

DEFORMED D1D5 CFT: A HOLOGRAPHIC PROBE OF QUANTUM
GRAVITY

by

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A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Physics
University of Toronto

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Abstract

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Doctor of Philosophy

Graduate Department of Physics

University of Toronto

2017

One of the big unsolved questions in gravity research is the black hole information problem. This problem, which pits the unitarity of quantum field theory against smooth classical spacetime, must have a solution in a complete theory of quantum gravity.

This thesis will explore aspects of one approach to this problem in the context of string theory. The approach imagines black hole microstates as string theoretic objects. We look at a prototype system, the D1D5 system, and exploit holography to examine the dual conformal field theory (CFT). Specifically, we examine the CFT deformed from the free orbifold point, dual to a very stringy bulk, using a twisted operator that will take us towards the point with the supergravity description.

The effects of twisted operators in the CFT are key to understanding physical processes such as emission and thermalization in black hole microstates. We will propose a component twist method for examining the effects of bare twist operators for higher twists in the continuum limit. Our method builds higher twists from simple 2-cycle twists, whose effects are known. We will find that, in this limit, the coefficients describing general states will follow a conjectured general functional form.

We then explore the deformed CFT directly by examining operator mixing for un-twisted operators. We will exploit the operator product expansion on the covering space, where twist operators of the orbifold are resolved. We use this to examine the mixing of a general supergravity operator, specifically examine the dilaton, and finish with the mixing of a non-supersymmetric candidate operator. We conjecture that this method could be extended to include twisted operators.

We will also examine the mixing of the non-supersymmetric candidate operator by examining three point functions. To automate the lengthy and repetitive computations, we wrote a Mathematica package to compute correlation functions and OPEs in the D1D5 CFT. We will explain some of the main functions of this package and how it can be applied to computations.

Finally, we will end with a short discussion on future directions.

To my loving family. My parents, Irene and Theodore, and my brother, Lucas.

Acknowledgements

First I would like to deeply thank Amanda W. Peet, my supervisor, for not only their physics guidance but also for their support and understanding.

I would also like to deeply thank Ben Burrington, for all the advice and support with my first paper.

I would also like to thank Zaq Carson for his aid with my second paper and for some very enjoyable conversations.

Thanks also to Pierre Savard and Michael Luke for serving on my committee.

I am grateful to past graduate students: Ida Zadeh for her great notes on the Lunin-Mathur method; Dan O'Keeffe for his camaraderie and support; and current student Jesse Cresswell for discussing preprints and insightful comments.

I would also like to mention Callum Quigley, for his work explaining (0,2) gauge theories and related topics.

I would not have been able to get to graduate school without the support and long time friendship of Jessica Randall, Calvin Luong, Drew Gilvary, and Kevin Bale.

Finally, I would like to thank my family. I thank my brother, Lucas Jardine, for the strong brotherly bond we share. I also thank my parents, Irene and Theodore Jardine, for all their hard work and care while raising me. Without them I would not be anywhere close to the man I am today.

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Chapter 1

Introduction

1.1 Black holes and the information problem

At the classical level, our universe is described by Einstein's general theory of relativity. This imagines spacetime to be warped by the presence of energy and these warps to influence the motion of matter within it. This interplay between matter and spacetime is captured in the beautiful Einstein equation [1],

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad (1.1)$$

where $g_{\mu\nu}$ is the metric of the space time, $R_{\mu\nu}$ is the Ricci tensor, R is the scalar curvature, Λ is the cosmological constant, and $T_{\mu\nu}$ is the stress-energy tensor. This equation has been very successful in describing not only basic aspects of astronomical observations but also in explaining cosmological evolution.

The first non-trivial solution to this equation was the Schwarzschild solution [2],

$$ds^2 = - \left(1 - \frac{2GM}{c^2r}\right) dt^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (1.2)$$

This solution is four dimensional and is one of the simplest solutions in a class of solutions known as black hole solutions. The key features of these solutions are the presence of horizons and the inner singularity. In Schwarzschild coordinates, the single horizon is at $r = r_s = 2GM/c^2$ and the singularity is at $r = 0$.

There have been a number of independent supporting astrophysical observations in favour of their existence. Most recently, the LIGO collaboration detected a number of gravitational wave patterns of binary black hole mergers [3]. This means we must deal with the existence of these objects in our universe and understand all aspects of their

description.

The horizon of the black hole is an interesting feature of these objects. They metric looks singular but this is a coordinate artefact and can be removed by a suitable coordinate transformation. Even then, the horizon still has physical consequences. This was shown with Hawking’s derivation of black hole radiation [4]. The toolset used was quantum field theory on a curved background. The semiclassical calculation found that through Hawking radiation a black hole will eventually evaporate away on a timescale $\sim G^{2/(D-3)} M^{(D-1)/(D-3)}$ with D the total spacetime dimension. There is also the problematic curvature singularity at the heart of a black hole, which is expected to be resolved somehow by quantum gravity and will not be our explicit focus here.

The evaporation of a black hole presents a bit of a problem. There is a theorem in four dimensional general relativity called the no-hair theorem [5–7]. This states that an asymptotically flat black hole can only support three types of conserved quantum numbers: their mass, angular momentum, and charges. The resultant thermal radiation after evaporation can only depend on these quantum numbers. If we form a black hole from gravitational collapse, any other information about the infalling matter is then lost. If we start with matter in a pure state and then collapse to a black hole and finally wait until evaporation is finished, we end up with a mixed state. This is a non-unitary process, which conflicts with the unitarity of quantum field theory. This conflict is one of the most important open problems in quantum gravity, called the black hole information problem.

Although there have been many proposals for correcting Hawking’s picture semiclassically, Mathur [8] made a broad-based argument that more major quantum gravity corrections will be needed to resolve the information problem. His argument hinged on only two assumptions. The first was that each Hawking pair created in the emission process are produced independently of previous pairs and the second that strong subadditivity, a quantum constraint on the entropies of subsystems, holds. The first assumption is dependent on the structure of the classical horizon and is therefore not changed in semi-classical arguments. The second is known to hold for any physical quantum mechanical system. So the only way to resolve the information problem would be with order one quantum gravity corrections.

Since then, the “firewall” arguments of Almheiri et al [9] considerably sharpened community thinking about the regime of validity of effective field theories involving Einstein gravity. They began with four main postulates. The first was unitarity, the second was the validity of semi-classical effective field theory with gravity, the third was that the Bekenstein entropy formula holds, and the last was smooth infall at the horizon. They

argued that these postulates together were incompatible, so the authors chose to give up the final one. They gave a very generic argument that the first and second postulates would imply a “firewall” at the horizon. This is a wall of very highly excited modes of Hawking radiation near the horizon that would destroy any infalling matter. Avoiding this result would require instead to give up a different of the four postulates. There are a number of proposals for solving the black hole information paradox and avoiding firewalls.

The “ER=EPR” proposal [10, 11] sees bipartite entanglement between particles (the EPR) as equivalent to a wormhole (the ER) that connects the particles. It was originally imagined for large black holes, but was also argued to hold for single particles with non-classical wormholes. These quantum wormholes allow new Hawking radiation to be connected to the black hole with a wormhole, which is connected via wormhole to earlier radiation. This picture is different from independent pair creation assumed in the Mathur argument and allows unitary to be restored. It remains unclear how these quantum wormholes would behave or the details of how multipartite entanglement configurations would be represented.

The state-dependent operator proposal [12–14] attempts to describe black holes by constructing mirror operators that would act as though there was a second asymptotic region behind the horizon. This proposal sees the Hilbert space of the interior of a black hole as a scrambled version of the exterior. State independent operators would have a number of problems, including violating locality, but their state dependent operators avoid these problems for low-point correlators. A number of papers argued against its validity, either showing non-unitary measurement processes [15] or violations of the Born rule [16]. Non-locality has been discussed in both semiclassical gravity [17] and string theory in a black hole background [18–21]. However, it is unclear how to incorporate non-locality manifestly in the calculations to see how the unitary of black hole evaporation is rescued.

In order to progress in finding an honest solution to the black hole information problem, we should focus in on a theory that could possibly describe quantum gravity in the ultraviolet (UV). In this thesis, we will focus on exploring the black hole information paradox through the lens of string theory. It is in string theory that we can construct setups that are well defined in the UV but could be used to describe the physics of certain types of black holes under strong theoretical control. In the next section, we will review some preliminaries of string theory to prepare for discussing these setups.

1.2 String theory as quantum gravity

String theory is a UV complete theory, which reduces in the low energy limit to supergravity. The emergence of gravity can be seen by studying the beta functions of the worldsheet sigma model description of strings, where spacetime fields like the metric arise as coupling functions of the quantum fields describing string motions. Having these beta functions vanish to preserve conformal invariance of string theory quantum mechanically will give back Einstein's equation of the target space (the actual spacetime we want to describe) and require that the number of spacetime dimensions is $D = 10$ for critical string theory. This is why the worldsheet description of string theory recruits techniques of conformal field theory (CFT). String theory is described by one coupling, g_s , and one length scale set by the string tension, $\alpha' = l_s^2$.

An important part of string theory is supersymmetry. There are plenty of introductions and reviews of supersymmetry, see for example [22, 23]. Supersymmetry relates bosonic degrees of freedom to fermionic degrees of freedom and has a corresponding charge, by Noether's theorem. A theory can have $\mathcal{N} > 1$ supersymmetry generators, at which point it is said to possess extended supersymmetry. The generators then have anticommutation relations that go, schematically, as

$$\{Q_\alpha^I, Q_\beta^J\} = \Gamma_{\alpha\beta}^\mu P_\mu \delta^{IJ} + C_{\alpha\beta} Z^{IJ} \quad (1.3)$$

where Q_α^I are spinor supercharges, P_μ is the momentum, and Z^{IJ} is the central charge. The actual form of the anticommutator depends on \mathcal{N} , d , and the conserved charges of the theory. This relation gives rise to the Bogomol'nyi-Prasad-Sommerfield (BPS) bound defined by $M \geq |z_i|$, where M is the mass of a state of the theory and $\{z_i\}$ are the eigenvalues of the central charge operator for the state, up to normalizations. Supersymmetry gives a significant amount of control, such as non-renormalization theorems, that can be exploited to compute various aspects of the theory.

String theory contains the eponymous strings but also nonperturbative Dp -branes, with tension proportional to $1/g_s$, where p is the number of spacetime dimensions that are parallel to the brane. A few classic textbooks for string theory are [24–26]. There are five distinct critical superstring theories in $D = 10$: Type I, Type IIA, Type IIB, Heterotic E ($E_8 \times E_8$), and Heterotic O ($SO(32)$). There also exists another description, M-theory which lives in $D = 11$. The exact spectrum of objects and fields in the low energy theory depends on the string theory and will split into two sectors depending on the periodicity of fermionic states. All types have a Neveu-Schwarz (NS) sector, with antiperiodic fermionic states, consisting of the string metric $G_{\mu\nu}$ and the dilaton ϕ , and also

two-index antisymmetric tensor potential $B_{\mu\nu}$. However, their Ramond (R) sectors, with periodic fermionic states, differ. For example, Type IIA has the 1-index and 3-index anti-symmetric potentials whereas Type IIB has a scalar, 2-index, and 4-index antisymmetric tensor potentials. For types I, IIA, and IIB, we have the D_p -branes, fundamental strings, and NS5-branes. The two heterotic theories have just the fundamental strings and the NS5-branes. Furthermore, D_p -branes couple to $(p+1)$ -index tensor potentials, so Type IIA has even dimensional stable D_p -branes carrying the pertinent charges whereas Type IIB has odd dimensional stable D_p -branes carrying charges. Stakes of straight parallel D_p -branes have solutions of the form [27]

$$\begin{aligned} dS^2 &= H_p^{-1/2}(-dt^2 + dx_{||}^2) + H_p^{1/2}dx_{\perp}^2, \\ C_{0\dots p} &= g_s^{-1}(1 - H_p^{-1}), \\ e^{\phi} &= H_p^{(3-p)/4}, \end{aligned} \tag{1.4}$$

where

$$H_p = 1 + \frac{C_p g_s N_p l_s^{7-p}}{r^{7-p}}. \tag{1.5}$$

Here $C_p = (2\sqrt{\pi})^{5-p} \Gamma[(7-p)/2]$ and r is the radius perpendicular to the brane worldvolume. Branes can then be put together to construct more complicated solutions in string theory. Generically to produce big smooth classical spacetimes, we need large numbers of branes to be used. We can see from H_p that $g_s N \ll 1$ will be valid for perturbative string theory and $g_s N \gg 1$ for a black brane description.

These different string theories are related by dualities, which act on both the objects in their spectra and couplings. The first duality discovered was T-duality. This acts on a compactified cycle and inverts its radius in string units while switching momentum and winding modes of strings wrapped or moving around the cycle. This duality exchanges Type IIA and IIB, exchanges Heterotic E with Heterotic O, and will take Type I to either IIA or IIB. It will also change D_p branes to D_{p+1} or D_{p-1} when acting perpendicular or parallel to the brane, respectively. Another important symmetry is S-duality, which is an electric-magnetic duality. In type IIB, it can be used to switch D1 branes for fundamental strings and D5 branes for NS5 branes. These dualities and others allow us to connect up the five 10D string theories with M theory in an interconnected web. Furthermore, it allows us to relate systems of strings and branes in one string theory to other systems in another theory where the construction may be easier to study technically.

1.3 Fuzzball proposal

An interesting and fruitful string theoretic approach to the black hole information problem in string theory has been the fuzzball program. See [28] for a good introduction for interested beginners. Its general framing is more top-down than bottom-up, and it focuses on black hole microstates, whose geometries are string theoretic constructions without horizons or singularities. The ultimate goal for this approach is to see how the sum over microstates captures the full physics of black holes. In a single microstate, there is no infinite throat of a black hole and instead there is a finite throat, ending in at the cap. This cap is a complicated, string theoretic arrangement of fluxes, compactified dimensions, and nonperturbative ingredients which identify the individual microstate. Order one corrections occur close to the cap and the ingredients of the setup will interact with infalling matter and radiate without information loss. In the literature there is normally a distinction drawn between a generic fuzzball, which may be a strongly coupled or stringy object, and a microstate geometry, a low energy supergravity solution that describes the fuzzball with a smooth geometry and no horizons.

One can use these fuzzballs to count the entropy from the bound state description and the black hole description and find agreement between the two descriptions. This was done first in the famous paper by Strominger and Vafa [29]. This was the first observation that string theoretic descriptions can be used to reproduce the physics of black holes. This microscopic counting was extended to non-extremal [30] and rotating black holes [31] and agreements were found in those cases as well.

The basic constructions involve using D-branes in compactified dimensions to create the black hole geometries. One of the simplest is the D1D5 system. We will describe this in detail in section 1.9. When fully compactified, this is a 5-dimensional black hole with a degenerate horizon. In order to get a non-degenerate horizon, one must add some momentum charge around the S^1 that the D1 branes wrap to get the three charge system D1D5P. This momentum then puffs up the horizon to macroscopic size and the results is a non-degenerate 5-dimensional black hole. To obtain a 4 dimensional black hole, one must use four charges. One such setup is the a D2D6NS5P solution. These are not the only such charged solutions, and in fact one can exploit the dualities of string theory to relate them to other solutions. For example, the D1D5 solution is dual to a fundamental string with a momentum charge. This can be useful for calculating quantities in a duality frame where it is easiest to compute. A review on the technical aspects of constructing these solutions can be found in [32]. A more generic description of how to construct fuzzball solutions is found in [33].

Some of the most simplistic microstates are also extremal states saturating the BPS bound. This means the black hole has zero temperature and it will not evaporate. In order to see emission from black holes, one needs non-extremal solutions. Radiation emission for a number of different types of near-extremal fuzzball geometries were reproduced beautifully using microscopic CFT methods [34–37]. However, astrophysical black holes are obviously not supersymmetric and far from extremal, and so there have been a number of works dedicated to constructing more complicated, more physical microstate geometries. The most famous of these is the JMaRT class [38] but there are other solutions that have been found as well [39, 40]. More recent works include finding an uncharged, rotating, but not stationary solution [41], and by exploiting the superstratum, a smooth horizonless supergravity solution parameterized by arbitrary functions of two variables [42, 43]. Most recently, a solution with arbitrarily small angular momentum has been found [44].

A question about the fuzzball proposal is how they might be formed from gravitational collapse. The typical size of these fuzzballs is roughly horizon sized, and so there must be some mechanism which prevents the infalling matter from forming the black hole. Early works [45] argued that the amplitude for tunneling, $\exp[-S_{\text{tunneling}}] = \exp[-kGM^2]$, where k is an order one number, could compete with the large phase space of fuzzballs $\exp[S_{\text{bek}}] = \exp[GM^2]$ and so there was an order one chance of tunneling into a fuzzball geometry upon collapse. A more detailed and intricate calculation of this was done more recently by Bena et al [46] by looking at a collapsing shell of branes. They considered a stepwise quantum tunneling process by considering some number of the original branes tunnelling into non-trivial fluxes that support horizon-sized fuzzball structure. Their results supported the initial arguments that such tunneling occurs.

So far we have indicated how fuzzballs can provide a microscopic description for black holes. We still need to discuss how they finesse the black hole information problem and the associated firewall argument. The solution to the black hole information problem is simple from the point of view of the microstates. They do not have a horizon and so there is no problem with ingoing and outgoing modes of Hawking radiation. A solution to the firewall problem has been proposed, in the form of fuzzball complementarity [47, 48]. The main idea is that highly energetic degrees of freedom – compared to the temperature of the corresponding black hole – which impact the fuzzball surface will have an effective description as falling through a classical horizon. The full fuzzball geometry avoids the firewall arguments by not having a horizon but one can still have a general relativity description for the infall problem.

Working out more complicated supergravity solutions for fuzzballs and their quantum

corrections will not be our focus here. We will focus on examining the prototypical D1D5 system which is best understood example of a fuzzball. However, we will not work in the supergravity picture. Instead we will describe the system by a dual conformal field theory (CFT) and study the system there, with strong string tension correction and weak string loop corrections. The context for this study is the famous AdS/CFT conjecture [49], will be described in the next section.

1.4 Holography and AdS/CFT

The holographic AdS/CFT correspondence [49–51] has been a fruitful laboratory for exploring many ideas involving quantum gravity, including the black hole information problem. In this framework, dynamics of quantum gravity in asymptotically anti de Sitter (AdS) spacetime are described in terms of a CFT without gravity living in one fewer dimensions on the boundary of the spacetime. Working models of holography were originally discovered by considering the near-horizon limit of a N stack of D3 branes, finding a correspondence between the $\text{AdS}_5 \times S^5$ and 4 dimensional, $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ and vanishing beta function. Each side of this duality has two theory parameters. Yang-Mills has the ‘t Hooft coupling $\lambda = g_{\text{YM}}^2 N$ and the rank of the gauge group N . On the quantum gravity side, we have the dimensionless ratio $R^2/\alpha' = \sqrt{4\pi g_s N}$ where R is radius of curvature of the AdS_5 and S^5 and the string coupling g_s . These two sides are then related by $\lambda = R^4/\alpha'^2$ and $N/\lambda = 1/(4\pi g_s)$. So this duality is strong-weak, meaning that strongly coupled AdS quantum gravity can be studied via weakly coupled gauge theory and vice versa. There have been further brane constructions engineered that lead to many other classes of dualities as well.

Many different quantities and aspects of the theories can be calculated on either side and shown to match. For example, in the $\text{AdS}_5/\text{CFT}_4$ case above, we can find that the bosonic symmetry group for either side is $SO(4, 2) \times SO(6)$ and that the various parameters have corresponding matches on the other side. Furthermore, one can relate the fields in the gravity theory to the operators in the CFT. This allows certain supersymmetric quantities that are protected by non-renormalization theorems to be compared between two weak coupling calculations from either side [52]. An example us chiral primary operators, which are the lowest weight operators that are annihilated by half the supersymmetry generators. The construction of these in the CFT side will match exactly to the dual gravity side.

The formal method for computing correlation functions in the CFT using the AdS bulk is done using holographic renormalization. For a detailed review of this scheme,

see [53]. The basic idea is that fields can be expanded in Fefferman-Graham coordinates, which is a set of coordinates that allows expansion of the metric and fields in terms of a radial parameter. Coefficients in this radial expansion will correspond to different data. For example, we can expand a field F as

$$F(x, \rho) = \rho^m (f_{(0)}(x) + f_{(2)}(x)\rho + \dots + \rho^n (f_{(2n)} + \log \rho \tilde{f}_{(2n)}) + \dots), \quad (1.6)$$

where x are the coordinates of the asymptotic AdS boundary, ρ is the radial coordinate, and n, m are non-negative integers that depend on the dimension of spacetime, spin, and mass of the field. Then we can find that $f_{(0)}$ corresponds to the source for the dual operator in the CFT and $f_{(2n)}$ is the vev of that operator. With this expansion, one can put in a cutoff away from the boundary and perform a renormalization scheme on the action of the bulk field. The renormalized action of the bulk can then be used to extract the CFT correlation functions for the operator dual to this bulk field.

This formalism has been useful in applying AdS/CFT to a number of contexts. Holography can be used in condensed matter to help model conformal theories arising in at the quantum critical point of a number of physical systems. For example, see recent reviews of “AdS/condensed matter” [54, 55]. AdS/CFT can also be used to model the quark-gluon plasma in quantum chromodynamics [56, 57]. There has also been work on extending the duality to de Sitter spaces and modeling cosmological features [58, 59]. A great general review of AdS/CFT can be found in [60].

An important intellectual shift has occurred in gravitational string theory in the last few years using the conformal symmetry and the structures that are generated in CFTs from it to study the AdS bulk indirectly. One of the first major approaches is to start by inspecting CFTs and asking what conditions a CFT should satisfy in order for it to have a well defined AdS gravity dual. These include a large central charge and a sparse spectrum of light operators [61]. Roughly, this can be connected to a large N limit in the bulk and having a hierarchy between the AdS and string scales. In the next section, we will see how recent work has focused on finding how the geometry of the bulk is stored in the degrees of freedom in the CFT.

1.5 The AdS in AdS/CFT

One of the key questions of active interest for researchers is how the dual gravity bulk emerges from data in the dual CFT. This is often referred to as the bulk reconstruction problem. The HKLL proposal [62–66] gave an early prescription for reconstructing bulk

fields. Its main feature was to propose reconstructing bulk fields from the dual boundary theory by smearing the dual operator on the boundary. The drawback of this approach is that the smearing is done with an integration kernel that depends on the equations of motion for the bulk field. Furthermore, applying this to black hole geometries requires complexifying the boundary and finite N corrections require giving up microcausality [67]. A number of other proposals for bulk reconstruction have been proposed more recently to try and improve on this method.

The main focus for these newer approaches is entanglement entropy, one of the main tools for studying entanglement. This has an entry in the holographic dictionary on both sides. This was conjectured by Ryu and Takayanagi [68, 69] and the formula relates the entanglement entropy of a region A (defined as $S_A = -\text{Tr}\rho_A \log \rho_A$ where ρ_A is the reduced density matrix) on the boundary to the area of the minimal surface in the bulk whose boundary matches the region. The Ryu-Takayanagi (RT) conjecture is fundamental to current efforts in exploring bulk reconstruction. The connection to bulk reconstruction began with van Raamsdonk in [70], where it was posited that the connectedness of classical spacetime is related to the entanglement structure of the dual CFT.

The extension of RT to a covariant formulation for weak time dependence was found by Hubeny, Rangamani, and Takayanagi (HRT) [71]. Proofs supporting the RT formula have been found. The first was for spherical regions [72, 73] where the entanglement entropy of the vacuum is mapped to a thermal state on a hyperbolic geometry. This method is still often used when computing spherical entanglement entropies. A more detailed supporting argument for the RT and the HRT formula were worked out in [74, 75]. This was done by examining classical Euclidean gravity solutions with a boundary which computes the entanglement entropy through the gravitational entropy. Looking at $1/N$ corrections was done by [76–78]. They found that the next order correction is given by the entanglement entropy in the bulk between the two regions separated by the original RT minimal surface. An alternative approach to computing the entanglement entropy using bit threads (a divergenceless norm-bounded vector field) and reformulates the minimal area condition as the maximum flux of the threads out of a region [79].

More aspects of entanglement have been studied than just the plain entanglement entropy. For example, more general Rényi entropies were worked out in [80]. It also found that RT surfaces have a phase transition when considering the entanglement entropy of two disconnected regions. The minimal surface when the two regions are close goes between the two regions but they separate into two separate minimal surfaces with sufficient separation, similar to soap bubbles.

It has also been argued that bipartite entanglement is not enough to fully determine the bulk geometry [81, 82]. Another quantity, entwinement, has been conjecture to be important in determining the full geometry for black holes [83–85]. Entwinement in the bulk is captured by non-minimal but still extremal surfaces. For example, in the three dimensional conical defect geometry we can have non-minimal geodesics that wind around the defect. These are still extremal geodesics, but they are not captured by the entanglement entropy. However, they are captured by the fractionated degrees of freedom in the CFT generated by the defect.

A related quantity, complexity, was examined in the series of works in [86–93]. The complexity is another quantum information measure, defined in quantum mechanics as the minimum number of unitary gates needed to create a state from a reference state. Various authors have suggested a number of proposals for the bulk realization for this. The main two competing proposals are the complexity-volume proposal, where complexity is dual to the volume of the spacetime behind the horizon, and the complexity-action proposal, where the complexity is dual to the action of the Wheeler-deWitt patch. One can study the divergence structure, as in the holographic renormalization scheme, and find the leading order divergences agree, though subleading terms do not [94, 95]. However, the CFT computation is not yet known, so it is not possible to determine which gives the correct entry in the holographic dictionary.

There has also been a lot of work towards finding gravity from the CFT more directly. It was shown in [96–99] that entanglement entropy defined via the Ryu-Takayanagi formula satisfies a first law relationship with the modular Hamiltonian (defined as $H_A = -\log(\rho_A/\text{tr}\rho_A)$) giving rise to the linearized Einstein equations. This work was extended to include sources and argued that the only non-linear completion would be Einstein gravity [100].

Others have focused on how much of the bulk can be obtained from a finite region on the boundary [101–106]. This program focused on a quantity known as differential entropy and using a family of regions in AdS_3 and their entanglement entropies to reconstruct curves. Specifically they found one could reproduce the circumference of a curve from a formula integrating the differences of the entanglement entropies of this family (hence the name “differential” entropy). This program was eventually upgraded to studying kinematic space using integral geometry methods [107–111]. What they found is that various geometric objects in the bulk, such as points and curves, are reorganized into an auxiliary space which they called kinematic space. It is a space of boundary causal diamonds, which are the defined as the intersection of the causal cones of two time-like separated points. The causal diamonds define spacelike regions on the boundary, which

can be used to reproduce the entanglement entropy in the CFT and the bulk minimal surfaces. It is hoped that this auxiliary kinematic space will act as a useful translator for uncovering further entries in the holographic dictionary.

Other quantum information tools have been brought to bear on studying how CFT degrees of freedom are organized. The concept of distinguishability of quantum states is important in quantum information. There are a number of ways to measure this, for example the Zamolodchikov metric [112] or the Fisher information metric [113, 114]. These are related holographically to the relative entropy and the canonical energy, respectively. One can also examine what inequalities the entropies of holographic states satisfy. These may include general inequalities known to quantum information theorists [115] or more novel ones [116–118]. These inequalities can have non-trivial consequences for the dual gravity theory, such as enforcing the null energy condition [119–122]. Other authors have investigated the utility of quantum error correcting codes. Quantum error correcting codes show up in quantum information theory as methods for protecting against the loss of qubits in quantum communication. Some aspects of the entanglement structure, such as bounds and gauge invariance of boundary operator dual to a bulk field, can be explained by these holographic error correcting codes [123–127]. Finally, the concept of tensor networks was been considered as a useful way of discretizing and studying quantum gravity [108, 128–132]. These networks are built up of connected tensors that are imbued with properties to reproduce entanglement structure and other aspects of the CFT.

One can also try to extract more detailed information about the entangling surface or the geometry. The dependence on the shape of the entanglement region has been explored in many works such as [133–136]. The authors in [137–139] studied details of entanglement entropy in the D1D5 system both in the CFT and the bulk. This required extending the RT formula to work for product spaces, specifically $\text{AdS}_3 \times S^3$. They found that subleading corrections to the RT formula were sensitive to the fuzzball geometry and saw this in both the bulk and boundary calculations.

Overall, we can see that there is a lot of information about gravity stored in CFTs. We will also want to examine how we can exploit holographic CFTs to explore questions foremost in the minds of gravitational physicists. Before we do, we will present a quick review of CFT structures.

1.6 Basic introduction to CFTs

In this section, we will discuss several definitions for 2d CFTs, although many of the objects discussed here will have analogues in higher dimensions. For a great review of 2d CFTs, see [140]. In two dimensions, the symmetry group for CFTs is governed by the infinite dimensional Virasoro algebra [141]. Suppose we start with our CFT on a cylinder. We can map the cylinder to the plane and see the Fourier modes in powers of z, \bar{z} . First recall our cylinder can be parametrized as $\sigma \in [0, 2\pi]$ and $\tau \in [-\infty, \infty]$. To get to the complex plane, we simply conformally transform by

$$z = e^{\tau - i\sigma} \tag{1.7}$$

$$\bar{z} = e^{\tau + i\sigma} \tag{1.8}$$

Now we can see how the time is mapped. For $\tau = -\infty$, we see $z = 0$ and for $\tau = \infty$, we see $z = \infty$. Furthermore, lines of constant τ are circles around the origin. So to define a time ordered product, we now do a radially ordered product.

There are several different types of fields we can work with, the most basic of which is called a primary field. Primary field transform as tensors under conformal transforms:

$$O(z, \bar{z}) \rightarrow O'(z, \bar{z}) = \left(\frac{\partial z'}{\partial z} \right)^h \left(\frac{\partial \bar{z}'}{\partial \bar{z}} \right)^{\bar{h}} O(z', \bar{z}'). \tag{1.9}$$

We call h and \bar{h} conformal weights. The scaling dimension of a field is given by $\Delta = h + \bar{h}$ and the conformal spin is given by $l = h - \bar{h}$. These indicate how field behave under conformal transformations, similar to the mass and spin under the Lorentz group. Holomorphic functions have $\bar{h} = 0$ and anti-holomorphic functions have $h = 0$. We can also define quasi-primary fields which transform as in equation (1.9) only under the global subgroup of the full Virasoro symmetry of the 2d CFT.

Now suppose we have energy momentum tensor components $T_{zz} = T(z)$ and $T_{\bar{z}\bar{z}} = \tilde{T}(\bar{z})$. The energy-momentum tensor can be used to generate conformal transformations. We can see this by noting that in a 2d theory with conformal symmetry has a traceless stress energy tensor which, together with stress-energy conservation, implies that $T(z)$ and $\tilde{T}(\bar{z})$ are conserved currents. Therefore the conformal symmetry is generated by the stress tensor.

We can use the definition of a primary field together with the generator to find the

operator product expansion (OPE) between the stress tensor and a primary field as

$$T(z)O(w) = \frac{hO(w)}{(z-w)^2} + \frac{\partial_w O(w)}{z-w} + \text{regular.} \quad (1.10)$$

Here regular refers to terms that are non-singular.

Now we expect to be able to expand T similarly. We can determine the OPE from the requirement of the stress energy tensor as the generator of conformal transformations. We can find

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{regular.} \quad (1.11)$$

Note that we do not need to have the term proportional to c for the OPE to give the correct conformal transformations. However, it does have the correct scaling dimension and symmetry to appear. This term is called the central charge and is a term that comes from the Weyl anomaly. If the central charge is not vanishing, then the stress energy tensor is not a full primary, only a quasi-primary. In higher dimensions the stress tensor OPE may contain other terms central charges corresponding to other anomalies.

So this brings us to our more general idea of a product expansion of two operators. Suppose that $\{O_i\}$ are a complete set of local operators with definite scaling dimensions. We can then expand a product of these two operators in terms of the others. We expand as

$$O_i(z)O_j(w) = \sum_k C_{ijk}(z-w)^{h_k-h_i-h_k} O_k(w). \quad (1.12)$$

It is important to note that the (anti)commutators of the operators are determined by the singularities in the operator product expansion. The coefficients C_{ijk} are the structure constants, which define the CFT. Furthermore, in CFTs, the OPE is a convergent series, unlike in regular QFTs where it is only an asymptotic series.

The energy-momentum tensor generating conformal transformations implies that the mode expansion of the stress tensor satisfy the Virasoro algebra.

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \quad (1.13)$$

L_n are generators of conformal transformations for $z \rightarrow z + z^{n+1}$. A important note is that L_{-1}, L_0, L_1 are the generators of $SL(2, \mathbb{R})$, the maximal closed subalgebra of the conformal group called the global conformal group. If we add $\bar{L}_{-1}, \bar{L}_0, \bar{L}_1$, then we get $SL(2, \mathbb{C})$.

We could also consider the effect of a finite conformal transformation, $z \rightarrow f(z)$, on

$T(z)$. We can use our OPE for $T(z)T(w)$ to find the result,

$$T(z) \rightarrow T'(z) = (\partial f(z))^2 T(f(z)) + \frac{c}{12} D(f)_z. \quad (1.14)$$

Note that $D(f)$ is the Schawrzian derivative, which is given by

$$D(f)_z = \frac{\partial f(z) \partial^3 f(z) - (3/2)(\partial^2 f(z))^2}{(\partial f(z))^2}. \quad (1.15)$$

We expect that we can now complete the Hilbert space by using our raising operators, L_{-k} , on primary operators. Somewhat counter-intuitively, the resulting states are called descendant states. For every primary ϕ_j , we can find a representation of the Virasoro algebra labelled by h_j . It is called a Verma module or Virasoro conformal family and it consists of all the states that have the form of

$$O_j^{k_1 \dots k_m}(z) = \hat{L}_{-k_1} \dots \hat{L}_{-k_m} O_j(z). \quad (1.16)$$

Note this only makes sense if $k_i > 0$. These states will have conformal weight of $h_j + \sum_i k_i$. Note that these states are secondary operators and, by conformal invariance, are contained in the operator product of any two operators that contains the primary operator ϕ_j . These will appear in less singular terms than its ancestor primary.

Now that we have our definitions of fields and operators sorted, we would like to investigate how the correlation functions behave. To do this we will restrict ourselves to fields called quasi-primary fields. Demanding invariance under these restricts the forms that are allowed for the correlation functions. The results are as follows,

$$\langle O_i(z) \rangle = 0 \quad (1.17)$$

$$\langle O_i(z_1) O_j(z_2) \rangle = \frac{C_{ij}}{z_{12}^{2h_i}} \delta_{h_i, h_j} \quad (1.18)$$

$$\langle O_i(z_1) O_j(z_2) O_k(z_3) \rangle = \frac{C_{ijk}}{z_{12}^{h_i+h_j-h_k} z_{13}^{h_i+h_k-h_j} z_{23}^{h_k+h_j-h_i}} \quad (1.19)$$

$$\langle O_1(z_1) O_2(z_2) O_3(z_3) O_4(z_4) \rangle = \frac{z_{13}^{h_2+h_4} z_{24}^{h_1+h_3}}{z_{12}^{h_1+h_2} z_{23}^{h_2+h_3} z_{34}^{h_3+h_4} z_{14}^{h_1+h_4}} f\left(\frac{z_{12} z_{34}}{z_{13} z_{24}}\right) \quad (1.20)$$

$$\langle T(z) O_1(w_1) \dots O_n(w_n) \rangle = \sum_{j=1}^n \left(\frac{h_j}{(z-w_j)^2} + \frac{\partial_{w_j}}{z-w_j} \right) \langle O_1(w_1) \dots O_n(w_n) \rangle, \quad (1.21)$$

where $z_{ij} = z_i - z_j$. Note C_{ij} are normalization constants that we usually set to unity. Furthermore, f is a function of the $(n-3)$ independent cross ratios which cannot be

specified further by conformal invariance and will depend both on the CFT and on the operators in the correlator. The final correlation identity is the unintegrated form of the conformal Ward identity.

Let us consider a very simple example of a free boson CFT to showcase the concepts here. In 2d, a free scalar has a two point function defined by

$$\langle \varphi(z) \varphi(w) \rangle = -\log(z - w). \quad (1.22)$$

This is not the correct form for a conformal primary, see equation (1.18), but we can see that $\partial\phi$ will be a primary. We can extract from this the OPE,

$$\partial\varphi(z)\partial\varphi(w) = -\frac{1}{(z - w)^2}. \quad (1.23)$$

Therefore $\partial\varphi(z)$ is weight one. We can then define the stress-energy tensor as

$$T(z) = -\frac{1}{2} : \partial\varphi(z)\partial\varphi(z) : \quad (1.24)$$

and show it satisfies all the equations we had above by using Wick contractions.

With the basics covered, we can now use the considerable structure and power of CFTs to explore properties of holographic CFTs and how that connects to the physics of quantum gravity.

1.7 The CFT in AdS/CFT

One approach to studying holographic CFTs is to look at the information stored in conformal blocks. The broad definition of a conformal block is the contribution from a single conformal family to a four-point function of primary operators. These can be defined in any dimensions but they can be extended to Virasoro blocks in $d = 2$ when one considers the full Virasoro descendants of the operator in the family. For example, we can consider a four point function of scalar primaries,

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle = \sum_{\mathcal{O} \in \phi_1 \times \phi_2} \sum_{\alpha} \langle 0 | \phi_3(x_3) \phi_4(x_4) | \alpha \rangle \langle \alpha | \phi_1(x_1) \phi_2(x_2) | 0 \rangle, \quad (1.25)$$

where $\mathcal{O} \in \phi_1 \times \phi_2$ are primary operators in the OPE of $\phi_1 \times \phi_2$ and α are the descendants of \mathcal{O} , including itself. If we consider a single term in this sum corresponding to a single

\mathcal{O} , then we can write the conformal block as

$$g_{\mathcal{O}}(u, v) = \frac{1}{C_{\phi_1\phi_2\mathcal{O}}C_{\phi_3\phi_4\mathcal{O}}} \left(\frac{x_{13}^2}{x_{14}^2}\right)^{\Delta_{34}/2} \left(\frac{x_{14}^2}{x_{24}^2}\right)^{\Delta_{12}/2} x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4} \times \sum_{\alpha} \langle 0 | \phi_3(x_3) \phi_4(x_4) | \alpha \rangle \langle \alpha | \phi_1(x_1) \phi_2(x_2) | 0 \rangle. \quad (1.26)$$

Here we have the structure constants $C_{\phi_1\phi_2\mathcal{O}}, C_{\phi_3\phi_4\mathcal{O}}$, the notation $x_{ij} = x_i - x_j$, $\Delta_{ij} = \Delta_i - \Delta_j$, and the cross ratios $u = (x_{12}^2 x_{34}^2)/(x_{13}^2 x_{24}^2)$, $u = (x_{14}^2 x_{23}^2)/(x_{13}^2 x_{24}^2)$. The main point of the conformal blocks is that they are fixed by the conformal symmetry and only depend on the dimensions and spins of \mathcal{O} and ϕ_i . This property makes them extremely useful for studying universal properties of CFTs and extracting holographic physics.

We know that in order to connect to quantum physics, we must perform $1/N$ corrections. These corrections correspond to $1/c$ corrections in the CFT. So if we can work out the conformal blocks and examine $1/c$ corrections, this is tantamount to doing considering quantum corrections in the bulk. Serious study of this expansion in the holographic context began with [142], with expressions in more general dimensional CFTs. The followup to this paper [143] focused on 2d CFTs. In this paper, they considered a four point correlator of two heavy and two light correlators where the conformal weights of the heavy operators where $h \sim c$ and the light operators have $h \ll c$. In this limit, the authors imagined the heavy operators as creating a black hole background in the bulk on which the light operators acted as probe fields in bulk. They showed that one can implement this in the CFT using conformal transformation to a non-trivial metric. This related the full Virasoro blocks to the global block in the new metric. This was the first in a series of works dedicated to extracting corrections in $1/c$ and h/c [144–148]. These papers used a variety of different techniques to extract the corrections to the Virasoro blocks. These corrections lead to a final extension showing that exact Virasoro blocks can be extracted from $\text{sl}(2)$ Chern-Simons Wilson lines [149].

A parallel series of developments examined the role of geodesic Witten diagrams. These were first suggested in [150] as the bulk duals to the conformal blocks. Standard Witten diagrams [51, 151, 152] are make up one method of computing holographic n-point functions. These diagrams are generated from the CFT path integral, similar to well known Feynman diagrams. The holographic correspondence then relates this path integral to the bulk path integral with the supergravity action as well as the operators in the CFT to fields in the bulk. So one considers a set of operators on the boundary and then connect the boundary insertions with bulk-to-boundary and bulk-to-bulk propagators. For standard Witten diagrams, one must integrate this point of contact over

the full bulk [153]. A geodesic Witten diagram is similar but has a big advantage. Here we consider a four point function and do the Witten diagram, except the integration is only done over a geodesic connecting two operators. The paper then went through the computation for a scalar conformal and showed the correspondence. This was extended to Virasoro blocks in [154]. In the large c limit, the leading contribution is the global block and so this paper allowed the geodesics to back-react on the AdS in order to extract $1/c$ corrections. The next series of papers took this further, working out the details on how to get conformal blocks for operators with spin [155, 156].

Although it is interesting to study conformal blocks from the point of view of the structure of CFTs, we are more interested on how this can be applied to study black holes. An important connection between conformal blocks and the black hole information problem was found in [147]. They focus on examining a specific CFT manifestation of the information problem. They first review that two point correlators in a black hole background decay at large time, as the information is lost. This behaviour is also found in thermal correlators of CFTs at large central charge. Then examining the problem would be to see how $1/c$ corrections may correct this. They take it further by pointing out that in the heavy-light limit, one can examine a thermal two point correlator as a four point correlator of two light and two heavy operators as explained earlier. Then the periodicity of the Euclidean time ($t_E \rightarrow t_E + \beta$) gives singularities that are forbidden in the full four point correlator at finite central charge. So examining these singularities will give another probe for information loss in the CFT. The papers [157, 158] argued that it is non-perturbative contributions (in $1/c$) to the correlators that are required to resolve the singularities and the late time decay. This was supported by a model CFT calculation of black hole collapse [159]. These contributions correspond to subleading saddles. The authors of [160] show explicitly one infinite class of these geometries do not have late time decaying correlators, giving evidence that such non-perturbative corrections may resolve the problem.

There are still other properties one can harness for holographic CFTs. For example, one can use causality in the CFT to constrain terms in a bulk Lagrangian. This was done in [161] to show how the causality, encoded in the crossing symmetry of correlators, can be used to derive the sign constraint on a $(\partial\phi)^4$ term. This was extended for light spinning operators in [162] which reproduces non-trivial bounds found earlier in [163, 164]. Another property is studying chaos as done in [165]. This paper examined the bound in [166] and found that ordinary holographic CFTs saturate the bound and rules out some other holographic theories with finite towers of higher spins.

Finding CFTs that satisfy holographic restrictions is not easy: they appear to be

rare in the space of all CFTs. One fruitful line of investigation is to focus on permutation orbifold CFTs, which are more likely to be holographic [167–171]. Some of these permutation orbifolds will possess the required sparse light spectrum, while also giving the desired growth at large energies consistent with existence of black holes in the dual theory. Of course, not all permutation orbifolds will necessarily give CFTs with a classical AdS dual. They might instead require full string theory to describe them. In higher dimensions that $d = 2$, there will also be further constraints on which seed theories can be used in the orbifold [171]. Accordingly, investigating the general structure of orbifold CFTs and the states in them is likely to be important to understanding holography more generally.

One can try to find the universal behaviour of the spectrum in more detail. For example, the requirement of a sparse light spectrum can be used to place restrictions on the partition function [172, 173]. This is accomplished using modular invariance, a symmetry of the partition function that can be used to relate $Z(\beta) = Z(\frac{4\pi^2}{\beta})$. One can define similar restrictions on the elliptic genus, as done in [169, 174]. There the authors try to use a few well known examples to show how the constraints can be satisfied (for example in the case of a symmetric orbifold) or not (more generally).

We will study aspects of the D1D5 symmetric permutation orbifold in detail in chapters 2 and 3. Before we get there, we will discuss in the next section some of our own explorations into trying to find a 2d (0,2) supersymmetric theory that would satisfy these bounds on the elliptic genus and show how difficult it is to combine all these properties together. See [175] for some review on 2d (0,2) supersymmetry and its relationship to (2,2).

1.8 (0,2) Holography

As an example of how to examine CFTs for holography, we mention here some of our work that ultimately failed to find class of holographic theories with reduced supersymmetry. We will outline why it is difficult to find a theory that works with all the restrictions but also can be examined with known techniques. Our goal was to find some CFTs with 2d (0,2) supersymmetry that were holographic.

There has been previous work to find holographic 2d (0,2) SCFTs [176, 177] through brane constructions. We wished to find more general holographic theories in the vein of [61], by examining CFTs with large central charge and sparse light spectrum. Specifically, we are looking for gauge theories that flow in the infrared to CFTs that have central charges that scales as $c \sim N^2$ for large N . Furthermore, we want to put constraints on

the sparseness of the spectrum with the results of [169, 174]. The bonus of using these is that the elliptic genus is invariant on the moduli space of the gauge theory and we can find the results without having to find the infrared CFT. Instead, we can construct the elliptic genus where we know the gauge theory and, if it satisfies the constraints, then the infrared CFT would satisfy them as well.

We began by considering the CFT presented in [178], which was suspected to possibly have a CFT dual. However, there is an obvious problem that one hits. The global symmetry of the model has multiple $SU(N) \times U(1)$ factors. In the holographic dictionary, global symmetries are related to currents in the bulk. At large N this would imply there would be a large number of massless currents, and so it would definitely fail the sparse low-lying spectrum condition for a Einstein gravity dual (though it still could have a string theory dual).

To continue this path, we need to first construct large N , $(0, 2)$ gauge theories with global symmetries that do not scale with N . Then we will need to identify a way to compute the elliptic genus and compare it to the constraints in [169, 174].

To construct $(0, 2)$ gauge theories, we can examine quiver gauge theories which are derived from quiver diagrams. There is some review of these diagrams in [179, 180]. The basic construction is that one has vertices and lines. To each vertex, we associate a vector multiplet and a gauge group V and directed links from V_1 to V_2 will represent a complex scalar transforming in the $\bar{V}_1 \times V_2$ representation. Depending on the supersymmetry at play, these can be used to form chiral or hypermultiplets. For our $(0, 2)$ gauge theories, we will have chiral multiplets. We can also have Fermi multiplets, where the only on-shell degree of freedom is left-moving fermion. These can be included into the diagrams as well. These diagrams will also represent a superpotential term for all closed loops. For the quiver gauge theories to make sense, one has to be careful of gauge anomalies and there are conditions on the fields at each vertex for these to cancel [181]. Furthermore, one then needs to go through a process of c-extremization in order to determine the R charges of the fields [182].

To illustrate the calculations a bit more, we will cover one of the simpler quiver diagrams that we examined. This is show in figure 1.1. Here we have m Fermi multiplets transforming in the adjoint representation of $SU(N)_1$, m Fermi multiplets transforming in the adjoint representation of $SU(N)_2$, n chiral multiplets transforming in the (\bar{N}, N) representation, and n chiral multiplets transforming in the (N, \bar{N}) representation. There are also global $U(1)$ symmetries as well, arising in the loops that give rise to the superpotentials. To cancel anomalies, we will have to take the combination $m + 1 = n$. Going through the c-extremization procedure, we can determine the R-charges of the multiplets

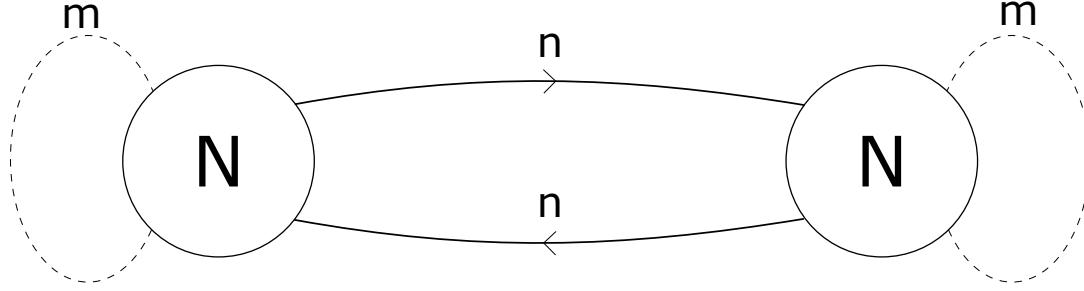


Figure 1.1: Simple (0,2) quiver diagram. Here we have $2m$ Fermi multiplets, $2n$ chiral multiplets, and 2 $SU(N)$ gauge groups.

as

$$R_\Phi = 1 + \frac{2m(N^2 - 1)}{nN^2 - 4m(N^2 - 1)}, R_\Psi = \frac{nN^2}{4m(N^2 - 1) - nN^2} \quad (1.27)$$

and we have a right central charge of

$$c_R = 6N^2 \left(\frac{nN^2 + m(n-4)(N^2 - 1)}{4m(N^2 - 1) - nN^2} \right) \quad (1.28)$$

which scales as N^2 at large N as required.

Now that we have the ability to construct gauge theories relatively simply, we turn to computing the elliptic genus using the techniques outlined in [183, 184]. The details are very technical and not important for what we wish to cover here, but a rough outline of the method is as follows. We start by considering localization. Suppose we have a supersymmetric theory and invariant supercharge Q . Now we wish to compute

$$Z = \int \mathcal{D}\Phi e^{-S[\Phi]}. \quad (1.29)$$

If we can find a functional W such that $\{Q, W\}$ is semi-definite and Q -invariant, then we can consider

$$Z_t = \int \mathcal{D}\Phi e^{-S[\Phi] + t\{Q, W\}}, \quad (1.30)$$

and find that

$$\frac{dZ_t}{dt} = 0. \quad (1.31)$$

Note that t is just a variational parameter. So we just need to construct the correct W to extract the contributions to the elliptic genus. Going through this correctly will yield

$$Z = - \sum_i \oint_{u=u_i} du Z_{1-loop}(u). \quad (1.32)$$

Here u are the various fugacities of the elliptic genus. This technique has been used in other contexts in holography to show some exact results in the Aharony, Bergman, Jafferis, Maldacena (ABJM) gauge/gravity duality [185, 186], a holographic setup of N M2 branes with a $\text{AdS}_4/\text{CFT}_3$ duality.

So if we can find the 1-loop determinant for our quiver gauge theory, we can find the full elliptic genus. We can find to give the contributions to the 1-loop partition function from any type of multiplet we may have in our gauge theory, using our knowledge of the representations of the multiplets under the gauge and global symmetries. These contributions are in terms of ratios of theta functions. Once we have our expression for the 1-loop expression in terms of theta functions, we can use the Jeffrey-Kirwan method [187] to evaluate the residue integrals in equation (1.32). For our simple example, the result for the $Z_{1\text{-loop}}$ is

$$Z_{1l} = (2\pi\eta(q)^2)^{2(N-1)} \prod_{i \neq j}^N \left[\frac{\theta_1(\tau|u_i - u_j)}{\eta(q)} \frac{\theta_1(\tau|v_i - v_j)}{\eta(q)} \right] \prod_{i,j=1}^N \left[\left(\frac{\eta(q)}{\theta_1(\tau|u_i - v_j - z)} \right)^{m+1} \right. \\ \left. \times \left(\frac{\eta(q)}{\theta_1(\tau|v_i - u_j - z)} \right)^{m+1} \left(\frac{\theta_1(\tau|u_i - u_j + 2z)}{\eta(q)} \frac{\theta_1(\tau|v_i - v_j + 2z)}{\eta(q)} \right)^m \right] \\ \times \delta\left(\sum_i u_i\right) \delta\left(\sum_i v_i\right) d^N u d^N v. \quad (1.33)$$

Here we have τ is the modular parameter of the elliptical genus, $q = \exp[2\pi i\tau]$, u_i and v_i are the i th component of the fugacities for the Cartan elements of $SU(N)_1$ and $SU(N)_2$ respectively, z is the fugacity for the global $U(1)$, and η , θ_1 are the eta and theta functions defined in [183, 184]. Note that, despite the many terms, one can pick out the contributions from the various fields from their representations under the gauge symmetry. Now actually evaluating these integrals is quite difficult, especially when we want to go to large N . So we can instead consider some of the low N cases, compute their elliptic genus, and try to extrapolate to large N .

The difficulty with this is we end up hitting elliptic genera that do not have the correct expansion to be compared to the works of Benjamin et al [169, 174]. The constraints that these authors work out are on coefficients of an expansion of the elliptic genus. They implicitly assume in their expansion that their gauge theories have a compact moduli space, allowing the elliptic genus to have a polynomial expansion in the fugacities. However, the theories we considered had rational expansions, indicative of non-compact moduli space. We attempted to avoid this problem by considering more complex quiver theories but we were unable to find a model that was able to work with our method.

Overall, it is clear that holographic CFTs are a rarity on the space of CFTs [188]. We

would need to have a theory that is unitary, has a large N expansion, sparse low-lying spectrum, and a compact moduli space. We shall now turn to one of these CFTs in the next section, which we will work with for the rest of this thesis.

1.9 D1D5 system

Our work in this thesis will focus on studying the properties of the D1D5 system, a prototype system for string theory fuzzballs. In this section, we will describe how the brane system is constructed and the relationship between the two sides of the AdS/CFT duality. Then we will focus on the D1D5 CFT at the orbifold point, giving a description of the operators involved, and describe how we can try to connect the CFT description to the supergravity description through the deformation operator.

1.9.1 D1D5 construction

The D1D5 system is constructed in Type IIB string theory, with a $S^1 \times T^4$ compactification in the full 10 dimensions. There are N_1 D1-branes wrapped on the S^1 and smeared along over the T^4 . Additionally, N_5 D5-branes are wrapped on the full $S^1 \times T^4$. This is usually explained visually in a table,

	0	1	2	3	4	5	6	7	8	9
D1	—	·	·	·	·	—	~	~	~	~
D5	—	·	·	·	·	·	—	—	—	—

Table 1.1: Configuration for the D1D5 system. Dashes represent parallel directions, dots represent perpendicular directions, tildes are smearing directions.

This setup can be described in Type IIB supergravity by the following solution [189],

$$ds^2 = f_1^{-1/2} f_5^{-1/2} (-dt^2 + dx_5^2) + f_1^{1/2} f_5^{1/2} dx_i dx^i + f_1^{1/2} f_5^{-1/2} dx_a dx^a, \quad (1.34)$$

$$C_{05}^{(2)} = -(f_1^{-1} - 1)/2, \quad (1.35)$$

$$e^{-2\phi} = f_5 f_1^{-1}, \quad (1.36)$$

$$f_{1,5} = 1 + r_{1,5}^2/r^2 \quad (1.37)$$

where $r_1^2 = (2\pi)^4 g_s l_s^6 N_1 / V_{T^4}$, $r_5^2 = g_s l_s^2 N_5$, x^a are the coordinates along the T^4 , and x^i are the non-compact directions.

One can show that in the near horizon limit, the D1D5 system becomes $AdS_3 \times S^3 \times T^4$. Note that the volume of the T^4 in this limit is near the string scale whereas the $AdS_3 \times S^3$

is large. So then a truncation to supergravity to $AdS_3 \times S^3$, ignoring KK modes on the torus, is well defined. The dynamics of the system can also be described (by the AdS/CFT correspondence) as a (1+1)-dimensional SCFT living on (x^0, x^5) .

Due to the holographic duality, we can already determine the symmetries of this CFT. The isometries of AdS_3 correspond to the global part of the Virasoro algebra of the CFT, the isometries of the S^3 will be the R-symmetry of the CFT, and the torus will provide a global symmetry.

Where the supergravity description is valid, the dual CFT will be strongly coupled. Conversely, a free CFT description has a bulk dual only described by full string theory. The D1D5 system is described by supergravity and a free CFT, but these descriptions are only valid at different points in its moduli space. The Type IIB on T^4 has 25 moduli, 5 of which are fixed by the near horizon limit geometry of $AdS_3 \times S^3$ [189, 190]. This leaves 20 moduli corresponding to 20 marginal operators. There are four of these in the twisted sector, corresponding to various blow up modes in the bulk. More details of the moduli space can be found in the comprehensive review [189].

To connect to black hole physics, we wish to study the marginal deformation operator in the free CFT which is a singlet of the various $SU(2)$ s; this particular blow up mode corresponds in the bulk to the field $a_1 C_0 + a_2 C_{ijkl}$. To begin, we will describe the free CFT and then finish the section with a discussion of how the marginal deformation operator appears in the CFT.

1.9.2 D1D5 CFT: introduction

The D1D5 CFT is a well known CFT describing the D1D5 system. For early papers on the system, see [189, 191–199]. It can be described by a $\mathcal{N} = (4, 4)$ 1+1d free SCFT with the target space $(T^4)^{N_1 N_5} / S_{N_1 N_5}$ at a point in its moduli space. We will sometimes abbreviate this symmetric orbifold as $(T^4)^N / S_N$ where $N = N_1 N_5$. This SCFT has R symmetry $SU(2)_L \times SU(2)_R$ and another symmetry from the symmetry of the T^4 , $SO(4) \cong SU(2)_1 \times SU(2)_2$. This CFT then has a central charge of $c = 6N_1 N_5$. One can write the action as

$$S = \frac{1}{4\pi} \sum_{a=1}^{N_1 N_5} \int dz d\bar{z} [\partial X_i^{(a)}(z) \bar{\partial} X^{(a),i}(\bar{z}) - \psi^{(a),\alpha\dot{A}}(z) \partial \psi_{\alpha\dot{A}}^{(a)}(z) - \tilde{\psi}^{(a),\dot{\alpha}\dot{A}}(\bar{z}) \bar{\partial} \tilde{\psi}_{(a),\dot{\alpha}\dot{A}}(\bar{z})], \quad (1.38)$$

where (a) are the copy indices of the fields, i are the directions of the torus, and $\alpha, \dot{\alpha}$ are the indicates of the $SU(2)_L \times SU(2)_R$, respectively.

For the moment, let us focus on a single copy. We can rewrite the scalar fields to

transform under the torus symmetry as,

$$X^{\dot{A}A} = \frac{1}{\sqrt{2}}(\sigma^i)^{\dot{A}A}X_i, \quad (1.39)$$

where A, \dot{A} are the indices of the $SU(2)_1 \times SU(2)_2$. We take the conventions for the Pauli matrices and epsilon tensors where

$$(\sigma^2)^{\dot{2}1} = i, \epsilon_{12} = \epsilon^{21} = 1, \quad (1.40)$$

and use the epsilon tensor to raise and lower indices of the fields. The fields have Hermitian conjugates of the form

$$(\psi^{\alpha\dot{A}})^\dagger = -\epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}}\psi^{\beta\dot{B}} = -\psi_{\alpha\dot{A}}, \quad (1.41)$$

$$(X^{\dot{A}A})^\dagger = -\epsilon_{\dot{A}\dot{B}}\epsilon_{AB}X^{\dot{B}B}. \quad (1.42)$$

These fields then have OPEs of the form

$$\partial X^{\dot{A}A}(z)\partial X^{\dot{B}B}(w) \sim \frac{\epsilon^{\dot{A}\dot{B}}\epsilon^{AB}}{(z-w)^2} \quad (1.43)$$

$$\psi^{\alpha\dot{A}}(z)\psi^{\beta\dot{B}}(w) \sim -\frac{\epsilon^{\alpha\beta}\epsilon^{\dot{A}\dot{B}}}{(z-w)} \quad (1.44)$$

These fields can then be used to build the supercharge, current, and energy momentum tensor. These are given respectively by

$$G^{\alpha A} = \epsilon_{\dot{A}\dot{B}}\psi^{\alpha\dot{A}}\partial X^{\dot{B}A}, \quad (1.45)$$

$$J^a = \frac{1}{4}\epsilon_{\dot{A}\dot{B}}\epsilon_{\alpha\beta}(\sigma^{*a})_\gamma^\beta\psi^{\alpha\dot{A}}\psi^{\gamma\dot{B}}, \quad (1.46)$$

$$T = \frac{1}{2}\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\partial X^{\dot{A}A}\partial X^{\dot{B}B} + \frac{1}{2}\epsilon_{\dot{A}\dot{B}}\epsilon_{\alpha\beta}\psi^{\alpha\dot{A}}\partial\psi^{\beta\dot{B}}, \quad (1.47)$$

with similar expressions for the anti-holomorphic side. These satisfy various commutation relations which can be found in [200]. When we have more than one copy, the supercharge, current, and energy momentum tensor will be given by a sum over all of these copies, which is S_N invariant.

1.9.3 D1D5 CFT: bosonization

Since we have a free CFT, we can avoid some of the hassle of dealing with fermions by bosonization. In general, it is easier to compute correlators of bosons than fermions. However, fermions have anti-commuting properties not captured by bosons which need to be enforced using cocycles. This has been studied extensively for the D1D5 system in [201]; here, we will only mention results that we require.

Calculating correlators with spin and fermion fields can be difficult. One way of simplifying these calculations is by bosonizing these fermionic fields into the form,

$$C_k e^{k \cdot \Phi}, \quad (1.48)$$

where C_k is the cocycle. It is responsible for restoring the fermionic statistics of the original fermionic field. We will use the notation and conventions from [201]. We take $C_k = \exp(i\pi k \cdot (M\alpha_0))$, where α_0 are momentum operators and M is a matrix of constants which are constrained by the anticommutation relations and OPEs of the fermions and spin fields. We also have $k \cdot \Phi = k_5 \phi_5 + k_6 \phi_6 + k_{\tilde{5}} \tilde{\phi}_5 + k_{\tilde{6}} \tilde{\phi}_6$ where the tildes represent the antiholomorphic bosonic fields. The commutation for the momentum and the fields is $[\phi_k, \alpha_0^j] = i\delta_k^j$. The paper cited above then provides lists of the bosonized forms of ψ , S , and J .

It also provides some constraints that must be satisfied by the matrix M for this to work out. These come from enforcing the anticommuting conditions on the fermions and the OPE structure of the spin fields. For example,

$$\psi^{+1}(z)\psi^{+2}(w) = e^{-i\pi M_{6i}\alpha_0^i}(e^{-i\phi_6})(z)e^{i\pi(M_{5i})\alpha_0^i}(e^{i\phi_5})(w), \quad (1.49)$$

$$= e^{\pi[\phi_6, M_{5i}\alpha_0^i]} e^{\pi[M_{6i}\alpha_0^i, \phi_5]} e^{i\pi M_{5i}\alpha_0^i}(e^{i\phi_5})(w)e^{i\pi M_{6i}\alpha_0^i}(e^{-i\phi_6})(z), \quad (1.50)$$

$$= e^{i\pi(M_{56} - M_{65})} \psi^{+2}(w)\psi^{+1}(z), \quad (1.51)$$

since we wish for these to anticommute, we then require that $M_{56} - M_{65} \equiv 1 \pmod{2}$. When all the anticommutation relations and OPE structures for the spin fields are accounted for, the conditions are $A_{ij} \pm 2 \equiv -A_{ij} \pmod{4}$, where $A_{ij} = M_{ij} - M_{ji}$. For this work, we took the solution of these constraints as $(A_{56}, A_{5\tilde{5}}, A_{5\tilde{6}}, A_{6\tilde{5}}, A_{6\tilde{6}}, A_{\tilde{5}\tilde{6}}) = (-1, -1, 1, 1, 1, 1)$.

1.9.4 D1D5 CFT: twisted sector

Our CFT is a $(T^4)^{N_1 N_5} / S_{N_1 N_5}$, meaning we have $N_1 N_5$ copies of the CFT described in subsection 1.9.2, and the orbifold means that all well defined fields must be invariant under permutations of the symmetric group, $S_{N_1 N_5}$. Regular fields, which we will call untwisted fields, can satisfy this by being simple sums over the copies. In addition to the regular fields, we also have a twisted sector from the action of the orbifold. These involve twisted boundary condition for fields permutes the copies of the CFT.

Operators in the twisted sector involve a bare twist operator, which enforce the boundary conditions at the insertion point. For example, we can consider the specific bare twist operator $\hat{\sigma}_{(12\dots n)} = \hat{\sigma}_n$ which will permute fields as we go from $z \rightarrow z e^{2\pi i}$ in the following manner,

$$\hat{\sigma}_n : \phi^{(1)} \rightarrow \phi^{(2)} \rightarrow \dots \rightarrow \phi^{(n)} \rightarrow \phi^{(1)}. \quad (1.52)$$

Here ϕ can either be the ∂X of ψ fields with copy index (i) . The bare twist operators then have a well known conformal weight,

$$\frac{1}{4}(n - \frac{1}{n}), \quad (1.53)$$

which can be argued by considering that a twist operator will take n copies of a vacuum to a single circle n times longer. So then it will change the energy from $-nc/24$ to $-c/24n$. For a single copy, $c = 6$ and so we have the conformal weight as above.

Of course, since we are orbifolding by the symmetric group, these twist operators can be defined by any permutation of length n where $n = 2$ to $N_1 N_5$. The length of the permutation of the twist operator will split the twisted sectors into different levels. We will call an operator, built off a twist operator of length n , a twist n operator. For operators not in the twisted sector, we will just call them untwisted operators. Well defined operators must be fully S_N invariant. A single permutation is not S_N invariant and so these twisted operators must be summed over the conjugacy class of the representative permutation given in the bare twist. This will require one to consider

$$\sigma_n = \frac{\sqrt{n(N-n)!}}{n(N-n)!\sqrt{N!}} \sum_g \sigma_{g(12\dots n)g^{-1}}, \quad (1.54)$$

where g are the group elements of S_N . Notice that this is mostly just amounts to a combinatorial factor out front and it is usually sufficient to just consider a representative twist operator $\sigma_{(12\dots n)}$. However, the S_N invariance is important in restricting the form of correlators with operators of fractional weights and branch cuts as explained in [201].

These twist fields allow construction of fractionally moded operators,

$$\mathcal{O}_{-m/n} = \oint \frac{dz}{2\pi i} \sum_{k=1}^n \mathcal{O}_{-m,(k)} e^{-2\pi im(k-1)/n} z^{h-m/n-1}. \quad (1.55)$$

where h is the conformal weight of the \mathcal{O} field. These are only well defined in the presence of the twist n operator, so that the integral is periodic around the insertion. With these in hand, we can build chiral primaries in the twisted sector. We can build the lowest weight chiral primary operator as for odd n as

$$\sigma_n = J_{-(n-2)/n}^+ \dots J_{-1/n}^+ \hat{\sigma}_n. \quad (1.56)$$

These then have conformal weight,

$$\frac{1}{n} + \dots + \frac{n-2}{n} + \frac{1}{4}(n - \frac{1}{n}) = \frac{(n-1)}{2}. \quad (1.57)$$

For even n , we have to be a bit more careful. Fermions have half integer weight and so will behave differently when we permute copies. For us to have the expected behaviour in equation (1.52), we will need to include a spin field to insure that the fermionic copies do not pick up a negative sign when we go around the insertion. So then we will have for even n the chiral primaries

$$\sigma_n = J_{-(n-2)/n}^+ \dots J_{-2/n}^+ S^+ \hat{\sigma}_n. \quad (1.58)$$

These then have conformal weight,

$$\frac{n-2}{n} + \dots + \frac{2}{n} + \frac{1}{4n} + \frac{1}{4}(n - \frac{1}{n}) = \frac{(n-1)}{2}. \quad (1.59)$$

We see that the spin field contribution $1/4n$ was crucial to the correct result. Note that a spin field usually has a weight of $1/4$ but we must consider the twist operator, which will lead to a weight of $1/4n$. Note that the bare twist for even twists should always come with this spin field, otherwise they will not have the correct behaviour.

We can create higher weight chiral primaries by applying $\psi_{-1/2}^{+\dot{A}}$ and J_{-1}^+ modes. These will have weights $n/2$ and $(n+1)/2$ respectively. This means that at each twist level, we can construct four holomorphic and four antiholomorphic chiral primaries, totalling 16 chiral primaries. We can also create the anti-chiral primaries with the same construction, but with opposite charges. Finally, we note that for $n = 2$, the bare twist with the spin field is the chiral primary operator.

With all this technology in hand, we can finally describe the deformation operator in the CFT. The operator takes the form of

$$\mathcal{O}_D = \epsilon_{AB} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} G_{-1/2}^{\alpha A} \tilde{G}_{-1/2}^{\dot{\alpha}\dot{B}} \sigma_2^{\beta\dot{\beta}}. \quad (1.60)$$

This is a singlet under the $SU(2)$ s and is a twist 2 operator. The deformation operator has been studied in a wide range of contexts [137, 202–216].

This deformation is the main focus of this thesis. We will focus on studying the how the CFT is affected by the deformation operator and what results we obtain when we perturb away from the orbifold point. To accomplish this, we will need to know how to compute correlation functions with twisted boundary conditions, as the deformation operator is a twisted operator.

1.10 Lunin-Mathur

The method for dealing with correlators with twisted operators was first introduced by Lunin and Mathur [217, 218]. The idea is to lift correlators to a covering space, using a map that properly ramifies the twisted boundary conditions for each of the twisted operators. The twisted operators will then be replaced by an identity operator. For example, consider the simple two point function,

$$\langle \sigma_n(b, \bar{b}) \sigma_n(0, 0) \rangle. \quad (1.61)$$

We can lift this correlator with the map

$$z(t) = \frac{bt^n}{t^n - (t - 1)^n}. \quad (1.62)$$

This map take the operators on the base space, z , covering space where we map the points $z = 0$ and $z = b$ to $t = 0$ and $t = b$, respectively. This satisfies the requirement that $z \sim (-1)^{n+1}bt^n$ around $z = 0, t = 0$ and $(z - b) \sim b(t - 1)^n$ around $z = b, t = 1$. Away from these ramified points, each point in the base will correspond to n images in the cover. These will correspond with the n copies that are involved in the twist. See figure 1.2 for an image of the base space mapped to the cover by (1.62) with $n = 2$. For some more detail on how the copies, images, and patches of the covering space come together, refer to [219].

In theory, we can define multiple covering space maps with the correct ramification. What is role played by the genus of the covering surface partition function? We are

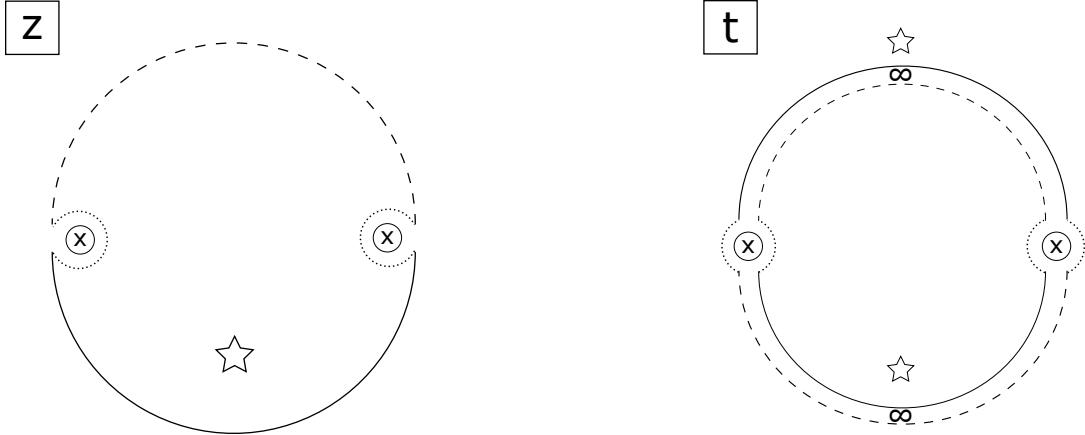


Figure 1.2: Left: Diagram of base space for two twist two insertions (circled x), with a closed path around both twist operators. Right: Lift of base space to the cover. Note the two images of general non-ramified points (star).

considering a large N expansion in the AdS/CFT expansion and higher genus partitions functions correspond to higher orders in $1/N$ [217]. So we focus those of spherical genus. Suppose we have a correlator of k twist operators, each with ramification order $n_j - 1$, and that a generic point will be mapped to s sheets of the covering surface. Then the genus is given by the Riemann-Hurwitz formula,

$$g = \frac{1}{2} \sum_j r_j - s + 1. \quad (1.63)$$

The maps we consider are for genus zero. It is an interesting open question how the general results of the lifting procedure would depend on a choice of map.

After lifting, the correlator will have two contributions, one from the now non-twisted correlator and one from the Liouville term. This term appears because our CFT has a large central charge and hence a conformal anomaly. So the lifting procedure untwists the twisted fields, but we must keep track of this lifting through a Liouville field. The contribution from the Liouville field reflects the contribution from the bare twist operators themselves, since they do have non-trivial conformal dimension. Specifically, if we lift our partition function $Z^{(z)}$ to the covering space, then we find

$$Z^{(z)} = e^{S_L} Z^{(t)}, \quad (1.64)$$

where the action of the Liouville field is

$$S_L = \frac{c}{96\pi} \int d^2t \sqrt{-g^{(t)}} [\partial_\mu \phi \partial_\nu \phi g^{(t)\mu\nu} + 2R^{(t)}\phi]. \quad (1.65)$$

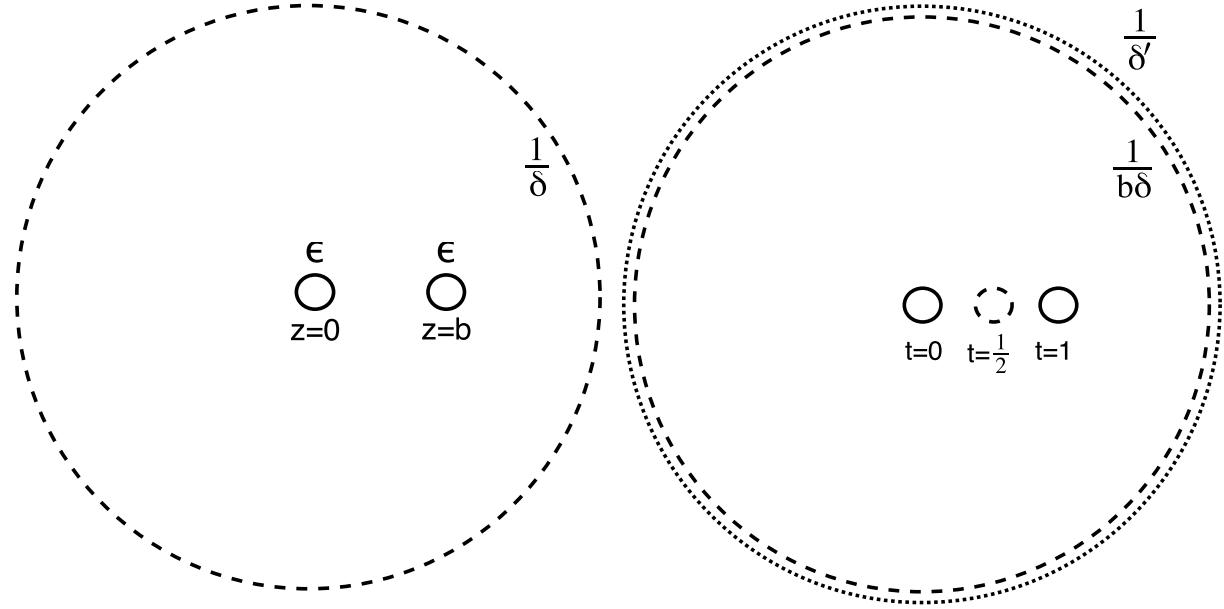


Figure 1.3: Left: regulator diagram on the base. Right: regulator diagram after lifting to the cover. The term “pizza diagrams” come from these regulators, the smaller ones looking like pepperoni and larger ones the crust.

To compute the contribution of the Liouville field, we will have to regulate around infinities and around the twist insertions themselves. This leads to “pizza diagrams” which help identify the various regulation patches and their copies on the covering space. See figure 1.3 for the case of the two point function of twist $n = 2$. The action of the Liouville field will give three distinct contributions to the correlation function. Two of these are just regulation dependent and cancel with proper normalization of the correlation functions. The important term comes from boundary of the holes cut around the twist insertions. Around these holes, we can rewrite the action of the Liouville field as

$$\frac{c}{96\pi} i \int dt [\phi \partial_t \phi + c.c.]. \quad (1.66)$$

In our example of a two point function, we can use the fact that

$$\phi = \log \left| \frac{dz}{dt} \right|^2, \quad (1.67)$$

to find that the contribution from both the images of the twist operators is given by

$$S_L = -\frac{(n-1)c}{6} \left[\frac{1}{n} \log |b| + \log(n\epsilon^{(n-1)/n}) \right], \quad (1.68)$$

where ϵ is the regulator for the twist insertions. With proper normalization, the regulator dependence drops out and we are left with

$$\langle \sigma_n(b, \bar{b}) \sigma_n(0, 0) \rangle = |b|^{-\frac{c}{6}(n - \frac{1}{n})}. \quad (1.69)$$

We can then read off the conformal weight as $\frac{c}{24}(n - \frac{1}{n})$, the correct conformal weight for bare twists!

The Liouville contribution together with the contribution of the non-twist part will yield the full correlation function. For the work we do here, we can use the results from [200] Appendix D, which has general expressions for the bare twist contributions using the Lunin-Mathur technique.

Chapter 2

Component twist method for higher twists in D1D5 CFT

2.1 Introduction

The deformation operator, written out in equation (1.60), is the key to understanding black hole physics in the D1D5 system. Towards this goal, we wish to see how thermalization occurs in the D1D5 system. At the orbifold point, the field theory is free and so it is not possible to see thermalization, as the information of any localized excitation cannot be scrambled without interactions. When we deform away from the orbifold point, the twisted deformation operator introduces interactions between the copies and this allows excitations to thermalize in the CFT. This is similar to the story of the emission of supersymmetric quanta from fuzzballs in CFT language [37]. In that scenario, the vertex operator for emission in the CFT was a twisted operator, with a number of dressings. This vertex operator can reproduce emission for a variety of non-supersymmetric fuzzball solutions, even with differing methods of emissions such as Hawking radiation and superradiance from the black hole and ergoregion emission from the smooth solutions. The common theme here is that both thermalization and emission stem from the physics of the twisted sector in the D1D5 CFT. So understanding the action of the twist operator is important in investigating the details of how the black hole information problem might be resolved.

One physical property known for free orbifold theories in 1+1d is that acting on the vacuum with twist operators will produce a squeezed state. The rough argument is that in the free theory we have a quadratic action, which means that the ground state is a Gaussian. Each twist introduces a change of basis, but since the wavefunction was

Gaussian to begin with, the result is a squeezed state. This behavior holds even when one applies an arbitrary number of twists [204, 220]. We can then try to find the Bogoliubov matrices describing these squeezed states. We described the existing technique for computing quantities with twisted operators in free symmetric orbifold CFTs, the Lunin-Mathur method in section 1.10. Recall that this method involves mapping the orbifold CFT into a covering space where all fields are single-valued. These maps from the base to the cover must later be inverted to obtain physics answers of interest. When enough twist operators are applied, the maps will become quintic and higher-order polynomials which cannot be inverted. This technical obstruction severely limits the utility of these techniques for higher numbers of twist operators. Our method will avoid using the Lunin-Mathur method but will require working directly in the continuum limit.

We continue from previous investigations, which we will describe in the next section, to study general twisted states. Our method, using twist 2 operators to build up general twist configurations, reproduces previous results obtainable using Lunin-Mathur technology in the continuum limit. We show this by computing a configuration of twisting and untwisting two copies. We also compute a new second-order twist 2 configuration by twisting three copies by first twisting two and then twisting the third into the doubly wound string. We find that the coefficients defining the squeezed state in this new scenario scale as conjectured in [213]. Our methodology gives further evidence that these general scalings should hold in the continuum limit for arbitrary configurations of twist operators.

The remainder of this chapter is organized as follows. In Section 2.2, we will review some of the other methods that have been used to study thermalization in the D1D5 CFT. In Section 2.3, we present the method in detail for twist operators in orbifold CFTs. In Section 2.4 we discuss briefly the particular orbifold CFT arising from the D1D5 black hole construction. We also discuss here the specific supersymmetry relations that relate the bosonic and fermionic sectors of this theory. In the subsequent two sections, we apply our method to two different twist configurations in the D1D5 CFT. In Section 2.5 we tackle a previously-studied configuration and find good agreement with known results, while in Section 2.6 we tackle a new twist configuration. Section 2.7 concludes with a few closing remarks. Our work builds heavily on [208], which presented the basics of this method entirely in the context of the D1D5 CFT.

This chapter mostly follows the work presented in the paper [216].

2.2 Review of previous methods

There have been a number of previous works dedicated to studying thermalization in the D1D5 system [137, 204, 205, 207–210, 212–214]. These results have been obtained by four main methods: direct calculations, the Ward Identity, entanglement entropy, and the continuum limit. Let us briefly review these results to give context for our original contribution.

The papers [204, 205, 207, 209, 212–214] focused on computing the effect of the twist 2 operator and the dressed deformation operator to first and second order. These papers focused on looking at the effect of the bare twist 2 operator, σ_2 , or the deformation operator \mathcal{O}_D . To first order, the twist 2 operator will twist together two copies into a squeezed state. The second order will twist the two copies together then split them apart. The copies can either be in the vacuum or a simple excited state, and one can set up correlation functions to measure the Bogoliubov coefficient describing the states.

The paper [210] took a different approach, using the Ward identity together with the Lunin-Mathur technique to work out the effect of the deformation operator. Here they used the Ward identity to extract the Bogoliubov coefficient. Since the stress tensor acts as a derivative on the squeezed state there is no need to compute the correlator, as the stress tensor can extract off the coefficient directly. One still needs to lift the Ward identity to the cover in order to remove the twisted boundary conditions and analyze the two sides of the Ward identity.

The paper [137] instead examined the entanglement entropy for the D1D5 system after a local quench by the deformation operator. Going through the definition of the entanglement entropy in terms of a reduced density matrix, they employed the replica trick in order to write out the expression in the CFT. Their entropy calculation reduced to computing a four point correlator of twist operators, two twist n operators from the replicas and two twist 2 operators from the state after the application of the deformation.

The Lunin-Mathur method is used in all three of the methods above. There are some technical details on how to set up the correlators, such as being careful about relating the modes on the cylinder to the plane, using spectral flow to remove the spin fields from the correlators, and lifting the modes to the cover. In the end, this method gives exact expressions. However, the Lunin-Mathur based approach will not work to higher orders in the twist operator. Part of the lifting method is the map, which is known for 2, 3, and some 4 point correlators of twist operators. But going higher than this, the map will require quintic or higher polynomials. Since higher point correlators will have some unfixed coordinates (SL(2,R) global conformal invariance cannot fix all the points)

one will need to be able to invert these maps to relate the cover and the base. Inverting quintics or higher polynomials is not generally solvable making the Lunin-Mathur method unable to handle higher order effects of the deformation.

A qualitatively different approach was taken in the paper [208], which studied the Bogoliubov coefficients of the squeezed state directly in the continuum limit. This limit is where we take the CFT to live on the line and not the circle. The results here are what we shall build on with our component twist method. The benefit of using this method is that we can avoid Lunin-Mathur map invertibility trouble and hence we will have results that could go to arbitrarily high order. The cost is that we only get a result valid in the continuum limit, not in general. However, there are a number of reasons to think that the continuum limit is where we expect to see black hole dynamics. First, the AdS/CFT correspondence says if we look at the Poincaré patch of a black hole, then it will be dual to a CFT living on \mathbb{R}^d rather than S^d . Second, we know that the long string, created by high twist operators, is the generic state of the black hole and these very long strings will act as if they effectively live on a line. Finally, when we have the CFT at the black hole point, there will be strong twist interactions between the copies. So any quantum on the S^1 will not travel long before being affected by a twist and the quantum will act as if it is living on a line. In the continuum limit, [208] used basic knowledge of quantum field theory and mode expansions to work out the expressions for the coefficients describing the squeezed state.

We will take these results for a single twist 2 operator and show how we can build up more general configurations at higher order. We will do this by noting that we can build up a general length n permutation by recruiting a number of 2-cycles and so the knowledge of any twist 2 operator can be used to build any generic configuration of twists. Then we will analyze the squeezed states by multiplying the matrices describing the twist 2 squeezed state together. We will call this method the component twist method and explain the details of how this works in the rest of the chapter.

2.3 Overview of the component twist method

Given a free orbifold CFT in $1 + 1$ dimensions, consider any arbitrary combination of individual twist operators σ_n , which together we will call $\hat{\sigma}$. This operator takes a free theory living on one set of windings to a free theory on another set of windings, with the total winding number remaining constant. For the moment, consider a scalar field ϕ exists on all windings. The allowed excitation modes for this field will differ between the

pre-and post-twist configurations. The result is

$$\begin{aligned}\hat{\sigma}|0\rangle &= C \exp \left[\sum_{m,n>0} \sum_{(j)(j')} \gamma_{mn}^{(j)(j')} a_m^{(j)\dagger} a_n^{(j')\dagger} \right] |0'\rangle \equiv |\chi(\hat{\sigma})\rangle \\ \hat{\sigma} a_m^{(i)\dagger} |0\rangle &= \sum_{n>0} \sum_{(j)} f_{mn}^{(i)(j)} a_n^{(j)\dagger} |\chi(\hat{\sigma})\rangle.\end{aligned}\quad (2.1)$$

The new parenthetical indices refer to the various CFT copies of the theory. For clarity, we use the index (i) exclusively for CFT copies before the twist operators and the index (j) exclusively for copies after the twist operators. The index (k) will be used when speaking of an arbitrary copy without specifying one side of the twist operators (including copy structures between the twist operators). Primes are used for additional copies of the same type. For calculational convenience, the mode indices m and n indicate the energy of each excitation. They can be fraction-valued for multiwound copies. The excitation level of each excitation is its mode index times the winding number of the CFT copy on which the excitation lives.

A method for handling the bosonic sector of these twist operators was explored in [208] in the context of the D1D5 CFT, though the need to invert infinite matrices inhibited any useful generalization. We present a straightforward workaround for obtaining the required inverse matrices. For the remainder of this chapter, we will work in the continuum limit of our orbifold CFT. In the next subsection, we will describe how to get out the $f^{(i)(j)}$ and $\gamma^{(j)(j')}$ using knowledge of Bogoliubov transformations.

2.3.1 Bogoliubov approach for arbitrary orbifold CFTs

Consider a particular component CFT copy (i) with winding number $N_{(i)}$ before the twist insertions. On this copy, the field ϕ may be expanded as:

$$\phi(x) = \sum_{mN_{(i)} \in \mathbb{Z}^+} (h^{(i)}(x)a_m^{(i)} + h_m^{(i)*}(x)a_m^{(i)\dagger}), \quad (2.2)$$

Now consider a particular copy (j) with winding number N_j after the twist insertions. Here ϕ may be expanded as:

$$\phi(x) = \sum_{mN_{(j)} \in \mathbb{Z}^+} (h^{(j)}(x)a_m^{(j)} + h_m^{(j)*}(x)a_m^{(j)\dagger}). \quad (2.3)$$

Since both of these expressions describe the same field, we can relate these two expansions with a linear relationship:

$$a_m^{(i)} = \sum_{nN_{(j)} \in Z^+} (\alpha_{mn}^{(i)(j)} a_n^{(j)} + \beta_{mn}^{(i)(j)} a_n^{(j)\dagger}). \quad (2.4)$$

The two matrices appearing here are also related to the coefficients of (2.1).

$$\begin{aligned} f_{mn}^{(i)(j)} &= (\alpha_{nm}^{(i)(j)})^{-1} \\ \gamma_{mn}^{(j)(j')} &= \sum_{kN_{(i)} \in Z^+} \sum_{(i)} (\alpha_{mk}^{(i)(j)})^{-1} \beta_{kn}^{(i)(j')}. \end{aligned} \quad (2.5)$$

It will be convenient to occasionally write these relations in their matrix form.

$$f = (\alpha^{-1})^T, \quad \gamma = \alpha^{-1} \beta = f^T \beta. \quad (2.6)$$

The Bogoliubov matrices are straightforward to calculate. They can be expressed in terms of the positive frequency solutions to the wave equation used in the expansions (2.2) and (2.3). The functions h form a complete orthonormal basis with respect to the inner product:

$$(h, g) \equiv -i \int_{\Sigma} d\Sigma^{\mu} (f \partial_{\mu} g^* - g^* \partial_{\mu} f), \quad (2.7)$$

where Σ is the Cauchy hypersurface where the domains of f and g overlap. Applying this orthonormality to (2.2) and (2.3) and comparing to (2.4), one finds:

$$\begin{aligned} \alpha_{mn}^{(i)(j)} &= (h_m^{(i)}, h_n^{(j)}) \\ \beta_{mn}^{(i)(j)} &= ((h_m^{(i)})^*, h_n^{(j)}). \end{aligned} \quad (2.8)$$

The material presented so far in this section is generic and was used in previous calculations. Once one has calculated the Bogoliubov matrices, it is necessary to then invert α in order to compute γ . This task is far from trivial. The Bogoliubov matrices are infinite matrices, so there is no mathematical guarantee that an inverse even exists. This is where our component twist method will come in and we will describe it in the next subsection.

2.3.2 Obtaining α^{-1} using component twists

To proceed, we need to find a way to obtain α^{-1} directly without having to invert α . We saw that in equation (2.6) that α^{-1} gives the unique physical transition matrix f . So

rather than inverting α directly, we present a direct method of computing f for arbitrary twist configurations based on the behaviour of the single component twists. In principle one then need calculate only the β matrix using the methods in the previous subsection to find γ .

Let us consider the twist configuration $\hat{\sigma}$ again. While we allowed this configuration to contain any combination of twists σ_n , each such component can itself be written in terms of two-twist operators σ_2 . We thus decompose $\hat{\sigma}$ in terms of these two-twist operators.

$$\hat{\sigma} = \prod_{q=1}^Q \sigma_2^{\{k_q\}}(x_q), \quad (2.9)$$

where each $\{k_q\}$ is an ordered pair of component CFTs. Since two-twist operators do not commute, it is important to specify that we will choose indices such that the twists act in order of increasing q .

We now act on a state containing only a single excitation of ϕ . The left hand side is given by (2.1). On the right, we will pass each twist through one at a time.

$$\begin{aligned} \hat{\sigma} a_m^{(i)\dagger} |0\rangle &= \sum_{n,(j)} f_{mn}^{(i)(j)} a_n^{(j)\dagger} |\chi(\hat{\sigma})\rangle \\ &= \prod_{q=1}^Q \sigma_2^{\{k_q\}}(x_q) a_m^{(i)\dagger} |0\rangle \\ &= \left(\prod_{q=2}^Q \sigma_2^{\{k_q\}}(x_q) \right) \sum_{n_1,(k_1)} f_{m,n_1}^{(i)(k_1)} a_{n_1}^{(k_1)\dagger} |\chi_1\rangle \\ &= \left(\prod_{q=3}^Q \sigma_2^{\{k_q\}}(x_q) \right) \sum_{n_2,(k_2)} \left(\sum_{n_1,(k_1)} f_{n_1,n_2}^{(k_1)(k_2)} f_{m,n_1}^{(i)(k_1)} \right) a_{n_2}^{(k_2)\dagger} |\chi_2\rangle \\ &= \left(\prod_{q=4}^Q \sigma_2^{\{k_q\}}(x_q) \right) \sum_{n_3,(k_3)} \left(\sum_{n_2,(k_2)} f_{n_2,n_3}^{(k_2)(k_3)} \left(\sum_{n_1,(k_1)} f_{n_1,n_2}^{(k_1)(k_2)} f_{m,n_1}^{(i)(k_1)} \right) \right) a_{n_3}^{(k_3)\dagger} |\chi_3\rangle \\ &\dots \\ &= \sum_{\{n_i\},\{(k_i)\}} \left(f_{n_Q,n_{Q-1}}^{(k_Q)(k_{Q-1})} \left(f_{n_{Q-1},n_{Q-2}}^{(k_{Q-1})(k_{Q-2})} \left(\dots \left(f_{n_2,n_1}^{(k_2),(k_1)} f_{n_1,m}^{(k_1),(i)} \right) \right) \right) \right) a_{n_Q}^{(k_Q)\dagger} |\chi(\hat{\sigma})\rangle \end{aligned}$$

Here each $|\chi_i\rangle$ is the state obtained by acting with the first i twists on the vacuum. The parentheses in the last line are used to present the correct order for the infinite sums.

The fact that this order matters is equivalent to the non-associativity of infinite matrix multiplication. We have however played a little loose with our copy notation. We have used k_i to indicate both a pair of copies defining a twist operator (when in brackets) and an intermediate CFT copy index (when in parentheses).

By construction, the sum over (k_Q) in the last line of (2.10) is the same as the sum over (j) the first line. Similarly, the sum over n_Q in the last line is the same as the sum over n in the first line. We thus find:

$$f_{mn}^{(i)(j)} = \sum_{\{n_i\}, \{(k_i)\}} \left(f_{n,n_{N-1}}^{(j)(k_{N-1})} \left(f_{n_{N-1},n_{N-2}}^{(k_{N-1})(k_{N-2})} \left(\dots \left(f_{n_2,n_1}^{(k_2),(k_1)} f_{n_1,m}^{(k_1),(i)} \right) \right) \right) \right). \quad (2.11)$$

In matrix form, this gives

$$f(\hat{\sigma}) = (f_Q(f_{Q-1}(\dots(f_2f_1)))), \quad (2.12)$$

where each f_i is the transition matrix for the i^{th} two-twist in the decomposition (2.9).

Now that we have an expression for the transition matrix, and by extension α^{-1} , for arbitrary twist configurations, the generalization of [208] is straightforward. Calculate the transition matrix and Bogoliubov matrices as described in this section. The matrix α can be used to check the transition matrix calculation by verifying $\alpha f^T = f^T \alpha = \mathbb{I}$. One then finds the matrix characterizing the squeezed state via $\gamma = f^T \beta$. All of this would apply for a generic free orbifold. To show this method explicitly, we turn to the D1D5 CFT.

2.4 Application to the D1D5 CFT

We have reviewed the D1D5 CFT in section (1.9) and we will now employ that here. This is a free orbifold theory, so our work in the previous section applies. However, the fields in this CFT carry global symmetries. We may worry that we will have a proliferation of characteristic coefficients from the $SU(2)$ structures. This is not the case. Aside from the fermion zero modes, there is only a single linearly independent γ and f matrix for each copy combination. In practice, however, it is convenient to write different matrices for bosons and fermions and to also add an $SU(2)_L$ index to the fermion transition matrix. One then has the particular $SU(2)$ structures:

$$\begin{aligned} |\chi(\hat{\sigma})\rangle &= C \exp \left[\gamma_{mn}^{B(j)(j')} \left(-a_{++,m}^{\dagger(j)} a_{--,n}^{\dagger(j')} + a_{+-,m}^{\dagger(i)} a_{-+,n}^{\dagger(j)} \right) \right] \\ &\quad \times \exp \left[\gamma_{mn}^{F(j)(j')} \left(d_m^{\dagger(j)++} d_n^{\dagger(j)--} - d_m^{\dagger(j)+-} d_n^{\dagger(j)-+} \right) \right] |0\rangle \end{aligned} \quad (2.13)$$

The transition matrices are independent of all $SU(2)$ charges except $SU(2)_L$. We thus add an index α to the fermionic piece.

$$f_{mn}^{B(i)(j)}, \quad f_{mn}^{F\alpha(i)(j)}. \quad (2.14)$$

Our method only allows us to calculate the bosonic coefficients directly. However, the nontrivial relationships between the fermionic and bosonic coefficients were shown in [212], [213]. We can thus access the fermionic sector (except zero modes).¹

Our method was derived for canonically normalized modes, but it has become standard for work in the D1D5 CFT to make use of modes that are not canonically normalized. We thus provide a translation between the standard mode normalizations and our canonical normalizations.

$$\begin{aligned} \alpha_{A\dot{A},-n}^{(k)} &= \sqrt{n} a_{A\dot{A},n}^{(k)\dagger} \\ \tilde{d}_{A\dot{A},-n}^{(k)} &= \sqrt{N_{(k)}} d_{A\dot{A},-n}^{(k)}, \end{aligned} \quad (2.15)$$

where the standard modes are on left-hand side. Because of this, our characteristic coefficients are also different. For the squeezed state we have

$$\begin{aligned} \tilde{\gamma}_{mn}^{B(j)(j')} &= \sqrt{mn} \gamma_{mn}^{B(j)(j')} \\ \tilde{\gamma}_{mn}^{F(j)(j')} &= \sqrt{N_{(j)} N_{(j')}} \gamma_{mn}^{F(j)(j')}, \end{aligned} \quad (2.16)$$

while for the transition matrix we have

$$\begin{aligned} \tilde{f}_{mn}^{B(i)(j)} &= \sqrt{\frac{n}{m}} f_{mn}^{B(i)(j)} \\ \tilde{f}_{mn}^{F\alpha(i)(j)} &= \sqrt{\frac{N_{(j)}}{N_{(i)}}} f_{mn}^{F(i)(j)}. \end{aligned} \quad (2.17)$$

We also translate the relationships between the bosonic and fermionic coefficients from

¹Our inability to access fermion zero modes means that our method is insensitive to the particular $SU(2)$ structure of the Ramond vacuum. This insensitivity is expected in the continuum limit.

[212], [213].

$$\begin{aligned}
 \tilde{\gamma}_{mn}^{F(i)(j)} &= -m \frac{\sqrt{mn}}{\sqrt{N_{(i)} N_{(j)}}} \gamma_{mn}^{B(i)(j)} \\
 \tilde{f}_{mn}^{F-, (i)(j)} &= \sqrt{\frac{n}{m}} \sqrt{\frac{N_{(i)}}{N_{(j)}}} f_{mn}^{B(i)(j)} \\
 \tilde{f}_{mn}^{F+, (i)(j)} &= \sqrt{\frac{m}{n}} \sqrt{\frac{N_{(j)}}{N_{(i)}}} f_{mn}^{B(i)(j)}. \tag{2.18}
 \end{aligned}$$

In all of the above expressions, the traditional forms are written on the left-hand side.

With the main relationships reviewed, we can now turn to using our method to reproduce results obtained using the Lunin-Mathur method to show that our method is sound. We will examine a twist and untwist set-up in the next section.

2.5 Reproduction of $1 + 1 \rightarrow 2 \rightarrow 1 + 1$ results

We now look at the case of two singly wound copies twisted into one doubly wound copy at $w_1 = \tau_1 + i\sigma_1$ and then untwisted back into two singly wound copies at $w_2 = \tau_2 + i\sigma_2$. For convenience, we will use translation and rotation invariance to select $\tau_1 = \tau_2 = 0$ and $\sigma_2 > \sigma_1$. This means that our Cauchy hypersurface Σ is a collection of σ contours.

Since we have two singly-wound CFTs in both the initial and final states, we will place primes on the explicit final-state copy indices. So the initial copies are (1) and (2) while the final copies are (1') and (2').

2.5.1 Calculating the Bogoliubov matrices

We begin by writing the positive frequency solutions to the wave equation.

$$\begin{aligned}
 h_m^{(k)} &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2m}} e^{im(\sigma-\tau)} \\
 h_{\bar{m}}^{(k)} &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\bar{m}}} e^{-i\bar{m}(\sigma+\tau)}, \tag{2.19}
 \end{aligned}$$

where m is an integer. The functional form is the same for all copies since each copy has the same winding number. However, different copies are valid over different regions. Importantly, copies (1) and (2) have no domain overlap, and similarly for copies (1') and (2'). What we need now is the domain of overlap for each initial-final copy pair. There is in fact some ambiguity here, owing to the freedom to define our copies however we

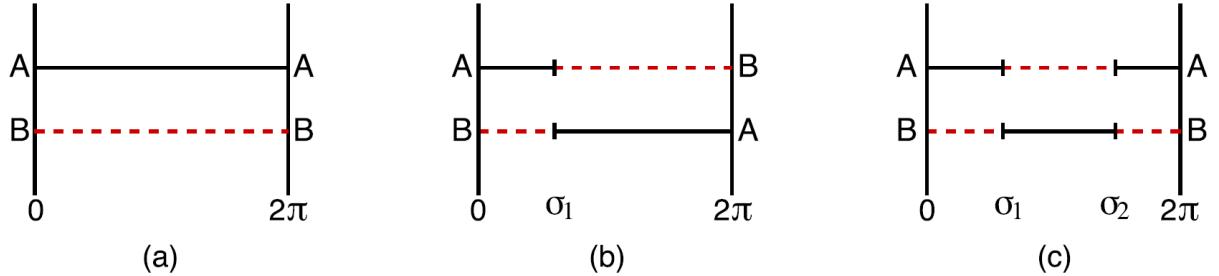


Figure 2.1: The layout of the copies as we apply the twists. Copy (1) is the black line, copy (2) is red. Here the A,B ends are identified under the $\sigma = 0 \sim 2\pi$ identification of the cylinder. We perform all calculations at $\tau = 0$, though we separate the windings here visually for clarity. (a) Copies before twists. (b) Application of $\sigma_{(12)}$ at σ_1 , twisting the copies together. (c) Application of $\sigma_{(21)}$ at σ_2 , splitting into two new copies (1') (top) and (2') (bottom)

choose. We will follow the conventions of [212]. The intermediate doubly-wound CFT has its interval $[0, \sigma_1]$ taken from copy (1), while the second twist takes the first 2π interval of this intermediate CFT into copy (1'). The initial-final domains of overlap can then be seen in figure 2.1.

$$\begin{aligned} (1), (1') \text{ or } (2), (2') &\implies \Sigma = [0, \sigma_1] \cup [\sigma_2, 2\pi] \\ (1), (2') \text{ or } (2), (1') &\implies \Sigma = [\sigma_1, \sigma_2]. \end{aligned} \quad (2.20)$$

Since the expansion functions are the same for all copies, each matrix's (1, 1') and (2, 2') components are identical, as are (1, 2') and (2, 1') components. This is expected, as the physics is invariant under the combination of interchanges $1 \leftrightarrow 2$ and $1' \leftrightarrow 2'$.

We can now calculate our Bogoliubov matrices. Starting with α , we have

$$\begin{aligned} \alpha_{mn}^{(1)(1')} &= -i \left(\int_0^{\sigma_1} + \int_{\sigma_2}^{2\pi} \right) \frac{1}{4\pi\sqrt{mn}} \left(e^{im(\sigma-\tau_0)} in e^{-in(\sigma-\tau_0)} \right. \\ &\quad \left. - e^{-in(\sigma-\tau)} (-im) e^{im(\sigma-\tau_0)} \right) d\sigma \\ &= \frac{1}{4\pi\sqrt{mn}} \left(\int_0^{\sigma_1} + \int_{\sigma_2}^{2\pi} \right) (m+n) e^{i(m-n)\sigma} d\sigma. \end{aligned} \quad (2.21)$$

The result is piecewise. For $m = n$, we have

$$\begin{aligned}\alpha_{mm}^{(1)(1')} &= \frac{1}{4\pi} \frac{1}{m} 2m(\sigma_1 + 2\pi - \sigma_2) \\ &= 1 - \frac{\sigma_1 - \sigma_2}{2\pi} \\ &= 1 - \frac{\Delta w}{2\pi i}.\end{aligned}\tag{2.22}$$

For $m \neq n$, we find

$$\begin{aligned}\alpha_{mn}^{(1)(1')} &= \frac{1}{4\pi\sqrt{mn}} \frac{m+n}{i(m-n)} (e^{-i(m-n)\sigma_1} - 1 + 1 - e^{-i(m-n)\sigma_2}) \\ &= \frac{1}{2\pi} \frac{m+n}{\sqrt{mn}(m-n)} e^{i(m-n)\frac{\sigma_1+\sigma_2}{2}} \frac{1}{2i} \left(e^{i(m-n)\frac{\sigma_1-\sigma_2}{2}} - e^{-i(m-n)\frac{\sigma_1-\sigma_2}{2}} \right) \\ &= \frac{1}{2\pi} \frac{m+n}{\sqrt{mn}(m-n)} e^{i(m-n)\frac{\sigma_1+\sigma_2}{2}} \sin\left((m-n)\frac{\Delta w}{2i}\right).\end{aligned}\tag{2.23}$$

We now use translation invariance to set the σ midpoint to zero. This eliminates the phase.

$$\alpha_{mn}^{(1)(1')} = \frac{1}{2\pi} \frac{m+n}{\sqrt{mn}(m-n)} \sin\left((m-n)\frac{\Delta w}{2i}\right).\tag{2.24}$$

As noted earlier, this is also the purely left-moving $(2, 2')$ portion of the α matrix. The calculation for $(1, 2')$ is identical up to a change in integration limits. The result is

$$\alpha_{mn}^{(1)(2')} = \begin{cases} \frac{\Delta w}{2\pi i} & m = n \\ -\alpha_{mn}^{(1)(1')} & m \neq n, \end{cases}\tag{2.25}$$

which is also the purely left-moving $(2, 1')$ result. As expected by their general decoupling, the purely right-moving results are identical to their purely left-moving counterparts. There are also parts of the α matrix with mixed holomorphicity. Since we expect the left and right movers to be decoupled at the free orbifold point, these matrices are nonphysical. This is reflected in vanishing of terms with them. This is reflecting the fact that there is an ambiguity in the definition of the α , as infinite matrices do not have unique inverses and f is a physical quantity. For example, we have found

$$\alpha_{m\bar{n}}^{(1)(1')} = \frac{1}{2\pi\sqrt{m\bar{n}}} \sin\left[(m-\bar{n})\frac{\Delta w}{2}\right]\tag{2.26}$$

We can multiply this by a f_{mp} , evaluating numerically, and find it will vanish. This

observation is discussed in more detail in Appendix B of [208]. We thus have all the physically-relevant parts of α .

The calculation for β is analogous. It gives:

$$\begin{aligned}\beta_{mn}^{(1)(1)} &= \frac{1}{2\pi} \frac{m-n}{\sqrt{mn(m+n)}} \sin\left((m+n)\frac{\Delta w}{2i}\right) \\ \beta_{mn}^{(1)(2)} &= -\beta_{mn}^{(1)(1)}.\end{aligned}\quad (2.27)$$

Conveniently, there is no need for a piecewise expression. Each component simply vanishes on the diagonal. Again, we have similar expressions for the purely right-moving results and the mixed holomorphicity end up vanishing in any physical terms.

With the α and β matrices, we can now turn to comparing to previous results.

2.5.2 Comparison to previous results

An analytic form of the transition matrix for this twist configuration in the continuum limit was identified in [213]. Shifting to canonical modes, for $0 < \Delta w \leq 2\pi i$ one has:

$$\begin{aligned}f_{mn}^{(1)(1')} &= f_{mn}^{(2)(2')} = \begin{cases} 1 - \frac{\Delta w}{2i} & m = n \\ -\frac{1}{\pi(m-n)} \sin\left((m-n)\frac{\Delta w}{2i}\right) & m \neq n \end{cases} \\ f_{mn}^{(1)(2')} &= f_{mn}^{(2)(1')} = \begin{cases} \frac{\Delta w}{2i} & m = n \\ \frac{1}{\pi(m-n)} \sin\left((m-n)\frac{\Delta w}{2i}\right) & m \neq n \end{cases}.\end{aligned}\quad (2.28)$$

We can multiply this matrix by α to check our calculations. Using a cutoff of $\Lambda = 2000$ for the sums, the result is \mathbb{I} to within 0.1% for mode indices of order 10 or larger.

One can also check the method of building this two-twist transition amplitude out of the behaviour of each component twist. Each component transition amplitude also scales as $1/(m-n)$, so the summand behaves as an inverse square. The convergence is fast. $\Lambda = 2000$ is sufficient for 0.2% accuracy for mode indices of order 10 or larger.

Like the transition matrix, the matrix γ has the following copy relations:

$$\gamma_{mn}^{(1')(1')} = \gamma_{mn}^{(2')(2')} = -\gamma_{mn}^{(1')(2')} = -\gamma_{mn}^{(2')(1')}.\quad (2.29)$$

We therefore compute only the $(1'), (1')$ sector. This computation is more arduous. The summand scales with a power of $-3/2$ so convergence is slower than checking inverses. Furthermore, larger mode indices are required to maintain the validity of the continuum

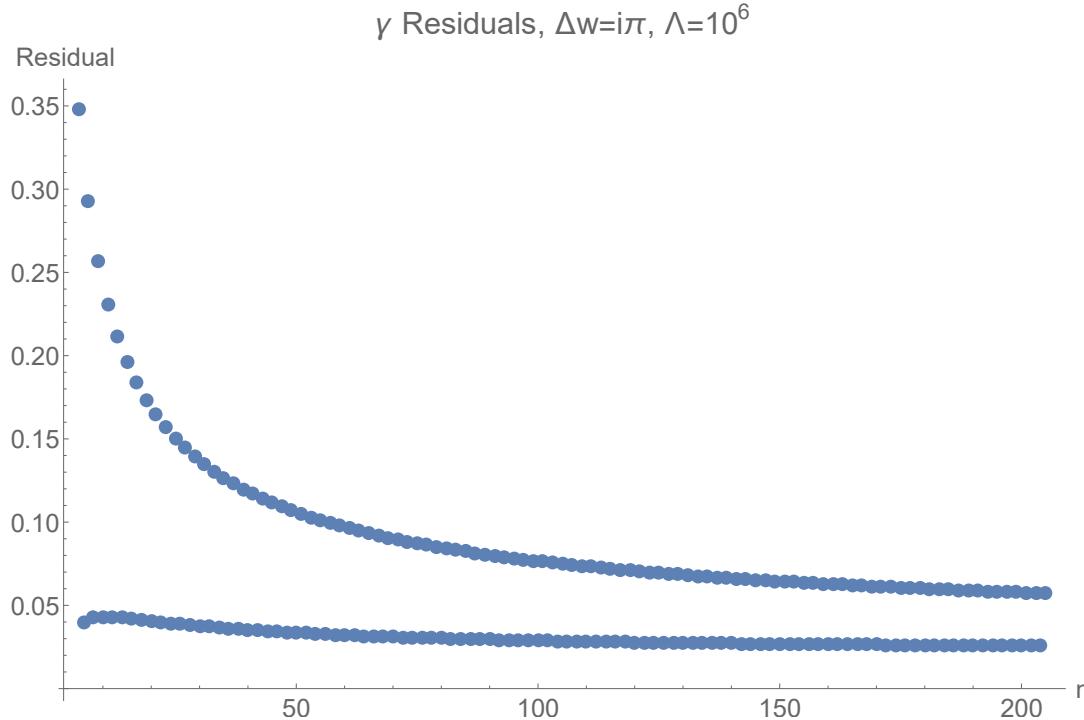


Figure 2.2: Residual vs n for $\gamma_{n,n}^{(1')(1')}$ at $\Delta w = \pi i$. Note that this is a single curve with even and odd behaving slightly differently.

limit approximations. This is in keeping with [213], wherein a general form of the falloff of γ with rising mode indices was not valid for low-lying modes.

Figure 2.2 shows the percent error of our calculation for $\Lambda = 10^6$ for various excitation levels. These high-cutoff calculations were performed on SciNet [221]. Computations with lower cutoffs were performed on personal computers. Figure 2.3 and 2.4 shows the cutoffs required for a desired accuracy. To get a feel for the accuracy/computation tradeoff, see Figures 2.5 and 2.6.

2.6 New twist configuration: $1 + 1 + 1 \rightarrow 2 + 1 \rightarrow 3$

We now turn to a new twist configuration which has not yet been addressed in the literature. Here we begin with three singly-wound CFT copies and through two two-twist operators join them into a single triply-wound copy. We can again use rotation and translation invariance to set $\tau_1 = \tau_2 = 0$ and $\sigma_2 > \sigma_1$. The twist at σ_1 will join copies (1) and (2) while the twist at σ_2 joins copies (2) and (3). There is only one final copy so we drop its index.

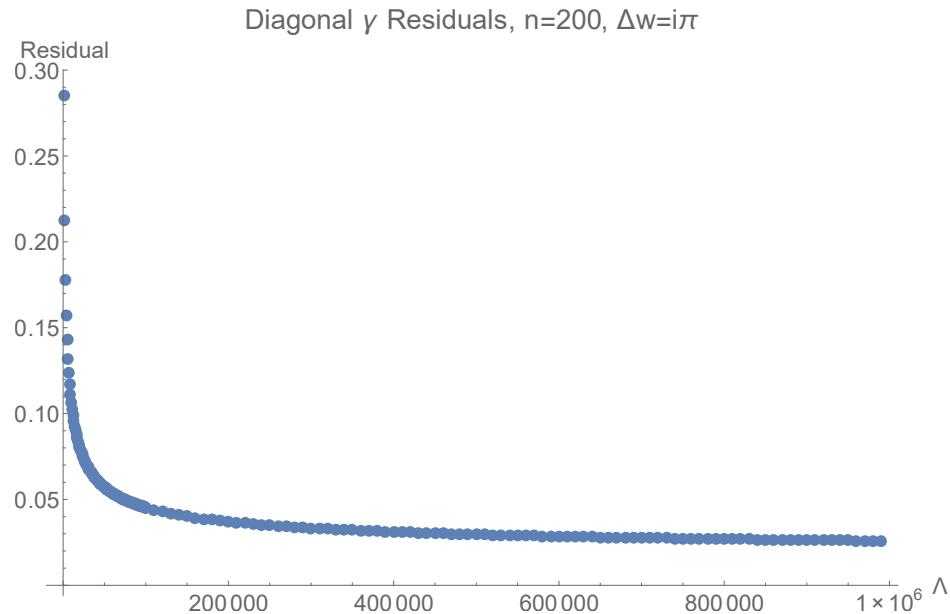


Figure 2.3: Residual vs cutoff for $\gamma_{200,200}^{(1')(1')}$ at $\Delta w = \pi i$.

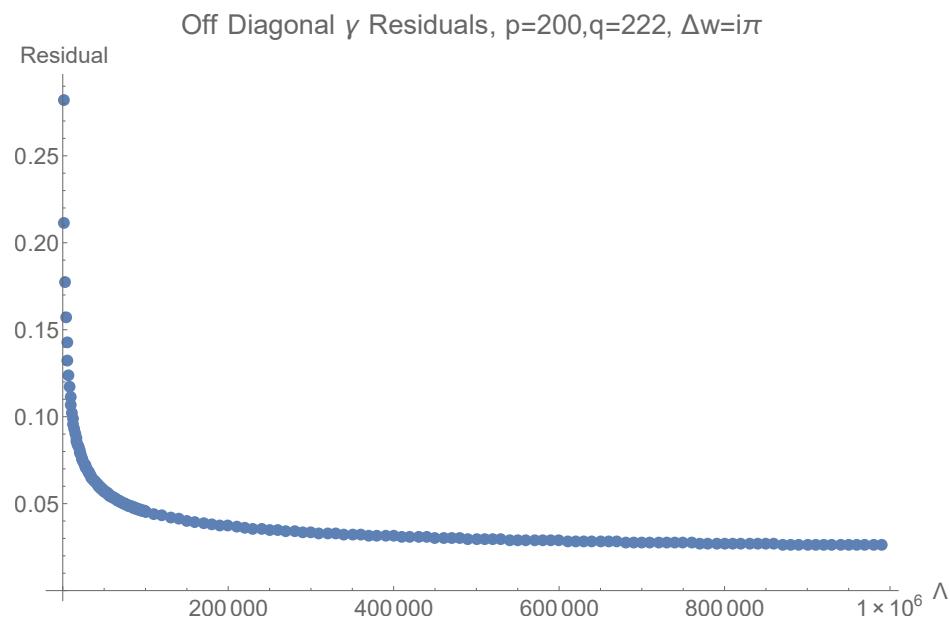


Figure 2.4: Residual vs cutoff for $\gamma_{200,222}^{(1')(1')}$ at $\Delta w = \pi i$.

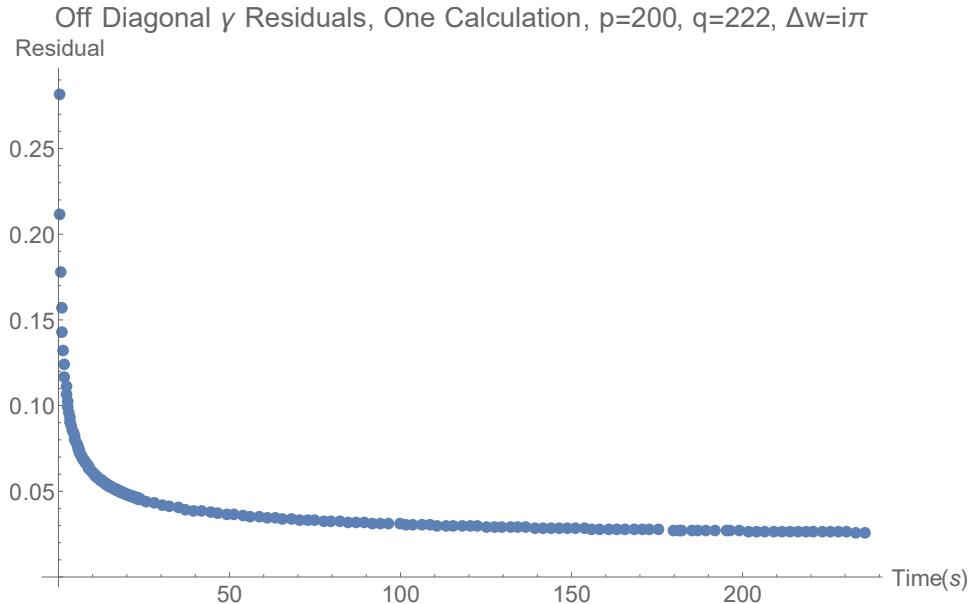


Figure 2.5: Residual vs computation time for $\gamma_{200,222}^{(1')(1')}$ at $\Delta w = \pi i$.

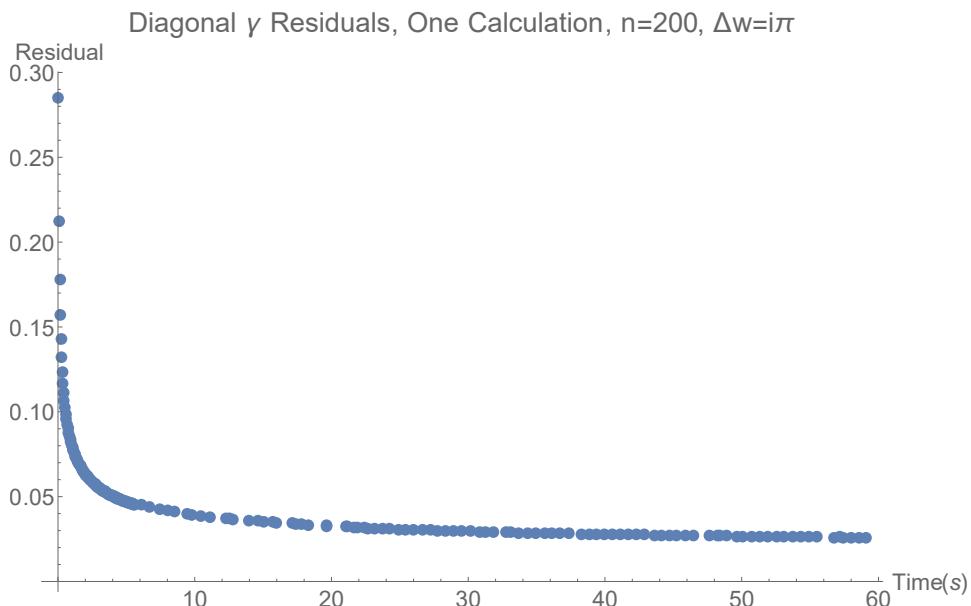


Figure 2.6: Residual vs computation time for $\gamma_{200,200}^{(1')(1')}$ at $\Delta w = \pi i$.

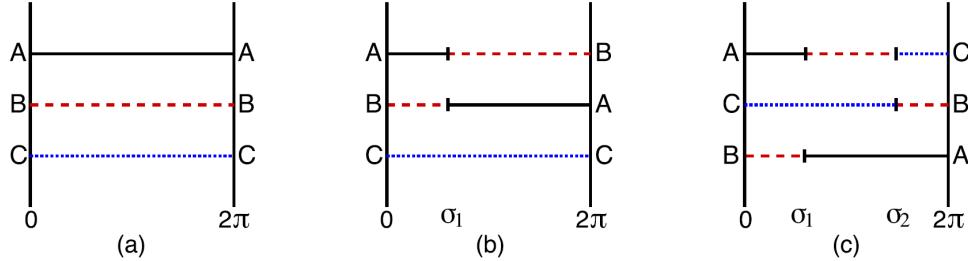


Figure 2.7: The layout of the copies as we apply the twists. Copy (1) is the black line, copy (2) is red, copy (3) is blue. Here the A, B, C, ends are identified under the $\sigma = 0 \sim 2\pi$. We perform all calculations at $\tau = 0$, though we separate the windings here visually for clarity. (a) Copies before twists. (b) Application of $\sigma_{(12)}$ at σ_1 , twisting the copies together. (c) Application of $\sigma_{(23)}$ at σ_2 , which twists all three together.

2.6.1 Calculating the Bogoliubov matrices

We begin by writing the positive frequency solutions to the wave equation. For each of the three initial copies, we have:

$$\begin{aligned} h_m^{(i)} &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2m}} e^{im(\sigma-\tau)} \\ h_{\bar{m}}^{(i)} &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\bar{m}}} e^{-i\bar{m}(\sigma+\tau)}, \end{aligned} \quad (2.30)$$

with integer m . For the final copy, we instead have:

$$\begin{aligned} h_s &= \frac{1}{\sqrt{6\pi}} \frac{1}{\sqrt{2m}} e^{is(\sigma-\tau)} \\ h_{\bar{s}} &= \frac{1}{\sqrt{6\pi}} \frac{1}{\sqrt{2\bar{m}}} e^{-is(\sigma+\tau)}, \end{aligned} \quad (2.31)$$

where s is a multiple of $1/3$ since the copy is triply wound. We have dropped the copy index to reduce index clutter. We now identify our domains of overlap, which are diagrammed in figure 2.7.

$$\begin{aligned} (1) \implies \Sigma &= [0, \sigma_1] \cup [4\pi + \sigma_1, 6\pi] \\ (2) \implies \Sigma &= [\sigma_1, \sigma_2] \cup [2\pi + \sigma_2, 4\pi + \sigma_1] \\ (3) \implies \Sigma &= [\sigma_2, 2\pi + \sigma_2], \end{aligned} \quad (2.32)$$

where each line is the listed copy's domain overlap with the unique final copy.

Before we proceed further, let us pause to identify an important point. Note that all of our wavefunctions are identical after shifts of $\sigma \rightarrow \sigma + 6\pi$. The domain overlap for

copy (1) can thus be rewritten as:

$$(1) \implies \Sigma = [4\pi + \sigma_1, 6\pi + \sigma_1] \quad (2.33)$$

It is now clear that copies (1) and (3) both have contiguous domain overlaps while copy (2) does not. This will manifest in significant similarity between all of copy (1) and copy (3) matrix components, while the copy (2) components are significantly different. This is physically expected, as copies (1) and (3) each only see one two-twist operator while copy (2) sees both operators.

We can now proceed with the calculation of α and β for each of our three initial copies. The mathematics behaves similarly to the previous case, so we present only the results. For convenience, we identify a commonly-occurring factor:

$$\mu_s \equiv 1 - e^{4\pi i s}, \quad (2.34)$$

which vanishes for integer s . We won't go over all the calculations, as they are similar to what was shown in the previous case, but we will show the calculation of (2) which will have a different behaviour. So we setup the integrals as before,

$$\begin{aligned} \alpha_{ms}^{(1)(1')} &= -i \left(\int_{\sigma_1}^{\sigma_2} + \int_{\sigma_2+2\pi}^{\sigma_1+4\pi} \right) \frac{1}{4\sqrt{3}\pi\sqrt{ms}} (e^{is(\sigma-\tau_0)} i m e^{-im(\sigma-\tau_0)} - e^{-im(\sigma-\tau)} (-is) e^{is(\sigma-\tau_0)}) d\sigma \\ &= \frac{(m+s)}{4\pi\sqrt{ms}} \left(\int_{\sigma_1}^{\sigma_2} + \int_{\sigma_2+2\pi}^{\sigma_1+4\pi} \right) e^{i(s-m)\sigma} d\sigma. \end{aligned} \quad (2.35)$$

Now if $m = s$,

$$\begin{aligned} \alpha_{ms}^{(1)(1')} &= \frac{1}{4\sqrt{3}\pi s} \frac{1}{s} 2s(\sigma_1 + 4\pi - \sigma_2 - 2\pi + \sigma_2 - \sigma_1) \\ &= \frac{1}{\sqrt{3}} \end{aligned} \quad (2.36)$$

For $m \neq s$, we find

$$\begin{aligned} \alpha_{ms}^{(1)(1')} &= \frac{1}{4\pi\sqrt{ms}} \frac{m+s}{i(s-m)} (e^{i(s-m)\sigma_1} e^{4\pi i s} - e^{i(s-m)\sigma_2} e^{2\pi i s} + e^{i(s-m)\sigma_2} - e^{i(s-m)\sigma_1}) \\ &= \frac{-i}{4\sqrt{3}\pi\sqrt{sm}} \frac{s+m}{s-m} (\mu_{-s} e^{i(s-m)\sigma_2} - \mu_s e^{i(s-m)\sigma_1}) \end{aligned} \quad (2.37)$$

Note that $s \in \mathbb{Z}/3$, so we can convert $1 - e^{2\pi i s} = 1 - e^{6\pi i s - 4\pi i s} = 1 - e^{-4\pi i s} = \mu_{-s}$.

So we can do the α matrices for the (1) and (2) copies and overall have the results.

$$\begin{aligned}\alpha_{ms}^{(1)} &= \begin{cases} \frac{1}{\sqrt{3}}\delta_{ms} & s \in \mathbb{Z} \\ -\frac{i}{4\pi\sqrt{3sm}} \frac{s+m}{s-m} \mu_s e^{i(s-m)\sigma_1} & s \notin \mathbb{Z} \end{cases} \\ \alpha_{ms}^{(2)} &= \begin{cases} \frac{1}{\sqrt{3}}\delta_{ms} & s \in \mathbb{Z} \\ -\frac{i}{4\pi\sqrt{3sm}} \frac{s+m}{s-m} (\mu_{-s} e^{i(s-m)\sigma_2} - \mu_s e^{i(s-m)\sigma_1}) & s \notin \mathbb{Z} \end{cases} \\ \alpha_{ms}^{(3)} &= \begin{cases} \frac{1}{\sqrt{3}}\delta_{ms} & s \in \mathbb{Z} \\ \frac{i}{4\pi\sqrt{3sm}} \frac{s+m}{s-q} \mu_{-s} e^{i(s-m)\sigma_2} & s \notin \mathbb{Z}. \end{cases}\end{aligned}\tag{2.38}$$

The result for the β matrix is:

$$\begin{aligned}\beta_{ms}^{(1)} &= \frac{i}{4\pi\sqrt{3sm}} \frac{s-m}{s+m} \mu_{-s} e^{-i(s+m)\sigma_1} \\ \beta_{ms}^{(2)} &= \frac{i}{4\pi\sqrt{3sm}} \frac{s+m}{s-m} (\mu_s e^{-i(s+m)\sigma_2} - \mu_{-s} e^{-i(s+m)\sigma_1}) \\ \beta_{ms}^{(3)} &= -\frac{i}{4\pi\sqrt{3sm}} \frac{s+m}{s-m} \mu_s e^{-i(s-m)\sigma_2}.\end{aligned}\tag{2.39}$$

As expected from our configuration, the results for copies (1) and (3) are identical to the behaviour of a single two-twist operator, which was solved in general in [208]. It is the behaviour of copy (2) here which is new.

Now we have both the α and β matrices, we turn to obtaining the α^{-1} using our component twist method.

2.6.2 Calculating the transition matrix

While this twist configuration has not been solved before, the behaviour of copies (1) and (3) is the same as a single two-twist configuration since each copy only interacts with one

of our twists. We can thus identify their transition matrices from [208].

$$\begin{aligned} f_{ms}^{(1)} &= \begin{cases} \frac{1}{\sqrt{3}}\delta_{ms} & s \in \mathbb{Z} \\ -\frac{1}{2\pi i \sqrt{3}} \frac{1}{s-m} \mu_{-s} e^{-i(s-m)\sigma_1} & s \notin \mathbb{Z} \end{cases} \\ f_{ms}^{(3)} &= \begin{cases} \frac{1}{\sqrt{3}}\delta_{ms} & s \in \mathbb{Z} \\ \frac{1}{2\pi i \sqrt{3}} \frac{1}{s-m} \mu_s e^{-i(s-m)\sigma_2} & s \notin \mathbb{Z}. \end{cases} \end{aligned} \quad (2.40)$$

We now turn to $f^{(2)}$, which is where we must apply our method of combining transition behaviours for each individual two-twist operator. Noting that the intermediate state is on a doubly-wound CFT, we have:

$$f_{ms}^{(2)} = \sum_{q \in \mathbb{Z}^+/2} f_{mq}^{[(1),(2)]}(\sigma_1) f_{qs}^{[(1+2),(3)]}(\sigma_2), \quad (2.41)$$

where here the copy indices in brackets are used to denote which copies are being twisted together. The expression $(1 + 2)$ refers to the intermediate doubly-wound CFT that results from the application of the twist at σ_1 . All first-order transition matrices are known for two-twist operators², so we simply refer to [208] for their continuum limit approximations.

$$\begin{aligned} f_{mq}^{[(1),(2)]} &= \begin{cases} \frac{1}{\sqrt{2}}\delta_{mq} & q \in \mathbb{Z} \\ \frac{1}{2\pi i \sqrt{2}} \frac{1}{q-m} \mu_q e^{-i(q-m)\sigma_1} & q \notin \mathbb{Z} \end{cases} \\ f_{qs}^{[(1+2),(3)]} &= \begin{cases} \sqrt{\frac{2}{3}}\delta_{qs} & s \in \mathbb{Z} \\ -\frac{1}{2\pi i \sqrt{6}} \frac{1}{s-q} \mu_s e^{-i(s-q)\sigma_2} & s \notin \mathbb{Z}. \end{cases} \end{aligned} \quad (2.42)$$

We now break our computation into two cases.

Case 1: $s \in \mathbb{Z}$

Here $f_{qs}^{[(1+2),(3)]}$ is nonzero only for $q = s$, which means $q \in \mathbb{Z}$. Thus $f_{mq}^{[(1),(2)]}$ is nonzero only for $m = q = s$. This leaves only one nonzero term in the sum.

$$[f_{ms}^{(2)}]_{s \in \mathbb{Z}} = \frac{1}{\sqrt{3}}\delta_{ms}. \quad (2.43)$$

²Of positive $SU(2)_L$ charge

This seems like the correct result for f given the results for the α matrices in the previous section.

Case 2: $s \notin \mathbb{Z}$

Here we cannot have $s = q$ since the allowed values for the two indices only coincide at integers. We can however still have $q = m$. We separate out this term, which is the only nonzero term for integer q . This gives:

$$[f_{qs}^{(2)}]_{s \notin \mathbb{Z}} = \frac{\mu_s}{4\pi\sqrt{3}} \left(\frac{i}{s-m} e^{-i(s-m)\sigma_2} - \frac{1}{\pi} e^{im\sigma_1} e^{-is\sigma_2} \sum_{q \in \mathbb{Z} + \frac{1}{2}, q > 0} \frac{e^{iq(\sigma_2 - \sigma_1)}}{(s-q)(q-m)} \right) \quad (2.44)$$

This does not have a nice analytic result. So, as before, we will tackle this sum by using a large but finite cutoff.

A quick check

Since there is no previous result to compare $f^{(2)}$ to, we instead multiply by $\alpha^{(2)}$ and use a large but finite cutoff to handle the sum. The product quickly reaches the identity thanks to the inverse square scaling of the summand. As with the previous configuration, a cutoff of $\Lambda = 2000$ is sufficient for 0.01% accuracy.

The behaviour of $f_{ms}^{(2)}$ is shown in Figures 2.8 and 2.9. As anticipated by [213], the continuum limit behaviour for $s \neq m$ scales as (for canonical modes):

$$f_{ms}^{(2)} \sim \frac{1}{(m-s)} h_f(m, s, \Delta w), \quad (2.45)$$

where h_f is a function oscillating in Δw with amplitude independent of m and s . This index falloff has been shown to hold for all cases of a single two-twist operator as well as for both second-order cases considered so far. We recommend this form of transition matrices as a first-guess fit for numeric work for any twist configurations in the continuum limit of the D1D5 CFT.

With the transition matrix in hand, we can now use it and the β matrix from the previous subsection to compute the γ in the next subsection.

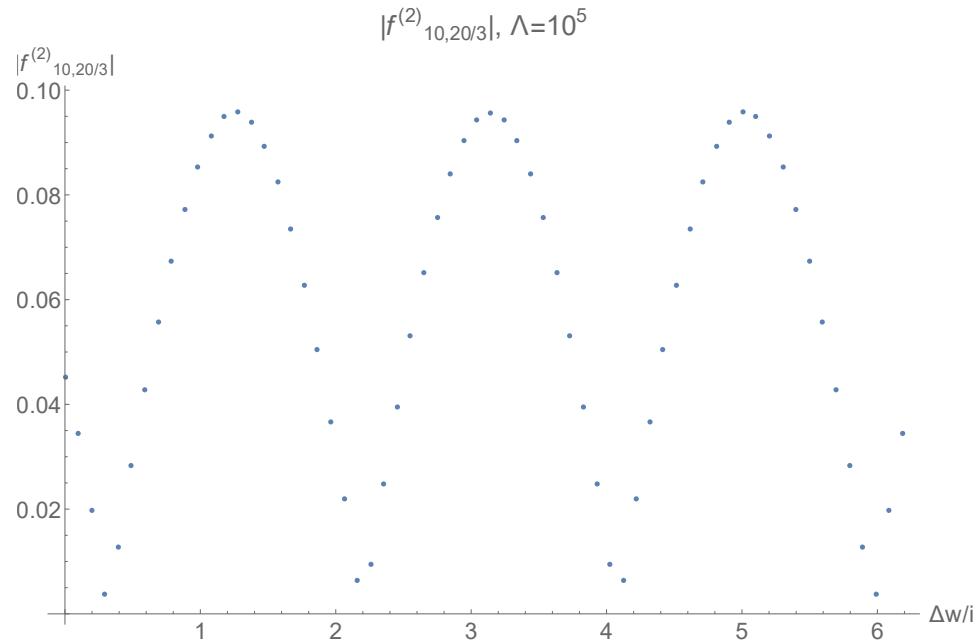


Figure 2.8: Transition amplitude as a function of separation for $f^{(2)}_{10,20/3}$.

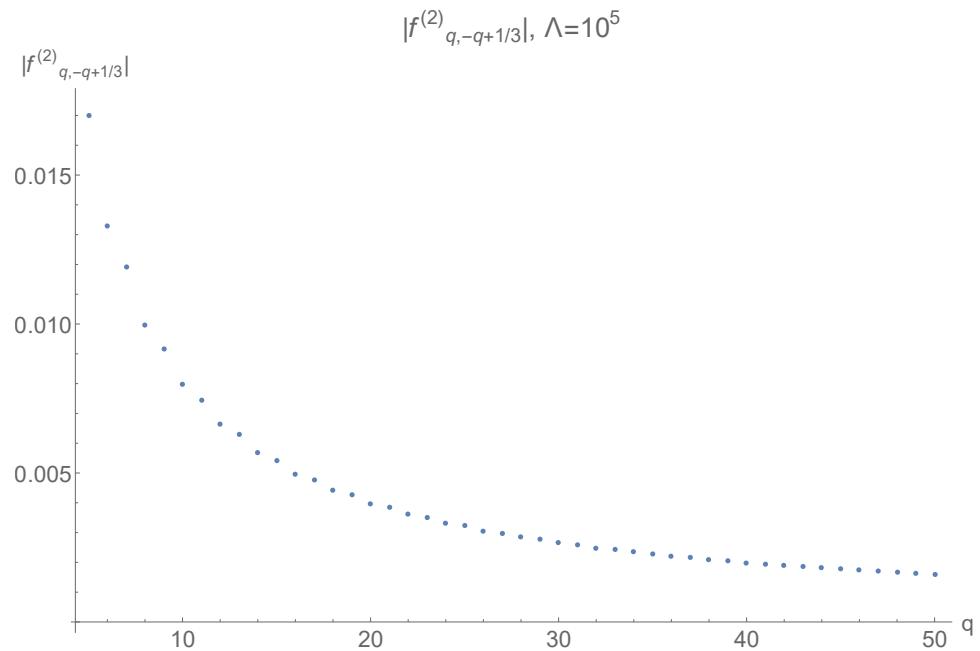


Figure 2.9: Transition amplitude as a function of its index for $f^{(2)}_{q,-q+1/3}$ at $\Delta w = \pi i$.

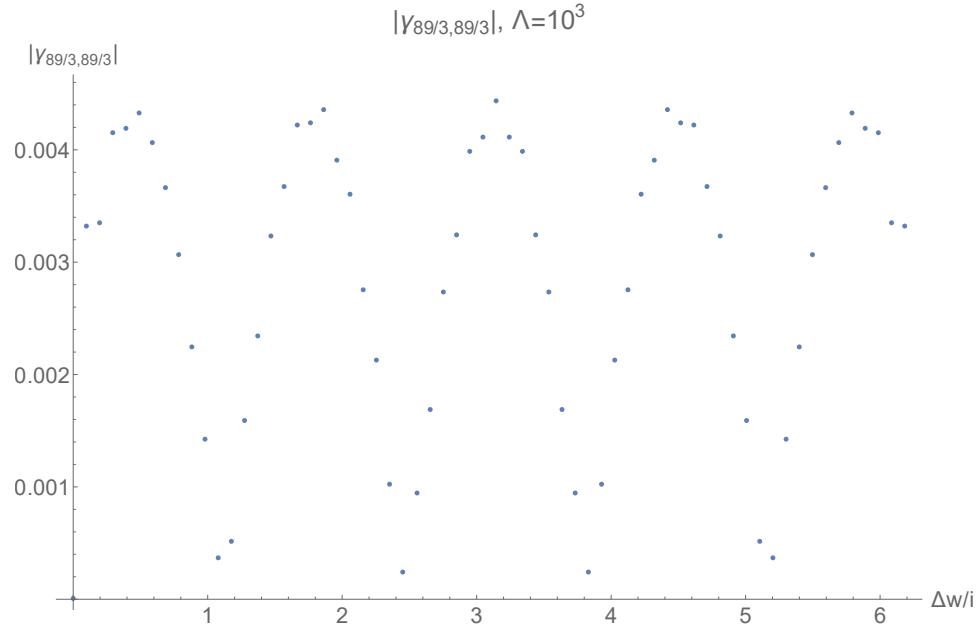


Figure 2.10: Gamma as a function of separation for $\gamma_{89/3,89/3}$.

2.6.3 The squeezed state matrix

To compute the γ matrix, we use the formula introduced earlier and we also must be careful of the sum over copies. All three initial copies contribute.

$$\gamma_{ss'} = [f^T \beta]_{ss'} = \sum_{m>0} \sum_{(i)} f_{ms}^{(i)} \beta_{ms'}^{(i)}. \quad (2.46)$$

There are technically two sums at work here, as a sum is required for the intermediate calculation of $f^{(2)}$. As usual we terminate each sum at some large cutoff. The resulting $\gamma_{ss'}$ behaviour is shown in Figures 2.10 and 2.11. Once again, the anticipated falloff from [213] is a good approximation for the continuum limit.

$$\gamma_{ss'} \sim \frac{1}{s + s'} h_\gamma(s, s', \Delta w), \quad (2.47)$$

where h_γ is a different function oscillating in Δw with amplitude independent of s and s' . We therefore recommend this falloff form for the squeezed state coefficient as a first-guess fit for numeric work with any twist configurations in the continuum limit of the D1D5 CFT.

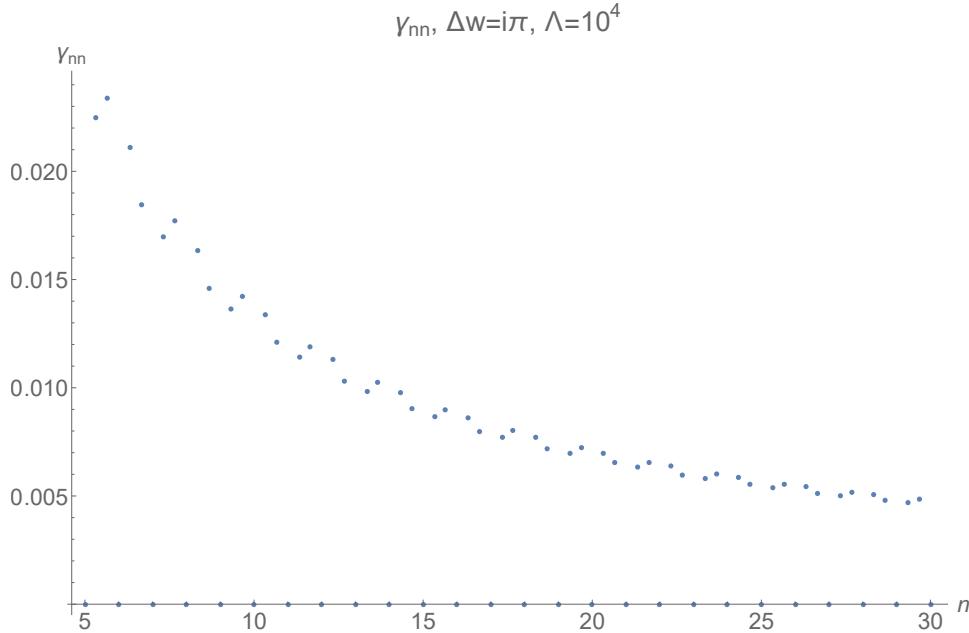


Figure 2.11: Gamma as a function of its index for $\gamma_{s,s}$ at $\Delta w = \pi i$.

2.7 Discussion

Our goal here was to further investigate the effect of twist operators on quantum states in the D1D5 orbifold CFT, in order to bring us closer to the general goal of understanding how thermalization works in the microscopic picture of black holes. We wished to find results in the continuum limit that went beyond the already known twist 2 results found previously. The method we presented here, using component twists, has the advantage of being applicable to higher twist configurations, unlike the Lunin-Mathur method.

We used knowledge of Bogoliubov transformations and the fact that twist two operators $\hat{\sigma}_2$ can be used to build up higher-twist operators $\hat{\sigma}_n$ to find the transition matrices f and squeezed state coefficient matrices γ associated to states created by $\hat{\sigma}_n$. To show our method works, we checked our component twist method by comparing to a previous result calculated with Lunin-Mathur technology.

With our method verified, we explicitly calculated a new configuration and found that the scaling predicted in [213] holds true: we see good agreement with equations (2.45) for the transition matrices f and (2.47) for the squeezed state coefficients γ . We expect that the form of these matrices will hold in the continuum limit for any general twist operator configurations. This conclusion is based on our component twist method, the forms of the twist 2 transition matrices, and simple power counting.

We have seen, using calculations in the D1D5 orbifold CFT, that it is possible to

build up the f matrices from components and to obtain the γ matrices from them. The only part that made our calculations specific to this CFT was the form of the twist two transition amplitudes. Twist operators in other orbifold CFTs might have a similar decomposition.

Twist operators also show up in other orbifold CFTs that are not themselves holographic but are useful for computing quantities of physical interest in holographic CFTs. In particular, cyclic orbifolds show up when using the replica trick for computing entanglement entropy via Renyi entropies in quantum field theories [84, 138, 222–224]. The entanglement entropy of multiple regions is written as a correlation function of multiple pairs of twist operators. Better understanding the effect of applying twist operators in the prototype D1D5 orbifold CFT may assist with learning about information theoretic quantities of interest for quantum gravity.

Chapter 3

Operator mixing from the OPE in deformed D1D5 CFT

3.1 Introduction

In both the introduction and chapter 2, we have discussed the fact that the D1D5 system has a moduli space; the supergravity description and the free orbifold description lie at two different points in it. In the previous chapter, we looked at the continuum limit to study the how to deform the free orbifold CFT towards the supergravity limit. There we only studied the bare twist operator to build up other, more general, twist configurations. With the correct dressing, this would correspond to higher orders in the deformation operator. In this chapter, we will instead focus on a first order calculation but with the full deformation operator.

Working with the D1D5 CFT requires unique techniques to deal with operators in the twisted sector. One of the most important of these is the Lunin-Mathur technique, which we described in section 1.10. Numerous applications have included matching correlators to bulk calculations (e.g. emission rates [36]), entanglement entropies and other related quantities [80, 84], thermalization [204, 209], and information loss [225]. Despite this large body of work, the full description of the D1D5 CFT, especially away from the free orbifold point, is still not fully understood.

One important aspect to the deformed theory is the anomalous dimensions. Studying the anomalous dimension of the operators at the orbifold point in the D1D5 CFT will give us a glimpse at how far away the supergravity description lies in the CFT moduli space. We expect that as we move towards the supergravity point, all operators not protected by supersymmetry will be lifted out of the low lying spectrum to have an effective

supergravity field theory. This is probed perturbatively by the anomalous dimension.

Another interesting body of work that plays a role here is the connection between higher spin theories and the orbifold point of the D1D5 CFT. Higher spin theories are toy models for working with quantum gravity, as they have a tower of massless higher spin fields without a lot of the complications that can arise in full string theory. Higher spin theories are generally forbidden in asymptotically flat spacetimes by well known no-go theorems (see [226] for a review). The general rule of thumb is that there are no interacting Lorentz invariant field theories involving spin greater than 2. In the case of AdS, pertinent to AdS/CFT holography, we can have interacting higher spin theories, known as Vasiliev higher spin theories. See [227–229] for some reviews of these theories in the context of holography.

Relatively recent work has shown that at the D1D5 CFT at the orbifold point has a closed subsector described by Vasiliev higher spin theory [230–232]. The main result of these papers was to show that the orbifold point is the tensionless limit of the string theory. Recall the two coupling constants on the AdS side of the AdS/CFT correspondence, R^2/α' and g_s . The tensionless limit is the limit where we take the coupling constant R^2/α' to zero, or $\alpha' \rightarrow \infty$. This limit would have very long and floppy strings and so higher spin excitation on the strings would be massless. Deformation away from the orbifold point acts like as a Higgs mechanism, giving mass to these higher spin fields. This is important, as the higher spin fields are not in the supergravity description and so these should lift out of the low energy spectrum to have the effective field theory be Type IIB supergravity.

To get closer towards the goal of getting to a supergravity description, we continue to develop the covering space techniques of [206, 219] to learn about conformally perturbing away from the orbifold point. In earlier work [206], Burrington, Peet, and Zadeh considered certain four point functions which contain two copies of our candidate operator and two copies of the deformation operator. Taking a coincidence limit of such a four point function allowed them to read off structure constants and show that at first order in perturbation theory a specific non-protected operator mixed with an unknown operator of the same conformal dimension, and in this case both operators acquire an anomalous dimension. One would have to find the exact form of the operator that mixes with the candidate operator, and iterate this procedure to find all operators that mix in this block. At each stage a four point function could be computed to find if there was a new operator that mixes, and if so, many three point functions would need to be computed to find out the proper combination that mixes. This is a fairly straightforward but extremely laborious technique, as four point functions have functions of cross ratios that are not

constrained.

In this chapter, we take another approach. The covering space techniques focus on the computation of n -point functions, particularly three and four point functions. The three point functions of quasi-primaries give the structure constants. The structure constants are the main building blocks for any CFT, and using bootstrap techniques allows the building of general correlation functions through crossing symmetry. A modern review of the bootstrap program can be found here [233]. There is of course another mainstay of CFT that contains the information about the structure constants: the operator product expansion (OPE). The symmetric orbifold CFT on the covering surface is particularly simple: it is a free theory.

Thus, from previous work, we may say the following:

- If we know what the lift of an operator is to the cover, then we know everything about the operator. We have enough to compute any correlator. Further, if we can find it for one map, then we can reconstruct it for any map by re-expressing the operator on the cover as appropriate modes of fields acting on twists that would lead to that particular operator for that particular map.
- The operator product expansion contains all information about three point functions because the OPE is written in terms of the structure constants.
- The OPE of free theories is simple. The free orbifold theory is a little more complicated, but on the covering surface the difficulties of boundary conditions are removed. The twisted boundary conditions are accounted for in the map from the base space to the covering surface. Thus, the OPE on the cover is simple.

Given these statements, there is a natural question: Does the knowledge of the OPE on the covering surface descend in some way to knowledge of the OPE on the base?

It is the purpose of this chapter to offer evidence that the answer to this question is “yes,” and outline how to do this in the special case where one of the operators is in the untwisted sector. We do this by considering specific operators as examples. We lay out the challenges of general twist-twist OPEs in the Discussion.

The eventual goal of these techniques would be to work out all the mixing operators for a candidate operator and then use conformal perturbation theory to compute the anomalous dimensions of the operators after the deformation. This would illuminate how the free CFT point is connected to the supergravity point in moduli space. Using the OPE on the cover and simply reading off the operator at the next stage of mixing is much more efficient than computing the plethora of three point functions at each stage.

The layout of the chapter is as follows. Section 3.2 contains details about how we examine operator mixing in a CFT through the structure constants. Section 3.3 will

show using the OPE and the lift to the cover that the exactly marginal supergravity operators do not have an anomalous dimension, as expected. We will also show how one can reproduce the leading singularity of the four point function for the dilaton operator and deformation operator with the OPE. Section 3.4 will use the OPE method to find the operator that mixes with a (2,2) untwisted operator and contributes to its anomalous dimension. We will also show how this result can be verified by computing three point functions and confirm these results by reproducing the coefficient for the mixing found in the four point function. We finish with the Discussion, including some directions for future work, in Section 3.5.

3.2 Operator mixing and the structure constants

In this chapter, we are studying mixing of operators with an eye to computing anomalous dimensions. We will use conformal perturbation theory to find these anomalous dimensions. This was discussed in more detail in the series of papers [206, 219]. We will give a lightning review it here. Consider a set of quasi-primary fields, \mathcal{O}_i . Assume, for now, no degeneracy in the conformal weights of an operator. To see the anomalous dimension, one adds a small perturbation to our free SCFT in the form

$$\delta S = \delta S = \lambda \int d^2 z \mathcal{O}_D(z, \bar{z}) + \text{h.c.} \quad (3.1)$$

where λ is the coupling constant and \mathcal{O}_D is the deformation operator. For a two point correlation functions without the perturbation, conformal invariance fixes the form as

$$\langle \mathcal{O}_i(z_1, \bar{z}_1) \mathcal{O}_j(z_2, \bar{z}_2) \rangle_0 = \frac{\delta_{ij}}{z_{12}^{h_i} \bar{z}_{12}^{\bar{h}_i}}. \quad (3.2)$$

With the perturbation to first order, the holographic conformal dimensions change as $h_i(\lambda) = h_i + \lambda \frac{\partial h_i}{\partial \lambda}(\lambda)$ and similarly for the anti-holomorphic conformal dimensions. Substituting into the above gives

$$\frac{\partial}{\partial \lambda} \langle \mathcal{O}_i(z_1, \bar{z}_1) \mathcal{O}_j(z_2, \bar{z}_2) \rangle_\lambda = \left(-2 \frac{\partial h_i(\lambda)}{\partial \lambda} \ln(z_{12}) - 2 \frac{\partial \bar{h}_i(\lambda)}{\partial \lambda} \ln(\bar{z}_{12}) \right) \langle \mathcal{O}_i(z_1, z_2) \mathcal{O}_j(z_2, \bar{z}_2) \rangle_0. \quad (3.3)$$

One may also look at the path integral of the theory. At first order,

$$\langle \mathcal{O}_i(z_1, \bar{z}_1) \mathcal{O}_j(z_2, \bar{z}_2) \rangle_{\lambda_A} = \langle \mathcal{O}_i(z_1, \bar{z}_1) \mathcal{O}_j(z_2, \bar{z}_2) \rangle_0 + \lambda \int d^2 z \langle \mathcal{O}_i(z_1, \bar{z}_1) \mathcal{O}_D(z, \bar{z}) \mathcal{O}_j(z_2, \bar{z}_2) \rangle. \quad (3.4)$$

Regulating the integral and completing it will lead to the condition $h_j - \tilde{h}_j = h_i - \tilde{h}_i$. This leads to a key relation,

$$\frac{\partial}{\partial \lambda} \langle \mathcal{O}_i(z_1, \bar{z}_1) \mathcal{O}_j(z_2, \bar{z}_2) \rangle_{\lambda} = \begin{cases} (2\pi C_{iDi} \ln(z_{12}) + 2\pi C_{iDi} \ln(\bar{z}_{12}) - 2\pi C_{iDi} \ln(\epsilon^2)) \langle \mathcal{O}_i(z_1, \bar{z}_1) \mathcal{O}_i(z_2, \bar{z}_2) \rangle_0, h_i = h_j \\ -2\pi \delta_{s_i, s_j} C_{iDj} \left(\frac{\epsilon^{d_j - d_i}}{d_j - d_i} \frac{1}{z_{12}^{2h_j} \bar{z}_{12}^{2\tilde{h}_j}} + \frac{\epsilon^{d_i - d_j}}{d_i - d_j} \frac{1}{z_{12}^{2h_i} \bar{z}_{12}^{2\tilde{h}_i}} \right), \quad h_i \neq h_j \end{cases} \quad (3.5)$$

where $s_i = h_i - \tilde{h}_i$, $d_i \equiv h_i + \tilde{h}_i$, and C_{iAj} are structure constants of the OPE of the CFT. With an appropriate renormalization

$$\mathcal{O}_i \rightarrow \mathcal{O}_i + \lambda \pi \ln(\epsilon^2) C_{iDi} \mathcal{O}_i + 2\pi \lambda \sum_j \delta_{s_i, s_j} \frac{\epsilon^{d_j - d_i}}{d_j - d_i} C_{iDj} \mathcal{O}_j, \quad (3.6)$$

and comparing to the earlier result leads to the formula for the anomalous dimension at first order in perturbation theory,

$$\frac{\partial h_i}{\partial \lambda} = -\pi C_{iAi}, \quad (3.7)$$

and similarly for the anti-holomorphic sector.

To lift the restriction of no degeneracy, we must diagonalize C_{iDk} in the entire block of fields with same conformal dimension. These mixing operators are what participate in the computations of the anomalous dimension. Other operators with non-vanishing structure constants will only end up affecting the wavefunction renormalization.

So the challenge for finding anomalous dimensions is to compute the structure constants C_{iDk} and the operators \mathcal{O}_k , as was alluded to in the Introduction. To do this through three point functions, we must first consider all operators that would possibly participate in the mixing. First, they must have the correct conformal weights. Then they also must satisfy any selection rules from the symmetries. Then each of these operators \mathcal{O}_k would then need to be put into the three point function $\langle \mathcal{O}_i \mathcal{O}_D \mathcal{O}_k \rangle$ and computed. If this vanishes, then the operator does not mix; if it is non-zero, then it must be included in the block diagonalization. These mixing operators would themselves mix with other operators, \mathcal{O}_l , and these also must be included to complete the diagonalization. This

would continue until all operators have been found.

This process would be tedious to do by hand. As mentioned in Appendix A for a simple (2,2) operator may require up to 287 operators and therefore 287 three point functions to compute. By hand, a single three point function could take a couple of weeks to compute, making this process extremely length. To help speed up the computations, we developed a Mathematica package to help with lifting, determining candidate mixing operators, and computing correlation functions using conformal Wick contractions. We will describe this package in Chapter 4. We ran some of the computations using the SciNet cluster located at the University of Toronto [221]. This code can compute a three point function within a few hours or days.

However, finding the structure constants by brute force is a lengthy process, even with the package as more and more operators enter the mixing. Moreover, there are a lot of operators that would have no mixing. So there is a lot of time spent computing vanishing three point functions. Instead, we can consider using the OPE. Recall the basic definition of the OPE,

$$\mathcal{O}_i(z, \bar{z})\mathcal{O}_D(w, \bar{w}) = \sum_k (z-w)^{h_k-h_i-h_D}(\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_D} C_{iDk} \mathcal{O}_k(w, \bar{w}) \quad (3.8)$$

So if we could compute the OPE of the deformation operator with an operator, we could extract both the full mixing operator and structure constant directly without having to compute a large number of three point functions. If we further specialize to $h_i = h_k$, $\bar{h}_i = \bar{h}_k$, and recalling that $h_D = \bar{h}_D = 1$, then we would only need the one term in the OPE,

$$|z-w|^{-2} C_{iDk} \mathcal{O}_k(w, \bar{w}) \quad (3.9)$$

So instead of computing a multitude of three point functions we can instead just compute the OPE and get the one operator that accounts for the full mixing, \mathcal{O}_k and get the required structure constant as well. One would still need to iterate this procedure to find all mixing operators, but this still cuts down significantly on the computational workload.

Previous work [206, 219] focused on computing the four point functions in order to study mixing of the dilaton operator and a (2,2) untwisted candidate operator. Taking coincidence limits of these 4 point functions would indicate mixing and would determine the full mixing. This limit does not give the mixing operator, but it does provide a useful diagnosis of mixing. The results found in these papers are important for our work here, so we will briefly review the method and results here.

The general idea of this approach is that in the coincident limit, we will access infor-

mation of the structure constants. To start, let us consider the four point function,

$$\langle \mathcal{O}_i(a_1, \bar{a}_1) \mathcal{O}_D(b, \bar{b}) \mathcal{O}_D(0, \bar{0}) \mathcal{O}_i(a_2, \bar{a}_2) \rangle, \quad (3.10)$$

where \mathcal{O}_i is a general operator of weight h and \mathcal{O}_D is the deformation operator. We now take the coincidence limit, $(a_1, \bar{a}_1) \rightarrow (0, 0)$ and $(a_2, \bar{a}_2) \rightarrow (b, \bar{b})$. In this case, can expand out the four point function using the OPE,

$$\langle \left(\sum_k (a_2 - b)^{-h-1+h_k} (\bar{a}_2 - \bar{b})^{-\bar{h}-1+\bar{h}_k} C_{iDk} \mathcal{O}_k \right) \left(\sum_p a_1^{-h-1+h_p} \bar{a}_1^{-\bar{h}-1+\bar{h}_p} C_{iDp} \mathcal{O}_p \right) \rangle. \quad (3.11)$$

We see now how information can be extracted from this coincidence limit. The most singular term of the four point function will correspond to the operator in the OPE with the smallest weight that shows up in the OPE. By definition, this will be a quasi-primary since descendants have higher weights. Then we see that the most singular term would be

$$\langle \left((a_2 - b)^{-h-1+h_1} (\bar{a}_2 - \bar{b})^{-\bar{h}-1+\bar{h}_1} C_{iD1} \mathcal{O}_1 \right) \left(a_1^{-h-1+h_1} \bar{a}_1^{-\bar{h}-1+\bar{h}_1} C_{iD1} \mathcal{O}_1 \right) \rangle, \quad (3.12)$$

$$= \frac{C_{iD1}^2}{(a_2 - b)^{h+1-h_1} (\bar{a}_2 - \bar{b})^{\bar{h}+1-\bar{h}_1} a_1^{h+1-h_1} \bar{a}_1^{\bar{h}+1-\bar{h}_1}}, \quad (3.13)$$

where we have assumed that the lowest weight operator is properly normalized. Higher orders in this expansion will then correspond to quasi-primary operators that show up in the OPE with higher weight and follow similarly. So we would be able to extract the structure constants from this limit. However, these higher orders would also contain contributions from descendants of the lower weight quasi-primaries in the OPE. So these must be removed so that only the contributions from quasi-primaries show up. For example if we wanted to study the order of $(a_2 - b)^{-h-1+h_1+1} (\bar{a}_2 - \bar{b})^{-\bar{h}-1+\bar{h}_1+1} a_1^{-h-1+h_1+1} \bar{a}_1^{-\bar{h}-1+\bar{h}_1+1}$, then we would have contributions from quasi-primaries of weight $(h_1 + 1, \bar{h}_1 + 1)$ plus the contribution from the descendants of weight (h_1, \bar{h}_1) , $(h_1 + 1, \bar{h}_1)$, or $(h_1, \bar{h}_1 + 1)$ operator which are also of weight $(h_1 + 1, \bar{h}_1 + 1)$.

This process was applied to the two operators in [206]. For the dilaton, the results showed that there was no anomalous dimension by showing that the singularity of an operator of weight $(1,1)$ is not present. Then the authors turned to looking at a $(2,2)$ untwisted operator and found that there was a singularity corresponding to a weight $(2,2)$ operator was present. This singularity was not the leading singularity, so they first showed the first singularity was from mixing with the deformation. Then the descendant

of the deformation contributes to the mixing singularity and this was removed.

The usefulness of this method is that it will compute the structure constants directly by computing the four point function. However, it will not give the operator that it will mix into. This must be recovered either through computing three point functions or studying the OPE directly. With these two approaches explained, we turn to showing how these would work with a simpler example in the next section before getting into a much more meaty calculation in Section 3.4.

3.3 Results for supergravity operator

3.3.1 OPE and lack of anomalous dimension

In order to compute structure constants, we must focus in on specific operators. To start off, we take a quick look at how we can show how the SUGRA modes do not have any mixing that contributes to their anomalous dimensions. So consider the OPE,

$$\mathcal{O}_s(z, \bar{z})\mathcal{O}_D(0, 0) \quad (3.14)$$

Where \mathcal{O}_D is the deformation operator and we have the general SUGRA mode,

$$\sum_i \partial X_{\dot{C}C,(i)}(z) \bar{\partial} X_{\dot{D}D,(i)}(\bar{z}) \quad (3.15)$$

where (i) are the copy indices and are summed over. Note we do not sum over the $SU(2)$ indices, and instead keep them general. These correspond to SUGRA modes in the D1D5, as shown in equation (6.16) of [189].

Now we consider lifting this operator. Since we are lifting copies separately, we have to be careful about the patches on the double cover and choices of branches. We are lifting with $z = -t^2$, then we lift with copy (1) having $t_1 = \sqrt{-z}$ and copy (2) in the patch defined by $t_2 = -\sqrt{-z}$. So we can lift the operator as

$$\sum_i \partial X_{\dot{C}C,(i)}(z) \bar{\partial} X_{\dot{D}D,(i)}(\bar{z}) \rightarrow \sum_i \left(\frac{dz}{dt} \right)^{-1}(t_i) \left(\frac{d\bar{z}}{d\bar{t}} \right)^{-1}(\bar{t}_i) \partial X_{\dot{C}C}(t_i) \bar{\partial} X_{\dot{D}D}(\bar{t}_i) \quad (3.16)$$

$$= \sum_i \frac{1}{4t_i \bar{t}_i} \partial X_{\dot{C}C}(t_i) \bar{\partial} X_{\dot{D}D}(\bar{t}_i) \quad (3.17)$$

Then we can use the fact that $\mathcal{O}_D \rightarrow -i\epsilon^{AB} \partial X_{\dot{A}A} \bar{\partial} X_{\dot{B}B} S^{\dot{A}\dot{B}}$. Then we can put these

together and find

$$\sum_i \frac{1}{4t_i \bar{t}_i} \partial X_{\dot{C}C}(t_i) \bar{\partial} X_{\dot{D}D}(\bar{t}_i) (-i \epsilon^{AB} \partial X_{\dot{A}A} \bar{\partial} X_{\dot{B}B} S^{\dot{A}\dot{B}}) \quad (3.18)$$

$$\begin{aligned} &= \sum_i (-\epsilon^{AB} \frac{i}{4t_i \bar{t}_i} (\frac{\epsilon_{\dot{C}\dot{A}} \epsilon_{CA}}{t_i^2} + \partial X_{\dot{C}C}(t_i) \partial X_{\dot{A}A}) (\frac{\epsilon_{\dot{D}\dot{B}} \epsilon_{DB}}{\bar{t}_i^2} + \bar{\partial} X_{\dot{D}D}(\bar{t}_i) \bar{\partial} X_{\dot{B}B}) S^{\dot{A}\dot{B}}) \\ &= \sum_i (-\frac{i}{4} \epsilon^{AB} (\frac{\epsilon_{\dot{C}\dot{A}} \epsilon_{CA}}{t_i^3} + \frac{1}{t_i} \partial X_{\dot{C}C}(t_i) \partial X_{\dot{A}A}) (\frac{\epsilon_{\dot{D}\dot{B}} \epsilon_{DB}}{\bar{t}_j^3} + \frac{1}{\bar{t}_j} \bar{\partial} X_{\dot{D}D}(\bar{t}_j) \bar{\partial} X_{\dot{B}B}) S^{\dot{A}\dot{B}}) \quad (3.19) \end{aligned}$$

We can immediately see that we do not have anything that goes like $z^{-1} \bar{z}^{-1} = t^{-2} \bar{t}^{-2}$. The most singular term, t^{-3} , does not have any Taylor expansion to make it into a t^{-2} and the rest is too non singular to contribute. So then we have no mixing that contributes to anomalous dimensions, as we expect.

We can also see that the OPE is non-zero, so there is still mixing. These contributions will not affect the anomalous dimensions, but will affect the wavefunction renormalization to first order in the deformation. We will show in the next Section how this will work.

3.3.2 OPE and wavefunction mixing

For the moment, let us specialize to the dilaton. The four point function of two dilaton and two deformations was found in [206]. We can now try to reproduce the leading singularity to provide a non-trivial check on the OPE calculation in the previous subsection. The dilaton is found from the general SUGRA operator when there is a simple contraction of the $SU(2)$ indices. Then we have

$$\mathcal{O}_{dil}(z, \bar{z}) = -\epsilon^{\dot{C}\dot{D}} \epsilon^{CD} \sum_i \partial X_{\dot{C}C,(i)}(z) \bar{\partial} X_{\dot{D}D,(i)}(\bar{z}) \quad (3.20)$$

The OPE then becomes

$$\begin{aligned} &\mathcal{O}_{dil}(z, \bar{z}) \mathcal{O}_D(0, 0) \quad (3.21) \\ &\rightarrow \frac{i}{2} \epsilon^{\dot{C}\dot{D}} \epsilon^{CD} \epsilon^{AB} (\frac{\epsilon_{\dot{C}\dot{A}} \epsilon_{CA}}{t^3} + \frac{1}{t} \partial X_{\dot{C}C}(t) \partial X_{\dot{A}A}) (\frac{\epsilon_{\dot{D}\dot{B}} \epsilon_{DB}}{\bar{t}^3} + \frac{1}{\bar{t}} \bar{\partial} X_{\dot{D}D}(\bar{t}) \bar{\partial} X_{\dot{B}B}) S^{\dot{A}\dot{B}} \end{aligned}$$

Note here that the full sum over copies will only involve two copies for any representative twist we might consider, so we end up with a factor of two for any term where we do not have a difference in sign. In general, one should do the Taylor series first then do the summing over images. This would lead so some terms canceling in the end. For our interests, this subtlety will not affect our results. So for now, the negative signs that

come from the differing patches can be ignored, as the holomorphic and antiholomorphic as summed together and so the signs end up canceling and we can write it in terms of a generic t . Continuing, we can expand this out and simplify

$$\begin{aligned} \mathcal{O}_{dil}(z, \bar{z})\mathcal{O}_D(0, 0) & (3.22) \\ \rightarrow \frac{i}{|t|^6} \epsilon_{\dot{A}\dot{B}} S^{\dot{A}\dot{B}} + \frac{i}{2\bar{t}^2|t|^2} \epsilon^{AB} \partial X_{\dot{B}B}(t) \partial X_{\dot{A}A} S^{\dot{A}\dot{B}} + \epsilon^{AB} \frac{i}{2t^2|t|^2} \bar{\partial} X_{\dot{A}A}(\bar{t}) \bar{\partial} X_{\dot{B}B} S^{\dot{A}\dot{B}} \\ + \frac{i}{2|t|^2} \epsilon^{AB} \epsilon^{\dot{C}\dot{D}} \epsilon^{CD} \partial X_{\dot{C}C}(t) \partial X_{\dot{A}A} \bar{\partial} X_{\dot{D}D}(\bar{t}) \bar{\partial} X_{\dot{B}B} S^{\dot{A}\dot{B}} \end{aligned}$$

Now let us consider just the first term,

$$\frac{i}{|t|^6} \epsilon_{\dot{A}\dot{B}} S^{\dot{A}\dot{B}} \quad (3.23)$$

We can write this in terms of a power series in z by using our relationship, $z = -t^2$. We see then that we have the powers $z^{-3/2}\bar{z}^{-3/2}$. If we go back to our original idea of the OPE and the fact that both the deformation and the SUGRA operators are (1,1), then this operator would be a power of $z^{h-1-1} = z^{-3/2}$ which implies $h = 1/2$. So this operator is a (1/2,1/2) on the base. We note this is exactly the weight of the operator that mentioned as creating the leading singularity in equation (3.20) of [206].

Note that we did not explicitly include the contribution from the bare twist using the Lunin-Mathur technique. We want to relate the result of the OPE to something from the base. Since both sides of the OPE are lifted with the same map, they would have the exact same contribution when lifting. So we do not need to explicitly include it to find our results. This will no longer be true in the case of lifting the OPE of operators where both are in the twisted sector. We will include more details about this in the Discussion.

So we seem to have got the correct mixing operator, but it would be nice to check that this is the correct result. For this, we turn to the four point function,

$\langle \mathcal{O}_{dil}(a_1, \bar{a}_1)\mathcal{O}_D(b, \bar{b})\mathcal{O}(0, 0)\mathcal{O}_{dil}(a_2, \bar{a}_2) \rangle$, and consider the coincidence limit $(a_1, \bar{a}_1) \rightarrow (0, 0)$ and $(a_2, \bar{a}_2) \rightarrow (b, \bar{b})$. The leading singularity in the coincidence limit took the form of

$$\frac{2^{-4}}{|a_1|^3|a_2 - b|^3|b|^2} \quad (3.24)$$

This should be recoverable from our OPE. Our coefficient from the OPE is just 1. However, we must normalize the operators involved to get the structure constant correct. The normalization of the dilaton will give a factor of $(4)^{-1/2}$ and the deformation $(8)^{-1/2}$. Then we also need the normalization of the operator itself. Luckily, this is super easy to

calculate. Consider,

$$\langle \epsilon_{\dot{A}\dot{B}} S^{\dot{A}\dot{B}}(1,1) \epsilon_{\dot{C}\dot{D}} S^{\dot{C}\dot{D}}(0,0) \rangle = \epsilon_{\dot{A}\dot{B}} \epsilon_{\dot{C}\dot{D}} \epsilon^{\dot{A}\dot{C}} \epsilon^{\dot{B}\dot{D}} = 2 \quad (3.25)$$

So then we must multiply the operator by $2^{1/2}$ to get it properly normalized. Then we find

$$\frac{i}{|t|^6} \epsilon_{\dot{A}\dot{B}} S^{\dot{A}\dot{B}} = \frac{1}{z^{3/2} \bar{z}^{3/2}} (4)^{-1/2} (8)^{-1/2} 2^{1/2} \mathcal{O}_{\text{leading}} = \frac{2^{-2}}{z^{3/2} \bar{z}^{3/2}} \mathcal{O}_{\text{leading}} \quad (3.26)$$

where $\mathcal{O}_{\text{leading}}$ is the normalized mixing operator. So in the end we find we get the result $C_{dDl} = 2^{-2}$.

To match up with the coefficient of the singularity, we must square this result. The leading singularity in the coincidence limit of the four point function would be the two point function of the normalized $\mathcal{O}_{\text{leading}}$ and together with its coefficient. So we would find that the coefficient of the leading singularity would be $C_{dDl}^2 = 2^{-4}$. This agrees with the coincidence limit of the four point function computed previously.

These results are relatively simple to work out. Now we turn to computing the mixing for a non-SUGRA mode where previous results had indicated there would be a non-trivial anomalous dimension.

3.4 Results for candidate operator

3.4.1 OPE and structure constants

Let us consider our main candidate operator,

$$\mathcal{O}_C(z, \bar{z}) = \epsilon_{AB} \epsilon_{\dot{A}\dot{B}} \epsilon_{EF} \epsilon_{\dot{E}\dot{F}} \sum_k \partial X_{-1,(k)}^{\dot{A}A} \partial X_{-1,(k)}^{\dot{B}B} \bar{\partial} X_{-1,(k)}^{\dot{E}E} \bar{\partial} X_{-1,(k)}^{\dot{F}F}, \quad (3.27)$$

Operators that mix with it must be part of the twist 2 sector, as the three point function with the target operator and the deformation operator involves one twist 2 (from the deformation), the candidate mixing operator must have a complementary twist 2 to get an S_N invariant correlator.

Our main approach is to analyze the OPE themselves, with the hope of finding a more efficient algorithm for computing mixing. However, this approach will hit an immediate problem. We are looking for weight (2,2) quasi-primary operators for the computation of the anomalous dimension. In general, the result of the OPE will include a sum over quasi-primaries and their descendants. So we will need to find a way to project out the descendants. We will discuss a simple procedure to do this in the Appendix section 3.6.

To show how this will help, let us consider doing the OPE of our candidate operator with the deformation operator. We will lift our OPE with the map $z = -t^2$, which will have the correct ramification of the twist 2 operator. Then we are looking at doing

$$\mathcal{O}_c(z, \bar{z})\mathcal{O}_D(0, 0) \rightarrow \sum_{(t_{\pm}, \bar{t}_{\pm})} \mathcal{O}_c(t_{\pm}, \bar{t}_{\pm})\mathcal{O}_D^{(t)}(0, 0) = 2\mathcal{O}_c(t, \bar{t})\mathcal{O}_D^{(t)}(0, 0) \quad (3.28)$$

Note here we have take the sum over images as just a simple multiple of 2. This is because the candidate operator just involves bosonic variables so the two images, which differ only by a sign, affects the odd orders of t and will not matter in the end for our calculations. So to simplify the presentation, we will just skip that complication and give the OPE in terms of a generic t .

Moving forward, we note that lifting with our map the candidate operator lifts to

$$\mathcal{O}_c(t, \bar{t}) = \frac{1}{16t^2\bar{t}^2} \left(\epsilon^{\dot{C}\dot{D}}\epsilon^{CD}\partial X_{\dot{C}C}(t)\partial X_{\dot{D}D}(t) + \frac{1}{t^2} \right) \left(\epsilon^{\dot{E}\dot{F}}\epsilon^{EF}\bar{\partial} X_{\dot{E}E}(\bar{t})\bar{\partial} X_{\dot{F}F}(\bar{t}) + \frac{1}{\bar{t}^2} \right) \quad (3.29)$$

and the deformation operator lifts to

$$\mathcal{O}_D^{(t)}(0, 0) = -i\epsilon^{AB}\partial X_{\dot{A}A}\bar{\partial} X_{\dot{B}B}S^{\dot{A}\dot{B}} \quad (3.30)$$

Note that in [206] the authors had worked out that the deformation operator mixes with the candidate operator. Let's first show that we can reproduce this here. Since the deformation operator is weight $(1, 1)$ and the candidate has $(2, 2)$, we will be looking at order $z^{1-1-2}\bar{z}^{1-1-2} = z^{-2}\bar{z}^{-2}$ on the cover this corresponds to $t^{-4}\bar{t}^{-4}$ So then expanding out the OPE gives us

$$-\frac{i}{16t^2\bar{t}^2}\epsilon^{AB}\epsilon^{\dot{C}\dot{D}}\epsilon^{CD}\epsilon^{\dot{E}\dot{F}}\epsilon^{EF} \left(\frac{1}{t^2}\epsilon_{\dot{D}\dot{A}}\epsilon_{DA}\partial X_{\dot{C}C}(t) + \frac{1}{\bar{t}^2}\epsilon_{\dot{C}\dot{A}}\epsilon_{CA}\bar{\partial} X_{\dot{D}D}(\bar{t}) + \frac{\epsilon_{\dot{C}\dot{D}}\epsilon_{CD}}{4t^2}\partial X_{\dot{A}A} \right) \quad (3.31)$$

$$\begin{aligned} &\times \left(\frac{1}{\bar{t}^2}\epsilon_{\dot{F}\dot{B}}\epsilon_{FB}\bar{\partial} X_{\dot{E}E}(\bar{t}) + \frac{1}{t^2}\epsilon_{\dot{E}\dot{B}}\epsilon_{EB}\bar{\partial} X_{\dot{F}F}(\bar{t}) + \frac{\epsilon_{\dot{E}\dot{F}}\epsilon_{EF}}{4\bar{t}^2}\bar{\partial} X_{\dot{B}B} \right) S^{\dot{A}\dot{B}} \\ &= -\frac{9i}{16t^4\bar{t}^4}\epsilon^{AB}\partial X_{\dot{A}A}\bar{\partial} X_{\dot{B}B}S^{\dot{A}\dot{B}} + \dots \end{aligned} \quad (3.32)$$

Here we have suppressed the other terms in the OPE. We also want to include the normalization 2^{-4} for the candidate operator. So then we have in total

$$\mathcal{O}_c(z, \bar{z})\mathcal{O}_D(0, 0) = 2 \times 2^{-4} \times \frac{9}{2^4 z^2 \bar{z}^2} \mathcal{O}_D(0, 0) + \dots = \frac{9}{2^7 z^2 \bar{z}^2} \mathcal{O}_D(0, 0) + \dots \quad (3.33)$$

This agrees with previous results, where the mixing of the deformation operator was found for the candidate as the leading singularity of the four point function. Recall the first factor of two is from the sum over images.

Next we can focus on looking for the mixing operator contributing to the anomalous dimension and see if the success holds. Again our OPE would yield the results

$$-\frac{i}{16t^2\bar{t}^2}\epsilon^{AB}\left(\frac{2}{t^2}\partial X_{\dot{A}A}(t) + \frac{1}{t^2}\partial X_{\dot{A}A} + \epsilon^{\dot{C}\dot{D}}\epsilon^{CD}\partial X_{\dot{C}C}(t)\partial X_{\dot{D}D}(t)\partial X_{\dot{A}A}\right) \quad (3.34)$$

$$\begin{aligned} &\times \left(\frac{2}{\bar{t}^2}\bar{\partial}X_{\dot{B}B}(t) + \frac{1}{\bar{t}^2}\bar{\partial}X_{\dot{B}B} + \epsilon^{\dot{E}\dot{F}}\epsilon^{EF}\bar{\partial}X_{\dot{E}E}(t)\bar{\partial}X_{\dot{F}F}(t)\bar{\partial}X_{\dot{B}B}\right) S^{\dot{A}\dot{B}} \\ &= \frac{i}{16t^2\bar{t}^2}\epsilon^{AB}(\partial X_{\dot{A}A}\epsilon^{\dot{C}\dot{D}}\epsilon^{CD}\partial X_{\dot{C}C}\partial X_{\dot{D}D} + \partial^3 X_{\dot{A}A}) \\ &\quad \times (\bar{\partial}X_{\dot{B}B}\epsilon^{\dot{E}\dot{F}}\epsilon^{EF}\bar{\partial}X_{\dot{E}E}\bar{\partial}X_{\dot{F}F} + \bar{\partial}^3 X_{\dot{B}B})S^{\dot{A}\dot{B}} + \dots \end{aligned} \quad (3.35)$$

Again we have suppressed the other terms in the OPE. This operator is not a quasi-primary. So we can use our procedure to find the quasi-primary contribution to the OPE at this order.

For the moment let us define some operator \mathcal{A} which lifts on the cover to

$$\begin{aligned} \mathcal{A}_\uparrow &= \epsilon^{AB}(\partial X_{\dot{A}A}\epsilon^{\dot{C}\dot{D}}\epsilon^{CD}\partial X_{\dot{C}C}\partial X_{\dot{D}D} + \partial^3 X_{\dot{A}A})(\bar{\partial}X_{\dot{B}B}\epsilon^{\dot{E}\dot{F}}\epsilon^{EF}\bar{\partial}X_{\dot{E}E}\bar{\partial}X_{\dot{F}F} + \bar{\partial}^3 X_{\dot{B}B})S^{\dot{A}\dot{B}} \\ &= \epsilon^{AB}(\partial X_{\dot{A}A}\epsilon^{\dot{C}\dot{D}}\epsilon^{CD}\partial X_{\dot{C}C}\partial X_{\dot{D}D} + \partial^3 X_{\dot{A}A})\tilde{Q}_{\dot{B}B}S^{\dot{A}\dot{B}} \end{aligned} \quad (3.36)$$

We have suppressed the antiholomorphic parts for simplicity. We want to compute $(1 - L_{-1}L_1)\mathcal{A}$. Let us start with finding

$$L_1\mathcal{A} \rightarrow \quad (3.37)$$

$$\begin{aligned} &\oint \frac{dt}{2\pi i} z(t)^2 \left(\frac{dz}{dt}\right)^{-1} (T(t) - \frac{1}{2}\{z, t\})\epsilon^{AB}(\partial X_{\dot{A}A}\epsilon^{\dot{C}\dot{D}}\epsilon^{CD}\partial X_{\dot{C}C}\partial X_{\dot{D}D} + \partial^3 X_{\dot{A}A})\tilde{Q}_{\dot{B}B}S^{\dot{A}\dot{B}} \\ &= -\frac{1}{2}\oint \frac{dt}{2\pi i} t^3 T(t)\epsilon^{AB}(\partial X_{\dot{A}A}\epsilon^{\dot{C}\dot{D}}\epsilon^{CD}\partial X_{\dot{C}C}\partial X_{\dot{D}D} + \partial^3 X_{\dot{A}A})\tilde{Q}_{\dot{B}B}S^{\dot{A}\dot{B}} \end{aligned} \quad (3.38)$$

First note that the Schwarzian term will integrate to zero regardless, so we dropped this. Working out the OPE and taking the residue, we find

$$L_1\mathcal{A} \rightarrow -6\epsilon^{AB}\partial X_{\dot{A}A}\tilde{Q}_{\dot{B}B}S^{\dot{A}\dot{B}} \quad (3.39)$$

Then we can look at

$$L_{-1}L_1\mathcal{A} \rightarrow \quad (3.40)$$

$$\begin{aligned} & \oint \frac{dt}{2\pi i} \left(\frac{dz}{dt} \right)^{-1} (T(t) - \frac{1}{2}\{z, t\}) (-6\epsilon^{AB} \partial X_{\dot{A}A} \tilde{Q}_{\dot{B}B} S^{\dot{A}\dot{B}}) \\ &= \frac{3}{2} \oint \frac{dt}{2\pi i} t^{-1} T(t) \epsilon^{AB} \partial X_{\dot{A}A} \tilde{Q}_{\dot{B}B} S^{\dot{A}\dot{B}} \end{aligned} \quad (3.41)$$

Again the Schwarzian does not contribute. We can work out the OPE, making sure that Taylor expand the singular terms of the OPE to correctly get the non singular terms. This will give us

$$L_{-1}L_1\mathcal{A} \rightarrow \quad (3.42)$$

$$\frac{3}{2} \epsilon^{AB} \left(\partial X_{\dot{A}A} \epsilon^{\dot{C}\dot{D}} \epsilon^{CD} \partial X_{\dot{C}C} \partial X_{\dot{D}D} + \partial^3 X_{\dot{A}A} + 2\partial X_{\dot{A}A} \partial^2 - \frac{1}{2} \partial X_{\dot{A}A} (\partial\phi^5 - \partial\phi^6)^2 \right) \tilde{Q}_{\dot{B}B} S^{\dot{A}\dot{B}}$$

So then we find

$$(1 - \frac{1}{2}L_{-1}L_1)\mathcal{A} \rightarrow \quad (3.43)$$

$$\frac{1}{8} \epsilon^{AB} \left(2\partial X_{\dot{A}A} \epsilon^{\dot{C}\dot{D}} \epsilon^{CD} \partial X_{\dot{C}C} \partial X_{\dot{D}D} + 2\partial^3 X_{\dot{A}A} - 12\partial X_{\dot{A}A} \partial^2 + 3\partial X_{\dot{A}A} (\partial\phi^5 - \partial\phi^6)^2 \right) \tilde{Q}_{\dot{B}B} S^{\dot{A}\dot{B}}$$

We now repeat the procedure with the antiholomorphic part of the operator as well. Putting this back into our OPE we find

$$\begin{aligned} & \frac{i}{16t^2\bar{t}^2} \frac{1}{8} \epsilon^{AB} \left(2\partial X_{\dot{A}A} \epsilon^{\dot{C}\dot{D}} \epsilon^{CD} \partial X_{\dot{C}C} \partial X_{\dot{D}D} + 2\partial^3 X_{\dot{A}A} - 12\partial X_{\dot{A}A} \partial^2 + 3\partial X_{\dot{A}A} (\partial\phi^5 - \partial\phi^6)^2 \right) \\ & \quad (3.44) \end{aligned}$$

$$\begin{aligned} & \left(2\bar{\partial} X_{\dot{B}B} \epsilon^{\dot{E}\dot{F}} \epsilon^{EF} \bar{\partial} X_{\dot{E}E} \bar{\partial} X_{\dot{F}F} + 2\bar{\partial}^3 X_{\dot{B}B} - 12\bar{\partial} X_{\dot{B}B} \bar{\partial}^2 + 3\bar{\partial} X_{\dot{B}B} (\bar{\partial}\tilde{\phi}^5 - \bar{\partial}\tilde{\phi}^6)^2 \right) S^{\dot{A}\dot{B}} \\ &= \frac{1}{2^{11}t^2\bar{t}^2} \times \\ & \left[2i\epsilon^{AB} \left(2\partial X_{\dot{A}A} \epsilon^{\dot{C}\dot{D}} \epsilon^{CD} \partial X_{\dot{C}C} \partial X_{\dot{D}D} + 2\partial^3 X_{\dot{A}A} - 12\partial X_{\dot{A}A} \partial^2 + 3\partial X_{\dot{A}A} (\partial\phi^5 - \partial\phi^6)^2 \right) \right. \\ & \left. \left(2\bar{\partial} X_{\dot{B}B} \epsilon^{\dot{E}\dot{F}} \epsilon^{EF} \bar{\partial} X_{\dot{E}E} \bar{\partial} X_{\dot{F}F} + 2\bar{\partial}^3 X_{\dot{B}B} - 12\bar{\partial} X_{\dot{B}B} \bar{\partial}^2 + 3\bar{\partial} X_{\dot{B}B} (\bar{\partial}\tilde{\phi}^5 - \bar{\partial}\tilde{\phi}^6)^2 \right) S^{\dot{A}\dot{B}} \right] \end{aligned} \quad (3.45)$$

The operator in the brackets is the mixing operator we denote as \mathcal{O}_{n2A} . Now we can include the normalizations to find the structure constant. The candidate operator has

the normalization $(2^8)^{-1/2} = 2^{-4}$, the deformation $(2^3)^{-1/2} = 2^{-3/2}$. The normalization of the mixing operator is $3 \times 2^{19/2}$, which was found by computing the two point function of \mathcal{O}_{n2A} using our code. Together, the contributions work out to an extra factor of 3×2^4 . We find that

$$\mathcal{O}_c(z, \bar{z})\mathcal{O}_D(0, 0) = 2 \times (3 \times 2^4) \times \frac{1}{2^{11} z \bar{z}} \mathcal{O}_{n2A}(0, 0) + \dots = \frac{3}{2^6 z \bar{z}} \mathcal{O}_{n2A}(0, 0) + \dots \quad (3.46)$$

This result agrees with what we find by computing three point functions. In the next Section, we will discuss the details of the results of the three point functions worked out with our Mathematica package.

3.4.2 Three point functions

We undertook a search of quasi-primaries of weight (2,2) in the twist 2 sector for the three point function computations. The operators that will contribute to the anomalous dimension of our target operator are the quasi-primaries of weight (2,2) and singlets under the $SU(2)$ s. This involves determining the effect of L_0, \bar{L}_0 , to find the weight and then the effect of L_1, \bar{L}_1 to see if the operator of interest would vanish as is expected for quasi-primaries. Of course, it is easier to check this on the cover, so we lift the computations to the cover where we would have no twists. Even with these constraints, there are still 34 operators that might participate in mixing. Doing this by hand would be very tedious, so we developed a Mathematica code to handle both the checking of operators being quasi-primary and the correlation functions. The length of calculations becomes considerably greater as we iterate the anomalous dimension procedure, and we will expand on this point in the Discussion.

From here we determine their three point functions to find our mixing operators. Not all of the operators that satisfy the restrictions will end up participate in mixing. In the

end, we found that 9 operators contribute to mixing.

$$\begin{aligned} \mathcal{O}_1 = & \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{\dot{B}\dot{C}} \epsilon_{BC} \epsilon_{\dot{F}\dot{G}} \epsilon_{FG} \\ & \partial X_{-1/2}^{\dot{B}B} \partial X_{-1/2}^{\dot{C}C} \partial X_{-1/2}^{\dot{A}A} \psi_0^{\alpha\dot{D}} \bar{\partial} X_{-1/2}^{\dot{F}F} \bar{\partial} X_{-1/2}^{\dot{G}G} \bar{\partial} X_{-1/2}^{\dot{E}E} \tilde{\psi}_0^{\dot{\alpha}\dot{H}} \sigma_2^{\beta\dot{\beta}} \end{aligned} \quad (3.47)$$

$$\mathcal{O}_2 = \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{\dot{B}\dot{C}} \epsilon_{BC} \partial X_{-1/2}^{\dot{B}B} \partial X_{-1/2}^{\dot{C}C} \partial X_{-1/2}^{\dot{A}A} \psi_0^{\alpha\dot{D}} \bar{\partial} X_{-3/2}^{\dot{E}E} \tilde{\psi}_0^{\dot{\alpha}\dot{H}} \sigma_2^{\beta\dot{\beta}} \quad (3.48)$$

$$\mathcal{O}_3 = \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{\dot{B}\dot{C}} \epsilon_{BC} \partial X_{-1/2}^{\dot{B}B} \partial X_{-1/2}^{\dot{C}C} \partial X_{-1/2}^{\dot{A}A} \psi_0^{\alpha\dot{D}} \bar{\partial} X_{-1/2}^{\dot{E}E} \tilde{\psi}_{-1}^{\dot{\alpha}\dot{H}} \sigma_2^{\beta\dot{\beta}} \quad (3.49)$$

$$\mathcal{O}_4 = \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{\dot{F}\dot{G}} \epsilon_{FG} \partial X_{-3/2}^{\dot{A}A} \psi_0^{\alpha\dot{D}} \bar{\partial} X_{-1/2}^{\dot{F}F} \bar{\partial} X_{-1/2}^{\dot{G}G} \bar{\partial} X_{-1/2}^{\dot{E}E} \tilde{\psi}_0^{\dot{\alpha}\dot{H}} \sigma_2^{\beta\dot{\beta}} \quad (3.50)$$

$$\mathcal{O}_5 = \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \partial X_{-3/2}^{\dot{A}A} \psi_0^{\alpha\dot{D}} \bar{\partial} X_{-3/2}^{\dot{E}E} \tilde{\psi}_0^{\dot{\alpha}\dot{H}} \sigma_2^{\beta\dot{\beta}} \quad (3.51)$$

$$\mathcal{O}_6 = \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \partial X_{-3/2}^{\dot{A}A} \psi_0^{\alpha\dot{D}} \bar{\partial} X_{-1/2}^{\dot{E}E} \tilde{\psi}_{-1}^{\dot{\alpha}\dot{H}} \sigma_2^{\beta\dot{\beta}} \quad (3.52)$$

$$\mathcal{O}_7 = \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{\dot{F}\dot{G}} \epsilon_{FG} \partial X_{-1/2}^{\dot{A}A} \psi_{-1}^{\alpha\dot{D}} \bar{\partial} X_{-1/2}^{\dot{F}F} \bar{\partial} X_{-1/2}^{\dot{G}G} \bar{\partial} X_{-1/2}^{\dot{E}E} \tilde{\psi}_0^{\dot{\alpha}\dot{H}} \sigma_2^{\beta\dot{\beta}} \quad (3.53)$$

$$\mathcal{O}_8 = \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \partial X_{-1/2}^{\dot{A}A} \psi_{-1}^{\alpha\dot{D}} \bar{\partial} X_{-3/2}^{\dot{E}E} \tilde{\psi}_0^{\dot{\alpha}\dot{H}} \sigma_2^{\beta\dot{\beta}} \quad (3.54)$$

$$\mathcal{O}_9 = \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \partial X_{-1/2}^{\dot{A}A} \psi_{-1}^{\alpha\dot{D}} \bar{\partial} X_{-1/2}^{\dot{E}E} \tilde{\psi}_{-1}^{\dot{\alpha}\dot{H}} \sigma_2^{\beta\dot{\beta}} \quad (3.55)$$

However, these are not quasi-primaries on their own. So we take a linear combination of them, $\sum_i a_i \mathcal{O}_i$. Applying the L_1 operator to the linear combination and setting the equation to zero lead the equations $6a_1 + 3a_4 + a_7 = 0$, $6a_2 + 3a_5 + a_8 = 0$, and $6a_3 + 3a_6 + a_9 = 0$. Applying the \bar{L}_1 operator to the linear combination and setting the equation to zero lead the equations $6a_1 + 3a_2 + a_3 = 0$, $6a_4 + 3a_5 + a_6 = 0$, and $6a_7 + 3a_8 + a_9 = 0$. These equations will lead to four independent solutions. We originally considered the combination,

$$\begin{aligned} \mathcal{O}_{11} = & \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} (\epsilon_{\dot{B}\dot{C}} \epsilon_{BC} \partial X_{-1/2}^{\dot{B}B} \partial X_{-1/2}^{\dot{C}C} \partial X_{-1/2}^{\dot{A}A} \psi_0^{\alpha\dot{D}} - 6 \partial X_{-1/2}^{\dot{A}A} \psi_{-1}^{\alpha\dot{D}}) \\ & (\epsilon_{\dot{F}\dot{G}} \epsilon_{FG} \bar{\partial} X_{-1/2}^{\dot{F}F} \bar{\partial} X_{-1/2}^{\dot{G}G} \bar{\partial} X_{-1/2}^{\dot{E}E} \tilde{\psi}_0^{\dot{\alpha}\dot{H}} - 6 \bar{\partial} X_{-1/2}^{\dot{E}E} \tilde{\psi}_{-1}^{\dot{\alpha}\dot{H}}) \sigma_2^{\beta\dot{\beta}} \end{aligned} \quad (3.56)$$

$$\begin{aligned} \mathcal{O}_{12} = & \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} (\epsilon_{\dot{B}\dot{C}} \epsilon_{BC} \partial X_{-1/2}^{\dot{B}B} \partial X_{-1/2}^{\dot{C}C} \partial X_{-1/2}^{\dot{A}A} \psi_0^{\alpha\dot{D}} - 6 \partial X_{-1/2}^{\dot{A}A} \psi_{-1}^{\alpha\dot{D}}) \\ & (\bar{\partial} X_{-3/2}^{\dot{E}E} \tilde{\psi}_0^{\dot{\alpha}\dot{H}} - 3 \bar{\partial} X_{-1/2}^{\dot{E}E} \tilde{\psi}_{-1}^{\dot{\alpha}\dot{H}}) \sigma_2^{\beta\dot{\beta}} \end{aligned} \quad (3.57)$$

$$\begin{aligned} \mathcal{O}_{21} = & \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} (\partial X_{-3/2}^{\dot{A}A} \psi_0^{\alpha\dot{D}} - 3 \partial X_{-1/2}^{\dot{A}A} \psi_{-1}^{\alpha\dot{D}}) \\ & (\epsilon_{\dot{F}\dot{G}} \epsilon_{FG} \bar{\partial} X_{-1/2}^{\dot{F}F} \bar{\partial} X_{-1/2}^{\dot{G}G} \bar{\partial} X_{-1/2}^{\dot{E}E} \tilde{\psi}_0^{\dot{\alpha}\dot{H}} - 6 \bar{\partial} X_{-1/2}^{\dot{E}E} \tilde{\psi}_{-1}^{\dot{\alpha}\dot{H}}) \sigma_2^{\beta\dot{\beta}} \end{aligned} \quad (3.58)$$

$$\mathcal{O}_{22} = \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} (\partial X_{-3/2}^{\dot{A}A} \psi_0^{\alpha\dot{D}} - 3 \partial X_{-1/2}^{\dot{A}A} \psi_{-1}^{\alpha\dot{D}}) (\bar{\partial} X_{-3/2}^{\dot{E}E} \tilde{\psi}_0^{\dot{\alpha}\dot{H}} - 3 \bar{\partial} X_{-1/2}^{\dot{E}E} \tilde{\psi}_{-1}^{\dot{\alpha}\dot{H}}) \sigma_2^{\beta\dot{\beta}} \quad (3.59)$$

Now we can consider the operator $\mathcal{O} = c_{11} \mathcal{O}_{11} + c_{12} \mathcal{O}_{12} + c_{21} \mathcal{O}_{21} + c_{22} \mathcal{O}_{22}$. We wish to use this operator in the three point function. To compute the three point function,

$\langle \mathcal{O}_C(a_1, \bar{a}_1) \mathcal{O}_d(b, \bar{b}) \mathcal{O}(0, 0) \rangle$, we will lift to the covering space with the map

$$z(t) = \frac{bt^2}{2t - 1} \quad (3.60)$$

This map has the correct ramifications for twist 2 operators around $z = b, t = 1$ and $z = 0, t = 0$. Lifting to the cover and computing the three point function with the Lunin-Mathur technique, we find

$$\langle \mathcal{O}_C(a_1, \bar{a}_1) \mathcal{O}_D(b, \bar{b}) \mathcal{O}(0, 0) \rangle = \frac{288(4c_{11} + 2c_{12} + 2c_{21} + c_{22})}{a_1^3(a_1 - b)b\bar{a}_1^3(\bar{a}_1 - \bar{b})\bar{b}}. \quad (3.61)$$

This result includes the contributions from the bare twists. However, our original set of operators turn out to not be orthogonal. The orthogonal combinations can be obtained by the usual Gram-Schmidt procedure. We can work out the general two point function of the operators, \mathcal{O} and $\tilde{\mathcal{O}} = b_{11}\mathcal{O}_{11} + b_{12}\mathcal{O}_{12} + b_{21}\mathcal{O}_{21} + b_{22}\mathcal{O}_{22}$. Computing this can be done with the same map in equation (3.60), where we simply set $b = 1$. This will give us,

$$\begin{aligned} \langle \mathcal{O}(1, 1) \tilde{\mathcal{O}}(0, 0) \rangle &= 460800c_{11}b_{11} + 115200c_{21}b_{21} + 115200c_{12}b_{12} + 28800c_{22}b_{22} \quad (3.62) \\ &+ 138240(c_{11}b_{21} + b_{11}c_{21} + b_{12}c_{11} + c_{11}b_{12}) + 41472(c_{11}b_{22} + c_{22}b_{11} + c_{12}b_{21} + c_{21}b_{12}) \\ &+ 34560(c_{22}b_{21} + b_{22}c_{21} + b_{12}c_{22} + c_{22}b_{12}) \end{aligned}$$

Using this two point function as the inner product for Gram-Schmidt will give the results,

$$\tilde{\mathcal{O}}_{11} = \mathcal{O}_{11} \quad (3.63)$$

$$\tilde{\mathcal{O}}_{12} = \mathcal{O}_{12} - \frac{3}{10}\mathcal{O}_{11} \quad (3.64)$$

$$\tilde{\mathcal{O}}_{21} = \mathcal{O}_{21} - \frac{3}{10}\mathcal{O}_{11} \quad (3.65)$$

$$\tilde{\mathcal{O}}_{22} = \mathcal{O}_{22} - \frac{3}{10}\mathcal{O}_{21} - \frac{3}{10}\mathcal{O}_{12} + \frac{9}{100}\mathcal{O}_{11}. \quad (3.66)$$

We still need to normalize this by the square root of the two point function of the mixing operator, $\langle \mathcal{O}(1, 1) \mathcal{O}(0, 0) \rangle$. We then can use our results in equation (3.62) to find,

$$\begin{aligned} \langle \mathcal{O}(1, 1) \mathcal{O}(0, 0) \rangle &= 460800c_{11}^2 + 276480c_{11}c_{21} + 276480c_{11}c_{12} + 115200c_{21}^2 \quad (3.67) \\ &+ 82944c_{11}c_{22} + 82944c_{21}c_{12} + 69120c_{21}c_{22} + 69120c_{12}c_{22} + 115200c_{12}^2 + 28800c_{22}^2. \end{aligned}$$

Using equation (3.67) to normalize our mixing operator together with the normalizations of the two point functions of the deformation, $\langle \mathcal{O}_D(1, 1)\mathcal{O}_D(0, 0) \rangle = 8$ and the candidate, $\langle \mathcal{O}_C(1, 1)\mathcal{O}_C(0, 0) \rangle = 2^8$, will lead to the four results,

$$\langle \mathcal{O}_C(a_1, \bar{a}_1)\mathcal{O}_D(b, \bar{b})\tilde{\mathcal{O}}_{11}(0, 0) \rangle = \frac{3}{80a_1^3\bar{a}_1^3(a_1 - b)(\bar{a}_1 - \bar{b})b\bar{b}} \quad (3.68)$$

$$\langle \mathcal{O}_C(a_1, \bar{a}_1)\mathcal{O}_D(b, \bar{b})\tilde{\mathcal{O}}_{12}(0, 0) \rangle = \frac{3}{160a_1^3\bar{a}_1^3(a_1 - b)(\bar{a}_1 - \bar{b})b\bar{b}} \quad (3.69)$$

$$\langle \mathcal{O}_C(a_1, \bar{a}_1)\mathcal{O}_D(b, \bar{b})\tilde{\mathcal{O}}_{21}(0, 0) \rangle = \frac{3}{160a_1^3\bar{a}_1^3(a_1 - b)(\bar{a}_1 - \bar{b})b\bar{b}} \quad (3.70)$$

$$\langle \mathcal{O}_C(a_1, \bar{a}_1)\mathcal{O}_D(b, \bar{b})\tilde{\mathcal{O}}_{22}(0, 0) \rangle = \frac{3}{320a_1^3\bar{a}_1^3(a_1 - b)(\bar{a}_1 - \bar{b})b\bar{b}}. \quad (3.71)$$

We can simplify our expression by changing our basis of operators here to have only one operator that accounts for the full mixing and we can also normalize it. This operator that captures the mixing is given by

$$\mathcal{O}_{n2A} = (\mathcal{O}_{11} + 2\mathcal{O}_{12} + 2\mathcal{O}_{21} + 4\mathcal{O}_{22})/(1536\sqrt{2}). \quad (3.72)$$

The structure constant for this operator is

$$C_{iD(n2A)} = \frac{3}{64}. \quad (3.73)$$

This structure constant agrees with the result obtained from the OPE.

Let us now show that the operator found by the procedure agrees as well. If we write out \mathcal{O}_A in the original nine operators, we find the linear combination (up to normalization),

$$\mathcal{O}_A = \mathcal{O}_1 + 2\mathcal{O}_2 - 12\mathcal{O}_3 + 2\mathcal{O}_4 + 4\mathcal{O}_5 - 24\mathcal{O}_6 - 12\mathcal{O}_7 - 24\mathcal{O}_8 + 144\mathcal{O}_9. \quad (3.74)$$

Now we can lift the operators to the cover. For the correlation functions, we use the map $z(t) = bt^2/(2t - 1)$ to the cover. However, we want to compare to results in section 3.4.1, we will need to use a map of the form $z = -t^2$.

Let us look at doing this lift. For the bosonic terms, this is quite easy. They are

$$\partial X_{-n/2}^{\dot{A}A} \rightarrow \oint \frac{dt}{2\pi i} z(t)^{-n/2} \partial X^{\dot{A}A}, \quad (3.75)$$

$$= \oint \frac{dt}{2\pi i} (-1)^{-n/2} t^{-n} \partial X^{\dot{A}A}, \quad (3.76)$$

$$= (-1)^{-n/2} \partial^{n+1} X^{\dot{A}A} / n!, \quad (3.77)$$

and we will also have

$$\partial X_{-1/2}^{\dot{A}A} \partial X_{-1/2}^{\dot{B}B} \partial X_{-1/2}^{\dot{C}C} \rightarrow (-1)^{-3/2} \partial X^{\dot{A}A} \partial X^{\dot{B}B} \partial X^{\dot{C}C}. \quad (3.78)$$

Here terms generated by the OPE between the ∂X s does not contribute with this map, but will for when we lift for the correlators.

With this, we can consider the fermions. This is a lot more difficult to do, as the bosonization and cocycles can be quite complicated. Still, we forge ahead and find

$$\psi_{-n/2}^{\alpha\dot{A}} \psi_{-m/2}^{\dot{\alpha}\dot{B}} \sigma^{\beta\dot{\beta}} \quad (3.79)$$

$$\begin{aligned} & \rightarrow \oint \frac{dt}{2\pi i} \oint \frac{d\bar{t}}{2\pi i} z(t)^{-(n+1)/2} \bar{z}(\bar{t})^{-(m+1)/2} \left(\frac{dz}{dt} \right)^{1/2} \left(\frac{d\bar{z}}{d\bar{t}} \right)^{1/2} \psi^{\alpha\dot{A}}(t) \psi^{\dot{\alpha}\dot{B}}(\bar{t}) S^{\beta\dot{\beta}}, \\ & = 2(-1)^{-(n+m)/2} \oint \frac{dt}{2\pi i} \oint \frac{d\bar{t}}{2\pi i} t^{-(n+1)+1/2} \bar{t}^{-(m+1)+1/2} \psi^{\alpha\dot{A}}(t) \psi^{\dot{\alpha}\dot{B}}(\bar{t}) S^{\beta\dot{\beta}}. \end{aligned} \quad (3.80)$$

Even with the derivatives, we have to keep in mind that the derivatives that would come from the residues would act just on the fermions. Because of this, it is difficult to write down a compact notation with the cocycles explicit. Note we would get a $t^{-1/2} \bar{t}^{-1/2}$ from the OPE, so then the integral is well defined.

To be able to present this in a managed way, we ran the code and had it expand the results for the lifting. We then wrote down the compact notation we believed it would correspond to and had the code expand that and compare the two expressions. From

this, we have

$$\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\psi_0^{\alpha\dot{A}}\psi_0^{\dot{\alpha}\dot{B}}\sigma^{\beta\dot{\beta}} \rightarrow -8iS^{\dot{A}\dot{B}}, \quad (3.81)$$

$$\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\psi_{-1/2}^{\alpha\dot{A}}\psi_0^{\dot{\alpha}\dot{B}}\sigma^{\beta\dot{\beta}} \rightarrow -8\partial S^{\dot{A}\dot{B}}, \quad (3.82)$$

$$\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\psi_{-1}^{\alpha\dot{A}}\psi_0^{\dot{\alpha}\dot{B}}\sigma^{\beta\dot{\beta}} \rightarrow 4i(\partial^2 - (\partial\phi_5 - \partial\phi_6)^2/4)S^{\dot{A}\dot{B}}, \quad (3.83)$$

$$\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\psi_0^{\alpha\dot{A}}\psi_{-1/2}^{\dot{\alpha}\dot{B}}\sigma^{\beta\dot{\beta}} \rightarrow -8\bar{\partial} S^{\dot{A}\dot{B}}, \quad (3.84)$$

$$\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\psi_{-1/2}^{\alpha\dot{A}}\psi_{-1/2}^{\dot{\alpha}\dot{B}}\sigma^{\beta\dot{\beta}} \rightarrow 8i\partial\bar{\partial} S^{\dot{A}\dot{B}}, \quad (3.85)$$

$$\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\psi_{-1}^{\alpha\dot{A}}\psi_{-1/2}^{\dot{\alpha}\dot{B}}\sigma^{\beta\dot{\beta}} \rightarrow 4(\partial^2 - (\partial\phi_5 - \partial\phi_6)^2/4)\bar{\partial} S^{\dot{A}\dot{B}}, \quad (3.86)$$

$$\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\psi_0^{\alpha\dot{A}}\psi_{-1}^{\dot{\alpha}\dot{B}}\sigma^{\beta\dot{\beta}} \rightarrow 4i(\bar{\partial}^2 - (\bar{\partial}\tilde{\phi}_5 - \bar{\partial}\tilde{\phi}_6)^2/4)S^{\dot{A}\dot{B}}, \quad (3.87)$$

$$\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\psi_{-1/2}^{\alpha\dot{A}}\psi_{-1}^{\dot{\alpha}\dot{B}}\sigma^{\beta\dot{\beta}} \rightarrow 4\partial(\bar{\partial}^2 - (\bar{\partial}\tilde{\phi}_5 - \bar{\partial}\tilde{\phi}_6)^2/4)S^{\dot{A}\dot{B}}, \quad (3.88)$$

$$\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\psi_{-1}^{\alpha\dot{A}}\psi_{-1}^{\dot{\alpha}\dot{B}}\sigma^{\beta\dot{\beta}} \rightarrow -2i(\partial^2 - (\partial\phi_5 - \partial\phi_6)^2/4)(\bar{\partial}^2 - (\bar{\partial}\tilde{\phi}_5 - \bar{\partial}\tilde{\phi}_6)^2/4)S^{\dot{A}\dot{B}}. \quad (3.89)$$

The phases and expressions could also be found with some work using the Gamma matrices from [201].

With all these pieces, we can look at the nine operators we had listed at the beginning. So lifting with the map $z = -t^2$, we get expressions on the cover like

$$\mathcal{O}_1 = 8i\epsilon^{AB}(\partial X_{\dot{A}A}\epsilon^{\dot{C}\dot{D}}\epsilon^{CD}\partial X_{\dot{C}C}\partial X_{\dot{D}D})(\bar{\partial} X_{\dot{B}B}\epsilon^{\dot{E}\dot{F}}\epsilon^{EF}\bar{\partial} X_{\dot{E}E}\bar{\partial} X_{\dot{F}F})S^{\dot{A}\dot{B}}, \quad (3.90)$$

$$\mathcal{O}_2 = 4i\epsilon^{AB}(\partial X_{\dot{A}A}\epsilon^{\dot{C}\dot{D}}\epsilon^{CD}\partial X_{\dot{C}C}\partial X_{\dot{D}D})(\bar{\partial}^3 X_{\dot{B}B})S^{\dot{A}\dot{B}}, \quad (3.91)$$

$$\mathcal{O}_3 = 4i\epsilon^{AB}(\partial X_{\dot{A}A}\epsilon^{\dot{C}\dot{D}}\epsilon^{CD}\partial X_{\dot{C}C}\partial X_{\dot{D}D})(\bar{\partial} X_{\dot{B}B}\bar{\partial}^2 - \bar{\partial} X_{\dot{B}B}(\bar{\partial}\tilde{\phi}_5 - \bar{\partial}\tilde{\phi}_6)^2/4)S^{\dot{A}\dot{B}}, \quad (3.92)$$

$$\mathcal{O}_4 = 4i\epsilon^{AB}(\bar{\partial}^3 X_{\dot{A}A})(\bar{\partial} X_{\dot{B}B}\epsilon^{\dot{E}\dot{F}}\epsilon^{EF}\bar{\partial} X_{\dot{E}E}\bar{\partial} X_{\dot{F}F})S^{\dot{A}\dot{B}}, \quad (3.93)$$

$$\mathcal{O}_5 = 2i\epsilon^{AB}(\bar{\partial}^3 X_{\dot{A}A})(\bar{\partial}^3 X_{\dot{B}B})S^{\dot{A}\dot{B}} \quad (3.94)$$

$$\mathcal{O}_6 = 2i\epsilon^{AB}(\bar{\partial}^3 X_{\dot{A}A})(\bar{\partial} X_{\dot{B}B}\bar{\partial}^2 - \bar{\partial} X_{\dot{B}B}(\bar{\partial}\tilde{\phi}_5 - \bar{\partial}\tilde{\phi}_6)^2/4)S^{\dot{A}\dot{B}}, \quad (3.95)$$

$$\mathcal{O}_7 = 4i\epsilon^{AB}(\partial X_{\dot{A}A}\partial^2 - \partial X_{\dot{A}A}(\partial\phi_5 - \partial\phi_6)^2/4)(\bar{\partial} X_{\dot{B}B}\epsilon^{\dot{E}\dot{F}}\epsilon^{EF}\bar{\partial} X_{\dot{E}E}\bar{\partial} X_{\dot{F}F})S^{\dot{A}\dot{B}}, \quad (3.96)$$

$$\mathcal{O}_8 = 2i\epsilon^{AB}(\partial X_{\dot{A}A}\partial^2 - \partial X_{\dot{A}A}(\partial\phi_5 - \partial\phi_6)^2/4)(\bar{\partial}^3 X_{\dot{B}B})S^{\dot{A}\dot{B}}, \quad (3.97)$$

$$\mathcal{O}_9 = 2i\epsilon^{AB}(\partial X_{\dot{A}A}\partial^2 - \partial X_{\dot{A}A}(\partial\phi_5 - \partial\phi_6)^2/4)(\bar{\partial} X_{\dot{B}B}\bar{\partial}^2 - \bar{\partial} X_{\dot{B}B}(\bar{\partial}\tilde{\phi}_5 - \bar{\partial}\tilde{\phi}_6)^2/4)S^{\dot{A}\dot{B}}. \quad (3.98)$$

Now using the fact that

$$\mathcal{O}_A = \mathcal{O}_1 + 2\mathcal{O}_2 - 12\mathcal{O}_3 + 2\mathcal{O}_4 + 4\mathcal{O}_5 - 24\mathcal{O}_6 - 12\mathcal{O}_7 - 24\mathcal{O}_8 + 144\mathcal{O}_9, \quad (3.99)$$

we find

$$\begin{aligned} \mathcal{O}_A = & 2i\epsilon^{AB}(2\partial X_{\dot{A}A}\epsilon^{\dot{C}\dot{D}}\epsilon^{CD}\partial X_{\dot{C}C}\partial X_{\dot{D}D} + 2\partial^3 X_{\dot{A}A} - 12\partial X_{\dot{A}A}\partial^2 + 3\partial X_{\dot{A}A}(\partial\phi_5 - \partial\phi_6)^2) \\ & \times (2\bar{\partial}X_{\dot{B}B}\epsilon^{\dot{E}\dot{F}}\epsilon^{EF}\bar{\partial}X_{\dot{E}E}\bar{\partial}X_{\dot{F}F} + 2\bar{\partial}^3 X_{\dot{B}B} - 12\bar{\partial}X_{\dot{B}B}\bar{\partial}^2 + 3\bar{\partial}X_{\dot{B}B}(\bar{\partial}\tilde{\phi}_5 - \bar{\partial}\tilde{\phi}_6)^2)S^{\dot{A}\dot{B}}. \end{aligned} \quad (3.100)$$

Note this agrees (up to normalization) with the results of the OPE in the previous section.

3.4.3 Four point function: coincidence limit

A non-trivial check on both methods is to examine the four point function of two candidates and two deformations. The coincidence limit will indicate the mixing for the operator. The four point function for the candidate operator had previously been worked out,

$$\begin{aligned} \langle \mathcal{O}_C(a_1, \bar{a}_1) \mathcal{O}_D(b, \bar{b}) \mathcal{O}_D(0, \bar{0}) \mathcal{O}_C(a_2, \bar{a}_2) \rangle = & \frac{|b|^4}{2^{12}} \lambda^2 |a_1|^{-4} |a_2|^{-4} |a_1 - b|^{-4} |a_2 - b|^{-4} \left\{ \frac{25}{4} \right. \\ & + 5 \left(\frac{(R+1)^2}{(R-1)^2} + \frac{(\bar{R}+1)^2}{(\bar{R}-1)^2} \right) + 16 \frac{R^{1/2}(R+1)}{(R-1)^2} \frac{\bar{R}^{1/2}(\bar{R}+1)}{(\bar{R}-1)^2} + 4 \left(\frac{(R+1)^2}{(R-1)^2} \frac{(\bar{R}+1)^2}{(\bar{R}-1)^2} \right. \\ & + 10 \left(\frac{R(R^2+6R+1)}{(R-1)^4} + \frac{\bar{R}(\bar{R}^2+6\bar{R}+1)}{(\bar{R}-1)^4} \right) + 16 \frac{R(R^2+6R+1)}{(R-1)^4} \frac{\bar{R}(\bar{R}^2+6\bar{R}+1)}{(\bar{R}-1)^4} \\ & + 64 \left(\frac{R^{1/2}(R+1)}{(R-1)^2} \frac{\bar{R}^{3/2}(\bar{R}+1)}{(\bar{R}-1)^4} + \frac{R^{3/2}(R+1)}{(R-1)^4} \frac{\bar{R}^{1/2}(\bar{R}+1)}{(\bar{R}-1)^2} \right) \\ & + 256 \frac{R^{3/2}(R+1)}{(R-1)^4} \frac{\bar{R}^{3/2}(\bar{R}+1)}{(\bar{R}-1)^4} \\ & \left. \left. + 8 \left(\frac{(R+1)^2}{(R-1)^2} \frac{\bar{R}(\bar{R}^2+6\bar{R}+1)}{(\bar{R}-1)^4} + \frac{R(R^2+6R+1)}{(R-1)^4} \frac{(\bar{R}+1)^2}{(\bar{R}-1)^2} \right) \right\}, \right. \end{aligned} \quad (3.101)$$

where $R = a_1(a_2 - b)(a_2(a_1 - b))$.

We wish to look at the coincidence limit, $(a_1, \bar{a}_1) \rightarrow (0, 0)$ and $(a_2, \bar{a}_2) \rightarrow (b, \bar{b})$. There will be an number of singularities in this limit, but we are only interested in the mixing from a weight (2,2), as mixing with other operators will not contribute to the anomalous dimension. To see what comes from mixing, one first has to look at (1,1) operators that contribute and subtract the weight (2,2) descendants' contribution, as mixing will only come from quasi-primaries not descendants. In the previous work [206], it was found that the only operator at weight (1,1) was the deformation operator itself. We also found this using our OPE method. Subtracting its descendant's contribution, the term that

indicates mixing with quasi-primary operators is given by

$$\frac{9}{2^{12}a_1(a_2-b)\bar{a}_1(\bar{a}_2-\bar{b})|b|^8} \quad (3.102)$$

This is the singularity that indicates the mixing with weight (2,2) operators. Using $C_{iD(n2A)} = 3/2^6$, we find that $C_{iD(n2A)}^2 = 9/2^{12}$ gives us the coefficient in the four point function. This indicates we have found the correct and complete mixing at twist 2.

3.5 Discussion

To fully compute the anomalous dimension, we would need to consider computing the mixing for higher twist operators. This is due to the fact that a twist 2 operator can mix with a twist 3 under the deformation and so on. Fixing the conformal dimension will put a cutoff on how high the twists can go, as the bare twist fields carry conformal weight themselves. A simple calculation will show that $n = 8$ is the highest twist allowed for a (2,2) operator.

Doing this procedure for higher twists using correlation functions is well defined. However, due to the fractional moding of operators, the number of operators at each twist level will generally increase. This means computing increasing numbers of correlators, which will also have more complicated structure. In total, we believe there would be a total of 288, including the candidate operator, that might be involved in the mixing. The code we have developed can do these computations as well, but it takes an unreasonably long time. Our preliminary calculations with the code showed that there were 9 independent mixing operators at twist 3 and 36 independent mixing operators at twist 4, and twist 5 was just as complicated. Finishing the computations by brute force was unappetizing owing to the enormous length of expressions obtained.

To avoid this problem, we would like to use our OPE algorithm to find the higher twist operators that participate in mixing. Then we would immediately recover the mixing operator from the one computation. However, this would require understanding how to properly lift the OPE to the cover when there are two twisted operators, and it turns out that there is a really interesting subtlety to this story. On the one side, we would have an OPE like

$$\mathcal{O}_D(z_0, \bar{z}_0)\mathcal{O}(0, 0) \quad (3.103)$$

where \mathcal{O}_D is the deformation operator and \mathcal{O} is the operator we are investigating the mixing for (which for our candidate is not in the twisted sector but in general may be

a twisted operator with twist n). To get the ramifications correct when lifting this, we could use the map

$$z = -nt^n \left(t - \frac{n+1}{n} t_0 \right) \quad (3.104)$$

This has the correct ramification for the twist 2 deformation operator at $t = t_0$ and the correct ramification for a twist n operator at $t = 0$. Note that $z_0 = t_0^{n+1}$, so the image of z_0 becomes $e^{2\pi i k/(n+1)} t_0$ on the cover. A sum over these will ensure that will have correct power series expansion of z_0 . However, the end result of the OPE would be an operator of twist $n+1$ at $t = 0$, which the map does not correctly ramify. We could consider the limit of the map $t_0 \rightarrow 0$, which would give us a map $z = -nt^{n+1}$. This has the correct ramification. However, in this process, a ramified point in the map has been moved. In the Lunin-Mathur covering space technique, there are extra contributions coming from a Liouville action contribution [217], and these contributions are localized at ramified points in the covering space. We checked this problem by computing the $\mathcal{O}_D(z_0, \bar{z}_0) \mathcal{O}_C(0, 0)$ OPE and found that, not being careful about the map changing leads to an incorrect result.

Another way to see the same difficulty is looking at the action of translation on a generic operator. At leading order,

$$\mathcal{O}(z_0 + \delta z_0) = \mathcal{O}(z_0) + \partial \mathcal{O}(z_0) \delta z_0 + \dots = \mathcal{O}(z_0) + (L_{-1} \mathcal{O})(z_0) \delta z_0 + \dots \quad (3.105)$$

The change $(L_{-1} \mathcal{O})(z_0) \delta z_0$ may be lifted to the covering surface by a map $z = f(t)$ where $z_0 = f(t_0)$ for some t_0 , and we find

$$(L_{-1} \mathcal{O})(z_0) \rightarrow \oint_{t_0} \frac{dt}{2\pi i} \left(\frac{\partial f(t)}{\partial t} \right)^{-1} \left(T(t) - \frac{c}{12} \{ f(t), t \} \right) \mathcal{O}_\uparrow(t_0) \quad (3.106)$$

We see here that if the map is ramified at $t = t_0$ with ramification n the function $\partial f(t)/\partial t$ will have an n^{th} order 0 at t_0 , leading to Virasoro generators L_{-k} with $k \geq 2$ on the cover. Thus, simple translation on the base space does not relate so easily to simple translation on the cover. Simple translation on the cover is only recovered for non-ramified points in the map, i.e. only applies to untwisted operators.

We plan to return to this issue in a future paper. We still expect that the knowledge of the mixing is contained in the OPE on the cover, along with some additional information coming from a careful treatment of the Liouville dressing terms.

Another point to keep in mind is that the correlators and structure constants computed here were done using a representative correlator. To get a fully S_N invariant

correlator in the orbifold CFT, we would need to sum over the conjugacy classes of the representative twist $\sigma_{(12)}$. This would lead to some combinatorial factors that we have suppressed thus far using a particular prescription for stripping these terms. To avoid the complications of these combinatorial factors, and to compare to the results of earlier work, we chose to follow the conventions of [206].

There are a number of interesting further potential applications of this D1D5 technology. The most obvious would be to look at other operators than the candidate studied here. However, this would involve larger numbers of possible operators in the mixing as one increases the conformal weight. This would make the three point function searches unfeasible but one could study the OPE of any untwisted and twisted operator just as easily as we have here. Going further, one could extend the methods here used for compute mixing to computing the general structure constants of other operators with other perturbations of the D1D5 system.

Another application would be to continue to study the deformation from the point of view of higher spin theory. There has been work done on finding the anomalous dimensions of the generators of these higher spin symmetries [211]. It would be interesting to see how using OPEs might help explore this further, or how the OPE method fits into the context of W_∞ representation theory.

There are other recent works done using D1D5 technology that might be able to be taken further with our code and/or our methods. One can consider a toy model [234] of infall in this D1D5 system. It may be interesting to study other such infalls using more complicated states (such as ones in [235] or using the duals of arbitrary angular momenta 3-charge system [44]) other than the Ramond vacuum. This would involve computing complicated correlators, which our code should be able to accomplish.

There has also been a large amount of work done studying the conformal blocks in CFT [150] [154] and OPE blocks using kinematic space [107] [110]. Since we have a free theory, we can calculate these conformal blocks directly. It might be interesting to show some explicit examples of these new methods for calculating conformal blocks, especially if they could be used to generate conformal blocks and by extension correlators in the D1D5 CFT more efficiently than the methods presented here.

As we continue to explore the D1D5 CFT, we are hoping that new tools will allow us to come to a better understanding of this important model.

3.6 Appendix: Quasi-primary projection procedure

In this appendix, we want to detail how we can take a general non quasi-primary state and project to a quasi-primary. There are a number of different methods to do this, but we will present two methods here and show they are equivalent. Another method was the one we used in section 3.4.2, where we summed we use the coefficients of a few non-quasi primaries to fix the combination that is a quasi primary. This method agrees with the two methods presented here.

Start with a state $|\phi_n\rangle$ with conformal weight h and suppose the state satisfies

$$L_1^{n+1}|\phi_n\rangle = 0, L_1^n|\phi_n\rangle \neq 0. \quad (3.107)$$

Now construct the following state

$$|\phi_{n-1}\rangle = \left(1 - \frac{1}{N(n)}L_{-1}L_1\right)|\phi_n\rangle, \quad (3.108)$$

where we have

$$N(n) = n(2h - (n + 1)). \quad (3.109)$$

We claim that this satisfies $L_1^n|\phi_{n-1}\rangle = 0$. To show this, consider

$$L_1^n|\phi_{n-1}\rangle = \left(L_1^n - \frac{1}{N(n)}L_1^nL_{-1}L_1\right)|\phi_n\rangle. \quad (3.110)$$

We recall that the general commutator for the Virasoro algebra is

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \quad (3.111)$$

We can use this to show that $[L_{-1}, L_1] = -2L_0$. Then we can rewrite equation (3.110) as

$$L_1^n|\phi_{n-1}\rangle = \left((1 + \frac{2h}{N(n)})L_1^n - \frac{1}{N(n)}L_1^{n+1}L_{-1}\right)|\phi_n\rangle. \quad (3.112)$$

Next we can use the general commutator expression with some basic identities to work out the following commutator,

$$[L_1^{n+1}, L_{-1}] = 2(n + 1)L_1^nL_0 - n(n + 1)L_1^n. \quad (3.113)$$

Plugging this in and simplifying, we find

$$L_1^n |\phi_{n-1}\rangle = \left(1 + \frac{2h}{N(n)} - \frac{2h(n+1)}{N(n)} + \frac{n(n+1)}{N(n)}\right) L_1^n |\phi_n\rangle - \frac{1}{N(n)} L_{-1} L_1^{n+1} |\phi_n\rangle. \quad (3.114)$$

If we use the fact that $N(n) = n(2h - (n+1))$, then the first term vanishes and the second term vanish by $L_1^{n+1} |\phi_n\rangle = 0$. So then we find $L_1^n |\phi_{n-1}\rangle = 0$ as claimed.

We may iterate this process to obtain

$$|\psi\rangle = \left(1 - \frac{1}{N(1)} L_{-1} L_1\right) \left(1 - \frac{1}{N(2)} L_{-1} L_1\right) \cdots \left(1 - \frac{1}{N(n)} L_{-1} L_1\right) |\phi_n\rangle, \quad (3.115)$$

where $L_1 |\psi\rangle = 0$, and so $|\psi\rangle$ is quasi-primary. So in general, this will be the approach we will take to get a quasi-primary out of a non-QP.

We could also construct the state

$$|\phi'_{n-1}\rangle \equiv \left(1 - \frac{1}{N'(n)} L_{-1}^n L_1^n\right) |\phi_n\rangle, \quad (3.116)$$

where

$$N'(n) = n! \prod_{p=1}^n (2h - 2n + p - 1). \quad (3.117)$$

The state $|\phi_{n-1}\rangle$ is a state that satisfies $L_1^n |\phi_{n-1}\rangle = 0$. Again we can iterate this and find

$$|\psi'\rangle = \left(1 + \frac{1}{N(1)} L_{-1} L_1\right) \left(1 + \frac{1}{N(2)} L_{-1}^2 L_1^2\right) \cdots \left(1 - \frac{1}{N(n)} L_{-1}^n L_1^n\right) |\phi_n\rangle \quad (3.118)$$

If we do this iterative process, we will find $|\psi'\rangle = |\psi\rangle$. The intermediate steps may not agree but these final results do. First, note that one can expand out the iterated projections, with the leading term being 1. All of the other terms will have an L_{-1} to the left. Hence, $|\psi\rangle$ is of the form

$$|\psi\rangle = |\phi_n\rangle + L_{-1} |\delta\rangle \quad (3.119)$$

for some state $|\delta\rangle$, and

$$|\psi'\rangle = |\phi_n\rangle + L_{-1} |\delta'\rangle \quad (3.120)$$

for some state $|\delta'\rangle$. Both $|\psi\rangle$ and $|\psi'\rangle$ are QP, and so their difference is as well. Thus the state

$$|\psi\rangle - |\psi'\rangle = L_{-1} (|\delta\rangle - |\delta'\rangle) \equiv L_{-1} |\Delta\rangle \quad (3.121)$$

is a QP as well (we have defined the state $|\Delta\rangle \equiv |\delta\rangle - |\delta'\rangle$). So, $L_{-1}|\Delta\rangle$ is a QP, and so

$$L_1 L_{-1} |\Delta\rangle = 0. \quad (3.122)$$

This proves that that

$$\langle \Delta | L_1 L_{-1} | \Delta \rangle = \left\| L_{-1} |\Delta\rangle \right\|^2 = 0. \quad (3.123)$$

For a unitary CFT constructed using irreducible representations of the Virasoro algebra, the only state with zero norm is the zero state. Therefore

$$L_{-1} |\Delta\rangle = 0 \quad (3.124)$$

and so

$$|\psi\rangle = |\psi'\rangle. \quad (3.125)$$

This gives evidence that our construction for give the quasi-primary contribution in a non quasi-primary operator.

Chapter 4

D1D5 orbifold CFT package for Mathematica

4.1 Introduction and motivation

This chapter will describe the code that we developed to compute OPEs and correlators in the D1D5 orbifold CFT. First we will discuss the motivations behind writing this code and discuss the previous Mathematica package which we extended to our CFT. Later we will discuss the extension of the base package to include the fields and OPEs in our CFT, how the code will do the lifting procedure to the cover, the main computational power behind the code doing the Wick contractions, how it computes the correlators, and finally how it was applied to calculations of interest.

Since our CFT is a free CFT, we can compute both the OPEs and correlators in the theory using simple Wick contractions once we lift to the cover. There, without the complication of twists, it is straightforward to carry out the contractions. The complication of this method is due to the number and forms of the mixing operators.

Let's examine what operators would mix with our original candidate operator in equation (3.27). As mentioned before, these mixing operators must be weight (2,2) and singlet under all the $SU(2)$ symmetries. Satisfying both these conditions is constraining, but there are still a number of operators that must be considered. The one reason this is even a tractable problem is that we do not consider arbitrarily twisted operators. The weight (2,2) condition means that the operators that can mix only go up to twist 8. This can be seen by considering lifting an generic twisted operator to the cover and comparing conformal weights. Consider the twist n operator \mathcal{O} with weight h_z and lift it under the

map $z = t^n$,

$$L_0 \mathcal{O} \rightarrow \oint \frac{dt}{2\pi i} \left(\frac{dz}{dt} \right)^{1-2} z(t)^{0+2-1} (T(t) - \frac{1}{2}\{z, t\}) \mathcal{O}^{(t)} \quad (4.1)$$

$$= \oint \frac{dt}{2\pi i} \frac{t}{n} (T(t) + \frac{n^2 - 1}{4}) \mathcal{O}^{(t)} \quad (4.2)$$

$$= \frac{h_t}{n} + \frac{1}{4}(n - \frac{1}{n}) \quad (4.3)$$

However, we also know that $L_0 \mathcal{O} = h_z \mathcal{O} \rightarrow h_z \mathcal{O}^{(t)}$. So then we have the relationship

$$h_z = \frac{h_t}{n} + \frac{1}{4}(n - \frac{1}{n}) \quad (4.4)$$

Note that we might have guessed this form from the beginning. We have the contribution from the bare twist, $(n^2 - 1)/(4n)$, which would become an identity on the cover and we have a factor of n^{-1} between the weight of the base on the cover from the mapping. So if we set $h_z = 2$ we get a formula, $4h_t = 8n - n^2 + 1$. we also know that $h_t \geq 0$ by unitarity of the CFT on the covering space, so we have a constraint on the twist level n as $8n - n^2 + 1 \geq 0$. Since n is an positive integer, this implies that $n \leq 8$ as we claimed. This is one of the reasons we chose a lower weight operator to study. The higher the weight, the more twist levels enter in and the more operators that might participate in the mixing.

In Appendix A, we lay out a number of operators in just the holomorphic sector that have weight 2, are singlets under the $SU(2)_L$, and can be made singlets under the others with the inclusion of an appropriate antiholomorphic operator. This list was generated by considering fractional modes of the base boson and fermion. Note we did not include possible fraction modes of J or other excitations from G or L . This is because these modes are composites of the ∂X and ψ fields, and so we do not get any new operators from considering these assuming we have exhausted the list of operators with ∂X and ψ fractional modes.

As mentioned earlier in section 3.5, we can count these operators and find 288 operators to consider. Of course, we do have the restriction that only neighbouring twist levels can be non-vanishing and that some of these will not mix, but this is a significant number of operators to go through. Additionally, due to fractional moding, we end up getting more complex operators in the middle of the twist levels. One could see this from $4h_t = 8n - n^2 + 1$, where the maximum conformal weight on the cover is for the middle values between $n = 2$ and $n = 8$. So as we consider higher twists, we get more operators and they are more complicated. Not all would mix but we would still need to consider

all of them when computing three point functions.

We can see that computing these all by hand would be tedious. This is why in chapter 3 we focused on looking at the OPEs directly. However, the OPEs themselves are computed using Wick contractions. So even there, we can use the code to help work out the results of OPEs with bosonized fermions and spin fields, which are tedious to do. This code can handle both correlation functions and OPEs in the CFT, making it a powerful tool for probing the D1D5 CFT.

With the motivation covered, we now turn to how we first constructed the code. We built our package off of the previous package, `ope.math`, developed by Akira Fujitsu [236]. This package was designed to work for free CFTs, which was our main goal. However, this was written for a generic free CFT. It does not include lifting, our bosonization conventions, and it does not distinguish between holomorphic and antiholomorphic fields. On the plus side, it did include a way to implement Wick contractions which is what we needed for our work. We worked to include the missing features and change the Wick contraction method to work for correlation functions. In the next section, we will start to describe the code by explaining how we included the D1D5 CFT operators.

4.2 The D1D5 fields and the OPEs

Before we can do the Wick contractions, we need to introduce our fields and our basic OPEs into the code. The OPE of currents and charges of the D1D5 CFT can be derived from the OPEs of the basic bosons and fermions of the CFT and so we do not need to include them explicitly. The basic boson and fermion OPEs come directly from the action 1.38, and so we must include these OPEs explicitly in the code. We also have to include the OPE of the bosonized operators, as the ϕ fields are not conformal fields by themselves, so we must teach the code how to deal the exponential of these field which is a conformal field. Of course, we also have a number of other definitions to include, such as the epsilon tensors and the Pauli matrices as defined in (1.40). So below I will discuss how we included the fields,

```
dx[Adot_,A_,z_,j_,h_]
```

These are the basic bosonic fields. The `Adot,A` are the symmetry indices, `z` is the position, `j` is the number of additional derivatives, `h` is the (anti)holomorphic flag when `(-1)`. This translates to $\partial^{j+1} X^{\dot{A}A}(z)$ when `h=1` and $\bar{\partial}^{j+1} X^{\dot{A}A}(\bar{z})$ when `h=-1`. This can be used in anything, including OPEs, correlations, and lifting.

```
fr[alpha_,Adot_,z_,j_,h_]
```

These are the basic fermionic fields. The `alpha,Adot` are the symmetry indices, `z` is

the position, j is the number of derivatives, h is the (anti)holomorphic flag when (-)1. It translates to $\partial^j \psi^{\alpha\dot{A}}(z)$ when $h=1$ and $\bar{\partial}^j \tilde{\psi}^{\dot{\alpha}\dot{A}}(\bar{z})$ when $h=-1$. For the most part, we do not use this in most of our calculations as we bosonize beforehand. Note **fr** is for fermion.

tc[j_-, q_-, z_-, h_-]

This is what we use for bosonized fermions. The j is the number of derivatives, q is a ordered list (with curly braces) representing the vector in the exponential, z is the position, and h is the (anti)holomorphic flag when (-)1. This refers to $\partial^j : \exp(iq \cdot \phi)(z) :$ for $h=1$ and $\bar{\partial}^j : \exp(iq \cdot \tilde{\phi})(\bar{z}) :$ for $h=-1$. Note that the original package also has operators like this, but it does not distinguish between holomorphic and antiholomorphic fields, which we need for correlation functions.

bo[$q_-, qbar_-, z_-, zbar_-, i_-, j_-$]

This is similar to **tc** but includes both holomorphic and antiholomorphic pieces together. This allows spin fields to be represented as bosonized operators. Additionally, this allows us to better define the cocycles, which were defined in the base package but did not include holomorphic and antiholomorphic pieces. This $\partial^i \bar{\partial}^j : \exp(iq \cdot \phi + i\bar{q} \cdot \tilde{\phi})(z, \bar{z}) :$

ph[j_-, q_-, z_-, h_-]

This is a free bosonic field, used in the bosonization and important when we expand out derivatives of the bosonized fermion field. The j is the number of derivatives, q is the index, z is the position, and h is the (anti)holomorphic flag when (-)1. This represents $\partial^{j+1} \phi_q(z)$ when $h=1$ and $\bar{\partial}^{j+1} \tilde{\phi}_q(\bar{z})$ when $h=-1$. Again, the original code also has operators like this, but it does not distinguish between holomorphic and antiholomorphic fields.

These are a lot of fields to put into the code. It presents a bit of a technical challenge for computing the Wick contractions, because each would require a separate rule to deal with them. To simplify the way the code implements the contractions, we organize the non-bosonized fields into an array. Then we can index the array to specify a specific field. In our code, we have the array **nff**[k, i, j, z, h]. Here k is the index for the field in the array, i and j are symmetry indices the physical meaning of which will depend on the field, z is the position, and h is the (anti)holomorphic flag. For example, we have **dx**[$Adot, A, z, j, h$] represented by **nff**[$0, Adot, A, z, j, h$].

In addition to the base operators, we can define composite operators with normal ordering. This is required in order to properly get the singularity structure out of OPEs.

For example, let us consider the full OPE of two bosonic fields,

$$\partial X^{\dot{A}A}(z) \partial X^{\dot{B}B}(w) = \frac{\epsilon^{\dot{A}\dot{B}} \epsilon^{\dot{B}B}}{(z-w)^2} + : \partial X^{\dot{A}A}(z) \partial X^{\dot{B}B}(w) : \quad (4.5)$$

$$= \frac{\epsilon^{\dot{A}\dot{B}} \epsilon^{\dot{B}B}}{(z-w)^2} + \sum_{j=0}^{\infty} \frac{(z-w)^j}{j!} : \partial^{j+1} X^{\dot{A}A}(w) \partial X^{\dot{B}B}(w) : \quad (4.6)$$

We can see that the first term is the singular piece but in general we will also have an infinite number of non-singular pieces that come with normal ordered operators. We have suppressed the normal ordering notation in chapter 3 for notational convenience but we must be explicit in the code. We represent normal ordered operators by including them inside of the function `N1[]`. For example, `: \partial^{j+1} X^{\dot{A}A}(w) \partial X^{\dot{B}B}(w) :` would read `N1[dx[Adot,A,w,j,1],dx[Bdot,B,w,0,1]]`. The code can determine the non-singular pieces of an OPE if needed. It is sometimes important to keep normal ordering and non-singular terms in the code. For example, when we consider lifting fields to the cover. The residue integrals from the lift will often pick out some of the non-singular terms from an OPE and so simply dropping these terms leads to an incorrect answer.

One aspect for OPEs needed is the cocycles for bosonized operators. In the code, the cocycles are not explicitly included but implicitly defined. So their effects will show up when we need to commute bosonized fields or when we do a simple OPE. These factors are defined generically in the code and we must explicitly include the definitions to agree with the conventions of bosonization we defined in section 1.9.3.

To demonstrate the way the cocycles are handled in the code, consider the following OPE,

$$\begin{aligned} e^{i\pi q^i M_{ij} \alpha_0^j} : e^{iq \cdot \phi}(z) : e^{i\pi k^i M_{ij} \alpha_0^j} : e^{ik \cdot \phi}(w) : \\ \sim (z-w)^{q \cdot k} \exp(i[q \cdot \phi, \pi k^i M_{ij} \alpha_0^j]) e^{i\pi(q+k)^i M_{ij} \alpha_0^j} : e^{iq \cdot \phi}(z) e^{ik \cdot \phi}(w) : \end{aligned} \quad (4.7)$$

So if we include the cocycles $e^{i\pi q^i M_{ij} \alpha_0^j}$, $e^{i\pi q^i M_{ij} \alpha_0^j}$, and $e^{i\pi(q+k)^i M_{ij} \alpha_0^j}$ implicitly, then the effect of the cocycles show up as a factor of $\exp(i[q \cdot \phi, \pi k^i M_{ij} \alpha_0^j])$ in the OPE. This is defined in the code as `C02[q,0,k,0]`, as the momentum vectors in the exponentials are the defining parts in this phase. In general, the cocycle in the code will take the form `C02[q,qbar,k,kbar]`, as we will also have cocycles for the spin fields that will depend on both the holographic and antiholomorphic parts. These expressions are general, so we will then need to use the definitions and choices made in section 1.9.3 to specify to our choice of cocycles. These are then able to reproduce the correct multiplicative factors required to compute OPEs with bosonized fermions and bosonized spin fields. The full

definitions of the bosonized operators also have generic phases to defined the fermions and spin fields. These are very straight forward to include in Mathematica.

With all the required ingredients defined, we can implement the OPEs of the fields that we laid out in equations (1.43) and (1.44). We also need to include the basic OPEs for the bosonized fields. These are the basic OPEs defined as follows,

$$\partial\phi_i(z)\partial\phi_j(w) \sim \frac{\delta_{ij}}{(z-w)^2}, \quad (4.8)$$

$$\partial\phi_j(z)e^{iq\cdot\phi}(w) \sim \frac{iq_j}{(z-w)}e^{iq\cdot\phi}(w), \quad (4.9)$$

along with the OPEs like the one in equation (4.7). Finally, we would need to define OPEs for operators which do not have an OPE with each other. For example, a ∂X and a ψ have no singular pieces in their OPEs, neither do any holomorphic fields with anti-holomorphic fields, etc.

All of these basic OPEs are defined in the code using the `Vev[]` function as these OPEs do come from the vacuum expectation value using the action. For example, we can consider the following,

```
Vev[nff[0,Adot_,A_,z_,d_,h_],nff[0,Bdot_,B_,w_,e_,m_],z1_,w1_]:=  
If[TrueQ[z==w1],Vev[nff[0,Bdot_,B,w,e,m],nff[0,Adot_,A,z,d,h],z1,w1],  
(-1)^(e+1) D[Log[dz],{dz,d+e+2}] ESYMBOLU[Adot,Bdot]*  
ESYMBOLU[A,B] N1[]]/;(Not[TrueQ[z==w]]&&m==h);
```

Note `ESYMBOLU[Adot,Bdot]` is simply the epsilon tensor with indices up, $\epsilon^{\dot{A}\dot{B}}$.

We can see here that this rule is carried out when we have the OPE of operators at different points (`Not[TrueQ[z==w]]`) and they are have the same holomorphicity (`m==h`). A separate rule will force this to vanish if these conditions are not satisfied. If they are, then we hit an if statement. Here we can see that if `z==w1`, we switch the order of the fields. The purpose of this is for easier simplification of the OPE later in the code. The more likely route is that we will carry out the OPE,

$$\partial^{d+1}X^{\dot{A}A}(z)\partial^{e+1}X^{\dot{B}B}(w) \sim \frac{(-1)^e\epsilon^{\dot{A}\dot{B}}\epsilon^{AB}}{(z-w)^{2+e+d}}. \quad (4.10)$$

This rule and other `Vev` rules are then used as Wick contractions to compute OPEs of more complex composite operators.

To actually compute the OPE in the package, one either calls `ope[]` or `OPEsort[]`. The first function `ope[]` compute OPEs when the fields in question are purely holomorphic or purely antiholomorphic. This function was in the base package, but we

extended it to work with the above fields. The second function, `OPEsort[]`, will first sort out the fields into their holomorphic and antiholomorphic parts before performing the OPEs. For example, to compute the OPE in (4.10) we would use the function `ope[dx[Adot,A,z,d,h],dx[Bdot,B,w,e,h]]`.

There are a number of other small parts of the code that are important to getting out the results, but we will not cover them. These mostly are devoted to turning input into easier notation for the code and then transforming the results back into something readable for the user. So we will now turn to the main pieces of the code that implement the Wick contractions fore OPEs.

4.3 Wick contractions

The main workhorse of the code for OPEs is the `Monomial` function. This function, together with other functions, will implement Wick contractions. This function was part of the base package, although we made a few minor modifications. We review it here as we used this as a base in implementing correlation function calculations. The main rule of `Monomial` can be shown in the one rule below.

```
Monomial[OP[a___,N1[b___,nff[c___]],VN1[N1[d___]]],z_,w_]:=  
Monomial[OP[a,N1[b,nff[c]],VN1[N1[d]]],z,w]=  
Monomial[OP[a,N1[b],VN1[N1[nff[c],d]]],z,w] +  
Module[{i0},Sum[Exp[Pi I Simplify[  
stsum[Drop[N1[d],-Length[N1[d]]+i0-1]] nff[c][[1]]  
]] Monomial[OP[a,N1[b],VN1[N1[Drop[N1[d],-Length[N1[d]]+i0-1],  
Expand[Vev[nff[c],N1[d][[i0]],z,w]],Drop[N1[d],i0]]]],z,w],  
{i0,1,Length[N1[d]]}]]];
```

There is a lot of functions here, but we will unpack the various parts that are part of our package and not the ones part of the base Mathematica code. To begin, we notice the focus of the rule is the field `nff[c]`, explained in the previous section. Note that `a` and `b` are going to be some unspecified number of other fields and `z,w` are the two coordinates involved in the OPE. Next we have the `stsum` function. This is simply a way to keep track of fermions. When we do a Wick contraction with fermions, we must be careful of anticommutation relations. For example, if we contract two fermions with a fermion in between them then we must include a negative sign. This function returns either an even or odd integer which will then, inside of the `Exp[Pi I]`, give the correct sign for the contraction. We recognize the `N1` function and the `VN1` function is similar. It keeps a track of what is left after the contractions, which are normal ordered. Finally,

we note that `OP` is just a useful tool for organizing various parts of the code.

Now we can give a overview of how these functions come together. The first two lines are part of the definition. This way of defining a function will save the result in RAM, which is important for saving time with the iterative procedure. This does have the cost of the code being very RAM intensive. When this rule is applied, it replaces the original `Monomial` function with two terms summed together. The first term simply transfers the operator over to the `VN1`, doing no contractions. The second term is the sum over all the contractions of `nff[c]` with all the other fields in `VN1[]`, as can be seen by the `Vev` function showing up in this term. This rule, and the related rules, will continue to be applied until all the fields are inside the `VN1[]`. This iterative approach will ensure we capture all the contractions without overcounting and obtain the non-singular normal order terms of the OPE as well.

To show this more explicitly, let's consider a simple example of computing the OPE of $\partial X^{\dot{A}A}(z) : \partial X^{\dot{B}B}(w) \partial X^{\dot{C}C}(w) :.$ Putting this in our code, it would be

```
ope[dx[Adot,A,z,0,1],N1[dx[Bdot,B,w,0,1],dx[Cdot,C,w,0,1]]].
```

After some transformations, this will eventually be reduced to considering

```
Monomial[OP[N1[dx[Adot,A,z,0,1]],N1[dx[Bdot,B,w,0,1],dx[Cdot,C,w,0,1]]],VN1[N1[],z,w].
```

We can now start applying the rule. The first step will simply move the `dx[Cdot,C,w,0,1]` into the `VN1[]` as there are no fields to contract with yet. So we then have

```
Monomial[OP[N1[dx[Adot,A,z,0,1]],N1[dx[Bdot,B,w,0,1]]],VN1[N1[dx[Cdot,C,w,0,1]],z,w].
```

Now we apply the rule again. This time we will have to consider both the terms in the rule, as there could be a contraction between `dx[Bdot,B,w,0,1]` and `dx[Cdot,C,w,0,1]`. However, we know that we should not contract two fields inside of a normal ordered expression, so the contraction here vanishes. So we just have the term

```
Monomial[OP[N1[dx[Adot,A,z,0,1]],VN1[N1[dx[Bdot,B,w,0,1],dx[Cdot,C,w,0,1]]],z,w].
```

Note that we have dropped the empty `N1[]` that would have been there. This is taken care of in a different rule that we have not covered. Now, when we apply the rule again, we get to the non-trivial part. Here we do have contractions which come into the mix. So after applying the rule we would get something of the form

```
Monomial[OP[VN1[N1[dx[Adot,A,z,0,1],dx[Bdot,B,w,0,1],dx[Cdot,C,w,0,1]]],z,w]+ESYMBOLU[Adot,Bdot]*ESYMBOLU[A,B]*Monomial[OP[dz^(-2) VN1[N1[dx[Cdot,C,w,0,1]]],z,w]+ESYMBOLU[Adot,Cdot]*ESYMBOLU[A,C]*
```

```
Monomial[OP[ $dz^{-2}$ ] VN1[N1[dx[Bdot,B,w,0,1]]],z,w].
```

Note that dz^{-2} is the representation of $(z - w)^{-2}$.

We can see now how the code captures both the singular terms of the OPE and the non-singular parts. In general the non-singular terms will be dropped unless explicitly requested in the `ope[]` function (i.e. terms where dz^k with k positive are forced to vanish). To keep the order n non-singular terms, one can use the form `ope[-n,N1[a__],N1[b__]]`. The code then multiplies everything in the `Monomial` by dz^{-n} and the code will keep all singular terms. It will then be up to the user to multiply the end result by $(z - w)^n$ to get back the correct expansion.

Once the `Monomial` function is finished, a number of other rules are applied to make it readable on the Mathematica interface. For our example, we would find the result,

$$(dx[Cdot, C, w, 0, 1] ESYMBOLU[A, B] ESYMBOLU[Adot, Bdot])/(-w + z)^2 + (dx[Bdot, B, w, 0, 1] ESYMBOLU[A, C] ESYMBOLU[Adot, Cdot])/(-w + z)^2.$$

This is agrees with the correct result,

$$\partial X^{\dot{A}A}(z) : \partial X^{\dot{B}B}(w) \partial X^{\dot{C}C}(w) : \sim \frac{\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\partial X^{\dot{C}C}}{(z - w)^2} + \frac{\epsilon^{AC}\epsilon^{\dot{A}\dot{C}}\partial X^{\dot{B}B}}{(z - w)^2}. \quad (4.11)$$

Of course, the snippet for `Monomial` included here is not the only rule and there are rules for `Monomial` to deal with bosonized fields in the code as well. These are contained in the array `nvf[]`. These must be handled differently, as we can see in equations (4.7) and (4.9) that a contraction with a bosonized field will generally not be a simple singularity with no field dependence. Formally, we would compute these OPEs using the fact that $e^{k\phi} = \sum_{n=0}^{\infty} (n!)^{-1} (k\phi)^n$ and do the contractions with the field ϕ . However, ϕ is not a conformal field and, furthermore, doing this explicitly while remaining analytic would be difficult in the code. So we must use equations (4.7) and (4.9) directly and be careful of the ordering of the contractions with other fields.

With these rules in hand, we can compute OPEs in the D1D5 CFT. Of course, we should be a bit careful. We have not explicitly included the copy indices in the fields. So our code will only apply either for a single copy of the D1D5 CFT or after an operator has been lifted to the covering space. The next section will discuss how we implement lifting in the code.

4.4 Lifting operators with the code

To continue, we need to consider how to use our code with twisted operators. In general, correlation functions and OPEs with twist operators will not be calculable on the base,

so we must lift them to the covering space where we no longer have the twisted boundary conditions to deal with. Recall that in equation (1.55) we defined fractional modes. We can lift this fractionally moded operator to the cover using the appropriate map. We can now discuss how we do this with our package.

So we can lift a fractional moded operator of a primary field to

$$\mathcal{O}_{-m/n} O_n \rightarrow \oint \frac{dt}{2\pi i} \left(\frac{dz}{dt} \right)^{1-h} z(t)^{-m/n+h-1} \mathcal{O}^t(t) O_n^t. \quad (4.12)$$

where \mathcal{O} is a field of the D1D5 and O_n is a twist n operator. To perform this integral, we will need to have an number of different aspects included in the code. The first is employing the OPE for $\mathcal{O}^t(t) O_n^t$. This will have some non-trivial expansion in t , which would affect the residue integral. We have covered OPEs in the previous sections. The second aspect is the residue integral itself. Note that after performing the OPE, we will have operators in the code that are normal ordered with `N1[]`. This presents a bit of a technical problem in the code, as the built-in Mathematica residue function will consider `N1[]` as a function with derivatives. So we had to define our own function `residue` which would ignore the normal ordering for the purpose of doing the residue integral.

Things get a bit more complicated when we have both holomorphic and antiholomorphic fields, especially when we have spin fields which will involve cocycles. This is where our main function for lifting operators, `LiftOMt` is used for. It will evaluate the residue integral in equation (4.12), making sure to separate out correctly the various holomorphic and antiholomorphic pieces and operators correctly and making sure the OPE results are done before the residue integral is evaluated. We can see this in the one of the main rules for this function,

```
LiftOMt[N1[f_],N1[x_],N1[a_],z_,t_,t0_,p_,zbar_,tbar_,tbar0_,pbar_]:=  
residue[residue[Expand[SeparateCoordOPt[Expand[SeparateCoordOPMt[  
LiftIntegralCoord[z,t]*LiftONew[N1[f],t,t0],LiftIntegralCoord[zbar,tbar]*  
LiftONew[N1[x],tbar,tbar0],N1[a],coordHolo[],coordAholo[],z,t,t0,p,  
zbar,tbar,tbar0,pbar]],coord[],z,t,t0,zbar,tbar,tbar0]],  
Powerdt[],Collate[],{t,t0,dt}],Powerdt[],Collate[],{tbar,tbar0,dtbar}]];
```

Again, there are a lot of functions here, but we will focus on the important parts. To begin, we will explain some of the input arguments for `LiftOMt`. `N1[f_]` and `N1[x_]` will contain any holomorphic and antiholomorphic fractional moded operators we might want to apply, respectively. These are applied to the operator `N1[a_]`, which should already have been lifted to the cover. Additionally, we have the coordinates involved,

`z` and `zbar` on the base plus `t` and `tbar` on the cover. The two coordinates `t0` and `tbar0` are the location of the twist operator. Finally, `p` and `pbar` are used for including Schwarzian derivatives for lifting quasi-primaries like the energy momentum tensor.

With the inputs explained, we will turn to discussing the functions. First, we can focus on the integrand of the integral. We have `LiftIntegralCoord`, which is just the Jacobian from the integral factor as we lift to the cover. We also have `LiftONew`, which simply adds in the correct factors of $(\frac{dz}{dt})^{-h} z(t)^{-m/n+h-1}$ for the fractionally moded operators. Next, we focus on the OPE and doing the correct expansion around the twist for the residue integral. `SeparateCoordOPMt` first separates out the holomorphic and antiholomorphic pieces, then checks what non-singular terms in the OPE we must keep by examining the order of the series expansion, and then doing the OPE. `SeparateCoordOPt` will take in the results from `SeparateCoorOPMt` and then work out the series expansion to use in the `residue` function in both holomorphic and antiholomorphic sectors. Finally, we can recognize the function `residue`, which appears twice to compute the holomorphic and antiholomorphic integrals. The `Powerdt[]` and `Collate[]` arguments in the residue are just the way to sort out the powers of $(t - t_0)$ as defined with `{t, t0, dt}`. This is what will evaluate the residue integral at the end. Put all together, we have all the pieces required to compute the integral in equation (4.12).

We can also use the same covering space technology to evaluate whether a twisted operator is a quasi-primary or not. We can use our code to determine the form of the operator on the cover, which we can then use together with the lift of the L_1 mode to the cover,

$$L_1 \mathcal{O}_n \rightarrow \oint \frac{dt}{2\pi i} \left(\frac{dz}{dt} \right)^{-1} z(t)^2 [T(t) - \frac{1}{2} \{z, t, \}] \mathcal{O}_n^t. \quad (4.13)$$

If this vanishes, then it will be a quasi-primary. This was the same method employed in chapter 3 and we can use the code to extract this information as well. We can do this with the function `LCommute[m_, Tmom_, N1[f___] z_, t_, t0_, h_]`, which can handle general L_m modes acting on fields, and will act similarly to how `LiftOMt` does above.

4.5 Correlation functions

Now that we have both lifting and OPEs included in the code, we can now turn to how the code handles correlation functions. The inclusion of Wick contractions for correlators is a new addition to the package. To begin, we should point out that the code does not compute the bare twist contribution to a correlation function. This is handled by the Lunin-Mathur technology and is not implemented in the code. The code only com-

putes the nontwist part of the correlator on the cover by performing the required Wick contractions. The bare twist must be included by hand at the end.

Implementing the Wick contractions for correlation functions is very similar to how we do it with OPEs, except we have multiple fields at multiple points involved. The main rule for the contractions are contained in the following rule.

```
corMonomial[OP[a___,N1[b___,nff[c___]],VN1[N1[d___]]]]:=
corMonomial[OP[a,N1[b,nff[c]],VN1[N1[d]]]]=
corMonomial[OP[a,N1[b],VN1[N1[nff[c],d]]]] +
Module[{i0},Sum[Exp[Pi I Simplify[stsumcor[
Drop[N1[d],-Length[N1[d]]+i0-1],nff[c][[1]]]]]*

corMonomial[OP[a,N1[b],VN1[N1[Drop[N1[d],-Length[N1[d]]+i0-1],

Expand[corContract[nff[c],N1[d][[i0]],nff[c][[4]],N1[d][[i0]][[4]]],Drop[N1[d],i0]]]]],{i0,1,Length[N1[d]]}]]];
```

We see this has a very similar form to the `Monomial` function with a few differences. First there is no reference to `z` and `w`. This is simply due to the fact that we will have multiple operators at multiple positions and so we must extract the position from the field itself, `nff[c][[4]]`. The other change is that we have replace `Vev` with `corContract`. They operate the same but `corContract` has no explicit coordinate dependence. Specifically, whereas `Monomial` represented $(z - w)$ as `dz` as we only had the one difference for the OPE, `corMonomial` must instead work directly with various different coordinates.

Of course we do have an additional complication. Unlike with OPEs, generally there will not be any field dependence left over at the end. For most fields, this just translates to the fact that their one point function will vanish (i.e. has zero vacuum expectation value). So we set most of these terms to vanish. The exception is the exponential fields, which must be handled slightly differently like with the OPEs. Once the contractions are all calculated, we will be left with correlations with just exponential fields, as their OPEs always return an exponential field. There is a well known formula to handle these,

$$\left\langle \prod_{i=1}^n : \exp[i(k \cdot \phi + \tilde{k} \cdot \tilde{\phi})](z_i, \bar{z}_i) : \right\rangle = \delta^D \left(\sum_{i=1}^n k_i \right) \delta^D \left(\sum_{i=1}^n \tilde{k}_i \right) \prod_{i,j=1, i < j}^n z_{ij}^{k_i k_j} \bar{z}_{ij}^{\tilde{k}_i \tilde{k}_j}, \quad (4.14)$$

where we are considering this correlation function to be properly normalized so we can drop any proportionality constants. We can implement this with another iterative rule shown below.

```
corVertex[a___,N1[c___,bo[q_,qbar_,z_,zbar_,0,0]],done[e_],
momentumHolo[p_],momentumAHolo[pbar_]:=C02[q,qbar,p,pbar]*

Product[Abs[z-done[e][[i]][[1]][[3]]]^(done[e][[i]][[1]][[1]].q),
```

```

{i, 1, Length[done[e]]}]*
Product[Abs[zbar-done[e][[i]][[1]][[4]]]^ (done[e][[i]][[1]][[2]].qbar),
{i, 1, Length[done[e]]}]*
corVertex[a,N1[c],done[N1[bo[q,qbar,z,zbar,0,0]],e],
momentumHolo[p+q],momentumAHolo[pbar+qbar]];

```

The inputs for `corVertex` are relatively simple. Here both `a` and `c` are going to be `bo[]` fields. This function focuses on dealing with the one bosonized field `bo[q,qbar,z,zbar,0,0]`. Here `done[]` is what it appears to be, a place to put all the bosonized fields we are finished considering. We also have the `momentumHolo` and `momentumAHolo` functions, which store the total of the holomorphic and antiholomorphic momentum vectors, respectively. This function will take the vector arguments, `q`, `qbar`, of the bosonized field and will get the correct coordinate dependence from this field with

$$(z\text{-done}[e][[i]][[1]][[3]])^{\text{(done}[e][[i]][[1]][[1]].q)}.$$

This is part of a product over all the bosonized fields in the `done` function. The code then move the field into the `done[]` function and adds the two momentum vectors to the two totals. We also have to be careful of the cocycles, which is expressed with `C02[q,qbar,p,pbar]`. This iterative approach ensures that we do not double count. Once we have extracted the coordinate dependence from the bosonized fields, we can then enforce the delta functions very easily. This is done by

```

corVertex[done[e___],momentumHolo[q_],momentumAHolo[p_]]:=
KroneckerDelta[zerolist[q],q]KroneckerDelta[zerolist[p],p];

```

`KroneckerDelta` is a build in function for enforcing delta functions and we simply compare the totals for the vectors in the correlator to a vector of the same length with zeros as entries.

Now we have all the pieces we need to compute correlation functions. To compute the correlation functions, we use `cor[]`. For example, to compute the correlator $\langle \partial X^{\dot{A}A}(z_1) \partial X^{\dot{B}B}(z_2) \partial X^{\dot{C}C}(z_3) \partial X^{\dot{D}D}(z_4) \rangle$ we would use `cor[dx[Adot,A,z1,0,1],dx[Bdot,B,z2,0,1],dx[Cdot,C,z3,0,1],dx[Ddot,D,z4,0,1]]`. This finishes the summary of the code that we have developed.

4.6 Applications of the code

To finish off this chapter, we will put everything we have discussed so far together and show how we were able to compute the three point functions presented in chapter 3. We will also present some preliminary results for twist 3 and twist 4 that were computed with the code.

To start, we must first consider the correct lifting map. This was stated in equation 3.60, which would have the required ramification. With this, we can consider lifting the deformation operator and the candidate operator to the cover. The general lift for these operators was already done in previous work but we can also use the code to do this. These lift with our map as,

$$\mathcal{O}_D(b, \bar{b}) \rightarrow -i|b|^{-5/4} \epsilon^{AB} \partial X_{\dot{A}A}(1) \bar{\partial} X_{\dot{B}B}(1) S^{\dot{A}\dot{B}}(1, 1), \quad (4.15)$$

$$\begin{aligned} \mathcal{O}_C(a_1, \bar{a}_1) \rightarrow & \frac{|b|^4 |t_{\pm 1}|^4 |t_{\pm 1} - 1|^4}{16|a_1|^4 |a_1 - b|^4} \left(\epsilon^{\dot{C}\dot{D}} \epsilon^{CD} \partial X_{\dot{C}C}(t_{\pm 1}) \partial X_{\dot{D}D}(t_{\pm 1}) + \frac{1}{t_{\pm 1}^2 (t_{\pm 1} - 1)^2} \right) \times \\ & \left(\epsilon^{\dot{E}\dot{F}} \epsilon^{EF} \bar{\partial} X_{\dot{E}E}(\bar{t}_{\pm 1}) \bar{\partial} X_{\dot{F}F}(\bar{t}_{\pm 1}) + \frac{1}{\bar{t}_{\pm 1}^2 (\bar{t}_{\pm 1} - 1)^2} \right), \end{aligned} \quad (4.16)$$

where $t_{\pm 1} = (a_1 \pm \sqrt{a_1(a_1 - b)})/b$.

With these in hand, we can now turn to considering operators that might mix at the twist 2 level. We take each of the operators at twist 2 that are listed in Appendix A, lift them to the cover, and test each one in the correlation function $\langle \mathcal{O}_C(a_1, \bar{a}_1) \mathcal{O}_D(b, \bar{b}) \mathcal{O}(0, 0) \rangle$. In general, the forms of these operators on the cover are not as simple as shown in chapter 3. This is due the fact that the map in (3.60) has two ramified points and so there are additional pieces when lifting to the cover. We can include an example of some of this by looking at the following lift,

$$\begin{aligned} \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \partial X_{-3/2}^{\dot{A}A} \psi_0^{\alpha\dot{D}} \bar{\partial} X_{-3/2}^{\dot{E}E} \tilde{\psi}_0^{\dot{\alpha}\dot{H}} \sigma_2^{\beta\dot{\beta}} \rightarrow & \frac{2}{|b|^{1/4}} \epsilon_{AE} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{A}\dot{D}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \times \\ & \left(\frac{1}{2} \partial^3 X^{\dot{A}A} - 3\partial^2 X^{\dot{A}A} + \frac{3}{2} \partial X^{\dot{A}A} \right) \left(\frac{1}{2} \bar{\partial}^3 X^{\dot{E}E} - 3\bar{\partial}^2 X^{\dot{E}E} + \frac{3}{2} \bar{\partial} X^{\dot{E}E} \right) S^{\dot{D}\dot{H}}. \end{aligned} \quad (4.17)$$

There are a few things to note here. First, in addition to the $\partial^3 X$ terms we saw in chapter 3, we also have a few other terms like $\partial^2 X$ and ∂X . These show up because of the Taylor expansion of the map inside residue integral. We also have a normalization of $|b|^{-1/4}$. This is from a normalization needed for the spin fields on the base. When we lift with our map, we have a behaviour of $z \sim -bt^2$ around $z = 0$, and so this transformation must be reflected when transforming the spin field in the twist 2. It has weight $h = (1/4)/2$ for reasons explained in section 1.9.4. Furthermore, when we have non-trivial fermionic excitation, the cocycles in the code does not simplify to simple SU(2) index structures, as the code cannot recognize them. This leads to, at times, very large expressions for operators on the cover.

Despite this, the code does compute things correctly. In fact, we can double check

that we are doing the code correctly by considering the three point function,

$$\langle \mathcal{O}_C^{(t)}(t_{\pm 1}, \bar{t}_{\pm 1}) \mathcal{O}_D^{(t)}(1, 1) \mathcal{O}^{(t)}(0, 0) \rangle, \quad (4.18)$$

where we define $\mathcal{O}^{(t)}$ to be the lift of $\sum_i a_i \mathcal{O}_i$ with \mathcal{O}_i are defined in equations (3.47)-(3.55). We find the result,

$$\begin{aligned} & \frac{9(a_9|t_{\pm 1}|^4 - 2a_7t_{\pm 1}^2(4 - 8\bar{t}_{\pm 1} + \bar{t}_{\pm 1}^2) - a_8t^2(4 - 8\bar{t}_{\pm 1} + \bar{t}_{\pm 1}^2) - 2a_3\bar{t}_{\pm 1}^2(4 - 8t_{\pm 1} + t_{\pm 1}^2))}{|a_1|^4|a_1 - b|^4|b|^{1/2}|t_{\pm 1}|^4} \\ & + \frac{9(4a_1|4 - 8t_{\pm 1} + t_{\pm 1}^2|^2 + 2a_2|4 - 8t_{\pm 1} + t_{\pm 1}^2|^2 - a_6\bar{t}_{\pm 1}^2(4 - 8t_{\pm 1} + t_{\pm 1}^2))}{|a_1|^4|a_1 - b|^4|b|^{1/2}|t_{\pm 1}|^4} \\ & + \frac{9(2a_4|4 - t_{\pm 1} + t_{\pm 1}^2|^2 + a_5|4 - t_{\pm 1} + t_{\pm 1}^2|^2)}{|a_1|^4|a_1 - b|^4|b|^{1/2}|t_{\pm 1}|^4}. \end{aligned} \quad (4.19)$$

This is not very clear and does not look like a proper three point function. However, recall that in chapter 3 we mentioned that $\sum_i a_i \mathcal{O}_i$ is not a quasi-primary. Using the conditions that $6a_1 + 3a_4 + a_7 = 0$, $6a_2 + 3a_5 + a_8 = 0$, $6a_1 + 3a_2 + a_3 = 0$, $6a_4 + 3a_5 + a_6 = 0$, and $6a_7 + 3a_8 + a_9 = 0$, we can simplify this down to

$$\langle \mathcal{O}_C^{(t)}(t_{\pm 1}, \bar{t}_{\pm 1}) \mathcal{O}_D^{(t)}(1, 1) \mathcal{O}^{(t)}(0, 0) \rangle = \frac{144(4c_{11} + 2c_{12} + 2c_{21} + c_{22})|1 - t_{\pm 1}|^4}{|a_1|^4|a_1 - b|^4|b|^{1/2}|t_{\pm 1}|^4}, \quad (4.20)$$

where we now have $\mathcal{O} = c_{11}\mathcal{O}_{11} + c_{12}\mathcal{O}_{12} + c_{21}\mathcal{O}_{21} + c_{22}\mathcal{O}_{22}$ with \mathcal{O}_{11} , \mathcal{O}_{12} , \mathcal{O}_{21} , and \mathcal{O}_{22} defined in equations (3.56)-(3.59). There are still a few things left to do to get the final result. First, we must sum over the images, like in chapter 3. This gives

$$\frac{288(4c_{11} + 2c_{12} + 2c_{21} + c_{22})}{a_1^3(a_1 - b)b^{1/4}\bar{a}_1^3(\bar{a}_1 - \bar{b})b^{1/4}}. \quad (4.21)$$

The final piece to finish the calculation would be to include the contribution from the bare twists. For our correlation function, we have the properly normalized contribution of

$$\langle \sigma_2(b, \bar{b}) \sigma_2(0, 0) \rangle = \frac{1}{b^{3/4}\bar{b}^{3/4}} \quad (4.22)$$

This then leads to the result,

$$\langle \mathcal{O}_C(a_1, \bar{a}_1) \mathcal{O}_D(b, \bar{b}) \mathcal{O}(0, 0) \rangle = \frac{288(4c_{11} + 2c_{12} + 2c_{21} + c_{22})}{a_1^3(a_1 - b)b\bar{a}_1^3(\bar{a}_1 - \bar{b})\bar{b}}. \quad (4.23)$$

This was the result we discussed in section 3.4.2. Note that all of this was handled in the code, from the lifting to the computation of the correlator on the cover. We only

needed to include the two point function (4.22) and sum over the copies by hand, as the code does not handle this itself. Not only does the structure constant match, but we also have the correct form for a three point function. This is a non-trivial check that our code works properly.

We can use the code to go further. Although the OPE method examined in chapter 3 does not yet work for operators with twists, the three point functions are still calculable. To do this, we consider the correlators

$$\langle \mathcal{O}_{n+1}(0,0)\mathcal{O}_D(1,1)\mathcal{O}_n(\infty,\bar{\infty}) \rangle, \quad (4.24)$$

where \mathcal{O}_{n+1} and \mathcal{O}_n are operators of twist $n+1$ and twist n respectively. To have a well defined limit as we approach infinity, we must normalize by the two point function $\langle \mathcal{O}_n(0,0)\mathcal{O}_n(\infty,\bar{\infty}) \rangle$. We are suppressing the various other normalizations that would also need to be included as well. Since we have the mixing operator of twist 2, we can use the above correlator to find the mixing with twist 3. With twist 3, we can find mixing with twist 4 and so on. The general map to lift this correlator, we can use the map

$$z(t) = \frac{t^n}{nt - (n-1)}. \quad (4.25)$$

This map has the ramifications, $z(t) \sim t^n/(1-n)$ for $z=0=t$, $z-1 \sim n(n-1)(t-1)^2/2$ for $z=1=t$, and $z \sim t^{n-1}/n$ for $z,t \rightarrow \infty$. These are the correct ramifications for our correlator. We can use the code to compute the lifting and to compute the correlator. There is a small issue with lifting around infinity, but we get around this by transforming the residue integral to bring infinity to zero and then compute as we normally would. The results of these calculations for twist 3 and twist 4 can be found in Appendix B.

In theory, the code could complete the rest of the calculations up to twist 8 to find all the mixing operators but we run into a problem as we go to higher twists. The higher twist levels have large number and more complicated operators leading to significantly larger expressions for the operators and more pieces for the code to compute. As we mentioned in section 4.3, we end up saving a lot of results for the contractions. This means that the more complex and larger the operator, the more Mathematica must store in the RAM.

Despite this limitation, the code is still of great use. It can be used with OPE method explained in chapter 3 to check mixing more directly. The code could also be used to compute correlation functions not related to operator mixing. Since correlators are used in probing many different aspects of the D1D5 CFT, there are many possibilities for

future work with the code. Overall, this code can be useful in cutting down a lot of the annoying aspects of computing OPEs and correlation functions in the D1D5 CFT.

Chapter 5

Conclusions

5.1 Summary

The overall theme of the thesis is captured in the title. We view CFTs as a way of defining and probing quantum gravity. We mentioned a number of different ways in the introduction about how holographic CFTs capture a number of features of AdS quantum gravity without needing a specific string theoretic construction. Our system, chosen for its relative calculability and for its firm UV standing, is just one of many ways to approach quantum gravity. We can expect a number of important physical phenomena to be present in a well defined quantum gravity. Our motivation for this thesis stemmed from a central thorny problem in the study of black holes, the information problem. This problem has yet to be solved, but there has been significant recent progress in understanding where semiclassical physics fails. The trick is to show how quantum gravity corrects semiclassical predictions.

To study this problem in detail, we chose the context of string theory as a well known theory of quantum gravity. The purpose of this is to have a UV theory with which we can make concrete calculations without resorting to ancient semiclassical approximations. String theory can be used to construct microstates of the black hole. Studying these microstates can be done using AdS/CFT. This duality relates the gravitational degrees of freedom in an asymptotically AdS space to those in the degrees of freedom of CFT living on a space conformal to the boundary.

We studied the D1D5 system, a prototype system describing a two-charge 5 dimensional black hole. The dual CFT has a description in its moduli space as a free orbifold theory, $(T^4)^N/S_N$. The restriction introduced by the orbifold allows the construction of operators with twisted boundary conditions. These operators are important for reaching towards the supergravity description from this very stringy orbifold point. We focused on

deforming this orbifold CFT with a deformation operator built from a twist 2 operator dual to a blow up mode resolving the orbifold. The main purpose of this thesis was to explore the deformed D1D5 CFT through the OPE, correlators, and other techniques.

Chapter 2 of this thesis demonstrated how we can extract the effect of generic twist operators from the D1D5 CFT in the continuum limit. We noted that twist operators are important for thermalization. In black holes, thermalization is an important process that will scramble information. Thermalization does not show up in a free theory, and so we must introduce interactions between the copies by twist operators. To study how the twist operators act, we exploit the property that twist operators acting on a vacuum will create a squeezed state. Earlier work focused on studying these states by computing correlators in the D1D5 CFT with the Lunin-Mathur method. This works for low order calculations, but inverting the map to the covering space will in general fail for higher order calculations. We avoided this by presenting a novel composite twist method, in which we build more general twist operators out of twist 2 operators in the continuum limit. This is the limit where the gap between momentum excitations is parametrically small, which matches the expectations of black holes. In this limit, the expressions for the squeezed state are known and we can build up the required matrices to compute the characteristic coefficients of the squeezed state from these composite twists. We confirmed the validity of this method by comparing its results to those previously obtained with the Lunin-Mathur method. We then examined a new configuration of twists. The results for this computation, plus some general power-counting arguments, led us to confirm a set of conjectured scaling laws for the effect of generic twist operators.

The method here could be of use to considering thermalization and scrambling more directly. If we consider creating some excitation on a single strand, we could then track how this excitation is spread as we interact the copies using twist operators. Though one can consider this to first or second order with previous methods, our method here could give some indication how thermalization looks at more general orders.

Chapter 3 of the thesis showed how the OPE can be used as a calculational tool in addition to correlation functions to study the deformed D1D5 CFT. We considered the D1D5 CFT near the orbifold point and developed methods for computing the mixing of untwisted operators to first order by using the OPE on the covering surface. We argued that the OPE on the cover encodes both the structure constants for the orbifold CFT and the explicit form of the mixing operators. We began by considering a family of operators dual to supergravity modes, and showed that the OPE implies that there is no anomalous dimension to first order, as expected. We then specialized to the operator dual to the dilaton, and showed that the leading order singularity in the OPE reproduces the

correct structure constant obtained from the four point function. Finally, we considered an unprotected candidate low-lying string state operator of conformal dimension (2,2), and showed that the leading order singularity and one of the subleading singularities both reproduced the correct structure constant. We checked that the operator produced at subleading order using the OPE method is correct by calculating a large number of three point functions using a Mathematica package we developed.

The work discussed here is an important step forward in examining the deformed D1D5 CFT. The OPE on the cover is a far easier thing to deal with, and allows us to consider the structure constants and mixing operators more directly than just dealing with the correlation functions. This could be used to further compute the anomalous dimensions of operators, but it may also find use in examining other aspects of the D1D5 CFT such as the conformal blocks and their behaviours.

Chapter 4 of the thesis reviewed work done in order to put into code both the OPE and the correlation functions for the D1D5 CFT. This Mathematica package was highly important for our work in the third chapter. In order to implement the required functionality into the package, we needed to include the basic definitions of fields and their fundamental OPEs defining the CFT. This included programming in the bosonization and cocycles required for the CFT, and implementing Wick contractions. Since the CFT at the orbifold point is free, we could use simple Wick contractions to compute the OPEs of composite operators. This then allowed us to lift operators to the covering space as in the Lunin-Mathur method, including operators with fractional moding which require care in performing residue integrals. Finally, we were able to adapt the code to compute correlation functions. We used this ability to analyze the mixing of the candidate low-lying string state in chapter 3 and we were able to obtain some preliminary results for mixing at higher twists.

Overall, we have learned a number of aspects about how to analyze the deformed D1D5 CFT but there is still a lot left to do. In the next section I will outline a few possible detailed directions to consider for the future.

5.2 Future directions

5.2.1 Mixing in deformed D1D5 from OPEs

One possible next step would be to follow up on some of the discussion in section 3.5. There we outlined how we would generalize our results to higher twist levels. We discussed how the more general approach to using the OPE to analyzing mixing with a perturbation

of the deformation operator was to examine the OPE

$$\mathcal{O}_D(z_0, \bar{z}_0)\mathcal{O}(0, 0), \quad (5.1)$$

and lift with

$$z = -nt^n \left(t - \frac{n+1}{n}t_0 \right). \quad (5.2)$$

The difficulty is that this map does not have the correct ramification for the results of the OPE. This means we must somehow be able to move the location of the twist insertion, which means studying how translation generators act on the cover. We noted how, around ramified points, the lift of the L_{-1} mode on the base is non-trivial on the cover.

There are a number of ways to attempt to approach the problem. The first is to see the moving of the twist insertion directly through the Lunin-Mathur method. One could examine Liouville contribution and see how the action changes under a conformal transformation that implements a translation on the base. This would be a direct way to probe how the twist operator would change.

A different approach is to examine how the L_{-1} lifts to the cover more carefully. As discussed earlier, if we define a L_1 around a ramified point, it lifts to terms with L_{-k} where $k \geq 2$. So how do we generate L_{-1} modes on the cover? Normally if we have fractionally moded operators, $\mathcal{O}_{-k/n}$ we find the lift to the cover as \mathcal{O}_{-k} . But we cannot define $L_{-1/2}$ due to the nature of the conformal symmetry group. So we must examine $\mathcal{O}_D(z_0 + \delta z_0)$ and $\mathcal{O}_{D^\dagger}(t_0 + \delta t_0)$. In other words, how does an infinitesimal translation on the base correspond to an infinitesimal translation on the cover?

These questions are interesting not only because it would allow us to use the OPE to study mixing but it would also clarify how the covering space in the Lunin-Mathur method organizes the structures of the base space. Concepts such as translations exist and are well defined on the base space and studying it would give us a better understanding of this method.

Of course, with this method we could examine other operators other than just the candidate operator we studied. For example, the fermionic version of the operator might be interesting to study and see any connections to this. As already mentioned, the higher spin currents have already been studied in [211]. However, there may be other operators that have well defined representations under these higher spin operators other than the currents that might provide a clue to how the higher spin symmetry is Higgsed. With the OPE method, we could more efficiently study operators with even higher conformal weights without worrying about the number of calculations getting out of hand.

5.2.2 Infall and chaos

Of course, there are other aspects of the D1D5 system other than just mixing under the deformation operator. Our code can compute correlation functions in this system that may show us other physics.

Studying infall in the D1D5 system is difficult. This is due to the fact that if we study operators that are not protected by supersymmetry non-renormalization theorems, it is difficult to compare to results from the orbifold point and the supergravity point. However, that does not mean we are unable to study the infall process at all. The work presented in [234] presented a model of absorption and emission in an extremal D1D5. They study this with the correlator

$$\langle S_{N-l,l}(v)S_{N-k,k}^\dagger(w)[(J_0^-)^p\sigma_n^P](0)[(J_0^+)^q\sigma_n^{Q\dagger}](a)\rangle \quad (5.3)$$

setting $l = 0$ by global $SU(2)$. Here the spin fields set up the black hole background and the other two is the absorption and later emission of a chiral primary. They consider a simple black hole background here, mainly so that they could do the dual calculation to match relatively easily. To go further, we would want to study a more interesting state. We could consider some perturbation of them [235] or complicated states with different charges [43, 44, 139]. Alternatively, one could use something more complex than the simple chiral primary. However, like the case of a more complex state, we would need to consider an interesting field to calculate with. Moreover, emission of anything that is not a supergravity field would not have a dual computation to compare to.

Another area of interest is studying chaos in conformal field theories. Like with the case with thermalization, we do not expect to see chaos in at the orbifold point. This is due to the fact that the D1D5 system at this point in its moduli space is integrable. However, the authors of [237] studied a time-dependent coarse graining at the orbifold point to simulate what chaos in the D1D5 system might look like away from the orbifold point. One could instead just study this phenomenon directly by introducing the deformation operator. The drawback of this that we would have a six point correlator to deal with, two twisted operators to create the initial and final states, two twisted deformation operators, and two untwisted probe operators. In theory this could be done either by hand or with the code, but it would be a long calculation either way. Furthermore, chaos may not be evident at this level of perturbation theory and may require higher orders of deformations. Regardless, examining this deformation might provide some clue as to how chaos behaves with the D1D5 system and clarify if the authors model correctly captures the physics.

5.2.3 Conformal blocks and entanglement

A useful idea to explore would be the conformal blocks of the D1D5 system and their relation to black hole information. We discussed the details of this in section 1.7, that looking at $1/c$ corrections to conformal blocks is related to studying the black hole information problem. It was already shown in [225] that in the heavy-light limit and large c there are no singularities indicative of information loss. They study this limit both in the gravity and CFT by looking at protected operators. Their results were done with very specific operator and they argued that their results should generalize. However, it might be interesting to check this with other operators such as those that are purely higher spin fields as suggested in the paper. Alternatively it might be interesting to think about this the other way and try to see how these singularities would arise from the fuzzball complementarity picture.

Entanglement entropy has been well studied in the D1D5 system [137–139]. It would be difficult to extend this much further, but there are still a few unanswered questions. The first, identified by [139], would be to study how the conformal blocks of primary operators other than the identity contribute the entanglement entropy of the fuzzball states. Another direction would be to study the geometries of [44], which are valid for arbitrary small angular momentum. Finding the relationship between the entanglement entropy of the CFT states as a function of the angular momentum of the dual geometry may lead to some insight on how well the minimal geodesics are able to probe the geometries of fuzzballs.

The D1D5 system could also be used to study other interesting aspects of holography and black holes such as complexity and entwinement. A general description of how complexity may fit into the fuzzball program was laid out in [238]. Here, infall in the gravity picture is described from the dual picture as states evolving to higher complexity states. It would be interesting to see if this model of fuzzball complementarity could be connected to the calculations for the complexity in terms of the Wheeler-deWitt patch [90]. Without a horizon in a fuzzball geometry, it is not possible to define a volume behind a horizon, but the Wheeler-deWitt patch, defined as the union of all spatial slices anchored at a given boundary time, should be well defined. It may provide a clearer definition to extend the model presented in [238]. Alternatively, we could consider other methods for defining complexity, as discussed in [239, 240], where the complexity is defined on the CFT side through some quantum information definitions. Entwinement was directly related to the D1D5 system in the paper [85]. Here they showed a construction of entwinement in symmetric orbifold CFTs and then directly used their method for calculating the entwinement and comparing it to the non-minimal

geodesics of the corresponding supergravity state. They look at very simple states, so it might be possible to consider more general states and see if the correspondence still holds.

We do not need to fully restrict ourselves to the D1D5 system to study black holes. Without a specific system to study, we can turn to looking for more universal results like the results that have been obtained using quantum information theory. One direction we have been thinking about is taking a closer look at kinematic space for Bañados, Teitelboim, and Zanelli (BTZ) black holes. The BTZ black hole is a solution for (2+1)-dimensional Einstein gravity with a negative cosmological constant. The kinematic space of BTZ has been examined from the bulk side in [241] and from the point of view of tensor networks in [108], but there has not been much development in looking at the kinematic space from the CFT side. One can start with the simpler case of a conical defect defined by AdS_3 quotiented by \mathbb{Z}_N , which may provide clues on how to handle BTZ [242]. If one could solve this simpler problem, then the generalization to the BTZ case may become more clear. Furthermore this would clarify the role of the minimal and non-minimal geodesics in the kinematic space.

5.3 Final words

Quantum gravity is a complicated and difficult thing to study. The approach taken for this thesis, seeing conformal field theories as defining quantum gravities, is just one of many approaches and the D1D5 system itself is only one of many holographic systems. The work presented here is a small part of the larger and more ambitious programme. However small, the two methods described here allow us to push further along towards understanding both the D1D5 system and, more generally, string theory as a theory of quantum gravity.

Appendix A

Table of mixing operators

Below is a table of operators that have weight 2 and can be made into singlets of the $SU(2)$ s when paired with the appropriate anti-holomorphic part. The first column will contain the form of the operator. The second will give the twist level. The next four will be the number of doublet indices from the operator for each of the $SU(2)$ s. The final column will be a generic symbol with indices left over in the holomorphic part and show how many indices must be contracted with the antiholomorphic to make a full singlet. Recall $\hat{\sigma}$ is the bare twist, whereas σ are the bare twist including the spin field taken into account for the lift (but not including any fractionally moded J s, so not the chiral primary for twist > 2). Also recall that we use α for index of $SU(2)_L$, $\dot{\alpha}$ for index of $SU(2)_R$, A for index of $SU(2)_1$, and \dot{A} for $SU(2)_2$.

Assuming these all are involved with mixing, the full, final operator count (including both left and right pieces that can be made into singlets) at each twist level is:

Twist 2: 34

Twist 3: 41

Twist 4: 128

Twist 5: 41

Twist 6: 24

Twist 7: 8

Twist 8: 1

Total: 287

Operator	Twist	$SU(2)_L$	$SU(2)_1$	$SU(2)_2$
$(\partial X_{-\frac{1}{6}})^3 \sigma$	6	$\underline{1}$	$\underline{2} \times \underline{2} \times \underline{2} \rightarrow \underline{2}$	$\underline{2} \times \underline{2} \times \underline{2} \times \underline{2} \rightarrow \underline{1}$
$\partial X_{-\frac{1}{6}} \partial X_{-\frac{1}{3}} \sigma$	6	$\underline{1}$	$\underline{2} \times \underline{2} \rightarrow \underline{1}$	$\underline{2} \times \underline{2} \times \underline{2} \rightarrow \underline{2}$
$\partial X_{-\frac{1}{2}} \sigma$	6	$\underline{1}$	$\underline{2}$	$\underline{2} \times \underline{2} \rightarrow \underline{1}$
$(\partial X_{-\frac{1}{6}})^2 \psi_{-\frac{1}{6}} \sigma$	6	$\underline{2} \times \underline{2} \rightarrow \underline{1}$	$\underline{2} \times \underline{2} \rightarrow \underline{1}$	$\underline{2} \times \underline{2} \times \underline{2} \rightarrow \underline{2}$
$\partial X_{-\frac{1}{6}} (\psi_{-\frac{1}{6}})^2 \sigma$	6	$\underline{2} \times \underline{2} \rightarrow \underline{1}$	$\underline{2}$	$\underline{2} \times \underline{2} \times \underline{2} \times \underline{2} \rightarrow \underline{1}$
$\partial X_{-\frac{1}{3}} \psi_{-\frac{1}{6}} \sigma$	6	$\underline{2} \times \underline{2} \rightarrow \underline{1}$	$\underline{2}$	$\underline{2} \times \underline{2} \rightarrow \underline{1}$
$\partial X_{-\frac{1}{6}} \psi_{-\frac{1}{3}} \sigma$	6	$\underline{2} \times \underline{2} \rightarrow \underline{1}$	$\underline{2}$	$\underline{2} \times \underline{2} \rightarrow \underline{1}$
$\psi_{-\frac{1}{3}} \psi_{-\frac{1}{6}} \sigma$	6	$\underline{2} \times \underline{2} \rightarrow \underline{1}$	$\underline{1}$	$\underline{2} \times \underline{2} \times \underline{2} \rightarrow \underline{2}$
$(\psi_{-\frac{1}{6}})^3 \sigma$	6	$\underline{2} \times \underline{2} \times \underline{2} \rightarrow \underline{2}$	$\underline{1}$	$\underline{2} \times \underline{2} \times \underline{2} \rightarrow \underline{2}$
$(\partial X_{-\frac{1}{7}})^2 \hat{\sigma}$	7	$\underline{1}$	$\underline{2} \times \underline{2} \rightarrow \underline{1}$	$\underline{2} \times \underline{2} \rightarrow \underline{1}$
$\partial X_{-\frac{1}{7}} \hat{\sigma}$	7	$\underline{1}$	$\underline{2}$	$\underline{2}$
$(\psi_{-\frac{1}{14}})^2 \partial X_{-\frac{1}{7}} \hat{\sigma}$	7	$\underline{2} \times \underline{2} \rightarrow \underline{1}$	$\underline{2} \times \underline{2} \rightarrow \underline{1}$	$\underline{2} \times \underline{2} \times \underline{2} \rightarrow \underline{2}$
$(\psi_{-\frac{1}{14}})^4 \hat{\sigma}$	7	$\underline{2} \times \underline{2} \times \underline{2} \times \underline{2} \rightarrow \underline{1}$	$\underline{1}$	$\underline{2} \times \underline{2} \times \underline{2} \times \underline{2} \rightarrow \underline{1}$
σ	8	$\underline{1}$	$\underline{1}$	$\underline{2}$

Appendix B

Higher twist mixing operators

The twist 2 operator found in (3.74) could mix with operators at twist 3. The results here were obtained using the Mathematica package. Looking for operators that mix with this operator we find the operators that mix are given by

$$\begin{aligned}
\mathcal{O}_{n3u1b1} &= \epsilon_{\dot{D}\dot{C}}\epsilon_{DC}\epsilon_{\dot{B}\dot{A}}\epsilon_{BA}\epsilon_{\dot{H}\dot{G}}\epsilon_{HG}\epsilon_{\dot{F}\dot{E}}\epsilon_{FE} \\
&\quad \partial X_{-1/3}^{\dot{D}D}\partial X_{-1/3}^{\dot{C}C}\partial X_{-1/3}^{\dot{B}B}\partial X_{-1/3}^{\dot{A}A}\bar{\partial} X_{-1/3}^{\dot{H}H}\bar{\partial} X_{-1/3}^{\dot{G}G}\bar{\partial} X_{-1/3}^{\dot{F}F}\bar{\partial} X_{-1/3}^{\dot{E}E}\hat{\sigma}_3 \\
\mathcal{O}_{n3u1b4} &= \epsilon_{\dot{D}\dot{C}}\epsilon_{DC}\epsilon_{\dot{B}\dot{A}}\epsilon_{BA}\epsilon_{\dot{F}\dot{E}}\epsilon_{FE}\partial X_{-1/3}^{\dot{D}D}\partial X_{-1/3}^{\dot{C}C}\partial X_{-1/3}^{\dot{B}B}\partial X_{-1/3}^{\dot{A}A}\bar{\partial} X_{-2/3}^{\dot{F}F}\bar{\partial} X_{-2/3}^{\dot{E}E}\hat{\sigma}_3 \\
\mathcal{O}_{n3u1b7} &= \epsilon_{\dot{D}\dot{C}}\epsilon_{DC}\epsilon_{\dot{B}\dot{A}}\epsilon_{BA}\epsilon_{HG}\epsilon_{\dot{H}\dot{F}}\epsilon_{\dot{G}\dot{E}}\epsilon_{\dot{\lambda}\dot{\gamma}} \\
&\quad \partial X_{-1/3}^{\dot{D}D}\partial X_{-1/3}^{\dot{C}C}\partial X_{-1/3}^{\dot{B}B}\partial X_{-1/3}^{\dot{A}A}\bar{\partial} X_{-2/3}^{\dot{H}H}\bar{\partial} X_{-1/3}^{\dot{G}G}\tilde{\psi}_{-1/6}^{\dot{\lambda}\dot{F}}\tilde{\psi}_{-1/6}^{\dot{\gamma}\dot{E}}\hat{\sigma}_3 \\
\mathcal{O}_{n3u4b1} &= \epsilon_{\dot{B}\dot{A}}\epsilon_{BA}\epsilon_{\dot{H}\dot{G}}\epsilon_{HG}\epsilon_{\dot{F}\dot{E}}\epsilon_{FE}\partial X_{-2/3}^{\dot{B}B}\partial X_{-2/3}^{\dot{A}A}\bar{\partial} X_{-1/3}^{\dot{H}H}\bar{\partial} X_{-1/3}^{\dot{G}G}\bar{\partial} X_{-1/3}^{\dot{F}F}\bar{\partial} X_{-1/3}^{\dot{E}E}\hat{\sigma}_3 \\
\mathcal{O}_{n3u4b4} &= \epsilon_{\dot{B}\dot{A}}\epsilon_{BA}\epsilon_{\dot{F}\dot{E}}\epsilon_{FE}\partial X_{-2/3}^{\dot{B}B}\partial X_{-2/3}^{\dot{A}A}\bar{\partial} X_{-2/3}^{\dot{F}F}\bar{\partial} X_{-2/3}^{\dot{E}E}\hat{\sigma}_3 \\
\mathcal{O}_{n3u4b7} &= \epsilon_{\dot{B}\dot{A}}\epsilon_{BA}\epsilon_{HG}\epsilon_{\dot{H}\dot{F}}\epsilon_{\dot{G}\dot{E}}\epsilon_{\dot{\lambda}\dot{\gamma}}\partial X_{-2/3}^{\dot{B}B}\partial X_{-2/3}^{\dot{A}A}\bar{\partial} X_{-2/3}^{\dot{H}H}\bar{\partial} X_{-1/3}^{\dot{G}G}\tilde{\psi}_{-1/6}^{\dot{\lambda}\dot{F}}\tilde{\psi}_{-1/6}^{\dot{\gamma}\dot{E}}\hat{\sigma}_3\hat{\sigma}_3 \\
\mathcal{O}_{n3u7b1} &= \epsilon_{\dot{H}\dot{G}}\epsilon_{HG}\epsilon_{\dot{F}\dot{E}}\epsilon_{FE}\epsilon_{DC}\epsilon_{\dot{D}\dot{B}}\epsilon_{\dot{C}\dot{A}}\epsilon_{\beta\alpha} \\
\mathcal{O}_{n3u7b1} &= \epsilon_{\dot{H}\dot{G}}\epsilon_{HG}\epsilon_{\dot{F}\dot{E}}\epsilon_{FE}\epsilon_{DC}\epsilon_{\dot{D}\dot{B}}\epsilon_{\dot{C}\dot{A}}\epsilon_{\beta\alpha} \\
&\quad \partial X_{-2/3}^{\dot{D}D}\partial X_{-1/3}^{\dot{C}C}\psi_{-1/6}^{\beta\dot{B}}\psi_{-1/6}^{\alpha\dot{A}}\bar{\partial} X_{-1/3}^{\dot{H}H}\bar{\partial} X_{-1/3}^{\dot{G}G}\bar{\partial} X_{-1/3}^{\dot{F}F}\bar{\partial} X_{-1/3}^{\dot{E}E}\hat{\sigma}_3 \\
\mathcal{O}_{n3u7b4} &= \epsilon_{\dot{F}\dot{E}}\epsilon_{FE}\epsilon_{DC}\epsilon_{\dot{D}\dot{B}}\epsilon_{\dot{C}\dot{A}}\epsilon_{\beta\alpha}\partial X_{-2/3}^{\dot{D}D}\partial X_{-1/3}^{\dot{C}C}\psi_{-1/6}^{\beta\dot{B}}\psi_{-1/6}^{\alpha\dot{A}}\bar{\partial} X_{-2/3}^{\dot{F}F}\bar{\partial} X_{-2/3}^{\dot{E}E}\hat{\sigma}_3 \\
\mathcal{O}_{n3u7b7} &= \epsilon_{HG}\epsilon_{\dot{H}\dot{F}}\epsilon_{\dot{G}\dot{E}}\epsilon_{DC}\epsilon_{\dot{D}\dot{B}}\epsilon_{\dot{C}\dot{A}}\epsilon_{\dot{\lambda}\dot{\gamma}}\epsilon_{\beta\alpha} \\
&\quad \partial X_{-2/3}^{\dot{D}D}\partial X_{-1/3}^{\dot{C}C}\bar{\partial} X_{-2/3}^{\dot{H}H}\bar{\partial} X_{-1/3}^{\dot{G}G}\psi_{-1/6}^{\beta\dot{B}}\tilde{\psi}_{-1/6}^{\dot{\lambda}\dot{F}}\psi_{-1/6}^{\alpha\dot{A}}\tilde{\psi}_{-1/6}^{\dot{\gamma}\dot{E}}\hat{\sigma}_3
\end{aligned} \tag{B.1}$$

These are all independent. Diagonalizing like before, we find

$$\begin{aligned}
 \mathcal{O}_{n3A} = & \frac{27\mathcal{O}_{n3u1b1}}{128\sqrt{140041}} + \frac{15}{64}\sqrt{\frac{3}{280082}}\mathcal{O}_{n3u1b4} + \frac{3}{16}\sqrt{\frac{3}{140041}}\mathcal{O}_{n3u1b7} + \frac{15}{64}\sqrt{\frac{3}{280082}}\mathcal{O}_{n3u4b1} \\
 & + \frac{25\mathcal{O}_{n3u4b4}}{64\sqrt{140041}} + \frac{5\mathcal{O}_{n3u4b7}}{8\sqrt{280082}} + \frac{3}{16}\sqrt{\frac{3}{140041}}\mathcal{O}_{n3u7b1} + \frac{5\mathcal{O}_{n3u7b4}}{8\sqrt{280082}} - \frac{1}{2}\sqrt{\frac{5}{140041}}\mathcal{O}_{n3u7b7}
 \end{aligned} \tag{B.2}$$

This has a structure constant $C_{iDA} = -(94/\sqrt{140041})$. This twist 3 operator will mix with operators at twist 4. At twist 4, we find 36 mixing operators. These are

Again, we can rediagonalize using the structure constants. The results are then

$$\begin{aligned}
& - ((-375(1632174696960885657034471\sqrt{2} + 621163548073744078648500\sqrt{3})\mathcal{O}_{n4u10b10} \\
& + 175(1863490644221232235945500 + 1632174696960885657034471\sqrt{6})\mathcal{O}_{n4u10b12} \\
& - (46587266105530805898637500 + 40804367424022141425861775\sqrt{6})\mathcal{O}_{n4u10b15} \\
& + (242253783748760190672915000\sqrt{5} + 212182710604915135414481230\sqrt{30})\mathcal{O}_{n4u10b2} \\
& + (367239306816199272832755975\sqrt{2} + 139761798316592417695912500\sqrt{3})\mathcal{O}_{n4u10b6} \\
& - (489652409088265697110341300 + 93174532211061611797275000\sqrt{6})\mathcal{O}_{n4u10b9} \\
& + (326110862738715641290462500 + 285630571968154989981032425\sqrt{6})\mathcal{O}_{n4u12b10} \\
& - (399882800755416985973445395\sqrt{2} + 152185069278067299268882500\sqrt{3})\mathcal{O}_{n4u12b12} \\
& + (57126114393630997996206485\sqrt{2} + 21740724182581042752697500\sqrt{3})\mathcal{O}_{n4u12b15} \\
& - (297055794846881189580273722\sqrt{10} + 113051765749421422314027000\sqrt{15})\mathcal{O}_{n4u12b2} \\
& - (195666517643229384774277500 + 171378343180892993988619455\sqrt{6})\mathcal{O}_{n4u12b6} \\
& + (130444345095486256516185000\sqrt{2} + 228504457574523991984825940\sqrt{3})\mathcal{O}_{n4u12b9} \\
& - (46587266105530805898637500 + 40804367424022141425861775\sqrt{6})\mathcal{O}_{n4u15b10} \\
& + (57126114393630997996206485\sqrt{2} + 21740724182581042752697500\sqrt{3})\mathcal{O}_{n4u15b12} \\
& + (8160873484804428285172355\sqrt{2} + 3105817740368720393242500\sqrt{3})\mathcal{O}_{n4u15b15} \\
& + (42436542120983027082896246\sqrt{10} + 16150252249917346044861000\sqrt{15})\mathcal{O}_{n4u15b2}
\end{aligned}$$

$$\begin{aligned}
& + (27952359663318483539182500 + 24482620454413284855517065\sqrt{6})\mathcal{O}_{n4u15b6} \\
& + (18634906442212322359455000\sqrt{2} + 32643493939217713140689420\sqrt{3})\mathcal{O}_{n4u15b9} \\
& + (242253783748760190672915000\sqrt{5} + 212182710604915135414481230\sqrt{30})\mathcal{O}_{n4u2b10} \\
& - (297055794846881189580273722\sqrt{10} + 113051765749421422314027000\sqrt{15})\mathcal{O}_{n4u2b12} \\
& + (42436542120983027082896246\sqrt{10} + 16150252249917346044861000\sqrt{15})\mathcal{O}_{n4u2b15} \\
& - (1103350095145558704155302396\sqrt{2} + 419906558497850997166386000\sqrt{3})\mathcal{O}_{n4u2b2} \\
& - (145352270249256114403749000\sqrt{5} + 127309626362949081248688738\sqrt{30})\mathcal{O}_{n4u2b6} \\
& + (96901513499504076269166000\sqrt{10} + 169746168483932108331584984\sqrt{15})\mathcal{O}_{n4u2b9} \\
& + (367239306816199272832755975\sqrt{2} + 139761798316592417695912500\sqrt{3})\mathcal{O}_{n4u6b10} \\
& - (195666517643229384774277500 + 171378343180892993988619455\sqrt{6})\mathcal{O}_{n4u6b12} \\
& + (27952359663318483539182500 + 24482620454413284855517065\sqrt{6})\mathcal{O}_{n4u6b15} \\
& - (145352270249256114403749000\sqrt{5} + 127309626362949081248688738\sqrt{30})\mathcal{O}_{n4u6b2} \\
& - (220343584089719563699653585\sqrt{2} + 8385707898995450617547500\sqrt{3})\mathcal{O}_{n4u6b6} \\
& + (167714157979910901235095000 + 146895722726479709133102390\sqrt{6})\mathcal{O}_{n4u6b9} \\
& - (489652409088265697110341300 + 93174532211061611797275000\sqrt{6})\mathcal{O}_{n4u9b10} \\
& + (130444345095486256516185000\sqrt{2} + 228504457574523991984825940\sqrt{3})\mathcal{O}_{n4u9b12} \\
& + (18634906442212322359455000\sqrt{2} + 32643493939217713140689420\sqrt{3})\mathcal{O}_{n4u9b15} \\
& + (96901513499504076269166000\sqrt{10} + 169746168483932108331584984\sqrt{15})\mathcal{O}_{n4u9b2} \\
& + (167714157979910901235095000 + 146895722726479709133102390\sqrt{6})\mathcal{O}_{n4u9b6} \\
& + (195860963635306278844136520\sqrt{2} + 74539625768849289437820000\sqrt{3})\mathcal{O}_{n4u9b9}) \\
& /((160(6^{3/4})\sqrt{1185481(434997000 + 289939321\sqrt{6})} \\
& /(96440412218155034761 + 5809127089939992000\sqrt{6}))) \tag{B.4}
\end{aligned}$$

This has a structure constant of

$$\begin{aligned}
& - ((2(2^{1/4})(3481428628617569106454526643 + 551675047572779352077651198\sqrt{2} \\
& + 209953279248925498583193000\sqrt{3} + 662470924020648059878625250\sqrt{6})) \tag{B.5} \\
& /((3^{3/4})\sqrt{1185481(434997000 + 289939321\sqrt{6})}) \\
& /(96440412218155034761 + 5809127089939992000\sqrt{6})).
\end{aligned}$$

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