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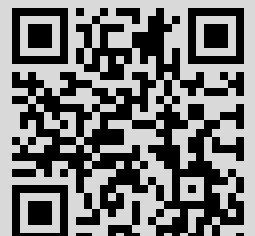
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A STUDY OF THE RIEMANN CURVATURE TENSOR AND PETROV CLASSIFICATION IN GENERAL RELATIVITY

*Z. Ahsan***Abstract**

In this article, we present and support by examples a criterion for the existence of gravitational radiation in terms of the invariants of the Riemann curvature tensor. We give a classification of spacetimes in terms of electric and magnetic parts of the Weyl tensor and discuss some examples of spacetimes having purely magnetic and purely electric Weyl tensors. The Lanczos potential is studied using the method of general observers and tetrad formalisms. We obtain the Lanczos potentials for perfect fluid spacetimes, Gödel cosmological model, and Kerr black hole. The work also considers the space-matter tensor, introduced by Petrov, and the perfect-fluid spacetimes with the divergence-free space-matter tensor.

Key words: Weyl tensor, Lanczos potential, tetrad formalisms, Gödel model, Kerr black hole.

Introduction

The general theory of relativity is a theory of gravitation in which gravitation emerges as the property of the space-time structure through the metric tensor g_{ij} . The metric tensor determines another object (of tensorial nature) known as *Riemann curvature tensor*. At any given event, this tensorial object provides all information about the gravitational field in the neighbourhood of the event. It may, in real sense, be interpreted as describing the curvature of the spacetime. The Riemann curvature tensor is the simplest non-trivial object one can build at a point; its vanishing is the criterion for the absence of genuine gravitational fields and its structure determines the relative motion of the neighbouring test particles via the equation of geodesic deviation. These discussions clearly illustrate the importance of the Riemann curvature tensor in general relativity and it is for these reasons, a study of this curvature tensor has been made here.

1. The invariants of Riemann tensor

The Riemann curvature tensor R_{ijl}^k is defined, for a covariant vector field A_k , through the Ricci identity [1]

$$A_{i;jl} - A_{i;l j} = R_{ijl}^k A_k, \quad (1)$$

where $R_{ijl}^k = \frac{\partial}{\partial x^j} \Gamma_{il}^k - \frac{\partial}{\partial x^l} \Gamma_{ij}^k + \Gamma_{il}^m \Gamma_{mj}^k - \Gamma_{ij}^m \Gamma_{ml}^k$.

The Riemann curvature tensor can be decomposed as [1]

$$R_{ijkl} = C_{ijkl} + E_{ijkl} + G_{ijkl}, \quad (2)$$

where C_{ijkl} is the Weyl tensor, $E_{ijkl} = -\frac{1}{2}(g_{ik}S_{jl} + g_{jl}S_{ik} - g_{il}S_{jk} - g_{jk}S_{il})$ is the Einstein curvature tensor, with $S_{ij} \equiv R_{ij} - \frac{1}{4}g_{ij}R$ being the traceless tensor

and $G_{ijkl} \equiv -\frac{R}{12}(g_{ik}g_{jl} - g_{il}g_{jk})$. The Ricci tensor R_{ij} is defined by $R_{ij} \equiv R_{ijk}^k$ and $R \equiv g^{ij}R_{ij}$ is the Ricci scalar. These equations lead to a more convenient decomposition of the Riemann tensor as [1]

$$R_{ijkl} = C_{ijkl} + \frac{1}{2}(g_{il}R_{jk} + g_{jk}R_{il} - g_{ik}R_{jl} - g_{jl}R_{ik}) - \frac{R}{6}(g_{il}g_{jk} - g_{ik}g_{jl}). \quad (3)$$

The Riemann curvature tensor has fourteen invariants. There is the Ricci scalar R . There are four invariants of the Weyl tensor C_{ijkl} . There are three invariants of the Einstein curvature tensor E_{ijkl} and six invariants of the combined Weyl and Einstein curvature tensors. In empty spacetimes, there are four invariants of Riemann tensor which are given by

$$A_1 = R_{ijkl}R^{ijkl}, \quad A_2 = R_{ijkl}^*R^{ijkl}, \\ B_1 = \frac{4}{3}R_{ijmn}R^{mnr}R_{rs}^{ij}, \quad B_2 = \frac{4}{3}R_{ijmn}^*R^{mnr}R_{rs}^{ij}.$$

These invariants have been calculated for the classifications of Riemann tensor according to Sharma and Husain [2] and Petrov [3]. It has been found that A_1 , A_2 , B_1 , and B_2 are all equal to zero for cases III(a) and III(b) or Petrov type III. This has led to the following criterion for the existence of gravitational radiation:

If $R_{abcd} \neq 0$ and $A_1 = A_2 = B_1 = B_2 = 0$, then the gravitational radiation is present; otherwise, there is no gravitational radiation.

The validity of this assertion has been checked by considering the following metrics (cf., [4]):

(i) *Takeo's plane wave solution*

$$ds^2 = -A dx^2 - 2D dx dy - B dy^2 - dz^2 + dt^2;$$

(ii) *Einstein-Rosen metric*

$$ds^2 = e^{2\gamma-2\psi}(dt^2 - dr^2) - r^2 e^{-2\psi} d\phi^2 - e^{2\psi} dz^2,$$

where γ and ψ are functions of r and t only, $\psi = 0$ and $\gamma = \gamma(r - t)$;

(iii) *The Peres metric*

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 - 2f(dx_4 + dx_3)^2 + dx_4^2;$$

(iv) *The Schwarzschild exterior solution*

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + \left(1 - \frac{2m}{r}\right) dt^2.$$

It is found that for the metrics (i)–(iii), all the four invariants of the Riemann tensor vanish and thus correspond to the state of gravitational radiation. While for the Schwarzschild exterior solution $A_1 \neq 0$, $B_1 \neq 0$, $A_2 = 0$, $B_2 \neq 0$; Schwarzschild solution, being a Petrov type D solution, is known to be non-radiative.

2. The electric and magnetic spacetimes

It is known that a physical field is always produced by a source, which is termed as its charge. Manifestation of fields when charges are at rest is called electric and magnetic when they are in motion. This general feature is exemplified by the Maxwell's theory of electromagnetism from which the terms of electric and magnetic are derived. This

decomposition can be adapted in general relativity and the Weyl tensor can be decomposed into electric and magnetic parts. Based on this decomposition, a classification of spacetimes is given here which is supported by a number of examples.

An observer with time like 4-velocity vector u is said [5] to measure the electric and magnetic components, E_{ac} and H_{ac} respectively, of the Weyl tensor C_{abcd} by

$$E_{ac} = C_{abcd} u^b u^d, \quad H_{ac} = {}^* C_{abcd} u^b u^d, \quad (4)$$

where the dual is defined to be ${}^* C_{abcd} = \frac{1}{2} \epsilon_{abef} C_{cd}^{ef}$. It is possible to choose a null tetrad in NP formalism such that the invariants of curvature tensor can be expressed in terms of the electric and magnetic parts of the Weyl tensor. Electric and magnetic Weyl tensors of types I and D have been considered by McIntosh et al. [6]. For the remaining Petrov types, we have

Theorem 1. *For type II, the Weyl tensor is purely electric (magnetic) if and only if ψ_2 (or λ) is real (imaginary).*

Theorem 2. *Types III and N Weyl tensors are neither purely electric nor purely magnetic.*

It is seen [7] that plane-fronted gravitational waves and Robinson–Trautman types III and N metrics are neither purely electric nor purely magnetic. While, the Robinson–Trautman type II metric, the Schwarzschild and the Reissner–Nordstrom solutions are purely electric.

3. Lanczos spin tensor

It is known that an electromagnetic field can be generated by a potential, the question then arises that whether it is possible to generate the gravitational field through a potential. The answer is affirmative: this indeed can be done through the covariant differentiation of a tensor field L_{ijk} [8]. This tensor field is now known as Lanczos potential and the Weyl tensor C_{hijk} is generated by L_{ijk} through the equation [9, 10]

$$\begin{aligned} C_{hijk} = & L_{hij;k} - L_{hik;j} + L_{jkh;i} - L_{jki;h} + \frac{1}{2}(L_i^p{}_{j;p} + L_j^p{}_{i;p}) g_{hk} + \\ & + \frac{1}{2}(L_h^p{}_{k;p} + L_k^p{}_{h;p}) g_{ij} - \frac{1}{2}(L_h^p{}_{j;p} + L_j^p{}_{h;p}) g_{ik} - \frac{1}{2}(L_i^p{}_{k;p} + L_k^p{}_{i;p}) g_{hj}. \end{aligned} \quad (5)$$

This equation is known as Weyl–Lanczos equation.

For a gravitational field with perfect fluid source, the basic covariant variables are: the fluid scalars θ (expansion), $\tilde{\rho}$ (energy density), p (pressure); the fluid spatial vectors u_i (4-acceleration), w_i (vorticity); the spatial trace-free symmetric tensors σ_{ij} (fluid shear), the electric (E_{ij}) and the magnetic (H_{ij}) parts of the Weyl tensor; the projection tensor h_{ij} which projects orthogonal to the fluid 4-velocity vector u_i . We have expressed these quantities and the equations satisfied by them in terms of the Newman–Penrose formalism and in the process have obtained the Lanczos potential for perfect fluid spacetimes. In fact we have proved the following [11]:

Theorem 3. *If in a given spacetime there is a field of observers u^i that is shear-free, irrotational and expansion-free, then the Lanczos potential is given by*

$$L_{ijk} = -\bar{\kappa} \left\{ m_{[i} u_{j]} u_k - \frac{1}{3} m_{[i} g_{j]k} \right\} - \kappa \left\{ \bar{m}_{[i} u_{j]} u_k - \frac{1}{3} \bar{m}_{[i} g_{j]k} \right\}, \quad (6)$$

where $u^i = \frac{1}{\sqrt{2}}(l^i + n^j)$ and the Lanczos scalars $L_i (i = 0, 1, \dots, 7)$ in this case are $L_o = -\frac{1}{2}\kappa$, $L_2 = -\frac{1}{3}\bar{L}_o$, $L_5 = \frac{1}{3}L_o$, $L_7 = -\bar{L}_o$, $L_1 = L_3 = L_4 = L_6 = 0$.

Theorem 4. *If in a given spacetime there is a field of observers u^i which is geodesic, shear-free, expansion-free and the vorticity vector is covariantly constant (i.e., $a_i = \theta = \sigma_{ij} = 0, \omega_{i;j} = 0$), then the Lanczos potential is given by*

$$L_{ijk} = \frac{\sqrt{2}}{9} \rho \{2(m_i \bar{m}_j - \bar{m}_i m_j)u_k + (m_i \bar{m}_k - \bar{m}_i m_k)u_j - (m_j \bar{m}_k - \bar{m}_j m_k)u_i\}, \quad (7)$$

where $u^i = \frac{1}{\sqrt{2}}(l^i + n^i)$ and the Lanczos scalars $L_i (i = 0, 1, \dots, 7)$ are $L_1 = L_6 = \frac{1}{9}\rho$, $L_o = L_2 = L_3 = L_4 = L_5 = L_7 = 0$.

It may be noted here that the hypothesis of Theorem 4 are in fact the conditions of the Gödel solution and thus, through Eq. (7), a Lanczos potential for the Gödel solution is obtained.

The two-parameter family of solutions which describe the spacetime around black holes is the Kerr family discovered by Roy Patrick Kerr in July 1963. The two parameters are the mass and angular momentum of the black hole. Using GHP formalism (a tetrad formalism), we have obtained Lanczos potential for Kerr spacetime as [11]

$$L_1 = \frac{1}{3} \left(\frac{\psi_2}{M} \right)^{1/3}, \quad L_2 = \frac{-A}{3} \left(\frac{\psi_2}{M} \right)^{2/3},$$

which shows that Lanczos potential of Kerr spacetime is related to the mass parameter of the Kerr black hole and the Coulomb component of the gravitational field. Here A is a constant.

4. Space-matter tensor

Petrov [12] introduced a fourth rank tensor which satisfies all the algebraic properties of the Riemann curvature tensor and is more general than the Weyl conformal curvature tensor. This tensor is defined as

$$P_{abcd} = R_{abcd} - A_{abcd} + \sigma(g_{ac}g_{bd} - g_{ad}g_{bc}), \quad (8)$$

where $A_{abcd} = \frac{\lambda}{2}(g_{ac}T_{bd} + g_{bd}T_{ac} - g_{ad}T_{bc} - g_{bc}T_{ad})$ and T_{ab} is given by the Einstein's field equations $R_{ab} - \frac{1}{2}Rg_{ab} = \lambda T_{ab}$. Here λ is a constant and T_{ab} is the energy-momentum tensor. The tensor P_{abcd} is known as space-matter tensor. The first part of this tensor represents the curvature of the space and the second part represents the distribution and motion of the matter. From the equations of Section 1, we have

$$P_{abcd} = C_{abcd} + (g_{ad}R_{bc} + g_{bc}R_{ad} - g_{ac}R_{bd} - g_{bd}R_{ac}) + \left(\frac{2}{3}R + \sigma \right) (g_{ac}g_{bd} - g_{ad}g_{bc}), \quad (9)$$

which can also be expressed as

$$P_{bcd}^h = C_{bcd}^h + (\delta_d^h R_{bc} - \delta_c^h R_{bd} + g_{bc}R_d^h + g_{bd}R_c^h) + \left(\frac{2}{3}R + \sigma \right) (\delta_c^h g_{bd} - \delta_d^h g_{bc}). \quad (10)$$

The algebraic properties (including spinor equivalent) and the classification of the space-matter tensor have been studied by Ahsan [13–15]. The concept of matter collineation, defined in terms of the space-matter tensor, has also been introduced by Ahsan [16], who obtained the necessary and sufficient conditions under which a spacetime, including electromagnetic fields, may admit such collineation. In this section, the divergence of the space-matter tensor has been expressed in terms of the energy-momentum and Ricci tensors. Perfect-fluid spacetimes with divergence-free space-matter tensor have also been considered.

The space matter tensor can also be written as

$$P_{bcd}^h = R_{bcd}^h + \frac{1}{2}(\delta_c^h T_{bd} - \delta_d^h T_{bc} + g_{bd} T_c^h - g_{bc} T_d^h) + \sigma(\delta_c^h g_{bd} - \delta_d^h g_{bc}) \quad (11)$$

so that the divergence of P_{bcd}^h is given by

$$P_{bcd;h}^h = R_{bcd;h}^h + \frac{1}{2}(T_{bd;c} - T_{bc;d}) + \frac{1}{2}(g_{bd} T_{c;h}^h - g_{bc} T_{d;h}^h) + \sigma_{,c} g_{bd} - \sigma_{,d} g_{bc}. \quad (12)$$

which on using the contracted Bianchi identities and $T_{;b}^{ab} = 0$ reduces to

$$P_{bcd;h}^h = \frac{1}{2}(T_{bc;d} - T_{bd;c}) + \frac{1}{2}[(T + 2\sigma)_{,c} g_{bd} - (T + 2\sigma)_{,d} g_{bc}]. \quad (13)$$

We thus have

Theorem 5. *If T_{ab} is a Codazzi tensor and $T = -2\sigma$, then the space-matter tensor is divergence-free.*

While in terms of Ricci tensor, the divergence of the space-matter tensor takes the form

$$P_{bcd;h}^h = R_{bc;d} - R_{bd;c} - \frac{1}{4}(R_{,c} g_{bd} - R_{,d} g_{bc}). \quad (14)$$

We thus have

Theorem 6. *For Einstein spaces with $\sigma = 0$, the divergence of space-matter tensor vanishes.*

For a perfect-fluid distribution, the energy-momentum tensor is given by

$$T_{ab} = (\mu + p) u_a u_b + p g_{ab}, \quad (15)$$

where μ is the energy density, p is the isotropic pressure and u_a is the fluid four velocity vector. Consider now the fluid spacetime with divergence-free space-matter tensor and we have

Theorem 7. *If $\sigma = -\frac{1}{2}T$ and the fluid spacetime has divergence-free space-matter tensor, then either $(\mu + p) = 0$ (that is, the perfect fluid spacetime satisfies the vacuumlike equation of state) or the spacetime represents a Friedmann–Robertson–Walker cosmological model with $(\mu - 3p)$ as constant.*

For the proofs of these theorems and other related results, see [17].

Резюме

3. *Ahsan*. Исследование риманова тензора кривизны и классификации Петрова в общей теории относительности.

На основе инвариантов риманова тензора кривизны представлен и проиллюстрирован различными примерами критерий существования гравитационного излучения. Дана классификация пространств-времен исходя из электрической и магнитной составляющих тензора Вейля. Рассмотрены некоторые примеры пространств-времен, у которых тензор Вейля содержит только магнитную или только электрическую часть. Проведено исследование потенциала Ланцоша с помощью метода общих наблюдателей и тетрадного формализма, получен потенциал Ланцоша для пространств-времен идеальной жидкости. Кроме того, получен потенциал Ланцоша для космологической модели Гёделя и черной дыры Керра. Рассмотрены тензор пространства-материи, введенный Петровым, и пространства-времени идеальной жидкости с тензором пространства-материи, не содержащим дивергентной части.

Ключевые слова: тензор Вейля, потенциал Ланцоша, тетрадный формализм, модель Гёделя, черная дыра Керра.

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