma_3\,Q_3\,\sigma_2\,Q_2 + 32\,{\sigma_3}^2 Q_2\,Q_4 - 24\,\sigma_3\,Q_4\,\sigma_4\,Q_3 - 9\,{Q_3}^2\sigma_2\,\sigma_4 + 12\,{\sigma_3}^2{Q_3}^2 \\ +16\,Q_4\,{\sigma_2}^2) \tag{8.63}$$

$$\begin{aligned} \frac{\partial Q_4}{\partial \sigma_3} &= \frac{1}{N_4}(12\,\sigma_4\,{Q_2}^4\sigma_3 + 27\,{\sigma_4}^2{Q_2}^3Q_3 + 8\,\sigma_2\,{Q_2}^3\sigma_3 - 72\,\sigma_4\,Q_4\,{Q_2}^2\sigma_3 + 42\,\sigma_2\,{Q_2}^2\sigma_4\,Q_3 \\ &-54\,{\sigma_4}^2Q_4\,Q_2\,Q_3 - 30\,\sigma_3\,Q_2\,\sigma_4\,{Q_3}^2 + 14\,{\sigma_2}^2Q_2\,Q_3 - 32\,Q_4\,\sigma_2\,\sigma_3\,Q_2 \\ &+27\,{\sigma_4}^2{Q_3}^3 + 96\,\sigma_3\,\sigma_4\,{Q_4}^2 - 48\,\sigma_4\,Q_4\,\sigma_2\,Q_3 - 6\,\sigma_3\,{Q_3}^2\sigma_2 + 16\,{\sigma_3}^2Q_4\,Q_3) \end{aligned} \tag{8.64}$$

$$\begin{aligned} \frac{\partial Q_4}{\partial \sigma_4} &= \frac{1}{N_4}(-6\,{Q_2}^4\sigma_4\,\sigma_2 - 8\,{Q_2}^4{\sigma_3}^2 + 36\,{Q_2}^3{\sigma_4}^2Q_4 - 18\,{Q_2}^3\sigma_3\,\sigma_4\,Q_3 - 4\,{Q_2}^3{\sigma_2}^2 \\ &+60\,{Q_2}^2\sigma_4\,Q_4\,\sigma_2 - 28\,{Q_2}^2\sigma_3\,Q_3\,\sigma_2 + 16\,{Q_2}^2{\sigma_3}^2Q_4 - 72\,Q_2\,{\sigma_4}^2{Q_4}^2 \\ &+12\,Q_2\,\sigma_4\,Q_3\,\sigma_3\,Q_4 - 6\,Q_2\,\sigma_4\,{Q_3}^2\sigma_2 + 24\,Q_2\,{\sigma_2}^2Q_4 + 20\,Q_2\,{\sigma_3}^2{Q_3}^2 \\ &+36\,{Q_3}^2Q_4\,{\sigma_4}^2 - 18\,\sigma_4\,\sigma_3\,{Q_3}^3 - 48\,\sigma_4\,{Q_4}^2\sigma_2 - 8\,\sigma_3\,Q_3\,Q_4\,\sigma_2 + 3\,{Q_3}^2{\sigma_2}^2) \end{aligned} \tag{8.65}$$

where we have defined:

$$\begin{aligned} N_4 \equiv\; & -4\,{\sigma_2}^3 - 8\,{\sigma_3}^3Q_3 + 36\,\sigma_3\,Q_3\,\sigma_2\,\sigma_4 + 8\,{\sigma_3}^2\sigma_2\,Q_2 - 18\,{\sigma_2}^2Q_2\,\sigma_4 \\ & -48\,{\sigma_3}^2Q_4\,\sigma_4 + 36\,{\sigma_4}^2Q_4\,\sigma_2 - 27\,{\sigma_4}^3{Q_3}^2 + 18\,\sigma_3\,Q_2\,{\sigma_4}^2Q_3 \\ & +24\,{Q_2}^2{\sigma_3}^2\sigma_4 - 36\,\sigma_2\,{Q_2}^2{\sigma_4}^2 + 54\,{\sigma_4}^3Q_4\,Q_2 - 27\,{\sigma_4}^3{Q_2}^3 \end{aligned} \tag{8.66}$$

Observing the following pairs of equations: (8.58,8.60), (8.59,8.63) and (8.64,8.62) we observe the following integrability conditions:

$$\frac{\partial Q_2}{\partial \sigma_3} = \frac{\partial Q_3}{\partial \sigma_2} \tag{8.67}$$

$$\frac{\partial Q_2}{\partial \sigma_4} = \frac{\partial Q_4}{\partial \sigma_2} \tag{8.68}$$

$$\frac{\partial Q_4}{\partial \sigma_3} = \frac{\partial Q_3}{\partial \sigma_4} \tag{8.69}$$

From 8.21, for $N = 4$ we explicitly have:

$$I_{\text{os}} = -k\,(\sigma_2\,Q_2 + 2\sigma_3\,Q_3 + 3\sigma_4\,Q_4) \tag{8.70}$$

Now we differentiate (8.70) only once with respect to each of the chemical potentials, and then we plug the expressions (8.57) - (8.65) on it, and we get:

$$\frac{\partial I_{\text{os}}}{\partial \sigma_2} = kQ_2 \tag{8.71}$$

$$\frac{\partial I_{\text{os}}}{\partial \sigma_3} = kQ_3 \tag{8.72}$$

$$\frac{\partial I_{\text{os}}}{\partial \sigma_4} = kQ_4 \tag{8.73}$$

$$\tag{8.74}$$

which coincides with 8.14 at $N = 4$.

Finally, we can remark that in a completely analogous way, in any of these examples the relations can be reverted, i.e., considering the chemical potential σ_i as functions of the charges $\sigma_i(Q_j)$. Thus, considering this dependence we also use the holonomy equations, but now in the computations instead of use the (grand) canonical on shell action (8.21), one uses the micro-canonical action (8.22) to find consistency with (8.16).

8.5 Entropy for the Higher spin black holes

In gravitational theories, the entropy for black hole solutions is often given by horizon valued boundary terms coming from the on shell Hamiltonian action. This boundary term arises due to the necessity of regularize an infinite value for the Hamiltonian action at the horizon.

On the other hand, in the case of a CS theory, we can compute the Hamiltonian action, for this we make a 2+1 decomposition on the torus, but now using time to foliate it, getting:

$$I_{\text{Hamiltonian}} = \frac{k}{4\pi}\int dt \int dr d\phi (A_r \partial_t A_\phi + A_t F_{r\phi}) + \frac{k}{4\pi}\left(\int_\infty dt d\phi \,\text{Tr}(A_t A_\phi)\right) - B_+ \tag{8.75}$$

The boundary term B_+ in (8.75) comes from the need of introduce a virtual boundary that surrounds the inner ring at that center of the torus (at the point $r = 0$) where the vector field ∂_t along which we make the temporal foliation has a fixed point, i.e., all the leaves of the foliation intersect at the ring $r = 0$, thus the foliation is degenerated.

Evaluating this action in a black hole-like solution, which must be static and spherically symmetric, the bulk term vanish and we get:

$$I^0_{\text{Hamiltonian}} = \frac{k}{4\pi}\left(\int_\infty dt d\phi \,\text{Tr}(A_t A_\phi)\right) - B_+ \tag{8.76}$$

On the other hand if we evaluate the CS action decomposed along the angular coordinate (8.17) in a black hole-like solution, due that the solution is circularly symmetric, i.e., independent from ϕ, we get:

$$I^0_{\text{angular}} = -\frac{k}{4\pi}\int_\infty dt d\phi \,\text{Tr}(A_t A_\phi) \tag{8.77}$$

The action (8.75) is the same that the action (8.17), where just a different 2+1 dimensional split has been done. Also these two actions are the same as the fully convariant (8.7). Thus equating its on shell values given by (8.76) and (8.77) and solving for the boundary term B_+, we get (see [52]):

$$B_+ = \frac{k}{2\pi}\int_\infty dt d\phi \,\text{Tr}(A_t A_\phi) \tag{8.78}$$

And using the equations of motion (8.3) with (8.5), and using the definitions (8.6), we get an explicit expression for the boundary term of the CS action, given by:

$$B_+ = k\sum_{n=2}^{N} n Q_n \sigma_n \tag{8.79}$$

Now, considering that the formulation of the gravitational theory is given by the substraccion of two CS actions with the same level, then the entropy for the higher spin black holes, which should be given by the total boundary term of the higher spin gravitational theory, is given by:

$$S = B_{+}^{(1)}[A] - B_{+}^{(2)}[\bar{A}] = k\sum_{n=2}^{N} nQ_n\sigma_n - k\sum_{n=2}^{N} n\bar{Q}_n\bar{\sigma}_n \tag{8.80}$$

8.6 Conclusions

We have shown that it is the on-shell CS action the functional, which solves the integrability conditions, whose existence was spotted in the literature. In the process, we also have shown that this functional posses a natural description in terms of the gauge invariant Casimirs of the angular components of the gauge connections as the physical degrees of freedom of the boundary theory. Also, in the process, chemical potentials were introduced by solving the equations of motion, to then analyse the on-shell variational principle to find that both kind of variables, i.e.: Casimirs and chemical potentials, turns out to be canonically conjugated at the boundary. This gave us the knolewdge about the simplectic structure at the boundary due to these conjugated variables. Furthermore, studying the consistency of the on-shell variational principle, i.e., allowing only variations of the gauge fields which are solutions of the equations of motion, it was shown that the impossition of trivial holonomy conditions around the contractile cycle of the torus, turns out to be fundamental in order to have consistent picture of the variational principle.

Chapter 9

Higher Spin Black Holes

9.1 Introduction

Previous attempts to build explicit black hole solutions with higher spin charges has been carried out in [49, 40, 39, 50], however all of these solutions belongs to the non-rotating case and therefore they are charaterised by only two parameters, i.e., only one of spin 2 nature, and only one of spin 3 nature. Even more, the solutions constructed in [40, 39], by construction belongs to the diagonal embedding which means that they carry no fundamental higher spin charges. Furthermore, in [51] it has been argued that solutions found in [49] also belongs to the principal embedding. In this work we will construct rotating black hole solutions possessing higher spin charges, which at the same time posses a regular smooth horizon.

In the last chapter (8), we have found the action for higher spin black holes in three dimensions, which leads to a consistent 'thermodynamics'. There, we have also learnt which are the relevant canonically conjugated variables involved in its 'thermodynamic' description. In the process we learnt that this action is completely topological and also independent on the radial coordinate, and as such, a regularization and renormalization process has not been needed. In fact in the process we have only needed to take care about a well settled Dirichlet problem. However, regarding the solutions found in the last chapter, as we have not said too much about the radial coordinate itself, which should somehow be involved in a explicit metric-like black hole description, one may think that it could be too early to call them black holes. These prospective black hole solutions are allowed to exist, at this time of the discussion, just because we have made a topologically non-trivial characterization of the CS flat solutions which lives on the non-trivial solid torus manifold. In fact, in the previous analysis of the on shell CS action, the Casimirs invariants of the angular components of the gauge field were identified as the gauge invariant, and thus physical, boundary degrees of freedom (and as such, being free). This was supported by the restriction made on the angular holonomies (around the non-contractile cycle) to be non-trivial. On the other hand, the trivial temporal holonomies (around the contractile cycle) were fundamental to gives us the right regularity conditions [1], between the relevant canonically conjugated variables at infinity (the boundary). Thus, just as we saw in chapter (6) in the simple scalar field example, and also as we saw and confirmed in chapter (8) for our

[1] which are imposed at the gauge field level, which being radially independent, are also also globally defined and purely topological ones

higher spin setting, the regularity conditions once again are needed for the consistency of the holographic correspondence.

In chapter (5), we have mentioned an argument which says that in a metric-like fully interacting (non-linearised) theory, non-linearised higher spin (with spin $s_{\rm H} > 2$) gauge transformations should act on the lower spin (with spin $s_{\rm L} < s_{\rm H}$) fields. This is because the higher spin gauge parameters (with are symmetric tensor possessing a spin $s_{\rm H} - 1$) also carry lower spin invariant subspaces up to spin $s_h - 1$. ...which at the end of the day, should mean that the deformed[2] gauge symmetries of the full interacting theory, must somehow acts on the lower spin fields through the deformed fully non linear gauge transformation of the fully interacting system. Furthermore, in the frame formulation of the $SL(3, \Re)$ theory in chapter (5), we explicitly saw how a higher spin gauge transformation acts on the spin 2 metric. This has as consequence, that the meaningful coordinate invariants quantities (i.e., under non-linear spin 2 gauge transformations) that exist in general relativity, when are analysed inside the higher spin ($s > 2$) setting, they lose its fully gauge invariant meaning, because they are no longer invariant under the higher spin diffeomorphisms which are the extended symmetries of the system. Certainly, this is problematic in the higher spin setting, if we are planing to keep the concept of a (higher spin) black hole as an object characterized by a singularity expressed in the (spin 2) curvature invariants.

However, as respect with what should be black holes solutions in the higher spin context, in an euclidean formulation besides the condition of having pairs of canonically conjugated thermodynamic variables to describe them, which we know it is possible, one further condition that one would like to have for a black hole, is the existence of an horizon, or something similar, perhaps as an extended concept, maybe a 'higher spin horizon'. Of course, at the present the lacking of knowledge about a full metric-like formulation of the theory, and also the lacking of knowledge of higher spin geometric concepts such as, e.g., prospective higher spin curvatures, prospective higher spin coordinate transformations (i.e., full non-linear completion of higher spin diffeomorphism), etc., one can imagine, but it would be very difficult to prove, an hypothetical relation between this hypothetical higher spin horizon and some hypothetical singularity in a higher spin curvature invariant surrounded by the higher spin horizon. Thus, at the moment, an attempt to tackle the problem by this route should be completely abandoned because it is pure speculation.

However, in the $SL(2, \Re)$ theory, the spin 2 euclidean BTZ black hole, posses an horizon but not a curvature singularity. Due to its topological construction on the solid torus, it rather may posses only a coordinate singularity, which reflects itself as coming from a bad choice in the temporal period which leads to a topological conical singularity. But having properly chosen the value of temporal period, the conical singularity will be absent, leading to a completely regular surface at the horizon, which is located at the center of the topological manifold, i.e., the solid torus, corresponding to the ring at $r = 0$.

In the three dimensional euclidean higher spin context, we will also make a topological construction of the solutions, and in analogy with the $SL(2, \Re)$ case, this construction spots that higher spin black holes should be described by bulk fields which near the horizon are described

[2]Deformed from the free theory

by completely regular surfaces, and regular 'higher spin surfaces', without having to refer this regularity to some still unknown higher spin curvature extensions of geometry. Thus, in the higher spin theory, just regularity of the bulk fields at the 'horizon', will be a mandatory concept to construct what we will call as a regular 'black hole solution'.

It is better to stress that in trying to build regular solutions with spin 3 (or with higher spin charges), one mayor 'problem' that we have faced, is that in the construction of regular black holes solutions, in general we have found that they do not posses a 'desired' asymptotic behaviour at infinity. Firstly, this was thought only as a technical problem in the construction, because as we will see, there is a lot of gauge freedom involved in their construction, which leads one to think that by somehow choosing the 'right gauge fixing' conditions one could achieve the desired asymptotic behaviour. Also, the lacking of a well settled knowledge about higher spin geometrical concepts, makes this situation as an even worst scenario to do work. However, after an enormous amount of attempts in trying to find regular solutions with the 'desired' asymptotic behaviour, we have found a theorem, which is beyond the scope of this thesis and thus it wont be shown here, rather it will be shown in a future publication. That result which is worth to mention basically states that in the higher spin theory ($N > 2$), for regular solutions the 'desired behaviour' it is ruled out, i.e., do not exist. A partial discussion of what it implies will be done in some of the next sections.

9.2 Importance of a regular solution in the AdS/CFT context

In chapter (6), we saw that in order to have a consistent picture of the AdS/CFT conjecture along the bulk, we need to construct regular bulk fields solutions at the interior 'point' of the bulk manifold. Thus, a regular bulk field solution at the interior, turns out to be the required configuration for the whole consistency of the holographic picture.
In the context of euclidean higher spin black holes which lives on the solid torus, in the following, we will call the interior point of the topological bulk manifold as the 'horizon' which is the ring at the center of the solid torus. And this will be the 'point', where we will demand that the bulk field solutions be regular (see figure 8.1).

9.3 General considerations: Static and circularly symmetric solutions

In order to build black hole solutions, we should consider to start considering the set of static and circularly symmetric CS solutions, which in the chapter (8) were shown to be the appropriate gauge connections for black hole physics. The most general solution to CS equations of motion fulfilling these conditions are:

$$\begin{aligned} A_r &= g_1^{-1}\partial_r g_1 \quad , \quad & B_r &= g_2^{-1}\partial_r g_2 \\ A_\varphi &= g_1^{-1}a_\varphi g_1 \quad , \quad & B_\varphi &= g_2^{-1}b_\varphi g_2 \\ A_t &= g_1^{-1}a_t g_1 \quad , \quad & B_t &= g_2^{-1}b_t g_2 \end{aligned} \tag{9.1}$$

where $g_1 = g_1(r)$, $g_2 = g_2(r)$ are fully arbitrary radial-dependent $SL(N,\Re)$ group elements, and the lower case matrices a_t, a_φ, b_t, b_φ are constant $SL(N,\Re)$ algebra matrices, which satisfy the equations:

$$[a_t, a_\varphi] = 0 \ , \ [b_t, b_\varphi] = 0 \tag{9.2}$$

in order that the equations of motion be satisfied. Recall that they can be solved as it was done in chapter (8), and as was discussed there, given a_φ we will have $a_t = f(a_\varphi)$ in which the functional form of f, by Cayley-Hamilton theorem, is uniquely fixed by the introduction of $N-1$ arbitrary parameters (chemical potentials), and thus the form of a_t will depend on these, and also in the explicit form that a_φ posses. Note that given a_φ, and thus a given a_t, and considering a general unconstrained but only radial dependent gauge group element $g_1(\mathrm{r})$, the fields A_μ represents itself, also the set of all the gauge transformed fields with a radial dependent group element, which preserves the static and circular symmetry. Of course these discussions holds for the B_μ fields.

As was described in chapter (5), the dreibein and spin connection are constructed as:

$$e_\mu = A_\mu - B_\mu \ , \ \omega_\mu = A_\mu + B_\mu \tag{9.3}$$

and the $N-1$ metric fields are:

$$\begin{aligned} g_{\mu_1\mu_2} &= \tfrac{1}{f_N} \ tr(e_{(\mu_1}e_{\mu_2)}) \\ g_{\mu_1\mu_2\mu_3} &= tr(e_{(\mu_1}e_{\mu_2}e_{\mu_3)}) \\ &\vdots \\ g_{\mu_1\ldots\mu_N} &= tr(e_{(\mu_1}\cdots e_{\mu_N)}) \end{aligned} \tag{9.4}$$

Where f_N is just a normalizing factor given in the appendix.

9.4 Definition of a horizon

Given a solution, we define the horizon as the point in the radial coordinate where temporal component of the dreibein '$e_t = A_t - B_t$ ' vanish. This means:

$$e_t(0) \equiv A_t(0) - B_t(0) = g_1^{-1}(0)a_t g_1(0) - g_2^{-1}(0)b_t g_2(0) = 0 \tag{9.5}$$

This definition of horizon implies that all the components of the metric-like fields given by (9.4), that at least posses one temporal index will vanish at the horizon. We take this definition as the very starting point to solve the big problem that would be to try to define the horizon without having the frame formalism, by just doing it in the metric formulation of the theory which, given that we don't even know what is the complete metric-like formulation of it, it would be insane to try to do this.

By looking at (9.5), it is straight forward to see that in order for this definition of horizon be able to exist, the following conditions has to be satisfied:

- 1) The lower case matrices a_t and b_t has to be in the same class $a_t \sim b_t$:
  These conditions on the lower case gauge connections, can be translated into conditions between the chemical potentials and the would be physical degrees of freedom (a_ϕ casimirs) which are present in the thermodynamic description of the system in an euclidean formulation. Let's say, to put a_t in some given class, we adjust it's chemical potentials in terms of the it's charges. By the other side, the same is true for b_t. Of course we do this, by taking care that we have put both a_t and b_t in the same class.

- 2) Given a_t and b_t in the same class, the existence of a well defined similarity matrix $U = g_2(r_0)g_1^{-1}(r_0)$ which relates a_t and b_t as $b_t = Ua_tU^{-1}$ has to be guaranteed:
  This condition partially fix the form of the group elements $g_1(r_0), g_2(r_0)$ at the horizon denoted by r_0.

9.5 Euclidean solutions and holonomies

Now if we want to have euclidean solutions (after a Wick rotation) with a smooth horizon we have to put a_t and b_t in the trivial class, this means that trivial temporal holonomies has to be satisfied. Of course, the very definition of a horizon as $e_t = 0$ is further justifyied considering the fact that for euclidean solutions, the euclidean temporal differential dt is an angle, and thus it explodes at the horizon, which is easy to see in Cartesian coordinates where we have: $dt \sim \frac{1}{\rho}$, thus e_t must vanish at least as $e_t \sim \rho$.

9.6 Constrainning the group elements along the bulk

Considering the above discussion, if we plan to build euclidean regular solutions, we need that the trivial temporal holonomies be satisfied. These are conditions on the chemical potentials to be solved in terms of the charges. Also, the group elements $g_1 = g_1(r)$, $g_2 = g_2(r)$ has to satisfy some regularity conditions at the horizon. However, the group element, still has a lot of gauge freedom along the bulk, and in order to have a sensible solution one has to constraint it using some sensible criteria.

The criteria used to constraint the group elements has been the following ones:

1. The metric-like must posses a Fefferman-Graham form which is somehow desired if one want to do holography. Explicitly it has been demanded that the metric posses a FG form in radial proper coordinates: i.e. a form with the aspect: $g_{\rho\rho} = 1, g_{\rho t} = 0, g_{\rho\varphi} = 0$.
2. It has been demanded that some spin 3 metric-like components be null, g.e: $g_{ttt} = g_{\rho t\varphi} = g_{\rho\rho t} = g_{\rho\varphi\varphi} = g_{\rho\rho\rho} = g_{\rho tt} = 0$. This form for the spin 3 metric like field, is somehow desired because the resulting field posses a form as

$$d\phi \times \text{Black Hole} \tag{9.6}$$

Having imposed these conditions, one expects, and in fact it turns out to be, that the following conditions are automatically satisfied:

1. The solutions posses a smooth horizon, in which the holonomy conditions on gauge fields imply Hawking periodicity conditions on metric-like fields.
2. The solutions posses a BTZ as limit when spin 3 charges (and chemical potentials) are turned off.

However, it has been observer two class of solutions in which it has been observed the following phenomena:
Class A solutions:

1. Metric-like solutions, as expected, are described completely in terms of four independent charges.

2. When spin 3 charges are on, the solutions asymptotes to a different curvature radius than when spin 3 charges are absent.

Class B solutions:

1. Metric-like solutions, contrary to what one expect, are described completely in terms of three independent charges.

2. Cosmological constant do not change.

It is worth to mention that there has been implemented so many methods to look for solutions with the characteristics we were expecting to see, but from the workable ones, essentially all of them give the same answers. However it has been only one method that has allowed us to manage in a controlled and systematic way the large expressions which are usually involved in the computations. This method is the one which will be presented in the next sections.

9.7 Method to construct euclidean regular solutions

With the setting as is (9.1), one choose a_φ (and b_φ) as the appropriate gauge connections in the principal embedding, which posses explicit W_3 symmetries. Then we chose a_t (and b_t respectively) by Cayley-Hamilton introducing chemical potentials as in (8.5). Then one solves the holonomy conditions fixing the chemicals potentials in terms of the charges. Doing this one is putting a_t and b_t in the trivial class. By the discussion done above, being a_t and b_t in the same class, there must exists groups elements such at the horizon fulfils the condition of vanishing e_t.

Being a_t and b_t in the trivial class (which satisfy the trivial temporal holonomies), and given that originally a_t, which is constructed from a_ϕ trough the addition of chemical potentials, and by other side the construction of b_t is analogous, but taking into account that the parameters of both copies a_i, b_i are unrelated to each other, one conclude there must exist a charge independent matrix C as a representative of the trivial class. One way to think about this is from (9.5), as e.g.:

$$e_t(0) \equiv A_t(0) - B_t(0) = C - C = g_1^{-1}(0)a_t g_1(0) - g_2^{-1}(0)b_t g_2(0) = 0 \tag{9.7}$$

Having $g_1^{-1}(0)a_t g_1(0) = C$, $g_2^{-1}(0)b_t g_2(0) = C$ and thus $a_t \sim C$ and $b_t \sim C$ which of course implies: $a_t \sim b_t$.

Due to the structure of the equations of motion in (9.1) and given that at this point of the discussion g_1 and g_2 in (9.1) are arbitrary up to some regularity conditions, one can invert the problem and choose to start with:

$$\begin{aligned} A_r &= h_1^{-1}\partial_r h_1 \quad , \quad B_r = h_2^{-1}\partial_r h_2 \\ A_t &= h_1^{-1}C h_1 \quad , \quad B_t = h_2^{-1}C h_2 \\ A_\varphi &= h_1^{-1}C_\varphi h_1 \quad , \quad B_\varphi = h_2^{-1}\bar{C}_\varphi h_2 \end{aligned} \tag{9.8}$$

where now C is a trivial class charge independent $SL(N,\Re)$ matrix, and the horizon existence condition implies that at the horizon, the group elements are the identity matrix: $h_1(0) = h_2(0) = I_{N\times N}$. Of course in order for the equations of motion holds:

$$[C, C_\phi] = 0 \ , \ [C, \bar{C}_\phi] = 0 \tag{9.9}$$

has to be satisfied using Cayley-Hamilton theorem to introduce $N-1$ new parameters in the game, which are not the chemical potentials, because having started with the trivial class matrix C the trivial holonomies are already satisfied. Rather, they are just new parameters in order to reparameterise the $(N-1)$-th Casimirs charges of the angular gauge connections. Of course, it is not difficult to convince oneself, that if originally we have chosen to begin with a particular form for the a_ϕ field in (9.1), lets say e.g. as the principally embedded form of the solution given by (8.35), when passing to this construction in (9.8), we will have that the C_ϕ field constructed from the solution of (9.9) will be in the same class that the original a_ϕ field, where the similarity matrix is the same similarity matrix that relates a_t and C. Analogously, this discussion is also valid for the $\bar{C}_\phi$ and b_ϕ fields.

This is one of the most sensible and workable methods we have used to look for regular solutions, and using it has proven to be the most easy way to make sensible ansatz for the group element $h_1(r), h_2(r)$ in order to fix the radial gauge freedom. This is what will be used in the next sections and, furthermore, it will prove to be useful to unravel an ansatz structure extensible to build regular solutions for all N.

9.8 The BTZ example

Let's begin applying the method developed in this thesis, with a simple computation, in order to build the BTZ black hole [31, 32] in the $SL(2,\Re)$ theory. Let's use the gauge connections:

$$a_t = b_t = C = \frac{q\sqrt{2}}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{q}{\sqrt{2}}(-L_1 + L_{-1}) \tag{9.10}$$

which already satisfy the (anti) trivial [3] holonomy conditions $\mathrm{Tr}(a_t^2) = \mathrm{Tr}(b_t^2) = q^2$ provided that $q = \pm\sqrt{2}\,\pi$. Note that for a Minkowskian computation, the value of $q \neq 0$ is left arbitrary, but we still need that $\mathrm{Tr}(a_t^2) = \mathrm{Tr}(b_t^2) = q^2$ be satisfied, which means that a_t and b_t are in the same class. Moreover in the method we will use here we will demand they be equal as stated in (9.10).

The solution to the equations of motion $[a_t, a_\phi] = 0$ and $[b_t, b_\phi] = 0$, by Cayley-Hamilton theorem are given by:

$$a_\phi = \mu_1 a_t \ , \ b_\phi = \mu_2 b_t \tag{9.11}$$

[3] In fact to be precise, the regular euclidean BTZ gauge connection satisfy $Pe^{\oint a_t dt} = -I_d$ (and similarly for b_ϕ) (see [39]), which basically means that after two complete temporal cycles, the parallel transported matrix comes back to its original value, or equivalently, after two temporal cycles is like if it has transformed by the identity matrix. One possible way to understand this fact, is because in this example, we have built BTZ by using the fundamental representation of $SL(2,\Re)$ which is 2×2 dimensional representation, i.e., a spin $\frac{1}{2}$ representation provided that the representation posses dimension $d = 2s+1$.

Where now the Casimirs of a_ϕ, b_ϕ will be given in terms of the new parameters μ_1, μ_2, as:

$$Q_2 \equiv \frac{1}{2}\text{Tr}(a_\phi^2) = \frac{1}{2}q^2\mu_1^2 \ , \quad \bar{Q}_2 \equiv \frac{1}{2}\text{Tr}(b_\phi^2) = \frac{1}{2}q^2\mu_2^2 \tag{9.12}$$

At this point of the discussion, it is not necessary to fulfil our goal, but surely it is very instructive to stress that with this new reparameterisation (9.12) of the charges, the holonomy equation (8.31) (and the analogous equation for b_t) can be solved for the spin 2 chemical potential as:

$$\sigma_2 = \frac{1}{\mu_1} \ , \quad \bar{\sigma}_2 = -\frac{1}{\mu_2} \tag{9.13}$$

Of course, we do not need do this at this time because we have already started with an holonomy fulfilling fields a_t (and b_t).

Having started with (9.10), the idea is give radial dependence to the fields trough (9.1), using a group element which gives the identity when valued at the horizon, at which $e_t(0) = g^{-1}(0)a_t g(0) - g(0)b_t g^{-1}(0) = a_t - b_t = 0$. Demanding this it will be guaranteed that all the fields with at least one temporal index in (9.4) vanish at the horizon.

One further restriction one want to impose is to fix the radial gauge freedom by the vanishing of the components $g_{\rho t} = g_{\rho\phi} = 0$ along all the bulk, and also made the choice of a proper radial coordinate $g_{\rho\rho} = l^2$, i.e., this is to choose a FG [53] coordinate system. The choice of this system is made by choosing the group element given by:

$$g(\rho) = \begin{bmatrix} e^{\frac{\rho}{2}} & 0 \\ 0 & e^{-\frac{\rho}{2}} \end{bmatrix} = e^{\rho L_0} \tag{9.14}$$

Then trough the first line in (9.4) using $f_2 = \frac{1}{2}$ (see appendix), the metric is given by the arc length:

$$ds^2 = l^2 d\rho^2 - \frac{l^2q^2}{2}\left(e^{-2\rho} + e^{2\rho} - 2\right)dt^2 - \frac{l^2q^2}{2}(\mu_1 - \mu_2)\left(e^{-2\rho} + e^{2\rho} - 2\right)d\phi\, dt$$
$$+ \frac{l^2q^2}{2}\left({\mu_2}^2 + {\mu_1}^2 + \mu_1\, e^{-2\rho}\mu_2 + e^{2\rho}\mu_2\,\mu_1\right)d\phi^2 \tag{9.15}$$

In this coordinates the ranges goes as $0 < \rho < \infty$, $0 < \phi < 2\pi$, $-\infty < t < \infty$, and if we are demanding an Euclidean continuation, as $t \to i\tau$ in order to give sense to the trivial temporal holonomies, we need that the temporal coordinate be identified as $\tau \sim \tau + m$, such that $0 < \tau < 1$.

In order to prove that this metric is BTZ is useful to do the parametrization given by:

$$r_\pm = \frac{1}{2}q\sqrt{2}\,(\mu_1 \pm \mu_2)\,l \tag{9.16}$$

and performing the change of coordinates given by:

$$t \to \quad \frac{1}{\beta}t \quad \text{where:} \quad \beta = \frac{\sqrt{2}\,r_+\,ql^2}{(-r_- + r_+)\,(r_- + r_+)} \tag{9.17}$$

$$\rho \to \quad r \quad \text{where:} \quad \cosh^2(\rho)\,r_+^2 - \sinh^2(\rho)\,r_-^2 = r^2 \tag{9.18}$$

we are left with a metric in the form:

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{N^2} + r^2 \left(d\phi + N_\phi \, dt\right)^2 \tag{9.19}$$

where now the ranges of coordinates goes as $r_+ < r < \infty$, $0 < \phi < 2\pi$, $0 < t < \beta$, and where:

$$N_\phi = -\frac{r_-}{r_+ l} + \frac{r_- r_+}{r^2 l} \quad N^2 = \frac{(r - r_-)(r + r_-)(r - r_+)(r + r_+)}{r^2 l^2} \tag{9.20}$$

Note that the outer horizon is located at r_+, which is the point where the lapse N and also the regular shift N_ϕ [4] vanish.

Finally, using the relations:

$$r_+^2 + r_-^2 = 8GMl^2 \ , \ r_+ r_- = 4GJl \tag{9.21}$$

it is clear that this metric is BTZ.

On the other hand, using (9.16) and (9.12), is clear that the relations between the asymptotic BTZ conserved charges (mass M and angular momentum J) and the Casimir invariants of the gauge connections a_ϕ, b_ϕ are:

$$Ml = k(Q_2 + \bar{Q}_2), J = k(Q_2 - \bar{Q}_2) \text{ , where we have used: } k = \frac{l}{4G} \tag{9.22}$$

From here on, we will refer to the regular BTZ by the metric given in (9.15), where we can straightforwardly see that it describes a regular surface at the horizon $\rho = 0$. In fact, passing to the euclidean formulation doing $t \to i\,t$ and fixing $q = \sqrt{2}\,\pi$ in (9.15) we can compute the Hawking periodicity condition:

$$\frac{1}{\rho^2} \frac{g_{tt}}{g_{\rho\rho}}|_0 = 4\,\pi^2 \tag{9.23}$$

which is the right periodicity condition for a regular surface, i.e., a plane near the horizon $\rho = 0$ in the polar coordinates: (ρ, t).

Finally, note that this method, from which we directly obtain the regular BTZ metric as appear in the form given by (9.15), only describes BTZ in the outer region where $\rho > 0$, or equivalently $r > r_+$ in (9.19). However, if we allow ourselves to extend the range of coordinates in (9.19) we will be describing BTZ also into the inner region of the black hole.

The same method used here to describe BTZ by falling in (9.15), will be extended and then used for the construction of the regular rotating spin 3 black hole solution in the outer horizon region. However, we will restrict ourselves to work only in the coordinates ρ, t, ϕ as in (9.15). This means that we will not to attempt to make an analogous higher spin extension of the passage from the coordinates in $\rho \to r$ as was done when we pass from (9.15) to (9.19) in the

[4] This 'regular' angular shift can be understood as a shift constructed from the usual shift $N_\phi = \frac{r_- r_+}{r^2}$ that in the literature BTZ posses, by performing a change of coordinates given by $d\phi \to d\phi - \frac{r_-}{r_+} dt$ such that the metric in the new coordinates is regular. Of course with the construction made here the metric automatically is regular.

BTZ case, this is because in the higher spin case, this passage is not well understood, thus rather than simply be an ordinary coordinate transformation, it may imply to perform something that would be like a higher spin diffeomorphism but we do not know how this must act directly on metrics.

9.9 Extending the method for $SL(3,\Re)$: a naive solution

For the $SL(3,\Re)$ theory we choose the trivial class matrix C_d given by:

$$a_t = b_t = C_d = \frac{q}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{9.24}$$

which satisfy $Tr(a_t^2) = Tr(b_t^2) = q^2$ and $Tr(a_t^3) = Tr(b_t^3) = 0$. Fixing $q = 2\sqrt{2}\pi$ this matrix is on the trivial class.

The solution to the equations of motion $[a_t, a_\phi] = 0$ and $[b_t, b_\phi] = 0$, by Cayley-Hamilton theorem are given by:

$$a_\phi = \mu_1 a_t + \nu_1 (a_t^2 - \frac{I_3}{3} tr(a_t^2)) \quad , \quad b_\phi = -\mu_2 b_t - \nu_2 (b_t^2 - \frac{I_3}{3} tr(b_t^2)) \tag{9.25}$$

Where now the quadratic and cubic Casimirs of a_ϕ (b_ϕ) will be given in terms of the new parameters μ_1, ν_1 (μ_2, ν_2), as:

$$Q_2 \equiv \frac{1}{2}\mathrm{Tr}(a_\phi^2) = \frac{q^2}{12}\left({\nu_1}^2 q^2 + 6\,{\mu_1}^2\right) \quad , \quad \bar{Q}_2 \equiv \frac{1}{2}\mathrm{Tr}(b_\phi^2) = \frac{q^2}{12}\left({\nu_2}^2 q^2 + 6\,{\mu_2}^2\right) \tag{9.26}$$

$$Q_3 \equiv \frac{1}{3}\mathrm{Tr}(a_\phi^3) = \frac{\nu_1\, q^4}{108}\left(-{\nu_1}^2 q^2 + 18\,{\mu_1}^2\right) \quad , \quad \bar{Q}_3 \equiv \frac{1}{3}\mathrm{Tr}(b_\phi^3) = \frac{\nu_2\, q^4}{108}\left({\nu_2}^2 q^2 - 18\,{\mu_2}^2\right) \tag{9.27}$$

We chose the group elements as:

$$g_1 = g_2^{-1} = e^{\rho(\sqrt{1-2x^2}L_0 + x(W_1 + W_{-1}))} \tag{9.28}$$

We construct the two-index metric and three-index metric by using (9.4), using $f_3 = 2$ (see appendix). The non-null two index metric components are:

$$g_{\rho\rho} = l^2 \tag{9.29}$$

$$g_{tt} = -\frac{1}{8} l^2 q^2 (x-1)^2 (x+1)^2 \left(e^{2\rho} - e^{-2\rho}\right)^2 + \frac{1}{4} l^2 q^2 x^2 \left(-1+2x^2\right) \left(e^{\rho} - e^{-\rho}\right)^2 \tag{9.30}$$

$$\begin{aligned} g_{t\phi} = & -\frac{l^2 q^2}{16} \left(\left(\sqrt{2}\,(-\nu_2+\nu_1)\, q - 2\,\mu_2 + 2\,\mu_1\right) x^2 + 2\,\mu_2 - 2\,\mu_1\right) (x-1)\,(x+1) \left(e^{2\rho} - e^{-2\rho}\right)^2 \\ & -\frac{l^2 q^2}{16} \left(\sqrt{2}\,(-\nu_2+\nu_1)\, q - 2\,\mu_2 + 2\,\mu_1\right) \left(2\,x + \sqrt{2}\right) \left(-2\,x + \sqrt{2}\right) x^2 \left(e^{\rho} - e^{-\rho}\right)^2 \end{aligned} \tag{9.31}$$

$$\begin{aligned} g_{\phi\phi} = & \frac{l^2 q^2}{32} \left(\sqrt{2}\nu_1\, q x^2 - 2\,\mu_1 + 2\,x^2 \mu_1\right) \left(\sqrt{2}\nu_2\, q x^2 + 2\,x^2 \mu_2 - 2\,\mu_2\right) \left(e^{2\rho} - e^{-2\rho}\right)^2 \\ & + \frac{l^2 q^2}{32} \left(\sqrt{2}\nu_1\, q + 2\,\mu_1\right) \left(\sqrt{2}\nu_2\, q + 2\,\mu_2\right) \left(2\,x + \sqrt{2}\right) \left(-2\,x + \sqrt{2}\right) x^2 \left(e^{\rho} - e^{-\rho}\right)^2 \\ & + \frac{l^2 q^2}{48} \left((\nu_2+\nu_1)^2 q^2 + 6\,(\mu_1+\mu_2)^2\right) \end{aligned} \tag{9.32}$$

The three index metric is left with the form:

$$ds^3 = d\phi \times \left(g_{\phi\rho\rho} d\rho^2 + g_{\phi tt} dt^2 + g_{\phi\phi t} d\phi dt + g_{\phi\phi\phi} d\phi^2\right) \tag{9.33}$$

but for the purpose of this discussion the expressions for its components are very long and not worth displaying. It is enough to report that the two-index metric like field depends on the four independent parameters $\mu_1, \mu_2, \nu_1, \nu_2$ and it also depends explicitly on the parameter x which comes from the group element. The metric is asymptotically AdS.

If we set $\nu_1 = 0, \nu_2 = 0$ we are turning off the cubic Casimirs $Q_3 = 0, \bar{Q}_3 = 0$ and the metric asymptotes AdS with the same radius. The BTZ limit of this solution requires, besides from vanishing cubic Casimirs $\nu_1 = 0, \nu_2 = 0$, it requires that $x = 0$, in which for the two index metric we recover BTZ and the three index metric vanish.

However, if we a priori we set $x = 0$, then the two and three index metrics only sees three independent combinations of the four parameters: $\mu_1, \mu_2, \nu_1, \nu_2$, these combinations are given by $(\mu_1 + \mu_2)$, $(\mu_1 - \mu_2)$, $(\nu_1 + \nu_2)$. We will see that this happens because this solution, instead of being a solution belonging to the principal embedding is a solution that belongs to the diagonal embedding.

9.10 Remark on different embeddings

It was shown in the previous chapter (8) that the boundary degrees of freedom are completely encoded inside the lower case connections a_φ (b_φ). Thus, we should not expect such a determining behaviour as the above depending on the presence or absence of a parameter coming from the group element. As was explained in chapter (5), for $SL(3,\Re)$ connections there exist only two inequivalent embeddings: Principal and diagonal. From these, only the principal embedding describes higher spins, with one spin 2 and one spin 3 field as a field content. Meanwhile the diagonal embedding describes one spin 2, one spin 1, and two spin $\frac{3}{2}$ fields. Due to this fact, the diagonal embedding will not be of interest to us, simply because, even if it is certainly possible to build a metric-like field which posses three space-time index, this three index metric-like field will not be a fundamental higher spin field, rather it will be some field composed by lower spin fields.

9.11 $SL(3,\Re)$ solution in the diagonal embedding

The solution found in point 9.9 with the required condition that the BTZ limit exist: i.e., $x=0$, is a solution which belongs to the diagonal embedding, even if is certainly that there exist a similarity matrix which transforms C_ϕ by conjugation into a principally embedded a_ϕ (and analogously for $\bar{C}_\phi$ and b_ϕ), it can be shown, and it is very easy to convince himself, that this similarity matrix it is not connected with the identity matrix on its parameters space, i.e., there is not exist values of the parameters for which the similarity matrix be the identity matrix. Thus, as it is disconnected from the identity, it cannot be thought as a $SL(3,\Re)$ Lie group element produced by the exponentiation of a $SL(3,\Re)$ algebra element. Rather, the matrices C_ϕ (and $\bar{C}_\phi$) are in the diagonal embedding because they can be brought into a diagonally embedded a_ϕ (and b_ϕ respectively) with null spin $\frac{3}{2}$ fields, by using identity connected similarity matrices.

In fact, the matrix C_d can be constructed as:

$$C_d = \frac{q}{\sqrt{2}}(-L_1^d + L_{-1}^d) \tag{9.34}$$

where the generators L_1^d and L_{-1}^d corresponds to the diagonally embedded $SL(2,\Re)$ generators $L_1^d \equiv W_2$ and $L_{-1}^d \equiv W_{-2}$ respectively. Also, with the BTZ limit choice of the extra parameter $x=0$, the group element is simply given by $g = e^{\rho L_0}$.

With $x=0$ fixed a priori, the metric is left as:

$$\begin{aligned} ds^2 &= l^2 d\rho^2 - \frac{l^2 q^2}{8}\left(e^{2\rho} - e^{-2\rho}\right)^2 dt^2 - \frac{l^2 q^2}{8}\left(e^{2\rho} - e^{-2\rho}\right)^2 (-\mu_2 + \mu_1)\, d\phi\, dt \\ &\quad + \frac{l^2 q^2}{8}\left(\frac{1}{6}(\nu_2 + \nu_1)^2 q^2 + (\mu_1 + \mu_2)^2 + \mu_1 \mu_2 \left(e^{2\rho} - e^{-2\rho}\right)^2\right) d\phi^2 \end{aligned} \tag{9.35}$$

And the three index metric is left as:

$$\begin{aligned} ds^3 &= \frac{l^3 q^2}{2}(\nu_2 + \nu_1)\, d\rho^2 d\phi - \frac{l^3 q^4}{16}\left(e^{2\rho} - e^{-2\rho}\right)^2 (\nu_2 + \nu_1)\, d\phi\, dt^2 \\ &\quad - \frac{l^3 q^4}{16}\left(e^{2\rho} - e^{-2\rho}\right)^2 (\nu_2 + \nu_1)(-\mu_2 + \mu_1)\, d\phi^2 dt \\ &\quad + \frac{q^4 l^3}{16}(\nu_2 + \nu_1)\left(-\frac{1}{18}(\nu_2 + \nu_1)^2 q^2 + \left(e^{2\rho} - e^{-2\rho}\right)^2 \mu_1 \mu_2 + (\mu_1 + \mu_2)^2\right) d\phi^3 \end{aligned} \tag{9.36}$$

9.12 $SL(3,\Re)$ principal embedding solution

In analogy with the $SL(2,\Re)$ case, we start with the temporal components of the gauge fixed connections chosen to be given by a principally embedded C_p matrix as:

$$a_t = b_t = C_p = \frac{q}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{q}{2\sqrt{2}}(-L_1 + L_{-1}) \tag{9.37}$$

where L_1 and L_{-1} are principally embedded $SL(2,\Re)$ generators. The parameter q in euclidean solutions is chosen to be $q = 2\sqrt{2}\pi$, such that C_p already satisfy the trivial [5] holonomy conditions $\mathrm{Tr}(a_t^2) = \mathrm{Tr}(b_t^2) = q^2$ and $\mathrm{Tr}(a_t^3) = \mathrm{Tr}(b_t^3) = 0$.

The solution to the equations of motion $[a_t, a_\phi] = 0$ and $[b_t, b_\phi] = 0$, by Cayley-Hamilton theorem are given by:

$$a_\phi = \mu_1 a_t + \nu_1 (a_t^2 - \frac{I_3}{3} tr(a_t^2)) \ , \ b_\phi = -\mu_2 b_t - \nu_2 (b_t^2 - \frac{I_3}{3} tr(b_t^2)) \tag{9.38}$$

Where now the quadratic and cubic Casimirs of a_ϕ (b_ϕ) will be given in terms of the new parameters μ_1, ν_1 (μ_2, ν_2), as:

$$Q_2 \equiv \frac{1}{2}\mathrm{Tr}(a_\phi^2) = \frac{q^2}{12}\left({\nu_1}^2 q^2 + 6\,{\mu_1}^2\right) \ , \ \bar{Q}_2 \equiv \frac{1}{2}\mathrm{Tr}(b_\phi^2) = \frac{q^2}{12}\left({\nu_2}^2 q^2 + 6\,{\mu_2}^2\right) \tag{9.39}$$

$$Q_3 \equiv \frac{1}{3}\mathrm{Tr}(a_\phi^3) = \frac{\nu_1\, q^4}{108}\left(-{\nu_1}^2 q^2 + 18\,{\mu_1}^2\right) \ , \ \bar{Q}_3 \equiv \frac{1}{3}\mathrm{Tr}(b_\phi^3) = \frac{\nu_2\, q^4}{108}\left({\nu_2}^2 q^2 - 18\,{\mu_2}^2\right) \tag{9.40}$$

Again at this point it is instructive, but not necessary for the discussion, to stress that with this reparameterisation of the charges we can solve the holonomy equations (8.39) for the chemical potentials in what we will call the BTZ branch[6] as:

$$\begin{aligned} \sigma_2 &= \frac{6\,{\mu_1}^2 - {\nu_1}^2 q^2}{3\mu_1\,(2\,{\mu_1}^2 - {\nu_1}^2 q^2)} \ , \quad \sigma_3 = \frac{-2\nu_1}{\mu_1\,(2\,{\mu_1}^2 - {\nu_1}^2 q^2)} \\ \bar{\sigma}_2 &= -\frac{6\,{\mu_2}^2 - {\nu_2}^2 q^2}{3\mu_2\,(2\,{\mu_2}^2 - {\nu_2}^2 q^2)} \ , \quad \bar{\sigma}_3 = \frac{-2\nu_2}{\mu_2\,(2\,{\mu_2}^2 - {\nu_2}^2 q^2)} \end{aligned} \tag{9.41}$$

We choose the group element as being given by:

[5] In this case, as we are working with a 3×3 dimensional spin 1 representation of $SL(2,\Re)$, we have that after only one complete temporal cycle, the parallel transported a_t matrix comes back to its original value $Pe^{\oint a_t d_t} = +I_d$ (and similarly for b_ϕ)

[6] Recall that being the $SL(N,\Re)$ holonomy equations a highly coupled system of polynomial equations, in general there will be several branches as solutions. In particular for the $SL(3,\Re)$ in (8.39) there exist three branches, from which we will call the BTZ branch as the branch which posses the limit of null spin 3 chemical potential $\sigma_3 = 0$ as the spin 3 charge goes to zero $Q_3 = 0$. Note that in (9.41) this limit is controlled by ν_1, being achieved when $\nu_1 = 0$.

$$g = \begin{bmatrix} e^{\rho} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\rho} \end{bmatrix} = e^{\rho L_0} \tag{9.42}$$

Using (9.4), with the normalising factor as $f_3 = 2$ (see appendix), then the metric is left as:

$$\begin{aligned} ds^2 &= l^2 d\rho^2 - \frac{l^2 q^2}{8}\left(e^{\rho} - e^{-\rho}\right)^2 dt^2 - \frac{l^2 q^2}{8}\left(e^{\rho} - e^{-\rho}\right)^2 (\mu_1 - \mu_2)\, d\phi\, dt \qquad (9.43) \\ &+ \frac{l^2 q^2}{8}\left(\mu_1 \mu_2 \left(e^{\rho} - e^{-\rho}\right)^2 + \frac{q^2}{8}\nu_1 \nu_2 \left(e^{2\rho} - e^{-2\rho}\right)^2 + \frac{q^2}{6}(\nu_1 + \nu_2)^2 + (\mu_1 + \mu_2)^2\right) d\phi^2 \end{aligned}$$

and the spin 3 metric-like field is left as:

$$ds^3 = d\phi \times \left(g_{\phi\rho\rho} d\rho^2 + g_{\phi tt} dt^2 + g_{\phi\phi t} d\phi dt + g_{\phi\phi\phi} d\phi^2\right) \tag{9.44}$$

where:

$$g_{\phi\rho\rho} = -\frac{l^3 q^2}{4}(\nu_1 + \nu_2) \tag{9.45}$$

$$g_{\phi tt} = \frac{l^3 q^4}{128}(\nu_1 + \nu_2)\left(-8\left(e^{\rho} - e^{-\rho}\right)^2 + 3\left(e^{2\rho} - e^{-2\rho}\right)^2\right) \tag{9.46}$$

$$g_{\phi\phi t} = \frac{l^3 q^4}{64}\left(4(\nu_1 + \nu_2)(\mu_2 - \mu_1)\left(e^{\rho} - e^{-\rho}\right)^2 + 3(\mu_1 \nu_2 - \mu_2 \nu_1)\left(e^{2\rho} - e^{-2\rho}\right)^2\right) \tag{9.47}$$

$$\begin{aligned} g_{\phi\phi\phi} &= \frac{1}{2304} l^3 q^4 \left(144 \mu_1 \mu_2 (\nu_1 + \nu_2)\left(e^{\rho} - e^{-\rho}\right)^2\right) \\ &+ \frac{1}{2304} l^3 q^4 \left(\left(-9\nu_1 \nu_2 (\nu_1 + \nu_2) q^2 + 54 {\mu_1}^2 \nu_2 + 54 {\mu_2}^2 \nu_1\right)\left(e^{2\rho} - e^{-2\rho}\right)^2\right) \\ &+ \frac{1}{2304} l^3 q^4 \left(-8(\nu_1 + \nu_2)^3 q^2 + 144(\mu_1 + \mu_2)^2 (\nu_1 + \nu_2)\right) \end{aligned} \tag{9.48}$$

The range of these coordinates goes as $0 < \rho < \infty$, $0 < \phi < 2\pi$, $-\infty < t < \infty$. Note that going into the Euclidean section we do $t \to i\tau$, and identify $\tau = \tau + m$ with m integer, such that $0 < \tau < 1$. Then in order to fulfil trivial temporal holonomies of the gauge connections, we set: $q = 2\sqrt{2}\pi$, and then from the spin 2 metric we can compute the Hawking periodicity of the fields, which is given by:

$$\frac{1}{\rho^2}\frac{g_{tt}}{g_{\rho\rho}}|_0 = 4\pi^2 \tag{9.49}$$

and we see it posses the correct periodicity for a plane in polar coordinates, i.e., we have:

$$ds^2 \sim l^2\left(d\rho^2 + \rho^2 (2\pi)^2 d\tau^2\right) + \{\ldots\}_{\text{regular terms}} \tag{9.50}$$

Therefore, holonomies provides the right periodicity conditions for euclidean regular fields.

Analogously, as the spin 3 metric-like field posses a Black hole form times the 'regular' angular differential $d\,\phi$, we expect to have near the horizon $\rho = 0$, which its location is shared with the location of the horizon in the spin 2 metric, we can compute the Hawking periodicity condition in these coordinates obtaining:

$$\frac{1}{\rho^2}\frac{g_{\phi tt}}{g_{\phi\rho\rho}}|_0 = 4\,\pi^2 \tag{9.51}$$

which tell us that, near the horizon the spin 3 metric-like field is described by a regular object which is: (plane) $\times d\phi$ as:

$$ds^3 \sim d\phi \times \left(d\rho^2 + \rho^2(2\pi)^2 d\tau^2 + \{\ldots\}_{\text{regular terms}}\right) \tag{9.52}$$

These metric-like fields that has been found are thus completely regular. A further computation of the curvature shows that they do not posses, at least spin 2 curvature, singularities. Using spin 2 technology one can compute the curvature invariants for the two index metric-like field and for the black hole part of the three index metric like fields.

However they represent mayor conceptual challenges because a new phenomena occurs which is absent in the $SL(2,\Re)$ theory where we have pure gravity coupled to a cosmological constant. And this conceptual challenge, firstly spotted in [49], it is the fact that in (9.43) the spin 3 related parameters, i.e., ν_1, ν_2 which controls the presence ($\nu_1 \neq 0$, $\nu_2 \neq 0$) or absence ($\nu_1 = 0$, $\nu_2 = 0$) of spin 3 charges (see $Q_3, \bar{Q}_3$ in (9.40)), also controls asymptotically ($\rho \to \infty$) dominant terms ($\sim e^{4\rho}$) in the spin 2 metric (see (9.43)). In order to explain why this behaviour is problematic conceptually speaking, lets analyse the following: In the scenario of lower spins coupled to gravity, the terms which are asymptotically dominant (leading) in the metric use to be associated with the order of the background, such fluctuations around the background will be given by asymptotically sub-leading terms, in some cases as e.g. regular rotating BTZ, fluctuations can be found at most at the same order of the background. In this way at large distances (radial coordinate), fluctuations can be considered small compared to the background, and we can say that the full solution departs softly from the background.

The inverse of this situation is precisely what happens in the higher spin case. Higher spin fluctuations are dominant in such a way that for large radius they cannot be considered small and thus the picture of small higher spin fluctuations around a background fails.

However, it is important to mention that this asymptotic behaviour of the solution it is not due to the particular gauge fixing criteria settled in 9.6. This 'problematic behaviour' it is not of a gauge fixing nature. In a future publication it will be shown a mathematical proof of this claim, which basically states that starting with static and spherically symmetric fields as in (9.1), in which the lower case connections are in the principal embedding, and a_t and b_t fields are holonomy fulfilling, then there is no exist group elements $g_1(r)$, $g_1(r)$ in (9.1) for which this behaviour can be avoided. That result, which is beyond the scope of this report, sets an incompatibility in the higher spin theory, between regular solutions and an expected asymptotic behaviour similar in nature to the lower spin scenario as 'small fluctuations around a background'.

This method, to build regular solutions, can be easily extended for general N. It just consist in to pick a gauge in which a_t and b_t are equal to a charge independent similarity matrix C which belongs to the (anti)-trivial class for N (even) odd, respectively, but which goes only along the principally embedded L_1, L_{-1} generators. Then by Cayley-Hamilton theorem introduce $N-1$ independent parameters along the a_ϕ component, and others $N-1$ parameters along b_ϕ components. In each copy, the $N-1$ parameters introduced will give rise to the $N-1$ independent Casimirs of the angular component. Finally, pick the group elements as $g_1 = g_2^{-1} = e^{\rho L_0}$ generators. After that the metric like fields are constructed with (9.4).

9.13 Justification of the presence of a strong back-reaction

This behaviour was heuristically explained by Gutperle and Krauss in [49] and the physical argument goes as follows: Standing at the CFT side, when we have deformed the CFT action as in (7.1), we have coupled a spin 3 operator $\widehat{\mathcal{W}} \sim Q_3$ to the CFT, which by definition posses a conformal dimension $(3,0)$ and thus also a scaling dimension $\Delta = 3$, which means that its mass dimensions is 3. Then the spin 3 operator, considering that the CFT lives in two dimensions, by mass dimension counting turns to be non-renormalisable. As such, it must be UV relevant from the CFT point of view. This leads one to think that it must be irrelevant at the IR CFT, i.e., not affecting the IR CFT physics. Then, according to the weak-strong nature of AdS/CFT duality, and regarding to the interpretation given to the radial coordinate as an energy scale at which we are looking the boundary theory (see chapter (6)), the CFT has its minimum length (high energy) scale at the boundary, because from the boundary point of view, we are looking the CFT theory near the boundary. This implies that the UV CFT regime must belongs to the boundary. On the other side, when we are looking the bulk theory from the bulk point of view, we measure distances from the centre of the bulk manifold, i.e., as if we were stand on the horizon. Therefore, the boundary from this point of view it is located at large distances (low energy). In particular this means that the IR BULK physics must be located at the boundary, which is where the UV CFT physics live. Therefore the spin 3 operator, being an UV CFT relevant operator, it must affect the UV CFT physics, and this statement at the BULK theory side should be understood as it must affect the IR BULK physics which is precisely located at the boundary. Furthermore, in lower spins AdS/CFT, it is usual to think that, at the BULK theory side, the IR BULK physics is described asymptotically, i.e., at large distances, being far away from the centre (and thus near the asymptotic boundary), which is the regime where the physical BULK fields can be seen as small fluctuations around an asymptotic fixed background. This asymptotic background, being asymptotically defined, is considered as living on the boundary, where all the fluctuation of the fields can be thought as small because are highly suppressed by the long radial coordinate approaching the boundary. It is precisely this picture of small fluctuations around an asymptotic background, the one which fails in the higher spin $N \geq 3$ scenario, because, being the spin $N \geq 3$ operator, an UV CFT relevant operator, it relevantly affects the IR BULK physics according the conjecture (see figure (9.13) for a schematic commutative diagram).

$$\begin{array}{ccc} Q_3 \text{ irrelevant en IR CFT} & \longrightarrow & Q_3 \text{ irrelevant en UV BULK} \\ \downarrow & & \uparrow \\ Q_3 \text{ relevant en UV CFT} & \longrightarrow & Q_3 \text{ relevant en IR BULK} \end{array}$$

Please also note, that the in the first line of the above diagram, this scheme also justify the solution found in for the spin 3 field in the principal embedding, because just as this line states, the irrelevant at the IR CFT spin 3 operator, must be irrelevant at the UV bulk regime which is located near the horizon. This is confirmed with the fact that in the near horizon $\rho \sim 0$ analysis, the expressions (9.50) and (9.52) concerned to the study of possible divergencies, are independent on the spin 3 operator, and thus the near horizon behaviour it is not affected by this operator. (see also (9.70),(9.76) and (9.78) concerning the spin 4 case).

9.14 $SL(4, \Re)$ solution in the principal embedding

Just as before, we use a principally embedded charge-independent matrix given by:

$$a_t = b_t = C_p = q(-L_1 + L_{-1}) \tag{9.53}$$

where L_1, L_{-1} are the 4×4 dimensional $SL(2, \Re)$ generators which are principally embedded into $SL(4, \Re)$ (see appendix for conventions). Then, the C_p matrix satisfy:

$$Tr(C_p^2) = 20q^2, Tr(C_p^3) = 0, Tr(C_p^4) = 164q^4 \tag{9.54}$$

If we choose to fix $q = \pi$, then C_p satisfy the (anti)-trivial holonomy conditions, i.e.,:

$$e^{ia_t} = e^{ib_t} = e^{iC_p} = -I_d \tag{9.55}$$

where I_d is the 4×4 identity matrix.

After that, by Cayley-Hamilton theorem the fields a_ϕ and b_ϕ are constructed as:

$$a_\phi = \mu_1 C_p + \nu_1 (C_p^2 - \frac{1}{4} I_d Tr(C_p^2)) + \varepsilon_1 (C_p^3 - \frac{1}{4} I_d Tr(C_p^3)) \tag{9.56}$$

$$b_\phi = -\mu_2 C_p - \nu_2 (C_p^2 - \frac{1}{4} I_d Tr(C_p^2)) - \varepsilon_2 (C_p^3 - \frac{1}{4} I_d Tr(C_p^3)) \tag{9.57}$$

With this new parameterisation the spin 2, 3, and 4 charges (Casimirs) are given as:

$$Q_2 = 2\,q^2 \left(16\,{\nu_1}^2 q^2 + 5\,{\mu_1}^2 + 82\,q^2 \mu_1\,\varepsilon_1 + 365\,q^4 {\varepsilon_1}^2\right) \tag{9.58}$$

$$Q_3 = 64\,\nu_1\,q^4 \left(\mu_1 + 7\,\varepsilon_1\,q^2\right)\left(\mu_1 + 13\,\varepsilon_1\,q^2\right) \tag{9.59}$$

$$\begin{aligned} Q_4 = {} & 265721\,q^{12} {\varepsilon_1}^4 + 118100\,\mu_1\,q^{10} {\varepsilon_1}^3 + 6\,q^8 \left(3281\,{\mu_1}^2 + 5840\,{\nu_1}^2 q^2\right) {\varepsilon_1}^2 \\ & + 4\,q^6 \mu_1 \left(1968\,{\nu_1}^2 q^2 + 365\,{\mu_1}^2\right) \varepsilon_1 + q^4 \left(41\,{\mu_1}^4 + 256\,{\nu_1}^4 q^4 + 480\,{\mu_1}^2 {\nu_1}^2 q^2\right) \end{aligned} \tag{9.60}$$

and analogously, for the other copy we have:

$$\bar{Q}_2 = 2\,q^2\left(16\,{\nu_2}^2q^2 + 5\,{\mu_2}^2 + 82\,q^2\mu_2\,\varepsilon_2 + 365\,q^4{\varepsilon_2}^2\right) \tag{9.61}$$

$$\bar{Q}_3 = -64\,q^4\nu_2\left(\mu_2 + 13\,\varepsilon_2\,q^2\right)\left(\mu_2 + 7\,\varepsilon_2\,q^2\right) \tag{9.62}$$

$$\begin{aligned}\bar{Q}_4 = {}& 265721\,q^{12}{\varepsilon_2}^4 + 118100\,\mu_2\,q^{10}{\varepsilon_2}^3 + 6\,q^8\left(3281\,{\mu_2}^2 + 5840\,{\nu_2}^2q^2\right){\varepsilon_2}^2 \\ &+4\,q^6\mu_2\left(1968\,{\nu_2}^2q^2 + 365\,{\mu_2}^2\right)\varepsilon_2 + q^4\left(41\,{\mu_2}^4 + 256\,{\nu_2}^4q^4 + 480\,{\mu_2}^2{\nu_2}^2q^2\right)\end{aligned} \tag{9.63}$$

Again, we can say that our a_t and b_t fields already satisfy the (anti) trivial temporal holonomies, but supposing that this were not the case, performing this reparameterization of the charges, we could have solved the holonomy conditions in the BTZ branch with:

$$\begin{aligned}\sigma_2 = {}& \frac{1}{S}\times(74620\,{\varepsilon_1}^5q^{10} + 41243\,q^8{\varepsilon_1}^4\mu_1 + \left(-6582\,{\nu_1}^2q^8 + 9010\,q^6{\mu_1}^2\right){\varepsilon_1}^3 \\ &+\left(964\,q^4{\mu_1}^3 - 3690\,{\nu_1}^2q^6\mu_1\right){\varepsilon_1}^2 + \left(-594\,{\nu_1}^2q^4{\mu_1}^2 - 160\,{\nu_1}^4q^6 + 50\,q^2{\mu_1}^4\right)\varepsilon_1 \\ &-32\,{\nu_1}^4q^4\mu_1 - 30\,q^2{\mu_1}^3{\nu_1}^2 + {\mu_1}^5)\end{aligned} \tag{9.64}$$

$$\sigma_3 = -\frac{\nu_1}{S}\left(32\,\varepsilon_1\,q^4{\nu_1}^2 + 15\,q^2{\mu_1}^2\varepsilon_1 + 27\,q^4\mu_1\,{\varepsilon_1}^2 - 275\,q^6{\varepsilon_1}^3 + {\mu_1}^3\right) \tag{9.65}$$

$$\sigma_4 = \frac{1}{S}(-91\,q^4{\varepsilon_1}^3 - {\mu_1}^2\varepsilon_1 - 20\,q^2\mu_1\,{\varepsilon_1}^2 + 10\,\varepsilon_1\,q^2{\nu_1}^2 + 2\,{\nu_1}^2\mu_1) \tag{9.66}$$

where:

$$S = \left(\mu_1 + 9\,\varepsilon_1\,q^2\right)\left(\mu_1 + \varepsilon_1\,q^2\right)\left(\left(\mu_1 + 7\,\varepsilon_1\,q^2\right)^2 - 4\,{\nu_1}^2q^2\right)\left(\left(\mu_1 + 13\,\varepsilon_1\,q^2\right)^2 - 16\,{\nu_1}^2q^2\right) \tag{9.67}$$

and for the other copy, we could have solved the holonomy conditions with analogous expressions for the chemical potentials $\bar{\sigma}_2$, $\bar{\sigma}_3$, $\bar{\sigma}_4$ in terms of the other parameters $\mu_2, \nu_2, \varepsilon_2$, which for brevity we omit.

Once more, please note that BTZ branch is the branch which has the BTZ values for the chemical potentials $\sigma_2 = \frac{1}{\mu_1}$, $\sigma_3 = 0$, $\sigma_4 = 0$ in the limit case when $\nu_1 = 0$, $\varepsilon_1 = 0$. Also, note that in this limit case we will have $Q_3 = 0$, and $Q_2 \neq 0$ but also we have $Q_4 \neq 0$. The last, it is not supposed to be a problem given that already in the simple fundamental $SL(2,\Re)$ theory there exist non-vanishing quartic Casimirs invariants given as multiples of the quadratic Casimirs invariants.

The group elements are chosen as $g_1 = g_2^{-1} = e^{\rho L_0}$, and the two index metric-like field, trough (9.4) with normalising factor $f_4 = \frac{1}{5}$, are given as (omitting combinatoric factors):

$$ds^2 = g_{\rho\rho}d\rho^2 + g_{tt}dt^2 + g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 \tag{9.68}$$

with the components:

$$
\begin{aligned}
g_{\rho\rho} &= l^2 \\
g_{tt} &= -l^2 q^2 \left(e^{\rho} - e^{-\rho}\right)^2 \\
g_{t\phi} &= \left(-\frac{41}{5}\, l^2 \left(\varepsilon_1 - \varepsilon_2\right) q^4 - l^2 \left(-\mu_2 + \mu_1\right) q^2\right) \left(e^{\rho} - e^{-\rho}\right)^2 \\
g_{\phi\phi} &= l^2 q^2 \left(\frac{347}{5}\, \varepsilon_2\, \varepsilon_1\, q^4 + \frac{41}{5}\, \left(\varepsilon_2\, \mu_1 + \mu_2\, \varepsilon_1\right) q^2 + \mu_2\, \mu_1\right) \left(e^{\rho} - e^{-\rho}\right)^2 \\
&\quad + \frac{12\, l^2 q^4}{5}\, \nu_1\, \nu_2\, \left(e^{2\,\rho} - e^{-2\,\rho}\right)^2 + \frac{18\, l^2 q^6}{5}\, \varepsilon_2\, \varepsilon_1\, \left(e^{3\,\rho} - e^{-3\,\rho}\right)^2 \\
&\quad + \frac{2\, l^2 q^4}{5}\, \left(41 (\varepsilon_2\, \mu_1 + \mu_2\, \varepsilon_2 + \mu_1\, \varepsilon_1 + \mu_2\, \varepsilon_1) + 8\, (\nu_1 + \nu_2)^2\right) \\
&\quad + 73\, l^2 q^6 \left(\varepsilon_1 + \varepsilon_2\right)^2 + l^2 q^2 \left(\mu_1 + \mu_2\right)^2
\end{aligned} \tag{9.69}
$$

This two index metric-like field depends on the six charge-related parameters $\mu_1, \mu_2, \nu_1, \nu_1, \varepsilon_1, \varepsilon_2$ independently. Posses an horizon at $\rho = 0$ where al the components with at least one temporal label vanish. The horizon is smooth as near the horizon $\rho \sim 0$ we get:

$$
ds^2 = l^2 \left(d\rho^2 + \rho^2 (2\pi)^2 d\tau + \{\ldots\}_{\text{regular terms}}\right) \tag{9.70}
$$

Note that the strong back-reaction it is again present in (9.69), but in a pronounced way, where dominant leading terms in $g_{\phi\phi}$ now goes as $\sim e^{6\rho}$.
Also, in the limit $\nu_1 = 0$, $\nu_2 = 0$, $\varepsilon_1 = 0$, $\varepsilon_2 = 0$, the metric (9.69) falls in BTZ.

The three index metric like field is given as:

$$
ds^3 = d\phi \times \left(g_{\phi\rho\rho} d\rho^2 + g_{\phi tt} dt^2 + g_{\phi\phi t} d\phi dt + g_{\phi\phi\phi} d\phi^2\right) \tag{9.71}
$$

with the components:

$$g_{\rho\rho\phi} = -12\, l^3 q^2 (\nu_1 + \nu_2) \quad (9.72)$$

$$g_{tt\phi} = 3\, l^3 q^4 (\nu_1 + \nu_2) \left(-8 \left(e^{\rho} - e^{-\rho}\right)^2 + 3 \left(e^{2\rho} - e^{-2\rho}\right)^2\right) \quad (9.73)$$

$$\begin{aligned} g_{t\phi\phi} = & -l^3 q^4 \{3\,(71(\varepsilon_1\nu_2 - \varepsilon_2\nu_1) + 80(\nu_1\varepsilon_1 - \varepsilon_2\nu_2))\, q^2 + 24\,(\nu_1+\nu_2)\,(\mu_1-\mu_2)\}\left(e^{\rho} - e^{-\rho}\right)^2 \\ & + l^3 q^4 \left(-180\,(\varepsilon_2\,\nu_1 - \varepsilon_1\,\nu_2)\, q^2 + 18\,(\mu_1\,\nu_2 - \mu_2\,\nu_1)\right)\left(e^{2\rho} - e^{-2\rho}\right)^2 \\ & + 27\, l^3 q^6 (\varepsilon_2\,\nu_1 - \varepsilon_1\,\nu_2)\left(e^{3\rho} - e^{-3\rho}\right)^2 \end{aligned} \quad (9.74)$$

$$\begin{aligned} g_{\phi\phi\phi} = & \{2049\, l^3 \varepsilon_2\,\varepsilon_1\,(\nu_1+\nu_2)\, q^8 + 3\, l^3 (80\,\nu_1\,\varepsilon_1\,\mu_2 + 71\,\nu_1\,\mu_1\,\varepsilon_2 + 80\,\mu_1\,\nu_2\,\varepsilon_2 + 71\,\mu_2\,\varepsilon_1\,\nu_2)\, q^6 \\ & + 24\, l^3 \mu_2\,\mu_1\,(\nu_1+\nu_2)\, q^4\} \times \left(e^{\rho} - e^{-\rho}\right)^2 \\ & + 9 l^3 q^4 \{91\,\left({\varepsilon_2}^2\nu_1 + {\varepsilon_1}^2\nu_2\right) q^4 + 20\,(\mu_2\,\varepsilon_2\,\nu_1 + \mu_1\,\varepsilon_1\,\nu_2)\, q^2 \\ & + {\mu_2}^2\nu_1 + {\mu_1}^2\nu_2\} \times \left(e^{2\rho} - e^{-2\rho}\right)^2 \\ & + \left(135\, l^3 \varepsilon_2\,\varepsilon_1\,(\nu_1+\nu_2)\, q^8 + 27\, l^3 (\mu_2\,\varepsilon_1\,\nu_2 + \nu_1\,\mu_1\,\varepsilon_2)\, q^6\right) \times \left(e^{3\rho} - e^{-3\rho}\right)^2 \\ & + 2184\, l^3 (\varepsilon_1+\varepsilon_2)^2 (\nu_1+\nu_2)\, q^8 + 480\, l^3 (\varepsilon_1+\varepsilon_2)(\nu_1+\nu_2)(\mu_1+\mu_2)\, q^6 \\ & + 24\, l^3 (\mu_1+\mu_2)^2 (\nu_1+\nu_2)\, q^4 \end{aligned} \quad (9.75)$$

This three index metric-like field depends on the six charge-related parameters $\mu_1, \mu_2, \nu_1, \nu_1, \varepsilon_1, \varepsilon_2$ independently. In the BTZ limit: $\nu_1 = 0$, $\nu_2 = 0$, $\varepsilon_1 = 0$, $\varepsilon_2 = 0$ this field vanish. Also this field posses an horizon at $\rho = 0$ where all the components with at least one temporal label vanish. The horizon is smooth as near the horizon $\rho \sim 0$, up to a global numerical factor, we get:

$$ds^3 \sim d\phi \times \left(d\rho^2 + \rho^2 (2\pi)^2 d\tau^2 + \{\ldots\}_{\text{regular terms}}\right) \quad (9.76)$$

The four index metric-like field, posses the form (omitting combinatoric factors):

$$\begin{aligned} ds^4 = & \, g_{\rho\rho\rho\rho} d\rho^4 + g_{\rho\rho tt} d\rho^2 dt^2 + g_{\rho\rho t\phi} d\rho^2 dt d\phi + g_{\rho\rho\phi\phi} d\rho^2 d\phi^2 \\ & + g_{tttt} dt^4 + g_{ttt\phi} dt^3 d\phi + g_{tt\phi\phi} dt^2 d\phi^2 + g_{t\phi\phi\phi} dt d\phi^3 + g_{\phi\phi\phi\phi} d\phi^4 \end{aligned} \quad (9.77)$$

The components of the 4-index metric-like field are given by extremely huge expressions which are not worth to show them. However, it is important to stress that this field in general does not factorizes as two black holes, nor even as $d\phi^2 \times$ (BH), nor a combination of the two. But we can certainly say that this metric-like field near the horizon $\rho \sim 0$, expanding it at order $\mathcal{O}(\rho^4)$ (as there is a component g_{tttt}), has a structure given by:

$$ds^4 = \left(d\rho^2 + \rho^2 (2\pi)^2 d\tau^2\right) \times \left(d\rho^2 + \rho^2 (2\pi)^2 d\tau^2 + \{\ldots\}_{\text{regular terms}}\right) + \{\ldots\}_{\text{regular terms}} \quad (9.78)$$

which means it posses a smooth horizon, because near from it, it is given by the product of two regular surfaces.

Also, in the BTZ limit ($\nu_1 = 0$, $\nu_2 = 0$, $\varepsilon_1 = 0$, $\varepsilon_2 = 0$) for this field we have a factorization as:

$$ds^4 \sim (BTZ) \times (BTZ) \tag{9.79}$$

with BTZ given as in (9.15).

9.15 Conclusion

We have shown a way to construct regular bulk solutions inside a theory of higher spin fields coupled to gravity. To this end, we have used the criteria of the existence of an horizon which has been demanded by imposing $e_t = 0$ at this point. In particular, for static and spherically symmetric solutions of the equations of motion, the existence of an horizon implies that the constant lower case matrices a_t and b_t must be in the same class. Also, looking for solutions that posses spin 3 charges and a BTZ limit when these spin 3 charges are turned off, we have used solutions belonging to the principal embedding. In order to do this, inspired by the simplest example given by $N = 2$, an automatic way to ensure that we have a solution in the principal embedding has been given by using as a starting point some particular choices for the fields as $a_t = b_t = C$, where C is a charge independent constant matrix which belongs to the principal embedding. Also, we have shown that if we want euclidean metric-like solutions with smooth horizons, the C matrix must belong to the class which posses trivial (for N odd) or anti-trivial (for N even) holonomies. Meanwhile the geometry of a regular spin 2 metric near the horizon is a plane, we have shown that the geometry of the regular higher spin metric-like fields, near the horizon is described, in the spin 3 case, by the direct product of the regular angular 1-form with a plane, i.e.: '$d\phi\times$ plane'. Similarly, in the spin 4 case we have shown that near the horizon the geometry is given as a 'plane $\times$ plane'. In both cases, for $N = 3$ and $N = 4$ we have found that the solutions posses an asymptotic behaviour which strongly departs from the background. This is consistent with the fact that in this theory an irrelevant IR CFT spin $N > 2$ operator has been coupled to the system, being relevant for the IR bulk physics. The method used here to construct regular bulk solutions can be straightforwardly generalised for all N.

Appendices

Appendix A

Algebra Generators

A.1 $SL(3,\Re)$ generators representation, first convention

In chapters 5 and 7, we have strictly followed conventions used in [27] for the fundamental representation of the $SL(3,\Re)$ algebra:

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, L_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, L_{-1} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \tag{A.1}$$

$$W_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, W_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, W_0 = \frac{2}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{A.2}$$

$$W_{-1} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, W_{-2} = \begin{bmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A.3}$$

These eight generators satisfy the $SL(3,\Re)$ algebra given by:

$$[L_i, L_j] = (i-j)L_{i+j} \tag{A.4}$$

$$[L_i, W_m] = (2i-m)W_{i+m} \tag{A.5}$$

$$[W_m, W_n] = -\frac{1}{3}(m-n)(2m^2+2n^2-mn-8)L_{m+n} \tag{A.6}$$

The generators (L_1, L_0, L_{-1}) are the generators of $SL(2,\Re)$ principally embedded into $SL(3,\Re)$. They transform as a spin 1 triplet under themselves, meanwhile the generators $(W_2, W_1, W_0, W_{-1}, W_{-2})$ transforms as a five component spin 2 multiplet under (L_1, L_0, L_{-1}).

The generators (W_2, L_0, W_{-2}) are the generators of $SL(2,\Re)$ diagonally embedded into $SL(3,\Re)$. They transform as a spin 1 triplet under themselves. On the other hand under a transformation with (W_2, L_0, W_{-2}), the generator W_0 transform as a spin 0, meanwhile the generators (W_1, L_{-1}) and (W_{-1}, L_1) transforms as two spin $\frac{1}{2}$ doublets.

A.2 Principally embedded $SL(2,\Re)$ generators into $SL(N,\Re)$ convention

In chapter 9, we have strictly followed the convention used in Castro et al work [54]. An explicit representation for the principally embedded $SL(2,\Re)$ generators into the fundamental representation of the $SL(N,\Re)$ algebra, is given by:

$$L_1 = -\begin{bmatrix} 0 & \cdots & & & & & 0 \\ \sqrt{N-1} & 0 & \cdots & & & & \vdots \\ 0 & \sqrt{2(N-2)} & 0 & \cdots & & & \\ \vdots & 0 & \ddots & 0 & \cdots & & \\ & 0 & 0 & \sqrt{|i(N-i)|} & 0 & \cdots & \\ & 0 & 0 & 0 & \ddots & 0 & \\ 0 & \cdots & & & & \sqrt{N-1} & 0 \end{bmatrix}, \quad \text{(A.7)}$$

$$L_{-1} = \begin{bmatrix} 0 & \sqrt{N-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2(N-2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{|i(N-i)|} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{N-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{(A.8)}$$

$$L_0 = \frac{1}{2}\begin{bmatrix} (N-1) & 0 & \cdots & & & & 0 \\ 0 & (N-3) & 0 & \cdots & & & \\ \vdots & 0 & \ddots & 0 & 0 & 0 & 0 \\ & 0 & 0 & (N+1-2i) & 0 & 0 & 0 \\ & 0 & 0 & 0 & \ddots & 0 & 0 \\ & 0 & 0 & 0 & 0 & -(N-3) & 0 \\ 0 & \cdots & & & & & -(N-1) \end{bmatrix}, \quad \text{(A.9)}$$

The above generators satisfy the $SL(2,\Re)$ commutation relation:

$$[L_i, L_j] = (i-j)L_{i+j} \quad \text{(A.10)}$$

From them we have the normalization factor:

$$f_N = tr(L_0 L_0) = \frac{1}{12} N(N^2 - 1) \tag{A.11}$$

Although not explicitly used in this thesis, for completion we can say that higher spin generators can be explicitly constructed by taking products of the principally embedded $SL(2, \Re)$ generators as:

$$W_m^{(s)} = (-1)^{s+m-1} \frac{(s+m-1)!}{(2s-2)!} [L_{-1}, [L_{-1}, \dots [L_{-1}, L_1^{s-1}] \dots] \tag{A.12}$$

These higher spin generators satisfy the relation commutation:

$$[L_i, W_m^{(s)}] = (i(s-1) - m) W_{i+m}^{(s)} \tag{A.13}$$

In the $N = 2$ case, there exist only one embedding which through (A.7) is explicitly given by:

$$L_1 = -\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, L_{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, L_0 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{A.14}$$

For the $N = 3$ case ($SL(3, \Re)$), the principally embedded $SL(2, \Re)$ generators in (A.7) are explicitly given by:

$$L_1 = -\begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}, L_{-1} = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}, L_0 = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \tag{A.15}$$

For the $N = 4$ case ($SL(4, \Re)$), the principally embedded $SL(2, \Re)$ generators in (A.7) are explicitly given by:

$$L_1 = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}, L_{-1} = \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}, L_0 = \frac{1}{2}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \tag{A.16}$$

Appendix B

Chern-Simons with Boundary

Here, we will give a brief review of the CS theory defined on a manifold with boundary. This exposition follows the lines that were developed in [55].

In the CS theory defined on the solid torus manifold, we have the torus surface as a boundary. To study the consequences of having a boundary, we can study its Hamiltonian formulation, to this we do a $(2+1)$ splitting of the coordinates as:

$$A = A_t dt + A_i dx^i \tag{B.1}$$

with this splitting the CS action reads as:

$$S_{\rm CS} = \frac{k}{4\pi}\int_{\mathcal{M}} dt \wedge dx^i \wedge dx^j tr(\dot{A}_i A_j + A_t F_{ij}) + \frac{k}{4\pi}\int_{\partial\mathcal{M}} dx^i tr(A_t A_i) + B_T \tag{B.2}$$

where B_T is an extra boundary term that guaranty the differentiability of the CS action.

Expanding the above expression explicitly in the gauge algebra generators one is left with:

$$S_{\rm CS} = \frac{k}{4\pi}\int_{\mathcal{M}} dt \wedge dx^i \wedge dx^j g_{ab}(\dot{A}_i^a A_j^b + A_t^a F_{ij}^b) + \frac{k}{4\pi}\int_{\partial\mathcal{M}} dx^i g_{ab}(A_t^a A_i^b) + B_T \tag{B.3}$$

where g_{ab} is the Killing metric of the Lie algebra, and the indices $a, b : 1, \ldots, N$ stands for label the different generators of the algebra, where N is the gauge group dimension.

After space and time splitting, this action is automatically left as a Hamiltonian action, where A_i^a are N dynamical fields and its conjugated momentum π_a^i is given by $\pi_a^i = \frac{k}{4\pi}\epsilon^{ij} g_{ab} A_j^b$. Although nobody use this directly, it is worth to mention it.

Also, A_t^a are N Lagrange multipliers while F_{ij}^a gives rise to N first class constraints which comes from vary with respect to the Lagrange multiplier A_t^a as:

$$\phi_a(x) = \frac{k}{4\pi} g_{ab}\varepsilon^{ij} F_{ij}^b(x) \tag{B.4}$$

For two functionals P_1, P_2 defined on the phase space, its equal-time Poisson brackets is given by:

$$\{P_1, P_2\} = \frac{2\pi}{k}\int dx^i \wedge dx^j tr(\frac{\delta P_1}{\delta A_i(x)}\frac{\delta P_2}{\delta A_j(x)}) \tag{B.5}$$

Using the Poisson bracket, we see that ϕ_a satisfy a first class algebra given by:

$$\{\phi_a(x), \phi_b(x')\} = f^c_{ab}\phi_c(x)\delta(x - x') \tag{B.6}$$

To study how the gauge transformations acts on field configurations, it is useful to define an extended generator of gauge transformations as a distribution smeared with a test gauge function Λ as:

$$\begin{aligned} \Phi[\Lambda] &= \frac{k}{4\pi}\int_\sigma dx^2 tr(\Lambda\phi) + Q[\Lambda] \\ &= \frac{k}{4\pi}\int_\sigma dx^i \wedge dx^j tr(\Lambda F_{ij}) + Q[\Lambda] \end{aligned} \tag{B.7}$$

where $Q[\Lambda]$ is a boundary term chosen in such a way that ensures functional differentiability of $\Phi[\Lambda]$. This boundary term is crucial to understand from where does it comes the physical degrees of freedom of the theory.

For gauge transformations using (B.7) in which the parameter Λ is independent of the fields, we have:

$$Q[\Lambda] = -\frac{k}{2\pi}\int dx^i tr(\Lambda A_i) \tag{B.8}$$

Using this kind of parameter-independent gauge transformations, we can compute the Poisson bracket of the extended generators of gauge transformation $\Phi[\Lambda_1]$ and $\Phi[\Lambda_2]$ getting:

$$\{\Phi[\Lambda_1], \Phi[\Lambda_2]\} = \Phi[[\Lambda_1, \Lambda_2]] + \frac{k}{2\pi}\int dx^i tr(\Lambda_1\partial_i\Lambda_2) \tag{B.9}$$

Also, for the transformations performed with $\Phi[\Lambda]$, after imposing the constraints, in the weak equality we have $\Phi[\Lambda] \simeq Q[\Lambda]$. Now, if Λ is such that $Q[\Lambda] \neq 0$, then the gauge transformations performed with $\Phi[\Lambda]$ will give rise to global symmetries which transforms a physical state of the boundary into another physical state of the boundary, which are not gauge equivalent. In particular this means that for some values of Λ in which $Q[\Lambda] \neq 0$, true gauge symmetry coming from the bulk is really lost at the boundary, and thus $Q[\Lambda]$ receives the name of global charges of the 'would be' the boundary theory. Is straightforward to see, after solving the constraints, that the global charges satisfy an algebra similar to (B.9), but now with the Dirac brackets of the reduced phase space:

$$\{Q[\Lambda_1], Q[\Lambda_2]\}_D = Q[[\Lambda_1, \Lambda_2]] + \frac{k}{2\pi}\int dx^i tr(\Lambda_1\partial_i\Lambda_2) \tag{B.10}$$

Using (B.8) inside (B.10), with the A^a_i fields expanded on modes on the circle, one finally arrives at the Kac-Moody Algebra. If one imposes the correct constraints on the boundary values of A^a_i fields, one can fall into the extended conformal W_n algebras.

Appendix C

Quasi-Primary and Primary Operators

In the language of holomorphic operator product expansions (OPE) of the CFT, a quasi-primary operator '$\widehat{\mathcal{O}}(\omega,\bar{\omega})$' is one that under a transformation made with the spin 2 current 'energy momentum' operator (denoted in its holomorphic components as '$\widehat{\mathcal{L}}(z), \widehat{\bar{\mathcal{L}}}(\bar{z})$'), transforms, in the OPE formalism, as:

$$\widehat{\mathcal{L}}(z)\widehat{\mathcal{O}}(\omega,\bar{\omega}) = \ldots + h\frac{\widehat{\mathcal{O}}(\omega,\bar{\omega})}{(z-\omega)^2} + \frac{\partial\widehat{\mathcal{O}}(\omega,\bar{\omega})}{(z-\omega)} + \ldots \tag{C.1}$$

$$\widehat{\bar{\mathcal{L}}}(\bar{z})\widehat{\mathcal{O}}(\omega,\bar{\omega}) = \ldots + \bar{h}\frac{\widehat{\mathcal{O}}(\omega,\bar{\omega})}{(\bar{z}-\bar{\omega})^2} + \frac{\bar{\partial}\widehat{\mathcal{O}}(\omega,\bar{\omega})}{(\bar{z}-\bar{\omega})} + \ldots \tag{C.2}$$

where $(h,\bar{h})$ are called the conformal weights of the operator $\widehat{\mathcal{O}}(\omega,\bar{\omega})$. These two parameters can be related to the spin 's' and scaling 'Δ' parameters of the operator $\widehat{\mathcal{O}}(\omega,\bar{\omega})$ through:

$$s = h - \bar{h} \tag{C.3}$$

$$\Delta = h + \bar{h} \tag{C.4}$$

which are, respectively, the eigenvalues of the rotation operator $R = z\partial - \bar{z}\bar{\partial}$ and dilatations operator $D = z\partial + \bar{z}\bar{\partial}$.

The dots on the left hand side of (C.1) (and (C.2)), means higher powers of the singular terms which are at orders higher than $\frac{1}{(z-w)^2}$ (and $\frac{1}{(\bar{z}-\bar{w})^2}$ for the second line), meanwhile the dots on the right hand side means regular terms.

A primary operator is a quasi-primary operator, for which the terms denoted by the dots of left hand side of (C.1) (and (C.2)) are completely absent, i.e, the series truncates at order $\frac{1}{(z-w)^2}$.

The above OPE transformation (C.1), for a primary operator (without the presence of higher order singular terms in the dots of the left hand side), after quantum averages has been taken, produces a behaviour in the averaged quantities as a general smooth conformal transformation with $\delta z = \varepsilon(z)$:

$$\delta\mathcal{O}(\omega,\bar{\omega}) = \varepsilon\mathcal{O}'(\omega,\bar{\omega}) + h\varepsilon'\mathcal{O}(\omega,\bar{\omega}) \tag{C.5}$$

An analogous expression holds for the anti-holomorphic transformation $\delta\bar{z} = \bar{\varepsilon}(\bar{z})$ produced with the anti-holomorphic component $\widehat{\mathcal{L}}(\bar{z})$ of the energy-momentum tensor.

If the transformed operator is a non-primary one, as is the case of the energy-momentum itself, using $\mathcal{O} = \mathcal{L}(z)$ in (C.1), then a higher order singular term of the left side of (C.1) is present, being proportional to the central charge. This term give rise to the term that goes as ϵ''' in (5.89). Thus, in principle, without possessing explicitly the OPES, but knowing the transformation rules for an operator, one can indirectly see that there must exist singular terms of orders higher than $\frac{1}{(z-w)^2}$ in (C.1) and thus deduce that the operator in question will be a quasi-primary one.

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