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An initial-boundary value problem for the general three-component nonlinear Schrödinger equations on a finite interval

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The general three-component nonlinear Schrödinger (gtc-NLS) equations are completely integrable and contain the self-focusing, defocusing and mixed cases, which are applied in many physical fields. In this paper, we would like to use the Fokas method to explore the initial-boundary value (IBV) problem for the gtc-NLS equations with a 4×4 matrix Lax pair on a finite interval based on the inverse scattering transform. The solutions of the gtc-NLS equations can be expressed using the solution of a 4×4 matrix Riemann–Hilbert (RH) problem constructed in the complex k-plane. The jump matrices of the RH problem can be explicitly found in terms of three spectral functions related to the initial data, and the Dirichlet–Neumann boundary data, respectively. The global relation between the distinct spectral functions is also proposed to derive two distinct but equivalent types of representations of the Dirichlet–Neumann boundary value problems. Particularly, the relevant formulae for the boundary value problems on the finite interval can generate ones on the half-line as the length of the interval closes to infinity. Finally, we also analyse the linearizable boundary conditions for the Gel'fand–Levitan–Marchenko representation. These results will be useful to further study the solution properties of the IBV problem of the gtc-NLS system by using the Deift–Zhou's nonlinear steepest descent method and some numerical methods.

Keywords: general three-component nonlinear Schrödinger equations; initial-boundary value problem; inverse scattering; Riemann–Hilbert problem; global relation; Dirichlet and Neumann problems; Gel'fand–Levitan–Marchenko representation.

1. Introduction

In the field of nonlinear integrable systems, there are some powerful approaches to study their integrable properties. Particularly, in 1967, Gardner *et al.* (1967) first presented the celebrated inverse scattering transform (IST, also called the nonlinear Fourier transform) to analytically study the initial value (IV) problem of the integrable Korteweg–de Vries (KdV) equation. After that, the IST has been applied to the IV problems of other many integrable nonlinear partial differential equations (PDEs) starting from their matrix Lax pairs (Lax, 1968), such as the nonlinear Schrödinger (NLS) equation, modified KdV equation, sine-Gordon equation, AKNS system, KP equation, etc. (see, e.g. Ablowitz & Segur, 1981; Ablowitz & Clarkson, 1991; Zakharov & Shabat, 1972, 1973, and the references therein). In 1992–1993, Deift & Zhou (1992, 1993) used the IST to present the nonlinear steepest descent method (an asymptotic method) to explicitly explore the long-time asymptotics of the Cauchy problems of (1+1)-dimensional integrable nonlinear PDEs in terms of matrix Riemann–Hilbert (RH) problems, defined as (Deift & Zhou, 1992, 1993; Its, 2003) follows: let Σ be an oriented contour in the complex k-plane (Σ might have points of self-intersection and might have more than one connected component), and $J(k)$ being a map from Σ into an invertible matrix. The RH problem given by (Σ, J) is to find a 2×2 matrix-valued meromorphic function $M(k)$ satisfying the following three properties:

- *Analyticity: $M(k)$ is analytic for k in $k \in \mathbb{C} \setminus \Sigma$*;
- *Jump condition: $M^{+}(k) = M^{-}(k)J(k)$ for $k \in \Gamma$, where the limits $M^{+}(k)$ and $M^{-}(k)$ of $M(k)$ from the plus and minus sides of Σ, respectively*;
- *Normalization condition: $M(k) \to \mathbb{I}$ as $k \to \infty$.*

In 1997, inspired by the well-known IST, Fokas (1997) introduced a new unified approach (alias the Fokas method) studying the initial-boundary value (IBV) problems of both linear and integrable nonlinear PDEs on the half-line and the finite interval (Fokas, 2000, 2002, 2008; Pelloni, 2015). The Fokas method was regarded as a significant extension of IST from the IV problems to the IBV problems and further developed by him and his collaborators (see Fokas, 2008, and the reference therein). The main three steps of the Fokas method for the analysis of an IBV problem for an integrable nonlinear PDE for $u(x, t)$ with the Lax pair on a finite region $\Omega = \{(x, t)|x \in [0, L],\ t \in [0, T]\}$ are listed as follows (see, e.g. Fokas, 2008; Boutet de Monvel *et al.*, 2006):

- *Step 1: Based on the simultaneous spectral analysis of the associated Lax pair, a RH formulation is given under the assumption of existence and can be determined by the distinct spectral functions arising from the various initial and boundary value conditions, where these spectral functions are not independent, but they satisfy some identity, i.e. the so-called global relation. As a result, $u(x, t)$ can be expressed by the solution of the RH problem;*
- *Step 2: Existence under the assumption that the spectral functions satisfy the global relation. As a result, one can have $u(x, t)$ is defined globally for all $0 < t < T$, $0 < x < L$, solves the given nonlinear PDE and satisfies the initial value conditions.*
- *Step 3: The key study of the global relation generating the Dirichlet–Neumann map. For the given a subset of the boundary values (e.g. Dirichlet boundary conditions or Neumann boundary conditions) as boundary conditions, one can use the global relation to characterize the remaining part of the boundary values (e.g. Neumann boundary conditions or Dirichlet boundary conditions) by the appropriate symmetries of spectral functions and the use of Cauchy's theorem and Jordan's lemma from complex analysis.*

Notice that the results for the IBV conditions in the limit $L \to \infty$ can reduce to the corresponding ones on the half-line (see, e.g. Fokas, 2002, 2008; Fokas *et al.*, 2005; Boutet de Monvel *et al.*, 2004; Lenells, 2008, 2011a, and the references therein).

The Fokas method used in the integrable nonlinear PDEs with the matrix Lax pairs admits the two major breakthroughs (Fokas, 1997, 2008):

- *On the one hand, the Fokas method was based on the idea of integrating the associated matrix Lax pairs simultaneously, as opposed to separately what was performed in the classical IST method* (Gardner *et al.*, 1967; Ablowitz & Segur, 1981).
- *On the other hand, the Fokas method was the exploitation of the complex spectral plane as a means of eliminating certain unknown boundary values that arise in the solution representation formulae by integrating the matrix Lax pairs.*

Since the Fokas method was first announced in 1997, it has been applied to a variety of distinct settings regarding the type of linear and integrable nonlinear PDEs with distinct domains and IBV conditions. For example, it can be used to study some physically significant integrable nonlinear PDEs with 2×2 matrix Lax pairs on the half-line and the finite interval (e.g. the NLS equation, Fokas, 1997; Fokas *et al.*, 2005; Fokas & Its, 1996; Kamvissis, 2003; Fokas & Its, 2004; the derivative NLS equation, Lenells, 2008, 2011a; the sine-Gordon equation, Fokas & Its, 1992; Pelloni, 2005; the KdV equation, Fokas & Its, 1994; the mKdV equation, Boutet de Monvel *et al.*, 2006, 2004, 2003a; Ernst equations, Lenells, 2011b; Lenells & Fokas, 2011; and etc., Boutet de Monvel & Kotlyarov, 2000; Boutet de Monvel *et al.*, 2003b; Fokas, 2005; Treharne & Fokas, 2008; Fokas & Lenells, 2012; Lenells & Fokas, 2012a,b). Most notably, it has been successfully applied to integrable nonlinear PDEs with 3×3 matrix Lax pairs like the Degasperis–Procesi equation on the half-line (Lenells, 2012, 2013). After that, the idea for the 3×3 Lax pairs was also applied to other integrable nonlinear PDEs with 3×3 matrix Lax pairs, such as the Sasa–Satsuma equation (Xu & Fan, 2013), the coupled nonlinear Schrödinger equations (Biondini & Bui, 2012; Geng *et al.*, 2015; Xu & Fan, 2016a; Tian, 2017) and the Ostrovsky–Vakhnenko equation (Xu & Fan, 2016b). More recently, it was also successfully used to the integrable nonlinear PDEs with 4×4 matrix Lax pairs such as the spin-1 Gross–Pitaeviskii system (Yan, 2017, 2019).

As is well known, as a universal physical model, the standard dimensionless NLS equation

$$iq_t + q_{xx} - 2\sigma |q|^2 q = 0, \quad \sigma = \pm 1, \quad q(x,t) \in \mathbb{C}[x,t] \tag{1}$$

can be used to describe the propagation of slowly varying nonlinear wave envelopes in dispersive media and appears in various backgrounds, such as nonlinear optics, deep ocean, Bose–Einstein condensates, acoustics, plasma physics and even finance (Ablowitz & Segur, 1981; Ablowitz & Clarkson, 1991; Zakharov, 1972; Sulem & Sulem, 1999; Agrawal, 2013; Osborne, 2009; Pitaevskii & Stringari, 2016; Yan, 2010). Eq. (1) is completely integrable and possesses a 2×2 Lax pair (Zakharov & Shabat, 1972). If the wave propagations in an isotropic medium are extended to ones in an anisotropic medium, and q is a sum of two left- and right-hand polarized waves q_1 and q_2, then the coupled NLS equations (also the Manakov system) were presented by Manakov (1974)

$$\begin{cases} iq_{1t} + q_{1xx} - 2(s_1|q_1|^2 + s_2|q_2|^2)q_1 = 0, \\ iq_{2t} + q_{2xx} - 2(s_1|q_1|^2 + s_2|q_2|^2)q_2 = 0, \quad s_{1,2} = \pm 1, \end{cases} \tag{2}$$

which contain the self-phase modulation (SPM, e.g. $|q_j|^2 q_j$) and cross-phase modulation (XPM, e.g. $|q_j|^2 q_{3-j}$) and can be used to describe the nonlinear pulse propagation in birifrangent, single-mode fibers (Menyuk, 1987), an ultrashort light pulse or continuous wave beam propagating in a monomode optical fibre (Agrawal, 2013), matter waves in the two-component Bose–Einstein condensates (Pitaevskii & Stringari, 2016) and the evolution of slowly varying two-phase standing waves in deep water (Roskes, 1976a, 1984). System (2) is also completely integrable and admits a 3×3 matrix Lax pair (Manakov, 1974). Moreover, Manakov (1974) also presented the three propagating waves in a nonlinear fiber described by a three-component ($n = 3$) NLS system

$$i\mathbf{q}_t + \mathbf{q}_{xx} - 2(\mathbf{q}\Lambda\mathbf{q}^\dagger)\mathbf{q} = 0, \quad \mathbf{q} = (q_1, q_2, q_3), \quad \Lambda = \mathrm{diag}(s_1, s_2, s_3), \; s_j \in \mathbb{R}, \tag{3}$$

and even the n-component NLS system (Roskes, 1976b; Ablowitz *et al.*, 2004) defined by Eq. (3) with $\mathbf{q} = (q_1, q_2, ..., q_n)$ and $\Lambda = \mathrm{diag}(s_1, s_2, ..., s_n)$ with $s_j \in \mathbb{R}$. System (3) can also be used to discuss the launching and propagation of waves along the three spines of an alpha helix in protein (Scott, 1982, 1984). The three-component and n-component NLS systems are both completely integrable.

Recently, except for the effects of the SPM and XPM appearing in the above-mentioned systems (2) and (3), if the pair- and three-tunneling effects (Albiez *et al.*, 2005; Fölling *et al.*, 2007) are also considered, then the dynamics of a three-component system can be described by a general three-component nonlinear Schrödinger (gtc-NLS) system (Wang *et al.*, 2010)

$$\begin{cases} iq_{1t} + q_{1xx} - 2\left[\alpha_{11}|q_1|^2 + \alpha_{22}|q_2|^2 + \alpha_{33}|q_3|^2 + 2\,\mathrm{Re}\,(\alpha_{12}\bar{q}_1q_2 + \alpha_{13}\bar{q}_1q_3 + \alpha_{23}\bar{q}_2q_3)\right] q_1 = 0, \\ iq_{2t} + q_{2xx} - 2\left[\alpha_{11}|q_1|^2 + \alpha_{22}|q_2|^2 + \alpha_{33}|q_3|^2 + 2\,\mathrm{Re}\,(\alpha_{12}\bar{q}_1q_2 + \alpha_{13}\bar{q}_1q_3 + \alpha_{23}\bar{q}_2q_3)\right] q_2 = 0, \\ iq_{3t} + q_{3xx} - 2\left[\alpha_{11}|q_1|^2 + \alpha_{22}|q_2|^2 + \alpha_{33}|q_3|^2 + 2\,\mathrm{Re}\,(\alpha_{12}\bar{q}_1q_2 + \alpha_{13}\bar{q}_1q_3 + \alpha_{23}\bar{q}_2q_3)\right] q_3 = 0, \end{cases} \quad (4)$$

where $q_j = q_j(x,t), j = 1,2,3$ denote the complex-valued fields, the overbar denotes the complex conjugate, $\mathrm{Re}\,(\cdot)$ denotes the real part of a function and the six complex-valued coefficients α_{ij}'s $(1 \le i \le j \le 3)$ of interactions form a 3×3 Hermitian-unitary matrix

$$\mathcal{M} = \mathcal{M}^{\dagger} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \bar{\alpha}_{12} & \alpha_{22} & \alpha_{23} \\ \bar{\alpha}_{13} & \bar{\alpha}_{23} & \alpha_{33} \end{pmatrix}, \quad \mathcal{M}^2 = \mathbb{I}. \quad (5)$$

The gtc-NLS system contains the group velocity dispersion (GVD, i.e. q_{jxx}), self-phase modulation (SPM, e.g. $|q_j|^2q_j$), cross-phase modulation (XPM, e.g. $|q_j|^2q_s, j \neq s$), pair-tunneling modulation (PTM, e.g. $q_j^2\bar{q}_s, j \neq s$) and three-tunneling modulation (TTM, e.g. $q_1\bar{q}_2q_3$). System (4) can be used to describe the wave propagations of three modes with four types of modulations containing the SPM, XPM, PTM and TTM in the nonlinear optics, acoustics, deep ocean, Bose–Einstein condensates, etc. (Sulem & Sulem, 1999; Agrawal, 2013; Pitaevskii & Stringari, 2016; Roskes, 1976b; Ablowitz *et al.*, 2004; Whitham, 1999). Particularly, the tunneling modulations could make the system generate the distinct wave structures (Wang *et al.*, 2010). It is easy to see that system (4) is an extension of system (3). System (4) can reduce to the distinct models for the choices of six parameters α_{ij} $(1 \le i \le j \le 3)$, such as

- The three-component focusing NLS system for $\alpha_{jj} = -1$ and $\alpha_{ij} = 0$ with $i < j$ (Manakov, 1974; Scott, 1982, 1984);
- The three-component defocusing NLS system for $\alpha_{jj} = 1$ and $\alpha_{ij} = 0$ with $i < j$ (Biondini *et al.*, 2016);
- The three-component mixed NLS system for $(\alpha_{11} = -1, \alpha_{22} = \alpha_{33} = 1)$ or $(\alpha_{11} = 1, \alpha_{22} = \alpha_{33} = -1)$ and $\alpha_{ij} = 0$ with $i < j$ (Zhang & Yan, 2018);
- The three-component NLS system with only tunneling modulations for $\alpha_{jj} = 0$ (Wang *et al.*, 2010);
- Other types of general three-component NLS systems;

- The general two-component NLS system containing the Manakov system (Manakov, 1974) for $q_3 \equiv 0$.

Recently, the three-component defocusing NLS equations with nonzero boundary conditions were studied via the IST (Biondini *et al.*, 2016). More recently, the novelly integrable nonlocal version of Eq. (4) were found (Yan, 2015, 2016, 2018). It is still an important subject to study the solutions of the gtc-NLS system (4).

To the best of our knowledge, there was no report on the IBV problems of the integrable gtc-NLS system (4) with 4×4 matrix Lax pairs on the finite interval before even if we recently investigated the IBV problems for the spin-1 Gross–Pitaevskii system with 4×4 matrix Lax pairs on the half-line (Yan, 2017) or the finite interval (Yan, 2019). The aim of this paper is to apply the Fokas method to analyse the IBV problems for the integrable gtc-NLS system with a 4×4 matrix Lax pair on a finite region $\Omega = \{(x,t) | x \in [0, L],\ t \in [0, T]\}$, with $L > 0$ being the length of the interval and $T > 0$ being the fixed finite time. The extension will contain some novelties from the 2×2 and 3×3 matrix Lax pairs to 4×4 ones, but the two key steps of this Fokas method (Fokas, 1997, 2000, 2002, 2008) are similar: (i) finding an integral representation of the solution in terms of a matrix RH problem formulated in the complex k-plane (k is a spectral parameter of the associated Lax pair). The integral representation in general contains the unknown boundary data such that this expression of the solution is not effective yet; (ii) applying a global relation between the distinct spectral function matrices to find the unknown boundary value conditions. The representation of the unknown boundary values in general involves the solution of a nonlinear problem. But this problem for the linearizable boundary conditions can be ignored since the unknown boundary values can be avoided in terms of only algebraic operations (Fokas, 2008).

In this paper, we would like to investigate the gtc-NLS system (4) with the following IBV problems:

$$\begin{cases} \textit{Initial conditions:} & q_j(x, t=0) = q_{0j}(x), \quad j = 1,2,3, \\ \textit{Dirichlet boundary conditions:} & q_j(x=0, t) = u_{0j}(t), \quad q_j(x=L, t) = v_{0j}(t), \quad j = 1,2,3, \\ \textit{Neumann boundary conditions:} & q_{jx}(x=0, t) = u_{1j}(t), \quad q_{jx}(x=L, t) = v_{1j}(t), \quad j = 1,2,3, \end{cases} \tag{6}$$

where the initial data $q_{0j}(x)$ and Dirichlet and Neumann boundary data $u_{0j}(t)$, $v_{0j}(t)$ and $u_{1j}(t)$, $v_{1j}(t)$, $j = 1,2,3$ are sufficiently smooth and compatible at points $(x,t) = (0,0)$, $(L,0)$, respectively. Of course, the relevant formulae for the IBV problems on the finite interval can generate ones on the half-line as the length L of the interval approaches to infinity. It should be pointed out that the results about the 4×4 Lax pair differ from the previous ones for the cases of 2×2 and 3×3 matrix Lax pairs, and our results can reduce to the corresponding ones of some above-mentioned special three-component NLS systems, and the known results of some special cases such as the Manakov system and other general two-component coupled NLS equations.

The rest of this paper is organized as follows. In Sec. 2, we investigate the spectral analysis of the associated 4×4 matrix Lax pair of Eq. (4), such as the eigenfunctions, the spectral functions arising from the initial conditions and Dirichlet–Neumann boundary conditions, the jump matrices and the global relation between the distinct spectral functions. Sec. 3 gives the corresponding 4×4 matrix RH problem by means of the jump matrices obtained in Sec. 2. Moreover, we can find the solution (q_1, q_2, q_3) given by Eq. (57) can be expressed by using the obtained 4×4 matrix RH problem. The solution (57) will be useful to further study the long-time asymptotics for the solution of the gtc-NLS system via the Deift–Zhou method (Deift & Zhou, 1992, 1993), or the numerical method (Trogdon, 2013) starting from the above-obtained RH problem. In Sec. 4, a global relation is used to establish the map between the Dirichlet and Neumann boundary values. Particularly, the relevant formulae for boundary value problems on the finite interval can reduce to ones on the half-line as the length L of the interval approaches to infinity. In Sec. 5, we present the Gelfand–Levitan–Marchenko (GLM) representation of the eigenfunctions in terms of the global relation. Moreover, we also show that the GLM representation is equivalent to one obtained in Sec. 4. Moreover, we also give the linearizable boundary conditions for the GLM representation. Finally, some conclusions and discussions are given in Sec. 6.

2. The spectral analysis of a 4×4 matrix Lax pair

2.1 *The definition and boundedness of modified eigenfunctions*

The gtc-NLS system (4) is just the compatible condition, $\psi_{xt} = \psi_{tx}$, of a set of linear PDEs (also called a 4×4 matrix Lax pair) (Ablowitz *et al.*, 2004; Wang *et al.*, 2010)

$$\begin{cases} \psi_x + ik\sigma_4\psi = U(x,t)\psi, \\ \psi_t + 2ik^2\sigma_4\psi = V(x,t,k)\psi, \end{cases} \tag{7}$$

where $\psi = \psi(x,t,k)$ is a complex 4×4 matrix-valued or 4×1 vector-valued eigenfunction, $k \in \mathbb{C}$ is a spectral parameter, $\sigma_4 = \mathrm{diag}(1,1,1,-1)$ and the 4×4 matrices U and V are defined as

$$U(x,t) = \begin{pmatrix} 0 & 0 & 0 & q_1(x,t) \\ 0 & 0 & 0 & q_2(x,t) \\ 0 & 0 & 0 & q_3(x,t) \\ p_1(x,t) & p_2(x,t) & p_3(x,t) & 0 \end{pmatrix}, \quad V(x,t,k) = 2kU(x,t) + V_0(x,t), \tag{8}$$

with $p_1(x,t) = \alpha_{11}\bar{q}_1 + \bar{\alpha}_{12}\bar{q}_2 + \bar{\alpha}_{13}\bar{q}_3$, $p_2(x,t) = \alpha_{12}\bar{q}_1 + \alpha_{22}\bar{q}_2 + \bar{\alpha}_{23}\bar{q}_3$, $p_3(x,t) = \alpha_{13}\bar{q}_1 + \alpha_{23}\bar{q}_2 + \alpha_{33}\bar{q}_3$, and

$$V_0(x,t) = -i(U_x + U^2)\sigma_4 = -i\begin{pmatrix} q_1p_1 & q_1p_2 & q_1p_3 & -q_{1x} \\ q_2p_1 & q_2p_2 & q_2p_3 & -q_{2x} \\ q_3p_1 & q_3p_2 & q_3p_3 & -q_{3x} \\ p_{1x} & p_{2x} & p_{3x} & -(q_1p_1 + q_2p_2 + q_3p_3) \end{pmatrix}, \tag{9}$$

Let the modified eigenfunction $\mu(x,t,k)$ be

$$\mu(x,t,k) = \psi(x,t,k)e^{i(kx+2k^2t)\sigma_4}, \tag{10}$$

such that the Lax pair (7) becomes the equivalent form for $\mu(x,t,k)$

$$\begin{cases} \mu_x + ik[\sigma_4,\mu] = U(x,t)\mu, \\ \mu_t + 2ik^2[\sigma_4,\mu] = V(x,t,k)\mu, \end{cases} \tag{11}$$

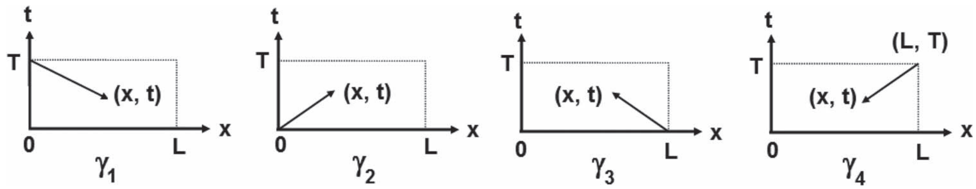

FIG. 1. Contours $\gamma_j (j = 1, ..., 4)$ from (x_j, t_j) to (x, t) in the region $\Omega = \{(x, t)|x \in [0, L], t \in [0, T]\}$.

where $[\sigma_4, \mu] \equiv \sigma_4\mu - \mu\sigma_4$. Let $\widehat{\sigma}_4$ denote the commutator with respect to σ_4 and the operator acting on a 4×4 matrix X by $\hat{\sigma}_4 X = [\sigma_4, X]$ such that $e^{\widehat{\sigma}_4}X = e^{\sigma_4}Xe^{-\sigma_4}$, then the Lax pair (11) can be written as a full derivative form

$$d\left[e^{i(kx+2k^2t)\widehat{\sigma}_4}\mu(x, t, k)\right] = W(x, t, k), \tag{12}$$

where the exact one-form $W(x, t, k)$ is of the form

$$W(x, t, k) = e^{i(kx+2k^2t)\widehat{\sigma}_4}[U(x, t)dx + V(x, t, k)dt]\mu(x, t, k). \tag{13}$$

For any point (x, t) in the region $\Omega = \{(x, t)|x \in [0, L],\ t \in [0, T]\}$, let $\{\gamma_j\}_1^4$ be four contours connecting fours vertexes $(x_1, t_1) = (0, T)$, $(x_2, t_2) = (0, 0)$, $(x_3, t_3) = (L, 0)$, $(x_4, t_4) = (L, T)$ to (x, t), respectively (see Fig. 1). Then, one can have the following inequalities on these contours:

$$\begin{aligned}
\gamma_1 :\ & (0, T) \to (x, t),\ x - \xi \geq 0,\ t - \tau \leq 0,\\
\gamma_2 :\ & (0, 0) \to (x, t),\ x - \xi \geq 0,\ t - \tau \geq 0,\\
\gamma_3 :\ & (L, 0) \to (x, t),\ x - \xi \leq 0,\ t - \tau \geq 0,\\
\gamma_4 :\ & (L, T) \to (x, t),\ x - \xi \leq 0,\ t - \tau \leq 0.
\end{aligned} \tag{14}$$

By means of the Volterra integral equation, it follows from Eqs. (12) and (13) that the four eigenfunctions $\{\mu_j\}_1^4$ on the four contours $\{\gamma_j\}_1^4$ can be written as

$$\mu_j(x, t, k) = \mathbb{I}_4 + \int_{(x_j,t_j)}^{(x,t)} e^{-i(kx+2k^2t)\widehat{\sigma}_4} W_j(\xi, \tau, k), \tag{15}$$

where $\mathbb{I}_4$ is a 4×4 identity matrix, the integral is over a piecewise smooth curve from (x_j, t_j) to (x, t) and $W_j(x, t, k)$ is given by Eq. (13) with $\mu(x, t, k)$ replaced by $\mu_j(x, t, k)$. Since the one-form W_j's are closed, thus μ_j's are independent of the path of integration, and we now choose the paths of integration to be parallel to the x and t axes. It follows from Eq. (15) with the chosen paths of integration that the four columns of the matrix $\mu_j(x, t, k)$ contain the following exponentials:

$$[\mu_j]_s :\ e^{2ik(x-\xi)+4ik^2(t-\tau)},\ j = 1, 2, 3, 4; s = 1, 2, 3, \tag{16a}$$

$$[\mu_j]_4 :\ e^{-2ik(x-\xi)-4ik^2(t-\tau)},\ e^{-2ik(x-\xi)-4ik^2(t-\tau)},\ e^{-2ik(x-\xi)-4ik^2(t-\tau)},\ j = 1, 2, 3, 4. \tag{16b}$$

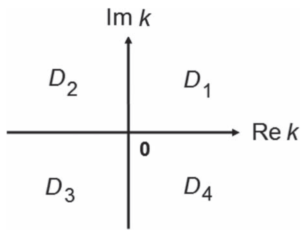

FIG. 2. The domains D_n ($n = 1, 2, 3, 4$) in the complex k-plane.

To analyse the bounded regions of these obtained eigenfunctions $\{\mu_j\}_1^4$, we need to use the curve $\{k \in \mathbb{C} | (\mathrm{Re} f(k))(\mathrm{Re}\, g(k)) = 0, f(k) = ik,\, g(k) = ik^2\}$ to separate the complex k-plane into four regions (see Fig. 2):

$$\begin{aligned} D_1 &= \{k \in \mathbb{C} \,|\, \mathrm{Re} f(k) < 0,\quad \mathrm{Re}\, g(k) < 0\}, \qquad & D_2 &= \{k \in \mathbb{C} \,|\, \mathrm{Re} f(k) < 0,\quad \mathrm{Re}\, g(k) > 0\},\\ D_3 &= \{k \in \mathbb{C} \,|\, \mathrm{Re} f(k) > 0,\quad \mathrm{Re}\, g(k) < 0\}, \qquad & D_4 &= \{k \in \mathbb{C} \,|\, \mathrm{Re} f(k) > 0,\quad \mathrm{Re}\, g(k) > 0\}, \end{aligned} \tag{17}$$

which imply that D_1 and D_3 (D_2 and D_4) are symmetric about the origin.

Thus it follows from Eqs. (14), (16) and (17) that the regions are presented below:

$$\begin{cases} \mu_1 : (f_- \cap g_+, f_- \cap g_+, f_- \cap g_+, f_+ \cap g_-) := (D_2, D_2, D_2, D_3),\\ \mu_2 : (f_- \cap g_-, f_- \cap g_-, f_- \cap g_-, f_+ \cap g_+) := (D_1, D_1, D_1, D_4),\\ \mu_3 : (f_+ \cap g_-, f_+ \cap g_-, f_+ \cap g_-, f_- \cap g_+) := (D_3, D_3, D_3, D_2),\\ \mu_4 : (f_+ \cap g_+, f_+ \cap g_+, f_+ \cap g_+, f_- \cap g_-) := (D_4, D_4, D_4, D_1), \end{cases} \tag{18}$$

where the different columns of eigenfunctions $\{\mu_j\}_1^4$ are bounded and analytic in the complex k-plane, $f_+ := \mathrm{Re} f(k) > 0$, $f_- := \mathrm{Re} f(k) < 0$, $g_+ := \mathrm{Re}\, g(k) > 0$ and $g_- := \mathrm{Re}\, g(k) < 0$.

2.2 *The new matrix-valued functions M_n's and jump matrices*

To construct the jump matrix in a RH problem, we introduce the solutions $M_n(x, t, k)$ ($n = 1, 2, 3, 4$) of Eq. (11) as

$$(M_n)_{sj}(x, t, k) = \delta_{sj} + \int_{(\gamma^n)_{sj}} \left(e^{-i(kx+2k^2t)\widehat{\sigma}_4} W_n(\xi, \tau, k)\right)_{sj}, \quad k \in D_n, \quad s, j = 1, 2, 3, 4, \tag{19}$$

where $\delta_{sj} = 1$ for $s = j$, $\delta_{sj} = 0$ for $s \neq j$, $W_n(x, t, k)$ is given by Eq. (13) with $\mu(x, t, k)$ replaced with $M_n(x, t, k)$, and the contours $(\gamma^n)_{sj}$'s are given by

$$(\gamma^n)_{sj} = \begin{cases} \gamma_1, & \text{as } \mathrm{Re} f_s(k) > \mathrm{Re} f_j(k) \text{ and } \mathrm{Re}\, g_s(k) \leq \mathrm{Re}\, g_j(k), \\ \gamma_2, & \text{as } \mathrm{Re} f_s(k) > \mathrm{Re} f_j(k) \text{ and } \mathrm{Re}\, g_s(k) > \mathrm{Re}\, g_j(k), \\ \gamma_3, & \text{as } \mathrm{Re} f_s(k) \leq \mathrm{Re} f_j(k) \text{ and } \mathrm{Re}\, g_s(k) \geq \mathrm{Re}\, g_j(k), \\ \gamma_4, & \text{as } \mathrm{Re} f_s(k) \leq \mathrm{Re} f_j(k) \text{ and } \mathrm{Re}\, g_s(k) \leq \mathrm{Re}\, g_j(k), \end{cases} \tag{20}$$

for $k \in D_n$, where $f_{1,2,3}(k) = -f_4(k) = -ik$, and $g_{1,2,3}(k) = -g_4(k) = -2ik^2$.

Notice that to distinguish $(\gamma^n)_{sj}$'s to be the contour γ_3 or γ_4 for the special cases, $\mathrm{Re} f_s(k) = \mathrm{Re} f_j(k)$ and $\mathrm{Re}\, g_s(k) = \mathrm{Re}\, g_j(k)$ in Eq. (20), we choose them in these cases as γ_3 (or γ_4), which must appear in the matrix γ^n; otherwise, we choose them in all these cases as the same γ_3 (or γ_4).

It follows from the definition (20) of $(\gamma^n)_{sj}$ that γ^n $(n = 1, 2, 3, 4)$ can be written explicitly as

$$\gamma^1 = \begin{pmatrix} \gamma_4 & \gamma_4 & \gamma_4 & \gamma_2 \\ \gamma_4 & \gamma_4 & \gamma_4 & \gamma_2 \\ \gamma_4 & \gamma_4 & \gamma_4 & \gamma_2 \\ \gamma_4 & \gamma_4 & \gamma_4 & \gamma_4 \end{pmatrix}, \quad \gamma^2 = \gamma^1\big|_{\{\gamma_4 \to \gamma_3, \gamma_2 \to \gamma_1\}}, \quad \gamma^3 = (\gamma^2)^T, \quad \gamma^4 = (\gamma^1)^T. \tag{21}$$

PROPOSITION 2.1 *For the matrix-valued functions $M_n(x, t, k)$ $(n = 1, 2, 3, 4)$ defined by Eq. (19) for $k \in \bar{D}_n$ and $(x, t) \in \Omega$, and any fixed point (x, t), $M_n(x, t, k)$'s are the bounded and analytic functions of $k \in D_n$ away from a possible discrete set of singularity $\{k_j\}$ at which the Fredholm determinants vanish. Moreover, $M_n(x, t, k)$'s admit the bounded and continuous extensions to $\bar{D}_n$ and*

$$M_n(x, t, k) = \mathbb{I}_4 + O\left(\frac{1}{k}\right), \ k \in D_n, \ k \to \infty, \ n = 1, 2, 3, 4. \tag{22}$$

Proof. Similarly to the proof for the case of the 3×3 matrix Lax pair in (Lenells, 2012, 2013), we can also show the boundedness and analyticity of M_n. The substitution of

$$\mu(x, t, k) = M_n(x, t, k) = M_n^{(0)}(x, t, k) + \sum_{j=1}^{\infty} \frac{M_n^{(j)}(x, t, k)}{k^j}, \quad k \to \infty,$$

into the x-part of the Lax pair (11) yields Eq. (22). □

The above-defined matrix-valued functions M_n's can be used to formulate a 4×4 matrix RH problem. We introduce the spectral functions $S_n(k)$ $(n = 1, 2, 3, 4)$ by

$$S_n(k) = M_n(x = 0, t = 0, k), \quad k \in D_n, \quad n = 1, 2, 3, 4. \tag{23}$$

Let $M(x,t,k)$ denote the sectionally analytic function on the Riemann k-sphere, which is equivalent to $M_n(x,t,k)$ for $k \in D_n$. Then $M(x,t,k)$ solves the jump equations

$$M_n(x,t,k) = M_m(x,t,k)J_{mn}(x,t,k), \quad k \in \bar{D}_n \cap \bar{D}_m, \quad n,m = 1,2,3,4, \quad n \neq m, \tag{24}$$

with the jump matrices $J_{mn}(x,t,k)$ defined by

$$J_{mn}(x,t,k) = e^{-i(kx+2k^2t)\sigma_4}[S_m^{-1}(k)S_n(k)]e^{i(kx+2k^2t)\sigma_4}. \tag{25}$$

2.3 *The minors or the transpose of the adjugates of eigenfunctions*

To conveniently calculate the spectral functions $S_n(k)$ in the following sections, we need to use the cofactor matrix X^A (or the transpose of the adjugate) of a 4×4 matrix X defined as

$$\operatorname{adj}(X)^T = X^A = \begin{pmatrix} m_{11}(X) & -m_{12}(X) & m_{13}(X) & -m_{14}(X) \\ -m_{21}(X) & m_{22}(X) & -m_{23}(X) & m_{24}(X) \\ m_{31}(X) & -m_{32}(X) & m_{33}(X) & -m_{34}(X) \\ -m_{41}(X) & m_{42}(X) & -m_{43}(X) & m_{44}(X) \end{pmatrix}, \tag{26}$$

where $m_{ij}(X)$ denotes the (ij)-th minor of X and $(X^A)^T X = \operatorname{adj}(X)X = \det X$.

It follows from the Lax pair (7) that the eigenfunction $\{\mu_j^A\}_1^4$ of the matrices $\{\mu_j(x,t,k)\}_1^4$ satisfy the Lax equation

$$\begin{cases} \mu_x^A - ik[\sigma_4, \mu^A] = -U^T(x,t)\mu^A, \\ \mu_t^A - 2ik^2[\sigma_4, \mu^A] = -V^T(x,t,k)\mu^A, \end{cases} \tag{27}$$

whose solutions can be written as

$$\mu_j^A(x,t,k) = \mathbb{I}_4 - \int_{\gamma_j} e^{i[k(x-\xi)+2k^2(t-\tau)]\widehat{\sigma}_4}\left[U^T(\xi,\tau)\mathrm{d}\xi + V^T(\xi,\tau,k)\mathrm{d}\tau\right]\mu_j^A(\xi,\tau,k), \quad j=1,2,3,4, \tag{28}$$

in terms of the Volterra integral equations.

It is easy to check that the regions of boundedness of μ_j^A are given by

$$\begin{cases} \mu_1^A(x,t,k) \ \text{ is } \ \text{bounded } \ \text{for } k \in (D_3, D_3, D_3, D_2), \\ \mu_2^A(x,t,k) \ \text{ is } \ \text{bounded } \ \text{for } k \in (D_4, D_4, D_4, D_1), \\ \mu_3^A(x,t,k) \ \text{ is } \ \text{bounded } \ \text{for } k \in (D_2, D_2, D_2, D_3), \\ \mu_4^A(x,t,k) \ \text{ is } \ \text{bounded } \ \text{for } k \in (D_1, D_1, D_1, D_4), \end{cases}$$

which are symmetric ones of μ_j about the $\operatorname{Re} k$-axis (cf. Eq. (18)).

2.4 *Symmetries of eigenfunctions*

Let

$$\check{U}(x,t,k) = -ik\sigma_4 + U(x,t), \quad \check{V}(x,t,k) = -2ik^2\sigma_4 + V(x,t,k). \tag{29}$$

in the Lax pair (7). Then we have

$$\mathcal{P}\overline{\check{U}(x,t,\bar{k})}\mathcal{P} = -\check{U}(x,t,k)^T, \quad \mathcal{P}\overline{\check{V}(x,t,\bar{k})}\mathcal{P} = -\check{V}(x,t,k)^T, \tag{30}$$

where the symmetric matrix $\mathcal{P}$ is taken as

$$\mathcal{P} = \begin{pmatrix} \alpha_{11} & \bar{\alpha}_{12} & \bar{\alpha}_{13} & 0 \\ \alpha_{12} & \alpha_{22} & \bar{\alpha}_{23} & 0 \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathcal{P}^2 = \mathbb{I}_4, \quad \mathcal{P} = \mathcal{P}^\dagger. \tag{31}$$

Notice that the used symmetric matrix $\mathcal{P}$ differs from the diag ones used in the cases of 3×3 Lax pairs (Biondini & Bui, 2012; Geng *et al.*, 2015; Xu & Fan, 2016a; Tian, 2017).

Based on Eqs. (27) and (30) we have the following proposition:

PROPOSITION 2.2 *The matrix-valued eigenfunctions $\psi(x,t,k)$ of the Lax pair (7) and $\mu_j(x,t,k)$ of the Lax pair (11) both possess the same symmetric relations*

$$\psi^{-1}(x,t,k) = \mathcal{P}\overline{\psi(x,t,\bar{k})}^T\mathcal{P}, \quad \mu_j^{-1}(x,t,k) = \mathcal{P}\overline{\mu_j(x,t,\bar{k})}^T\mathcal{P}, \quad j = 1,2,3,4, \tag{32}$$

Moreover, In the domains where μ_j is bounded, we have

$$\mu_j(x,t,k) = \mathbb{I}_4 + O\left(\frac{1}{k}\right), \quad k \to \infty, \quad j = 1,2,3,4 \tag{33}$$

and $\det[\mu_j(x,t,k)] = 1$ ($j = 1,2,3,4$) since the traces of the matrices $U(x,t,k)$ and $V(x,t,k)$ are zero.

2.5 *The relations between spectral functions and jump matrices J_{mn}*

Since these functions $\mu_j(x,t,k)$, $j = 1,2,3,4$ are dependent, thus one can define the three 4×4 matrix-valued functions $S(k)$, $s(k)$ and $\mathbb{S}(k)$ between μ_2 and μ_j, $j = 1,3,4$ in the forms (cf. Fig. 3)

$$\begin{aligned} \mu_1(x,t,k) &= \mu_2(x,t,k)e^{-i(kx+2k^2t)\widehat{\sigma}_4}S(k), \\ \mu_3(x,t,k) &= \mu_2(x,t,k)e^{-i(kx+2k^2t)\widehat{\sigma}_4}s(k), \\ \mu_4(x,t,k) &= \mu_2(x,t,k)e^{-i(kx+2k^2t)\widehat{\sigma}_4}\mathbb{S}(k). \end{aligned} \tag{34}$$

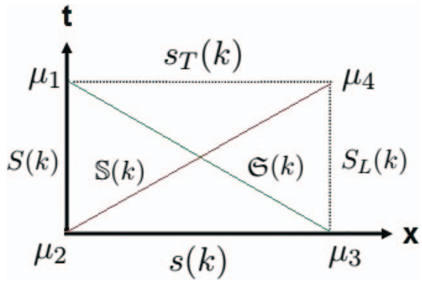

FIG. 3. The relations among $\mu_j(x,t,k)$, $j=1,2,3,4$.

Evaluating system (34) at $(x,t) = (0,0)$ and the three equations in system (34) at $(x,t) = (0,T), (L,0), (L,T)$, respectively, yields

$$\begin{aligned} S(k) &= \mu_1(0,0,k) = e^{2ik^2T\widehat{\sigma}_4}\mu_2^{-1}(0,T,k), \\ s(k) &= \mu_3(0,0,k) = e^{ikL\widehat{\sigma}_4}\mu_2^{-1}(L,0,k), \\ \mathbb{S}(k) &= \mu_4(0,0,k) = e^{i(kL+2k^2T)\widehat{\sigma}_4}\mu_2^{-1}(L,T,k), \end{aligned} \tag{35}$$

Except for the defined three relations, it follows from Eqs. (34) and (35) that we can find other three relations:

- The relation between $\mu_3(x,t,k)$ and $\mu_4(x,t,k)$:

$$\mu_4(x,t,k) = \mu_3(x,t,k)e^{-i[k(x-L)+2k^2(t-T)]\widehat{\sigma}_4}\mu_3^{-1}(L,T,k) = \mu_3(x,t,k)e^{-i[k(x-L)+2k^2t]\widehat{\sigma}_4}S_L(k)$$

with

$$S_L(k) = \mu_4(L,0,k) = e^{2ik^2T\widehat{\sigma}_4}\mu_3^{-1}(L,T,k), \tag{36}$$

- The relation between $\mu_1(x,t,k)$ and $\mu_4(x,t,k)$:

$$\mu_3(x,t,k) = \mu_1(x,t,k)e^{-i(kx+2k^2t)\widehat{\sigma}_4}\mathfrak{S}(k), \quad \mathfrak{S}(k) = S^{-1}(k)s(k), \tag{37}$$

- The relation between $\mu_1(x,t,k)$ and $\mu_4(x,t,k)$:

$$\mu_4(x,t,k) = \mu_1(x,t,k)e^{-i(kx+2k^2t)\widehat{\sigma}_4}s_T(k), \quad s_T(k) = S^{-1}(k)\mathbb{S}(k). \tag{38}$$

It follows from Eqs. (35) and (36) that we have the relation

$$\mathbb{S}(k) = s(k)e^{ikL\widehat{\sigma}_4}S_L(k). \tag{39}$$

The map of these relations among μ_j's is exhibited in Fig. 3.

According to the definitions (15) of μ_j's, Eq. (35) and (36) imply that

$$
\begin{aligned}
s(k) =&\mathbb{I}_4 - \int_0^L e^{ik\xi\widehat{\sigma}_4}(U\mu_3)(\xi,0,k)\mathrm{d}\xi = \left[\mathbb{I}_4 + \int_0^L e^{ikx'\widehat{\sigma}_4}(U\mu_2)(\xi,0,k)\mathrm{d}\xi\right]^{-1},\\
S(k) =&\mathbb{I}_4 - \int_0^T e^{2ik^2\tau\widehat{\sigma}_4}(V\mu_1)(0,\tau,k)\mathrm{d}\xi = \left[\mathbb{I}_4 + \int_0^T e^{2ik^2\tau\widehat{\sigma}_4}(V\mu_2)(0,\tau,k)\mathrm{d}\tau\right]^{-1},\\
S_L(k) =&\mathbb{I}_4 - \int_0^T e^{2ik^2\tau\widehat{\sigma}_4}(V\mu_4)(L,\tau,k)\mathrm{d}\tau = \left[\mathbb{I}_4 + \int_0^T e^{2ik^2\tau\widehat{\sigma}_4}(V\mu_3)(L,\tau,k)\mathrm{d}\tau\right]^{-1},\\
\mathbb{S}(k) =&\mathbb{I}_4 - \int_0^L e^{ik\xi\widehat{\sigma}_4}(U\mu_4)(\xi,0,k)\mathrm{d}\xi - e^{ikL\widehat{\sigma}_4}\int_0^T e^{2ik^2\tau\widehat{\sigma}_4}(V\mu_4)(L,\tau,k)\mathrm{d}\tau\\
=&\left[\mathbb{I}_4 + e^{2ik^2T\widehat{\sigma}_4}\int_0^L e^{ik\xi\widehat{\sigma}_4}(U\mu_2)(\xi,T,k)\mathrm{d}\xi + \int_0^T e^{2ik^2\tau\widehat{\sigma}_4}(V\mu_2)(0,\tau,k)\mathrm{d}\tau\right]^{-1},
\end{aligned}
\tag{40}
$$

which can lead to $\mathfrak{S}(k)$ and $s_T(k)$ in terms of Eqs. (37) and (38), where $\mu_{j_2}(0,t,k)$, $j_2 = 1,2$, $\mu_{j_3}(L,t,k)$, $j_3 = 3,4$, $\mu_{j_1}(x,0,k)$, $j_1 = 2,3,4$, $\mu_2(x,T,k)$, $0 < x < L$, $0 < t < T$ are defined by the integral equations

$$
\begin{aligned}
&\mu_1(0,t,k) = \mathbb{I}_4+\int_T^t e^{-2ik^2(t-\tau)\widehat{\sigma}_4}(V\mu_1)(0,\tau,k)\mathrm{d}\tau, \quad \mu_2(0,t,k)=\mathbb{I}_4+\int_0^t e^{-2ik^2(t-\tau)\widehat{\sigma}_4}(V\mu_2)(0,\tau,k)\mathrm{d}\tau,\\
&\mu_3(L,t,k) = \mathbb{I}_4+\int_0^t e^{-2ik^2(t-\tau)\widehat{\sigma}_4}(V\mu_3)(L,\tau,k)\mathrm{d}\tau, \quad \mu_4(L,t,k)=\mathbb{I}_4+\int_T^t e^{-2ik^2(t-\tau)\widehat{\sigma}_4}(V\mu_4)(L,\tau,k)\mathrm{d}\tau,\\
&\mu_2(x,0,k) = \mathbb{I}_4+\int_0^x e^{ik\xi\widehat{\sigma}_4}(U\mu_2)(\xi,0,k)\mathrm{d}\xi, \quad \mu_3(x,0,k) = \mathbb{I}_4 + \int_L^x e^{ik\xi\widehat{\sigma}_4}(U\mu_3)(\xi,0,k)\mathrm{d}\xi,\\
&\mu_4(x,0,k) = \mathbb{I}_4+\int_L^x e^{ik\xi\widehat{\sigma}_4}(U\mu_4)(\xi,0,k)\mathrm{d}\xi - e^{-ik(x-L)\widehat{\sigma}_4}\int_0^T e^{2ik^2\tau\widehat{\sigma}_4}(V\mu_4)(L,\tau,k)\mathrm{d}\tau,\\
&\mu_2(x,T,k) = \mathbb{I}_4+\int_0^x e^{-ik(x-\xi)\widehat{\sigma}_4}(U\mu_2)(\xi,T,k)\mathrm{d}\xi + e^{-ikx\widehat{\sigma}_4}\int_0^T e^{-2ik^2(T-\tau)\widehat{\sigma}_4}(V\mu_2)(0,\tau,k)\mathrm{d}\tau.
\end{aligned}
$$

It follows from the properties of μ_j and μ_j^A that the spectral functions $\{S(k),\ s(k),\ \mathbb{S}(k),\ S_L(k)\}$ and $\{S^A(k),\ s^A(k),\ \mathbb{S}^A(k),\ S_L^A(k)\}$ have the following boundedness:

$$
\begin{cases}
S(k), S_L(k), \text{and } S_L^A(k) \ \text{ are bounded for } k \in (D_2\cup D_4, D_2\cup D_4, D_2\cup D_4, D_1\cup D_3),\\
s(k) \ \text{ is bounded for } k \in (D_3\cup D_4, D_3\cup D_4, D_3\cup D_4, D_1\cup D_2),\\
\mathbb{S}(k) \ \text{ is bounded for } k \in (D_4, D_4, D_4, D_1),\\
S^A(k) \ \text{ is bounded for } k \in (D_1\cup D_3, D_1\cup D_3, D_1\cup D_3, D_2\cup D_4),\\
s^A(k) \ \text{ is bounded for } k \in (D_1\cup D_2, D_1\cup D_2, D_1\cup D_2, D_3\cup D_4),\\
\mathbb{S}^A(k) \ \text{ is bounded for } k \in (D_2, D_2, D_2, D_3).
\end{cases}
$$

Proposition 2.3 *The matrix-valued functions $S_n(x,t,k)$ $(n=1,2,3,4)$ defined by*

$$M_n(x,t,k)=\mu_2(x,t,k)e^{-i(kx+2k^2t)\widehat{\sigma}_4}S_n(k), \quad k\in D_n, \tag{41}$$

with M_n given by Eq. (19) can be determined by the entries of the data $S(k)=(S_{ij})_{4\times 4}$, $s(k)=(s_{ij})_{4\times 4}$, and $\mathbb{S}(k)=(\mathbb{S}_{ij})_{4\times 4}$ given by Eq. (35) as follows:

$$S_1(k)=\begin{pmatrix} \mathbb{S}_{11} & \mathbb{S}_{12} & \mathbb{S}_{13} & 0 \\ \mathbb{S}_{21} & \mathbb{S}_{22} & \mathbb{S}_{23} & 0 \\ \mathbb{S}_{31} & \mathbb{S}_{32} & \mathbb{S}_{33} & 0 \\ \mathbb{S}_{41} & \mathbb{S}_{42} & \mathbb{S}_{43} & \dfrac{1}{m_{44}(\mathbb{S})} \end{pmatrix}, \qquad S_2(k)=\begin{pmatrix} s_{11} & s_{12} & s_{13} & \dfrac{S_{14}}{(S^Ts^A)_{44}} \\ s_{21} & s_{22} & s_{23} & \dfrac{S_{24}}{(S^Ts^A)_{44}} \\ s_{31} & s_{32} & s_{33} & \dfrac{S_{34}}{(S^Ts^A)_{44}} \\ s_{41} & s_{42} & s_{43} & \dfrac{S_{44}}{(S^Ts^A)_{44}} \end{pmatrix},$$

$$S_3(k)=\begin{pmatrix} S_3^{(11)} & S_3^{(12)} & S_3^{(13)} & s_{14} \\ S_3^{(21)} & S_3^{(22)} & S_3^{(23)} & s_{24} \\ S_3^{(31)} & S_3^{(32)} & S_3^{(33)} & s_{34} \\ S_3^{(41)} & S_3^{(42)} & S_3^{(43)} & s_{44} \end{pmatrix}, \quad S_4(k)=\begin{pmatrix} \dfrac{n_{11,44}(\mathbb{S})}{\mathbb{S}_{44}} & \dfrac{n_{12,44}(\mathbb{S})}{\mathbb{S}_{44}} & \dfrac{n_{13,44}(\mathbb{S})}{\mathbb{S}_{44}} & \mathbb{S}_{14} \\ \dfrac{n_{21,44}(\mathbb{S})}{\mathbb{S}_{44}} & \dfrac{n_{22,44}(\mathbb{S})}{\mathbb{S}_{44}} & \dfrac{n_{23,44}(\mathbb{S})}{\mathbb{S}_{44}} & \mathbb{S}_{24} \\ \dfrac{n_{31,44}(\mathbb{S})}{\mathbb{S}_{44}} & \dfrac{n_{32,44}(\mathbb{S})}{\mathbb{S}_{44}} & \dfrac{n_{33,44}(\mathbb{S})}{\mathbb{S}_{44}} & \mathbb{S}_{34} \\ 0 & 0 & 0 & \mathbb{S}_{44} \end{pmatrix}, \tag{42}$$

where $n_{i_1j_1,i_2j_2}(X)$ denotes the determinant of the sub-matrix generated by choosing the cross elements of $i_{1,2}$th rows and $j_{1,2}$th columns of X (Yan, 2019), i.e.

$$n_{i_1j_1,i_2j_2}(X)=\begin{vmatrix} X_{i_1j_1} & X_{i_1j_2} \\ X_{i_2j_1} & X_{i_2j_2} \end{vmatrix},$$

and

$$\begin{cases} S_3^{(1l)}=\dfrac{m_{24}(S)n_{1l,24}(s)-m_{34}(S)n_{1l,34}(s)+m_{44}(S)n_{1l,44}(s)}{(s^TS^A)_{44}}, \\ S_3^{(2l)}=\dfrac{m_{14}(S)n_{2l,14}(s)-m_{34}(S)n_{2l,34}(s)+m_{44}(S)n_{2l,44}(s)}{(s^TS^A)_{44}}, \\ S_3^{(3l)}=\dfrac{m_{14}(S)n_{3l,14}(s)-m_{24}(S)n_{3l,24}(s)+m_{44}(S)n_{3l,44}(s)}{(s^TS^A)_{44}}, \\ S_3^{(4l)}=\dfrac{m_{14}(S)n_{4l,14}(s)-m_{24}(S)n_{4l,24}(s)+m_{34}(S)n_{4l,34}(s)}{(s^TS^A)_{44}}. \end{cases} \quad l=1,2,3,$$

Proof. We introduce the matrix-valued functions $R_n(k), S_n(k), T_n(k)$ and $P_n(k)$, $n = 1,2,3,4)$ by $M_n(x,t,k)$ and $\mu_j(x,t,k)$

$$\begin{aligned} M_n(x,t,k) &= \mu_1(x,t,k)e^{-i(kx+2k^2t)\widehat{\sigma}_4}R_n(k), \\ M_n(x,t,k) &= \mu_2(x,t,k)e^{-i(kx+2k^2t)\widehat{\sigma}_4}S_n(k), \\ M_n(x,t,k) &= \mu_3(x,t,k)e^{-i(kx+2k^2t)\widehat{\sigma}_4}T_n(k), \\ M_n(x,t,k) &= \mu_4(x,t,k)e^{-i(kx+2k^2t)\widehat{\sigma}_4}P_n(k). \end{aligned} \tag{43}$$

It follows from Eq. (43) that we have the relations

$$\begin{aligned} &R_n(k) = e^{2ik^2T\widehat{\sigma}_4}M_n(0,T,k), \quad S_n(k) = M_n(0,0,k), \\ &T_n(k) = e^{ikL\widehat{\sigma}_4}M_n(L,0,k), \quad P_n(k) = e^{i(kL+2k^2T)\widehat{\sigma}_4}M_n(L,T,k), \end{aligned} \tag{44}$$

and

$$\begin{aligned} S(k) &= \mu_1(0,0,k) = S_n(k)R_n^{-1}(k), \\ s(k) &= \mu_3(0,0,k) = S_n(k)T_n^{-1}(k), \\ \mathbb{S}(k) &= \mu_4(0,0,k) = S_n(k)P_n^{-1}(k), \end{aligned} \tag{45}$$

which can in general generate the functions $\{R_n, S_n, T_n, P_n\}$ for the given functions $\{s(k), S(k), \mathbb{S}(k)\}$.

Moreover, we can also determine some entries of $\{R_n, S_n, T_n, P_n\}$ as

$$\begin{aligned} &(R_n(k))_{ij} = 0, \quad \text{if } (\gamma^n)_{ij} = \gamma_1, \\ &(S_n(k))_{ij} = 0, \quad \text{if } (\gamma^n)_{ij} = \gamma_2, \\ &(T_n(k))_{ij} = \delta_{ij}, \quad \text{if } (\gamma^n)_{ij} = \gamma_3, \\ &(P_n(k))_{ij} = \delta_{ij}, \quad \text{if } (\gamma^n)_{ij} = \gamma_4, \end{aligned} \tag{46}$$

in terms of Eqs. (19) and (43). Thus it follows from systems (45) and (47) that we can obtain Eq. (42). □

2.6 *The global relation between the distinct spectral functions*

The definitions of the above-mentioned spectral functions $S(k), s(k), S_L(k)$ and $\mathbb{S}(k)$ imply that they are not independent. It follows from Eqs. (34) and (36) that

$$\begin{aligned} \mu_4(x,t,k) = &\ \mu_2(x,t,k)e^{-i(kx+2k^2t)\widehat{\sigma}_4}\mathbb{S}(k) = \mu_2(x,t,k)e^{-i(kx+2k^2t)\widehat{\sigma}_4}[s(k)e^{ikL\widehat{\sigma}_4}S_L(k)] \\ = &\ \mu_1(x,t,k)e^{-i(kx+2k^2t)\widehat{\sigma}_4}[S^{-1}(k)s(k)e^{ikL\widehat{\sigma}_4}S_L(k)], \end{aligned} \tag{47}$$

which leads to the global relation

$$c(T,k) = \mu_4(0,T,k) = e^{-2ik^2T\widehat{\sigma}_4}[S^{-1}(k)s(k)e^{ikL\widehat{\sigma}_4}S_L(k)], \tag{48}$$

by evaluating Eq. (47) at the point $(x,t) = (0,T)$ and using $\mu_1(0,T,k) = \mathbb{I}$.

2.7 *The residue conditions for M_n*

Since $\mu_2(x,t,k)$ is an entire function, it follows from Eq. (41) that $M_n(x,t,k)$ only have singularities at the points where the $S_n(k)$'s have singularities. We find from the expressions of $S_n(k)$ given by Eq. (42) that the possible singularities of M_n are listed as follows:

- $[M]_4$ could admit the poles in D_1 at the zeros of $m_{44}(\mathbb{S})(k)$;
- $[M]_4$ could have the poles in D_2 at the zeros of $(S^T s^A)_{44}(k)$;
- $[M]_l,\ l=1,2,3$ could possess the poles in D_3 at the zeros of $(s^T S^A)_{44}(k)$;
- $[M]_l,\ l=1,2,3$ could have the poles in D_4 at the zeros of $\mathbb{S}_{44}(k)$.

We introduce the above-mentioned possible zeros by $\{k_j\}_1^N$, and suppose that they satisfy the following condition.

Assumption 2.4 *Suppose that*

- $m_{44}(\mathbb{S})(k)$ has the n_1 possible simple zeros in D_1 denoted by $\{k_j\}_1^{n_1}$;
- $(S^T s^A)_{44}(k)$ has the n_2-n_1 possible simple zeros in D_2 denoted by $\{k_j\}_{n_1+1}^{n_2}$;
- $(s^T S^A)_{44}(k)$ has the n_3-n_2 possible simple zeros in D_3 denoted by $\{k_j\}_{n_2+1}^{n_3}$;
- $\mathbb{S}_{44}(k)$ has the $N-n_3$ possible simple zeros in D_4 denoted by $\{k_j\}_{n_3+1}^{N}$;

and that none of these zeros coincide. Moreover, none of these functions are assumed to have zeros on the boundaries of D_n's ($n=1,2,3,4$).

We can deduce the residue conditions at these zeros by the following expressions:

Proposition 2.5 *Let $\{M_n\}_1^4$ be the eigenfunctions given by Eq. (19) and suppose that the set $\{k_j\}_1^N$ of singularities is as the above-mentioned Assumption 2.4. Then we have the following residue conditions for M_n:*

$$\mathrm{Res}_{k=k_j}[M_1]_4=\frac{n_{12,23}(\mathbb{S})(k_j)[M_1(k_j)]_1-n_{11,23}(\mathbb{S})(k_j)[M_1(k_j)]_2+n_{11,22}(\mathbb{S})(k_j)[M_1(k_j)]_3}{\dot{m}_{44}(\mathbb{S})(k_j)m_{34}(\mathbb{S})(k_j)}e^{2\theta(k_j)},\quad \text{for}\quad 1\le j\le n_1,\quad k\in D_1, \tag{49}$$

$$\begin{aligned}\mathrm{Res}_{k=k_j}[M_2]_4=&\frac{[M_2(k_j)]_1[S_{14}(k_j)n_{22,43}(s)(k_j)-S_{24}(k_j)n_{12,43}(s)(k_j)+S_{44}(k_j)n_{12,23}(s)(k_j)]}{(\dot{S^T s^A})_{44}(k_j)m_{34}(s)(k_j)e^{-2\theta(k_j)}}\\&-\frac{[M_2(k_j)]_2[S_{14}(k_j)n_{21,43}(s)(k_j)-S_{24}(k_j)n_{11,43}(s)(k_j)+S_{44}(k_j)n_{11,23}(s)(k_j)]}{(\dot{S^T s^A})_{44}(k_j)m_{34}(s)(k_j)e^{-2\theta(k_j)}}\\&+\frac{[M_2(k_j)]_3[S_{14}(k_j)n_{21,42}(s)(k_j)-S_{24}(k_j)n_{11,42}(s)(k_j)+S_{44}(k_j)n_{11,22}(s)(k_j)]}{(\dot{S^T s^A})_{44}(k_j)m_{34}(s)(k_j)e^{-2\theta(k_j)}},\\&\text{for}\quad n_1+1\le j\le n_2,\quad k\in D_2,\end{aligned} \tag{50}$$

$$\begin{aligned}\mathrm{Res}_{k=k_j}[M_3]_l=&\frac{m_{14}(S)(k_j)n_{4l,14}(s)(k_j)-m_{24}(S)(k_j)n_{4l,24}(s)(k_j)+m_{34}(S)(k_j)n_{4l,34}(s)(k_j)}{(s^T\dot{S}^A)_{44}(k_j)s_{44}(k_j)e^{2\theta(k_j)}}\\ &\times[M_3k_j)]_4,\quad \text{for}\quad n_2+1\le j\le n_3,\quad k\in D_3,\quad l=1,2,3,\end{aligned} \tag{51}$$

$$\mathrm{Res}_{k=k_j}[M_4]_l=-\frac{\mathbb{S}_{4l}(k_j)}{\dot{\mathbb{S}}_{44}(k_j)}[M_4(k_j)]_4e^{-2\theta(k_j)},\quad \text{for}\quad n_3+1\le j\le N,\quad k\in D_4,\quad l=1,2,3, \tag{52}$$

where the overdot stands for the derivative with resect to the parameter k and $\theta=\theta(k)=-i(kx+2k^2t)$.

Proof. It follows from Eqs. (41) and (42) that the four columns of M_1 are given by the matrices μ_2 and $S_1(k)$

$$[M_1]_j=[\mu_2]_1\mathbb{S}_{1j}+[\mu_2]_2\mathbb{S}_{2j}+[\mu_2]_3\mathbb{S}_{3j}+[\mu_2]_4\mathbb{S}_{4j}e^{-2\theta},\quad j=1,2,3, \tag{53a}$$

$$[M_1]_4=\frac{[\mu_2]_4}{m_{44}(\mathbb{S})}, \tag{53b}$$

the four columns of M_2 are given by the matrices μ_2 and $S_2(k)$

$$[M_2]_j=[\mu_2]_1s_{1j}+[\mu_2]_2s_{2j}+[\mu_2]_3s_{3j}+[\mu_2]_4s_{4j}e^{-2\theta},\quad j=1,2,3, \tag{54a}$$

$$[M_2]_4=\frac{[\mu_2]_1S_{14}}{(S^Ts^A)_{44}}e^{2\theta}+\frac{[\mu_2]_2S_{24}}{(S^Ts^A)_{44}}e^{2\theta}+\frac{[\mu_2]_3S_{34}}{(S^Ts^A)_{44}}e^{2\theta}+\frac{[\mu_2]_4S_{44}}{(S^Ts^A)_{44}}, \tag{54b}$$

the four columns of M_3 are given by the matrices μ_2 and $S_3(k)$

$$[M_3]_j=[\mu_2]_1S_3^{(1j)}+[\mu_2]_2S_3^{(2j)}+[\mu_2]_3S_3^{(3j)}+[\mu_2]_4S_3^{(4j)}e^{-2\theta},\quad j=1,2,3, \tag{55a}$$

$$[M_3]_4=[\mu_2]_1s_{14}e^{2\theta}+[\mu_2]_2s_{24}e^{2\theta}+[\mu_2]_3s_{34}e^{2\theta}+[\mu_2]_4s_{44}, \tag{55b}$$

and the four columns of M_4 are given by the matrices μ_2 and $S_4(k)$

$$[M_4]_j=[\mu_2]_1\frac{n_{1j,44}(\mathbb{S})}{\mathbb{S}_{44}}+[\mu_2]_2\frac{n_{2j,44}(\mathbb{S})}{\mathbb{S}_{44}}+[\mu_2]_3\frac{n_{3j,44}(\mathbb{S})}{\mathbb{S}_{44}},\quad j=1,2,3, \tag{56a}$$

$$[M_4]_4=[\mu_2]_1\mathbb{S}_{14}e^{2\theta}+[\mu_2]_2\mathbb{S}_{24}e^{2\theta}+[\mu_2]_3\mathbb{S}_{34}e^{2\theta}+[\mu_2]_4\mathbb{S}_{44}, \tag{56b}$$

For the case that $k_j\in D_1$ is a simple zero of $m_{44}(\mathbb{S})(k)$, it follows from Eq. (53a) that we have $[\mu_2]_j$, $j=1,2,4$ and then substitute them into Eq. (53b) to yield

$$[M_1]_4=\frac{n_{12,23}(\mathbb{S})[M_1]_1-n_{11,23}(\mathbb{S})[M_1]_2+n_{11,22}(\mathbb{S})[M_1]_3}{m_{34}(\mathbb{S})m_{44}(\mathbb{S})}e^{2\theta}-\frac{[\mu_2]_3}{m_{34}(\mathbb{S})}e^{2\theta},$$

whose residue at k_j yields Eq. (49) for $k_j\in D_1$, respectively.

Similarly, we solve Eq (54a) for $[\mu_2]_j$, $j = 1, 2, 4$, and then substitute them into Eq (54b) to yield

$$
\begin{aligned}
[M_2]_4 = & \frac{[M_2]_1[S_{14}n_{22,43}(s) - S_{24}n_{12,43}(s) + S_{44}n_{12,23}(s)]}{(S^T s^A)_{44} m_{34}(s)} e^{2\theta} \\
& - \frac{[M_2]_2[S_{14}n_{21,43}(s) - S_{24}n_{11,43}(s) + S_{44}n_{11,23}(s)]}{(S^T s^A)_{44} m_{34}(s)} e^{2\theta} \\
& + \frac{[M_2]_3[S_{14}n_{21,42}(s) - S_{24}n_{11,42}(s) + S_{44}n_{11,22}(s)]}{(S^T s^A)_{44} m_{34}(s)} e^{2\theta} - \frac{[\mu_2]_3}{m_{34}(s)} e^{2\theta},
\end{aligned}
$$

whose residues at k_j yield Eq. (50) for $k_j \in D_2$, respectively. Similarly, we can show Eq. (51) for $k_j \in D_3$ and Eq. (52) for $k_j \in D_4$ by analysing Eqs. (55a)–(56b). □

3. The 4×4 matrix RH problem

By using the district contours γ_j ($j = 1, 2, 3, 4$), the integral solutions of the revised Lax pair (11) and S_n due to $\{S(k), s(k), \mathbb{S}(k), S_L(k)\}$, we have defined the sectionally analytic functions $M_n(x, t, k)$ ($n = 1, 2, 3, 4$), which solve a 4×4 matrix RH problem. This RH problem can be formulated on the basis of the initial and boundary data of the functions $q_1(x, t)$, $q_2(x, t)$ and $q_3(x, t)$. Thus the solution of Eq. (4) for all values of x, t can be refound by solving the RH problem.

THEOREM 3.1 *Let $(q_1(x, t), q_2(x, t), q_3(x, t))$ be a solution of Eq. (4) in the interval domain $\Omega = \{(x, t) | x \in [0, L],\ t \in [0, T]\}$. Then it can be reconstructed from the initial data $q_j(x, t = 0) = q_{0j}(x)$, $j = 1, 2, 3$,* Dirichlet boundary data $q_j(x = 0, t) = u_{0j}(t)$, $q_j(x = L, t) = v_{0j}(t)$, $j = 1, 2, 3$ and Neumann boundary data $q_{jx}(x = 0, t) = u_{1j}(t)$, $q_{jx}(x = L, t) = v_{1j}(t)$, $j = 1, 2, 3$. We can use the initial and boundary data to define the jump matrices $J_{mn}(x, t, k)$, $(n, m = 1, ..., 4)$ given by Eq. (25) as well as the spectral functions $S(k)$, $s(k)$ and $\mathbb{S}(k)$ defined by Eq. (35). Assume that the possible zeros $\{k_j\}_1^N$ of the functions $m_{44}(\mathbb{S})(k)$, $(S^T s^A)_{44}(k)$, $(s^T S^A)_{44}(k)$ and $\mathbb{S}_{44}(k)$ are as in Assumption 2.4. Then the solution $(q_1(x, t),\ q_2(x, t), q_3(x, t))$ of Eq. (4) is given by $M(x, t, k)$ in the form

$$
q_j(x, t) = 2i \lim_{k \to \infty} (kM(x, t, k))_{j4}, \quad j = 1, 2, 3, \tag{57}
$$

where $M(x, t, k)$ satisfies the following 4×4 matrix RH problem:

- $M(x, t, k)$ is sectionally meromorphic on the Riemann k-sphere with jumps across the contours $\bar{D}_n \cup \bar{D}_m$, $(n, m = 1, ..., 4)$ (see Fig. 2);
- Across the contours $\bar{D}_n \cup \bar{D}_m$ $(n, m = 1, ..., 4)$, $M(x, t, k)$ satisfies the jump condition (24);
- The residue conditions of $M(x, t, k)$ are satisfied in Proposition 2.5;
- $M(x, t, k) = \mathbb{I}_4 + O(1/k)$ as $k \to \infty$.

Proof. System (57) can be deduced from the large k asymptotics of the eigenfunctions. We can follow the similar ones in Fokas (2002); Fokas *et al.* (2005) to show the rest proof of the Theorem. □

REMARK 3.2 The result (57) will be useful to further study the long-time asymptotics for the solution of the gtc-NLS system via the Deift–Zhou method (Deift & Zhou, 1992, 1993), or the numerical method (Trogdon, 2013) starting from the above-obtained RH problem.

4. The nonlinearizable boundary conditions

The main difficulty of IBV problems is to find the boundary values for a well-posed problem. All boundary value conditions are required for the definition of $S(k)$ and $S_L(k)$, and hence for the formulation the RH problem. Our main conclusion exhibits the unknown boundary condition on basis of the prescribed boundary condition, the initial condition and the solution of a system of nonlinear integral equations.

4.1 *The generalized global relation*

By evaluating Eqs. (47) and (48) at the point $(x, t) = (0, t)$, we have the global relation between the spectral functions

$$c(t,k) = \mu_2(0,t,k)e^{-2ik^2t\widehat{\sigma}_4}[s(k)e^{ikL\widehat{\sigma}_4}S_L(k)], \tag{58}$$

which and Eq. (36) lead to the global relation in the form

$$\begin{aligned} c(t,k) &= \mu_2(0,t,k)e^{-2ik^2t\widehat{\sigma}_4}[s(k)e^{ikL\widehat{\sigma}_4}e^{2ik^2t\widehat{\sigma}_4}\mu_3^{-1}(L,t,k)] \\ &= \mu_2(0,t,k)[e^{-2ik^2t\widehat{\sigma}_4}s(k)][e^{ikL\widehat{\sigma}_4}\mu_3^{-1}(L,t,k)]. \end{aligned} \tag{59}$$

Thus, the column vectors $[c(t,k)]_j$, $j = 1, 2, 3$ are analytic and bounded in D_4 away from the possible zeros of $\mathbb{S}_{44}(k)$ and of order $O(\frac{1+e^{-2ikL}}{k})$ as $k \to \infty$, and the column vector $[c(t,k)]_4$ is analytic and bounded in D_1 away from the possible zeros of $m_{44}(\mathbb{S})(k)$ and of order $O(\frac{1+e^{2ikL}}{k})$ as $k \to \infty$,

4.2 *Asymptotic behaviors of eigenfunctions*

It follows from the Lax pair (11) that the eigenfunctions $\{\mu_j\}_1^4$ possess the following asymptotics (as $k \to \infty$):

$$\begin{aligned} \mu_j(x,t,k) &= \mathbb{I}_4 + \sum_{l=1}^{2} \frac{1}{k^l} \begin{pmatrix} \mu_{j,11}^{(l)} & \mu_{j,12}^{(l)} & \mu_{j,13}^{(l)} & \mu_{j,14}^{(l)} \\ \mu_{j,21}^{(l)} & \mu_{j,22}^{(l)} & \mu_{j,23}^{(l)} & \mu_{j,24}^{(l)} \\ \mu_{j,31}^{(l)} & \mu_{j,32}^{(l)} & \mu_{j,33}^{(l)} & \mu_{j,34}^{(l)} \\ \mu_{j,41}^{(l)} & \mu_{j,42}^{(l)} & \mu_{j,43}^{(l)} & \mu_{j,44}^{(l)} \end{pmatrix} + O\left(\frac{1}{k^3}\right) \\ &= \mathbb{I}_4 + \frac{1}{k} \begin{pmatrix} \int_{(x_j,t_j)}^{(x,t)} \Delta_{11}^{(1)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{12}^{(1)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{13}^{(1)} & -\frac{i}{2}q_1 \\ \int_{(x_j,t_j)}^{(x,t)} \Delta_{21}^{(1)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{22}^{(1)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{23}^{(1)} & -\frac{i}{2}q_2 \\ \int_{(x_j,t_j)}^{(x,t)} \Delta_{31}^{(1)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{32}^{(1)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{33}^{(1)} & -\frac{i}{2}q_3 \\ \frac{i}{2}p_1 & \frac{i}{2}p_2 & \frac{i}{2}p_3 & \int_{(x_j,t_j)}^{(x,t)} \Delta_{44}^{(1)} \end{pmatrix} \end{aligned}$$

$$
+\frac{1}{k^2}\begin{pmatrix} \int_{(x_j,t_j)}^{(x,t)} \Delta_{11}^{(2)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{12}^{(2)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{13}^{(2)} & \mu_{j,14}^{(2)} \\ \int_{(x_j,t_j)}^{(x,t)} \Delta_{21}^{(2)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{22}^{(2)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{23}^{(2)} & \mu_{j,24}^{(2)} \\ \int_{(x_j,t_j)}^{(x,t)} \Delta_{31}^{(2)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{32}^{(2)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{33}^{(2)} & \mu_{j,34}^{(2)} \\ \mu_{j,41}^{(2)} & \mu_{j,42}^{(2)} & \mu_{j,43}^{(2)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{44}^{(2)} \end{pmatrix} + O\left(\frac{1}{k^3}\right), \tag{60}
$$

where we have introduced the following functions:

$$
\begin{aligned}
\Delta_{jl}^{(1)} &= \frac{i}{2} q_j p_l dx + \frac{1}{2}(q_j p_{lx} - q_{jx} p_l) dt, \quad j, l = 1, 2, 3, \\
\Delta_{44}^{(1)} &= -\frac{i}{2}\sum_{j=1}^{3} q_j p_j dx + \frac{1}{2}\sum_{j=1}^{3}(p_j q_{jx} - p_{jx} q_j) dt,
\end{aligned} \tag{61}
$$

and

$$
\mu_{j,l4}^{(2)} = \frac{1}{4} q_{lx} + \frac{1}{2i} q_l \int_{(x_j,t_j)}^{(x,t)} \Delta_{44}^{(1)}, \quad \mu_{j,4l}^{(2)}(t) = \frac{1}{4} p_{lx} + \frac{i}{2}\sum_{s=1}^{3} p_s \int_{(x_j,t_j)}^{(x,t)} \Delta_{sl}^{(1)}, \quad l = 1, 2, 3,
$$

$$
\begin{aligned}
\Delta_{sl}^{(2)} &= \left[\frac{1}{4} q_s p_{lx} + \frac{i}{2} q_s \sum_{n=1}^{3} p_n \int_{(x_j,t_j)}^{(x,t)} \Delta_{nl}^{(1)}\right] \mathrm{d}x \\
&\quad + \left\{\frac{1}{4}\left[q_s p_{lx} + i q_{sx} p_{lx} - i q_s p_l \sum_{j=1}^{3} q_j p_j\right] + \frac{1}{2}\sum_{n=1}^{3}(q_s p_{nx} - q_{sx} p_n)\int_{(x_j,t_j)}^{(x,t)} \Delta_{nl}^{(1)}\right\} \mathrm{d}t, \ s, l = 1, 2, 3,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{44}^{(2)}(t) &= \left[\frac{1}{4}\sum_{l=1}^{3} p_l q_{lx} - \frac{i}{2}\sum_{l=1}^{3} p_l q_l \int_{(x_j,t_j)}^{(x,t)} \Delta_{44}^{(1)}\right] \mathrm{d}x \\
&\quad + \left\{\frac{1}{4}\left[\sum_{l=1}^{3}(p_l q_{lx} - i p_{lx} q_{lx}) + i\left(\sum_{l=1}^{3} p_l q_l\right)^2\right] + \frac{1}{2}\sum_{l=1}^{3}(p_l q_{lx} - p_{lx} q_l)\int_{(x_j,t_j)}^{(x,t)} \Delta_{44}^{(1)}\right\} \mathrm{d}t.
\end{aligned}
$$

The functions $\{\mu_{jl}^{(s)} = \mu_{jl}^{(s)}(x,t)\}_1^4$, $s = 1, 2$ are independent of k.

We define the functions $\{\Psi_{ij}(t,k)\}_{i,j=1}^4$ as

$$\mu_2(0,t,k) = (\Psi_{sj}(t,k))_{4\times 4} = \mathbb{I}_4 + \sum_{l=1}^{2} \frac{1}{k^l} \begin{pmatrix} \Psi_{11}^{(l)}(t) & \Psi_{12}^{(l)}(t) & \Psi_{13}^{(l)}(t) & \Psi_{14}^{(l)}(t) \\ \Psi_{21}^{(l)}(t) & \Psi_{22}^{(l)}(t) & \Psi_{23}^{(l)}(t) & \Psi_{24}^{(l)}(t) \\ \Psi_{31}^{(l)}(t) & \Psi_{32}^{(l)}(t) & \Psi_{33}^{(l)}(t) & \Psi_{34}^{(l)}(t) \\ \Psi_{41}^{(l)}(t) & \Psi_{42}^{(l)}(t) & \Psi_{43}^{(l)}(t) & \Psi_{44}^{(l)}(t) \end{pmatrix} + O\left(\frac{1}{k^3}\right). \tag{62}$$

Based on the asymptotics of Eq. (60) and the boundary data at $x=0$, we find

$$\begin{cases} \Psi_{j4}^{(1)}(t) = -\dfrac{i}{2}u_{0j}(t), \\ \Psi_{j4}^{(2)}(t) = \dfrac{1}{4}u_{1j} - \dfrac{i}{2}u_{0j}\Psi_{44}^{(1)}, \quad j=1,2,3, \\ \Psi_{41}^{(1)}(t) = \dfrac{i}{2}\left[\alpha_{11}\bar{u}_{01}(t) + \bar{\alpha}_{12}\bar{u}_{02}(t) + \bar{\alpha}_{13}\bar{u}_{03}(t)\right], \\ \Psi_{42}^{(1)}(t) = \dfrac{i}{2}\left[\alpha_{12}\bar{u}_{01}(t) + \alpha_{22}\bar{u}_{02}(t) + \bar{\alpha}_{23}\bar{u}_{03}(t)\right], \\ \Psi_{43}^{(1)}(t) = \dfrac{i}{2}\left[\alpha_{13}\bar{u}_{01}(t) + \alpha_{23}\bar{u}_{02}(t) + \alpha_{33}\bar{u}_{03}(t)\right], \\ \Psi_{44}^{(1)}(t) = \dfrac{1}{2}\displaystyle\int_0^t \Big\{u_{11}\left[\alpha_{11}\bar{u}_{01}(t) + \bar{\alpha}_{12}\bar{u}_{02}(t) + \bar{\alpha}_{13}\bar{u}_{03}(t)\right] + u_{12}\left[\alpha_{12}\bar{u}_{01}(t) + \alpha_{22}\bar{u}_{02}(t) + \bar{\alpha}_{23}\bar{u}_{03}(t)\right] \\ \qquad + u_{13}\left[\alpha_{13}\bar{u}_{01}(t) + \alpha_{23}\bar{u}_{02}(t) + \alpha_{33}\bar{u}_{03}(t)\right] - u_{01}\left[\alpha_{11}\bar{u}_{11}(t) + \bar{\alpha}_{12}\bar{u}_{12}(t) + \bar{\alpha}_{13}\bar{u}_{13}(t)\right] \\ \qquad - u_{02}\left[\alpha_{12}\bar{u}_{11}(t) + \alpha_{22}\bar{u}_{12}(t) + \bar{\alpha}_{23}\bar{u}_{13}(t)\right] - u_{03}\left[\alpha_{13}\bar{u}_{11}(t) + \alpha_{23}\bar{u}_{12}(t) + \alpha_{33}\bar{u}_{13}(t)\right]\Big\}\mathrm{d}t. \end{cases} \tag{63}$$

Thus we have the the boundary data at $x=0$:

$$u_{0j}(t) = 2i\Psi_{j4}^{(1)}(t), \quad u_{1j}(t) = 4\Psi_{j4}^{(2)}(t) + 2iu_{0j}(t)\Psi_{44}^{(1)}(t), \quad j=1,2,3. \tag{64}$$

Similarly, we assume that the asymptotic formula of $\mu_3(L,t,k) = \{\phi_{ij}(t,k)\}_{i,j=1}^4$ is of the form

$$\mu_3(L,t,k) = (\phi_{sj}(t,k))_{4\times 4} = \mathbb{I}_4 + \sum_{l=1}^{2} \frac{1}{k^l} \begin{pmatrix} \phi_{11}^{(l)}(t) & \phi_{12}^{(l)}(t) & \phi_{13}^{(l)}(t) & \phi_{14}^{(l)}(t) \\ \phi_{21}^{(l)}(t) & \phi_{22}^{(l)}(t) & \phi_{23}^{(l)}(t) & \phi_{24}^{(l)}(t) \\ \phi_{31}^{(l)}(t) & \phi_{32}^{(l)}(t) & \phi_{33}^{(l)}(t) & \phi_{34}^{(l)}(t) \\ \phi_{41}^{(l)}(t) & \phi_{42}^{(l)}(t) & \phi_{43}^{(l)}(t) & \phi_{44}^{(l)}(t) \end{pmatrix} + O\left(\frac{1}{k^3}\right). \tag{65}$$

By using the asymptotics of Eq. (60) and the boundary data at $x = L$, we find

$$\begin{cases} \phi_{j4}^{(1)}(t) = -\dfrac{i}{2}v_{0j}(t), \\ \phi_{j4}^{(2)} = \dfrac{1}{4}v_{1j} + \dfrac{1}{2i}v_{0j}\phi_{44}^{(1)}, \quad j = 1,2,3, \\ \phi_{41}^{(1)}(t) = \dfrac{i}{2}\left[\alpha_{11}\bar{v}_{01}(t) + \bar{\alpha}_{12}\bar{v}_{02}(t) + \bar{\alpha}_{13}\bar{v}_{03}(t)\right], \\ \phi_{42}^{(1)}(t) = \dfrac{i}{2}\left[\alpha_{12}\bar{v}_{01}(t) + \alpha_{22}\bar{v}_{02}(t) + \bar{\alpha}_{23}\bar{v}_{03}(t)\right], \\ \phi_{43}^{(1)}(t) = \dfrac{i}{2}\left[\alpha_{13}\bar{v}_{01}(t) + \alpha_{23}\bar{v}_{02}(t) + \alpha_{33}\bar{v}_{03}(t)\right], \\ \phi_{44}^{(1)} = \dfrac{1}{2}\displaystyle\int_0^t \Big\{ v_{11}\left[\alpha_{11}\bar{v}_{01}(t) + \bar{\alpha}_{12}\bar{v}_{02}(t) + \bar{\alpha}_{13}\bar{v}_{03}(t)\right] + v_{12}\left[\alpha_{12}\bar{v}_{01}(t) + \alpha_{22}\bar{v}_{02}(t) + \bar{\alpha}_{23}\bar{v}_{03}(t)\right] \\ \qquad + v_{13}\left[\alpha_{13}\bar{v}_{01}(t) + \alpha_{23}\bar{v}_{02}(t) + \alpha_{33}\bar{v}_{03}(t)\right] - v_{01}\left[\alpha_{11}\bar{v}_{11}(t) + \bar{\alpha}_{12}\bar{v}_{12}(t) + \bar{\alpha}_{13}\bar{v}_{13}(t)\right] \\ \qquad - v_{02}\left[\alpha_{12}\bar{u}_{11}(t) + \alpha_{22}\bar{v}_{12}(t) + \bar{\alpha}_{23}\bar{v}_{13}(t)\right] - v_{03}\left[\alpha_{13}\bar{v}_{11}(t) + \alpha_{23}\bar{v}_{12}(t) + \alpha_{33}\bar{v}_{13}(t)\right] \Big\} \mathrm{d}t, \end{cases} \tag{66}$$

which generates the following expressions for the boundary values at $x = L$:

$$v_{0j}(t) = 2i\phi_{j4}^{(1)}(t), \quad v_{1j}(t) = 4\phi_{j4}^{(2)}(t) + 2iv_{0j}(t)\phi_{44}^{(1)}(t), \quad j = 1,2,3. \tag{67}$$

For the vanishing initial values, it follows from Eq. (60) that we have the following asymptotics of the global relation $c_{j4}(t,k)$ and $c_{4j}(t,k), j = 1,2,3$.

PROPOSITION 4.1 *Let the initial and Dirichlet boundary conditions be compatible at points $x = 0, L$ (i.e. $q_{0j}(0) = u_{0j}(0)$ at $x = 0$ and $q_{0j}(L) = v_{0j}(0)$ at $x = L$, $j = 1,2,3$). Then, the global relation (60) with the vanishing initial data implies that the large k behaviors of $c_{j4}(t,k)$ and $c_{4j}(t,k), j = 1,2,3$ are of the forms*

$$\begin{aligned} c_{14}(t,k) = {} & \frac{\Psi_{14}^{(1)}}{k} + \frac{\Psi_{14}^{(2)} + \Psi_{14}^{(1)}\bar{\phi}_{44}^{(1)}}{k^2} + O\left(\frac{1}{k^3}\right) \\ & - \left\{ \frac{\alpha_{11}\bar{\phi}_{41}^{(1)} + \bar{\alpha}_{12}\bar{\phi}_{42}^{(1)} + \bar{\alpha}_{13}\bar{\phi}_{43}^{(1)}}{k} + \frac{1}{k^2}\Big[\alpha_{11}\bar{\phi}_{41}^{(2)} + \bar{\alpha}_{12}\bar{\phi}_{42}^{(2)} + \bar{\alpha}_{13}\bar{\phi}_{43}^{(2)} \right. \\ & + \Psi_{11}^{(1)}\left(\alpha_{11}\bar{\phi}_{41}^{(1)} + \bar{\alpha}_{12}\bar{\phi}_{42}^{(1)} + \bar{\alpha}_{13}\bar{\phi}_{43}^{(1)}\right) + \Psi_{12}^{(1)}\left(\alpha_{12}\bar{\phi}_{41}^{(1)} + \alpha_{22}\bar{\phi}_{42}^{(1)} + \bar{\alpha}_{23}\bar{\phi}_{43}^{(1)}\right) \\ & \left. + \Psi_{13}^{(1)}\left(\alpha_{13}\bar{\phi}_{41}^{(1)} + \alpha_{23}\bar{\phi}_{42}^{(1)} + \alpha_{33}\bar{\phi}_{43}^{(1)}\right)\Big] + O\left(\frac{1}{k^3}\right) \right\} e^{2ikL}, \quad k \to \infty, \end{aligned} \tag{68}$$

$$
\begin{aligned}
c_{24}(t,k) = {} & \frac{\Psi_{24}^{(1)}}{k} + \frac{\Psi_{24}^{(2)} + \Psi_{24}^{(1)}\bar{\phi}_{44}^{(1)}}{k^2} + O\left(\frac{1}{k^3}\right) \\
& - \left\{ \frac{\alpha_{12}\bar{\phi}_{41}^{(1)} + \alpha_{22}\bar{\phi}_{42}^{(1)} + \bar{\alpha}_{23}\bar{\phi}_{43}^{(1)}}{k} + \frac{1}{k^2}\left[\alpha_{12}\bar{\phi}_{41}^{(2)} + \alpha_{22}\bar{\phi}_{42}^{(2)} + \bar{\alpha}_{23}\bar{\phi}_{43}^{(2)} \right.\right. \\
& + \Psi_{21}^{(1)}\left(\alpha_{11}\bar{\phi}_{41}^{(1)} + \bar{\alpha}_{12}\bar{\phi}_{42}^{(1)} + \bar{\alpha}_{13}\bar{\phi}_{43}^{(1)}\right) + \Psi_{22}^{(1)}\left(\alpha_{12}\bar{\phi}_{41}^{(1)} + \alpha_{22}\bar{\phi}_{42}^{(1)} + \bar{\alpha}_{23}\bar{\phi}_{43}^{(1)}\right) \\
& \left.\left. + \ \Psi_{23}^{(1)}\left(\alpha_{13}\bar{\phi}_{41}^{(1)} + \alpha_{23}\bar{\phi}_{42}^{(1)} + \alpha_{33}\bar{\phi}_{43}^{(1)}\right)\right] + O\left(\frac{1}{k^3}\right)\right\} e^{2ikL}, \quad k \to \infty,
\end{aligned} \tag{69}
$$

$$
\begin{aligned}
c_{34}(t,k) = {} & \frac{\Psi_{34}^{(1)}}{k} + \frac{\Psi_{34}^{(2)} + \Psi_{34}^{(1)}\bar{\phi}_{44}^{(1)}}{k^2} + O\left(\frac{1}{k^3}\right) \\
& - \left\{ \frac{\alpha_{13}\bar{\phi}_{41}^{(1)} + \alpha_{23}\bar{\phi}_{42}^{(1)} + \alpha_{33}\bar{\phi}_{43}^{(1)}}{k} + \frac{1}{k^2}\left[\alpha_{13}\bar{\phi}_{41}^{(2)} + \alpha_{23}\bar{\phi}_{42}^{(2)} + \alpha_{33}\bar{\phi}_{43}^{(2)} \right.\right. \\
& + \Psi_{31}^{(1)}\left(\alpha_{11}\bar{\phi}_{41}^{(1)} + \bar{\alpha}_{12}\bar{\phi}_{42}^{(1)} + \bar{\alpha}_{13}\bar{\phi}_{43}^{(1)}\right) + \Psi_{32}^{(1)}\left(\alpha_{12}\bar{\phi}_{41}^{(1)} + \alpha_{22}\bar{\phi}_{42}^{(1)} + \bar{\alpha}_{23}\bar{\phi}_{43}^{(1)}\right) \\
& \left.\left. + \ \Psi_{33}^{(1)}\left(\alpha_{13}\bar{\phi}_{41}^{(1)} + \alpha_{23}\bar{\phi}_{42}^{(1)} + \alpha_{33}\bar{\phi}_{43}^{(1)}\right)\right] + O\left(\frac{1}{k^3}\right)\right\} e^{2ikL}, \quad k \to \infty,
\end{aligned} \tag{70}
$$

$$
\begin{aligned}
c_{41}(t,k) = {} & -\left\{ \frac{\alpha_{11}\bar{\phi}_{14}^{(1)} + \alpha_{12}\bar{\phi}_{24}^{(1)} + \alpha_{13}\bar{\phi}_{34}^{(1)}}{k} + \frac{1}{k^2}\left[\alpha_{11}\bar{\phi}_{14}^{(2)} + \alpha_{12}\bar{\phi}_{24}^{(2)} + \alpha_{13}\bar{\phi}_{34}^{(2)} \right.\right. \\
& \left.\left. +\Psi_{44}^{(1)}\left(\alpha_{11}\bar{\phi}_{14}^{(1)} + \alpha_{12}\bar{\phi}_{24}^{(1)} + \alpha_{13}\bar{\phi}_{34}^{(1)}\right)\right] + O\left(\frac{1}{k^3}\right)\right\} e^{-2ikL} \\
& + \frac{1}{k}\left[(\alpha_{11}^2 + |\alpha_{12}|^2 + |\alpha_{13}|^2)\Psi_{41}^{(1)} + (\alpha_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\bar{\alpha}_{23})\Psi_{42}^{(1)} \right. \\
& \left. +(\alpha_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33})\Psi_{43}^{(1)}\right] + \frac{1}{k^2}\left[(\alpha_{11}^2 + |\alpha_{12}|^2 + |\alpha_{13}|^2)\Psi_{41}^{(2)} \right. \\
& + (\alpha_{11}\alpha_{12} + \alpha_{12}\alpha_{22} + \alpha_{13}\bar{\alpha}_{23})\Psi_{42}^{(2)} + (\alpha_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33})\Psi_{43}^{(2)} \\
& + \Psi_{41}^{(1)}\left[\alpha_{11}\left(\alpha_{11}\bar{\phi}_{11}^{(1)} + \bar{\alpha}_{12}\bar{\phi}_{12}^{(1)} + \bar{\alpha}_{13}\bar{\phi}_{13}^{(1)}\right) + \alpha_{12}\left(\alpha_{11}\bar{\phi}_{21}^{(1)} + \bar{\alpha}_{12}\bar{\phi}_{22}^{(1)} + \bar{\alpha}_{13}\bar{\phi}_{23}^{(1)}\right)\right. \\
& \left. +\alpha_{13}\left(\alpha_{11}\bar{\phi}_{31}^{(1)} + \bar{\alpha}_{12}\bar{\phi}_{32}^{(1)} + \bar{\alpha}_{13}\bar{\phi}_{33}^{(1)}\right)\right] + \Psi_{42}^{(1)}\left[\alpha_{11}\left(\alpha_{12}\bar{\phi}_{11}^{(1)} + \alpha_{22}\bar{\phi}_{12}^{(1)} + \bar{\alpha}_{23}\bar{\phi}_{13}^{(1)}\right)\right. \\
& \left. +\alpha_{12}\left(\alpha_{12}\bar{\phi}_{21}^{(1)} + \alpha_{22}\bar{\phi}_{22}^{(1)} + \bar{\alpha}_{23}\bar{\phi}_{23}^{(1)}\right) + \alpha_{13}\left(\alpha_{12}\bar{\phi}_{31}^{(1)} + \alpha_{22}\bar{\phi}_{32}^{(1)} + \bar{\alpha}_{23}\bar{\phi}_{33}^{(1)}\right)\right] \\
& + \Psi_{43}^{(1)}\left[\alpha_{11}\left(\alpha_{13}\bar{\phi}_{11}^{(1)} + \alpha_{23}\bar{\phi}_{12}^{(1)} + \alpha_{33}\bar{\phi}_{13}^{(1)}\right) + \alpha_{12}\left(\alpha_{13}\bar{\phi}_{21}^{(1)} + \alpha_{23}\bar{\phi}_{22}^{(1)} + \alpha_{33}\bar{\phi}_{23}^{(1)}\right)\right. \\
& \left.\left. +\alpha_{13}\left(\alpha_{13}\bar{\phi}_{31}^{(1)} + \alpha_{23}\bar{\phi}_{32}^{(1)} + \alpha_{33}\bar{\phi}_{33}^{(1)}\right)\right]\right] + O\left(\frac{1}{k^3}\right), \quad k \to \infty,
\end{aligned} \tag{71}
$$

$$
\begin{aligned}
c_{42}(t,k) = &-\Bigg\{ \frac{\bar{\alpha}_{12}\bar{\phi}^{(1)}_{14} + \alpha_{22}\bar{\phi}^{(1)}_{24} + \alpha_{23}\bar{\phi}^{(1)}_{34}}{k} + \frac{1}{k^2}\Big[\bar{\alpha}_{12}\bar{\phi}^{(2)}_{14} + \alpha_{22}\bar{\phi}^{(2)}_{24} + \alpha_{23}\bar{\phi}^{(2)}_{34} \\
&+\Psi^{(1)}_{44}\left(\bar{\alpha}_{12}\bar{\phi}^{(1)}_{14} + \alpha_{22}\bar{\phi}^{(1)}_{24} + \alpha_{23}\bar{\phi}^{(1)}_{34}\right)\Big] + O\left(\frac{1}{k^3}\right)\Bigg\} e^{-2ikL} \\
&+ \frac{1}{k}\Big\{(|\alpha_{12}|^2 + \alpha_{22}^2 + |\alpha_{23}|^2)\Psi^{(1)}_{42} + (\alpha_{11}\bar{\alpha}_{12} + \bar{\alpha}_{12}\bar{\alpha}_{22} + \bar{\alpha}_{13}\alpha_{23})\Psi^{(1)}_{41} \\
&+ (\bar{\alpha}_{12}\alpha_{13} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33})\Psi^{(1)}_{43} + \frac{1}{k^2}\Big[(|\alpha_{12}|^2 + \alpha_{22}^2 + |\alpha_{23}|^2)\Psi^{(2)}_{42} \\
&+ (\alpha_{11}\bar{\alpha}_{12} + \bar{\alpha}_{12}\bar{\alpha}_{22} + \bar{\alpha}_{13}\alpha_{23})\Psi^{(2)}_{41} + (\bar{\alpha}_{12}\alpha_{13} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33})\Psi^{(2)}_{43} \\
&+ \Psi^{(1)}_{41}\Big[\bar{\alpha}_{12}\left(\alpha_{11}\bar{\phi}^{(1)}_{11} + \bar{\alpha}_{12}\bar{\phi}^{(1)}_{12} + \bar{\alpha}_{13}\bar{\phi}^{(1)}_{13}\right) + \alpha_{22}\left(\alpha_{11}\bar{\phi}^{(1)}_{21} + \bar{\alpha}_{12}\bar{\phi}^{(1)}_{22} + \bar{\alpha}_{13}\bar{\phi}^{(1)}_{23}\right) \\
&+\alpha_{23}\left(\alpha_{11}\bar{\phi}^{(1)}_{31} + \bar{\alpha}_{12}\bar{\phi}^{(1)}_{32} + \bar{\alpha}_{13}\bar{\phi}^{(1)}_{33}\right)\Big] + \Psi^{(1)}_{42}\Big[\bar{\alpha}_{12}\left(\alpha_{12}\bar{\phi}^{(1)}_{11} + \alpha_{22}\bar{\phi}^{(1)}_{12} + \bar{\alpha}_{23}\bar{\phi}^{(1)}_{13}\right) \\
&+\alpha_{22}\left(\alpha_{12}\bar{\phi}^{(1)}_{21} + \alpha_{22}\bar{\phi}^{(1)}_{22} + \bar{\alpha}_{23}\bar{\phi}^{(1)}_{23}\right) + \alpha_{23}\left(\alpha_{12}\bar{\phi}^{(1)}_{31} + \alpha_{22}\bar{\phi}^{(1)}_{32} + \bar{\alpha}_{23}\bar{\phi}^{(1)}_{33}\right)\Big] \\
&+ \Psi^{(1)}_{43}\Big[\bar{\alpha}_{12}\left(\alpha_{13}\bar{\phi}^{(1)}_{11} + \alpha_{23}\bar{\phi}^{(1)}_{12} + \alpha_{33}\bar{\phi}^{(1)}_{13}\right) + \alpha_{22}\left(\alpha_{13}\bar{\phi}^{(1)}_{21} + \alpha_{23}\bar{\phi}^{(1)}_{22} + \alpha_{33}\bar{\phi}^{(1)}_{23}\right) \\
&+\alpha_{23}\left(\alpha_{13}\bar{\phi}^{(1)}_{31} + \alpha_{23}\bar{\phi}^{(1)}_{32} + \alpha_{33}\bar{\phi}^{(1)}_{33}\right)\Big] + O\left(\frac{1}{k^3}\right), \quad k \to \infty,
\end{aligned}
\tag{72}
$$

$$
\begin{aligned}
c_{43}(t,k) = &-\Bigg\{ \frac{\bar{\alpha}_{13}\bar{\phi}^{(1)}_{14} + \bar{\alpha}_{23}\bar{\phi}^{(1)}_{24} + \alpha_{33}\bar{\phi}^{(1)}_{34}}{k} + \frac{1}{k^2}\Big[\bar{\alpha}_{13}\bar{\phi}^{(2)}_{14} + \bar{\alpha}_{23}\bar{\phi}^{(2)}_{24} + \alpha_{33}\bar{\phi}^{(2)}_{34} \\
&+\Psi^{(1)}_{44}\left(\bar{\alpha}_{13}\bar{\phi}^{(1)}_{14} + \bar{\alpha}_{23}\bar{\phi}^{(1)}_{24} + \alpha_{33}\bar{\phi}^{(1)}_{34}\right)\Big] + O\left(\frac{1}{k^3}\right)\Bigg\} e^{-2ikL} \\
&+ \frac{1}{k}\Big\{(|\alpha_{13}|^2 + |\alpha_{23}|^2 + \alpha_{33}^2)\Psi^{(1)}_{43} + (\alpha_{11}\bar{\alpha}_{13} + \bar{\alpha}_{12}\bar{\alpha}_{23} + \bar{\alpha}_{13}\alpha_{33})\Psi^{(1)}_{41} \\
&+ (\bar{\alpha}_{13}\alpha_{12} + \alpha_{22}\bar{\alpha}_{23} + \alpha_{13}\bar{\alpha}_{23})\Psi^{(1)}_{42} + \frac{1}{k^2}\Big[(|\alpha_{13}|^2 + |\alpha_{23}|^2 + \alpha_{33}^2)\Psi^{(2)}_{43} \\
&+ (\alpha_{11}\bar{\alpha}_{12} + \bar{\alpha}_{12}\bar{\alpha}_{22} + \bar{\alpha}_{13}\alpha_{23})\Psi^{(2)}_{41} + (\bar{\alpha}_{12}\alpha_{13} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33})\Psi^{(2)}_{43} \\
&+ \Psi^{(1)}_{41}\Big[\bar{\alpha}_{13}\left(\alpha_{11}\bar{\phi}^{(1)}_{11} + \bar{\alpha}_{12}\bar{\phi}^{(1)}_{12} + \bar{\alpha}_{13}\bar{\phi}^{(1)}_{13}\right) + \bar{\alpha}_{23}\left(\alpha_{11}\bar{\phi}^{(1)}_{21} + \bar{\alpha}_{12}\bar{\phi}^{(1)}_{22} + \bar{\alpha}_{13}\bar{\phi}^{(1)}_{23}\right) \\
&+\alpha_{33}\left(\alpha_{11}\bar{\phi}^{(1)}_{31} + \bar{\alpha}_{12}\bar{\phi}^{(1)}_{32} + \bar{\alpha}_{13}\bar{\phi}^{(1)}_{33}\right)\Big] + \Psi^{(1)}_{42}\Big[\bar{\alpha}_{13}\left(\alpha_{12}\bar{\phi}^{(1)}_{11} + \alpha_{22}\bar{\phi}^{(1)}_{12} + \bar{\alpha}_{23}\bar{\phi}^{(1)}_{13}\right) \\
&+\bar{\alpha}_{23}\left(\alpha_{12}\bar{\phi}^{(1)}_{21} + \alpha_{22}\bar{\phi}^{(1)}_{22} + \bar{\alpha}_{23}\bar{\phi}^{(1)}_{23}\right) + \alpha_{33}\left(\alpha_{12}\bar{\phi}^{(1)}_{31} + \alpha_{22}\bar{\phi}^{(1)}_{32} + \bar{\alpha}_{23}\bar{\phi}^{(1)}_{33}\right)\Big] \\
&+ \Psi^{(1)}_{43}\Big[\bar{\alpha}_{13}\left(\alpha_{13}\bar{\phi}^{(1)}_{11} + \alpha_{23}\bar{\phi}^{(1)}_{12} + \alpha_{33}\bar{\phi}^{(1)}_{13}\right) + \bar{\alpha}_{23}\left(\alpha_{13}\bar{\phi}^{(1)}_{21} + \alpha_{23}\bar{\phi}^{(1)}_{22} + \alpha_{33}\bar{\phi}^{(1)}_{23}\right) \\
&+\alpha_{33}\left(\alpha_{13}\bar{\phi}^{(1)}_{31} + \alpha_{23}\bar{\phi}^{(1)}_{32} + \alpha_{33}\bar{\phi}^{(1)}_{33}\right)\Big] + O\left(\frac{1}{k^3}\right), \quad k \to \infty.
\end{aligned}
\tag{73}
$$

Proof. This proof of this proposition is presented in **Appendix A.** □

4.3 *The relation between Dirichlet and Neumann boundary value problems*

In what follows we show that the spectral functions $S(k)$ and $S_L(k)$ can be expressed in terms of the prescribed Dirichlet and Neumann boundary data and the initial data using the solution of a system of integral equations. Introduce the new notations as

$$F_{\pm}(t,k) = F(t,k) \pm F(t,-k), \quad \Sigma_{\pm}(k) = e^{2ikL} \pm e^{-2ikL}. \tag{74}$$

The sign ∂D_j stands for the boundary of the jth quadrant D_j, oriented so that D_j lies to the left of ∂D_j. ∂D_3^0 denotes the boundary contour that does not contain the zeros of $\Sigma_-(k)$ and $\partial D_3^0 = -\partial D_1^0$.

PROPOSITION 4.2 *Let $q_{0j}(x) = q_j(x, t=0) = 0$, $j = 1,2,3$ be the initial data of Eq. (4) on the interval $x \in [0,L]$ and $T < \infty$. (i) For the Dirichlet problem, the boundary data $u_{0j}(t)$ and $v_{0j}(t)$ $(j = 1,2,3)$ on the interval $t \in [0,T)$ are sufficiently smooth and compatible with the initial data $q_{0j}(x)$, $(j = 1,2,3)$ at points $(x_2,t_2) = (0,0)$ and $(x_3,t_3) = (L,0)$, respectively, i.e. $u_{0j}(0) = q_{0j}(0)$, $v_{0j}(0) = q_{0j}(L)$, $j = 1,2,3$; (ii) For the Neumann problem, the boundary data $u_{1j}(t)$ and $v_{0j}(t)$ $(j = 1,2,3)$ on the interval $t \in [0,T)$ are sufficiently smooth and compatible with the initial data $q_{0j}(x)$, $(j = 1,2,3)$ at the origin $(x_2,t_2) = (0,0)$ and $(x_3,t_3) = (L,0)$, respectively.*

For simplicity, let $n_{33,44}(\mathbb{S})(k)$ have no zero in the domain D_1. Then the spectral functions $S(k)$ and $S_L(k)$ are defined by

$$S(k) = e^{2ik^2T\widehat{\sigma}_4}\left[\mathcal{P}\begin{pmatrix} \overline{\Psi_{11}(T,\bar{k})} & \overline{\Psi_{21}(T,\bar{k})} & \overline{\Psi_{31}(T,\bar{k})} & \overline{\Psi_{41}(T,\bar{k})} \\ \overline{\Psi_{12}(T,\bar{k})} & \overline{\Psi_{22}(T,\bar{k})} & \overline{\Psi_{32}(T,\bar{k})} & \overline{\Psi_{42}(T,\bar{k})} \\ \overline{\Psi_{13}(T,\bar{k})} & \overline{\Psi_{23}(T,\bar{k})} & \overline{\Psi_{33}(T,\bar{k})} & \overline{\Psi_{43}(T,\bar{k})} \\ \overline{\Psi_{14}(T,\bar{k})} & \overline{\Psi_{24}(T,\bar{k})} & \overline{\Psi_{34}(T,\bar{k})} & \overline{\Psi_{44}(T,\bar{k})} \end{pmatrix}\mathcal{P}\right], \tag{75}$$

$$S_L(k) = e^{2ik^2T\widehat{\sigma}_4}\left[\mathcal{P}\begin{pmatrix} \overline{\phi_{11}(T,\bar{k})} & \overline{\phi_{21}(T,\bar{k})} & \overline{\phi_{31}(T,\bar{k})} & \overline{\phi_{41}(T,\bar{k})} \\ \overline{\phi_{12}(T,\bar{k})} & \overline{\phi_{22}(T,\bar{k})} & \overline{\phi_{32}(T,\bar{k})} & \overline{\phi_{42}(T,\bar{k})} \\ \overline{\phi_{13}(T,\bar{k})} & \overline{\phi_{23}(T,\bar{k})} & \overline{\phi_{33}(T,\bar{k})} & \overline{\phi_{43}(T,\bar{k})} \\ \overline{\phi_{14}(T,\bar{k})} & \overline{\phi_{24}(T,\bar{k})} & \overline{\phi_{34}(T,\bar{k})} & \overline{\phi_{44}(T,\bar{k})} \end{pmatrix}\mathcal{P}\right], \tag{76}$$

where the matrix $\mathcal{P}$ is given by Eq. (31), and the complex-valued functions $\{\Psi_{ij}(t,k)\}_{i,j=1}^4$ have the following system of integral equations:

$$\Psi_{11,t}(t,k) = 1 + \int_0^t \big[(2ku_{01} + iu_{11})\Psi_{41} - i\Psi_{11}(\alpha_{11}|u_{01}|^2 + \bar{\alpha}_{12}u_{01}\bar{u}_{02} + \bar{\alpha}_{13}u_{01}\bar{u}_{03})$$
$$-i\Psi_{21}(\alpha_{12}|u_{01}|^2 + \alpha_{22}u_{01}\bar{u}_{02} + \bar{\alpha}_{23}u_{01}\bar{u}_{03}) - i\Psi_{31}(\alpha_{13}|u_{01}|^2 + \alpha_{23}u_{01}\bar{u}_{02} + \alpha_{33}u_{01}\bar{u}_{03})\big](\tau,k)\mathrm{d}\tau,$$
$$\Psi_{21,t}(t,k) = \int_0^t \big[(2ku_{02} + iu_{12})\Psi_{41} - i\Psi_{11}(\alpha_{11}u_{02}\bar{u}_{01} + \bar{\alpha}_{12}|u_{02}|^2 + \bar{\alpha}_{13}u_{02}\bar{u}_{03})$$
$$-i\Psi_{21}(\alpha_{12}u_{02}\bar{u}_{01} + \alpha_{22}|u_{02}|^2 + \bar{\alpha}_{23}u_{02}\bar{u}_{03}) - i\Psi_{31}(\alpha_{13}u_{02}\bar{u}_{01} + \alpha_{23}|u_{02}|^2 + \alpha_{33}u_{02}\bar{u}_{03})\big](\tau,k)\mathrm{d}\tau,$$
$$\Psi_{31,t}(t,k) = \int_0^t \big[(2ku_{03} + iu_{13})\Psi_{41} - i\Psi_{11}(\alpha_{11}u_{03}\bar{u}_{01} + \bar{\alpha}_{12}u_{03}\bar{u}_{02} + \bar{\alpha}_{13}|u_{03}|^2)$$
$$-i\Psi_{21}(\alpha_{12}u_{03}\bar{u}_{01} + \alpha_{22}u_{03}\bar{u}_{02} + \bar{\alpha}_{23}|u_{03}|^2) - i\Psi_{31}(\alpha_{13}u_{03}\bar{u}_{01} + \alpha_{23}u_{03}\bar{u}_{02} + \alpha_{33}|u_{03}|^2)\big](\tau,k)\mathrm{d}\tau,$$
$$\Psi_{41,t}(t,k) = \int_0^t e^{4ik^2(t-\tau)}\big\{\Psi_{11}[\alpha_{11}(2k\bar{u}_{01} - i\bar{u}_{11}) + \bar{\alpha}_{12}(2k\bar{u}_{02} - i\bar{u}_{12}) + \bar{\alpha}_{13}(2k\bar{u}_{03} - i\bar{u}_{13})]$$
$$+ \Psi_{21}[\alpha_{12}(2k\bar{u}_{01} - i\bar{u}_{11}) + \alpha_{22}(2k\bar{u}_{02} - i\bar{u}_{12}) + \bar{\alpha}_{23}(2k\bar{u}_{03} - i\bar{u}_{13})]$$
$$+ \Psi_{31}[\alpha_{13}(2k\bar{u}_{01} - i\bar{u}_{11}) + \alpha_{23}(2k\bar{u}_{02} - i\bar{u}_{12}) + \alpha_{33}(2k\bar{u}_{03} - i\bar{u}_{13})]$$
$$+ i\Psi_{41}[\alpha_{11}|u_{01}|^2 + \bar{\alpha}_{12}u_{01}\bar{u}_{02} + \bar{\alpha}_{13}u_{01}\bar{u}_{03} + \alpha_{12}u_{02}\bar{u}_{01}$$
$$+ \alpha_{22}|u_{02}|^2 + \bar{\alpha}_{23}u_{02}\bar{u}_{03} + \alpha_{13}u_{03}\bar{u}_{01} + \alpha_{23}u_{03}\bar{u}_{02} + \alpha_{33}|u_{03}|^2]\big\}(\tau,k)\mathrm{d}\tau, \tag{77}$$

$$\Psi_{12,t}(t,k) = \int_0^t \big[(2ku_{01} + iu_{11})\Psi_{42} - i\Psi_{12}(\alpha_{11}|u_{01}|^2 + \bar{\alpha}_{12}u_{01}\bar{u}_{02} + \bar{\alpha}_{13}u_{01}\bar{u}_{03})$$
$$-i\Psi_{22}(\alpha_{12}|u_{01}|^2 + \alpha_{22}u_{01}\bar{u}_{02} + \bar{\alpha}_{23}u_{01}\bar{u}_{03}) - i\Psi_{32}(\alpha_{13}|u_{01}|^2 + \alpha_{23}u_{01}\bar{u}_{02} + \alpha_{33}u_{01}\bar{u}_{03})\big](\tau,k)\mathrm{d}\tau,$$
$$\Psi_{22,t}(t,k) = 1 + \int_0^t \big[(2ku_{02} + iu_{12})\Psi_{42} - i\Psi_{12}(\alpha_{11}u_{02}\bar{u}_{01} + \bar{\alpha}_{12}|u_{02}|^2 + \bar{\alpha}_{13}u_{02}\bar{u}_{03})$$
$$-i\Psi_{22}(\alpha_{12}u_{02}\bar{u}_{01} + \alpha_{22}|u_{02}|^2 + \bar{\alpha}_{23}u_{02}\bar{u}_{03}) - i\Psi_{32}(\alpha_{13}u_{02}\bar{u}_{01} + \alpha_{23}|u_{02}|^2 + \alpha_{33}u_{02}\bar{u}_{03})\big](\tau,k)\mathrm{d}\tau,$$
$$\Psi_{32,t}(t,k) = \int_0^t \big[(2ku_{03} + iu_{13})\Psi_{42} - i\Psi_{12}(\alpha_{11}u_{03}\bar{u}_{01} + \bar{\alpha}_{12}u_{03}\bar{u}_{02} + \bar{\alpha}_{13}|u_{03}|^2)$$
$$-i\Psi_{22}(\alpha_{12}u_{03}\bar{u}_{01} + \alpha_{22}u_{03}\bar{u}_{02} + \bar{\alpha}_{23}|u_{03}|^2) - i\Psi_{32}(\alpha_{13}u_{03}\bar{u}_{01} + \alpha_{23}u_{03}\bar{u}_{02} + \alpha_{33}|u_{03}|^2)\big](\tau,k)\mathrm{d}\tau,$$
$$\Psi_{42,t}(t,k) = \int_0^t e^{4ik^2(t-\tau)}\big\{\Psi_{12}[\alpha_{11}(2k\bar{u}_{01} - i\bar{u}_{11}) + \bar{\alpha}_{12}(2k\bar{u}_{02} - i\bar{u}_{12}) + \bar{\alpha}_{13}(2k\bar{u}_{03} - i\bar{u}_{13})]$$
$$+ \Psi_{22}[\alpha_{12}(2k\bar{u}_{01} - i\bar{u}_{11}) + \alpha_{22}(2k\bar{u}_{02} - i\bar{u}_{12}) + \bar{\alpha}_{23}(2k\bar{u}_{03} - i\bar{u}_{13})]$$
$$+ \Psi_{32}[\alpha_{13}(2k\bar{u}_{01} - i\bar{u}_{11}) + \alpha_{23}(2k\bar{u}_{02} - i\bar{u}_{12}) + \alpha_{33}(2k\bar{u}_{03} - i\bar{u}_{13})]$$
$$+ i\Psi_{42}[\alpha_{11}|u_{01}|^2 + \bar{\alpha}_{12}u_{01}\bar{u}_{02} + \bar{\alpha}_{13}u_{01}\bar{u}_{03} + \alpha_{12}u_{02}\bar{u}_{01}$$
$$+ \alpha_{22}|u_{02}|^2 + \bar{\alpha}_{23}u_{02}\bar{u}_{03} + \alpha_{13}u_{03}\bar{u}_{01} + \alpha_{23}u_{03}\bar{u}_{02} + \alpha_{33}|u_{03}|^2]\big\}(\tau,k)\mathrm{d}\tau, \tag{78}$$

$$
\begin{aligned}
\Psi_{13,t}(t,k) =&\int_0^t \big[(2ku_{01}+iu_{11})\Psi_{43}-i\Psi_{13}(\alpha_{11}|u_{01}|^2+\bar{\alpha}_{12}u_{01}\bar{u}_{02}+\bar{\alpha}_{13}u_{01}\bar{u}_{03})\\
&-i\Psi_{22}(\alpha_{12}|u_{01}|^2+\alpha_{22}u_{01}\bar{u}_{02}+\bar{\alpha}_{23}u_{01}\bar{u}_{03})-i\Psi_{32}(\alpha_{13}|u_{01}|^2+\alpha_{23}u_{01}\bar{u}_{02}+\alpha_{33}u_{01}\bar{u}_{03})\big](\tau,k)\mathrm{d}\tau,\\
\Psi_{23,t}(t,k) =&\int_0^t \big[(2ku_{02}+iu_{12})\Psi_{42}-i\Psi_{13}(\alpha_{11}u_{02}\bar{u}_{01}+\bar{\alpha}_{12}|u_{02}|^2+\bar{\alpha}_{13}u_{02}\bar{u}_{03})\\
&-i\Psi_{23}(\alpha_{12}u_{02}\bar{u}_{01}+\alpha_{22}|u_{02}|^2+\bar{\alpha}_{23}u_{02}\bar{u}_{03})-i\Psi_{33}(\alpha_{13}u_{02}\bar{u}_{01}+\alpha_{23}|u_{02}|^2+\alpha_{33}u_{02}\bar{u}_{03})\big](\tau,k)\mathrm{d}\tau,\\
\Psi_{33,t}(t,k) =&1+\int_0^t \big[(2ku_{03}+iu_{13})\Psi_{43}-i\Psi_{13}(\alpha_{11}u_{03}\bar{u}_{01}+\bar{\alpha}_{12}u_{03}\bar{u}_{02}+\bar{\alpha}_{13}|u_{03}|^2)\\
&-i\Psi_{23}(\alpha_{12}u_{03}\bar{u}_{01}+\alpha_{22}u_{03}\bar{u}_{02}+\bar{\alpha}_{23}|u_{03}|^2)-i\Psi_{33}(\alpha_{13}u_{03}\bar{u}_{01}+\alpha_{23}u_{03}\bar{u}_{02}+\alpha_{33}|u_{03}|^2)\big](\tau,k)\mathrm{d}\tau,\\
\Psi_{43,t}(t,k) =&\int_0^t e^{4ik^2(t-\tau)}\big\{\Psi_{13}[\alpha_{11}(2k\bar{u}_{01}-i\bar{u}_{11})+\bar{\alpha}_{12}(2k\bar{u}_{02}-i\bar{u}_{12})+\bar{\alpha}_{13}(2k\bar{u}_{03}-i\bar{u}_{13})]\\
&+\Psi_{23}[\alpha_{12}(2k\bar{u}_{01}-i\bar{u}_{11})+\alpha_{22}(2k\bar{u}_{02}-i\bar{u}_{12})+\bar{\alpha}_{23}(2k\bar{u}_{03}-i\bar{u}_{13})]\\
&+\Psi_{33}[\alpha_{13}(2k\bar{u}_{01}-i\bar{u}_{11})+\alpha_{23}(2k\bar{u}_{02}-i\bar{u}_{12})+\alpha_{33}(2k\bar{u}_{03}-i\bar{u}_{13})]\\
&+i\Psi_{43}[\alpha_{11}|u_{01}|^2+\bar{\alpha}_{12}u_{01}\bar{u}_{02}+\bar{\alpha}_{13}u_{01}\bar{u}_{03}+\alpha_{12}u_{02}\bar{u}_{01}\\
&+\alpha_{22}|u_{02}|^2+\bar{\alpha}_{23}u_{02}\bar{u}_{03}+\alpha_{13}u_{03}\bar{u}_{01}+\alpha_{23}u_{03}\bar{u}_{02}+\alpha_{33}|u_{03}|^2]\big\}(\tau,k)\mathrm{d}\tau, \qquad (79)
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_{14,t}(t,k) =&\int_0^t e^{-4ik^2(t-\tau)}\big[(2ku_{01}+iu_{11})\Psi_{44}-i\Psi_{14}(\alpha_{11}|u_{01}|^2+\bar{\alpha}_{12}u_{01}\bar{u}_{02}+\bar{\alpha}_{13}u_{01}\bar{u}_{03})\\
&-i\Psi_{24}(\alpha_{12}|u_{01}|^2+\alpha_{22}u_{01}\bar{u}_{02}+\bar{\alpha}_{23}u_{01}\bar{u}_{03})-i\Psi_{34}(\alpha_{13}|u_{01}|^2+\alpha_{23}u_{01}\bar{u}_{02}+\alpha_{33}u_{01}\bar{u}_{03})\big](\tau,k)\mathrm{d}\tau,\\
\Psi_{24,t}(t,k) =&\int_0^t e^{-4ik^2(t-\tau)}\big[(2ku_{02}+iu_{12})\Psi_{44}-i\Psi_{14}(\alpha_{11}u_{02}\bar{u}_{01}+\bar{\alpha}_{12}|u_{02}|^2+\bar{\alpha}_{13}u_{02}\bar{u}_{03})\\
&-i\Psi_{24}(\alpha_{12}u_{02}\bar{u}_{01}+\alpha_{22}|u_{02}|^2+\bar{\alpha}_{23}u_{02}\bar{u}_{03})-i\Psi_{34}(\alpha_{13}u_{02}\bar{u}_{01}+\alpha_{23}|u_{02}|^2+\alpha_{33}u_{02}\bar{u}_{03})\big](\tau,k)\mathrm{d}\tau,\\
\Psi_{34,t}(t,k) =&\int_0^t e^{-4ik^2(t-\tau)}\big[(2ku_{03}+iu_{13})\Psi_{44}-i\Psi_{14}(\alpha_{11}u_{03}\bar{u}_{01}+\bar{\alpha}_{12}u_{03}\bar{u}_{02}+\bar{\alpha}_{13}|u_{03}|^2)\\
&-i\Psi_{24}(\alpha_{12}u_{03}\bar{u}_{01}+\alpha_{22}u_{03}\bar{u}_{02}+\bar{\alpha}_{23}|u_{03}|^2)-i\Psi_{34}(\alpha_{13}u_{03}\bar{u}_{01}+\alpha_{23}u_{03}\bar{u}_{02}+\alpha_{33}|u_{03}|^2)\big](\tau,k)\mathrm{d}\tau,\\
\Psi_{44,t}(t,k) =&1+\int_0^t \big\{\Psi_{14}[\alpha_{11}(2k\bar{u}_{01}-i\bar{u}_{11})+\bar{\alpha}_{12}(2k\bar{u}_{02}-i\bar{u}_{12})+\bar{\alpha}_{13}(2k\bar{u}_{03}-i\bar{u}_{13})]\\
&+\Psi_{24}[\alpha_{12}(2k\bar{u}_{01}-i\bar{u}_{11})+\alpha_{22}(2k\bar{u}_{02}-i\bar{u}_{12})+\bar{\alpha}_{23}(2k\bar{u}_{03}-i\bar{u}_{13})]\\
&+\Psi_{34}[\alpha_{13}(2k\bar{u}_{01}-i\bar{u}_{11})+\alpha_{23}(2k\bar{u}_{02}-i\bar{u}_{12})+\alpha_{33}(2k\bar{u}_{03}-i\bar{u}_{13})]\\
&+i\Psi_{44}[\alpha_{11}|u_{01}|^2+\bar{\alpha}_{12}u_{01}\bar{u}_{02}+\bar{\alpha}_{13}u_{01}\bar{u}_{03}+\alpha_{12}u_{02}\bar{u}_{01}\\
&+\alpha_{22}|u_{02}|^2+\bar{\alpha}_{23}u_{02}\bar{u}_{03}+\alpha_{13}u_{03}\bar{u}_{01}+\alpha_{23}u_{03}\bar{u}_{02}+\alpha_{33}|u_{03}|^2]\big\}(\tau,k)\mathrm{d}\tau. \qquad (80)
\end{aligned}
$$

The functions $\{\phi_{ij}(t,k)\}_{i,j=1}^4$ *are of the same integral equations (77)–(80) by replacing the functions* $\{u_{0j},\, u_{1j}\}$ *with* $\{v_{0j},\, v_{1j}\}$, $(j=1,2,3)$, *i.e.* $\phi_{ij}(t,k)=\Phi_{ij}(t,k)\big|_{\{u_{0l}(t)=v_{0l}(t),\, u_{1l}(t)=v_{1l}(t)\}}$, $(i,j=1,2,3,4;\, l=1,2,3)$

(i) For the given Dirichlet problem, the unknown Neumann boundary data $\{u_{1j}(t)\}_{j=1}^{3}$ and $\{v_{1j}(t)\}_{j=1}^{3}$, $0 < t < T$ can be given by

$$\begin{aligned}u_{11}(t) =&\int_{\partial D_3^0}\left[\frac{2\Sigma_+}{i\pi\Sigma_-}(k\Psi_{14-}+iu_{01})+u_{01}(2\Psi_{44-}-\bar{\phi}_{44-})\right]\mathrm{d}k\\&+\int_{\partial D_3^0}\frac{4}{\pi\Sigma_-}\big[\alpha_{11}\left(-ik\bar{\phi}_{41-}+\alpha_{11}v_{01}+\alpha_{12}v_{02}+\alpha_{13}v_{03}\right)\\&+\bar{\alpha}_{12}\left(-ik\bar{\phi}_{42-}+\bar{\alpha}_{12}v_{01}+\alpha_{22}v_{02}+\alpha_{23}v_{03}\right)+\bar{\alpha}_{13}\left(-ik\bar{\phi}_{43-}+\bar{\alpha}_{13}v_{01}+\bar{\alpha}_{23}v_{02}+\alpha_{33}v_{03}\right)\big]\,dk\\&+\int_{\partial D_3^0}\frac{4k}{i\pi\Sigma_-}\Big[\Psi_{14}(\bar{\phi}_{44}-1)e^{-2ikL}-(\Psi_{11}-1)(\alpha_{11}\bar{\phi}_{41}+\bar{\alpha}_{12}\bar{\phi}_{42}+\bar{\alpha}_{13}\bar{\phi}_{43})\\&-\Psi_{12}(\alpha_{12}\bar{\phi}_{41}+\alpha_{22}\bar{\phi}_{42}+\bar{\alpha}_{23}\bar{\phi}_{43})-\Psi_{13}(\alpha_{13}\bar{\phi}_{41}+\alpha_{23}\bar{\phi}_{42}+\alpha_{33}\bar{\phi}_{43})\Big]_-\,\mathrm{d}k,\end{aligned}\tag{81}$$

$$\begin{aligned}u_{12}(t)=&\int_{\partial D_3^0}\left[\frac{2\Sigma_+}{i\pi\Sigma_-}(k\Psi_{24-}+iu_{02})+u_{02}(2\Psi_{44-}-\bar{\phi}_{44-})\right]\mathrm{d}k\\&+\int_{\partial D_3^0}\frac{4}{\pi\Sigma_-}\big[\alpha_{12}\left(-ik\bar{\phi}_{41-}+\alpha_{11}v_{01}+\alpha_{12}v_{02}+\alpha_{13}v_{03}\right)\\&+\alpha_{22}\left(-ik\bar{\phi}_{42-}+\bar{\alpha}_{12}v_{01}+\alpha_{22}v_{02}+\alpha_{23}v_{03}\right)+\bar{\alpha}_{23}\left(-ik\bar{\phi}_{43-}+\bar{\alpha}_{13}v_{01}+\bar{\alpha}_{23}v_{02}+\alpha_{33}v_{03}\right)\big]\,\mathrm{d}k\\&+\int_{\partial D_3^0}\frac{4k}{i\pi\Sigma_-}\Big[\Psi_{24}(\bar{\phi}_{44}-1)e^{-2ikL}-\Psi_{21}(\alpha_{11}\bar{\phi}_{41}+\bar{\alpha}_{12}\bar{\phi}_{42}+\bar{\alpha}_{13}\bar{\phi}_{43})\\&-(\Psi_{22}-1)(\alpha_{12}\bar{\phi}_{41}+\alpha_{22}\bar{\phi}_{42}+\bar{\alpha}_{23}\bar{\phi}_{43})-\Psi_{23}(\alpha_{13}\bar{\phi}_{41}+\alpha_{23}\bar{\phi}_{42}+\alpha_{33}\bar{\phi}_{43})\Big]_-\,\mathrm{d}k,\end{aligned}\tag{82}$$

$$\begin{aligned}u_{13}(t) =&\int_{\partial D_3^0}\left[\frac{2\Sigma_+}{i\pi\Sigma_-}(k\Psi_{34-}+iu_{03})+u_{03}(2\Psi_{44-}-\bar{\phi}_{44-})\right]\mathrm{d}k\\&+\int_{\partial D_3^0}\frac{4}{\pi\Sigma_-}\big[\alpha_{13}\left(-ik\bar{\phi}_{41-}+\alpha_{11}v_{01}+\alpha_{12}v_{02}+\alpha_{13}v_{03}\right)\\&+\alpha_{23}\left(-ik\bar{\phi}_{42-}+\bar{\alpha}_{12}v_{01}+\alpha_{22}v_{02}+\alpha_{23}v_{03}\right)+\alpha_{33}\left(-ik\bar{\phi}_{43-}+\bar{\alpha}_{13}v_{01}+\bar{\alpha}_{23}v_{02}+\alpha_{33}v_{03}\right)\big]\,\mathrm{d}k\\&+\int_{\partial D_3^0}\frac{4k}{i\pi\Sigma_-}\Big[\Psi_{34}(\bar{\phi}_{44}-1)e^{-2ikL}-\Psi_{31}(\alpha_{11}\bar{\phi}_{41}+\bar{\alpha}_{12}\bar{\phi}_{42}+\bar{\alpha}_{13}\bar{\phi}_{43})\\&-\Psi_{32}(\alpha_{12}\bar{\phi}_{41}+\alpha_{22}\bar{\phi}_{42}+\bar{\alpha}_{23}\bar{\phi}_{43})-(\Psi_{33}-1)(\alpha_{13}\bar{\phi}_{41}+\alpha_{23}\bar{\phi}_{42}+\alpha_{33}\bar{\phi}_{43})\Big]_-\,\mathrm{d}k,\end{aligned}\tag{83}$$

and

$$v_{1j}(t) = 4\phi_{j4}^{(2)}+\frac{2}{\pi}\int_{\partial D_3^0} v_{0j}\phi_{44-}\mathrm{d}k, \quad j=1,2,3, \tag{84}$$

where

$$\begin{pmatrix}\phi_{14}^{(2)}\\ \phi_{24}^{(2)}\\ \phi_{34}^{(2)}\end{pmatrix}=\frac{1}{4\pi}\int_{\partial D_3^0}\left[\frac{2i\Sigma_+}{\Sigma_-}\begin{pmatrix}k\phi_{14-}+iv_{01}\\ k\phi_{24-}+iv_{02}\\ k\phi_{34-}+iv_{03}\end{pmatrix}-\Psi_{44-}^{(1)}\begin{pmatrix}v_{01}\\ v_{02}\\ v_{03}\end{pmatrix}\right]\mathrm{d}k+\frac{\mathcal{M}^T}{2i\pi}\begin{pmatrix}I_1(t)\\ I_2(t)\\ I_3(t)\end{pmatrix},$$

with

$$\begin{aligned}I_1(t)=&\int_{\partial D_3^0}\Big\{\frac{2}{\Sigma_-}\big[(\alpha_{11}^2+|\alpha_{12}|^2+|\alpha_{13}|^2)(k\bar{\Psi}_{41-}+i(\bar{\alpha}_{11}u_{01}+\alpha_{12}u_{02}+\alpha_{13}u_{03}))\\ &+(\alpha_{11}\bar{\alpha}_{12}+\bar{\alpha}_{12}\alpha_{22}+\bar{\alpha}_{13}\alpha_{23})(k\bar{\Psi}_{42-}+i(\bar{\alpha}_{12}u_{01}+\alpha_{22}u_{02}+\alpha_{23}u_{03}))\\ &+(\alpha_{11}\bar{\alpha}_{13}+\bar{\alpha}_{12}\bar{\alpha}_{23}+\bar{\alpha}_{13}\alpha_{33})(k\bar{\Psi}_{43-}+i(\bar{\alpha}_{13}u_{01}+\bar{\alpha}_{23}u_{02}+\alpha_{33}u_{03}))\big]\Big\}\mathrm{d}k\\ &+\int_{\partial D_3^0}\frac{2k}{\Sigma_-}\Big\{(1-\bar{\Psi}_{44})(\alpha_{11}\phi_{14}+\bar{\alpha}_{12}\phi_{24}+\bar{\alpha}_{13}\phi_{34})e^{2ikL}\\ &+\bar{\Psi}_{41}\big[\alpha_{11}(\alpha_{11}(\phi_{11}-1)+\alpha_{12}\phi_{12}+\alpha_{13}\phi_{13})+\bar{\alpha}_{12}(\alpha_{11}\phi_{21}+\alpha_{12}(\phi_{22}-1)+\alpha_{13}\phi_{23})\\ &+\bar{\alpha}_{13}(\alpha_{11}\phi_{31}+\alpha_{12}\phi_{32}+\alpha_{13}(\phi_{33}-1))\big]+\bar{\Psi}_{42}\big[\alpha_{11}(\bar{\alpha}_{12}(\phi_{11}-1)+\alpha_{22}\phi_{12}+\alpha_{23}\phi_{13})\\ &+\bar{\alpha}_{12}(\bar{\alpha}_{12}\phi_{21}+\alpha_{22}(\phi_{22}-1)+\alpha_{23}\phi_{23})+\bar{\alpha}_{13}(\bar{\alpha}_{12}\phi_{31}+\alpha_{22}\phi_{32}+\alpha_{23}(\phi_{33}-1))\big]\\ &+\bar{\Psi}_{43}\big[\alpha_{11}(\bar{\alpha}_{13}(\phi_{11}-1)+\bar{\alpha}_{23}\phi_{12}+\alpha_{33}\phi_{13})+\bar{\alpha}_{12}(\bar{\alpha}_{13}\phi_{21}+\bar{\alpha}_{23}(\phi_{22}-1)+\alpha_{33}\phi_{23})\\ &+\bar{\alpha}_{13}(\bar{\alpha}_{13}\phi_{31}+\bar{\alpha}_{23}\phi_{32}+\alpha_{33}(\phi_{33}-1))\big]\Big\}_-\mathrm{d}k,\end{aligned}\tag{85}$$

$$\begin{aligned}I_2(t)=&\int_{\partial D_3^0}\Big\{\frac{2}{\Sigma_-}\big[(\alpha_{12}^2+\alpha_{22}^2+|\alpha_{23}|^2)(k\bar{\Psi}_{42-}+i(\bar{\alpha}_{12}u_{01}+\alpha_{22}u_{02}+\alpha_{23}u_{03}))\\ &+(\alpha_{11}\alpha_{12}+\bar{\alpha}_{23}\alpha_{22}+\bar{\alpha}_{33}\alpha_{33})(k\bar{\Psi}_{41-}+i(\bar{\alpha}_{11}u_{01}+\alpha_{12}u_{02}+\alpha_{13}u_{03}))\\ &+(\alpha_{12}\bar{\alpha}_{13}+\alpha_{22}\bar{\alpha}_{23}+\bar{\alpha}_{23}\alpha_{33})(k\bar{\Psi}_{43-}+i(\bar{\alpha}_{13}u_{01}+\bar{\alpha}_{23}u_{02}+\alpha_{33}u_{03}))\big]\Big\}\mathrm{d}k\\ &+\int_{\partial D_3^0}\frac{2k}{\Sigma_-}\Big\{(1-\bar{\Psi}_{44})(\alpha_{12}\phi_{14}+\alpha_{22}\phi_{24}+\bar{\alpha}_{23}\phi_{34})e^{2ikL}\\ &+\bar{\Psi}_{41}\big[\alpha_{12}(\alpha_{11}(\phi_{11}-1)+\alpha_{12}\phi_{12}+\alpha_{13}\phi_{13})+\alpha_{22}(\alpha_{11}\phi_{21}+\alpha_{12}(\phi_{22}-1)+\alpha_{13}\phi_{23})\\ &+\bar{\alpha}_{23}(\alpha_{11}\phi_{31}+\alpha_{12}\phi_{32}+\alpha_{13}(\phi_{33}-1))\big]+\bar{\Psi}_{42}\big[\alpha_{12}(\bar{\alpha}_{12}(\phi_{11}-1)+\alpha_{22}\phi_{12}+\alpha_{23}\phi_{13})\\ &+\alpha_{22}(\bar{\alpha}_{12}\phi_{21}+\alpha_{22}(\phi_{22}-1)+\alpha_{23}\phi_{23})+\bar{\alpha}_{23}(\bar{\alpha}_{12}\phi_{31}+\alpha_{22}\phi_{32}+\alpha_{23}(\phi_{33}-1))\big]\\ &+\bar{\Psi}_{43}\big[\alpha_{12}(\bar{\alpha}_{13}(\phi_{11}-1)+\bar{\alpha}_{23}\phi_{12}+\alpha_{33}\phi_{13})+\alpha_{22}(\bar{\alpha}_{13}\phi_{21}+\bar{\alpha}_{23}(\phi_{22}-1)+\alpha_{33}\phi_{23})\\ &+\bar{\alpha}_{23}(\bar{\alpha}_{13}\phi_{31}+\bar{\alpha}_{23}\phi_{32}+\alpha_{33}(\phi_{33}-1))\big]\Big\}_-\mathrm{d}k,\end{aligned}\tag{86}$$

$$
\begin{aligned}
I_3(t) = &\int_{\partial D_3^0} \Big\{ \frac{2}{\Sigma_-} \big[(\alpha_{12}^2 + \alpha_{22}^2 + |\alpha_{23}|^2)(k\bar{\Psi}_{43-} + i(\bar{\alpha}_{13}u_{01} + \bar{\alpha}_{23}u_{02} + \alpha_{33}u_{03})) \\
&+ (\alpha_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \bar{\alpha}_{13}\alpha_{23})(k\bar{\Psi}_{41-} + i(\bar{\alpha}_{11}u_{01} + \alpha_{12}u_{02} + \alpha_{13}u_{03})) \\
&+ (\alpha_{13}\bar{\alpha}_{12} + \alpha_{22}\alpha_{23} + \bar{\alpha}_{13}\alpha_{23})(k\bar{\Psi}_{42-} + i(\bar{\alpha}_{12}u_{01} + \alpha_{22}u_{02} + \alpha_{23}u_{03}))\big]\Big\} \mathrm{d}k \\
&+ \int_{\partial D_3^0} \frac{2k}{\Sigma_-} \Big\{ (1 - \bar{\Psi}_{44})(\alpha_{13}\phi_{14} + \alpha_{23}\phi_{24} + \alpha_{33}\phi_{34}) e^{2ikL} \\
&+ \bar{\Psi}_{41} \big[\alpha_{13}(\alpha_{11}(\phi_{11} - 1) + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \alpha_{23}(\alpha_{11}\phi_{21} + \alpha_{12}(\phi_{22} - 1) + \alpha_{13}\phi_{23}) \\
&+ \alpha_{33}(\alpha_{11}\phi_{31} + \alpha_{12}\phi_{32} + \alpha_{13}(\phi_{33} - 1))\big] + \bar{\Psi}_{42}\big[\alpha_{13}(\bar{\alpha}_{12}(\phi_{11} - 1) + \alpha_{22}\phi_{12} + \alpha_{23}\phi_{13}) \\
&+ \alpha_{23}(\bar{\alpha}_{12}\phi_{21} + \alpha_{22}(\phi_{22} - 1) + \alpha_{23}\phi_{23}) + \alpha_{33}(\bar{\alpha}_{12}\phi_{31} + \alpha_{22}\phi_{32} + \alpha_{23}(\phi_{33} - 1))\big] \\
&+ \bar{\Psi}_{43}\big[\alpha_{13}(\bar{\alpha}_{13}(\phi_{11} - 1) + \bar{\alpha}_{23}\phi_{12} + \alpha_{33}\phi_{13}) + \alpha_{23}(\bar{\alpha}_{13}\phi_{21} + \bar{\alpha}_{23}(\phi_{22} - 1) + \alpha_{33}\phi_{23}) \\
&+ \alpha_{33}(\bar{\alpha}_{13}\phi_{31} + \bar{\alpha}_{23}\phi_{32} + \alpha_{33}(\phi_{33} - 1))\big]\Big\}_- \mathrm{d}k.
\end{aligned} \tag{87}
$$

(ii) For the known Neumann problem, the unknown Dirichlet boundary data $\{u_{0j}(t)\}_{j=1}^3$ and $\{v_{0j}(t)\}_{j=1}^3$, $0 < t < T$ can be determined by

$$
\begin{aligned}
u_{01}(t) = &\int_{\partial D_3^0} \frac{1}{\pi \Sigma_-} \big[\Sigma_+ \Psi_{14+} - 2(\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43})_+ \big] \mathrm{d}k \\
&+ \int_{\partial D_3^0} \frac{2}{\pi \Sigma_-} \Big\{ \Psi_{14}(\bar{\phi}_{44} - 1)e^{-2ikL} - \big[(\Psi_{11} - 1)(\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43}) \\
&+ \Psi_{12}(\alpha_{12}\bar{\phi}_{41} + \alpha_{22}\bar{\phi}_{42} + \bar{\alpha}_{23}\bar{\phi}_{43}) + \Psi_{13}(\alpha_{13}\bar{\phi}_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43})\big]\Big\}_+ \mathrm{d}k,
\end{aligned} \tag{88}
$$

$$
\begin{aligned}
u_{02}(t) = &\int_{\partial D_3^0} \frac{1}{\pi \Sigma_-} \big[\Sigma_+ \Psi_{24+} - 2(\alpha_{12}\bar{\phi}_{41} + \alpha_{22}\bar{\phi}_{42} + \bar{\alpha}_{23}\bar{\phi}_{43})_+ \big] \mathrm{d}k \\
&+ \int_{\partial D_3^0} \frac{2}{\pi \Sigma_-} \Big\{ \Psi_{24}(\bar{\phi}_{44} - 1)e^{-2ikL} - \big[\Psi_{21}(\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43}) \\
&+ (\Psi_{22} - 1)(\alpha_{12}\bar{\phi}_{41} + \alpha_{22}\bar{\phi}_{42} + \bar{\alpha}_{23}\bar{\phi}_{43}) + \Psi_{23}(\alpha_{13}\bar{\phi}_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43})\big]\Big\}_+ \mathrm{d}k,
\end{aligned} \tag{89}
$$

$$
\begin{aligned}
u_{03}(t) = &\int_{\partial D_3^0} \frac{1}{\pi \Sigma_-} \big[\Sigma_+ \Psi_{34+} - 2(\alpha_{13}\bar{\phi}_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43})_+ \big] \mathrm{d}k \\
&+ \int_{\partial D_3^0} \frac{2}{\pi \Sigma_-} \Big\{ \Psi_{34}(\bar{\phi}_{44} - 1)e^{-2ikL} - \big[\Psi_{31}(\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43}) \\
&+ \Psi_{32}(\alpha_{12}\bar{\phi}_{41} + \alpha_{22}\bar{\phi}_{42} + \bar{\alpha}_{23}\bar{\phi}_{43}) + (\Psi_{33} - 1)(\alpha_{13}\bar{\phi}_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43})\big]\Big\}_+ \mathrm{d}k,
\end{aligned} \tag{90}
$$

and

$$v_{01}(t) = 2i\phi_{14}^{(1)}, \quad v_{02}(t) = 2i\phi_{24}^{(1)}, \quad v_{03}(t) = 2i\phi_{34}^{(1)}, \tag{91}$$

where

$$\begin{pmatrix} \phi_{14}^{(1)} \\ \phi_{24}^{(1)} \\ \phi_{34}^{(1)} \end{pmatrix} = -\frac{1}{2i\pi}\int_{\partial D_3^0} \frac{\Sigma_+}{\Sigma_-} \begin{pmatrix} \phi_{14+} \\ \phi_{24+} \\ \phi_{34+} \end{pmatrix} \mathrm{d}k + \frac{\mathcal{M}^T}{2i\pi} \begin{pmatrix} J_1(t) \\ J_2(t) \\ J_3(t) \end{pmatrix},$$

with

$$\begin{aligned} J_1(t) =& \int_{\partial D_3^0} \frac{2}{\Sigma_-}\big[(\alpha_{11}^2 + |\alpha_{12}|^2 + |\alpha_{13}|^2)\bar{\Psi}_{41+} + (\alpha_{11}\bar{\alpha}_{12} + \bar{\alpha}_{12}\alpha_{22} + \bar{\alpha}_{13}\alpha_{23})\bar{\Psi}_{42+} \\ & +(\alpha_{11}\bar{\alpha}_{13} + \bar{\alpha}_{12}\bar{\alpha}_{23} + \bar{\alpha}_{13}\alpha_{33})\bar{\Psi}_{43+}\big]\mathrm{d}k \\ & + \int_{\partial D_3^0} \frac{2}{\Sigma_-}\Big\{(1-\bar{\Psi}_{44})(\alpha_{11}\phi_{14} + \bar{\alpha}_{12}\phi_{24} + \bar{\alpha}_{13}\phi_{34})e^{2ikL} \\ & +\bar{\Psi}_{41}\big[\alpha_{11}(\alpha_{11}(\phi_{11}-1) + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \bar{\alpha}_{12}(\alpha_{11}\phi_{21} + \alpha_{12}(\phi_{22}-1) + \alpha_{13}\phi_{23}) \\ & +\bar{\alpha}_{13}(\alpha_{11}\phi_{31} + \alpha_{12}\phi_{32} + \alpha_{13}(\phi_{33}-1))\big] + \bar{\Psi}_{42}\big[\alpha_{11}(\bar{\alpha}_{12}(\phi_{11}-1) + \alpha_{22}\phi_{12} + \alpha_{23}\phi_{13}) \\ & +\bar{\alpha}_{12}(\bar{\alpha}_{12}\phi_{21} + \alpha_{22}(\phi_{22}-1) + \alpha_{23}\phi_{23}) + \bar{\alpha}_{13}(\bar{\alpha}_{12}\phi_{31} + \alpha_{22}\phi_{32} + \alpha_{23}(\phi_{33}-1))\big] \\ & +\bar{\Psi}_{43}\big[\alpha_{11}(\bar{\alpha}_{13}(\phi_{11}-1) + \bar{\alpha}_{23}\phi_{12} + \alpha_{33}\phi_{13}) + \bar{\alpha}_{12}(\bar{\alpha}_{13}\phi_{21} + \bar{\alpha}_{23}(\phi_{22}-1) + \alpha_{33}\phi_{23}) \\ & +\bar{\alpha}_{13}(\bar{\alpha}_{13}\phi_{31} + \bar{\alpha}_{23}\phi_{32} + \alpha_{33}(\phi_{33}-1))\big]\Big\}_+ \mathrm{d}k, \end{aligned} \tag{92}$$

$$\begin{aligned} J_2(t) =& \int_{\partial D_3^0} \frac{2}{\Sigma_-}\big[(\alpha_{12}^2 + \alpha_{22}^2 + |\alpha_{23}|^2)\bar{\Psi}_{42+} + (\alpha_{11}\alpha_{12} + \bar{\alpha}_{23}\alpha_{22} + \bar{\alpha}_{33}\alpha_{33})\bar{\Psi}_{41+} \\ & +(\alpha_{12}\bar{\alpha}_{13} + \alpha_{22}\bar{\alpha}_{23} + \bar{\alpha}_{23}\alpha_{33})\bar{\Psi}_{43+}\big]\mathrm{d}k \\ & + \int_{\partial D_3^0} \frac{2}{\Sigma_-}\Big\{(1-\bar{\Psi}_{44})(\alpha_{12}\phi_{14} + \alpha_{22}\phi_{24} + \bar{\alpha}_{23}\phi_{34})e^{2ikL} \\ & +\bar{\Psi}_{41}\big[\alpha_{12}(\alpha_{11}(\phi_{11}-1) + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \alpha_{22}(\alpha_{11}\phi_{21} + \alpha_{12}(\phi_{22}-1) + \alpha_{13}\phi_{23}) \\ & +\bar{\alpha}_{23}(\alpha_{11}\phi_{31} + \alpha_{12}\phi_{32} + \alpha_{13}(\phi_{33}-1))\big] + \bar{\Psi}_{42}\big[\alpha_{12}(\bar{\alpha}_{12}(\phi_{11}-1) + \alpha_{22}\phi_{12} + \alpha_{23}\phi_{13}) \\ & +\alpha_{22}(\bar{\alpha}_{12}\phi_{21} + \alpha_{22}(\phi_{22}-1) + \alpha_{23}\phi_{23}) + \bar{\alpha}_{23}(\bar{\alpha}_{12}\phi_{31} + \alpha_{22}\phi_{32} + \alpha_{23}(\phi_{33}-1))\big] \\ & +\bar{\Psi}_{43}\big[\alpha_{12}(\bar{\alpha}_{13}(\phi_{11}-1) + \bar{\alpha}_{23}\phi_{12} + \alpha_{33}\phi_{13}) + \alpha_{22}(\bar{\alpha}_{13}\phi_{21} + \bar{\alpha}_{23}(\phi_{22}-1) + \alpha_{33}\phi_{23}) \\ & +\bar{\alpha}_{23}(\bar{\alpha}_{13}\phi_{31} + \bar{\alpha}_{23}\phi_{32} + \alpha_{33}(\phi_{33}-1))\big]\Big\}_+ \mathrm{d}k, \end{aligned} \tag{93}$$

$$
\begin{aligned}
J_3(t) = & \int_{\partial D_3^0} \frac{2}{\Sigma_-}\big[(\alpha_{12}^2 + \alpha_{22}^2 + |\alpha_{23}|^2)\bar{\Psi}_{43+} + (\alpha_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \bar{\alpha}_{13}\alpha_{23})\bar{\Psi}_{41+} \\
& +(\alpha_{13}\bar{\alpha}_{12} + \alpha_{22}\alpha_{23} + \bar{\alpha}_{13}\alpha_{23})\bar{\Psi}_{42+}\big]\mathrm{d}k \\
& + \int_{\partial D_3^0} \frac{2}{\Sigma_-}\Big\{(1-\bar{\Psi}_{44})(\alpha_{13}\phi_{14} + \alpha_{23}\phi_{24} + \alpha_{33}\phi_{34})e^{2ikL} \\
& +\bar{\Psi}_{41}\big[\alpha_{13}(\alpha_{11}(\phi_{11}-1) + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \alpha_{23}(\alpha_{11}\phi_{21} + \alpha_{12}(\phi_{22}-1) + \alpha_{13}\phi_{23}) \\
& +\alpha_{33}(\alpha_{11}\phi_{31} + \alpha_{12}\phi_{32} + \alpha_{13}(\phi_{33}-1))\big] + \bar{\Psi}_{42}\big[\alpha_{13}(\bar{\alpha}_{12}(\phi_{11}-1) + \alpha_{22}\phi_{12} + \alpha_{23}\phi_{13}) \\
& +\alpha_{23}(\bar{\alpha}_{12}\phi_{21} + \alpha_{22}(\phi_{22}-1) + \alpha_{23}\phi_{23}) + \alpha_{33}(\bar{\alpha}_{12}\phi_{31} + \alpha_{22}\phi_{32} + \alpha_{23}(\phi_{33}-1))\big] \\
& +\bar{\Psi}_{43}\big[\alpha_{13}(\bar{\alpha}_{13}(\phi_{11}-1) + \bar{\alpha}_{23}\phi_{12} + \alpha_{33}\phi_{13}) + \alpha_{23}(\bar{\alpha}_{13}\phi_{21} + \bar{\alpha}_{23}(\phi_{22}-1) + \alpha_{33}\phi_{23}) \\
& +\alpha_{33}(\bar{\alpha}_{13}\phi_{31} + \bar{\alpha}_{23}\phi_{32} + \alpha_{33}(\phi_{33}-1))\big]\Big\}_+ \mathrm{d}k,
\end{aligned} \tag{94}
$$

where $\Psi_{14} = \Psi_{14}(t,k)$, $\bar{\phi}_{44} = \overline{\phi_{44}(t,\bar{k})} = \bar{\phi}_{44}(t,\bar{k})$ and other functions have the similar expressions.

Proof. The proof of the Proposition is given in **Appendix B**. □

REMARK 4.3 It follows from Proposition 4.2 that the well-defined boundary value data of the integrable gtc-NLS system (4) are given, i.e. the generalized Dirichlet–Neumann map given by Eqs. (81)–(87) and the generalized Neumann–Dirichlet map given by Eqs. (88)–(94) are established, respectively. These obtained results can also provide one with more exact data to theoretically and numerically study the solutions of the IBV problem of system (4).

4.3 *The effective characterizations*

Substituting the perturbated expressions for eigenfunctions and initial boundary conditions

$$
\begin{cases}
\Psi_{ij}(t,k) = \Psi_{ij}^{[0]}(t,k) + \epsilon\Psi_{ij}^{[1]}(t,k) + \epsilon^2\Psi_{ij}^{[2]}(t,k) + \cdots, & i,j = 1,2,3,4, \\
\phi_{ij}(t,k) = \phi_{ij}^{[0]}(t,k) + \epsilon\phi_{ij}^{[1]}(t,k) + \epsilon^2\phi_{ij}^{[2]}(t,k) + \cdots, & i,j = 1,2,3,4, \\
u_{sj}(t) = \epsilon u_{sj}^{[1]}(t) + \epsilon^2 u_{sj}^{[2]}(t) + \cdots, & s = 0,1; j = 1,2,3, \\
v_{sj}(t) = \epsilon v_{sj}^{[1]}(t) + \epsilon^2 v_{sj}^{[2]}(t) + \cdots, & s = 0,1; j = 1,2,3,
\end{cases} \tag{95}
$$

into Eqs. (150)–(153), where $\epsilon > 0$ is a small parameter, we have these terms of $O(1)$ and $O(\epsilon)$ as

$$
O(1): \begin{cases}
\Psi_{jj}^{[0]} = 1, & j = 1,2,3,4, \\
\Psi_{ij}^{[0]} = 0, & i,j = 1,2,3,4,\ i \neq j,
\end{cases} \tag{96}
$$

$$O(\epsilon):\begin{cases}\Psi_{js}^{[1]}=\Psi_{44}^{[1]}=0,\quad j,s,=1,2,3,\\ \Psi_{j4}^{[1]}=\int_0^t e^{-4ik^2(t-\tau)}\left(2ku_{0j}^{[1]}+iu_{1j}^{[1]}\right)(\tau)\mathrm{d}\tau,\quad j=1,2,3,\\ \Psi_{41}^{[1]}=\int_0^t e^{4ik^2(t-\tau)}\left[\alpha_{11}\left(2k\bar{u}_{01}^{[1]}-i\bar{u}_{11}^{[1]}\right)+\bar{\alpha}_{12}\left(2k\bar{u}_{02}^{[1]}-i\bar{u}_{12}^{[1]}\right)+\bar{\alpha}_{13}\left(2k\bar{u}_{03}^{[1]}-i\bar{u}_{13}^{[1]}\right)\right](\tau)\mathrm{d}\tau,\\ \Psi_{42}^{[1]}=\int_0^t e^{4ik^2(t-\tau)}\left[\alpha_{12}\left(2k\bar{u}_{01}^{[1]}-i\bar{u}_{11}^{[1]}\right)+\alpha_{22}\left(2k\bar{u}_{02}^{[1]}-i\bar{u}_{12}^{[1]}\right)+\bar{\alpha}_{23}\left(2k\bar{u}_{03}^{[1]}-i\bar{u}_{13}^{[1]}\right)\right](\tau)\mathrm{d}\tau,\\ \Psi_{43}^{[1]}=\int_0^t e^{4ik^2(t-\tau)}\left[\alpha_{13}\left(2k\bar{u}_{01}^{[1]}-i\bar{u}_{11}^{[1]}\right)+\alpha_{23}\left(2k\bar{u}_{02}^{[1]}-i\bar{u}_{12}^{[1]}\right)+\alpha_{33}\left(2k\bar{u}_{03}^{[1]}-i\bar{u}_{13}^{[1]}\right)\right](\tau)\mathrm{d}\tau.\end{cases}\tag{97}$$

Similarly, we can also obtain the analogous expressions for $\{\phi_{ij}^{[l]}\}_{i,j=1}^4, l=0,1$ by means of the boundary values at $x=L$, i.e. $\{v_{ij}^{[l]}\}, i=0,1; j=1,2,3; l=0,1$.

If we assume that $m_{44}(\mathbb{S})$ has no zero, then we expand Eqs. (81)–(84) to have

$$\begin{aligned}u_{11}^{[n]}(t)=&\int_{\partial D_3^0}\left\{\frac{2\Sigma_+}{i\pi\Sigma_-}(k\Psi_{14-}^{[n]}+iu_{01}^{[n]})dk+\frac{4}{\pi\Sigma_-}\left[\alpha_{11}\Big(-ik\bar{\phi}_{41-}^{[n]}+\alpha_{11}v_{01}^{[n]}+\alpha_{12}v_{02}^{[n]}+\alpha_{13}v_{03}^{[n]}\Big)\right.\right.\\&+\bar{\alpha}_{12}\left(-ik\bar{\phi}_{42-}^{[n]}+\bar{\alpha}_{12}v_{01}^{[n]}+\alpha_{22}v_{02}^{[n]}+\alpha_{23}v_{03}^{[n]}\right)\\&\left.\left.+\bar{\alpha}_{13}\left(-ik\bar{\phi}_{43-}^{[n]}+\bar{\alpha}_{13}v_{01}^{[n]}+\bar{\alpha}_{23}v_{02}^{[n]}+\alpha_{33}v_{03}^{[n]}\right)\right]\right\}\mathrm{d}k+\mathrm{LOTs},\end{aligned}\tag{98}$$

$$\begin{aligned}u_{12}^{[n]}(t)=&\int_{\partial D_3^0}\left\{\frac{2\Sigma_+}{i\pi\Sigma_-}(k\Psi_{24-}^{[n]}+iu_{02}^{[n]})dk+\frac{4}{\pi\Sigma_-}\left[\alpha_{12}\Big(-ik\bar{\phi}_{41-}^{[n]}+\alpha_{11}v_{01}^{[n]}+\alpha_{12}v_{02}^{[n]}+\alpha_{13}v_{03}^{[n]}\Big)\right.\right.\\&+\alpha_{22}\left(-ik\bar{\phi}_{42-}^{[n]}+\bar{\alpha}_{12}v_{01}^{[n]}+\alpha_{22}v_{02}^{[n]}+\alpha_{23}v_{03}^{[n]}\right)\\&\left.\left.+\bar{\alpha}_{23}\left(-ik\bar{\phi}_{43-}^{[n]}+\bar{\alpha}_{13}v_{01}^{[n]}+\bar{\alpha}_{23}v_{02}^{[n]}+\alpha_{33}v_{03}^{[n]}\right)\right]\right\}\mathrm{d}k+\mathrm{LOTs},\end{aligned}\tag{99}$$

$$\begin{aligned}u_{13}^{[n]}(t)=&\int_{\partial D_3^0}\left\{\frac{2\Sigma_+}{i\pi\Sigma_-}(k\Psi_{34-}^{[n]}+iu_{03}^{[n]})dk+\frac{4}{\pi\Sigma_-}\left[\alpha_{13}\Big(-ik\bar{\phi}_{41-}^{[n]}+\alpha_{11}v_{01}^{[n]}+\alpha_{12}v_{02}^{[n]}+\alpha_{13}v_{03}^{[n]}\Big)\right.\right.\\&+\alpha_{23}\left(-ik\bar{\phi}_{42-}^{[n]}+\bar{\alpha}_{12}v_{01}^{[n]}+\alpha_{22}v_{02}^{[n]}+\alpha_{23}v_{03}^{[n]}\right)\\&\left.\left.+\alpha_{33}\left(-ik\bar{\phi}_{43-}^{[1]}+\bar{\alpha}_{13}v_{01}^{[n]}+\bar{\alpha}_{23}v_{02}^{[n]}+\alpha_{33}v_{03}^{[n]}\right)\right]\right\}\mathrm{d}k+\mathrm{LOTs},\end{aligned}\tag{100}$$

where the word 'LOTs' means 'lower order terms' standing for the result involving known terms of lower order.

The terms of $O(\epsilon^n)$ in Eqs. (77)–(80) and the similar equations for ϕ_{ij} yield

$$
\begin{aligned}
\Psi_{j4}^{[n]}(t,k) &= \int_0^t e^{-4ik^2(t-\tau)}\left(2ku_{0j}^{[n]} + iu_{1j}^{[n]}\right)(\tau)\mathrm{d}\tau + \text{LOTs}, \quad j = 1,2,3,\\
\bar{\phi}_{41}^{[n]}(t,\bar{k}) &= \int_0^t e^{-4ik^2(t-\tau)}\Big[\alpha_{11}(2kv_{01}^{[n]} + iv_{11}^{[n]}) + \alpha_{12}(2kv_{02}^{[n]} + iv_{12}^{[n]}) + \alpha_{13}(2kv_{03}^{[n]} + iv_{13}^{[n]})\Big](\tau)\mathrm{d}\tau + \text{LOTs},\\
\bar{\phi}_{42}^{[n]}(t,\bar{k}) &= \int_0^t e^{-4ik^2(t-\tau)}\Big[\bar{\alpha}_{12}(2kv_{01}^{[n]} + iv_{11}^{[n]}) + \alpha_{22}(2kv_{02}^{[n]} + iv_{12}^{[n]}) + \alpha_{23}(2kv_{03}^{[n]} + iv_{13}^{[n]})\Big](\tau)\mathrm{d}\tau + \text{LOTs},\\
\bar{\phi}_{43}^{[n]}(t,\bar{k}) &= \int_0^t e^{-4ik^2(t-\tau)}\Big[\bar{\alpha}_{13}(2kv_{01}^{[n]} + iv_{11}^{[n]}) + \bar{\alpha}_{23}(2kv_{02}^{[n]} + iv_{12}^{[n]}) + \alpha_{33}(2kv_{03}^{[n]} + iv_{13}^{[n]})\Big](\tau)\mathrm{d}\tau + \text{LOTs},
\end{aligned}
\tag{101}
$$

which can lead to

$$
\begin{aligned}
\Psi_{j4-}^{[n]}(t,k) &= 4k\int_0^t e^{-4ik^2(t-\tau)}u_{0j}^{[n]}(\tau)\mathrm{d}\tau + \text{LOTs}, \quad j = 1,2,3,\\
\bar{\phi}_{41-}^{[n]}(t,\bar{k}) &= 4k\int_0^t e^{-4ik^2(t-\tau)}\left(\alpha_{11}v_{01}^{[n]} + \alpha_{12}v_{02}^{[n]} + \alpha_{13}v_{03}^{[n]}\right)(\tau)\mathrm{d}\tau + \text{LOTs},\\
\bar{\phi}_{42-}^{[n]}(t,\bar{k}) &= 4k\int_0^t e^{-4ik^2(t-\tau)}\left(\bar{\alpha}_{12}v_{01}^{[n]} + \alpha_{22}v_{02}^{[n]} + \alpha_{23}v_{03}^{[n]}\right)(\tau)\mathrm{d}\tau + \text{LOTs},\\
\bar{\phi}_{43-}^{[n]}(t,\bar{k}) &= 4k\int_0^t e^{-4ik^2(t-\tau)}\left(\bar{\alpha}_{13}v_{01}^{[n]} + \bar{\alpha}_{23}v_{02}^{[n]} + \alpha_{33}v_{03}^{[n]}\right)(\tau)\mathrm{d}\tau + \text{LOTs}.
\end{aligned}
\tag{102}
$$

It follows from system (102) that $\Psi_{1j-}^{[n]}$ and $\phi_{4j-}^{[n]}$, $j = 1,2,3$ can be generated at each step from the known Dirichlet boundary data $u_{0j}^{[n]}$ and $v_{0j}^{[n]}$ such that we know that the Neumann boundary data $u_{1j}^{[n]}$ can be given by Eqs. (98)–(100). Similarly, we also show that the Neumann boundary data $v_{1j}^{[n]}$ can then be determined by the known Dirichlet boundary data $u_{0j}^{[n]}$ and $v_{0j}^{[n]}$.

Similarly, the substitution of Eq. (95) into Eqs. (88) and (89) yields the terms of $O(\epsilon^n)$ as

$$
u_{01}^{[n]}(t) = \int_{\partial D_3^0}\left[\frac{\Sigma_+}{\pi\,\Sigma_-}\Psi_{14+}^{[n]} - \frac{2}{\pi\,\Sigma_-}\left(\alpha_{11}\bar{\phi}_{41}^{[n]} + \bar{\alpha}_{12}\bar{\phi}_{42}^{[n]} + \bar{\alpha}_{13}\bar{\phi}_{43}^{[n]}\right)\right]\mathrm{d}k + \text{LOTs}, \tag{103a}
$$

$$
u_{02}^{[n]}(t) = \int_{\partial D_3^0}\left[\frac{\Sigma_+}{\pi\,\Sigma_-}\Psi_{24+}^{[n]} - \frac{2}{\pi\,\Sigma_-}\left(\alpha_{12}\bar{\phi}_{41}^{[n]} + \alpha_{22}\bar{\phi}_{42}^{[n]} + \bar{\alpha}_{23}\bar{\phi}_{43}^{[n]}\right)\right]\mathrm{d}k + \text{LOTs}, \tag{103b}
$$

$$
u_{03}^{[n]}(t) = \int_{\partial D_3^0}\left[\frac{\Sigma_+}{\pi\,\Sigma_-}\Psi_{34+}^{[n]} - \frac{2}{\pi\,\Sigma_-}\left(\alpha_{13}\bar{\phi}_{41}^{[n]} + \alpha_{23}\bar{\phi}_{42}^{[n]} + \alpha_{33}\bar{\phi}_{43}^{[n]}\right)\right]\mathrm{d}k + \text{LOTs}, \tag{103c}
$$

Eq. (101) implies that

$$
\begin{aligned}
\Psi_{j4+}^{[n]}(t,k) &= 2i\int_0^t e^{-4ik^2(t-\tau)}u_{1j}^{[n]}(\tau)\mathrm{d}\tau + \mathrm{LOTs}, \quad j=1,2,3,\\
\bar{\phi}_{41+}^{[n]}(t,\bar{k}) &= \int_0^t e^{-4ik^2(t-\tau)}\left(\alpha_{11}v_{11}^{[n]}+\alpha_{12}v_{12}^{[n]}+\alpha_{13}v_{13}^{[n]}\right)(\tau)\mathrm{d}\tau + \mathrm{LOTs},\\
\bar{\phi}_{42+}^{[n]}(t,\bar{k}) &= \int_0^t e^{-4ik^2(t-\tau)}\left(\bar{\alpha}_{12}v_{11}^{[n]}+\alpha_{22}v_{12}^{[n]}+\alpha_{23}v_{13}^{[n]}\right)(\tau)\mathrm{d}\tau + \mathrm{LOTs},\\
\bar{\phi}_{43+}^{[n]}(t,\bar{k}) &= \int_0^t e^{-4ik^2(t-\tau)}\left(\bar{\alpha}_{13}v_{11}^{[n]})+\bar{\alpha}_{23}v_{12}^{[n]}+\alpha_{33}v_{13}^{[n]}\right)(\tau)\mathrm{d}\tau + \mathrm{LOTs}.
\end{aligned}
\tag{104}
$$

It follows from system (104) that $\Psi_{j4+}^{[n]}$ and $\phi_{4j+}^{[n]}$, $j=1,2,3$ can be generated at each step from the known Neumann boundary data $u_{1j}^{[n]}$ and $v_{1j}^{[n]}$ such that we know that the Dirichlet boundary data $u_{0j}^{[n]}$ can then be given by Eqs. (103a)–(103c). Similarly, we also show that the Dirichlet boundary data $v_{0j}^{[n]}$ can then be determined by the known Neumann boundary data $u_{1j}^{[n]}$ and $v_{1j}^{[n]}$.

4.4 *The large L limit from the finite interval to the half-line*

The formulae for the initial and boundary value conditions $u_{0j}(t)$ and $u_{1j}(t)$, $j=1,2,2$ of Proposition 4.2 in the limit $L\to\infty$ can be extended to the corresponding ones on the half-line. Since when $L\to\infty$,

$$
v_{0j}\to 0,\quad v_{1j}\to 0,\quad j=1,2,3,\quad \phi_{ij}\to\delta_{ij},\quad \frac{\Sigma_+(k)}{\Sigma_-(k)}\to 1 \quad \text{as} \quad k\to\infty \quad \text{in} \quad D_3, \tag{105}
$$

Thus, according to Eq. (105), the $L\to\infty$ limits of Eqs. (81), (82), (88) and (89) yield the unknown Neumann boundary data

$$
u_{1j}(t) = \frac{2}{\pi}\int_{\partial D_3^0}\left[u_{0j}(\Psi_{44-}+1)-ik\Psi_{j4-}\right]\mathrm{d}k,\quad j=1,2,3, \tag{106}
$$

for the given Dirichlet boundary problem, and the unknown Dirichlet boundary data

$$
u_{0j}(t) = \frac{1}{\pi}\int_{\partial D_3^0}\Psi_{j4+}\mathrm{d}k,\quad j=1,2,3, \tag{107}
$$

for the given Neumann boundary problem.

5. The GLM representation and equivalence

In this section, we deduce the eigenfunctions $\Psi(t,k)$ and $\phi(t,k)$ in terms of the GLM approach (Boutet de Monvel & Kotlyarov, 2000; Boutet de Monvel *et al.*, 2003b; Fokas, 2005; Treharne & Fokas, 2008). Moreover, the global relation can be used to find the unknown Neumann (Dirichlet) boundary values from the given Dirichlet (Neumann) boundary values by means of the GLM representations.

Moreover, the GLM representations are shown to be equivalent to the ones obtained in Sec. 4. Finally, the linearizable boundary conditions are also presented for the GLM representations.

5.1 *The GLM representation*

PROPOSITION 5.1 *The eigenfunctions $\Psi(t,k)$ and $\phi(t,k)$ possess the GLM representation*

$$\Psi(t,k) = \mathbb{I}_4 + \int_{-t}^{t}\left[L(t,s) + \left(k + \frac{i}{2}U^{(0)}\sigma_4\right)G(t,s)\right]e^{-2ik^2(s-t)\sigma_4}\mathrm{d}s, \tag{108a}$$

$$\phi(t,k) = \mathbb{I}_4 + \int_{-t}^{t}\left[\mathcal{L}(t,s) + \left(k + \frac{i}{2}\mathcal{U}^{(L)}\sigma_4\right)\mathcal{G}(t,s)\right]e^{-2ik^2(s-t)\sigma_4}\mathrm{d}s, \tag{108b}$$

where the 4×4 matrix-valued functions $L(t,s) = (L_{ij})_{4\times 4}$ and $G(t,s) = (G_{ij})_{4\times 4}$, $-t \le s \le t$ satisfy a Goursat system

$$\begin{aligned} &L_t(t,s)+\sigma_4 L_s(t,s)\sigma_4 = i\sigma_4 U_x^{(0)}L(t,s) - \tfrac{1}{2}\left[(U^{(0)})^3 + i\dot{U}^{(0)}\sigma_4 + [U_x^{(0)}, U^{(0)}]\right]G(t,s),\\ &G_t(t,s) + \sigma_4 G_s(t,s)\sigma_4 = 2U^{(0)}L(t,s) + i\sigma_4 U_x^{(0)}G(t,s), \end{aligned} \tag{109}$$

with the initial conditions

$$\begin{aligned} &L_{lj}(t,-t) = L_{44}(t,-t) = G_{lj}(t,-t) = G_{44}(t,-t) = 0, \quad G_{j4}(t,t) = u_{0j}(t), \quad l,j = 1,2,3,\\ &G_{41}(t,t) = \alpha_{11}\bar{u}_{01}(t) + \bar{\alpha}_{12}\bar{u}_{02}(t) + \bar{\alpha}_{13}\bar{u}_{03}(t), \quad G_{42}(t,t) = \alpha_{12}\bar{u}_{01}(t) + \alpha_{22}\bar{u}_{02}(t) + \bar{\alpha}_{23}\bar{u}_{03}(t),\\ &G_{43}(t,t) = \alpha_{13}\bar{u}_{01}(t) + \alpha_{23}\bar{u}_{02}(t) + \alpha_{33}\bar{u}_{03}(t),\\ &L_{j4}(t,t) = \frac{i}{2}u_{1j}(t), \quad l,j = 1,2,3, \quad L_{41}(t,t) = -\frac{i}{2}(\alpha_{11}\bar{u}_{11}(t) + \bar{\alpha}_{12}\bar{u}_{12}(t) + \bar{\alpha}_{13}\bar{u}_{13}(t)),\\ &L_{42}(t,t) = -\frac{i}{2}(\alpha_{12}\bar{u}_{11}(t)+\alpha_{22}\bar{u}_{12}(t)+\bar{\alpha}_{23}\bar{u}_{13}(t)), \quad L_{43}(t,t) = -\frac{i}{2}(\alpha_{13}\bar{u}_{11}(t)+\alpha_{23}\bar{u}_{12}(t)+\alpha_{33}\bar{u}_{13}(t)), \end{aligned} \tag{110}$$

$$U^{(0)} = \begin{pmatrix} 0 & 0 & 0 & u_{01}(t)\\ 0 & 0 & 0 & u_{02}(t)\\ 0 & 0 & 0 & u_{03}(t)\\ p_{01}(t) & p_{02}(t) & p_{03}(t) & 0 \end{pmatrix}, \qquad U_x^{(0)} = U^{(0)}\big|_{\{p_{0j}(t)\to p_{1j}(t),\, u_{0j}(t)\to u_{1j}(t), j=1,2,3\}} \tag{111}$$

with

$$p_{01} = \alpha_{11}\bar{u}_{01} + \bar{\alpha}_{12}\bar{u}_{02} + \bar{\alpha}_{13}\bar{u}_{03},\ p_{02} = \alpha_{12}\bar{u}_{01} + \alpha_{22}\bar{u}_{02} + \bar{\alpha}_{23}\bar{u}_{03},\ p_{03} = \alpha_{13}\bar{u}_{01} + \alpha_{23}\bar{u}_{02} + \alpha_{33}\bar{u}_{03},$$
$$p_{11} = \alpha_{11}\bar{u}_{11} + \bar{\alpha}_{12}\bar{u}_{12} + \bar{\alpha}_{13}\bar{u}_{13},\ p_{12} = \alpha_{12}\bar{u}_{11} + \alpha_{22}\bar{u}_{12} + \bar{\alpha}_{23}\bar{u}_{13},\ p_{13} = \alpha_{13}\bar{u}_{11} + \alpha_{23}\bar{u}_{12} + \alpha_{33}\bar{u}_{13},$$

Similarly, $\mathcal{L}(t,s)$, $\mathcal{G}(t,s)$ satisfy the similar Eqs. (109) and (110) with $u_{0j}\to v_{0j}$, $u_{1j}\to v_{1j}$, $U^{(0)}\to \mathcal{U}^{(L)} = U^{(0)}\big|_{u_{0j}\to v_{0j}}$, $U_x^{(0)} \to \mathcal{U}_x^{(L)} = U_x^{(0)}\big|_{u_{1j}\to v_{1j}}$.

Proof. We assume that the function

$$\psi(t,k) = e^{-2ik^2t\sigma_4} + \int_{-t}^{t}[L_0(t,s) + kG(t,s)]e^{-2ik^2s\sigma_4}\mathrm{d}s, \tag{112}$$

satisfies the time-part of the Lax pair (7) with the boundary data $\psi(0,k) = \mathbb{I}_4$ at $x = 0$, where $L_0(t,s)$ and $G(t,s)$ are the unknown 4×4 matrix-valued functions. We substitute Eq. (112) into the time-part of Lax pair (7) with the boundary data (6) and use the identity

$$\int_{-t}^{t}F(t,s)e^{-2ik^2s\sigma_4}\mathrm{d}s = \frac{i}{2k^2}\left[F(t,t)e^{-2ik^2t\sigma_4} - F(t,-t)e^{2ik^2t\sigma_4} - \int_{-t}^{t}F_s(t,s)e^{-2ik^2s\sigma_4}\mathrm{d}s\right]\sigma_4, \tag{113}$$

where the function $F(t,s)$ is a 4×4 matrix-valued function. As a consequence, we find

$$\begin{aligned}
&L_0(t,-t) + \sigma_4L_0(t,-t)\sigma_4 = -iU^{(0)}G(t,-t)\sigma_4, \quad G(t,-t) + \sigma_4G(t,-t)\sigma_4 = 0,\\
&L_0(t,t) - \sigma_4L_0(t,t)\sigma_4 = iU^{(0)}G(t,t)\sigma_4 + V_0^{(0)}, \quad G(t,t) - \sigma_4G(t,t)\sigma_4 = 2U^{(0)},\\
&L_{0t}(t,s) + \sigma_4L_{0s}(t,s)\sigma_4 = -iU^{(0)}G_s(t,s)\sigma_4 + V_0^{(0)}L_0(t,s),\\
&G_t(t,s) + \sigma_4G_s(t,s)\sigma_4 = 2U^{(0)}L_0(t,s) + V_0^{(0)}G(t,s),
\end{aligned} \tag{114}$$

where $U^{(0)}$ is given by Eq. (111) and

$$V_0^{(0)} = -i(U_x^{(0)} + U^{(0)2})\sigma_4 = -i\begin{pmatrix} u_{01}p_{01} & u_{01}p_{02} & u_{01}p_{03} & -u_{11}\\ u_{02}p_{01} & u_{02}p_{02} & u_{02}p_{03} & -u_{21}\\ u_{03}p_{01} & u_{03}p_{02} & u_{03}p_{03} & -u_{31}\\ p_{11} & p_{21} & p_{31} & -(u_{01}p_{01}+u_{02}p_{02}+u_{03}p_{03})\end{pmatrix}.$$

To reduce system (114) we further introduce the new matrix $L(t,s)$ as

$$L(t,s) = L_0(t,s) - \frac{i}{2}U^{(0)}\sigma_4G(t,s), \tag{115}$$

such that the first four equations of system (114) become

$$\begin{aligned}
&L(t,-t) + \sigma_4L(t,-t)\sigma_4 = 0, \quad G(t,-t) + \sigma_4G(t,-t)\sigma_4 = 0,\\
&L(t,t) - \sigma_4L(t,t)\sigma_4 = V_0^{(0)}, \quad G(t,t) - \sigma_4G(t,t)\sigma_4 = 2U^{(0)},
\end{aligned}$$

which lead to Eq. (110), and from the last two equations of system (114) we have Eq. (109). By means of transformation (10), i.e. $\mu_2(0,t,k) = \Psi(t,k) = \psi(t,k)e^{2ik^2t\sigma_4}$, we know that $\Psi(t,k)$ is given by Eq. (108a). Similarly, we can also show that Eq. (108b) holds. □

For convenience, we rewrite a 4×4 matrix $C = (C_{ij})_{4\times 4}$ as

$$C = (C_{ij})_{4\times 4} = \begin{pmatrix}\tilde{C}_{3\times 3} & \tilde{C}_{j4}\\ \tilde{C}_{4j} & C_{44}\end{pmatrix},\ \tilde{C}_{3\times 3} = (C_{ij})_{3\times 3},\ \tilde{C}_{j4} = (C_{14}, C_{24}, C_{34})^T,\ \tilde{C}_{4j} = (C_{41}, C_{42}, C_{43}).$$

The Dirichlet and Neumann boundary values at $x = 0, L$ are simplified as

$$
\begin{aligned}
&u_j(t) = (u_{j1}(t), u_{j2}(t), u_{j3}(t)), \quad v_j(t) = (v_{j1}(t), v_{j2}(t), v_{j3}(t)), \ j = 1,2,3, \\
&w_{j0}(t) = (p_{j1}(t), p_{j2}(t), p_{j3}(t)), \quad w_{jL}(t) = (p_{j1}(t), p_{j2}(t), p_{j3}(t))|_{u_{sj}\to v_{sj}}, \ s = 0,1; j = 1,2,3.
\end{aligned} \tag{116}
$$

For a matrix-valued function $F(t,s)$, we introduce the $\hat{F}(t,k)$ by

$$\hat{F}(t,k) = \int_{-t}^{t} F(t,s) e^{2ik^2(s-t)} \mathrm{d}s. \tag{117}$$

Thus, the GLM expressions (108a) and (108b) of $\{\Psi_{ij}, \phi_{ij}\}$ can be simplified as

$$
\begin{aligned}
&\tilde{\Psi}_{3\times 3}(t,k) = \mathbb{I} + \hat{\tilde{L}}_{3\times 3} - \frac{i}{2} u_0^T(t) \hat{\tilde{G}}_{4j} + k\hat{\tilde{G}}_{3\times 3}, \quad \tilde{\Psi}_{44}(t,k) = 1 + \hat{\tilde{L}}_{44} + \frac{i}{2} \bar{u}_0(t) \mathcal{M} \hat{\tilde{G}}_{j4} + k\hat{\tilde{G}}_{44}, \\
&\tilde{\Psi}_{j4}(t,k) = \hat{\tilde{L}}_{j4} - \frac{i}{2} u_0^T(t) \hat{\tilde{G}}_{44} + k\hat{\tilde{G}}_{j4}, \quad \tilde{\Psi}_{4j}(t,k) = \hat{\tilde{L}}_{4j} + \frac{i}{2} \bar{u}_0(t) \mathcal{M} \hat{\tilde{G}}_{3\times 3} + k\hat{\tilde{G}}_{4j}, \quad j = 1,2,3,
\end{aligned} \tag{118a}
$$

$$
\begin{aligned}
&\tilde{\phi}_{3\times 3}(t,k) = \mathbb{I} + \hat{\tilde{\mathcal{L}}}_{3\times 3} - \frac{i}{2} v_0^T(t) \hat{\tilde{\mathcal{G}}}_{4j} + k\hat{\tilde{\mathcal{G}}}_{3\times 3}, \quad \tilde{\phi}_{44}(t,k) = 1 + \hat{\tilde{\mathcal{L}}}_{44} + \frac{i}{2} \bar{v}_0(t) \mathcal{M} \hat{\tilde{\mathcal{G}}}_{j4} + k\hat{\tilde{\mathcal{G}}}_{44}, \\
&\tilde{\phi}_{j4}(t,k) = \hat{\tilde{\mathcal{L}}}_{j4} - \frac{i}{2} v_0^T(t) \hat{\tilde{\mathcal{G}}}_{44} + k\hat{\tilde{\mathcal{G}}}_{j4}, \quad \tilde{\phi}_{4j}(t,k) = \hat{\tilde{\mathcal{L}}}_{4j} + \frac{i}{2} \bar{v}_0(t) \mathcal{M} \hat{\tilde{\mathcal{G}}}_{3\times 3} + k\hat{\tilde{\mathcal{G}}}_{4j}, \quad j = 1,2,3.
\end{aligned} \tag{118b}
$$

For the given Eqs. (118a) and (119b) we have the following proposition:

PROPOSITION 5.2

$$\lim_{\tau\to t} \int_{\partial D_1^0} \frac{k e^{4ik^2(t-\tau)}}{\Sigma_-} \left(\tilde{F}_{j4} e^{-2ikL}\right)_- \mathrm{d}k = \int_{\partial D_1^0} \left[\frac{ik}{2} u_0^T \left(\hat{\tilde{G}}_{44} - \bar{\hat{\tilde{\mathcal{G}}}}_{44}\right) + \frac{k}{\Sigma_-} \left(\tilde{F}_{j4} e^{-2ikL}\right)_-\right] \mathrm{d}k, \tag{119a}$$

$$\lim_{\tau\to t} \int_{\partial D_1^0} \frac{k e^{4ik^2(t-\tau)}}{\Sigma_-} \tilde{F}_{4j-} \mathrm{d}k = \int_{\partial D_1^0} \left[\frac{ik}{2} \mathcal{M}^T v_0^T \left(\hat{\tilde{\mathcal{G}}}_{44} - \bar{\hat{\tilde{G}}}_{44}\right) + \frac{k}{\Sigma_-} \tilde{F}_{4j-}\right] \mathrm{d}k, \tag{119b}$$

$$\lim_{\tau\to t} \int_{\partial D_1^0} \frac{e^{4ik^2(t-\tau)}}{\Sigma_-} \left(\tilde{F}_{j4} e^{-2ikL}\right)_+ \mathrm{d}k = \int_{\partial D_1^0} \frac{1}{\Sigma_-} \left(\tilde{F}_{j4} e^{-2ikL}\right)_+ \mathrm{d}k, \tag{119c}$$

$$\lim_{\tau\to t} \int_{\partial D_1^0} \frac{e^{4ik^2(t-\tau)}}{\Sigma_-} \tilde{F}_{4j+} \mathrm{d}k = \int_{\partial D_1^0} \frac{1}{\Sigma_-} \tilde{F}_{4j+} \mathrm{d}k, \tag{119d}$$

where the vector-valued functions $\tilde{F}_{j4}(t,k)$ *and* $\tilde{F}_{4j}(t,k)$ $(j = 1,2,3)$ *are defined by*

$$
\begin{aligned}
\tilde{F}_{j4} =& \frac{1}{2i} u_0^T \hat{\tilde{G}}_{44} + \frac{i}{2} \mathcal{M}^T \bar{\hat{\tilde{\mathcal{G}}}}{}_{3\times 3}^T \mathcal{M} v_0^T e^{2ikL} + \left(\hat{\tilde{L}}_{j4} - \frac{i}{2} u_0^T \hat{\tilde{G}}_{44} + k\hat{\tilde{G}}_{j4}\right) \left(\bar{\hat{\tilde{\mathcal{L}}}}_{44} - \frac{i}{2} \bar{\hat{\tilde{\mathcal{G}}}}{}_{j4}^T \mathcal{M} v_0^T + k\bar{\hat{\tilde{\mathcal{G}}}}_{44}\right) \\
&- \left(\hat{\tilde{L}}_{3\times 3} - \tfrac{i}{2} u_0^T \hat{\tilde{G}}_{4j} + k\hat{\tilde{G}}_{3\times 3}\right) \mathcal{M}^T \left(\bar{\hat{\tilde{\mathcal{L}}}}{}_{4j}^T - \tfrac{i}{2} \bar{\hat{\tilde{\mathcal{G}}}}{}_{3\times 3}^T \mathcal{M} v_0^T + k\bar{\hat{\tilde{\mathcal{G}}}}{}_{4j}^T\right) e^{2ikL},
\end{aligned} \tag{120}
$$

$$\begin{aligned}\tilde{F}_{4j}=&\tfrac{1}{2i}\bar{\hat{\tilde{G}}}^T_{3\times3}\mathcal{M}u_0^T+\tfrac{i}{2}\mathcal{M}^T v_0^T\hat{\tilde{\mathcal{G}}}_{44}e^{2ikL}+\mathcal{M}^T\left(\hat{\tilde{\mathcal{L}}}_{3\times3}-\tfrac{i}{2}v_0^T\hat{\tilde{\mathcal{G}}}_{4j}+k\hat{\tilde{\mathcal{G}}}_{3\times3}\right)\mathcal{M}^T\left(\bar{\hat{\tilde{L}}}^T_{4j}-\tfrac{i}{2}\bar{\hat{\tilde{G}}}^T_{3\times3}\mathcal{M}u_0^T+k\bar{\hat{\tilde{G}}}^T_{4j}\right)\\ &-\mathcal{M}^T\left(\hat{\tilde{\mathcal{L}}}_{j4}-\tfrac{i}{2}v_0^T\hat{\tilde{\mathcal{G}}}_{44}+k\hat{\tilde{\mathcal{G}}}_{j4}\right)\left(\bar{\hat{\tilde{L}}}_{44}-\tfrac{i}{2}\bar{\hat{\tilde{G}}}^T_{j4}\mathcal{M}u_0^T+k\bar{\hat{\tilde{G}}}_{44}\right)e^{2ikL}.\end{aligned}\tag{121}$$

Proof. Similarly to the proof of Lemma 4.3 in Lenells & Fokas (2012b), we here show Eq. (119a) in detail. Multiplying Eq. (120) by $\frac{k}{\Sigma_-}e^{4ik^2(t-\tau)}$ with $0<\tau<t$ and integrating along along ∂D_1^0 with respect to dk can yield

$$\begin{aligned}\int_{\partial D_1^0}\frac{k}{\Sigma_-}e^{4ik^2(t-\tau)}(\tilde{F}_{j4}e^{-2ikL})_-\mathrm{d}k=&\int_{\partial D_1^0}\frac{ik}{2}e^{4ik^2(t-\tau)}u_0^T\hat{\tilde{G}}_{44}\mathrm{d}k-\int_{\partial D_1^0}k^3e^{4ik^2(t-\tau)}\hat{\tilde{G}}_{j4}\bar{\hat{\tilde{\mathcal{G}}}}_{44}\mathrm{d}k\\ &-\int_{\partial D_1^0}ke^{4ik^2(t-\tau)}\left(\hat{\tilde{L}}_{j4}-\frac{i}{2}u_0^T\hat{\tilde{G}}_{44}\right)\left(\bar{\hat{\tilde{\mathcal{L}}}}_{44}-\frac{i}{2}\bar{\hat{\tilde{\mathcal{G}}}}^T_{j4}\mathcal{M}v_0^T\right)\mathrm{d}k\\ &+\int_{\partial D_1^0}\frac{k^2\Sigma_+}{\Sigma_-}e^{4ik^2(t-\tau)}\left[\left(\hat{\tilde{L}}_{j4}-\frac{i}{2}u_0^T\hat{\tilde{G}}_{44}\right)\bar{\hat{\tilde{\mathcal{G}}}}_{44}+\hat{\tilde{G}}_{j4}\left(\bar{\hat{\tilde{\mathcal{L}}}}_{44}-\frac{i}{2}\bar{\hat{\tilde{\mathcal{G}}}}_{j4}\mathcal{M}v_0^T\right)\right]\mathrm{d}k\\ &-\int_{\partial D_1^0}\frac{2k^2}{\Sigma_-}e^{4ik^2(t-\tau)}\left[\left(\hat{\tilde{L}}_{3\times3}-\frac{i}{2}u_0^T\hat{\tilde{G}}_{4j}\right)\mathcal{M}^T\bar{\hat{\tilde{\mathcal{G}}}}^T_{4j}+\hat{\tilde{G}}_{3\times3}\mathcal{M}^T\left(\bar{\hat{\tilde{\mathcal{L}}}}^T_{4j}-\frac{i}{2}\bar{\hat{\tilde{\mathcal{G}}}}^T_{3\times3}\mathcal{M}v_0^T\right)\right]\mathrm{d}k.\end{aligned}\tag{122}$$

To further analyse the above equation, the following identities are introduced:

$$\frac{4}{\pi}\int_{\partial D_1}ke^{4ik^2(t-\tau)}\hat{F}(t,k)\mathrm{d}k=\begin{cases}2F(t,2\tau-t), & 0<\tau<t,\\ F(t,t), & 0<\tau=t,\end{cases}\tag{123}$$

and

$$\int_{\partial D_1^0}\frac{k^2}{\Sigma_-}e^{4ik^2(t-\tau)}\hat{F}(t,k)\mathrm{d}k=2\int_{\partial D_1^0}\frac{k^2}{\Sigma_-}\left[\int_0^\tau e^{4ik^2(t-\tau)}\hat{F}(t,2\tau-t)\mathrm{d}\tau-\frac{F(t,2\tau-t)}{4ik^2}\right]\mathrm{d}k,\tag{124}$$

which also hold for the cases that $\frac{k^2}{\Sigma_-}$ is taken place by $\frac{k^2\Sigma_+}{\Sigma_-}$ or k^2.

It follows from the first integral on the RHS of Eq. (122) and Eq. (123) that we have

$$\lim_{\tau\to t}\int_{\partial D_1^0}\frac{ik}{2}e^{4ik^2(t-\tau)}u_0^T\hat{\tilde{G}}_{44}\mathrm{d}k=\lim_{\tau\to t}\frac{i\pi}{2}u_0^T\tilde{G}_{22}(t,2\tau-t)=\frac{i\pi}{4}u_0^T\tilde{G}_{44}(t,t),\tag{125a}$$

$$\lim_{\tau\to t}\int_{\partial D_1^0}\frac{ik}{2}e^{4ik^2(t-\tau)}u_0^T\hat{\tilde{G}}_{44}\mathrm{d}k=\int_{\partial D_1^0}\frac{ik}{2}u_0^T\hat{\tilde{G}}_{44}\mathrm{d}k=\frac{i\pi}{8}u_0^T\tilde{G}_{44}(t,t).\tag{125b}$$

Therefore, we know that the first integral on the RHS of Eq. (122) yields the following two terms:

$$\lim_{\tau\to t}\int_{\partial D_1^0}\frac{ik}{2}e^{4ik^2(t-\tau)}u_0^T\hat{\tilde{G}}_{44}\mathrm{d}k=\int_{\partial D_1^0}\frac{ik}{2}u_0^T\hat{\tilde{G}}_{44}\mathrm{d}k\Big|_{(125a)}+\int_{\partial D_1^0}\frac{ik}{2}u_0^T\hat{\tilde{G}}_{44}\mathrm{d}k\Big|_{(125b)}.\tag{126}$$

Nowadays we study the second integral on the RHS of Eq. (122). It follows from the second integral on the RHS of Eq. (122) and Eq. (124) that we have

$$
\begin{aligned}
-\int_{\partial D_1^0} k^3 e^{4ik^2(t-\tau)} \hat{\tilde{G}}_{j4} \hat{\bar{\tilde{\mathcal{G}}}}_{44} \mathrm{d}k &= -2 \int_{\partial D_1^0} k^3 \int_0^t e^{4ik^2(s-\tau)} \tilde{G}_{j4}(t, 2s-t) \hat{\bar{\tilde{\mathcal{G}}}}_{44} \mathrm{d}s \mathrm{d}k \\
&= -2 \int_{\partial D_1^0} k^3 \left[\int_0^\tau e^{4ik^2(s-\tau)} \tilde{G}_{j4}(t, 2\tau - t) \mathrm{d}s - \frac{\tilde{G}_{j4}(t, 2\tau - t)}{4ik^2} \right] \hat{\bar{\tilde{\mathcal{G}}}}_{44} \mathrm{d}k.
\end{aligned} \tag{127}
$$

Thus we take the limit $\tau \to t$ of Eq. (127) to have

$$
-\lim_{\tau \to t} \int_{\partial D_1^0} k^3 e^{4ik^2(t-\tau)} \hat{\tilde{G}}_{j4} \hat{\bar{\tilde{\mathcal{G}}}}_{44} \mathrm{d}k = -\int_{\partial D_1^0} k^3 \hat{\tilde{G}}_{j4} \hat{\bar{\tilde{\mathcal{G}}}}_{44} \mathrm{d}k + \int_{\partial D_1^0} \frac{k}{2i} u_0^T \hat{\bar{\tilde{\mathcal{G}}}}_{44} \mathrm{d}k.
$$

Finally, following the proof in Lenells & Fokas (2012b) we can show the limits $\tau \to t$ of the rest three integrals (i.e. the third, fourth and fifth integrals) of Eq. (122) can be deduced by simply making the limit $\tau \to t$ inside the every integral, i.e. no additional term arises in these integrals. For example,

$$
\begin{aligned}
&\lim_{\tau \to t} \int_{\partial D_1^0} k e^{4ik^2(t-\tau)} \left(\hat{\tilde{L}}_{j4} - \frac{i}{2} u_0^T \hat{\tilde{G}}_{44} \right) \left(\hat{\bar{\tilde{\mathcal{L}}}}_{44} - \frac{i}{2} \hat{\bar{\tilde{\mathcal{G}}}}_{j4}^T \mathcal{M} v_0^T \right) \mathrm{d}k \\
&= \int_{\partial D_1^0} k \left(\hat{\tilde{L}}_{j4} - \frac{i}{2} u_0^T \hat{\tilde{G}}_{44} \right) \left(\hat{\bar{\tilde{\mathcal{L}}}}_{44} - \frac{i}{2} \hat{\bar{\tilde{\mathcal{G}}}}_{j4}^T \mathcal{M} v_0^T \right) \mathrm{d}k.
\end{aligned}
$$

Thus, this completes the proof of Eq. (119a). Similarly, we can show that Eqs. (119b), (119c) and (119d) also hold. □

THEOREM 5.3 *Let $q_{0j}(x) = q_j(x, t=0) = 0$, $j = 1,2,3$ be the initial data of Eq. (4) on the interval $x \in [0, L]$ and $T < \infty$. For the Dirichlet problem, the boundary data $u_{0j}(t)$ and $v_{0j}(t)$ $(j = 1,2,3)$ on the interval $t \in [0, T)$ are sufficiently smooth and compatible with the initial data $q_{j0}(x)$ $(j = 1,2,3)$ at the points $(x_2, t_2) = (0,0)$ and $(x_3, t_3) = (L, 0)$, respectively. For the Neumann problem, the boundary data $u_{1j}(t)$ and $v_{1j}(t)$ $(j = 1,2,3)$ on the interval $t \in [0, T)$ are sufficiently smooth and compatible with the initial data $q_{0j}(x)$ $(j = 1,2,3)$ at the points $(x_2, t_2) = (0,0)$ and $(x_3, t_3) = (L, 0)$, respectively. For simplicity, let $n_{33,44}(\mathbb{S})(k)$ have no zero in the domain D_1. Then the spectral functions $S(k)$ and $S_L(k)$ are defined by Eqs. (75) and (76) with $\Psi(t,k)$ and $\phi(t,k)$ given by Eq. (108a) and (108b).*

(i) For the given Dirichlet boundary values $u_0(t)$ and $v_0(t)$, the unknown Neumann boundary values $u_1(t)$ and $v_1(t)$ are given by

$$
\begin{aligned}
u_1^T(t) = \frac{4}{i\pi} \int_{\partial D_1^0} \Bigg\{ &\frac{\Sigma_+}{\Sigma_-} \left[k^2 \hat{\tilde{G}}_{j4}(t,t) + \frac{i}{2} u_0^T(t) \right] - \frac{2\mathcal{M}^T}{\Sigma_-} \left[k^2 \hat{\bar{\tilde{\mathcal{G}}}}_{4j}^T(t,t) + \frac{i}{2} \mathcal{M} v_0^T(t) \right] \\
&+ \frac{ik}{2} u_0^T \left(\hat{\tilde{G}}_{44} - \hat{\bar{\tilde{\mathcal{G}}}}_{44} \right) + \frac{k}{\Sigma_-} \left[\tilde{F}_{j4} e^{-2ikL} \right]_- \Bigg\} \mathrm{d}k,
\end{aligned} \tag{128a}
$$

$$
\begin{aligned}
v_1^T(t) = \frac{4}{i\pi}\int_{\partial D_1^0}\Big\{&-\frac{\Sigma_+}{\Sigma_-}\left[k^2\hat{\tilde{\mathcal{G}}}_{j4}(t,t)+\frac{i}{2}v_0^T(t)\right]+\frac{2\mathcal{M}^T}{\Sigma_-}\left[k^2\bar{\hat{\tilde{G}}}{}_{4j}^T(t,t)+\frac{i}{2}\mathcal{M}u_0^T(t)\right]\\
&+\frac{ik}{2}v_0^T\left(\hat{\tilde{\mathcal{G}}}_{44}-\bar{\hat{\tilde{G}}}_{44}\right)+\frac{k}{\Sigma_-}\mathcal{M}^T\tilde{F}_{4j-}\Big\}\,\mathrm{d}k,
\end{aligned} \tag{128b}
$$

(ii) For the given Neumann boundary values $u_1(t)$ and $v_1(t)$, the unknown Dirichlet boundary values $u_0(t)$ and $v_0(t)$ are given by

$$
u_0^T(t) = \frac{2}{\pi}\int_{\partial D_1^0}\left[\frac{\Sigma_+}{\Sigma_-}\hat{\tilde{L}}_{j4}-\frac{2\mathcal{M}^T}{\Sigma_-}\bar{\hat{\tilde{\mathcal{L}}}}{}_{4j}^T+\frac{1}{\Sigma_-}\left(\tilde{F}_{j4}e^{-2ikL}\right)_+\right]\mathrm{d}k, \tag{129a}
$$

$$
v_0^T(t) = \frac{2}{\pi}\int_{\partial D_1^0}\left[\frac{2\mathcal{M}^T}{\Sigma_-}\bar{\hat{\tilde{L}}}{}_{j4}^T-\frac{1}{\Sigma_-}\hat{\tilde{\mathcal{L}}}_{4j}+\frac{\mathcal{M}^T}{\Sigma_-}\tilde{F}_{4j+}\right]\mathrm{d}k, \tag{129b}
$$

where $\tilde{F}_{j4}(t,k)$ and $\tilde{F}_{4j}(t,k)$ are defined by Eqs. (120) and (121).

Proof. The proof of this Theorem is given in **Appendix C**. □

5.2 *The equivalence of two distinct representations*

We now show that the above-mentioned GLM representation for the Dirichlet and Neumann boundary data in Theorem 5.3 is equivalent to one in Proposition 4.2.

Case i). From the Dirichlet boundary conditions to the Neumann boundary ones

It follows from Eqs. (118a) and (118b) that we obtain

$$
\hat{\tilde{G}}_{j4}=\frac{1}{2k}\tilde{\Psi}_{j4-},\quad \hat{\tilde{\mathcal{G}}}_{j4}=\frac{1}{2k}\tilde{\phi}_{j4-},\quad j=1,2,3,\quad \hat{\tilde{G}}_{44}=\frac{1}{2k}\tilde{\Psi}_{44-},\quad \hat{\tilde{\mathcal{G}}}_{44}=\frac{1}{2k}\tilde{\phi}_{44-}. \tag{130}
$$

Substituting Eqs. (120) and (130) into Eq. (128a) yields

$$
\begin{aligned}
u_1^T(t) &= \frac{4}{i\pi}\int_{\partial D_1^0}\Big\{\frac{\Sigma_+}{\Sigma_-}\left[k^2\hat{\tilde{G}}_{j4}(t,t)+\frac{i}{2}u_0^T(t)\right]-\frac{2\mathcal{M}^T}{\Sigma_-}\left[k^2\bar{\hat{\tilde{\mathcal{G}}}}{}_{4j}^T(t,t)+\frac{i}{2}\mathcal{M}v_0^T(t)\right]\\
&\quad+iku_0^T\hat{\tilde{G}}_{44}+\frac{k}{2i}u_0^T\bar{\hat{\tilde{\mathcal{G}}}}_{44}+\frac{k}{\Sigma_-}\left[\tilde{\Psi}_{j4}(\bar{\tilde{\phi}}_{44}-1)e^{-2ikL}-(\tilde{\Psi}_{3\times3}-\mathbb{I})\mathcal{M}^T\bar{\tilde{\phi}}{}_{4j}^T\right]_-\Big\}\,\mathrm{d}k\\
&=\int_{\partial D_1^0}\Big\{\frac{2\Sigma_+}{i\pi\,\Sigma_-}\left[k\tilde{\Psi}_{j4-}+iu_0^T(t)\right]+\frac{4i\mathcal{M}^T}{\pi\,\Sigma_-}\left[k\bar{\tilde{\phi}}{}_{4j-}^T+i\mathcal{M}v_0^T(t)\right]+\frac{1}{\pi}u_0^T(2\tilde{\Psi}_{44-}-\bar{\tilde{\phi}}_{44-})\\
&\quad+\frac{4k}{i\pi\,\Sigma_-}\left[\tilde{\Psi}_{j4}(\bar{\tilde{\phi}}_{44}-1)e^{-2ikL}-(\tilde{\Psi}_{3\times3}-\mathbb{I})\mathcal{M}^T\bar{\tilde{\phi}}{}_{4j}^T\right]_-\Big\}\,\mathrm{d}k.
\end{aligned} \tag{131}
$$

Since the integrand in Eq. (131) is an odd function about k, which makes sure that the contour ∂D_1^0 can be replaced by ∂D_3^0, thus we can find the same Neumann boundary data $u_{1j}(t)\,(j=1,2,3)$ at $x=0$ given by Eqs. (81)–(83) from Eq. (131). Similarly, we can also find the Neumann boundary data $v_{1j}(t)\,(j=1,2,3)$ at $x=L$ given by Eq. (84) from Eq. (128b).

Case ii). From the Neumann boundary conditions to the Dirichlet boundary ones
Eqs. (118a) and (118b) imply that

$$\hat{\hat{L}}_{j4} = \frac{1}{2}\tilde{\Psi}_{j4+}(t,k) + \frac{i}{2}u_0^T\hat{\hat{G}}_{44}, \quad \hat{\hat{\mathcal{L}}}_{4j}^T = \frac{1}{2}\bar{\tilde{\phi}}_{4j+}^T(t,k) + \frac{i}{2}\bar{\hat{\hat{\mathcal{G}}}}_{3\times 3}^T\mathcal{M}v_0^T. \tag{132}$$

The substitution of Eqs. (132) and (120) into Eq. (129a) yields

$$\begin{aligned} u_0^T(t) &= \frac{2}{\pi}\int_{\partial D_1^0}\left[\frac{\Sigma_+}{\Sigma_-}\hat{\hat{L}}_{j4} - \frac{2\mathcal{M}^T}{\Sigma_-}\bar{\hat{\hat{\mathcal{L}}}}_{4j}^T + \frac{1}{\Sigma_-}\left(\tilde{F}_{j4}e^{-2ikL}\right)_+\right]\mathrm{d}k \\ &= \int_{\partial D_1^0}\left\{\frac{\Sigma_+}{\pi\Sigma_-}\tilde{\Psi}_{j4+} - \frac{2\mathcal{M}^T}{\pi\Sigma_-}\bar{\tilde{\phi}}_{4j+}^T + \frac{2}{\pi\Sigma_-}\left[\tilde{\Psi}_{j4}(\bar{\tilde{\phi}}_{44}(t,\bar{k})-1)e^{-2ikL} - (\tilde{\Psi}_{3\times 3}-\mathbb{I}_4)\mathcal{M}^T\bar{\tilde{\phi}}_{4j}^T\right]_+\right\}\mathrm{d}k. \end{aligned} \tag{133}$$

Since the integrand in Eq. (133) is an odd function about k, which makes sure that the contour ∂D_1^0 can be replaced by ∂D_3^0, thus Eq. (133) yields the Dirichlet boundary values $u_{0j}(t)$, $j = 1, 2, 3$ again. Similarly, we can also deduce the Dirichlet boundary values $v_{0j}(t)$, $j = 1, 2, 3$ from Eq. (129b).

5.3 *Linearizable boundary conditions for the GLM representation*

In what follows we further explore the linearizable boundary conditions for the GLM representation given in Theorem 5.3.

PROPOSITION 5.4 *Let $q_j(x, t = 0) = q_{0j}(x)$, $j = 1, 2, 3$ be the initial conditions of the gtc-NLS equation (4) on the interval $x \in [0, L]$, and one of the following two boundary conditions: (i) Dirichlet boundary conditions at $x = 0, L$, $q_j(x = 0, t) = u_{0j}(t) = 0$ and $q_j(x = L, t) = v_{0j}(t) = 0$, $j = 1, 2, 3$; (ii) Robin boundary conditions $x = 0, L$, $q_{jx}(x = 0, t) - \chi q_j(x = 0, t) = u_{1j}(t) - \chi u_{0j}(t) = 0$, $j = 1, 2, 3$ and $q_{jx}(x = L, t) - \vartheta q_j(x = L, t) = v_{1j}(t) - \vartheta v_{0j}(t) = 0$, $j = 1, 2$, where χ and ϑ are both real parameters. Then the eigenfunctions $\Psi(t, k)$ and $\phi(t, k)$ can be expressed as*

(i)

$$\Psi(t,k) = \mathbb{I}_4 + \begin{pmatrix} \hat{\hat{L}}_{3\times 3} & \hat{\hat{L}}_{j4} \\ \hat{\hat{L}}_{4j} & \hat{\hat{L}}_{44} \end{pmatrix}, \tag{134a}$$

$$\phi(t,k) = \mathbb{I}_4 + \begin{pmatrix} \hat{\hat{\mathcal{L}}}_{3\times 3} & \hat{\hat{\mathcal{L}}}_{j4} \\ \hat{\hat{\mathcal{L}}}_{4j} & \hat{\hat{\mathcal{L}}}_{44} \end{pmatrix}, \tag{134b}$$

where the 4×4 matrix-valued function $L(t, s) = (L_{ij})_{4\times 4}$ satisfies a reduced Goursat system

$$\begin{aligned} &\tilde{L}_{3\times 3t} + \tilde{L}_{3\times 3s} = iu_1^T(t)\tilde{L}_{4j}, \quad \tilde{L}_{44t} + \tilde{L}_{44s} = -i\bar{u}_1(t)\mathcal{M}\tilde{L}_{j4}, \\ &\tilde{L}_{j4t} - \tilde{L}_{j4s} = iu_1^T(t)\tilde{L}_{44}, \quad \tilde{L}_{4jt} - \tilde{L}_{4js} = -i\bar{u}_1(t)\mathcal{M}\tilde{L}_{3\times 3}, \quad j = 1, 2, 3 \end{aligned} \tag{135}$$

with the initial data (cf. Eq. (110))

$$\tilde{L}_{3\times 3}(t,-t) = 0_{3\times 3}, \quad \tilde{L}_{44}(t,-t) = 0, \quad \tilde{L}_{j4}(t,t) = \frac{i}{2}u_1^T(t), \quad \tilde{L}_{4j}(t,t) = -\frac{i}{2}\bar{u}_1(t)\mathcal{M}. \tag{136}$$

Similarly, the 4×4 matrix-valued function $\mathcal{L}(t,s) = (\mathcal{L}_{ij})_{4\times 4}$ satisfies the analogous system (135) with $u_1(t)$ replaced by $v_1(t)$.

(ii)

$$\Psi(t,k) = \mathbb{I}_4 + \begin{pmatrix} \hat{\tilde{L}}_{3\times 3} & \hat{\tilde{L}}_{j4} \\ \hat{\tilde{L}}_{4j} & \hat{\tilde{L}}_{44} \end{pmatrix} + \begin{pmatrix} -\dfrac{i}{2} u_0^T(t)\hat{\tilde{G}}_{4j} & k\hat{\tilde{G}}_{j4} \\ k\hat{\tilde{G}}_{4j} & \dfrac{i}{2}\bar{u}_0(t)\mathcal{M}\hat{\tilde{G}}_{j4} \end{pmatrix}, \tag{137a}$$

$$\phi(t,k) = \mathbb{I}_4 + \begin{pmatrix} \hat{\tilde{\mathcal{L}}}_{3\times 3} & \hat{\tilde{\mathcal{L}}}_{j4} \\ \hat{\tilde{\mathcal{L}}}_{4j} & \hat{\tilde{\mathcal{L}}}_{44} \end{pmatrix} + \begin{pmatrix} -\dfrac{i}{2} v_0^T(t)\hat{\tilde{\mathcal{G}}}_{4j} & k\hat{\tilde{\mathcal{G}}}_{j4} \\ k\hat{\tilde{\mathcal{G}}}_{4j} & \frac{i}{2}\bar{v}_0(t)\mathcal{M}\hat{\tilde{\mathcal{G}}}_{j4} \end{pmatrix}, \tag{137b}$$

where the 4×4 matrix-valued functions $L(t,s) = (L_{ij})_{4\times 4}$ and $G(t,s) = (G_{ij})_{4\times 4}$ satisfy the reduced nonlinear Goursat system

$$\begin{aligned}
&\tilde{L}_{3\times 3t} + \tilde{L}_{3\times 3s} = i\chi u_0^T(t)\tilde{L}_{4j} + \frac{1}{2}\left[i\dot{u}_0^T(t) - u_0^T(t)\bar{u}_0(t)\mathcal{M}u_0^T(t)\right]\tilde{G}_{4j},\\
&\tilde{L}_{44t} + \tilde{L}_{44s} = -i\chi\bar{u}_0(t)\mathcal{M}\tilde{L}_{j4} - \frac{1}{2}\left[i\dot{\bar{u}}_0(t)\mathcal{M} + \bar{u}_0(t)\mathcal{M}u_0^T(t)\bar{u}_0(t)\mathcal{M}\right]\tilde{G}_{j4},\\
&\tilde{L}_{j4t} - \tilde{L}_{j4s} = i\chi u_0^T(t)\tilde{L}_{44}, \quad \tilde{L}_{4jt} - \tilde{L}_{4js} = -i\chi\bar{u}_0(t)\mathcal{M}\tilde{L}_{3\times 3},\\
&\tilde{G}_{j4t} - \tilde{G}_{j4s} = 2u_0^T(t)\tilde{L}_{44}, \quad \tilde{G}_{4jt} - \tilde{G}_{4js} = 2\bar{u}_0(t)\mathcal{M}\tilde{L}_{3\times 3},
\end{aligned} \tag{138}$$

with the initial data (cf. Eq. (110))

$$\begin{aligned}
&\tilde{L}_{3\times 3}(t,-t) = 0_{3\times 3},\ \tilde{L}_{44}(t,-t) = 0,\ \tilde{L}_{j4}(t,t) = \frac{i}{2}\chi u_0^T(t),\ \tilde{L}_{4j}(t,t) = -\frac{i}{2}\chi\bar{u}_0(t)\mathcal{M},\\
&\tilde{G}_{j4}(t,t) = u_0^T(t),\ \tilde{G}_{4j}(t,t) = \bar{u}_0(t)\mathcal{M}.
\end{aligned} \tag{139}$$

Similarly, the 4×4 matrix-valued functions $\mathcal{L}(t,s) = (\mathcal{L}_{ij})_{4\times 4}$ and $\mathcal{G}(t,s) = (\mathcal{G}_{ij})_{4\times 4}$ satisfy the similar system (138) with $u_0(t)$ and χ replaced by $v_0(t)$ and ϑ, respectively.

Proof. Let us show that the linearizable boundary data correspond to the special cases of Proposition 5.1.

Case i) The Dirichlet zero boundary data $q_j(x=0,t) = u_{0j}(t) = 0$. It follows from the second one of system (109) that $\tilde{G}_{ij}(t,s)$ satisfy

$$\begin{aligned}
&\tilde{G}_{3\times 3t} + \tilde{G}_{3\times 3s} = iu_1^T(t)\tilde{G}_{4j}, \quad \tilde{G}_{44t} + \tilde{G}_{44s} = -i\bar{u}_1(t)\mathcal{M}\tilde{G}_{j4},\\
&\tilde{G}_{j4t} - \tilde{G}_{j4s} = iu_1^T(t)\tilde{G}_{44}, \quad \tilde{G}_{4jt} - \tilde{G}_{4js} = -i\bar{u}_1(t)\mathcal{M}\tilde{G}_{3\times 3},\ j = 1,2,3,
\end{aligned} \tag{140}$$

with the initial data (cf. Eq. (110))

$$\tilde{G}_{3\times 3}(t,-t) = 0_{3\times 3}, \quad \tilde{G}_{44}(t,-t) = 0, \quad \tilde{G}_{j4}(t,t) = 0_{j4}, \quad \tilde{G}_{4j}(t,t) = 0_{4j},\ j = 1,2,3. \tag{141}$$

Thus the unique solution of Eq. (140) is trivial, i.e. $\tilde{G}_{3\times 3}(t,s) = 0$, $\tilde{G}_{4j}(t,s) = 0$, $\tilde{G}_{j4}(t,s) = 0$, $\tilde{G}_{44}(t,s) = 0$ such that Eq. (108a) reduces to Eq. (134a) and the condition (109) with (110) becomes (135) with (136). Similarly, for the Dirichlet zero boundary data $q_j(x=L,t) = v_{0j}(t) = 0$, $j = 1,2,3$, we can also show Eq. (134b).

Case ii) Consider the Robin boundary data $q_{jx}(x=0,t) - \chi q_j(x=0,t) = u_{1j}(t) - \chi u_{0j}(t) = 0$, $(j = 1,2,3)$, i.e. the Dirichlet and Neumann boundary data have the linear relation

$$u_1(t) = \chi u_0(t). \tag{142}$$

Let a 4×4 matrix $Q(t,s)$ be $Q(t,s) = 2L(t,s) - i\chi\sigma_4 G(t,s)$ be the linear combinations of L and G. Then

$$\begin{aligned}
&\tilde{Q}_{3\times 3}(t,s) = 2\tilde{L}_{3\times 3}(t,s) - i\chi \tilde{G}_{3\times 3}(t,s), \quad \tilde{Q}_{44}(t,s) = 2\tilde{L}_{44}(t,s) + i\chi \tilde{G}_{44}(t,s),\\
&\tilde{Q}_{j4}(t,s) = 2\tilde{L}_{j4}(t,s) - i\chi \tilde{G}_{j4}(t,s), \quad \tilde{Q}_{4j}(t,s) = 2\tilde{L}_{4j}(t,s) + i\chi \tilde{G}_{4j}(t,s),\ j=1,2,3.
\end{aligned} \tag{143}$$

It follows from Eq. (109) and (143) with Eq. (142) that $\tilde{Q}_{ij}(t,s)$, $\tilde{G}_{ij}(t,s)$, $i,j = 1,2$ satisfy

$$\begin{aligned}
&\tilde{Q}_{3\times 3t} + \tilde{Q}_{3\times 3s} = \left[i\dot{u}_0^T(t) - u_0^T(t)\bar{u}_0(t)\mathcal{M}u_0^T(t) + \chi^2 u_0^T(t)\right]\tilde{G}_{4j},\\
&\tilde{Q}_{j4t} - \tilde{Q}_{j4s} = \left[i\dot{u}_0^T(t) - u_0^T(t)\bar{u}_0(t)\mathcal{M}u_0^T(t) + \chi^2 u_0^T(t)\right]\tilde{G}_{44},\\
&\tilde{Q}_{4jt} - \tilde{Q}_{4js} = \left[-\bar{u}_0(t)\mathcal{M}u_0^T(t)\bar{u}_0(t)\mathcal{M} - i\dot{\bar{u}}_0(t)\mathcal{M} + \chi^2\bar{u}_0(t)\mathcal{M}\right]\tilde{G}_{3\times 3},\\
&\tilde{Q}_{44t} + \tilde{Q}_{44s} = \left[-\bar{u}_0(t)\mathcal{M}u_0^T(t)\bar{u}_0(t)\mathcal{M} - i\dot{\bar{u}}_0(t)\mathcal{M} + \chi^2\bar{u}_0(t)\mathcal{M}\right]\tilde{G}_{j4},\\
&\tilde{G}_{3\times 3t} + \tilde{G}_{3\times 3s} = u_0^T(t)\tilde{Q}_{4j}, \quad \tilde{G}_{44t} + \tilde{G}_{44s} = \bar{u}_0(t)\mathcal{M}\tilde{Q}_{j4},\\
&\tilde{G}_{j4t} - \tilde{G}_{j4s} = u_0^T(t)\tilde{Q}_{44}, \quad \tilde{G}_{4jt} - \tilde{G}_{4js} = \bar{u}_0(t)\mathcal{M}\tilde{Q}_{3\times 3},\ j=1,2,3,
\end{aligned} \tag{144}$$

with the initial data (cf. Eq. (110))

$$\begin{aligned}
&\tilde{G}_{3\times 3}(t,-t) = 0_{3\times 3}, \quad \tilde{G}_{44}(t,-t) = 0, \quad \tilde{G}_{j4}(t,t) = u_0^T(t), \quad \tilde{G}_{4j}(t,t) = \bar{u}_0(t)\mathcal{M},\\
&\tilde{Q}_{3\times 3}(t,-t) = 0_{3\times 3}, \quad \tilde{Q}_{44}(t,-t) = 0, \quad \tilde{Q}_{j4}(t,t) = 0_{j4}, \quad \tilde{Q}_{4j}(t,t) = 0_{4j},\ j=1,2,3,
\end{aligned} \tag{145}$$

Thus the unique solution of Eq. (144) is trivial, i.e. $\tilde{Q}_{j4}(t,s) = \tilde{Q}_{4j}(t,s) = \tilde{G}_{3\times 3}(t,s) = \tilde{G}_{44}(t,s) = 0$ such that Eq. (108a) reduces to Eq. (137a) and the condition (109) with Eq. (110) becomes Eq. (138) with Eq. (139). Similarly, for the Robin boundary data $q_{jx}(x=L,t) - \vartheta q_j(x=L,t) = v_{1j}(t) - \vartheta v_{0j}(t) = 0$, $j = 1,2,3$, i.e. $v_1(t) = \vartheta v_0(t)$, we can also show Eq. (137b). □

Based on the Theorem 5.3 and Proposition 5.4, we have the following Proposition.

PROPOSITION 5.5 *For the linearizable Dirichlet boundary data $u_0(t) = v_0(t) = 0$, we have the Neumann boundary data $u_1(t)$ and $v_1(t)$:*

$$u_1^T(t) = \frac{4i}{\pi}\int_{\partial D_1^0} k\tilde{\Psi}_{j4}(\bar{\tilde{\phi}}_{44} - \mathbb{I}_4)\mathrm{d}k, \quad v_1^T(t) = \frac{4i}{\pi}\int_{\partial D_1^0} k\tilde{\phi}_{j4}(\bar{\tilde{\Psi}}_{44} - \mathbb{I}_4)\mathrm{d}k, \tag{146}$$

where

$$\begin{aligned} &\tilde{\Psi}_{j4t} + 4ik^2\tilde{\Psi}_{j4} = iu_1^T(t)(\tilde{\Psi}_{44} + \mathbb{I}_4), \quad \tilde{\phi}_{j4t} + 4ik^2\tilde{\phi}_{j4} = iv_1^T(t)(\tilde{\phi}_{44} + \mathbb{I}_4), \quad j = 1, 2, 3, \\ &\tilde{\Psi}_{44t} = -i\bar{u}_1(t)\mathcal{M}\tilde{\Psi}_{j4}, \quad \tilde{\phi}_{44t} = -i\bar{v}_1(t)\mathcal{M}\tilde{\phi}_{j4}. \end{aligned} \tag{147}$$

REMARK 5.6 Although the Fokas method has been used to investigated some integrable nonlinear PDEs with 2×2 or 3×3 matrix Lax pairs, our results for the gtc-NLS equation (4) with a 4×4 matrix Lax pair differ from the known ones. Particularly, the found symmetry (31), the matrices S_n, the 4×4 RH problem and nonlinearizable boundary conditions all differ from ones appearing in the coupled NLS systems with 3×3 Lax pairs (Xu & Fan, 2016a; Tian, 2017).

6. Conclusions and discussions

In conclusion, the well-known inverse scattering transform can be used to solve the initial value problems of some integrable nonlinear PDEs with Lax pairs such as the KdV equation, NLS equation, mKdV equation, sine-Gordon equation, AKNS system, KP equation, Camassa–Holm equation, etc. The Fokas method is a powerful approach studying the IBV problems of linear and integrable nonlinear PDEs. In this paper, based on the IST, we have investigated the IBV problem for the gtc-NLS system with a 4×4 Lax pair on a finite interval by using the Fokas method. The 4×4 matrix Lax pair of the gtc-NLS system can be regarded as the 4×4 generalization of the 3×3 Lax pairs for the DP equation (Lenells, 2012, 2013), the Sasa–Satsuma equation (Xu & Fan, 2013) and two-component coupled NLS equations (Biondini & Bui, 2012; Geng *et al.*, 2015; Xu & Fan, 2016a; Tian, 2017). By the Fokas method, we find that the solutions of the gtc-NLS system can be formulated using the solution of a 4×4 matrix RH problem constructed in the complex k-plane. The relevant jump matrices with explicit (x, t)-dependence of the matrix RH problem can be explicitly found in terms of three spectral functions $\{s(k),\ S(k),\ S_L(k)\}$ arising from the initial data, and Dirichlet–Neumann boundary conditions at $x = 0, L$, respectively. The global relation is also proposed to derive two distinct but equivalent types of representations of the Dirichlet–Neumann boundary value problems. Particularly, the relevant formulae for the boundary value problems on the finite interval can generate ones on the half-line as the length of the interval closes to infinity. Finally, we also analyse the linearizable boundary conditions for the GLM representation.

Particularly, we have the following conclusions:

- For the given the initial conditions $q_j(x, 0) = q_{0j}(x)$, we have the spectral function matrix $s(k)$. For the given Dirichlet and Neumann boundary conditions at $x = 0$, $q_j(x = 0, t) = u_{0j}(t)$, $q_{jx}(x = 0, t) = u_{1j}(t)$, $j = 1, 2, 3$, we have the spectral function matrix $S(k)$. Similarly, the Dirichlet and Neumann boundary conditions at $x = L$, $q_j(x = L, t) = v_{0j}(t)$, $q_{jx}(x = L, t) = v_{1j}(t)$, $j = 1, 2, 3$, we have another spectral function matrix $S_L(k)$;

- Given the Dirichlet boundary conditions, $q_j(x=0,t)=u_{0j}(t)$, $q_j(x=L,t)=v_{0j}(t)$, $j=1,2,3$, one can determined the Neumann boundary conditions $q_{jx}(x=0,t)=u_{1j}(t)$, $q_{jx}(x=L,t)=v_{1j}(t)$, $j=1,2,3$ given by Eqs. (81)–(87) by means of the global relation between the distinct spectral function matrices. Conversely, the corresponding result also holds;
- For the given spectral function matrices $\{s(k),\ S(k),\ S_L(k)\}$, a matrix RH problem for $M(x,t,k)$ can be defined such that the functions $q_j(x,t)$, $j=1,2,3$ expressed by means of $M(x,t,k)$ can be shown to satisfy system (4) and the IBV conditions given by Eq. (6).

Based on the corresponding RH problem, these above-obtained results may be used to further study the long-time asymptotics of the solution by means of the Deift–Zhou's nonlinear steepest descent approach (Deift & Zhou, 1992, 1993) and some numerical methods (Trogdon, 2013). Moreover, the analogous analysis of the Fokas method can also be used to explore the IBV problems for other integrable nonlinear evolution PDEs with 4×4 matrix Lax pairs both on the the half-line and the finite interval, such as the three-component derivative NLS system, the three-component high-order NLS system and the three-component mKdV system. These important issues will be further implemented in future.

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A. The proof of Proposition 4.1

Proof. The global relation (59) under the vanishing initial data can be simplified as

$$\begin{aligned} c_{14}(t,k)=&\Psi_{14}(t,k)\bar{\phi}_{44}(t,\bar{k})-e^{2ikL}\big[\Psi_{11}(t,k)(\alpha_{11}\bar{\phi}_{41}(t,\bar{k})+\bar{\alpha}_{12}\bar{\phi}_{42}(t,\bar{k})+\bar{\alpha}_{13}\bar{\phi}_{43}(t,\bar{k}))\\ &+\Psi_{12}(t,k)(\alpha_{12}\bar{\phi}_{41}(t,\bar{k})+\alpha_{22}\bar{\phi}_{42}(t,\bar{k})+\bar{\alpha}_{23}\bar{\phi}_{43}(t,\bar{k}))\\ &+\Psi_{13}(t,k)(\alpha_{13}\bar{\phi}_{41}(t,\bar{k})+\alpha_{23}\bar{\phi}_{42}(t,\bar{k})+\alpha_{33}\bar{\phi}_{43}(t,\bar{k}))\big], \end{aligned} \tag{148a}$$

$$\begin{aligned} c_{24}(t,k)=&\Psi_{24}(t,k)\bar{\phi}_{44}(t,\bar{k})-e^{2ikL}\big[\Psi_{21}(t,k)(\alpha_{11}\bar{\phi}_{41}(t,\bar{k})+\bar{\alpha}_{12}\bar{\phi}_{42}(t,\bar{k})+\bar{\alpha}_{13}\bar{\phi}_{43}(t,\bar{k}))\\ &+\Psi_{22}(t,k)(\alpha_{12}\bar{\phi}_{41}(t,\bar{k})+\alpha_{22}\bar{\phi}_{42}(t,\bar{k})+\bar{\alpha}_{23}\bar{\phi}_{43}(t,\bar{k}))\\ &+\Psi_{23}(t,k)(\alpha_{13}\bar{\phi}_{41}(t,\bar{k})+\alpha_{23}\bar{\phi}_{42}(t,\bar{k})+\alpha_{33}\bar{\phi}_{43}(t,\bar{k}))\big], \end{aligned} \tag{148b}$$

$$\begin{aligned} c_{34}(t,k)=&\Psi_{34}(t,k)\bar{\phi}_{44}(t,\bar{k})-e^{2ikL}\big[\Psi_{31}(t,k)(\alpha_{11}\bar{\phi}_{41}(t,\bar{k})+\bar{\alpha}_{12}\bar{\phi}_{42}(t,\bar{k})+\bar{\alpha}_{13}\bar{\phi}_{43}(t,\bar{k}))\\ &+\Psi_{32}(t,k)(\alpha_{12}\bar{\phi}_{41}(t,\bar{k})+\alpha_{22}\bar{\phi}_{42}(t,\bar{k})+\bar{\alpha}_{23}\bar{\phi}_{43}(t,\bar{k}))\\ &+\Psi_{33}(t,k)(\alpha_{13}\bar{\phi}_{41}(t,\bar{k})+\alpha_{23}\bar{\phi}_{42}(t,\bar{k})+\alpha_{33}\bar{\phi}_{43}(t,\bar{k}))\big], \end{aligned} \tag{148c}$$

where $\bar{\phi}_{ij}(t,\bar{k})=\overline{\phi_{ij}(t,\bar{k})}$.

Recalling the time-part of the Lax pair (11)

$$\mu_t+2ik^2[\sigma_4,\mu]=V(x,t,k)\mu. \tag{149}$$

It follows from the first column of Eq. (149) with $\mu=\mu_2$ that we have

$$\begin{aligned} \Psi_{11,t}(t,k)=&(2ku_{01}+iu_{11})\Psi_{41}-i\Psi_{11}(\alpha_{11}|u_{01}|^2+\bar{\alpha}_{12}u_{01}\bar{u}_{02}+\bar{\alpha}_{13}u_{01}\bar{u}_{03})\\ &-i\Psi_{21}(\alpha_{12}|u_{01}|^2+\alpha_{22}u_{01}\bar{u}_{02}+\bar{\alpha}_{23}u_{01}\bar{u}_{03})-i\Psi_{31}(\alpha_{13}|u_{01}|^2+\alpha_{23}u_{01}\bar{u}_{02}+\alpha_{33}u_{01}\bar{u}_{03}),\\ \Psi_{21,t}(t,k)=&(2ku_{02}+iu_{12})\Psi_{41}-i\Psi_{11}(\alpha_{11}u_{02}\bar{u}_{01}+\bar{\alpha}_{12}|u_{02}|^2+\bar{\alpha}_{13}u_{02}\bar{u}_{03})\\ &-i\Psi_{21}(\alpha_{12}u_{02}\bar{u}_{01}+\alpha_{22}|u_{02}|^2+\bar{\alpha}_{23}u_{02}\bar{u}_{03})-i\Psi_{31}(\alpha_{13}u_{02}\bar{u}_{01}+\alpha_{23}|u_{02}|^2+\alpha_{33}u_{02}\bar{u}_{03}),\\ \Psi_{31,t}(t,k)=&(2ku_{03}+iu_{13})\Psi_{41}-i\Psi_{11}(\alpha_{11}u_{03}\bar{u}_{01}+\bar{\alpha}_{12}u_{03}\bar{u}_{02}+\bar{\alpha}_{13}|u_{03}|^2)\\ &-i\Psi_{21}(\alpha_{12}u_{03}\bar{u}_{01}+\alpha_{22}u_{03}\bar{u}_{02}+\bar{\alpha}_{23}|u_{03}|^2)-i\Psi_{31}(\alpha_{13}u_{03}\bar{u}_{01}+\alpha_{23}u_{03}\bar{u}_{02}+\alpha_{33}|u_{03}|^2),\\ \Psi_{41,t}(t,k)=&\Psi_{11}[\alpha_{11}(2k\bar{u}_{01}-i\bar{u}_{11})+\bar{\alpha}_{12}(2k\bar{u}_{02}-i\bar{u}_{12})+\bar{\alpha}_{13}(2k\bar{u}_{03}-i\bar{u}_{13})]\\ &+\Psi_{21}[\alpha_{12}(2k\bar{u}_{01}-i\bar{u}_{11})+\alpha_{22}(2k\bar{u}_{02}-i\bar{u}_{12})+\bar{\alpha}_{23}(2k\bar{u}_{03}-i\bar{u}_{13})]\\ &+\Psi_{31}[\alpha_{13}(2k\bar{u}_{01}-i\bar{u}_{11})+\alpha_{23}(2k\bar{u}_{02}-i\bar{u}_{12})+\alpha_{33}(2k\bar{u}_{03}-i\bar{u}_{13})]\\ &+i\Psi_{41}[4k^2+\alpha_{11}|u_{01}|^2+\bar{\alpha}_{12}u_{01}\bar{u}_{02}+\bar{\alpha}_{13}u_{01}\bar{u}_{03}+\alpha_{12}u_{02}\bar{u}_{01}\\ &+\alpha_{22}|u_{02}|^2+\bar{\alpha}_{23}u_{02}\bar{u}_{03}+\alpha_{13}u_{03}\bar{u}_{01}+\alpha_{23}u_{03}\bar{u}_{02}+\alpha_{33}|u_{03}|^2]. \end{aligned} \tag{150}$$

The second column of Eq. (149) with $\mu = \mu_2$ yields

$$
\begin{aligned}
\Psi_{12,t}(t,k) =& (2ku_{01} + iu_{11})\Psi_{42} - i\Psi_{12}(\alpha_{11}|u_{01}|^2 + \bar{\alpha}_{12}u_{01}\bar{u}_{02} + \bar{\alpha}_{13}u_{01}\bar{u}_{03}) \\
& - i\Psi_{22}(\alpha_{12}|u_{01}|^2 + \alpha_{22}u_{01}\bar{u}_{02} + \bar{\alpha}_{23}u_{01}\bar{u}_{03}) - i\Psi_{32}(\alpha_{13}|u_{01}|^2 + \alpha_{23}u_{01}\bar{u}_{02} + \alpha_{33}u_{01}\bar{u}_{03}), \\
\Psi_{22,t}(t,k) =& (2ku_{02} + iu_{12})\Psi_{42} - i\Psi_{12}(\alpha_{11}u_{02}\bar{u}_{01} + \bar{\alpha}_{12}|u_{02}|^2 + \bar{\alpha}_{13}u_{02}\bar{u}_{03}) \\
& - i\Psi_{22}(\alpha_{12}u_{02}\bar{u}_{01} + \alpha_{22}|u_{02}|^2 + \bar{\alpha}_{23}u_{02}\bar{u}_{03}) - i\Psi_{32}(\alpha_{13}u_{02}\bar{u}_{01} + \alpha_{23}|u_{02}|^2 + \alpha_{33}u_{02}\bar{u}_{03}), \\
\Psi_{32,t}(t,k) =& (2ku_{03} + iu_{13})\Psi_{42} - i\Psi_{12}(\alpha_{11}u_{03}\bar{u}_{01} + \bar{\alpha}_{12}u_{03}\bar{u}_{02} + \bar{\alpha}_{13}|u_{03}|^2) \\
& - i\Psi_{22}(\alpha_{12}u_{03}\bar{u}_{01} + \alpha_{22}u_{03}\bar{u}_{02} + \bar{\alpha}_{23}|u_{03}|^2) - i\Psi_{32}(\alpha_{13}u_{03}\bar{u}_{01} + \alpha_{23}u_{03}\bar{u}_{02} + \alpha_{33}|u_{03}|^2), \\
\Psi_{42,t}(t,k) =& \Psi_{12}[\alpha_{11}(2k\bar{u}_{01} - i\bar{u}_{11}) + \bar{\alpha}_{12}(2k\bar{u}_{02} - i\bar{u}_{12}) + \bar{\alpha}_{13}(2k\bar{u}_{03} - i\bar{u}_{13})] \\
& + \Psi_{22}[\alpha_{12}(2k\bar{u}_{01} - i\bar{u}_{11}) + \alpha_{22}(2k\bar{u}_{02} - i\bar{u}_{12}) + \bar{\alpha}_{23}(2k\bar{u}_{03} - i\bar{u}_{13})] \\
& + \Psi_{32}[\alpha_{13}(2k\bar{u}_{01} - i\bar{u}_{11}) + \alpha_{23}(2k\bar{u}_{02} - i\bar{u}_{12}) + \alpha_{33}(2k\bar{u}_{03} - i\bar{u}_{13})] \\
& + i\Psi_{42}[4k^2 + \alpha_{11}|u_{01}|^2 + \bar{\alpha}_{12}u_{01}\bar{u}_{02} + \bar{\alpha}_{13}u_{01}\bar{u}_{03} + \alpha_{12}u_{02}\bar{u}_{01} \\
& + \alpha_{22}|u_{02}|^2 + \bar{\alpha}_{23}u_{02}\bar{u}_{03} + \alpha_{13}u_{03}\bar{u}_{01} + \alpha_{23}u_{03}\bar{u}_{02} + \alpha_{33}|u_{03}|^2].
\end{aligned}
\tag{151}
$$

The third column of Eq. (149) with $\mu = \mu_2$ yields

$$
\begin{aligned}
\Psi_{13,t}(t,k) =& (2ku_{01} + iu_{11})\Psi_{43} - i\Psi_{13}(\alpha_{11}|u_{01}|^2 + \bar{\alpha}_{12}u_{01}\bar{u}_{02} + \bar{\alpha}_{13}u_{01}\bar{u}_{03}) \\
& - i\Psi_{23}(\alpha_{12}|u_{01}|^2 + \alpha_{22}u_{01}\bar{u}_{02} + \bar{\alpha}_{23}u_{01}\bar{u}_{03}) - i\Psi_{33}(\alpha_{13}|u_{01}|^2 + \alpha_{23}u_{01}\bar{u}_{02} + \alpha_{33}u_{01}\bar{u}_{03}), \\
\Psi_{23,t}(t,k) =& (2ku_{02} + iu_{12})\Psi_{43} - i\Psi_{13}(\alpha_{11}u_{02}\bar{u}_{01} + \bar{\alpha}_{12}|u_{02}|^2 + \bar{\alpha}_{13}u_{02}\bar{u}_{03}) \\
& - i\Psi_{23}(\alpha_{12}u_{02}\bar{u}_{01} + \alpha_{22}|u_{02}|^2 + \bar{\alpha}_{23}u_{02}\bar{u}_{03}) - i\Psi_{33}(\alpha_{13}u_{02}\bar{u}_{01} + \alpha_{23}|u_{02}|^2 + \alpha_{33}u_{02}\bar{u}_{03}), \\
\Psi_{33,t}(t,k) =& (2ku_{03} + iu_{13})\Psi_{43} - i\Psi_{13}(\alpha_{11}u_{03}\bar{u}_{01} + \bar{\alpha}_{12}u_{03}\bar{u}_{02} + \bar{\alpha}_{13}|u_{03}|^2) \\
& - i\Psi_{23}(\alpha_{12}u_{03}\bar{u}_{01} + \alpha_{22}u_{03}\bar{u}_{02} + \bar{\alpha}_{23}|u_{03}|^2) - i\Psi_{33}(\alpha_{13}u_{03}\bar{u}_{01} + \alpha_{23}u_{03}\bar{u}_{02} + \alpha_{33}|u_{03}|^2), \\
\Psi_{43,t}(t,k) =& \Psi_{13}[\alpha_{11}(2k\bar{u}_{01} - i\bar{u}_{11}) + \bar{\alpha}_{12}(2k\bar{u}_{02} - i\bar{u}_{12}) + \bar{\alpha}_{13}(2k\bar{u}_{03} - i\bar{u}_{13})] \\
& + \Psi_{23}[\alpha_{12}(2k\bar{u}_{01} - i\bar{u}_{11}) + \alpha_{22}(2k\bar{u}_{02} - i\bar{u}_{12}) + \bar{\alpha}_{23}(2k\bar{u}_{03} - i\bar{u}_{13})] \\
& + \Psi_{33}[\alpha_{13}(2k\bar{u}_{01} - i\bar{u}_{11}) + \alpha_{23}(2k\bar{u}_{02} - i\bar{u}_{12}) + \alpha_{33}(2k\bar{u}_{03} - i\bar{u}_{13})] \\
& + i\Psi_{43}[4k^2 + \alpha_{11}|u_{01}|^2 + \bar{\alpha}_{12}u_{01}\bar{u}_{02} + \bar{\alpha}_{13}u_{01}\bar{u}_{03} + \alpha_{12}u_{02}\bar{u}_{01} \\
& + \alpha_{22}|u_{02}|^2 + \bar{\alpha}_{23}u_{02}\bar{u}_{03} + \alpha_{13}u_{03}\bar{u}_{01} + \alpha_{23}u_{03}\bar{u}_{02} + \alpha_{33}|u_{03}|^2].
\end{aligned}
\tag{152}
$$

The fourth column of Eq. (149) with $\mu = \mu_2$ yields

$$\begin{aligned}
\Psi_{14,t}(t,k) =& (2ku_{01} + iu_{11})\Psi_{44} - i\Psi_{14}(4k^2 + \alpha_{11}|u_{01}|^2 + \bar{\alpha}_{12}u_{01}\bar{u}_{02} + \bar{\alpha}_{13}u_{01}\bar{u}_{03})\\
&-i\Psi_{24}(\alpha_{12}|u_{01}|^2 + \alpha_{22}u_{01}\bar{u}_{02} + \bar{\alpha}_{23}u_{01}\bar{u}_{03}) - i\Psi_{34}(\alpha_{13}|u_{01}|^2 + \alpha_{23}u_{01}\bar{u}_{02} + \alpha_{33}u_{01}\bar{u}_{03}),\\
\Psi_{24,t}(t,k) =& (2ku_{02} + iu_{12})\Psi_{44} - i\Psi_{14}(\alpha_{11}u_{02}\bar{u}_{01} + \bar{\alpha}_{12}|u_{02}|^2 + \bar{\alpha}_{13}u_{02}\bar{u}_{03})\\
&-i\Psi_{24}(4k^2+\alpha_{12}u_{02}\bar{u}_{01}+\alpha_{22}|u_{02}|^2+\bar{\alpha}_{23}u_{02}\bar{u}_{03})-i\Psi_{34}(\alpha_{13}u_{02}\bar{u}_{01}+\alpha_{23}|u_{02}|^2+\alpha_{33}u_{02}\bar{u}_{03}),\\
\Psi_{34,t}(t,k) =& (2ku_{03} + iu_{13})\Psi_{44} - i\Psi_{14}(\alpha_{11}u_{03}\bar{u}_{01} + \bar{\alpha}_{12}u_{03}\bar{u}_{02} + \bar{\alpha}_{13}|u_{03}|^2)\\
&-i\Psi_{24}(\alpha_{12}u_{03}\bar{u}_{01}+\alpha_{22}u_{03}\bar{u}_{02}+\bar{\alpha}_{23}|u_{03}|^2)-i\Psi_{34}(4k^2+\alpha_{13}u_{03}\bar{u}_{01}+\alpha_{23}u_{03}\bar{u}_{02}+\alpha_{33}|u_{03}|^2),\\
\Psi_{44,t}(t,k) =& \Psi_{14}[\alpha_{11}(2k\bar{u}_{01} - i\bar{u}_{11}) + \bar{\alpha}_{12}(2k\bar{u}_{02} - i\bar{u}_{12}) + \bar{\alpha}_{13}(2k\bar{u}_{03} - i\bar{u}_{13})]\\
&+\Psi_{24}[\alpha_{12}(2k\bar{u}_{01} - i\bar{u}_{11}) + \alpha_{22}(2k\bar{u}_{02} - i\bar{u}_{12}) + \bar{\alpha}_{23}(2k\bar{u}_{03} - i\bar{u}_{13})]\\
&+\Psi_{34}[\alpha_{13}(2k\bar{u}_{01} - i\bar{u}_{11}) + \alpha_{23}(2k\bar{u}_{02} - i\bar{u}_{12}) + \alpha_{33}(2k\bar{u}_{03} - i\bar{u}_{13})]\\
&+i\Psi_{44}[\alpha_{11}|u_{01}|^2 + \bar{\alpha}_{12}u_{01}\bar{u}_{02} + \bar{\alpha}_{13}u_{01}\bar{u}_{03} + \alpha_{12}u_{02}\bar{u}_{01} + \alpha_{22}|u_{02}|^2\\
&+\bar{\alpha}_{23}u_{02}\bar{u}_{03} + \alpha_{13}u_{03}\bar{u}_{01} + \alpha_{23}u_{03}\bar{u}_{02} + \alpha_{33}|u_{03}|^2].
\end{aligned} \tag{153}$$

Suppose that Ψ_{j1}'s, $j = 1, 2, 3, 4$ are of the forms

$$\begin{pmatrix}\Psi_{11}\\ \Psi_{21}\\ \Psi_{31}\\ \Psi_{41}\end{pmatrix} = \left(a_{10}(t) + \frac{a_{11}(t)}{k} + \frac{a_{12}(t)}{k^2} + \cdots\right) + \left(b_{10}(t) + \frac{b_{11}(t)}{k} + \frac{b_{12}(t)}{k^2} + \cdots\right)e^{4ik^2t}, \tag{154}$$

where the 4×1 column vector functions $a_{1j}(t), b_{1j}(t)$ $(j = 0, 1, ...,)$ are independent of k.

By substituting Eq. (154) into Eq.(150) and using the initial conditions

$$a_{10}(0) + b_{10}(0) = (1, 0, 0, 0)^T, \quad a_{11}(0) + b_{11}(0) = (0, 0, 0, 0)^T,$$

we have

$$\begin{aligned}
\begin{pmatrix}\Psi_{11}\\ \Psi_{21}\\ \Psi_{31}\\ \Psi_{41}\end{pmatrix} =& \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \sum_{s=1}^{2}\frac{1}{k^s}\begin{pmatrix}\Psi_{11}^{(s)}\\ \Psi_{21}^{(s)}\\ \Psi_{31}^{(s)}\\ \Psi_{41}^{(s)}\end{pmatrix} + O\left(\frac{1}{k^3}\right)\\
&+\left[\frac{1}{2ik}\begin{pmatrix}0\\0\\0\\ \alpha_{11}\bar{u}_{01}(0) + \bar{\alpha}_{12}\bar{u}_{02}(0) + \bar{\alpha}_{13}\bar{u}_{03}(0)\end{pmatrix} + O\left(\frac{1}{k^2}\right)\right]e^{4ik^2t}.
\end{aligned} \tag{155}$$

Similarly, it follows from Eqs. (151)–(153) that we have the asymptotic formulae for Ψ_{ij}, $i = 1, 2, 3, 4; j = 2, 3, 4$ in the form

$$\begin{pmatrix}\Psi_{12}\\ \Psi_{22}\\ \Psi_{32}\\ \Psi_{42}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \sum_{s=1}^{2}\frac{1}{k^s}\begin{pmatrix}\Psi_{12}^{(s)}\\ \Psi_{22}^{(s)}\\ \Psi_{32}^{(s)}\\ \Psi_{42}^{(s)}\end{pmatrix} + O\left(\frac{1}{k^3}\right)$$

$$+ \left[\frac{1}{2ik}\begin{pmatrix}0\\0\\0\\ \alpha_{12}\bar{u}_{01}(0) + \alpha_{22}\bar{u}_{02}(0) + \bar{\alpha}_{23}\bar{u}_{03}(0)\end{pmatrix} + O\left(\frac{1}{k^2}\right)\right] e^{4ik^2 t}, \tag{156}$$

$$\begin{pmatrix}\Psi_{13}\\ \Psi_{23}\\ \Psi_{33}\\ \Psi_{43}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \sum_{s=1}^{2}\frac{1}{k^s}\begin{pmatrix}\Psi_{13}^{(s)}\\ \Psi_{23}^{(s)}\\ \Psi_{33}^{(s)}\\ \Psi_{43}^{(s)}\end{pmatrix} + O\left(\frac{1}{k^3}\right)$$

$$+ \left[\frac{1}{2ik}\begin{pmatrix}0\\0\\0\\ \alpha_{13}\bar{u}_{01}(0) + \alpha_{23}\bar{u}_{02}(0) + \alpha_{33}\bar{u}_{03}(0)\end{pmatrix} + O\left(\frac{1}{k^2}\right)\right] e^{4ik^2 t} \tag{157}$$

and

$$\begin{pmatrix}\Psi_{14}\\ \Psi_{24}\\ \Psi_{34}\\ \Psi_{44}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \sum_{s=1}^{2}\frac{1}{k^s}\begin{pmatrix}\Psi_{14}^{(s)}\\ \Psi_{24}^{(s)}\\ \Psi_{34}^{(s)}\\ \Psi_{44}^{(s)}\end{pmatrix} + O\left(\frac{1}{k^3}\right) + \left[\frac{i}{2k}\begin{pmatrix}u_{01}(0)\\ u_{02}(0)\\ u_{03}(0)\\ 0\end{pmatrix} + O\left(\frac{1}{k^2}\right)\right] e^{-4ik^2 t}. \tag{158}$$

Similarly to Eqs. (150)–(153) for $\mu_2(0, t, k)$, we also know that the function $\mu(x, t, k) = \mu_3(L, t, k)$ at $x = L$ satisfies the t-part of Lax pair (149).

The first column of Eq. (149) with $\mu = \mu_3$ yields

$$\begin{aligned}
\phi_{11,t}(t,k) =& (2kv_{01} + iv_{11})\phi_{41} - i\phi_{11}(\alpha_{11}|v_{01}|^2 + \bar{\alpha}_{12}v_{01}\bar{v}_{02} + \bar{\alpha}_{13}v_{01}\bar{v}_{03}) \\
& -i\phi_{21}(\alpha_{12}|v_{01}|^2 + \alpha_{22}v_{01}\bar{v}_{02} + \bar{\alpha}_{23}v_{01}\bar{v}_{03}) - i\phi_{31}(\alpha_{13}|v_{01}|^2 + \alpha_{23}v_{01}\bar{v}_{02} + \alpha_{33}v_{01}\bar{v}_{03}), \\
\phi_{21,t}(t,k) =& (2kv_{02} + iv_{12})\phi_{41} - i\phi_{11}(\alpha_{11}v_{02}\bar{v}_{01} + \bar{\alpha}_{12}|v_{02}|^2 + \bar{\alpha}_{13}v_{02}\bar{v}_{03}) \\
& -i\phi_{21}(\alpha_{12}v_{02}\bar{v}_{01} + \alpha_{22}|v_{02}|^2 + \bar{\alpha}_{23}v_{02}\bar{v}_{03}) - i\phi_{31}(\alpha_{13}v_{02}\bar{v}_{01} + \alpha_{23}|v_{02}|^2 + \alpha_{33}v_{02}\bar{v}_{03}), \\
\phi_{31,t}(t,k) =& \ (2kv_{03} + iv_{13})\phi_{41} - i\phi_{11}(\alpha_{11}v_{03}\bar{v}_{01} + \bar{\alpha}_{12}v_{03}\bar{v}_{02} + \bar{\alpha}_{13}|v_{03}|^2) \\
& -i\phi_{21}(\alpha_{12}v_{03}\bar{v}_{01} + \alpha_{22}v_{03}\bar{v}_{02} + \bar{\alpha}_{23}|v_{03}|^2) - i\phi_{31}(\alpha_{13}v_{03}\bar{v}_{01} + \alpha_{23}v_{03}\bar{v}_{02} + \alpha_{33}|v_{03}|^2), \\
\phi_{41,t}(t,k) =& \phi_{11}[\alpha_{11}(2k\bar{v}_{01} - i\bar{v}_{11}) + \bar{\alpha}_{12}(2k\bar{v}_{02} - i\bar{v}_{12}) + \bar{\alpha}_{13}(2k\bar{v}_{03} - i\bar{v}_{13})] \\
& +\phi_{21}[\alpha_{12}(2k\bar{v}_{01} - i\bar{v}_{11}) + \alpha_{22}(2k\bar{v}_{02} - i\bar{v}_{12}) + \bar{\alpha}_{23}(2k\bar{v}_{03} - i\bar{v}_{13})] \\
& +\phi_{31}[\alpha_{13}(2k\bar{v}_{01} - i\bar{v}_{11}) + \alpha_{23}(2k\bar{v}_{02} - i\bar{v}_{12}) + \alpha_{33}(2k\bar{v}_{03} - i\bar{v}_{13})] \\
& +i\phi_{41}[4k^2 + \alpha_{11}|v_{01}|^2 + \bar{\alpha}_{12}v_{01}\bar{v}_{02} + \bar{\alpha}_{13}v_{01}\bar{v}_{03} + \alpha_{12}v_{02}\bar{v}_{01} \\
& +\alpha_{22}|v_{02}|^2 + \bar{\alpha}_{23}v_{02}\bar{v}_{03} + \alpha_{13}v_{03}\bar{v}_{01} + \alpha_{23}v_{03}\bar{v}_{02} + \alpha_{33}|v_{03}|^2].
\end{aligned} \tag{159}$$

The second column of Eq. (149) with $\mu = \mu_3$ yields

$$\begin{aligned}
\phi_{12,t}(t,k) =& (2kv_{01} + iv_{11})\phi_{42} - i\phi_{12}(\alpha_{11}|v_{01}|^2 + \bar{\alpha}_{12}v_{01}\bar{v}_{02} + \bar{\alpha}_{13}v_{01}\bar{v}_{03}) \\
& -i\phi_{22}(\alpha_{12}|v_{01}|^2 + \alpha_{22}v_{01}\bar{v}_{02} + \bar{\alpha}_{23}v_{01}\bar{v}_{03}) - i\phi_{32}(\alpha_{13}|v_{01}|^2 + \alpha_{23}v_{01}\bar{v}_{02} + \alpha_{33}v_{01}\bar{v}_{03}), \\
\phi_{22,t}(t,k) =& (2kv_{02} + iv_{12})\phi_{42} - i\phi_{12}(\alpha_{11}v_{02}\bar{v}_{01} + \bar{\alpha}_{12}|v_{02}|^2 + \bar{\alpha}_{13}v_{02}\bar{v}_{03}) \\
& -i\phi_{22}(\alpha_{12}v_{02}\bar{v}_{01} + \alpha_{22}|v_{02}|^2 + \bar{\alpha}_{23}v_{02}\bar{v}_{03}) - i\phi_{32}(\alpha_{13}v_{02}\bar{v}_{01} + \alpha_{23}|v_{02}|^2 + \alpha_{33}v_{02}\bar{v}_{03}), \\
\phi_{32,t}(t,k) =& \ (2kv_{03} + iv_{13})\phi_{42} - i\phi_{12}(\alpha_{11}v_{03}\bar{v}_{01} + \bar{\alpha}_{12}v_{03}\bar{v}_{02} + \bar{\alpha}_{13}|v_{03}|^2) \\
& -i\phi_{22}(\alpha_{12}v_{03}\bar{v}_{01} + \alpha_{22}v_{03}\bar{v}_{02} + \bar{\alpha}_{23}|v_{03}|^2) - i\phi_{32}(\alpha_{13}v_{03}\bar{v}_{01} + \alpha_{23}v_{03}\bar{v}_{02} + \alpha_{33}|v_{03}|^2), \\
\phi_{42,t}(t,k) =& \phi_{12}[\alpha_{11}(2k\bar{v}_{01} - i\bar{v}_{11}) + \bar{\alpha}_{12}(2k\bar{v}_{02} - i\bar{v}_{12}) + \bar{\alpha}_{13}(2k\bar{v}_{03} - i\bar{v}_{13})] \\
& +\phi_{22}[\alpha_{12}(2k\bar{v}_{01} - i\bar{v}_{11}) + \alpha_{22}(2k\bar{v}_{02} - i\bar{v}_{12}) + \bar{\alpha}_{23}(2k\bar{v}_{03} - i\bar{v}_{13})] \\
& +\phi_{32}[\alpha_{13}(2k\bar{v}_{01} - i\bar{v}_{11}) + \alpha_{23}(2k\bar{v}_{02} - i\bar{v}_{12}) + \alpha_{33}(2k\bar{v}_{03} - i\bar{v}_{13})] \\
& +i\phi_{42}[4k^2 + \alpha_{11}|v_{01}|^2 + \bar{\alpha}_{12}v_{01}\bar{v}_{02} + \bar{\alpha}_{13}v_{01}\bar{v}_{03} + \alpha_{12}v_{02}\bar{v}_{01} \\
& +\alpha_{22}|v_{02}|^2 + \bar{\alpha}_{23}v_{02}\bar{v}_{03} + \alpha_{13}v_{03}\bar{v}_{01} + \alpha_{23}v_{03}\bar{v}_{02} + \alpha_{33}|v_{03}|^2].
\end{aligned} \tag{160}$$

The third column of Eq. (149) with $\mu = \mu_3$ yields

$$\begin{aligned}
\phi_{13,t}(t,k) =& (2kv_{01} + iv_{11})\phi_{43} - i\phi_{13}(\alpha_{11}|v_{01}|^2 + \bar{\alpha}_{12}v_{01}\bar{v}_{02} + \bar{\alpha}_{13}v_{01}\bar{v}_{03}) \\
& - i\phi_{23}(\alpha_{12}|v_{01}|^2 + \alpha_{22}v_{01}\bar{v}_{02} + \bar{\alpha}_{23}v_{01}\bar{v}_{03}) - i\phi_{33}(\alpha_{13}|v_{01}|^2 + \alpha_{23}v_{01}\bar{v}_{02} + \alpha_{33}v_{01}\bar{v}_{03}), \\
\phi_{23,t}(t,k) =& (2kv_{02} + iv_{12})\phi_{43} - i\phi_{13}(\alpha_{11}v_{02}\bar{v}_{01} + \bar{\alpha}_{12}|v_{02}|^2 + \bar{\alpha}_{13}v_{02}\bar{v}_{03}) \\
& - i\phi_{23}(\alpha_{12}v_{02}\bar{v}_{01} + \alpha_{22}|v_{02}|^2 + \bar{\alpha}_{23}v_{02}\bar{v}_{03}) - i\phi_{33}(\alpha_{13}v_{02}\bar{v}_{01} + \alpha_{23}|v_{02}|^2 + \alpha_{33}v_{02}\bar{v}_{03}), \\
\phi_{33,t}(t,k) =& (2kv_{03} + iv_{13})\phi_{43} - i\phi_{13}(\alpha_{11}v_{03}\bar{v}_{01} + \bar{\alpha}_{12}v_{03}\bar{v}_{02} + \bar{\alpha}_{13}|v_{03}|^2) \\
& - i\phi_{23}(\alpha_{12}v_{03}\bar{v}_{01} + \alpha_{22}v_{03}\bar{v}_{02} + \bar{\alpha}_{23}|v_{03}|^2) - i\phi_{33}(\alpha_{13}v_{03}\bar{v}_{01} + \alpha_{23}v_{03}\bar{v}_{02} + \alpha_{33}|v_{03}|^2), \\
\phi_{43,t}(t,k) =& \phi_{13}[\alpha_{11}(2k\bar{v}_{01} - i\bar{v}_{11}) + \bar{\alpha}_{12}(2k\bar{v}_{02} - i\bar{v}_{12}) + \bar{\alpha}_{13}(2k\bar{v}_{03} - i\bar{v}_{13})] \\
& + \phi_{23}[\alpha_{12}(2k\bar{v}_{01} - i\bar{v}_{11}) + \alpha_{22}(2k\bar{v}_{02} - i\bar{v}_{12}) + \bar{\alpha}_{23}(2k\bar{v}_{03} - i\bar{v}_{13})] \\
& + \phi_{33}[\alpha_{13}(2k\bar{v}_{01} - i\bar{v}_{11}) + \alpha_{23}(2k\bar{v}_{02} - i\bar{v}_{12}) + \alpha_{33}(2k\bar{v}_{03} - i\bar{v}_{13})] \\
& + i\phi_{43}[4k^2 + \alpha_{11}|v_{01}|^2 + \bar{\alpha}_{12}v_{01}\bar{v}_{02} + \bar{\alpha}_{13}v_{01}\bar{v}_{03} + \alpha_{12}v_{02}\bar{v}_{01} \\
& + \alpha_{22}|v_{02}|^2 + \bar{\alpha}_{23}v_{02}\bar{v}_{03} + \alpha_{13}v_{03}\bar{v}_{01} + \alpha_{23}v_{03}\bar{v}_{02} + \alpha_{33}|v_{03}|^2].
\end{aligned} \tag{161}$$

The fourth column of Eq. (149) with $\mu = \mu_3$ yields

$$\begin{aligned}
\phi_{14,t}(t,k) =& (2kv_{01} + iv_{11})\phi_{44} - i\phi_{14}(4k^2 + \alpha_{11}|v_{01}|^2 + \bar{\alpha}_{12}v_{01}\bar{v}_{02} + \bar{\alpha}_{13}v_{01}\bar{v}_{03}) \\
& - i\phi_{24}(\alpha_{12}|v_{01}|^2 + \alpha_{22}v_{01}\bar{v}_{02} + \bar{\alpha}_{23}v_{01}\bar{v}_{03}) - i\phi_{34}(\alpha_{13}|v_{01}|^2 + \alpha_{23}v_{01}\bar{v}_{02} + \alpha_{33}v_{01}\bar{v}_{03}), \\
\phi_{24,t}(t,k) =& (2kv_{02} + iv_{12})\phi_{44} - i\phi_{14}(\alpha_{11}v_{02}\bar{v}_{01} + \bar{\alpha}_{12}|v_{02}|^2 + \bar{\alpha}_{13}v_{02}\bar{v}_{03}) \\
& - i\phi_{24}(4k^2 + \alpha_{12}v_{02}\bar{v}_{01} + \alpha_{22}|v_{02}|^2 + \bar{\alpha}_{23}v_{02}\bar{v}_{03}) - i\phi_{34}(\alpha_{13}v_{02}\bar{v}_{01} + \alpha_{23}|v_{02}|^2 + \alpha_{33}v_{02}\bar{v}_{03}), \\
\phi_{34,t}(t,k) =& (2kv_{03} + iv_{13})\phi_{44} - i\phi_{14}(\alpha_{11}v_{03}\bar{v}_{01} + \bar{\alpha}_{12}v_{03}\bar{v}_{02} + \bar{\alpha}_{13}|v_{03}|^2) \\
& - i\phi_{24}(\alpha_{12}v_{03}\bar{v}_{01} + \alpha_{22}v_{03}\bar{v}_{02} + \bar{\alpha}_{23}|v_{03}|^2) - i\phi_{34}(4k^2 + \alpha_{13}v_{03}\bar{v}_{01} + \alpha_{23}v_{03}\bar{v}_{02} + \alpha_{33}|v_{03}|^2), \\
\phi_{44,t}(t,k) =& \phi_{14}[\alpha_{11}(2k\bar{v}_{01} - i\bar{v}_{11}) + \bar{\alpha}_{12}(2k\bar{v}_{02} - i\bar{v}_{12}) + \bar{\alpha}_{13}(2k\bar{v}_{03} - i\bar{v}_{13})] \\
& + \phi_{24}[\alpha_{12}(2k\bar{v}_{01} - i\bar{v}_{11}) + \alpha_{22}(2k\bar{v}_{02} - i\bar{v}_{12}) + \bar{\alpha}_{23}(2k\bar{v}_{03} - i\bar{v}_{13})] \\
& + \phi_{34}[\alpha_{13}(2k\bar{v}_{01} - i\bar{v}_{11}) + \alpha_{23}(2k\bar{v}_{02} - i\bar{v}_{12}) + \alpha_{33}(2k\bar{v}_{03} - i\bar{v}_{13})] \\
& + i\phi_{44}[\alpha_{11}|v_{01}|^2 + \bar{\alpha}_{12}v_{01}\bar{v}_{02} + \bar{\alpha}_{13}v_{01}\bar{v}_{03} + \alpha_{12}v_{02}\bar{v}_{01} \\
& + \alpha_{22}|v_{02}|^2 + \bar{\alpha}_{23}v_{02}\bar{v}_{03} + \alpha_{13}v_{03}\bar{v}_{01} + \alpha_{23}v_{03}\bar{v}_{02} + \alpha_{33}|v_{03}|^2].
\end{aligned} \tag{163}$$

Similarly, we can also obtain the asymptotic formulae for ϕ_{ij}, $i,j = 1,2,3,4$. Substituting these formulae into Eq. (148a) and using the assumption that the initial and boundary data are compatible at $x = 0$ and $x = L$, we find the asymptotic result (68) of $c_{14}(t,k)$ for $k \to \infty$. Similarly we can also show that Eqs. (69) and (70) hold for $c_{24}(t,k)$ and $c_{34}(t,k)$ as $k \to \infty$.

Similarly, we have the global relation (59) under the vanishing initial data as

$$\begin{aligned}c_{41}(t,k) = &-\Psi_{44}(t,k)(\alpha_{11}\bar\phi_{14}+\alpha_{12}\bar\phi_{24}+\alpha_{13}\bar\phi_{34})e^{-2ikL}\\&+\Psi_{41}(t,k)\big[\alpha_{11}(\alpha_{11}\bar\phi_{11}+\bar\alpha_{12}\bar\phi_{12}+\bar\alpha_{13}\bar\phi_{13})+\alpha_{12}(\alpha_{11}\bar\phi_{21}+\bar\alpha_{12}\bar\phi_{22}+\bar\alpha_{13}\bar\phi_{23})\\&+\alpha_{13}(\alpha_{11}\bar\phi_{31}+\bar\alpha_{12}\bar\phi_{32}+\bar\alpha_{13}\bar\phi_{33})\big]+\Psi_{42}(t,k)\big[\alpha_{11}(\alpha_{12}\bar\phi_{11}+\alpha_{22}\bar\phi_{12}+\bar\alpha_{23}\bar\phi_{13})\\&+\alpha_{12}(\alpha_{12}\bar\phi_{21}+\alpha_{22}\bar\phi_{22}+\bar\alpha_{23}\bar\phi_{23})+\alpha_{13}(\alpha_{12}\bar\phi_{31}+\alpha_{22}\bar\phi_{32}+\bar\alpha_{23}\bar\phi_{33})\big]\\&+\Psi_{43}(t,k)\big[\alpha_{11}(\alpha_{13}\bar\phi_{11}+\alpha_{23}\bar\phi_{12}+\alpha_{33}\bar\phi_{13})+\alpha_{12}(\alpha_{13}\bar\phi_{21}+\alpha_{23}\bar\phi_{22}+\alpha_{33}\bar\phi_{23})\\&+\alpha_{13}(\alpha_{13}\bar\phi_{31}+\alpha_{23}\bar\phi_{32}+\alpha_{33}\bar\phi_{33})\big],\end{aligned}\tag{164}$$

$$\begin{aligned}c_{42}(t,k) = &-\Psi_{44}(t,k)(\bar\alpha_{12}\bar\phi_{14}+\alpha_{22}\bar\phi_{24}+\alpha_{23}\bar\phi_{34})e^{-2ikL}\\&+\Psi_{41}(t,k)\big[\bar\alpha_{12}(\alpha_{11}\bar\phi_{11}+\bar\alpha_{12}\bar\phi_{12}+\bar\alpha_{13}\bar\phi_{13})+\alpha_{22}(\alpha_{11}\bar\phi_{21}+\bar\alpha_{12}\bar\phi_{22}+\bar\alpha_{13}\bar\phi_{23})\\&+\alpha_{23}(\alpha_{11}\bar\phi_{31}+\bar\alpha_{12}\bar\phi_{32}+\bar\alpha_{13}\bar\phi_{33})\big]+\Psi_{42}(t,k)\big[\bar\alpha_{12}(\alpha_{12}\bar\phi_{11}+\alpha_{22}\bar\phi_{12}+\bar\alpha_{23}\bar\phi_{13})\\&+\alpha_{22}(\alpha_{12}\bar\phi_{21}+\alpha_{22}\bar\phi_{22}+\bar\alpha_{23}\bar\phi_{23})+\alpha_{23}(\alpha_{12}\bar\phi_{31}+\alpha_{22}\bar\phi_{32}+\bar\alpha_{23}\bar\phi_{33})\big]\\&+\Psi_{43}(t,k)\big[\bar\alpha_{12}(\alpha_{13}\bar\phi_{11}+\alpha_{23}\bar\phi_{12}+\alpha_{33}\bar\phi_{13})+\alpha_{22}(\alpha_{13}\bar\phi_{21}+\alpha_{23}\bar\phi_{22}+\alpha_{33}\bar\phi_{23})\\&+\alpha_{23}(\alpha_{13}\bar\phi_{31}+\alpha_{23}\bar\phi_{32}+\alpha_{33}\bar\phi_{33})\big],\end{aligned}\tag{165}$$

$$\begin{aligned}c_{43}(t,k) = &-\Psi_{44}(t,k)(\bar\alpha_{13}\bar\phi_{14}+\bar\alpha_{23}\bar\phi_{24}+\alpha_{33}\bar\phi_{34})e^{-2ikL}\\&+\Psi_{41}(t,k)\big[\bar\alpha_{13}(\alpha_{11}\bar\phi_{11}+\bar\alpha_{12}\bar\phi_{12}+\bar\alpha_{13}\bar\phi_{13})+\bar\alpha_{23}(\alpha_{11}\bar\phi_{21}+\bar\alpha_{12}\bar\phi_{22}+\bar\alpha_{13}\bar\phi_{23})\\&+\alpha_{33}(\alpha_{11}\bar\phi_{31}+\bar\alpha_{12}\bar\phi_{32}+\bar\alpha_{13}\bar\phi_{33})\big]+\Psi_{42}(t,k)\big[\bar\alpha_{13}(\alpha_{12}\bar\phi_{11}+\alpha_{22}\bar\phi_{12}+\bar\alpha_{23}\bar\phi_{13})\\&+\bar\alpha_{23}(\alpha_{12}\bar\phi_{21}+\alpha_{22}\bar\phi_{22}+\bar\alpha_{23}\bar\phi_{23})+\alpha_{13}(\alpha_{12}\bar\phi_{31}+\alpha_{22}\bar\phi_{32}+\bar\alpha_{23}\bar\phi_{33})\big]\\&+\Psi_{43}(t,k)\big[\bar\alpha_{13}(\alpha_{13}\bar\phi_{11}+\alpha_{23}\bar\phi_{12}+\alpha_{33}\bar\phi_{13})+\bar\alpha_{23}(\alpha_{13}\bar\phi_{21}+\alpha_{23}\bar\phi_{22}+\alpha_{33}\bar\phi_{23})\\&+\alpha_{33}(\alpha_{13}\bar\phi_{31}+\alpha_{23}\bar\phi_{32}+\alpha_{33}\bar\phi_{33})\big],\end{aligned}\tag{165}$$

where $\bar\phi_{ij}=\bar\phi_{ij}(t,\bar k)=\overline{\phi_{ij}(t,\bar k)}$, such that we can show that Eqs. (71)–(73) hold for $c_{4j}(t,k)$, $j=1,2,3$ as $k\to\infty$. □

B. The proof of Proposition 4.2

Proof. We can show that Eqs. (75) and (76) hold by means of Eqs. (35) and (36) with T replaced by t, i.e. $S(k)=e^{-2ik^2t\widehat{\sigma}_4}\mu_2^{-1}(0,t,k)$ and $S_L(k)=e^{-2ik^2t\widehat{\sigma}_4}\mu_3^{-1}(L,t,k)$ and the symmetry relation (32). Moreover, Eqs. (77)–(80) for $\Psi_{ij}(t,k)$, $i,j=1,2,3,4$ can be obtained by using the Volteral integral equations of $\mu_2(0,t,k)$. Similarly, the expressions of $\phi_{ij}(t,k)$, $(i,j=1,2,3,4)$ can be found by means of the Volteral integral equations of $\mu_3(L,t,k)$.

In what follows we show that Eqs. (81)–(91) hold, i.e. the maps hold between Dirichlet and Neumann boundary conditions.

(i) The Cauchy's theorem is employed to study Eq. (62) to generate

$$
\begin{aligned}
i\pi \Psi_{44}^{(1)}(t) &= -\left(\int_{\partial D_2}+\int_{\partial D_4}\right)[\Psi_{44}(t,k)-1]\mathrm{d}k = \left(\int_{\partial D_1}+\int_{\partial D_3}\right)[\Psi_{44}(t,k)-1]\mathrm{d}k \\
&= \int_{\partial D_3}[\Psi_{44}(t,k)-1]\mathrm{d}k - \int_{\partial D_3}[\Psi_{44}(t,-k)-1]\mathrm{d}k = \int_{\partial D_3}\Psi_{44-}(t,k)\mathrm{d}k,
\end{aligned} \tag{166}
$$

and

$$
\begin{aligned}
i\pi \Psi_{14}^{(2)}(t) &= \left(\textstyle\int_{\partial D_1}+\int_{\partial D_3}\right)\left[k\Psi_{14}(t,k)+\tfrac{i}{2}u_{01}(t)\right]\mathrm{d}k = \int_{\partial D_3}\left[k\Psi_{14}(t,k)+\frac{i}{2}u_{01}(t)\right]_{-}\mathrm{d}k \\
&= \int_{\partial D_3^0}\left\{k\Psi_{14}(t,k)+\frac{i}{2}u_{01}(t)+\frac{2e^{-2ikL}}{\Sigma_{-}(k)}\left[k\Psi_{14}(t,k)+\frac{i}{2}u_{01}(t)\right]\right\}_{-}\mathrm{d}k + C_1(t) \\
&= \int_{\partial D_3^0}\frac{\Sigma_{+}}{\Sigma_{-}}(k\Psi_{14-}+iu_{01})\mathrm{d}k + C_1(t),
\end{aligned} \tag{167}
$$

where we have introduced the function $C_1(t)$ as

$$
C_1(t) = -\int_{\partial D_3^0}\left\{\frac{2e^{-2ikL}}{\Sigma_{-}}\left[k\Psi_{14}(t,k)+\frac{i}{2}u_{01}(t)\right]\right\}_{-}\mathrm{d}k.
$$

We use the global relation (148a) to further reduce $C_1(t)$ in the form

$$
\begin{aligned}
C_1(t) =& -\int_{\partial D_3^0}\left\{\frac{2e^{-2ikL}}{\Sigma_{-}}\left[k\Psi_{14}(t,k)+\frac{i}{2}u_{01}(t)\right]\right\}_{-}\mathrm{d}k \\
&= \int_{\partial D_3^0}\left\{\frac{2e^{-2ikL}}{\Sigma_{-}}\left[-kc_{14}+\Psi_{14}^{(1)}+\frac{\Psi_{14}^{(1)}\bar{\phi}_{44}^{(1)}}{k}-(\alpha_{11}\bar{\phi}_{41}^{(1)}+\bar{\alpha}_{12}\bar{\phi}_{42}^{(1)}+\bar{\alpha}_{13}\bar{\phi}_{43}^{(1)})e^{2ikL}\right]\right\}_{-}\mathrm{d}k \\
&-\int_{\partial D_3^0}\left\{\frac{2e^{-2ikL}}{}\Sigma_{-}\left[\frac{\Psi_{14}^{(1)}\bar{\phi}_{44}^{(1)}}{k}+\left(\alpha_{11}(k\bar{\phi}_{41}-\bar{\phi}_{41}^{(1)})+\bar{\alpha}_{12}(k\bar{\phi}_{42}-\bar{\phi}_{42}^{(1)})+\bar{\alpha}_{13}(k\bar{\phi}_{43}-\bar{\phi}_{43}^{(1)})\right)e^{2ikL}\right]\right\}_{-}\mathrm{d}k \\
&+\int_{\partial D_3^0}\left\{\frac{2ke^{-2ikL}}{\Sigma_{-}}\left[\Psi_{14}(\bar{\phi}_{44}-1)-\left[(\Psi_{11}-1)(\alpha_{11}\bar{\phi}_{41}+\bar{\alpha}_{12}\bar{\phi}_{42}+\bar{\alpha}_{13}\bar{\phi}_{43})\right.\right.\right. \\
&\left.\left.\left.+\Psi_{12}(\alpha_{12}\bar{\phi}_{41}+\alpha_{22}\bar{\phi}_{42}+\bar{\alpha}_{23}\bar{\phi}_{43})+\Psi_{13}(\alpha_{13}\bar{\phi}_{41}+\alpha_{23}\bar{\phi}_{42}+\alpha_{33}\bar{\phi}_{43})\right]e^{2ikL}\right]\right\}_{-}\mathrm{d}k.
\end{aligned} \tag{168}
$$

By applying the Cauchy's theorem and asymptotics (68) to Eq. (168), we find that $C_1(t)$ can reduce to

$$\begin{aligned}C_1(t) = & -i\pi\Psi_{14}^{(2)} - \int_{\partial D_3^0}\Big\{\frac{i}{2}u_{01}\bar{\phi}_{44-} + \frac{2i}{\Sigma_-}\big[\alpha_{11}\left(-ik\bar{\phi}_{41-} + \alpha_{11}v_{01} + \alpha_{12}v_{02} + \alpha_{13}v_{03}\right) \\ & +\bar{\alpha}_{12}\big(-ik\bar{\phi}_{42-} + \bar{\alpha}_{12}v_{01} + \alpha_{22}v_{02} + \alpha_{23}v_{03}\big) + \bar{\alpha}_{13}\left(-ik\bar{\phi}_{43-} + \bar{\alpha}_{13}v_{01} + \bar{\alpha}_{23}v_{02} + \alpha_{33}v_{03}\right)\big]\Big\}\,\mathrm{d}k \\ & + \int_{\partial D_3^0}\frac{2k}{\Sigma_-}\Big[\Psi_{14}(\bar{\phi}_{44}-1)e^{-2ikL} - (\Psi_{11}-1)(\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43}) \\ & -\Psi_{12}(\alpha_{12}\bar{\phi}_{41} + \alpha_{22}\bar{\phi}_{42} + \bar{\alpha}_{23}\bar{\phi}_{43}) - \Psi_{13}(\alpha_{13}\bar{\phi}_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43})\Big]_-\,\mathrm{d}k.\end{aligned} \tag{169}$$

It follows from Eqs. (167) and (169) that we have

$$\begin{aligned}2i\pi\Psi_{13}^{(2)}(t) = & \int_{\partial D_3^0}\Big[\frac{\Sigma_+}{\Sigma_-}(k\Psi_{14-} + iu_{01}) - \frac{i}{2}u_{01}\bar{\phi}_{44-}\Big]\,\mathrm{d}k \\ & + \int_{\partial D_3^0}\frac{2i}{\Sigma_-}\big[\alpha_{11}\left(-ik\bar{\phi}_{41-} + \alpha_{11}v_{01} + \bar{\alpha}_{12}v_{02} + \bar{\alpha}_{13}v_{03}\right) + \bar{\alpha}_{12}(-ik\bar{\phi}_{42-} + \alpha_{12}v_{01} \\ & +\alpha_{22}v_{02} + \bar{\alpha}_{23}v_{03}) + \bar{\alpha}_{13}\left(-ik\bar{\phi}_{43-} + \alpha_{13}v_{01} + \alpha_{23}v_{02} + \alpha_{33}v_{03}\right)\big]\,\mathrm{d}k \\ & + \int_{\partial D_3^0}\frac{2k}{\Sigma_-}\Big[\Psi_{14}(\bar{\phi}_{44}-1)e^{-2ikL} - (\Psi_{11}-1)(\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43}) \\ & -\Psi_{12}(\alpha_{12}\bar{\phi}_{41} + \alpha_{22}\bar{\phi}_{42} + \bar{\alpha}_{23}\bar{\phi}_{43}) - \Psi_{13}(\alpha_{13}\bar{\phi}_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43})\Big]_-\,\mathrm{d}k.\end{aligned} \tag{170}$$

Thus substituting Eqs. (166) and (170) into the third one of system (64), we can get Eq. (81). Similarly, we can also show that Eqs. (82) and (83) hold.

To use Eq. (67) to verify Eq. (84) for $v_{11}(t)$ we need to find these functions $\phi_{44}^{(1)}(t,k)$ and $\phi_{14}^{(2)}(t,k)$. Applying the Cauchy's theorem to Eq. (65), we have

$$\begin{aligned} & i\pi[\alpha_{11}\phi_{14}^{(2)} + \bar{\alpha}_{12}\phi_{24}^{(2)} + \bar{\alpha}_{13}\phi_{34}^{(2)}] \\ & = \int_{\partial D_3}\Big[\alpha_{11}(k\phi_{14}(t,k) - \phi_{14}^{(1)}) + \bar{\alpha}_{12}(k\phi_{24}(t,k) - \phi_{24}^{(1)}) + \bar{\alpha}_{13}(k\phi_{34}(t,k) - \phi_{34}^{(1)})\Big]_-\,\mathrm{d}k, \\ & = \int_{\partial D_3^0}\Big\{\alpha_{11}\Big[k\phi_{14}(t,k) - \phi_{14}^{(1)} - \frac{2e^{2ikL}}{\Sigma_-}(k\phi_{14} - \phi_{14}^{(1)})\Big] + \bar{\alpha}_{12}\Big[k\phi_{24}(t,k) - \phi_{24}^{(1)} - \frac{2e^{2ikL}}{\Sigma_-}(k\phi_{24} - \phi_{24}^{(1)})\Big] \\ & \quad + \bar{\alpha}_{13}\Big[k\phi_{34}(t,k) - \phi_{34}^{(1)} - \tfrac{2e^{2ikL}}{\Sigma_-}(k\phi_{34} - \phi_{34}^{(1)})\Big]\Big\}_-\,\mathrm{d}k + C_2(t), \\ & = \int_{\partial D_3^0} -\frac{\Sigma_+}{\Sigma_-}\Big[\alpha_{11}(k\phi_{14-} - 2\phi_{14}^{(1)}) + \bar{\alpha}_{12}(k\phi_{24-} - 2\phi_{24}^{(1)}) + \bar{\alpha}_{13}(k\phi_{34-} - 2\phi_{34}^{(1)})\Big]\,\mathrm{d}k + C_2(t),\end{aligned} \tag{171}$$

where we have introduced the function $C_2(t)$ as

$$C_2(t) = \int_{\partial D_3^0}\Big\{\frac{2e^{2ikL}}{\Sigma_-}\Big[\alpha_{11}(k\phi_{14} - \phi_{14}^{(1)}) + \bar{\alpha}_{12}(k\phi_{24} - \phi_{24}^{(1)}) + \bar{\alpha}_{13}(k\phi_{34} - \phi_{34}^{(1)})\Big]\Big\}_-\,\mathrm{d}k.$$

We use the global relation (163) to further reduce $C_2(t)$ in the form

$$
\begin{aligned}
C_2(t) = &\int_{\partial D_3^0} \Bigg\{ \frac{2}{\Sigma_-} \Big[-k\bar{c}_{41}(t,\bar{k}) - \Big(\alpha_{11}\phi_{14}^{(1)} + \bar{\alpha}_{12}\phi_{24}^{(1)} + \bar{\alpha}_{13}\phi_{34}^{(1)}\Big) e^{2ikL} \\
&-\bar{\Psi}_{44}^{(1)} \Big(\alpha_{11}\phi_{14}^{(1)} + \bar{\alpha}_{12}\phi_{24}^{(1)} + \bar{\alpha}_{13}\phi_{34}^{(1)}\Big) \tfrac{e^{2ikL}}{k} + (\alpha_{11}^2 + |\alpha_{12}|^2 + |\alpha_{13}|^2)\bar{\Psi}_{41}^{(1)} \\
&+(\alpha_{11}\bar{\alpha}_{12} + \bar{\alpha}_{12}\alpha_{22} + \bar{\alpha}_{13}\alpha_{23})\bar{\Psi}_{42}^{(1)} + (\alpha_{11}\bar{\alpha}_{13} + \bar{\alpha}_{12}\bar{\alpha}_{23} + \bar{\alpha}_{13}\alpha_{33})\Psi_{43}^{(1)} \Big] \Bigg\}_- \mathrm{d}k \\
&+ \int_{\partial D_3^0} \Bigg\{ \frac{2}{\Sigma_-} \Bigg[\bar{\Psi}_{44}^{(1)} \Big(\alpha_{11}\phi_{14}^{(1)} + \bar{\alpha}_{12}\phi_{24}^{(1)} + \bar{\alpha}_{13}\phi_{34}^{(1)}\Big) \frac{e^{2ikL}}{k} \\
&+(\alpha_{11}^2 + |\alpha_{12}|^2 + |\alpha_{13}|^2)(k\bar{\Psi}_{41} - \bar{\Psi}_{41}^{(1)}) + (\alpha_{11}\bar{\alpha}_{12} + \bar{\alpha}_{12}\alpha_{22} + \bar{\alpha}_{13}\alpha_{23})(k\bar{\Psi}_{42} - \bar{\Psi}_{42}^{(1)}) \\
&+(\alpha_{11}\bar{\alpha}_{13} + \bar{\alpha}_{12}\bar{\alpha}_{23} + \bar{\alpha}_{13}\alpha_{33})(k\bar{\Psi}_{43} - \bar{\Psi}_{43}^{(1)}) \Bigg] \Bigg\}_- \mathrm{d}k \\
&+ \int_{\partial D_3^0} \frac{2k}{\Sigma_-} \Big\{ (1-\bar{\Psi}_{44})(\alpha_{11}\phi_{14} + \bar{\alpha}_{12}\phi_{24} + \bar{\alpha}_{13}\phi_{34})e^{2ikL} \\
&+\bar{\Psi}_{41}\big[\alpha_{11}(\alpha_{11}(\phi_{11}-1) + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \bar{\alpha}_{12}(\alpha_{11}\phi_{21} + \alpha_{12}(\phi_{22}-1) + \alpha_{13}\phi_{23}) \\
&+\bar{\alpha}_{13}(\alpha_{11}\phi_{31} + \alpha_{12}\phi_{32} + \alpha_{13}(\phi_{33}-1))\big] + \bar{\Psi}_{42}\big[\alpha_{11}(\bar{\alpha}_{12}(\phi_{11}-1) + \alpha_{22}\phi_{12} + \alpha_{23}\phi_{13}) \\
&+\bar{\alpha}_{12}(\bar{\alpha}_{12}\phi_{21} + \alpha_{22}(\phi_{22}-1) + \alpha_{23}\phi_{23}) + \bar{\alpha}_{13}(\bar{\alpha}_{12}\phi_{31} + \alpha_{22}\phi_{32} + \alpha_{23}(\phi_{33}-1))\big] \\
&+\bar{\Psi}_{43}\big[\alpha_{11}(\bar{\alpha}_{13}(\phi_{11}-1) + \bar{\alpha}_{23}\phi_{12} + \bar{\alpha}_{33}\phi_{13}) + \bar{\alpha}_{12}(\bar{\alpha}_{13}\phi_{21} + \bar{\alpha}_{23}(\phi_{22}-1) + \alpha_{33}\phi_{23}) \\
&+\bar{\alpha}_{13}(\bar{\alpha}_{13}\phi_{31} + \bar{\alpha}_{23}\phi_{32} + \alpha_{33}(\phi_{33}-1))\big]\Big\}_- \mathrm{d}k.
\end{aligned} \tag{172}
$$

We need to further reduce $C_2(t)$ by using the asymptotics (71) and the Cauchy's theorem such that we have $C_2(t)$ in the form

$$
C_2(t) = -i\pi[\alpha_{11}\phi_{14}^{(2)} + \bar{\alpha}_{12}\phi_{24}^{(2)} + \bar{\alpha}_{13}\phi_{34}^{(2)}] - \int_{\partial D_3^0} \frac{i}{2}\bar{\Psi}_{44-}^{(1)}(\alpha_{11}v_{01} + \bar{\alpha}_{12}v_{02} + \bar{\alpha}_{13}v_{03})\mathrm{d}k + I_1(t). \tag{173}
$$

It follows from Eqs. (171) and (173) that we have

$$
\begin{aligned}
2i\pi[\alpha_{11}\phi_{14}^{(2)} &+ \bar{\alpha}_{12}\phi_{24}^{(2)} + \bar{\alpha}_{13}\phi_{34}^{(2)}] \\
&= -\int_{\partial D_3^0} \frac{\Sigma_+(k)}{\Sigma_-(k)} \big[\alpha_{11}(k\phi_{14-} + iv_{01}) + \bar{\alpha}_{12}(k\phi_{24-} + iv_{02} + \bar{\alpha}_{13}(k\phi_{34-} + iv_{03})\big]\,\mathrm{d}k \\
&\quad - \int_{\partial D_3^0} \frac{i}{2}\bar{\Psi}_{44-}^{(1)}(\alpha_{11}v_{01} + \bar{\alpha}_{12}v_{02} + \bar{\alpha}_{13}v_{03})\mathrm{d}k + I_1(t),
\end{aligned} \tag{174}
$$

where $I_1(t)$ is given by Eq. (86). Similarly, we can also the expressions of $i\pi[\alpha_{12}\phi_{14}^{(2)} + \alpha_{22}\phi_{24}^{(2)} + \bar{\alpha}_{23}\phi_{34}^{(2)}]$ and $2i\pi[\alpha_{13}\phi_{14}^{(2)} + \alpha_{23}\phi_{24}^{(2)} + \alpha_{33}\phi_{34}^{(2)}]$ such that we can show that Eq. (84) holds.

(ii) We now deduce the Dirichlet boundary problem given by Eqs. (88)–(90) at $x = 0$ from the Neumann boundary problem. It follows from the first one of Eq. (64) that $u_{01}(t)$ can be expressed by means of $\Psi_{14}^{(1)}$. Applying the Cauchy's theorem to Eq. (62) yields

$$\begin{aligned} i\pi\Psi_{14}^{(1)}(t) &= \left(\int_{\partial D_1} + \int_{\partial D_3}\right)\Psi_{14}(t,k)\mathrm{d}k = \int_{\partial D_3}\Psi_{14-}(t,k)\mathrm{d}k \\ &= \int_{\partial D_3^0}\left[\Psi_{14-}(t,k) + \frac{2}{\Sigma_-(k)}(e^{-2ikL}\Psi_{14})_+\right]\mathrm{d}k + C_3(t) = \int_{\partial D_3^0}\frac{\Sigma_+(k)}{\Sigma_-(k)}\Psi_{14+}\mathrm{d}k + C_3(t), \end{aligned} \tag{175}$$

where $C_3(t)$ is defined by

$$C_3(t) = -\int_{\partial D_3^0}\frac{2}{\Sigma_-(k)}(e^{-2ikL}\Psi_{14})_+\mathrm{d}k. \tag{176}$$

By applying the global relation (148a), the Cauchy's theorem and asymptotics (68) to Eq. (176), we find

$$\begin{aligned} C_3(t) =& -\int_{\partial D_3^0}\frac{2}{\Sigma_-}(e^{-2ikL}\Psi_{14})_+\mathrm{d}k \\ =& \int_{\partial D_3^0}\frac{2}{\Sigma_-}\left[-c_{14}e^{-2ikL} - (\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43})\right]_+\mathrm{d}k \\ &+\int_{\partial D_3^0}\frac{2}{\Sigma_-}\Big[\Psi_{14}(\bar{\phi}_{44}-1)e^{-2ikL} - \big[(\Psi_{11}-1)(\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43}) \\ &+\Psi_{12}(\alpha_{12}\bar{\phi}_{41} + \alpha_{22}\bar{\phi}_{42} + \bar{\alpha}_{23}\bar{\phi}_{43}) + \Psi_{13}(\alpha_{13}\bar{\phi}_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43})\big]\Big]_+\mathrm{d}k \\ =& -i\pi\Psi_{14}^{(1)} - \int_{\partial D_3^0}\frac{2}{\Sigma_-}(\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43})_+\mathrm{d}k \\ &+\int_{\partial D_3^0}\frac{2}{\Sigma_-}\Big\{\Psi_{14}(\bar{\phi}_{44}-1)e^{-2ikL} - \big[(\Psi_{11}-1)(\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43}) \\ &+\Psi_{12}(\alpha_{12}\bar{\phi}_{41} + \alpha_{22}\bar{\phi}_{42} + \bar{\alpha}_{23}\bar{\phi}_{43}) + \Psi_{13}(\alpha_{13}\bar{\phi}_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43})\big]\Big\}_+\mathrm{d}k. \end{aligned} \tag{177}$$

Eqs. (175) and (178) imply that

$$\begin{aligned} 2i\pi\Psi_{14}^{(1)}(t) =& \int_{\partial D_3^0}\left[\frac{\Sigma_+(k)}{\Sigma_-(k)}\Psi_{14+} - \frac{2}{\Sigma_-}(\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43})_+\right]\mathrm{d}k \\ &+\int_{\partial D_3^0}\frac{2}{\Sigma_-}\Big\{\Psi_{14}(\bar{\phi}_{44}-1)e^{-2ikL} - \big[(\Psi_{11}-1)(\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43}) \\ &+\Psi_{12}(\alpha_{12}\bar{\phi}_{41} + \alpha_{22}\bar{\phi}_{42} + \bar{\alpha}_{23}\bar{\phi}_{43}) + \Psi_{13}(\alpha_{13}\bar{\phi}_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43})\big]\Big\}_+\mathrm{d}k. \end{aligned} \tag{178}$$

Thus, substituting Eq. (178) into the first one of Eq. (64) yields Eq. (88). Similarly, by applying the expressions of $\Psi_{24}^{(1)}(t)$ and $\Psi_{34}^{(1)}(t)$ to the second one of Eq. (64), we can show that Eqs. (89) and (90) hold.

Similarly we also can show that the Dirichlet boundary problem given by Eq. (91) at $x = L$ holds from the Neumann boundary problem. □

C. The proof of Theorem 5.3

Proof. By means of the global relation (59) and Proposition 5.1, we can show that the spectral functions $S(k)$ and $S_L(k)$ are defined by Eqs. (75) and (76) with $\Psi(t,k)$ and $\phi(t,k)$ given by Eq. (108a) and (108b), respectively.

(i) We firstly consider the Dirichlet problem. It follows from the global relation (59) with the vanishing initial data

$$c(t,k) = \mu_2(0,t,k)e^{ikL\hat{\sigma}_4}\mu_3^{-1}(L,t,k), \tag{179}$$

that we find

$$\tilde{c}_{j4}(t,k) = -\tilde{\Psi}_{3\times 3}\mathcal{M}^T\bar{\tilde{\phi}}_{4j}^T(t,\bar{k})e^{2ikL} + \tilde{\Psi}_{j4}\bar{\tilde{\phi}}_{44}(t,\bar{k}), \tag{180a}$$

$$\tilde{c}_{4j}(t,k) = \tilde{\Psi}_{4j}\mathcal{M}^T\bar{\tilde{\phi}}_{3\times 3}^T(t,\bar{k})\mathcal{M}^T - \tilde{\Psi}_{44}\bar{\tilde{\phi}}_{j4}^T(t,\bar{k})\mathcal{M}^Te^{-2ikL}, \tag{180b}$$

Substituting Eqs. (118a) and (118b) into Eq. (181a) yields

$$\mathcal{M}^T\bar{\hat{\tilde{\mathcal{L}}}}_{4j}^Te^{2ikL} - \hat{\tilde{L}}_{j4} = k\hat{\tilde{G}}_{j4} - k\mathcal{M}^T\bar{\hat{\tilde{\mathcal{G}}}}_{4j}^Te^{2ikL} + \tilde{F}_{j4}(t,k) - \tilde{c}_{j4}(t,k), \tag{181}$$

where $\tilde{F}_{j4}(t,k)$ is given by Eq. (120). Eq. (181) with $k \to -k$ further yields

$$\mathcal{M}^T\bar{\hat{\tilde{\mathcal{L}}}}_{4j}^Te^{-2ikL} - \hat{\tilde{L}}_{j4} = -k\hat{\tilde{G}}_{j4} + k\mathcal{M}^T\bar{\hat{\tilde{\mathcal{G}}}}_{4j}^Te^{-2ikL} + \tilde{F}_{j4}(t,-k) - \tilde{c}_{j4}(t,-k). \tag{182}$$

It follows from Eqs. (181) and (182) that we get

$$\hat{\tilde{L}}_{j4} = \frac{k\Sigma_+}{\Sigma_-}\hat{\tilde{G}}_{j4} - \frac{2k}{\Sigma_-}\mathcal{M}^T\bar{\hat{\tilde{\mathcal{G}}}}_{4j}^T + \frac{1}{\Sigma_-}\left\{[\tilde{F}_{j4}(t,k) - \tilde{c}_{j4}(t,k)]e^{-2ikL}\right\}_- . \tag{183}$$

Multiplying Eq. (183) by $ke^{4ik^2(t-\tau)}$ with $0 < \tau < t$ and integrating them along ∂D_1^0 with respect to dk, respectively, can yield

$$\begin{aligned}\int_{\partial D_1^0} ke^{4ik^2(t-\tau)}\hat{\tilde{L}}_{j4}\mathrm{d}k = &\int_{\partial D_1^0} e^{4ik^2(t-\tau)}\frac{k^2\Sigma_+}{\Sigma_-}\hat{\tilde{G}}_{j4}\mathrm{d}k - \int_{\partial D_1^0} e^{4ik^2(t-\tau)}\frac{2k^2}{\Sigma_-}\mathcal{M}^T\bar{\hat{\tilde{\mathcal{G}}}}_{4j}^T\mathrm{d}k \\ &+ \int_{\partial D_1^0}\frac{ke^{4ik^2(t-\tau)}}{\Sigma_-}[\tilde{F}_{j4}(t,k)e^{-2ikL}]_-\mathrm{d}k,\end{aligned} \tag{184}$$

where

$$\int_{\partial D_1^0} ke^{4ik^2(t-\tau)}\tilde{c}_{j4-}(t,k)\mathrm{d}k = \int_{\partial D_1^0} ke^{4ik^2(t-\tau)}(\tilde{c}_{j4}(t,k)e^{-2ikL})_-\mathrm{d}k = 0$$

in terms of their analytical properties in D_1^0.

Based on these conditions given by Eqs. (123) and (124), Eq. (184) can become

$$\begin{aligned}\frac{\pi}{2}\tilde{L}_{j4}(t,2\tau-t) =& 2\int_{\partial D_1^0}\frac{k^2\Sigma_+}{\Sigma_-}\left[\int_0^\tau e^{4ik^2(t-\tau)}\tilde{G}_{j4}(t,2s-t)\mathrm{d}s-\frac{\tilde{G}_{j4}(t,2\tau-t)}{4ik^2}\right]\mathrm{d}k\\ &-4\int_{\partial D_1^0}\frac{k^2\mathcal{M}^T}{\Sigma_-}\left[\int_0^\tau e^{4ik^2(t-\tau)}\bar{\tilde{\mathcal{G}}}_{4j}^T(t,2s-t)\mathrm{d}s-\frac{\bar{\tilde{\mathcal{G}}}_{4j}^T(t,2\tau-t)}{4ik^2}\right]\mathrm{d}k\\ &+\int_{\partial D_1^0}\frac{k}{\Sigma_-}e^{4ik^2(t-\tau)}[\tilde{F}_{j4}(t,k)e^{-2ikL}]_-\mathrm{d}k.\end{aligned}\tag{185}$$

We choose the limit $\tau\to t$ of Eq. (185) with the initial data (110) and use Proposition 5.2 to find

$$\begin{aligned}\frac{\pi}{2}\tilde{L}_{j4}(t,t) =& 2\lim_{\tau\to t}\int_{\partial D_1^0}\frac{k^2\Sigma_+}{\Sigma_-}\left[\int_0^\tau e^{4ik^2(t-\tau)}\tilde{G}_{j4}(t,2\tau-t)d\tau-\frac{\tilde{G}_{j4}(t,2\tau-t)}{4ik^2}\right]\mathrm{d}k\\ &-4\lim_{\tau\to t}\int_{\partial D_1^0}\frac{k^2\mathcal{M}^T}{\Sigma_-}\left[\int_0^\tau e^{4ik^2(t-\tau)}\bar{\tilde{\mathcal{G}}}_{4j}^T(t,2s-t)ds-\frac{\bar{\tilde{\mathcal{G}}}_{4j}^T(t,2\tau-t)}{4ik^2}\right]\mathrm{d}k\\ &+\lim_{\tau\to t}\int_{\partial D_1^0}\frac{k}{\Sigma_-}e^{4ik^2(t-\tau)}[\tilde{F}_{j4}(t,k)e^{-2ikL}]_-\mathrm{d}k\\ =&\int_{\partial D_1^0}\left\{\frac{\Sigma_+}{\Sigma_-}\left[k^2\hat{\tilde{G}}_{j4}(t,t)+\frac{i}{2}\tilde{G}_{j4}(t,t)\right]-\frac{2\mathcal{M}^T}{\Sigma_-}\left[k^2\hat{\bar{\tilde{\mathcal{G}}}}_{4j}^T(t,t)+\frac{i}{2}\bar{\tilde{\mathcal{G}}}_{4j}^T(t,t)\right]\right.\\ &\left.+\frac{ik}{2}u_0^T\left(\hat{\tilde{G}}_{44}-\hat{\bar{\tilde{\mathcal{G}}}}_{44}\right)+\frac{k}{\Sigma_-}\left(\tilde{F}_{j4}e^{-2ikL}\right)_-\right\}\mathrm{d}k.\end{aligned}\tag{186}$$

Since the initial data (110) are of the form

$$\tilde{L}_{j4}(t,t)=\frac{i}{2}u_1^T(t)=\frac{i}{2}(u_{11}(t),u_{12}(t),u_{13}(t))^T,\tag{187}$$

thus we know that Eq. (128a) holds by means of Eqs. (186) and (187).

To show Eq. (128b) we rewrite Eq. (180b) in the form

$$\bar{\tilde{c}}_{4j}^T(t,\bar{k})=\mathcal{M}^T\tilde{\phi}_{3\times 3}\mathcal{M}^T\bar{\tilde{\Psi}}_{4j}^T(t,\bar{k})-\mathcal{M}^T\tilde{\phi}_{j4}\bar{\tilde{\Psi}}_{44}^T(t,\bar{k})e^{2ikL}.\tag{188}$$

We substitute Eqs. (118a) and (118b) into Eq. (188) to have

$$-\hat{\bar{\tilde{L}}}_{4j}^T+\mathcal{M}^T\hat{\tilde{\mathcal{L}}}_{j4}e^{2ikL}=k\hat{\bar{\tilde{G}}}_{4j}^T-k\mathcal{M}^T\hat{\tilde{\mathcal{G}}}_{j4}e^{2ikL}+\tilde{F}_{4j}(t,k)-\bar{\tilde{c}}_{4j}^T(t,\bar{k}),\tag{189}$$

where $\tilde{F}_{4j}(t,k)$ is given by Eq. (121). Eq. (189) with $k\to -k$ yields

$$-\hat{\bar{\tilde{L}}}_{4j}^T+\mathcal{M}^T\hat{\tilde{\mathcal{L}}}_{j4}e^{-2ikL}=-k\hat{\bar{\tilde{G}}}_{4j}^T+k\mathcal{M}^T\hat{\tilde{\mathcal{G}}}_{j4}e^{-2ikL}+\tilde{F}_{4j}(t,-k)-\bar{\tilde{c}}_{4j}^T(t,-\bar{k}).\tag{190}$$

It follows from Eqs. (189) and (190) that we have

$$\mathcal{M}^T\hat{\tilde{\mathcal{L}}}_{j4}=\frac{2k}{\Sigma_-}\hat{\bar{\tilde{G}}}_{4j}^T-\frac{k\Sigma_+}{\Sigma_-}\mathcal{M}^T\hat{\tilde{\mathcal{G}}}_{j4}+\frac{1}{\Sigma_-}[\tilde{F}_{4j}(t,k)-\bar{\tilde{c}}_{4j}^T(t,\bar{k})]_-.\tag{191}$$

We multiply Eq. (191) by $ke^{4ik^2(t-\tau)}$ with $0<\tau<t$, integrate them along ∂D_1^0 with respect to dk and use these conditions given by Eqs. (123) and (124) to yield

$$\begin{aligned}\frac{\pi}{2}\mathcal{M}^T\tilde{\mathcal{L}}_{j4}(t,2\tau-t) =& -2\int_{\partial D_1^0}\frac{k^2\Sigma_+}{\Sigma_-}\mathcal{M}^T\left[\int_0^\tau e^{4ik^2(t-\tau)}\tilde{\mathcal{G}}_{j4}(t,2s-t)\mathrm{d}s-\frac{\tilde{\mathcal{G}}_{j4}(t,2\tau-t)}{4ik^2}\right]\mathrm{d}k\\ &+4\int_{\partial D_1^0}\frac{k^2}{\Sigma_-}\left[\int_0^\tau e^{4ik^2(t-\tau)}\bar{\tilde{G}}_{4j}^T(t,2s-t)\mathrm{d}s-\frac{\bar{\tilde{G}}_{4j}^T(t,2\tau-t)}{4ik^2}\right]\mathrm{d}k\\ &+\int_{\partial D_1^0}\frac{k}{\Sigma_-}e^{4ik^2(t-\tau)}\tilde{F}_{4j-}(t,k)\mathrm{d}k,\end{aligned}\tag{192}$$

where we have used the relation

$$\int_{\partial D_1^0}\frac{k}{\Sigma_-}e^{4ik^2(t-\tau)}\bar{\tilde{c}}_{4j-}^T(t,\bar{k})\mathrm{d}k=0$$

due to the analytical property of the integrand in D_1^0.

We consider the limit $\tau\to t$ of Eq. (192) with the initial data (110) and use Proposition 5.2 to find

$$\begin{aligned}\frac{\pi}{2}\mathcal{M}^T\tilde{\mathcal{L}}_{j4}(t,t) = \int_{\partial D_1^0}\Big\{&-\tfrac{\Sigma_+}{\Sigma_-}\mathcal{M}^T\left[k^2\hat{\tilde{\mathcal{G}}}_{j4}(t,t)+\tfrac{i}{2}\tilde{\mathcal{G}}_{j4}(t,t)\right]+\frac{2}{\Sigma_-}\left[k^2\hat{\bar{\tilde{G}}}_{4j}^T(t,t)+\frac{i}{2}\bar{\tilde{G}}_{4j}^T(t,t)\right]\\ &+\frac{ik}{2}\mathcal{M}^Tv_0^T\left(\hat{\tilde{\mathcal{G}}}_{44}-\hat{\bar{\tilde{G}}}_{44}\right)+\frac{k}{\Sigma_-}\tilde{F}_{4j-}(t,k)\Big\}\mathrm{d}k.\end{aligned}\tag{193}$$

Since the initial conditions are of the form

$$\tilde{\mathcal{L}}_{j4}(t,t)=\frac{i}{2}v_1^T(t)=\frac{i}{2}(v_{11}(t),v_{12}(t),v_{13}(t))^T,\tag{194}$$

thus we get Eq. (128b) by combining Eqs. (194) and (195).

(ii) We now turn to consider the Neumann problem. It follows from Eqs (181), (182), (189) and (190) that we have

$$\hat{\tilde{G}}_{j4}=\frac{1}{k\Sigma_-}\left\{\Sigma_+\hat{\tilde{L}}_{j4}-2\mathcal{M}^T\hat{\bar{\tilde{\mathcal{L}}}}_{4j}^T+\left[(\tilde{F}_{j4}(t,k)-\tilde{c}_{j4}(t,k))e^{-2ikL}\right]_+\right\},\tag{195a}$$

$$\hat{\tilde{\mathcal{G}}}_{j4}=\frac{1}{k\Sigma_-}\left\{2\mathcal{M}^T\hat{\bar{\tilde{L}}}_{j4}^T-\Sigma_+\hat{\tilde{\mathcal{L}}}_{j4}+\mathcal{M}^T\left[\tilde{F}_{4j}(t,k)-\bar{\tilde{c}}_{44}^T(t,\bar{k})\right]_+\right\}.\tag{195b}$$

We multiply Eqs. (195a) and (195b) by $ke^{4ik^2(t-\tau)}$ with $0<\tau<t$, integrate them along ∂D_1^0 with respect to dk and use these conditions given by Eqs. (123) and (124) to yield

$$\begin{aligned}
\frac{\pi}{2}\tilde{G}_{j4}(t,2\tau-t) =& \int_{\partial D_1^0} \frac{2\Sigma_+}{\Sigma_-}\left[\int_0^\tau e^{4ik^2(t-\tau)}\tilde{L}_{j4}(t,2s-t)\mathrm{d}s - \frac{\tilde{L}_{j4}(t,2\tau-t)}{4ik^2}\right]\mathrm{d}k \\
&- \int_{\partial D_1^0} \frac{4\mathcal{M}^T}{\Sigma_-}\left[\int_0^\tau e^{4ik^2(t-\tau)}\bar{\tilde{\mathcal{L}}}_{4j}^T(t,2s-t)\mathrm{d}s - \frac{\bar{\tilde{\mathcal{L}}}_{4j}^T(t,2\tau-t)}{4ik^2}\right]\mathrm{d}k \\
&+ \int_{\partial D_1^0} \frac{e^{4ik^2(t-\tau)}}{\Sigma_-}(\tilde{F}_{j4}e^{-2ikL})_+\mathrm{d}k,
\end{aligned} \tag{196a}$$

$$\begin{aligned}
\frac{\pi}{2}\tilde{\mathcal{G}}_{j4}(t,2\tau-t) =& \int_{\partial D_1^0} \frac{4\mathcal{M}^T}{\Sigma_-}\left[\int_0^\tau e^{4ik^2(t-\tau)}\bar{\tilde{L}}_{j4}^T(t,2s-t)\mathrm{d}s - \frac{\bar{\tilde{L}}_{j4}^T(t,2\tau-t)}{4ik^2}\right]\mathrm{d}k \\
&- \int_{\partial D_1^0} \frac{2}{\Sigma_-}\left[\int_0^\tau e^{4ik^2(t-\tau)}\tilde{\mathcal{L}}_{4j}(t,2s-t)\mathrm{d}s - \frac{\tilde{\mathcal{L}}_{4j}(t,2\tau-t)}{4ik^2}\right]\mathrm{d}k \\
&+ \int_{\partial D_1^0} \frac{\mathcal{M}^T}{\Sigma_-}e^{4ik^2(t-\tau)}\tilde{F}_{4j+}\mathrm{d}k,
\end{aligned} \tag{196b}$$

where we have used the analytical property of the matrix-valued functions

$$\int_{\partial D_1^0} \frac{1}{\Sigma_-}e^{4ik^2(t-\tau)}(\tilde{c}_{j4}(t,k)e^{-2ikL})_+\mathrm{d}k = \int_{\partial D_1^0} \frac{1}{\Sigma_-}e^{4ik^2(t-\tau)}\bar{\tilde{c}}_{4j+}^T(t,\bar{k})\mathrm{d}k = 0.$$

We consider the limits $\tau \to t$ of Eqs. (197a) and (197b) with the initial data (110) and use Proposition 5.2 to find

$$\frac{\pi}{2}\tilde{G}_{j4}(t,t) = \int_{\partial D_1^0}\left[\frac{\Sigma_+}{\Sigma_-}\hat{\tilde{L}}_{j4} - \frac{2\mathcal{M}^T}{\Sigma_-}\hat{\bar{\tilde{\mathcal{L}}}}_{4j}^T + \frac{1}{\Sigma_-}(\tilde{F}_{j4}e^{-2ikL})_+\right]\mathrm{d}k, \tag{197a}$$

$$\frac{\pi}{2}\tilde{\mathcal{G}}_{j4}(t,t) = \int_{\partial D_1^0}\left(\frac{2\mathcal{M}^T}{\Sigma_-}\hat{\bar{\tilde{L}}}_{j4}^T - \frac{1}{\Sigma_-}\hat{\tilde{\mathcal{L}}}_{4j} + \frac{\mathcal{M}^T}{\Sigma_-}\tilde{F}_{4j+}\right)\mathrm{d}k. \tag{197b}$$

Since the initial conditions are of the forms

$$\begin{aligned}
\tilde{G}_{j4}(t,t) &= u_0^T(t) = (u_{01}(t),u_{02}(t),u_{03}(t))^T, \\
\tilde{\mathcal{G}}_{j4}(t,t) &= v_0^T(t) = (v_{01}(t),v_{02}(t),v_{03}(t))^T,
\end{aligned} \tag{198}$$

thus we can find Eqs. (129a) and (129b) using Eqs. (198a) and (198b). This completes the proof of the Theorem. □