THE DECOMPOSITION OF DIRECT PRODUCTS OF IRREDUCIBLE REPRESENTATIONS OF SU (3)

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INTRODUCTION

In what follows, most of the information will be brought without proofs. Although the methods are exposed for cases relevant to the octet model only, they may be generalized for any semi-simple group in quite a simple fashion.

1. THE ADJOINT REPRESENTATION OF SU(3)

This group is generated by the following infinitesimal operators:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$E^1_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E^2_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E^1_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E^3_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E^3_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E^2_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The operators $T_z = (1/2)H_1$, $T_+ = E^1_2$ and $T_- = E^2_1$ generate the isospin **group.** The H's and the E's operate on covariant as well as on contravariant **vectors.** Let x^1, x^2 and x^3 be the basic contravariant vectors and y_1, y_2 and y_3 the basic covariant ones. The results are shown in Table I.

The results of all other operations are 0.

The representations of SU(3) on the x's and the y's are contravariant to each other.

Consider the direct product of the x-space and the y-space; this constitutes a basis for another representation of SU(3) which is, however, reducible. In order to carry on the reduction one has to know the way infinitesimal operators act on a product. The rule is:

$$O \cdot (v_1 v_2) = (O \cdot v_1) v_2 + v_1 (O \cdot v_2). \qquad (1)$$

O an infinitesimal operator similar to the operation of a derivative; the extension to products with any number of factors is obvious. Choose the following basis:

$$I = \sqrt{(1/3)}(x^1 y_1 + x^2 y_2 + x^3 y_3);$$

$$a_1 = x^1 y_3;\quad a_2 = x^2 y_3;\quad a_3 = x^1 y_2;\quad a_4 = \sqrt{(1/6)}(x^1 y_1 + x^2 y_2 - 2x^3 y_3);\quad (2)$$

$$a_5 = \sqrt{(1/2)}(x^1 y_1 - x^2 y_2);\quad a_6 = x^2 y_1;\quad a_7 = x^3 y_2;\quad a_8 = x^3 y_1.$$

TABLE I

EFFECT OF THE OPERATORS H AND E ON VECTORS x AND y

$E_3^1 x^3 = x^1$	$E_3^1 y_1 = -y_3$	$H_1 x^1 = x^1$	$H_1 y_1 = -y_1$
$E_3^2 x^3 = x^2$	$E_3^2 y_2 = -y_3$	$H_1 x^2 = -x^2$	$H_1 y_2 = y_2$
$E_2^1 x^2 = x^1$	$E_2^1 y_1 = -y_2$	$H_2 x^2 = x^2$	$H_2 y_2 = -y_2$
$E_1^2 x^1 = x^2$	$E_1^2 y_2 = -y_1$	$H_2 x^3 = -x^3$	$H_2 y_3 = y_3$
$E_2^3 x^2 = x^3$	$E_2^3 y_3 = -y_2$		
$E_1^3 x^1 = x^3$	$E_1^3 y_3 = -y_1$		

It follows from (1) that the vector I is invariant under SU(3). Similarly, the vectors $a_i (1 \le i \le 8)$ span a space which is irreducible under SU(3). The effect of the H's and the E's in this 8-dimensional space is shown in Table II.

TABLE II

EFFECT OF THE OPERATORS H AND E IN A 8-DIMENSIONAL SPACE

E_3^1	$a_4 \to -\sqrt{(3/2)}\, a_1$	E_3^2	$a_3 \to -a_1$	E_2^1	$a_2 \to a_1$
	$a_5 \to -\sqrt{(1/2)}\, a_1$		$a_4 \to -\sqrt{(3/2)}\, a_2$		$a_5 \to -\sqrt{2}\, a_3$
	$a_6 \to -a_2$		$a_5 \to \sqrt{(1/2)}\, a_2$		$a_6 \to \sqrt{2}\, a_5$
	$a_7 \to a_3$		$a_7 \to \sqrt{(3/2)}\, a_4 - \sqrt{(1/2)}\, a_5$		$a_8 \to -a_7$
	$a_8 \to \sqrt{(3/2)}\, a_4 + \sqrt{(1/2)}\, a_5$		$a_8 \to a_6$		
E_1^3	$a_1 \to -\sqrt{(3/2)}\, a_4 - \sqrt{(1/2)}\, a_5$	E_2^3	$a_1 \to -a_3$	E_1^2	$a_1 \to a_2$
	$a_2 \to -a_6$		$a_2 \to -\sqrt{(3/2)}\, a_4 + \sqrt{(1/2)}\, a_5$		$a_3 \to -\sqrt{2}\, a_5$
	$a_3 \to a_7$		$a_4 \to \sqrt{(3/2)}\, a_7$		$a_5 \to \sqrt{2}\, a_6$
	$a_4 \to \sqrt{(3/2)}\, a_8$		$a_5 \to -\sqrt{(1/2)}\, a_7$		$a_7 \to -a_8$
	$a_5 \to \sqrt{(1/2)}\, a_8$		$a_6 \to a_8$		

H_1	$a_1 \to a_1$;	$a_2 \to -a_2$;	$a_3 \to 2a_3$;	$a_6 \to -2a_6$;	$a_7 \to a_7$;	$a_8 \to -a_8$
H_2	$a_1 \to a_1$;	$a_2 \to 2a_2$;	$a_3 \to -a_3$;	$a_6 \to a_6$;	$a_7 \to -2a_7$;	$a_8 \to -a_8$

The results of all other operations are 0.

By the definition of weights, the correspondence between weights and vectors of this representation is:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
(1 1)	(2 -1)	(-1 2)	(0 0)	(0 0)	(1 -2)	(-2 1)	(-1 -1)

The representation is called "the adjoint representation" because its non-vanishing weights are the roots of the group SU(3). They correspond to the E's in the following way:

E_3^1	E_3^2	E_2^1	E_1^2	E_2^3	E_1^3
(1 1)	(2 -1)	(-1 2)	(1 -2)	(-2 1)	(-1 -1)

2. WEIGHT DIAGRAMS OF IRREDUCIBLE REPRESENTATIONS OF SU(3); MULTIPLICITY

Let $\mathcal{L}$ be the lattice of all points (λ,μ) in plane, where λ and μ are integers and λ-μ is divisible by 3. If both λ and μ are non-negative, there is an irreducible representation of SU(3), the highest weight* of which is (λ,μ), and this representation appears in the reduction of a certain product of the form:

$$(1\ 1) \times (1\ 1) \times \ldots \times (1\ 1).$$

Conversely, if an irreducible representation appears in the decomposition of such a product, its highest weight (λ,μ) is such that $\lambda,\mu \geq 0$ and λ-μ is divisible by 3.

Given (λ,μ), draw the hexagon defined by the points:

$$(\lambda,\mu), \quad (\lambda + \mu, -\mu), \quad (\mu, -\lambda - \mu), \quad (-\lambda, \lambda + \mu), \quad (-\lambda - \mu, \lambda), \quad (-\mu - \lambda).$$

It is readily seen that all these points belong to $\mathcal{L}$. Every point of $\mathcal{L}$ lying on the sides or inside the hexagon is a weight of the representation (λ, μ); no weight of the representation lies outside the hexagon. The hexagon shrinks into a triangle when either $\lambda = 0$ or $\mu = 0$; yet the statement remains true.

Different vectors of an irreducible representation may correspond to the same weight; e.g. a_4 and a_5 in the adjoint representation correspond both to (0 0). The number of independent vectors corresponding to a weight is called the multiplicity of this weight.

Suppose $\lambda \geq \mu > 0$, and consider the set weights:

$$(\lambda, \mu), (\lambda - 1, \mu - 1), (\lambda - 2, \mu - 2), \ldots. (\lambda - \mu, 0).$$

One may draw for each such weight a hexagon in a way similar to the original one; e.g., $(\lambda - 1, \mu - 1)$ determines the six points:

$$(\lambda - 1, \mu - 1), (\lambda + \mu - 2, -\mu + 1), (-\lambda + 1, \lambda + \mu - 2), (\mu - 1, -\lambda - \mu + 2),$$

$$(-\lambda - 1\mu + 2, \lambda - 1), (-\mu + 1, -\lambda + 1).$$

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* A weight (α, β) is positive when either $\alpha + \beta > 0$ or $\alpha + \beta = 0$, $\beta > 0$. (α, β) is higher than (γ, δ) when $(\alpha - \gamma, \beta - \delta)$ is positive.

The rule is that all weights lying on the $(\lambda - k, \mu - k)$ hexagon have multiplicity $k + 1$; and those lying on the innermost triangle or inside it have the multiplicity $\mu + 1$.

When $\mu \geq \lambda > 0$ we deal with a set of weights:

$$(\lambda, \mu), \quad (\lambda - 1, \mu - 1), \ldots \quad (0, \mu - \lambda);$$

yet the rule remains unchanged. Similarly, when either $\lambda = 0$ or $\mu = 0$, we have a triangle and the multiplicity of each weight is 1.

Example: (5, 2) defines a hexagon the vertices of which are (5, 2), (7, -2), (2, -7), (-5, 7), (-7, 5), (-2, -5). (Its weights diagram is drawn in Fig.1).

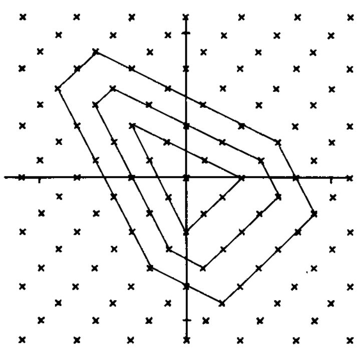

Fig. 1

The weights diagram of the representation (2 2)

3. THE CALCULATION OF THE REPRESENTATIONS

(a) Example

As weights have in general multiplicity > 1, one has to use additional quantum numbers in order to specify uniquely a vector corresponding to a given weight. It was shown by Racah that one needs $(N - 3\ell)/2$ such additional quantum numbers for a group of order N and rank ℓ. In the case of SU(3) we have $(N - 3\ell)/2 = 1$, and the labeling according to T^2 solves the problem.

The following lemma is important for calculation: If the vector $|m>$ corresponds to the weight $\underline{m}$, the vector $E_\alpha \, | m>$ corresponds to the weight $\underline{m} + \underline{\alpha}$.

Proof: $H_i E_\alpha | m> = [H_i E_\alpha] \, | m> + E_\alpha H_i | m> = (\alpha_i + m_i) E_\alpha | m>$ Q.E.D.

It follows that knowing a vector of the representation corresponding to a given weight, we may "walk" all over the diagram with the aid of the E's and get vectors corresponding to all other weights of this representation. The method seems to be best explained by an example. We have:

$$(1\ 1) \times (1\ 1) = (2\ 2) + (3\ 0) + (0\ 3) + (1\ 1) + (1\ 1) + (0\ 0) \ldots . \tag{3}$$

Let a_i and b_k $(1 \le i,\ k \le 8)$ span the bases of the two representations appearing on the left and suppose one has to calculate the basis of (2 2). The vectors belonging to this basis are combinations of products of the form $a_i\, b_k$ in such a way that each combination corresponds to a definite weight and has a definite T. It is readily seen that the weight corresponding to a product is the sum of the weights which correspond to the factors. Hence $a_1 b_1$ is the only product which corresponds to (2 2).

By Table II and using (1):

$$T_-(a_1 b_1) = a_2 b_1 + a_1 b_2; \qquad T_-(a_2 b_1 + a_1 b_2) = 2\ a_2 b_2 .$$

$\sqrt{(1/2)}(a_1 b_2 + a_2 b_1)$ corresponds to (3 0), $a_2 b_2$ to (4 -2). It can be seen from Fig. 2 that together with $a_1 b_1$ they form an isospin triplet, because

$$T_+(a_1 b_1) = T_-(a_2 b_2) = 0 .$$

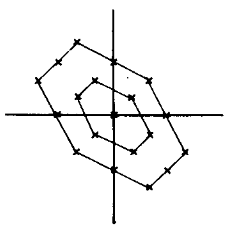

Fig. 2

The weights diagram of the representation (2 2)

The vector

$$\sqrt{(1/2)}E_2^3(a_1 b_1) = \sqrt{(1/2)}(a_1 b_3 + a_3 b_1)$$

corresponds to (0 3). Together with:

$$-\sqrt{(1/3)}T_-\sqrt{(1/2)}(a_1 b_3 + a_3 b_1) = \sqrt{(1/3)}(a_1 b_5 + a_5 b_1) - \sqrt{(1/6)}(a_2 b_3 + a_3 b_2),$$

$$\sqrt{(1/2)}T_-[\sqrt{(1/3)}(a_1 b_5 + a_5 b_1) - \sqrt{(1/6)}(a_2 b_3 + a_3 b_2)] = \sqrt{(1/6)}(a_1 b_6 + a_6 b_1) + \sqrt{(1/3)}(a_2 b_5 + a_5 b_2),$$

$$\sqrt{(1/3)}T_-[\sqrt{(1/6)}(a_1 b_6 + a_6 b_1) + \sqrt{(1/3)}(a_2 b_5 + a_5 b_2)] = \sqrt{(1/2)}(a_2 b_6 + a_6 b_2),$$

which correspond to (1 1), (2 -1) and (3 -3) respectively, they form an isospin quartet. However, the multiplicity of (1 1) (and of (2 -1)) is 2; i.e. (1 1) corresponds to another vector with a different T. Certainly, this T is 1/2. The vector

$$-E_1^3(a_1 b_1) = \sqrt{(3/2)}(a_1 b_4 + a_4 b_1) + \sqrt{(1/2)}(a_1 b_5 + a_5 b_1)$$

corresponds also to (1 1), and therefore it is a combination of the T = 3/2 and T = 1/2 vectors corresponding to (1 1). By the Grahm-Schmidt procedure one finds the T = 1/2 vector, which is orthogonal to the T = 3/2 one:

$$\sqrt{(1/30)}\{3\sqrt{(3/2)}(a_1 b_4 + a_4 b_1) + \sqrt{(1/2)}(a_1 b_5 + a_5 b_1) + (a_2 b_3 + a_3 b_2)\}$$

Operating on this by T_- we get the second member of the doublet:

$$\sqrt{(1/30)}\{(a_1 b_6 + a_6 b_1) + 3\sqrt{(3/2)}(a_2 b_4 + a_4 b_2) - \sqrt{(1/2)}(a_2 b_5 + a_5 b_2)\}.$$

which is the second vector corresponding to (2 -1).
The vector

$$-\sqrt{(1/2)}E_2^3\sqrt{(1/2)}(a_1 b_3 + a_3 b_1) = a_3 b_3$$

corresponds to (-2, 4). Together with $N^{(i)} T^i(a_3 b_3)$ where i = 1, 2, 3, 4 and where $N^{(i)}$ are normalization factors we get an isospin quintet. The T = 1 vector which corresponds to (- 1, 2) is obtained by operating with E_2^3 on the T = 1/2 vector corresponding to (1 1)* ;the other members of the triplet are obtained with the aid of T .

Operating with E_1^3 on the T = 1/2 vector corresponding to (1 1) we get a combination of the T = 1 and T = 0 vectors which correspond to (0 0). By the Grahm-Schmidt method we again pick out the T = 0 vector.

The continuation of the procedure is obvious (Fig. 3).

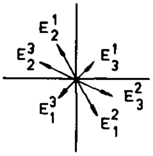

Fig. 3

The root diagram.

(b) The general method

In order to perform the decomposition of $(\lambda_1 \lambda_2) \times (\mu_1 \mu_2)$ one needs the following information:

(1) The representations $(\lambda_1 \lambda_2)$ and $(\mu_1 \mu_2)$ of the E's. The bases of these representations consist of polynomials in baryons, antibaryons, mesons and vector-mesons. Therefore rule (1) may be used to get these representations.**

* The reason is that E_2^3 (as well as E_1^3, E_3^2, E_3^1) can change T only by $\pm$.

** However, G. Racah calculated explicit formulae for the matrix-elements of the E's in any irreduble representation (private communication).

(2) Which representations appear in the decomposition. Define:

$$P(x) = (1/2)(|x| + x),$$

$$A = \mu_{in} \{\lambda_1 (1/3)[(\lambda_1 - \lambda_2) + (\mu_1 + 2\mu_2) - (\nu_1 - \nu_2)], (1/3)[(2\lambda_1 + \lambda_2) + (2\mu_1 + \mu_2) - (2\nu_1 + \nu_2)]\},$$

$$B = \mu_{o-x} \{P[(\lambda_1 - \nu_1) + (1/2)P((1/3)|(\lambda_1 + 2\lambda_2) - 2(\mu_1 - \mu_2) - (\nu_1 + 2\nu_2)| + (\nu_1 - \lambda_1))], (1/3)[2(\lambda_1 - \lambda_2) + (2\mu_1 + \mu_2) - (2\nu_1 + \nu_2)]\ (1/3)[(\lambda_1 - \lambda_2) + (\mu_1 + 2\mu_2) - (\nu_1 + 2\nu_2)]\}.$$

The number of times $(\nu_1\nu_2)$ appears in $(\lambda_1\lambda_2)\times(\mu_1\mu_2)$ is $P(A - B + 1)$.

(3) The vector corresponding to the maximal weight of such a representation. Let $\underline{\ell}$ be a weight of $(\lambda_1\lambda_2)$ and $\underline{m}$ a weight of $(\mu_1\mu_2)$ such that $\underline{\ell} + \underline{m} = (\nu_1\nu_2)$. If $|\underline{\ell}, T_1>$ is a vector of the basis of $(\lambda_1\lambda_2)$ corresponding to $\underline{\ell}$, and $|\underline{m}, T_2>$ a vector of the basis of $(\mu_1\mu_2)$ corresponding to $\underline{m}$, then the product $|\underline{\ell}, T_1>|\underline{m}, T_2>$ corresponds to $(\nu_1\nu_2)$.

We look for a linear combination of such products which is the vector corresponding to the highest weight of $(\nu_1\nu_2)$. Such a combination must vanish under E_3^1, E_3^2 and E_2^1, as follows from the lemma of part (a) (since $(\nu_1 + 1, \nu_2 + 1)$, $(\nu_1 + 2, \nu_2 - 1)$ and $(\nu_1 - 1, \nu_2 + 2)$ are not weights of the representation $(\nu_1\nu_2)$. The combination will also vanish under E_1^2, if $\nu_2 = 0$, and under E_2^3 if $\nu_1 = 0$. Considering the coefficients of the combination, we get for them a set of homogeneous equations when E_3^1, E_3^2 and E_2^1 (may be E_1^2, or E_2^3 also) are applied. If the solution is not unique, the representation $(\nu_1\nu_2)$ occurs several times, and one chooses an appropriate basis arbitrarily.

Example: The vector corresponding to the highest weight of (1 1) in the decomposition of $(1\ 1)\times(1\ 1)$. This vector is of the form:

$$\alpha(a_1 b_4) + \beta(a_1 b_5) + \gamma(a_2 b_3) + \delta(a_4 b_1) + \epsilon(a_5 b_1) + \zeta(a_3 b_2).$$

It will vanish under the operations of E_3^1, E_3^2 and E_2^1. Hence:

$$E_3^1 : -\sqrt{(3/2)}(\alpha + \delta)(a_1 b_1) - \sqrt{(1/2)}(\beta + \epsilon)(a_1 b_1) = 0,$$

$$E_3^2 : -\sqrt{(3/2)}\alpha(a_1 b_2) + \sqrt{(1/2)}\beta(a_1 b_2) - \gamma(a_2 b_1) - \sqrt{(3/2)}\delta(a_2 b_1), + \sqrt{(1/2)}\epsilon(a_2 b_1) - \zeta(a_1 b_2) = 0,$$

$$E_2^1 : -\sqrt{2}\,\beta(a_1 b_3) + \gamma(a_1 b_3) - \sqrt{2}\,\epsilon(a_3 b_1) + \zeta(a_3 b_1) = 0.$$

From these equations we get:

$$\sqrt{(3/2)}(\alpha + \delta) + \sqrt{(1/2)}(\beta + \epsilon) = 0,$$

$$-\sqrt{(3/2)}\alpha + \sqrt{(1/2)}\beta - \zeta = 0,$$

$$-\gamma - \sqrt{(3/2)}\delta + \sqrt{(1/2)}\epsilon = 0,$$

$$-\sqrt{2}\beta + \gamma = 0,$$

$$-\sqrt{2}\epsilon + \zeta = 0.$$

The set of solutions is two dimensional. Adding the condition $\gamma = \zeta$ or $\gamma = -\zeta$ one gets two mutually orthogonal solutions.

4. REMARKS

(1) According to the usual exposition of the Lie theory one should write $E_{(1\ 1)}$ instead of E^1_3, $E_{(2\ \dot{-}1)}$ instead of E^2_3 etc.. The present notation emphasizes the fact that the E's operate in a 3-dimensional vector space.

(2) The highest weights (λ,μ) are associated with Young-schemes. An irreducible representation of SU(3) may be characterized by a Young-scheme of not more than 3 rows. Denoting this scheme by $[a_1\, a_2\, a_3]$ (where a_i is the length of the i'th row) we have $\lambda = a_1 - a_2$, $\mu = a_2 - a_3$.

(3) As T_+ and T_- commute with the hypercharge, it follows that vectors belonging to the same isospin multiplet have the same hypercharge.

(4) The decomposition procedure described above does not determine any general phase convention.