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Exact massive and massless scalar quasibound states solutions of the Einstein–Maxwell-dilaton (EMD) black hole

David Senjaya[a]

Department of Physics, Mahidol University, 272 Phraram 6 Street, Ratchathewi, Bangkok 10400, Thailand

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Abstract In this letter, we will focus on the Klein–Gordon equation with static spherically symmetric black hole solution of the Einstein–Maxwell-dilaton (EMD) theory as its 3+1 background space-time. The Klein–Gordon equation represents quasibound states of both massive and massless scalar fields which are localized in the black hole potential well. By using the covariant Klein–Gordon equation, we investigate the behaviour of both massive and massless scalars in the EMD black hole space-time. We successfully exactly solved the relativistic wave equation and are going to present the novel exact results in this letter. The exact solutions, the wave functions and the energy levels, describe the decaying nature of the relativistic scalar field bound in the curved space-time. The massive scalar quasibound state has complex-valued energy levels where the real part is the massive scalar's energy while the imaginary part represents the decay. For the massless scalar quasibound state, pure imaginary energy levels are discovered. In this letter, by using the obtained exact scalar particle's wave functions, we also consider the Hawking radiation of the apparent horizon of the EMD black hole that is calculated via Damour–Ruffini method. In principle, the investigation of black hole quasibound states could provide possibility for laboratory testing of effects whose nature are absolutely related with quantum effects in gravity.

1 Introduction

Static chargeless black holes in general relativity was discovered by Karl Schwarzschild in 1916, one year after Einstein founded the general relativity. That discovery was in the same year with the Einstein's prediction of the gravitational waves. Later on, Reissner and Nordström [1,2] independently discover generalized Schwarzschild's static spherically symmetric chargeless black hole solution to include electrically charged source which later on known as the charged Reissner–Nordström metric. The Reissner–Nordström solution, thus, describes a static charged massive black hole solution which is characterized by these two parameters, the mass M and the charge Q. By setting the charge parameter to be zero, the Reissner–Nordström black hole becomes a static spherically symmetric Schwarzschild black hole.

With the rapid development of modified theory of gravity, especially the scalar tensor theory, in 1988, Gibbon and in 1991, Garfinkle [3,4] independently worked out generalization of the static charged black hole solution, which is called static charged black hole dilatonic solution, out of the Einstein–Maxwell-dilaton theory of gravity. The Lagrangian formulation of the theory is given as follows,

$$S = \int \left[R - 2\nabla^{\alpha}\varphi\nabla_{\alpha}\varphi - e^{-2a\varphi}F_{\mu\nu}F^{\mu\nu}\right]\sqrt{-g}d^4x, \quad (1)$$

where the scalar field φ is called dilaton field, R comes from Einstein theory of gravity, $F_{\mu\nu}$ is Maxwell tensor field and a is a dimensionless parameter where $a = 0$ recovers Reissner–Nordström solution and $a = 1$ is the value suggested by superstring theory [5].

However, it takes one full century to finally detect gravitational wave signal which directly came by from a binary black hole merger. When a black hole is perturbed, it undergoes damped oscillations which has complex valued frequencies. The quasibound states, quasinormal modes, and shadows of black holes are among the most interesting black hole's characteristics that are generated as particles crossing into the black hole [6]. The resonance spectra of the quasibound states have complex valued frequencies where the real part is associated as the scalar's energy and the imaginary part determines the stability of the system. Principally, it is possible to extract some information about the physics of black holes as well as to validate some modified theories of gravity from by detecting the quasibound states frequen-

[a] e-mail: davidsenjaya@protonmail.com (corresponding author)

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cies [6]. Also Levels transitions of the scalars around a black hole will emit gravitons, analogous to atomic transitions that emits photons [7]. Thus, the ability to calculate the exact quasibound states frequency analytically is very crucial.

There is an analytical results have been developed to investigate the quasibound states of static charged dilatonic black hole [8] where the analytical formula that resembles the Hydrogenic atom's energy levels up as its first order is found. By taking the small black hole limit $M_{blackhole} \ll \frac{m_{Plank}^2 c^2}{E_0}$-where m_{Plank} is Plank mass, E_0 is the scalar particle's rest energy, and c is the speed of light,-the imaginary part of the complex valued quasibound states' energy is suppressed [9–15].

In the present work, the massive and massless exact quasibound states solutions of the scalar particles around a EMD black hole with $a = 1$ and also the Hawking radiation of the black hole's apparent horizon are presented in detail. We discover that the exact solution of the wave functions and energy levels also the Hawking temperature are parametrized by the black hole's mass M and charge Q. And as the EMD black hole is a generalization the Schwarzschild black hole, the obtained exact wave functions and energy levels also the Hawking temperature can directly be used to obtain the quasibound states solutions and Hawking radiation of the Schwarzschild black holes by nulling the charge.

This letter is organized as follows: in Sect. 2, the metric of the EMD black hole is discussed. In Sect. 3, the Klein–Gordon equation in the EMD black hole background is constructed and the derivation of quasibound states solutions of massive and massless scalars are presented to obtain the exact solutions of the wave functions and the energy levels expressions. And in the Sect. 4, the Hawking radiation is investigated via the Damour–Ruffini method.

2 The metric

We start by writing the EMD metric in international unit as follows [5],

$$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 - \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2\left(1-\frac{2r_D}{r}\right)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \quad (2)$$

where,

$$r_s = \frac{2GM}{c^2}, \quad r_Q^2 = \frac{GQ^2}{4\pi\varepsilon_0 c^4}, \quad \frac{r_Q^2}{r_s} = r_D.$$

The EMD geometry is singular when $r = 0$, $r = r_s$ and $r = \frac{2r_Q^2}{r_s}$. The first is black hole singularity, the second is a surface of singularity and the third is found to be a curvature singularity after further investigation of the curvature tensor [16]. By setting the charge parameter, $Q = 0$, the Schwarzschild singularities, i.e. at $r = 0$ and $r = r_s$, are recovered.

3 The Klein–Gordon equation

In this section, we are going to construct and solve the relativistic Klein–Gordon equation with EMD black hole background. The quantum relativistic matter wave equation reads as follows,

$$-\hbar^2\nabla_\mu\nabla^\mu\psi + k^2c^2 = 0, \quad (3)$$

where ∇_μ is the covariant derivative.

Operating the covariant derivatives, we proceed as follows,

$$\nabla_\mu\nabla^\mu\psi = \nabla_\mu\partial^\mu\psi = \partial_\mu\partial^\mu\psi + \Gamma^\mu_{\mu\nu}\partial^\nu\psi, \quad (4)$$

where, we also can derive this following identity,

$$\begin{aligned}\Gamma^\alpha_{\alpha\beta} &= \frac{g^{\alpha\gamma}}{2}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}\right) = \frac{1}{2}g^{\alpha\gamma}\partial_\beta g_{\alpha\gamma} \\ &= \frac{1}{2}g^{\alpha\gamma}\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} = \frac{1}{2}g^{\alpha\gamma}\left(\frac{2}{\sqrt{-g}g^{\alpha\gamma}}\frac{\partial\sqrt{-g}}{\partial x^\beta}\right) \\ &= \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^\beta},\end{aligned} \quad (5)$$

so, with the help of the Christoffel symbol's property, we get,

$$\begin{aligned}\nabla_\mu\nabla^\mu\psi &= \partial_\mu\partial^\mu\psi + \left(\frac{1}{\sqrt{-g}}\partial_\nu\sqrt{-g}\right)\partial^\nu\psi \\ &= \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\psi\right) \\ &= \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right),\end{aligned} \quad (6)$$

and finally, we can express the Klein–Gordon equation in the terms of partial derivatives and the metric tensor components as follows,

$$\left\{-\hbar^2\left[\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\right] + k^2c^2\right\}\psi = 0, \quad (7)$$

where $E_0 = kc^2$ is the scalar's rest energy per unit mass. So, we have $k = 1$ for massive scalars and $k = 0$ for massless scalars. where the particle's rest energy per unit mass is $E_0 = kc^2$. Thus, $k = 1$ for massive particles and $k = 0$ for massless particles. The wave equation can be written explicitly as follows, by firstly, working out the Laplace–Beltrami operator one by one,

$$|g| = -r^4\left(1-\frac{2r_D}{r}\right)^2\sin^2\theta = -F^2(r)\sin^2\theta, \quad (8)$$

$$\frac{1}{\sqrt{-g}}\partial_0\sqrt{-g}g^{00}\partial_0 = -\frac{r}{(r-r_s)}\partial_{ct}^2, \tag{9}$$

$$\frac{1}{\sqrt{-g}}\partial_1\sqrt{-g}g^{11}\partial_1 = \left(1-\frac{r_s}{r}\right)\partial_r^2 + \left[\frac{2(r-r_D)}{r(r-2r_D)}\left(1-\frac{r_s}{r}\right)+\frac{r_s}{r^2}\right]\partial_r, \tag{10}$$

$$\frac{1}{\sqrt{-g}}\partial_2\sqrt{-g}g^{22}\partial_2 = \frac{1}{r(r-2r_D)}\frac{1}{\sin\theta}\partial_\theta \sin\theta\,\partial_\theta, \tag{11}$$

$$\frac{1}{\sqrt{-g}}\partial_3\sqrt{-g}g^{33}\partial_3 = \frac{1}{r(r-2r_D)}\frac{1}{\sin^2\theta}\partial_\theta^2. \tag{12}$$

Combining all in one equation, we get the full Klein–Gordon equation in a EMD black hole space-time,

$$\Bigg[-\frac{r}{(r-r_s)}\partial_{ct}^2+\left(1-\frac{r_s}{r}\right)\partial_r^2 + \left[\frac{2(r-r_D)}{r(r-2r_D)}\left(1-\frac{r_s}{r}\right)+\frac{r_s}{r^2}\right]\partial_r + \frac{1}{r(r-2r_D)}\left[\frac{1}{\sin\theta}\partial_\theta\sin\theta\,\partial_\theta+\frac{1}{\sin^2\theta}\partial_\phi^2\right] - \frac{k^2c^2}{\hbar^2}\Bigg]\psi = 0. \tag{13}$$

3.1 Exact angular solutions

In order to reduce the four dimensional relativistic wave equation above to be a single dimensional ordinary second order differential equation, we consider the symmetry of the whole Klein–Gordon equation and apply the separation of variables ansatz as follows,

$$\psi(t,r,\theta,\phi) = e^{-i\frac{E}{\hbar c}ct}R(r)\,Y_l^{m_l}(\theta,\phi), \tag{14}$$

where $Y_l^{m_l}(\theta,\phi)$ is a spherical harmonics function which is solution of this following spherical harmonics differential equation [17,18],

$$\nabla_{\Omega_2}^2 Y_l^{m_l}(\theta,\phi) = -l(l+1)\,Y_l^{m_l}(\theta,\phi). \tag{15}$$

3.2 The radial equation

Substituting the ansatz into the Klein–Gordon equation (13), followed by multiplying the whole with $\frac{r^2}{\psi(t,r,\theta,\phi)}$, we get a single dimensional radial equation as follows,

$$\Bigg[\frac{r}{(r-r_s)}\frac{E^2}{\hbar^2c^2}+\left(1-\frac{r_s}{r}\right)\partial_r^2 + \left[\frac{2(r-r_D)}{r(r-2r_D)}\left(1-\frac{r_s}{r}\right)+\frac{r_s}{r^2}\right]\partial_r - \frac{l(l+1)}{r(r-2r_D)} - \frac{E_0^2}{\hbar^2c^2}\Bigg]R = 0. \tag{16}$$

Multiplying the whole equation by $\frac{r}{(r-r_s)}$, we get,

$$\Bigg[\partial_r^2+\left[\frac{2(r-r_D)}{r(r-2r_D)}+\frac{r_s}{r(r-r_s)}\right]\partial_r - \frac{l(l+1)}{(r-2r_D)(r-r_s)} - \frac{E_0^2}{\hbar^2c^2}\frac{r}{(r-r_s)}+\frac{E^2}{\hbar^2c^2}\frac{r^2}{(r-r_s)^2}\Bigg]R = 0. \tag{17}$$

We are interested to find the solution outside the apparent horizon r_+. So, we define this following new radial variable,

$$\rho = r - r_s, \quad r_s - 2r_D = \delta_r, \tag{18}$$

that transforms the radial equation as follows,

$$\Bigg[\partial_\rho^2+\left[\frac{2(\rho+\delta_r+r_D)}{(\rho+r_s)(\rho+\delta_r)}+\frac{r_s}{\rho(\rho+r_s)}\right]\partial_\rho - \frac{l(l+1)}{(\rho+\delta_r)\rho} - \frac{E_0^2}{\hbar^2c^2}\frac{\rho+r_s}{\rho}+\frac{E^2}{\hbar^2c^2}\frac{(\rho+r_s)^2}{\rho^2}\Bigg]R = 0. \tag{19}$$

We want to simplify the radial equation by working with the fractional decompositions as follows,

$$\frac{2(\rho+\delta_r+r_D)}{(\rho+r_s)(\rho+\delta_r)} = -\frac{2r_D}{(\rho+\delta_r)(\delta_r-r_s)} \tag{20}$$

$$-\frac{2r_D}{(\rho+r_s)(\delta_r-r_s)}, \quad \frac{r_s}{\rho(\rho+r_s)} = \frac{1}{\rho}-\frac{1}{\rho+r_s}, \tag{21}$$

$$\frac{l(l+1)}{(\rho+\delta_r)\rho} = l(l+1)\left[-\frac{1}{\delta_r(\rho+\delta_r)}+\frac{1}{\delta_r\rho}\right], \tag{22}$$

$$\frac{\rho+r_s}{\rho} = 1+\frac{r_s}{\rho}, \tag{23}$$

$$\frac{(\rho+r_s)^2}{\rho^2} = 1+\frac{r_s^2}{\rho^2}+\frac{2r_s}{\rho}. \tag{24}$$

This allows us to rewrite the radial equation in terms of its canonical form as follows,

$$\Bigg[\partial_\rho^2+\left[\frac{1}{\rho+\delta_r}\left(-\frac{2r_D}{\delta_r-r_s}\right)+\frac{1}{\rho+r_s}\left(-\frac{2r_D}{\delta_r-r_s}-1\right)+\frac{1}{\rho}\right]\partial_\rho + \frac{1}{\rho}\left[-\frac{l(l+1)}{\delta_r}+\frac{2E^2-E_0^2}{\hbar^2c^2}r_s\right]+\frac{1}{\rho^2}\left[\frac{E^2}{\hbar^2c^2}r_s^2\right] + \frac{1}{\rho+\delta_r}\frac{l(l+1)}{\delta_r}+\frac{E^2-E_0^2}{\hbar^2c^2}\Bigg]R = 0. \tag{25}$$

Now, let us define these following dimensionless energy parameters, and also a new radial variable $\frac{\rho}{\delta_r} = -x$,

$$\Omega = \frac{Er_s}{\hbar c}, \quad \Omega_0 = \frac{E_0 r_s}{\hbar c}. \tag{26}$$

In term of the new radial variable x, the radial equation looks like as following,

$$\Bigg[\partial_x^2+\Bigg[\frac{1}{x-1}\underbrace{\left(\frac{2r_D}{r_s-\delta_r}\right)}_{=1}+\frac{1}{x}\Bigg]\partial_x$$

$$-\frac{1}{x}\left[-l\left(l+1\right)+2\Omega^2-\Omega_0^2\right]+\frac{\Omega^2}{x^2}$$
$$\left.-\frac{l\left(l+1\right)}{x-1}+\Omega^2-\Omega_0^2\right]R=0. \tag{27}$$

Here we are going to transform this linear second order differential equation into its normal form (see Appendix A) by firstly recognizing,

$$p=\frac{1}{x-1}+\frac{1}{x}, \tag{28}$$

$$q=-\frac{1}{x}\left[-l\left(l+1\right)+2\Omega^2-\Omega_0^2\right] \tag{29}$$

$$+\frac{\Omega^2}{x^2}-\frac{l\left(l+1\right)}{x-1}+\Omega^2-\Omega_0^2, \tag{30}$$

and this leads to,

$$-\frac{1}{2}\frac{dp}{dx}=\frac{1}{\left(x-1\right)^2}\left(\frac{1}{2}\right)+\frac{1}{2}\frac{1}{x^2}, \tag{31}$$

$$-\frac{1}{4}p^2=-\frac{1}{4}\frac{1}{\left(x-1\right)^2}-\frac{1}{4}\frac{1}{x^2}+\frac{1}{2}\frac{1}{x}-\frac{1}{2}\frac{1}{\left(x-1\right)}. \tag{32}$$

After a diligent work, finally we obtain this following result,

$$-\frac{1}{2}\frac{dp}{dx}-\frac{1}{4}p^2+q=\Omega^2-\Omega_0^2$$
$$-\frac{-l\left(l+1\right)-\frac{1}{2}+2\Omega^2-\Omega_0^2}{x}$$
$$+\frac{\Omega^2+\frac{1}{4}}{x^2}-\frac{l\left(l+1\right)+\frac{1}{2}}{x-1}+\frac{\frac{1}{4}}{\left(x-1\right)^2}. \tag{33}$$

So, we get a normal form,

$$\partial_x^2Y+\left[\Omega^2-\Omega_0^2-\frac{-l\left(l+1\right)-\frac{1}{2}+2\Omega^2-\Omega_0^2}{x}\right.$$
$$+\frac{\Omega^2+\frac{1}{4}}{x^2}$$
$$\left.-\frac{l\left(l+1\right)+\frac{1}{2}}{x-1}+\frac{\frac{1}{4}}{\left(x-1\right)^2}\right]Y=0, \tag{34}$$

$$Y\left(x\right)=R\left(x\right)e^{\frac{1}{2}\int pdx}=\left(x-1\right)^{\frac{1}{2}}x^{\frac{1}{2}}R\left(x\right). \tag{35}$$

3.2.1 Energy quantization

Comparing the normal form of the radial equation above with the Confluent Heun's normal form (see Appendix B), we find the explicit expressions of the Confluent Heun's parameters as follows,

$$\alpha=2i\sqrt{\Omega^2-\Omega_0^2}, \tag{36}$$

$$\beta=i2\Omega, \tag{37}$$

$$\gamma=0, \tag{38}$$

$$\delta=2\Omega^2-\Omega_0^2, \tag{39}$$

$$\eta=l\left(l+1\right)+\frac{1}{2}-2\Omega^2+\Omega_0^2. \tag{40}$$

And we also obtain the novel exact solution of the Klein–Gordon equation in the EMD black hole space-time can be written as follows,

$$\psi=\psi_0e^{i\frac{E}{\hbar c}ct}Y_l^{m_l}\left(\theta,\phi\right)e^{-\frac{1}{2}\alpha x}$$
$$\left[x^{\frac{1}{2}\beta}\left(x-1\right)^{\frac{1}{2}\gamma}HeunC\left(-x\right)\right], \tag{41}$$

where $x=-\frac{r-r_s}{\delta_r}$.

A quantized state exists if the radial solution has limited number of zeros, i.e. when the radial wave is obtained in terms of n-th order polynomial function. And for the Confluent Heun function, the polynomial condition is as the following (see Appendix B),

$$\frac{\delta}{\alpha}+\frac{\beta+\gamma}{2}=-n\quad n=n_r+1=1,2,3,\ldots, \tag{42}$$

substituting the parameters explicitly, we obtain a novel quantized energy expression that can only be discovered after exactly solving the radial equation,

$$\frac{\left(\Omega_0^2-2\Omega^2\right)}{2\sqrt{\Omega_0^2-\Omega^2}}+i\Omega=-n, \tag{43}$$

or, after simplification, we get,

$$\frac{\frac{E_0r_s}{\hbar c}\left(1-2\frac{E^2}{E_0^2}\right)}{2\left(1-\frac{E^2}{E_0^2}\right)^{\frac{1}{2}}}+i\frac{Er_s^2}{\hbar c}=-n. \tag{44}$$

In the small black hole limit $Er_s\rightarrow 0$, we obtain this following expression,

$$E=E_0\sqrt{1-\left(\frac{E_0r_s}{2n\hbar c}\right)^2}, \tag{45}$$

$$E\approx E_0\left(1-\frac{1}{2}\left(\frac{E_0r_s}{2n\hbar c}\right)^2\right). \tag{46}$$

The first order energy expression above is in agreement with these Refs. [7,10–12,14].

Now, let us consider the massless scalar around EMD black hole, whose rest energy $E_0=0$. Solving the (43), we obtain a pure imaginary energy levels as follows,

$$\frac{-2\Omega^2}{2\sqrt{-\Omega^2}}+i\Omega=-n, \tag{47}$$

$$E_n=i\frac{n\hbar c}{2r_s}. \tag{48}$$

3.2.2 Behaviour of the quasibound states

Now, let us investigate the behaviour of the exact radial solutions in two extreme regions, i.e. are the black hole's outer horizon, $r \to r_S$ and asymptotic behaviour far away from the black hole's horizon $r \to \infty$. Remember that the quasibound states are quantized states, thus, the Heun functions are now polynomial functions.

Let us first consider how the quasibound states behave very near to the apparent horizon by taking the limit $r \to r_s$. We investigate as follows, as $x = \frac{r-r_s}{\delta_r}$ is approaching $x = 0$, the Confluent Heun functions, $\text{HeunC}(0) = \text{HeunC}'(0) \approx 1$. Also the exponential $e^{-\frac{1}{2}\alpha\left(\frac{r-r_s}{r_s}\right)} \approx 1$. Thus, we get this following wave function of the quasibound states expression in the near horizon limit,

$$\psi_{\to r_s} = e^{i\frac{E}{\hbar c}ct} Y_\ell^{m_\ell}(\theta, \phi) \times \left[A\left(\frac{r-r_s}{\delta_r}\right)^{\frac{1}{2}\beta} + B\left(\frac{r-r_s}{\delta_r}\right)^{-\frac{1}{2}\beta} \right]. \tag{49}$$

Now, let us define a new radial variable $\frac{r-r_s}{\delta_r} = \zeta r - \zeta_0$ and expressing the Heun's β parameter following (37) or equivalently, $\beta = i|\beta|$ to get this following expression,

$$\psi_{\to r_s} = e^{i\frac{E}{\hbar c}ct} Y_\ell^{m_\ell}(\theta, \phi) \times \left[A(\zeta r - \zeta_0)^{\frac{i|\beta|}{2}} + B(\zeta r - \zeta_0)^{-\frac{i|\beta|}{2}} \right], \tag{50}$$

and using the complex identity,

$$z^i = e^{1\ln(z)}, \tag{51}$$

together with,

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \tag{52}$$

we get,

$$\psi_{\to r_s} = e^{i\frac{E}{\hbar c}ct} Y_\ell^{m_\ell}(\theta, \phi)\left[C\cos(\zeta r - \zeta_0)\right], \tag{53}$$

which represent a purely ingoing cosine wave. So, the quasibound states are purely ingoing waves very close to black hole's horizon and as $r \to \infty$, the exponential function $e^{-\frac{1}{2}\alpha\left(\frac{r-r_s}{r_s}\right)} \approx 1$ is definitely suppressing the Heun polynomials quenching the whole wave at the asymptotic infinity.

4 Hawking radiation

After successfully obtain the exact solutions of the radial wave, The Hawking radiation of the EMD black hole's apparent horizon can be investigated. We will start with the complete solution of the wave function as follows,

$$\psi = e^{i\frac{E}{\hbar c}ct} Y_\ell^{m_\ell}(\theta, \phi)\, e^{-\frac{1}{2}\alpha x} \times \left[A x^{\frac{1}{2}\beta}\, \text{HeunC}(x) + B x^{-\frac{1}{2}\beta}\, \text{HeunC}'(x) \right]. \tag{54}$$

Getting closer to the horizon $r \to r_s$, $x \to 0$, we could expand the wave function in the lowest order of r where $\text{HeunC}(0) = \text{HeunC}'(0) = 1$ also $e^{-\frac{1}{2}\alpha x} = 1$,

$$\psi = e^{i\frac{E}{\hbar c}ct} e^{-im_\ell \phi} Y_\ell^{m_\ell}\left[B x^{-\frac{1}{2}\beta} + A x^{\frac{1}{2}\beta} \right]. \tag{55}$$

The wave near horizon wave function (55) consists of two parts,

$$\psi = \begin{cases} \psi_{+in} = A e^{i\frac{E}{\hbar c}ct} Y_\ell^{m_\ell}(\theta, \phi)\left(\frac{r-r_s}{\delta_r}\right)^{\frac{1}{2}\beta} \\ \psi_{+out} = B e^{i\frac{E}{\hbar c}ct} Y_\ell^{m_\ell}(\theta, \phi)\left(\frac{r-r_s}{\delta_r}\right)^{-\frac{1}{2}\beta} \end{cases}, \tag{56}$$

where ψ_{+in} is the ingoing wave and ψ_{+out} is the outgoing wave.

Suppose an ingoing wave hits the apparent horizon r_+. The wave induces a particle-antiparticle pair where the particle will enhance the reflected wave and the antiparticle will become the transmitted wave going inside the black hole. Analytical continuation of the wave function $\psi\left(\frac{r-r_s}{\delta_r}\right)$ can be calculated as follows,

$$\left(\frac{r-r_s}{\delta_r}\right)^\lambda = \left(\frac{r_s}{\delta_r}\right)^\lambda \left(\frac{r}{r_s} - 1\right)^\lambda \to \left(\frac{r_s}{\delta_r}\right)^\lambda \left[\left(\frac{r}{r_s} - 1\right) + i\epsilon\right]^\lambda = \begin{cases} \left(\frac{r_s}{\delta_r}\right)^\lambda \left(\frac{r-r_s}{\delta_r}\right)^\lambda, & r > r_s \\ \left(\frac{r_s}{\delta_r}\right)^\lambda \left|\frac{r-r_s}{\delta_r}\right|^\lambda e^{i\lambda\pi}, & r < r_s \end{cases}. \tag{57}$$

We can get the $\psi_{-out} = \psi_{+out}\left(\left(\frac{r-r_s}{\delta_r}\right) \to \left(\frac{r-r_s}{\delta_r}\right) e^{i\pi}\right)$ simply by changing $\left(\frac{r-r_s}{\delta_r}\right) \to -\left(\frac{r-r_s}{\delta_r}\right) = \left(\frac{r-r_s}{\delta_r}\right) e^{i\pi}$ as follows,

$$\psi_{-out} = B e^{i\frac{E}{\hbar c}ct} Y_\ell^{m_\ell}(\theta, \phi)\left(\left(\frac{r-r_s}{\delta_r}\right) e^{i\pi}\right)^{-\frac{1}{2}\beta}, = \psi_{+out} e^{-\frac{1}{2}i\pi\beta} \tag{58}$$

$$\left|\frac{\psi_{-out}}{\psi_{+in}}\right|^2 = \left|\frac{\psi_{+out}}{\psi_{+in}}\right|^2 e^{-i2\pi\beta} = \left|\frac{\psi_{+out}}{\psi_{+in}}\right|^2 e^{4\pi\left[\frac{E}{\hbar c} r_s\right]}. \tag{59}$$

The relative probability $e^{-4\pi\frac{E r_s}{\hbar c}}$ above describes the amplitude of the pair production to occur. The absolute probability of the process to occur outside the horizon can be calculated by summing all of the probabilities of creating no pair, 1 pair, 2 pairs, etc must equal to 1,

$$C_\omega\left(1 + e^{-4p\frac{E r_s}{\hbar c}} + \left(e^{-4p\frac{E r_s}{\hbar c}}\right)^2 + \ldots\right) = 1 \to C_\omega = 1 - e^{-4p\frac{E r_s}{\hbar c}}. \tag{60}$$

The probability to create j pairs is represented by $C_\omega\left(e^{-4\pi\frac{E r_s}{\hbar c}}\right)^j = \left(1 - e^{-4\pi\frac{\omega r_s}{c}}\right) e^{-j4\pi\frac{\omega r_s}{c}}$. From here, we can obtain the distribution function of the pair production by summing all the possible pair productions that may occur as follows,

$$n(\omega) = \sum_{n=0}^{\infty} n\left(1 - e^{-4\pi\frac{\omega r_s}{c}}\right) e^{-n4\pi\frac{\omega r_s}{c}} = \frac{1}{e^{4\pi\frac{\omega r_s}{c}} - 1}. \quad (61)$$

The distribution function tells us the number of particles that is emitted from the horizon and is going outside. So, equivalently, we can also write,

$$\left\langle \frac{\psi_{out}}{\psi_{in}} \middle| \frac{\psi_{out}}{\psi_{in}} \right\rangle = 1 = \left|\frac{B}{A}\right|^2 \left|1 - e^{4\pi\left[\frac{E}{\hbar c} r_s\right]}\right|, \quad (62)$$

$$\left|\frac{B}{A}\right|^2 = \frac{1}{e^{\frac{E}{\hbar c} r_s} - 1}. \quad (63)$$

The Hawking temperature, T_H, is then obtained from this following modification,

$$\frac{E}{\hbar c} r_s = 4\pi\left[\frac{\omega}{c}(r_s)\right] = \frac{\hbar\omega}{\left[\frac{c\hbar}{4\pi r_s}\right]} = \frac{\hbar\omega}{k_B T_H}. \quad (64)$$

Finally, the EMD black hole's apparent horizon's temperature is obtained as follows,

$$T_H = \frac{c\hbar}{4\pi k_B r_s}. \quad (65)$$

5 Conclusions

In this work, we successfully solve the Klein–Gordon wave equation in the EMD black hole space-time background. The exact analytical massive and massless scalar quasibound states' quantized energy levels (43),(48) and their wave functions (41) are obtained. It is important to mention that since the obtained solutions are exact, the are valid for all region of interest, i.e. $r_s \leq r < \infty$. This is a remarkable improvement of the asymptotical method whose solutions solve only for either very close to the horizon region of very far away from the horizon.

By nulling the charge Q and taking small black hole limit, we can reproduce the real valued energy levels of the Schwarzschild massive quasibound state's as (46),

$$\frac{E_n}{E_0} \approx 1 - \frac{\kappa^2}{2n^2}, \kappa = \left(\frac{E_0 r_s}{\hbar c}\right)^2. \quad (66)$$

The Hydrogenic atom-like energy expression $\frac{1}{n^2}$ is also sound in many previously published works [9–14].

In additional, we also investigate the behaviour of the exact wave solutions in the two extreme regions, i.e. the near horizon and at infinity. Near the EMD black hole's horizon, the quasibound states behave like a purely ingoing wave (49) while at infinity, the quasibound states are vanishing.

With exact relativistic Klein–Gordon solutions in hand, the Damour–Ruffini method [19] is applied to investigate the Hawking temperature of the black hole's apparent horizon. The method uses the Klein pair production scenario where the pair production occurring at the horizon is induced by an incoming particle. The induced particle goes to infinity while the induced anti-particle goes towards the black hole. From there, we make a summation of all possible pair productions and obtain the radiation distribution function (61). Comparing it with the bosonic distribution function, the Hawking temperature of the EMD black hole's apparent horizon is obtained (4).

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Appendix A: Normal form

An ordinary differential equation is said to be in the normal form if is solved explicitly for the highest derivative [17]. One may start with a general form of the linear second order ordinary differential equation as follows,

$$\frac{d^2 y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0. \quad (67)$$

In order to bring the linear second order ordinary differential equation above to its normal form, we express the $y(x)$ in this particular form, which is specially designed to remove the first order derivative terms,

$$y = Y(x)e^{-\frac{1}{2}\int p(x)dx}, \quad (68)$$

$$\frac{dy}{dx} = \frac{dY}{dx}e^{-\frac{1}{2}\int p(x)dx} - \frac{1}{2}Ype^{-\frac{1}{2}\int p(x)dx}, \quad (69)$$

$$\frac{d^2y}{dx^2} = \frac{d^2Y}{dx^2}e^{-\frac{1}{2}\int p(x)dx} - \frac{1}{2}\frac{dY}{dx}pe^{-\frac{1}{2}\int p(x)dx}$$
$$-\frac{1}{2}Y\frac{dp}{dx}e^{-\frac{1}{2}\int p(x)dx} + \frac{1}{4}Yp^2e^{-\frac{1}{2}\int p(x)dx}. \tag{70}$$

Substituting the expressions for $y(x)$ into (67), a lot of things cancel out. And we obtain this following equation without the first order derivative,

$$\frac{d^2Y}{dx^2} + \left(-\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2 + q\right)Y = 0, \tag{71}$$
$$Y = ye^{\frac{1}{2}\int p(x)dx}. \tag{72}$$

If $Q(x) = -\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2 + q < 0$ and $Y(x)$ is the nontrivial solution of the normal form (71), then $Y(x)$ does not oscillate at all and has at most one zero. But, if $Q(x) = -\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2 + q > 0$ and $\int_1^\infty Q(x)dx = \infty$, then $Y(x)$ has infinitely many zeros on the positive x-axis [20].

Appendix B: Normal form of confluent Heun equation

Let us consider another ordinary linear second order differential equation so called the Confluent Heun differential equation [21],

$$\frac{d^2y}{dx^2} + \left(\alpha + \frac{\beta+1}{x} + \frac{\gamma+1}{x-1}\right)\frac{dy}{dx}$$
$$+\left(\frac{\mu}{x} + \frac{\nu}{x-1}\right)y = 0, \tag{73}$$
$$\nu = \frac{1}{2}(\alpha + \beta + \gamma + \alpha\beta + \beta\gamma) + \delta + \eta, \tag{74}$$
$$y = A\,\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, x)$$
$$+Bx^{-\beta}\,\text{HeunC}(\alpha, -\beta, \gamma, \delta, \eta, x), \tag{75}$$
$$y = A\,\text{HeunC}(x) + Bx^{-\beta}\,\text{HeunC}'(x), \tag{76}$$
$$\frac{\delta}{\alpha} + \frac{\beta+\gamma}{2} + 1 = -n_r, \quad n_r = 0, 1, 2, \ldots. \tag{77}$$

The Confluent Heun's differential equation can be brought to the normal form by recognizing p and q function (see Appendix A) as follows,

$$p = \alpha + \frac{\beta+1}{x} + \frac{\gamma+1}{x-1}, q = \frac{\mu}{x} + \frac{\nu}{x-1}, \tag{78}$$
$$y = \text{HeunC} = Y(x)e^{-\frac{1}{2}\alpha x}x^{-\frac{1}{2}(\beta+1)}(x-1)^{-\frac{1}{2}(\gamma+1)}, \tag{79}$$

and this leads to,

$$-\frac{1}{2}\frac{dp}{dx} = \frac{1}{x^2}\left(\frac{\beta+1}{2}\right) + \frac{1}{(x-1)^2}\left(\frac{\gamma+1}{2}\right), \tag{80}$$
$$-\frac{1}{4}p^2 = -\frac{\alpha^2}{4} - \frac{1}{x^2}\left(\frac{\beta^2+1+2\beta}{4}\right)$$
$$-\frac{1}{(x-1)^2}\left(\frac{\gamma^2+1+2\gamma}{4}\right) - \frac{2}{x}\left(\frac{\alpha\beta+\alpha}{4}\right)$$
$$-\frac{2}{x-1}\left(\frac{\alpha\gamma+\alpha}{4}\right) - \frac{2}{x(x-1)}\left(\frac{\beta\gamma+1+\beta+\gamma}{4}\right), \tag{81}$$
$$-\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2 + q = -\frac{\alpha^2}{4} + \frac{\frac{1}{2}-\eta}{x} + \frac{\frac{1}{4}-\frac{\beta^2}{4}}{x^2}$$
$$+\frac{-\frac{1}{2}+\delta+\eta}{x-1} + \frac{\frac{1}{4}-\frac{\gamma^2}{4}}{(x-1)^2}. \tag{82}$$

Combining everything, we get the Confluent Heun equation's normal form,

$$\frac{d^2Y}{dx^2} + \left(-\frac{\alpha^2}{4} + \frac{\frac{1}{2}-\eta}{x} + \frac{\frac{1}{4}-\frac{\beta^2}{4}}{x^2}\right. \tag{83}$$
$$\left.+\frac{-\frac{1}{2}+\delta+\eta}{x-1} + \frac{\frac{1}{4}-\frac{\gamma^2}{4}}{(x-1)^2}\right)Y = 0, \tag{84}$$
$$Y = e^{\frac{1}{2}\alpha x}x^{\frac{1}{2}(\beta+1)}(x-1)^{\frac{1}{2}(\gamma+1)}\,\text{HeunC}. \tag{85}$$

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