QUANTUM GROUPS [1]

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1. Introduction

1.1. History. The roots of quantum groups lie in physics. They are related with the studies of integrable systems in quantum field theory and statistical mechanics, using the quantum inverse scattering method, by Sklyanin, Kulish, Reshetikhin, Takhtajan, and Faddeev in the 1980s. Mathematical abstraction and the concept of the quantum groups (1985-1995) and the applications of quantum groups from different mathematical and physical points of view (after 1990) are given by Drinfeld, Jimbo, Manin, Woronowicz, Majid, Lustzig, Connes, Wess, Zumino, Macfarlane, Rosso, Biedenharn, etc.
At the end, in references, I shall give the list of important articles and books which would give more details and lead to the literature on the subject of quantum groups and related topics and their applications.

1.2. Notion of quantization. It is well-known that all physical systems are quantum mechanical. The quantum mechanical behavior is generally revealed only at the molecular and deeper level. At the macroscopic level of everyday experience quantum physics becomes classical physics as a good approximation. The quantum physics was discovered in the 20th century, as a consequence of failure of classical physics at the atomic level. In classical and quantum physics there exist two main concepts in the description of physical systems: **states and observables**. In classical physics the states of a system are the elements of some manifolds $\mathcal{M}$ and the observables are the (real) functions on $\mathcal{M}$, but in quantum physics the states are $1-$dimensional subspaces of some Hilbert space $\mathcal{H}$ and the observables are the self-adjoint operators on $\mathcal{H}$. It is easier to understand the connection between classical and quantum physics in terms of observables. In both cases the obsevables form an associative algebra which is **commutative in classical physics** and **non-commutative in quantum physics**. More precisely, let $\mathcal{O}$ be set of all "admissible" observables, then in both cases, $\mathcal{O}$ is a vector space with standard operations (addition of functions and multiplication of function by

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scalars). In the case of classical physics $\mathcal{O}$ is an associative commutative algebra with multiplication defined by

$$[f,g](p) = f(p)g(p) - g(p)f(p) = 0, \quad \forall f,g \in \mathcal{O}, m \in \mathcal{M}, \tag{1}$$

and in quantum physics $\mathcal{O}$ is an associative non-commutative algebra with multiplication defined by

$$[A,B](p) = A(B(p)) - B(A(p)) \quad \forall A,B \in \mathcal{O}, p \in \mathcal{H}. \tag{2}$$

The transition from classical physics to quantum physics can be mathematically described as a **process of deformation** of the classical physics in which the commuting classical observables of a physical system are replaced by non-commuting self-adjoint operators. This process is characterized by a very small deformation parameter known as the Planck constant $\hbar$,

$$[x, p_x] = \imath\hbar$$

and roughly speaking in the limit $\hbar \longrightarrow 0$ quantum physics become classical physics.
So, the notion of the quantization from mathematical point of view is the replacement of some commutative object with non-commutative one. This is main reason of the appearance of the word quantum in the name of our object. Quantum groups are not groups in ordinary sense, but essentially they are deformed enveloping algebras which are non-cocommutative.

1.3. A classical example. Let us consider a two dimensional classical vector space $V^2 = \{v = (x,y)^\tau \mid x,y \in \mathbb{R}\}$ (τ means transpose of a matrix) and let us make a linear transformation of the vector v which preserves orientation and area,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3}$$

then $a\,d - b\,c = \det M = 1$ so M is an element of the group $SL(2,\mathbb{R})$.
The group $SL(2,\mathbb{R})$ is a Lie group, it means that this set is a group (subgroup of $GL(2,\mathbb{R})$), manifold and the multiplication and inverse mapping are smooth mappings. It is well-known that the structure of such groups is determined (almost completely) by the structure of some neighborhoods of the unit element. One can show that on the dense set an element g of Lie group $SL(2,\mathbb{R})$, can be parameterized as

$$g = g(w,t,u) = \begin{pmatrix} e^t & e^t\,u \\ e^t\,w & e^{-t} + e^t\,u\,w \end{pmatrix} = e^{w\,F}\,e^{t\,H}\,e^{u\,E}, \tag{4}$$

where the parameters $\{w,t,u\}$ characterize the group element g and where the matrices $\left\{ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$, are constant.

When the group element g is close to the identity element $\mathbb{1} = g(0,0,0)$ i.e. when the parameters $\{w, t, u\}$ are sufficiently close to the 0, then one can write

$$g \approx I + w\,F + t\,H + u\,E. \tag{5}$$

It is well-known that the matrices $\{E, F, H\}$ are basis of Lie algebra $\mathfrak{sl}(2)$ and their commutators are

$$[H, E] = 2\,E, \qquad [H, F] = -\,2\,F, \qquad [E, F] = H. \tag{6}$$

2. Quantum groups and algebras

2.1. The universal enveloping algebra as Hopf algebra. $\mathfrak{U} = \mathfrak{U}\,(\mathfrak{sl}(2))$, of Lie algebra $\mathfrak{sl}(2)$ is the associative $\mathbb{C}-$algebra, which is generated by $\mathbb{1}, E, F, H$ and relations (6). It means that $\mathfrak{U}$ is a complex vector space together with $\mathbb{C}-$linear mappings $m : \mathfrak{U} \otimes \mathfrak{U} \longrightarrow \mathfrak{U}$ and $e : \mathbb{C} \longrightarrow \mathfrak{U}$ such that the following diagrams commute,

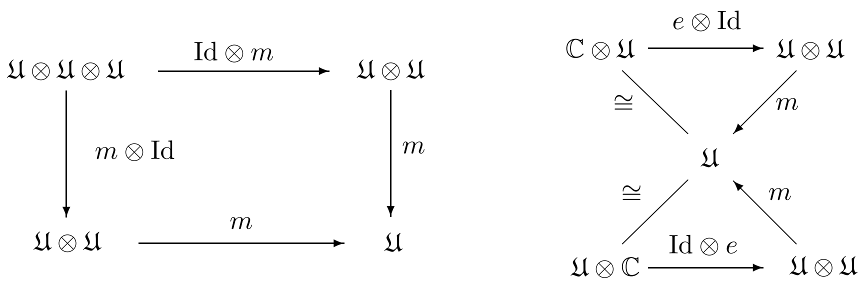

Fig. 1. Associativity of multiplication and properties of unit

The mapping m is multiplication and e is unit. Usually, we replace m by . Using relations (6), it is easy to see that the monomials,

$$E^n\,H^m\,F^l \qquad\qquad n, m, l \in \mathbb{N}_0,$$

form the basis (Poincaré-Birkhoff-Witt theorem) of the algebra $\mathfrak{U}\,(\mathfrak{sl}(2))$. So, $\mathfrak{U}\,(\mathfrak{sl}(2))$ as vector space is isomorphic to the algebra of polynomials (symmetric algebra) in the three variables E, H, F.

On the algebra $\mathfrak{U}$ is possible to introduce the two operations *comultiplication* *(*Δ*)* and *counit (*ϵ*)*:

$$\begin{array}{lll} \Delta : \mathfrak{U} \longrightarrow \mathfrak{U} \otimes \mathfrak{U}, & \Delta(X) = X \otimes \mathbb{1} + \mathbb{1} \otimes X, & \forall X \in \mathfrak{U}, \\ \epsilon : \mathfrak{U} \longrightarrow \mathbb{C}, & \epsilon(X) = 0, & \forall X \in \mathfrak{U}, \end{array} \tag{7}$$

which are $\mathbb{C}$-linear mappings such that the following diagrams commute,

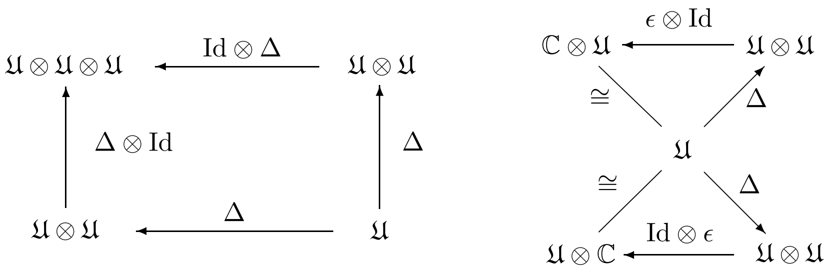

Fig. 2. Coassociativity of comultiplication and properties of counit

it means that on the $\mathfrak{U}$ we have the structure of **coalgebra**. We say that a $\mathbb{C}-$algebra (or coalgebra) $\mathbb{A}$ is commutative (cocommutative) if for the transposition map, $\sigma:\mathbb{A}\otimes\mathbb{A}\longrightarrow\mathbb{A}\otimes\mathbb{A}$, $\sigma(X\otimes Y)=Y\otimes X$, the following diagrams commute

Fig. 3. Commutativity (Cocommutativity) of a $\mathbb{C}-$algebra (or coalgebra) $\mathbb{A}$

It is clear that our algebra $\mathfrak{U}(\mathfrak{sl}(2))$ is not commutative and it is easy to see that it is cocommutative. The structures of algebra and coalgebra on $\mathfrak{U}$ are connected by commutativity of the following diagram:

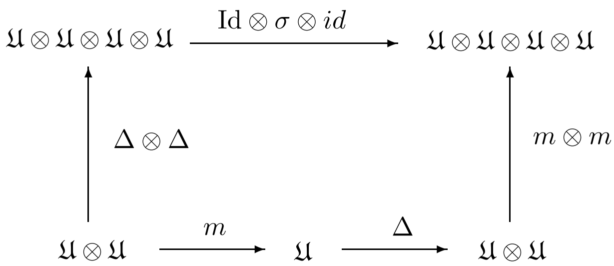

Fig. 4. Compatibility of algebra and coalgebra structures of $\mathfrak{U}$

We say that an algebra and coalgebra $\mathbb{A}$, together with property described by above diagram (Fig.4) is a **bialgebra**. So, $\mathfrak{U}$ is a bialgebra. One can easily check that the map given by

$$\mathcal{S}:\mathfrak{U}\longrightarrow\mathfrak{U},\qquad \mathcal{S}(X)=-\,X,\qquad \forall X\in\mathfrak{U}, \tag{8}$$

is an automorphism of $\mathfrak{U}$ ($\mathcal{S}(X\,Y)=\mathcal{S}(Y)\,\mathcal{S}(X)$) which is called **antipod**. In our case, the antipodal map has property described by the following commutative diagram,

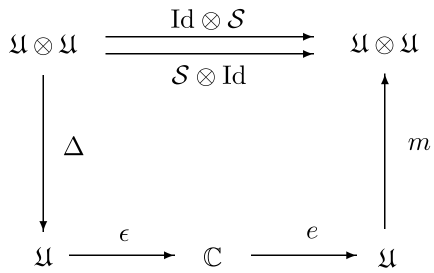

Fig. 5. Compatibility of antipod with bialgebra structure of $\mathfrak{U}$

which made of bialgebra $\mathfrak{U}$ a **Hopf algebra**. Taking in account the cocommutativity of $\mathfrak{U}$ we see that $\mathfrak{U}$ is a cocommutative Hopf algebra.
The commutative diagrams Fig.1-5 describe the following relations for all $X, Y, Z \in \mathfrak{U}$,

$$(m \circ (m \otimes \mathrm{Id}))(X \otimes Y \otimes Z) = (m \circ (\mathrm{Id} \otimes m))(X \otimes Y \otimes Z) \tag{9}$$
$$(m \circ (\mathrm{Id} \otimes e))(X \otimes 1) = \mathrm{Id}(X) = (m \circ (e \otimes \mathrm{Id}))(1 \otimes X) \tag{10}$$
$$((\Delta \otimes \mathrm{Id}) \circ \Delta)(X) = ((\mathrm{Id} \otimes \Delta) \circ \Delta)(X) \tag{11}$$
$$((\mathrm{Id} \otimes \epsilon) \circ \Delta)(X) = \mathrm{Id}(X) = ((\epsilon \otimes \mathrm{Id}) \circ \Delta)(X) \tag{12}$$
$$(\sigma \circ \Delta)(X) = \Delta(X) \tag{13}$$
$$((m \otimes m) \circ (\mathrm{Id} \otimes \sigma \otimes \mathrm{Id}) \circ (\Delta \otimes \Delta))(X \otimes Y) = (\Delta \circ m)(X \otimes Y) \tag{14}$$
$$(m \circ (\mathrm{Id} \otimes \mathcal{S}) \circ \Delta)(X) = e \circ \epsilon)(X) = (m \circ (\mathcal{S} \otimes \mathrm{Id}) \circ \Delta)(X) \tag{15}$$

2.2. Quantum plane. Let us recall classical two dimensional vector space V^2 and let the transformation of change of coordinates, $M \in SL(2, \mathbb{R})$ be as in (3). We assume that the coordinate x and y of a vector $v = (x, y)^\tau$ commute

$$x\,y = y\,x, \tag{16}$$

then on the space of function $\mathcal{O} = \{f(v) = f(x, y) \mid v \in V^2\}$ (observables) we define partial derivative operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ and the operations of multiplications by x and y. They define the differential calculus on $\mathcal{O}$ by the following relations

$$\begin{aligned} [x\,,\,y] \;&=\; 0\,, \qquad \left[\tfrac{\partial}{\partial x}\,,\,\tfrac{\partial}{\partial y}\right] = 0\,, \qquad \left[\tfrac{\partial}{\partial x}\,,\,y\right] \;=\; 0\,, \\ \left[\tfrac{\partial}{\partial y}\,,\,x\right] \;&=\; 0\,, \qquad \left[\tfrac{\partial}{\partial x}\,,\,x\right] = 1\,, \qquad \left[\tfrac{\partial}{\partial y}\,,\,y\right] \;=\; 1\,. \end{aligned} \tag{17}$$

Then, the new coordinates x' and y' and partial derivatives with respect to them, $\frac{\partial}{\partial x'}$ and $\frac{\partial}{\partial y'}$, also satisfy the similar relations as (3). One can easily checked that the following relation holds,

$$\begin{pmatrix} \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = (M^{-1})^\tau \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}. \tag{18}$$

We say that the differential calculus on the two dimensional (x, y)-plane is covariant under the group $SL(2, R)$.

Analogously with the process of quantizing the classical physics let us now quantize the classical vector space V^2 to get a quantum vector space V_q^2, assuming that the coordinates do not commute with each other at any point. We assume that the coordinates of any vector $v = (X, Y)^\tau$ of quantum plane, V_q^2, not commute and satisfy

$$X\,Y = q\,Y\,X\,, \tag{19}$$

where q is the deformation parameter (any nonzero complex number) and X and Y commute with scalars. Note that in the limit $q \longrightarrow 1$ the non-commuting quantum coordinates X and Y become commuting classical coordinates.

Example. Let us consider the following operators A_β and $B_{\theta/\beta}$ be operators acting on functions of a real variable x as follows

$$A_\beta \xi(x) = \xi(x - \beta)\,, \quad B_{\theta/\beta}\xi(x) = e^{\,i\theta x/\beta}\xi(x)\,.$$

Then, for any $\xi(x)$,

$$A_\beta B_{\theta/\beta}\xi(x) = e^{\,i\theta(x-\beta)/\beta}\xi(x-\beta) = e^{-\,i\theta} B_{\theta/\beta} A_\beta \xi(x)\,,$$

for a given fixed value of θ. So, the operators A_β and $B_{\theta/\beta}$ are non-commutative variables (in β) obeying the relation

$$A_\beta B_{\theta/\beta} = e^{-i\,\theta} B_{\theta/\beta} A_\alpha \; = q\, B_{\theta/\beta} A_\alpha\,,$$

with fixed value of θ.

Let us try to define a differential calculus on the two dimensional quantum (X, Y)-plane such that it will be covariant under some generalization of the classical group $SL(2)$, which will act on V_q^2.

Firstly, let us give a meaning to partial derivatives with respect to X and Y. These have to operate on the space of polynomials in X and Y, by which is possible to approximate any "sufficiently good" function in X, Y. We can write $f(X, Y) = \sum_{n,l} \alpha_{nl} X^n Y^l$ since any polynomial in X and Y, with coefficients commuting with X and Y, can be rewritten in this form using the commutation relation (19). If we take

$$\frac{\partial}{\partial X} X^n = n\, X^{n-1}\,, \quad \frac{\partial}{\partial Y} Y^l = l\, Y^{l-1}\,, \tag{20}$$

we would have a differential calculus in the quantum (X, Y)-plane, as desired. Now, we can calculate the remaining commutation relations between

X, Y, $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$ and we obtain (without any further details),

$$X\,Y = qY X\,, \quad \frac{\partial}{\partial X}\frac{\partial}{\partial Y} = q^{-1}\frac{\partial}{\partial Y}\frac{\partial}{\partial X}\,, \quad \frac{\partial}{\partial X}Y = qY\frac{\partial}{\partial X}\,, \tag{21}$$

$$\frac{\partial}{\partial Y}X = qX\frac{\partial}{\partial Y}\,, \quad \frac{\partial}{\partial Y}Y - q^2 Y\frac{\partial}{\partial Y} = 1\,, \tag{22}$$

$$\frac{\partial}{\partial X}X - q^2 X\frac{\partial}{\partial X} = 1 + (q^2-1)Y\frac{\partial}{\partial Y}\,. \tag{23}$$

This noncommutative differential calculus on the two-dimensional quantum plane is covariant under the transformations

$$\begin{aligned} \begin{pmatrix} X' \\ Y' \end{pmatrix} &= T\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} \\ \begin{pmatrix} \frac{\partial}{\partial X'} \\ \frac{\partial}{\partial Y'} \end{pmatrix} &= (T^{-1})^{\tau}\begin{pmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{pmatrix} = \begin{pmatrix} D & -\,q\,C \\ -\,q^{-1}B & A \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{pmatrix}, \end{aligned} \tag{24}$$

where

$$\begin{aligned} &A, B, C, \text{ and } D \text{ commute with } X \text{ and } Y \\ &A\,B = q\,B\,A\,, \quad C\,D = q\,D\,C\,, \quad A\,C = q\,C\,A\,, \quad B\,D = q\,D\,B\,, \quad (25) \\ &B\,C = C\,B\,, \quad A\,D - D\,A = \left(q - q^{-1}\right)B\,C\,, \qquad \text{and} \\ &A\,D - q\,B\,C = \det{}_q T = 1\,. \quad (26) \end{aligned}$$

The operators $X', Y', \frac{\partial}{\partial X'}$, and $\frac{\partial}{\partial Y'}$ satisfy same relations (3) and (18) (if one replace x with X, y with Y, x' with X' and y' with Y'). It is easy to see that $\det_q T$ defined in (26) commutes with all the matrix elements of T and that matrix

$$T^{-1} = \begin{pmatrix} D & -q^{-1}B \\ -q\,C & A \end{pmatrix}$$

is inverse of T, i.e.

$$T\,T^{-1} = T^{-1}\,T = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A matrix $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is called a 2×2 **quantum matrix** if its matrix elements $\{A, B, C, D\}$ satisfy the commutation relations (25), it practically means that the matrix elements of T may be ordinary classical matrices satisfying the relations (25).

Firstly note that the identity matrix, $\mathbb{1}$, is a quantum matrix and secondly if we take the limit $q \longrightarrow 1$ a quantum matrix T becomes a classical matrix with commuting elements.

Let $T_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ and $T_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$ be any two quantum matrices; *i.e.*, $\{A_1, B_1, C_1, D_1\}$ and $\{A_2, B_2, C_2, D_2\}$ satisfy the relations (25). Define the product

$$\begin{aligned} \Delta_{12}(T) = T_1 \dot{\otimes} T_2 &= \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \dot{\otimes} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1 \otimes A_2 + B_1 \otimes C_2 & A_1 \otimes B_2 + B_1 \otimes D_2 \\ C_1 \otimes A_2 + D_1 \otimes C_2 & C_1 \otimes B_2 + D_1 \otimes D_2 \end{pmatrix} \\ &= \begin{pmatrix} \Delta_{12}(A) & \Delta_{12}(B) \\ \Delta_{12}(C) & \Delta_{12}(D) \end{pmatrix} \end{aligned} \tag{27}$$

where $\otimes$ denotes product with the property $(P \otimes R)(Q \otimes S) = PQ \otimes RS$. Then one finds that the matrix elements of $\Delta_{12}(T)$, namely,

$$\begin{aligned} \Delta_{12}(A) &= A_1 \otimes A_2 + B_1 \otimes C_2\,, \quad \Delta_{12}(B) = A_1 \otimes B_2 + B_1 \otimes D_2\,, \\ \Delta_{12}(C) &= C_1 \otimes A_2 + D_1 \otimes C_2\,, \quad \Delta_{12}(D) = C_1 \otimes B_2 + D_1 \otimes D_2\,, \end{aligned} \tag{28}$$

also satisfy the commutation relations (25). In other words, $\Delta_{12}(T)$ is also a quantum matrix. This product, $\Delta_{12}(T) = T_1 \dot{\otimes} T_2$, is called the coproduct or comultiplication. But this product is not so good because there is no inverse. Under this coproduct the set of 2×2 quantum matrices $\mathbb{M}_q^2$ form a pseudomatrix group, sometimes called a **quantum group**, denoted by $SL_q(2)$.

2.3. Quantum group $\mathfrak{U}_q = \mathfrak{U}_q(\mathfrak{sl}(2))$. First, let us recall the q-numbers $[n]_q$, which were defined by Heine (1846).

$$\begin{aligned} &[n]_q = \frac{1-q^n}{1-q}\,, \qquad [0]_q! = 1 \\ &[n]_q! = [n]_q\,[n-1]_q\,[n-2]_q \ldots [2]_q\,[1]_q\,, \quad n = 1, 2, \ldots\,. \end{aligned} \tag{29}$$

Consider now the following $q-$generalization of the exponential function, known as q-exponential function

$$e_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}\,. \tag{30}$$

Note that

$$[n]_q \xrightarrow{q \to 1} n\,, \qquad e_q^z \xrightarrow{q \to 1} e^z\,.$$

In the theory of quantum groups we slightly modify the q-numbers, i.e we define

$$[\![n]\!]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\,. \tag{31}$$

The reason why new $q-$ numbers, $[\![n]\!]_q$ are introduced is in their symmetry with respect to the interchange of q and q^{-1}. Note that $[\![n]\!]_q$ also becomes n in the limit $q \to 1$.

Now, we follow same idea as in classical case, consider the 2-dimensional quantum matrix T parameterized as

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} e^t & e^t\, u \\ e^t\, w & e^{-t} + e^t\, u\, w \end{pmatrix}. \tag{32}$$

The condition that T have to be a quantum matrix (25) leads to the following requirements on the variable parameters $\{t, u, w\}$

$$[t, u] = (\ln q)\, u\,, \qquad [t, w] = (\ln q)\, w,\,, \qquad [u, w] = 0\,.$$

Then, one can write

$$T = e_{q^{-2}}^{w\,F}\, e^{t\,H}\, e_{q^2}^{u\,E}\,, \tag{33}$$

where F, H and E are same as in (5) and (6). By similar procedure as in classical case one can find out from (33), that generators $\{F, H, E\}$ have to obey the following relations:

$$[H,\, E] = 2\,E, \quad [H,\, F] = -\,2\,F, \quad [E,\, F] = \frac{q^H - q^{-H}}{q - q^{-1}} = [\![H]\!]_q\;, \tag{34}$$

where $q = e^h$ ($h \in \mathbb{C}$ and here plays role of the Planck constant) and this algebra is $q-$analog of $\mathfrak{sl}(2)$, which is also called the *quantum algebra,* $\mathfrak{sl}_q(2)$. The relations (34)(last one), imply that we lose the structure of Lie algebra on the vector space generated by $\{F, H, E\}$, but in the limit $q \to 1$ (or equivalently $h \to 0$), the last relation of (34) becomes classical one and we obtain the Lie algebra $\mathfrak{sl}(2)$. Let us now see what is happened at the level of universal enveloping algebra. Let us consider $\mathbb{C}-$associative algebra generated by $\mathbb{1}, F, H, E$ and relations (34). Of course, this object is an algebra which is the q-deformation of the universal enveloping algebra $\mathfrak{U}(\mathfrak{sl}(2))$ and is called **quantum group** of Lie algebra $\mathfrak{sl}(2)$. We denote it by $\mathfrak{U}_q = \mathfrak{U}_q(\mathfrak{sl}(2))$.
From relations (34) is clear that the algebra $\mathfrak{U}_q(\mathfrak{sl}(2))$ is generated by polynomials in $\{F, H, [\![H + 2\,k \cdot \mathbb{1}]\!]\ (k \in \mathbb{Z}), E\}$, so an analog of the Poincaré-Birkoff-Witt theorem holds for $\mathfrak{U}_q(\mathfrak{sl}(2))$. Moreover, algebra $\mathfrak{U}_q(\mathfrak{sl}(2))$ is also a Hopf algebra with respect to coproduct(Δ_q), counit(ϵ_q) and antipod($\mathcal{S}_q$) defined by:

$$\begin{aligned} &\Delta_q\,(H) = H \otimes \mathbb{1} + \mathbb{1} \otimes H, && \epsilon_q\,(H) = 0, && \mathcal{S}_q\,(H) = -\,H, \\ &\Delta_q\,(E) = E \otimes q^{H/2} + q^{-\,H/2} \otimes E, && \epsilon_q\,(E) = 0, && \mathcal{S}_q\,(E) = -\,q^{-1}E, \\ &\Delta_q\,(F) = F \otimes q^{H/2} + q^{-\,H/2} \otimes F, && \epsilon_q\,(F) = 0, && \mathcal{S}_q\,(F) = -\,q\,F. \end{aligned} \tag{35}$$

It means that coproduct(Δ_q), counit(ϵ_q) and antipod($\mathcal{S}_q$) are $\mathbb{C}-$linear mappings of $\mathfrak{U}_q$ and that relations (9-15) hold for those operations.
For example, one can easily verified that this comultiplication rule is an algebra isomorphism of $\mathfrak{U}_q$:

$$\begin{aligned} &[\Delta_q(H), \Delta_q(E)] = 2\,\Delta_q(E), \quad [\Delta_q(H), \Delta_q(F)] = -\,2\,\Delta_q(F), \\ &[\Delta_q(E), \Delta_q(F)] = [\![\Delta_q(H)]\!]_q . \end{aligned} \tag{36}$$

The most important property of this coproduct is its noncommutativity. So, as a **result of our quantization process we associate with cocommutative Hopf algebra $\mathfrak{U}(\mathfrak{sl}(2))$ the noncommutative one, namely, $\mathfrak{U}_q(\mathfrak{sl}(2))$.**

2.4. $\mathcal{R}-$matrix. Note that the algebra (34) is invariant under the interchange $q \leftrightarrow q^{-1}$ since $[\![2\mathcal{X}_0]\!]_q = [\![2\mathcal{X}_0]\!]_{q^{-1}}$, but the comultiplication (35) is not invariant under same interchange. It is obvious that the comultiplication obtained from (35) by an interchange $q \leftrightarrow q^{-1}$ should also be an equally good comultiplication. One can verified that the coproduct $\Delta_{q^{-1}}$

$$\begin{aligned} &\Delta_{q^{-1}}(H) = H \otimes \mathbb{1} + \mathbb{1} \otimes H, \quad \Delta_{q^{-1}}(E) = E \otimes q^{-H/2} + q^{H/2} \otimes E, \\ &\Delta_q(F) = F \otimes q^{-H/2} + q^{H/2} \otimes F, \end{aligned} \tag{37}$$

is also an algebra isomorphism of $\mathfrak{U}_q$. This coproduct, $\Delta_{q^{-1}}$, is called the opposite coproduct because of the obvious relation

$$\Delta_{q^{-1}}(X) = \sigma\left(\Delta_q(X)\right), \quad \text{where} \;\; \sigma(X \otimes Y) = Y \otimes X .$$

It is clear $\Delta_{q^{-1}} \neq \Delta_q$ and $\sigma \circ \Delta \neq \Delta$. So, the comultiplications Δ_q and $\Delta_{q^{-1}}$ of $\mathfrak{U}_q$ are noncommutative and moreover in the limit $q \to 1$, both of those comultiplications become the classical one of $\mathfrak{U}$ ($\Delta(X) = X \otimes \mathbb{1} + \mathbb{1} \otimes X$,) which is commutative.
It is well-known that those two comultiplications of $\mathfrak{U}_q$, Δ_q and $\Delta_{q^{-1}}$, are connected by an equivalence relation, namely, there exists a $\mathcal{R} \in \mathfrak{U}_q \otimes \mathfrak{U}_q$, called the **universal $\mathcal{R}$-matrix**, satisfying the relation

$$\Delta_{q^{-1}}(X) = \mathcal{R}\,\Delta_q(X)\,\mathcal{R}^{-1} . \tag{38}$$

The Hopf algebra which allow the existence of $\mathcal{R}$ matrix is called *quasitriangular Hopf algebra.* So, the quantum group $\mathfrak{U}_q$ is a quasitriangular Hopf algebra.
The universal $\mathcal{R}$-matrix is very important and it plays role of the central object of the quantum group theory. In our case it can be shown that

$$\mathcal{R} = q^{1/2\,(H \otimes H)} \sum_{n=0}^{\infty} \frac{(1-q^2)^n}{[\![n]\!]_q!}\, q^{n(n-1)/2} \left(q^{H/2}\, E \otimes q^{-H/2}\, F\right)^n . \tag{39}$$

If we replaced the matrices from (5) and (6) in above expression for $\mathcal{R}$ we get numerical R-matrix, the fundamental 4-dimensional R-matrix

$$R = \frac{1}{\sqrt{q}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & \left(q - q^{-1}\right) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} . \tag{40}$$

It is clear from (39), that any R we can rewrite in the form

$$R = \sum_i a_i \, u_i \otimes v_i \, .$$

If we put

$$R_{12} = R \otimes \mathbb{1} \, , \quad R_{13} = \sum_i a_i \, u_i \otimes \mathbb{1} \otimes v_i \, , \quad R_{23} = \mathbb{1} \otimes R \, .$$

Then, these satisfy the famous relation

$$R_{12} \, R_{13} \, R_{23} = R_{23} \, R_{13} \, R_{12} \, . \tag{41}$$

known as the quantum Yang-Baxter equation, or simply the Yang-Baxter equation (YBE).

Let us mention a few interesting propeties of quantum groups and their applications. Firstly, let us see how these things started, for a quantum matrix T we define

$$\begin{aligned} T_1 &= T \otimes \mathbb{1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \\ T_2 &= \mathbb{1} \otimes T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} A & B \\ C & D \end{pmatrix} . \end{aligned} \tag{42}$$

One can notice that

$$T_1 \, T_2 = \begin{pmatrix} A^2 & AB & BA & B^2 \\ AC & AD & BC & BD \\ CA & CB & DA & DB \\ C^2 & CD & DC & D^2 \end{pmatrix} \neq T_2 \, T_1 = \begin{pmatrix} A^2 & BA & AB & B^2 \\ CA & DA & CB & DB \\ AC & BC & AD & BD \\ C^2 & DC & CD & D^2 \end{pmatrix} , \tag{43}$$

because $\{A, B, C, D\}$ are noncommutative. The relation between $T_1 \, T_2$ and $T_2 \, T_1$ turns out to be

$$R \, T_1 \, T_2 = T_2 \, T_1 \, R \, . \tag{44}$$

The relations of this type was appeared in the quantum inverse scattering method approach to integrable models in quantum field theory and statistical mechanics. If we substitute in (44) R from (40), and T_1 and T_2 from (42), it is found that equation (44) is a compact way of stating the commutation relations (25) defining the quantum matrix T and $SL_q(2)$.

Similarly the defining commutation relations (34) of $\mathfrak{U}_q(\mathfrak{sl}(2))$ is possible to write in more compact and elegant way. Let it be

$$\begin{aligned} L^{(+)} &= \begin{pmatrix} q^{-H/2} & -\sqrt{q}\left(q-q^{-1}\right)F \\ 0 & q^{H/2} \end{pmatrix}, \\ L^{(-)} &= \begin{pmatrix} q^{H/2} & 0 \\ q^{-1/2}\left(q-q^{-1}\right)E & q^{-H/2} \end{pmatrix}, \\ L_1^{(\pm)} &= L^{(\pm)} \otimes \mathbb{1}, \quad L_2^{(\pm)} = \mathbb{1} \otimes L^{(\pm)}. \end{aligned} \tag{45}$$

Then, the commutation relations (34), between the generators of $\mathfrak{U}_q(\mathfrak{sl}(2))$, can be rewritten as

$$R^{-1} L_1^{(\pm)} L_2^{(\pm)} = L_2^{(\pm)} L_1^{(\pm)} R^{-1}, \quad R^{-1} L_1^{(+)} L_2^{(-)} = L_2^{(-)} L_1^{(+)} R^{-1}. \tag{46}$$

One can easily check that the $L^{(\pm)}$-matrices are special quantum matricies from $SL_q(2)$ (they satisfy relations (34)).

Let R be any solution of YBE, (41), if we define

$$S_1 = \check{R} \otimes \mathbb{1}, \qquad S_2 = \mathbb{1} \otimes \check{R},$$

where

$$\check{R} = P\,R, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{47}$$

then, the following relation holds

$$S_1\, S_2\, S_1 = S_2\, S_1\, S_2, \tag{48}$$

which is an alternative form of the YBE.

Let us mention here that the generators of symmetric group S_n, namely, the transpositions $\sigma_1 = (1\,2), \sigma_2 = (2\,3), \ldots, \sigma_{n-1} = (n-1\,n)$, satisfy relations

$$\sigma_i\,\sigma_j = \sigma_j\,\sigma_i, \qquad |i-j| > 1, \tag{49}$$

$$\sigma_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\sigma_{i+1}, \qquad i = 1, 2, \ldots, n, \tag{50}$$

$$\sigma_i^2 = \mathrm{Id}, \qquad i = 1, 2, \ldots, n.$$

So, the relation (48) is same as relations (50). The one of the most natural generalizations of symmetric group S_n is obtained if we consider the group generated by generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations (49) and (50). This generalization is called **braid group**. An element of the braid group B_n can be described as a system of n strings joining two sets of n points located on two parallel lines, say top and bottom, with over-crossings or under-crossings of the strings.

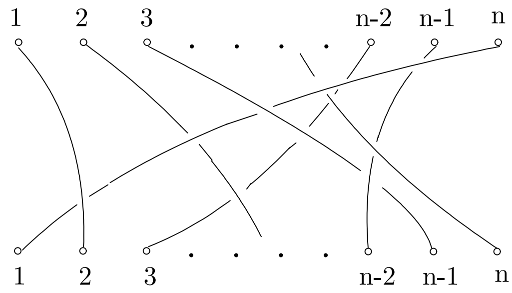

Fig. 6. An element of braid group B_n

The over-crossings and the under-crossings of the strings make B_n (see Fig. 6) an infinite group. Let us describe the generators of this group. Let i and $i+1$ be two consecutive points on the top and bottom lines, the string starting at i on the top line can reach $i+1$ on the bottom line by either over-crossing or under-crossing the string starting at $i+1$ on the top line and reaching i on the bottom line. The corresponding elements of the braid group are usually denoted by σ_i and σ_i^{-1}, respectively (Fig. 7).

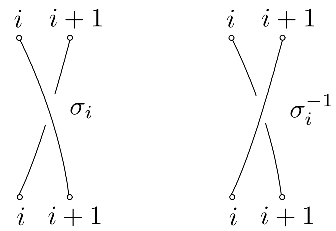

Fig. 7. σ_i and σ_i^{-1}

It is clear that the relations (48) and (50) are same and this fact implies that the solutions of the YBE (R-matrices), or the quantum groups, should play a central role in the theory of representations of braid groups. Braid groups have many applications, in mathematics (knot theory, complex functions of hypergeometric type having several variables) and in physics (statistical mechanics, two-dimensional conformal field theory, and so on).

Quantum and classical mechanics, up to recently, are based on the following two very important assumptions: the notion of continuous space-time and mutual commutativity of all coordinates (spacelike or timelike). It is natural to state the following question:

What will be happened if at some deeper level the space-time coordinates themselves are non-commutative?

It is obvious, that the theory of quantum groups and non-commutative differential calculus provide the necessary framework for dealing in such situation.

Example. The q-deformation of the quantum mechanical harmonic oscillator algebra, i.e. the boson algebra. The algebraic treatment of boson algebra include a creation operator $\left(a^{\dagger}\right)$, an annihilation operator (a), and a number operator (N), and the commutation relations,

$$\left[a\,,\,a^{\dagger}\right]=1\,,\qquad \left[N\,,\,a^{\dagger}\right]=a^{\dagger}\,, \tag{51}$$

where N is a hermitian operator and $a^{\dagger}$ is the hermitian conjugate of a. The eigenvalues of the Hamiltonian operator,

$$H=\frac{1}{2}\left(a^{\dagger}\,a+a\,a^{\dagger}\right)\,, \tag{52}$$

give the energy spectrum of the harmonic oscillator. Let $\left\{a_1,a_1^{\dagger},N_1\right\}$ and $\left\{a_2,a_2^{\dagger},N_2\right\}$ be the sets of oscillator operators, which are assumed to commute with each other, and defining

$$X_0=\frac{1}{2}\left(N_1-N_2\right)\,,\quad X_+=a_1^{\dagger}a_2\,,\quad X_-=a_2^{\dagger}a_1\,, \tag{53}$$

it is found that

$$X_0^{\dagger}=X_0\,,\quad X_+^{\dagger}=X_-\,, \tag{54}$$

and

$$\left[X_0\,,\,X_{\pm}\right]=\pm\,X_{\pm}\,,\quad \left[X_+\,,\,X_-\right]=2\,X_0\,. \tag{55}$$

The vector space spanned by $\{X_0,X_+,X_-\}$ and relations (55) is the Lie algebra $\mathfrak{sl}(2)$, and because of the hermiticity conditions (54) this Lie algebra is known as $\mathfrak{su}(2)$, the Lie algebra of the Lie group $SU(2)$. The $\mathfrak{su}(2)$ is known as the algebra of three dimensional rigid rotator, where $\{X_0,X_{\pm}\}$ represent the angular momentum operators. The coproduct rule

$$\Delta\left(X_0\right)=X_0\otimes\mathbb{1}+\mathbb{1}\otimes X_0\,,\quad \Delta\left(X_{\pm}\right)=X_{\pm}\otimes\mathbb{1}+\mathbb{1}\otimes X_{\pm}\,, \tag{56}$$

of enveloping algebra $\mathfrak{U}\left(\mathfrak{sl}(2)\right)$, represent the rule for addition of angular momenta. Then one can "quantize" algebra $\mathfrak{su}(2)$, i.e. we rewrite relations (34) as

$$\left[\mathcal{X}_0\,,\,\mathcal{X}_{\pm}\right]=\pm\mathcal{X}_{\pm}\,,\quad \left[\mathcal{X}_+\,,\,\mathcal{X}_-\right]=\left[\!\left[2\,X_0\right]\!\right]_q\,, \tag{57}$$

with the hermiticity conditions

$$\mathcal{X}_0^{\dagger}=\mathcal{X}_0\,,\quad \mathcal{X}_-^{\dagger}=\mathcal{X}_+\,, \tag{58}$$

represent the $U_q(\mathfrak{su}(2))$ algebra or $\mathfrak{su}_q(2)$ the q-deformed version of the $\mathfrak{su}(2)$. So, $\mathfrak{su}_q(2)$ is the algebra of the q-rotator. For the q-angular momentum operators there are two possible addition rules,

$$\Delta_{q^{\pm1}}\left(\mathcal{X}_0\right)=\mathcal{X}_0\otimes\mathbb{1}+\mathbb{1}\otimes\mathcal{X}_0\,,\quad \Delta_{q^{\pm1}}\left(\mathcal{X}_{\pm}\right)=\mathcal{X}_{\pm}\otimes q^{\pm\mathcal{X}_0}+q^{\mp\mathcal{X}_0}\otimes\mathcal{X}_{\pm}\,, \tag{59}$$

as seen from (35) and (37). Let us mention also that we have a realization of $\mathfrak{su}_q(2)$ generators given by

$$\mathcal{X}_0 = \frac{1}{2}\left(\mathcal{N}_1 - \mathcal{N}_2\right) , \quad \mathcal{X}_+ = A_1^\dagger A_2 , \quad \mathcal{X}_- = A_2^\dagger A_1 . \tag{60}$$

This is in complete analogy with the case of $\mathfrak{su}(2)$, where the two sets of operators $\left\{A_1, A_1^\dagger, \mathcal{N}_1\right\}$ and $\left\{A_2, A_2^\dagger, \mathcal{N}_2\right\}$ commute and obey the relations

$$AA^\dagger - qA^\dagger A = q^{-\mathcal{N}} , \qquad \left[\mathcal{N} , A^\dagger\right] = A^\dagger . \tag{61}$$

It is obvious that $\mathcal{N}$ is hermitian and $\{A , A^\dagger\}$ is a hermitian conjugate pair. The q-deformed oscillator algebra given by the relations (61) is known as the q-oscillator or the q-boson algebra. It is easy to see that when $q \longrightarrow 1$ the q-oscillator algebra (61) becomes the oscillator algebra (51).

3. Quantum group associate with Kac-Moody Lie algebra

3.1. Cartan matrix. Let $A = (a_{ij})_{i,j=1,\ldots,n}$ be a complex $n \times n$ matrix of rank l. A is called **generalized Cartan matrix** if

$$\begin{array}{lll} \text{(i1)} & a_{ij} \in \mathbb{Z}, & i,j = 1,2,\ldots,n, \\ \text{(i2)} & a_{ii} = 2, & i = 1,2,\ldots,n, \\ \text{(i3)} & a_{ij} \leq 0, & i \neq j, \quad i,j = 1,2,\ldots,n, \\ \text{(i4)} & a_{ij} = 0 \ \Rightarrow & a_{ji} = 0. \end{array} \tag{62}$$

If A satisfies the additional condition,

$$\text{(i5)} \quad \text{there exist } d_i \neq 0, \quad i = 1,2,\ldots,n \text{ such that}$$
$$d_i\, a_{ij} = d_j\, a_{ji}, \qquad i,j = 1,2,\ldots,n, \tag{63}$$

then we say that A is a *symmetrizable Cartan matrix.* The condition (63) means that it exists a regular diagonal matrix D such the matrix $D\,A$ is symmetric.

Firstly, to an arbitrary Cartan matrix one can associate a complex vector space $\mathfrak{h}$ and finite two subsets, namely $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \mathfrak{h}^*$ and $\Pi^v = \{H_1, H_2, \ldots, H_n\} \subseteq \mathfrak{h}$ such that

(i1) Π and Π^v are linearly independent,

(i2) $\langle H_i, \alpha_j\rangle = a_{ij}, \qquad i,j = 1,2,\ldots,n,$

(i3) $n - l = \dim\mathfrak{h} - n,$

and then to every generalized (symmetrizable) Cartan matrix one can associate a complex Lie algebra $\mathfrak{g}(A)$, as follows:

Let $\alpha_i(H) = a_{ij}$, $(i,j = 1,2,\dots,n)$, then Lie algebra $\mathfrak{g}(A)$ is generated by $\mathfrak{h}$, generators E_i, F_i, $(i = 1,\dots n)$ and relations

$$\begin{aligned}
&[E_i, F_j] = \delta_{ij} H_i, && i,j = 1,2,\dots,n,\\
&[H, H'] = 0, && H,\ H' \in \mathfrak{h},\\
&[H, E_i] = a_{ij}\, E_i, && i,j = 1,2,\dots,n;\ H \in \mathfrak{h},\\
&[H, F_i] = -\, a_{ij}\, E_i, && i,j = 1,2,\dots,n;\ H \in \mathfrak{h}.
\end{aligned} \tag{64}$$

If we consider the adjoint map $ad_X : \mathfrak{g}(A) \longrightarrow \mathfrak{g}(A)$, defined by formula $ad_X(Y) = [X,Y]$, than one can show that also hold following *Serre relations*:

$$\begin{aligned}
&(ad_{E_i})^{1-a_{ij}}(E_j) = 0, && i \neq j, \quad i,j = 1,2,\dots,n,\\
&(ad_{F_i})^{1-a_{ij}}(F_j) = 0, && i \neq j, \quad i,j = 1,2,\dots,n.
\end{aligned} \tag{65}$$

It is known that the deep theory of Lie algebras could be developed only for symmetrazible generalized Cartan matrices, because in that case is possible to define an invariant symmetric bilinear form (*standard invariant form*) on $\mathfrak{h}$. If A is non-decomposable generalized real Cartan matrix and if its rank is equal to n then we obtain classical finite dimensional Lie algebras, and when rank of A is $n-1$ we obtain very important class of *affine Lie algebras.*

Let us denote by $\mathfrak{n}_+$ ($\mathfrak{n}_-$) the subalgebras of $\mathfrak{g}(A)$ generated by $\{E_1, E_2 \dots, E_n\}$ ($\{F_1, F_2 \dots, F_n\}$), then we emphasize the following well-known facts

T1. $\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. (Cartan decomposition) (66)

T2. The mapping,
$E_i \to -\,F_i,\ F_i \to -\,E_i\,(i = 1,2,\dots,n),\ H \to -\,H\,(H \in \mathfrak{h})$,
are uniquely extendable to an involution ω of $\mathfrak{g}(A)$.

T3. The center of $\mathfrak{g}(A)$ is $\mathfrak{c} = \{H \in \mathfrak{h} \mid \alpha_i(H) = 0, i = 1,\dots,n\}$,
and $\dim \mathfrak{c} = n - l$.

The subalgebra $\mathfrak{h}$ of $\mathfrak{g}(A)$ is called *Cartan subalgebra*, the elements $\{E_i, F_i, i = 1,\dots,n\}$ are known as *Chevalley generators* of $\mathfrak{g}(A)$ and ω is known as *Chevalley involution.* For arbitrary Kac-Moody Lie algebra, $\mathfrak{g}(A)$, we define **universal enveloping algebra,** $\mathfrak{U}(\mathfrak{g}(A))$, on the following way. The universal enveloping algebra $\mathfrak{U}(\mathfrak{g}(A))$ is an associative $\mathbb{C}-$algebra generated by generators $E_i, F_i, H_j\,(i = 1,\dots n; j = 1,\dots,2n-l))$ and the relations (64) and (65). Let us remark that the Serre relations (65) take the form

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{pmatrix} 1-a_{ij} \\ \nu \end{pmatrix} E_i^{1-a_{ij}-\nu} E_j\, E_i^\nu = 0, \qquad i \neq j, \quad i,j = 1,2,\dots,n\,,$$

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{pmatrix} 1-a_{ij} \\ \nu \end{pmatrix} F_i^{1-a_{ij}-\nu} F_j\, F_i^\nu = 0, \qquad i \neq j, \quad i,j = 1,2,\dots,n\,,$$

Because of the Cartan decomposition of $\mathfrak{g}(A)$ (66), we obtain the following triangular decomposition of $\mathfrak{U}(\mathfrak{g}(A))$

$$\mathfrak{U}(\mathfrak{g}(A)) = \mathfrak{U}(\mathfrak{n}_-) \otimes \mathfrak{U}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{n}_+) \tag{67}$$

Example. The Cartan matrix of affine Lie algebra of the type $A_n^{(1)}$ $(n \geq 2)$, which is an affine generalization of $\mathfrak{sl}(n)$ is the following $n \times n$ matrix

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 & -1 \\ -1 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}. \tag{68}$$

The Cartan matrix of the Lie algebra $\mathfrak{sl}(n)$ is obtained from above matrix by dropping the first column and the first row. Let us remark that Cartan matrix is essentially the Gram matrix of the system of simple roots of algebra and that the Cartan matrix are in one-to-one correspondence with Dynkin diagrams.

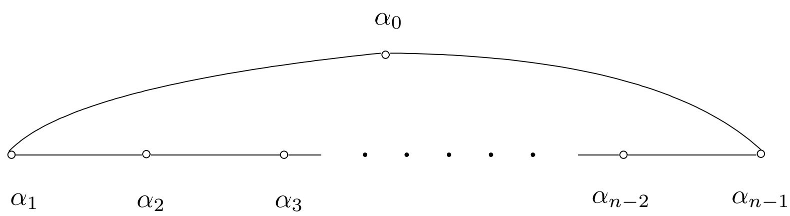

Fig. 8. Dynkin diagram of $A_n^{(1)}$ $(n \geq 2)$

3.2. Quantum group associated with $\mathfrak{g}(A)$. Following analogous construction as in the case of Lie algebra $\mathfrak{sl}(2)$ one can associated to arbitrary Lie algebra $\mathfrak{g}(A)$ (i.e $\mathfrak{U}(\mathfrak{g}(A))$) corresponding quantum group. This fact shows (from mathematical point of view) the importance of quantum groups, because this construction is applicable on very large and important class of objects such Kac-Moody algebras are.

Let us recall and introduce useful notation $q = e^h$, and for symmetrizable Cartan matrix A we define $q_i = q^{d_i}$ and the generalized binomial coefficient

$$\left[\left[\begin{array}{c} n \\ l \end{array} \right]\right]_q = \frac{[\![n]\!]_q\,!}{[\![l]\!]_q\,!\,[\![n-l]\!]_q\,!} = \frac{[\![n]\!]_q\,[\![n-1]\!]_q\cdots[\![n-l+1]\!]_q}{[\![l]\!]_q\,[\![l-1]\!]_q\cdots[\![2]\!]_q\,[\![1]\!]_q}, \quad n > l \geq 0,$$
$$\left[\left[\begin{array}{c} n \\ l \end{array} \right]\right]_q = 1, \qquad \text{for} \ \ n = l \ \ \text{or} \ \ l = 0. \tag{69}$$

If we define $a_{ij} = a_{ji} = 0$, for $i = 1, \ldots n; j = 1, \ldots, 2n-l$, then the quantum group, $\mathfrak{U}_q(\mathfrak{g}(A))$, which is associated to $\mathfrak{U}(\mathfrak{g}(A))$ is an associative $\mathbb{C}-$algebra generated by generators $E_i, F_i, H_j\,(i = 1, \ldots n; j = 1, \ldots, 2n-l)$ and the relations

$$\begin{aligned}
&[H_i, H_j] = 0, && i, j = 1, 2, \ldots, 2\,n - l, \\
&[H_j, E_i] = a_{ji}\,E_i, && i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, 2\,n - l, \\
&[H_j, F_i] = -\,a_{ji}\,F_i, && i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, 2\,n - l, \\
&[E_i, F_j] = \delta_{ji}\,\frac{q_i^{H_i} - q_i^{-\,H_i}}{q_i - q_i^{-1}}\,, && i, j = 1, 2, \ldots, n, \\
&\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \left[\left[\begin{array}{c} 1 - a_{ij} \\ \nu \end{array} \right]\right]_{q_i} E_i^{1-a_{ij}-\nu}\,E_j\,E_i^\nu = 0, && i, j = 1, \ldots, n,\ i \neq j \\
&\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \left[\left[\begin{array}{c} 1 - a_{ij} \\ \nu \end{array} \right]\right]_{q_i} F_i^{1-a_{ij}-\nu}\,F_j\,F_i^\nu = 0, && i, j = 1, \ldots, n,\ i \neq j.
\end{aligned} \tag{70}$$

Then, we take for coproduct, counit and antipod the maps

$$\begin{aligned}
&\Delta_q\,(H_i) = H_i \otimes \mathbb{1} + \mathbb{1} \otimes H_i, && \epsilon_q\,(H_i) = 0, \quad \mathcal{S}_q\,(H_i) = -\,H_i, \\
&\Delta_q\,(E_i) = E_i \otimes q_i^{H_i/2} + q_i^{-\,H_i/2} \otimes E_i, && \epsilon_q\,(E_i) = 0, \quad \mathcal{S}_q\,(E_i) = -\,q_i^{-1} E_i, \\
&\Delta_q\,(F_i) = F_i \otimes q_i^{H_i/2} + q_i^{-\,H_i/2} \otimes F_i, && \epsilon_q\,(F_i) = 0, \quad \mathcal{S}_q\,(F_i) = -\,q_i\,F_i.
\end{aligned} \tag{71}$$

One can directly verify that all of relations (9)-(15) hold in this case.

In the literature there exist a few slightly different definitions of quantum group. Usually, it means that some authors replace the deformation parameter $q = e^h$ by $e^{\alpha\,h}$ (where α is constant and usually equal to 1/2 or to 1/4), and the generators H_i by $K_i = q^{\beta H_i}$ (where β is equal to 1/2 or to 1/4). Let us mention one of the most interesting definitions of quantum group, due to Drinfeld and Jimbo. The "good" generators are also: $\mathbb{1}, X_{i_+} = E_i\,q_i^{H/2}$, $X_{i_-} = q_i^{-H/2}\,, F_i, K_i = q_i^{H_i}\,, K_i^{-1}$, then the relations (70) become

$$\begin{aligned}
&[K_i, K_j] = 0, \quad K_i\,K_i^{-1} = K_i\,K_i^{-1} = \mathbb{1}, && \forall i, j \\
&K_j\,X_{i_+}\,K_j^{-1} = q_i^{a_{ji}}\,X_{i_+}, && \forall\, i, j,
\end{aligned}$$

$$
\begin{aligned}
&K_j\, X_{i_-}\, K_j^{-1} = q_i^{-\,a_{ji}}\, X_{i_-}, && \forall\, i,j,\\
&[X_{i_+}, X_{j_-}] = \delta_{ji}\,\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, && \forall\, i,j,\\
&\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \left[\!\!\left[\begin{array}{c} 1-a_{ij} \\ \nu \end{array} \right]\!\!\right]_{q_i} X_{i_+}^{1-a_{ij}-\nu}\, X_{j_+}\, X_{i_+}^{\nu} = 0, && \forall\, i,j,\ i\neq j,\\
&\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \left[\!\!\left[\begin{array}{c} 1-a_{ij} \\ \nu \end{array} \right]\!\!\right]_{q_i} X_{i_-}^{1-a_{ij}-\nu}\, X_{j_-}\, X_{i_-}^{\nu} = 0, && \forall\, i,j,\ i\neq j.
\end{aligned}
\tag{72}
$$

and coproduct, counit and antipod are given by

$$
\begin{aligned}
&\Delta_q\left(K_i^{\pm 1}\right) = K_i^{\pm 1}\otimes K_i^{\pm 1}, && \epsilon_q\left(K_i^{\pm 1}\right) = \mathbb{1},\ \mathcal{S}_q\left(K_i^{\pm 1}\right) = K_i^{\mp 1},\\
&\Delta_q\left(X_{i_+}\right) = X_{i_+}\otimes K_i + \mathbb{1}\otimes X_{i_+}, && \epsilon_q\left(X_{i_+}\right) = 0,\ \mathcal{S}_q\left(X_{i_+}\right) = -\,X_{i_+}\, K_i^{-1},\\
&\Delta_q\left(X_{i_-}\right) = X_{i_-}\otimes \mathbb{1} + K_i^{-1}\otimes X_{i_-}, && \epsilon_q\left(X_{i_-}\right) = 0,\ \mathcal{S}_q\left(X_{i_-}\right) = -\,K_i\, X_{i_-}.
\end{aligned}
\tag{73}
$$

It is interesting to mention here that in the case of classical Lie algebras(regular Cartan matrix), their quantum groups satisfy triangular decomposition and the analog of PBW theorem hold. More precisely we have: Let $\mathfrak{U}_q(\mathfrak{g}(A))$ be a quantum group associated to the regular Cartan matrix, A, and let $\mathfrak{n}_+$, $\mathfrak{n}_-$ and $\mathfrak{h}$ be the subalgebras of $\mathfrak{U}(\mathfrak{g}(A))$ generated by $\{E_1, E_2 \ldots, E_n\}$ (or $\{X_{1+}, X_{2+} \ldots, X_{n+}\}$), $\{F_1, F_2 \ldots, F_n\}$ (or $\{X_{1-}, X_{2-} \ldots, X_{n-}\}$) and $\{H_1, H_2 \ldots, H_n\}$ (or $\{K_1^{\pm 1}, K_2^{\pm 1} \ldots, K_n^{\pm 1}\}$). Then we have

$$
\mathfrak{U}_q(\mathfrak{g}(A)) = \mathfrak{U}_q(\mathfrak{n}_-)\otimes \mathfrak{U}_q(\mathfrak{h})\otimes \mathfrak{U}_q(\mathfrak{n}_+),
$$

and $\mathfrak{U}_q(\mathfrak{g}(A))$ is, as the vector space, isomorphic to the algebra of polynomials in $\{E_i, F_i, K_i,\ i = 1, \ldots, n\}$.

4. Representation theory of quantum groups.

4.1. Basic facts about representation theory of Lie algebras. Let us recall some basic facts on the representations of complex finite dimensional Lie algebras. A representation of Lie algebra $\mathfrak{g}$ is a linear map $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, where V is a vector space, which satisfies defining relation for $\mathfrak{g}$. Sometimes is convenient to speak about representation as about $\mathfrak{g}-$modules. It is possible because a vector $\rho(X)(v)\,(X \in \mathfrak{g}, v \in V)$ one can write in more compact form as $X.v$ and then the algebra structure of $\mathfrak{g}$ implies that V is an $\mathfrak{g}-$modul. It means that the following axioms are satisfied:

$$
\begin{aligned}
&(\text{i1}) && X.(v+u) = X.v + X.u, && \forall\, X \in \mathfrak{g},\ u, v \in V,\\
&(\text{i2}) && X.(Y.v) = (X\,Y).v, && \forall\, X, Y \in \mathfrak{g},\ v \in V,\\
&(\text{i3}) && (X+Y).v = X.v + Y.v, && \forall\, X, Y \in \mathfrak{g},\ v \in V.
\end{aligned}
\tag{74}
$$

If ρ is a representation of $\mathfrak{g}$ in V and if there exists an invariant subspace $W \subseteq V$ $(\rho(X)(w) \in W, \forall X \in \mathfrak{g}, w \in W)$ then by

$$\begin{array}{lll} \varphi = \rho_{|W} : \mathfrak{g} \longrightarrow \mathfrak{gl}(W), & \varphi(X)(w) = \rho(X)(w), & \text{and} \\ \zeta = \rho_{V/W} : \mathfrak{g} \longrightarrow \mathfrak{gl}(V/W) & \zeta(v+W) = \rho(v) + W, & \end{array}$$

is well defined *subrepresentation*(*submodul*), φ, (in W) and *quotient representation*(*quotient modul*), ζ, (in V/W) of $\mathfrak{g}$, respectively. These construction of new representations work with given one, ρ, and spaces of new representations are the subspaces (or isomorphic to subspaces) of V. The representation which doesn't contain any subrepresentation is called irreducible representation of $\mathfrak{g}$ or simple $\mathfrak{g}-$modul. The irreducible representations(simple modules) in the set of all representations (modules) of Lie algebra, $\mathfrak{g}$, play similar role to the prime numbers in the sets of all integers. So, the purpose of the representation theory is to describe the set of all irreducible representations (simple modules), $\mathcal{I}r(\mathfrak{g})$ and then find the connections between an arbitrary representation of Lie algebra $\mathfrak{g}$ and the elements of $\mathcal{I}r(\mathfrak{g})$. The most important ways for producing new representations from the known ρ_1 in V_1 and ρ_2 in V_2 are direct sum and tensor product of representations:

$$\begin{array}{ll} (\rho_1 + \rho_2)(X)(v_1, v_2) = (\rho_1(X)(v_1), \rho_2(X)(v_2)), & \text{in } V_1 \oplus V_2, \\ (\rho_1 \otimes \rho_2)(X)\, v_1 \otimes v_2 = \rho_1(X)(v_1) \otimes v_2 + v_1 \otimes \rho_2(X)(v_2), & \text{in } V_1 \otimes V_2. \end{array}$$

The main idea in reaching the tasks of representation theory of Lie algebra, $\mathfrak{g}$, is based on the existence of Cartan subalgebra, $\mathfrak{h}$ (maximal commutative subalgebra of $\mathfrak{g}$). When we have such subalgebra, then *ad* is a representation of $\mathfrak{g}$ in $\mathfrak{gl}(\mathfrak{g})$ and there exists a basis of $\mathfrak{g}$, in which all elements of $\mathfrak{h}$ are represented by diagonal matrices. The most important consequence of this fact is Cartan decomposition (66). Now, if we consider a $\mathfrak{g}-$modul V (representation of $\mathfrak{g}$ in V), then for $\lambda \in \mathfrak{h}^*$, we call the subspace $V_\lambda = \{v \in V \mid H.v = \langle \lambda, H \rangle\, v \ \in \mathfrak{h}\} \neq \{0\}$ *weight subspace* of $\mathfrak{g}-$modul V. and λ is called *weight*.

The well-known facts about $\mathfrak{g}-$modules V are:

T1. For any $\mathfrak{g}-$modul V exists $v \in V$ such that $\mathfrak{n}_+.v = 0$ and $v \in V_\lambda$ for some weight λ. Such vector v is called the highest weight vector and λ is called highest weight.

T2. If V a is simple $\mathfrak{g}-$modul then such vector is unique up to scalar.

T3. The set of all simple modules of $\mathfrak{g}$ is parameterized by so-called *the lattice of dominant weights,* i.e

$$\Lambda = k_1\, \alpha_1^v + k_2\, \alpha_2^v + \ldots + k_n\, \alpha_n^v, \qquad k_i \in \mathbb{N},\ \forall\, i, \quad \text{where } n = \dim \mathfrak{h},$$

and where $\{\alpha_i^v, i = 1, \dots, n\} \subseteq \mathfrak{h}$ are dual basis to the basis of simple roots $\{\alpha_i, i = 1, \dots, n\} \subseteq \mathfrak{h}^*$.

T4. *Weyl theorem of complete reducibility,* which says that every $\mathfrak{g}-$modul V is direct sum of irreducible $\mathfrak{g}-$moduls V_{λ_i}, i.e.

$$V = \bigoplus_{i=1}^{l} V_{\lambda_i} \qquad V_{\lambda_i} \in \mathcal{I}r(\mathfrak{g}).$$

T5. The set of all finite dimensional representations of $\mathfrak{g}$ become a ring with direct sum of representation and tensor product as ring operations.

4.2. Representation of the quantum group $\mathfrak{U}_q(\mathfrak{sl}(2))$. Let $\mathfrak{g}$ be a complex finite dimensional algebra and q, the parameter of deformation, is not root of 1. First of all it is clear that the representations of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is in very close relation with the representations of corresponding Lie algebra $\mathfrak{g}$. Namely, PBW theorem enable us to find matrix which correspond to arbitrary element of $\mathfrak{U}(\mathfrak{g})$.
Secondly, it is clear that the representations of quantum group $\mathfrak{U}_q(\mathfrak{g})$ as deformed enveloping algebra $\mathfrak{U}(\mathfrak{g})$ have to be in very close connection to the representations of corresponding enveloping algebra. Same technic, as in the case of Lie algebras, could be applied in the theory of the representations of quantum groups and it is known that the same quantum analogs to the theorems (T1)-(T4) hold also in the case of representation of quantum group $\mathfrak{U}_q(\mathfrak{g})$. We will describe the case of the finite dimensional representation of quantum group $\mathfrak{U}(\mathfrak{sl}(2))$.

As we know the generators E, F and H satisfy the relations (34), and let V be a non-trivial $\mathfrak{U}_q(\mathfrak{sl}(2))-$modul, then if we take any eigenvector of H (such exists, because $\mathbb{C}$ is algebraically closed field) say v, then in the series $E.v, E^2.v, \dots$ take the last element different from null vector and this vector will be vector of the highest weight. If we denote that vector by v_0, its eigenvalue by λ and define inductively $v_n = \frac{1}{[\![n]\!]_q} F(v_{n-1})$, for $n \geq 0$ and $V_{-1} = 0$, then we have the following relations:

$$\begin{aligned} &\text{(i1)} \quad H\, v_n = (\lambda - 2n)\, v_n \\ &\text{(i2)} \quad F\, v_n = [\![n+1]\!]_q\, v_{n+1} \\ &\text{(i3)} \quad E\, v_n = [\![\lambda - n + 1]\!]_q\, v_{n-1} \end{aligned} \tag{75}$$

Now, from the relations (75) follow (because not all vectors of v_n can be different from 0) that there exists the greatest integer l, such that $E(v_l) \neq 0$ and $E(v_{l+1}) = 0$. Then the relation (i3) from (75) for $n = l+1$, implies that $[\![\lambda - n]\!]_q = 0$. Since q is not a root from 1, then

$$\lambda = l + \frac{k\,\pi\,\imath}{h} \qquad k \in \mathbb{Z}.$$

So, for each $l \in \mathbb{N}$ we obtain a countable series of non-isomorphic representations of dimension $l+1$. Let ρ_l be the representation with the highest weight $\lambda = l$, then all weights of ρ_l are, $l, l-2, \ldots, -l+2, -l$, and generators F, E and H are respectively represented in $\mathbb{C}^{l+1}$ by the matrices

$$\begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ [\![1]\!]_q & 0 & \cdots & \cdots & \vdots \\ 0 & [\![2]\!]_q & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & [\![l]\!]_q & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & [\![l]\!]_q & 0 & \cdots & 0 \\ 0 & 0 & [\![l-1]\!]_q & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & [\![1]\!]_q \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

$$\begin{pmatrix} \lambda & 0 & \cdots & \cdots & 0 \\ 0 & \lambda-2 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \cdots & -\lambda+2 & 0 \\ 0 & \cdots & \cdots & 0 & -\lambda \end{pmatrix}. \tag{76}$$

Let us mention the basic results on the representation theory of $\mathfrak{U}_q(\mathfrak{sl}(2))$.

T1. The set of all irreducible representation of $\mathfrak{U}_q(\mathfrak{sl}(2))$ is parameterized by two integers, $l \in \mathbb{N}_0$ and $k \in \mathbb{Z}$, it means that all irreducible representation have the highest weight of the form $\lambda_l^k = l + \frac{k\pi\imath}{h}$.

T2. The element $C_q = \frac{[\![(H+1)]\!]^2_{q^{1/2}}}{[\![2]\!]^2_{q^{1/2}}} + FE$ is an element of the center of $\mathfrak{U}_q(\mathfrak{sl}(2))$, and is known as *Casimir operator.* Let us mention that if $q \longrightarrow 1$ then $C_q \longrightarrow C = (\frac{H+1}{2})^2 + FE$, where C is the Casimir operator in $\mathfrak{U}(\mathfrak{sl}(2))$.

T3. Every finite dimensional representation of $\mathfrak{U}_q(\mathfrak{sl}(2))$ are completely reducible.

T4. The representation λ_l^k is equivalent with representations $\lambda_l^0 \otimes \lambda_0^k$ and $\lambda_0^k \otimes \lambda_l^0$.

T5. *Clebsch-Gordon's rule.* For integers $\lambda_1, \lambda_2 \geq 0$, we have

$$\lambda_{l_1} \otimes \lambda_{l_2} = \lambda_{l_1+l_2} \oplus \lambda_{l_1+l_2-2} \oplus \lambda_{l_1+l_2-4} \oplus \cdots \oplus \lambda_{|l_1-l_2|},$$

where $\lambda_l = \lambda_l^0$.

At the end let us tell few words on the representation of the quantum group, $\hat{\mathfrak{U}}_q(\mathfrak{sl}(2))$, which is generated by X_+, X_- and K relations (see (72) and (73))

$$K\,K^{-1} = K^{-1}\,K = \mathbb{1}, \tag{77}$$

$$K\,X_\pm\,K^{-1} = q^{\pm 2}\,X_\pm, \qquad (78)$$
$$[X_+, X_-] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Actually, this algebra $\hat{\mathfrak{U}}_q(\mathfrak{sl}(2))$ (which is also called quantum group) is a subalgebra of $\mathfrak{U}_q(\mathfrak{sl}(2))$. The representation theory of $\hat{\mathfrak{U}}_q(\mathfrak{sl}(2))$ is essentially same as the representation theory of $\mathfrak{U}_q(\mathfrak{sl}(2))$, and mentioned facts (T2)-(T5). Only, (T1) holds up to the set of irreducible representations in same dimension. Here for each $l \in \mathbb{N}_0$ we have just two irreducible representations, but in the case of algebra $\mathfrak{U}_q(\mathfrak{sl}(2))$ we have a countable family of irreducible representations. Namely, the difference between $\mathfrak{U}_q(\mathfrak{sl}(2))$ and $\hat{\mathfrak{U}}_q(\mathfrak{sl}(2))$ is in replacing diagonal element H with $K = q^H$, and because of that the highest weights of $\hat{\mathfrak{U}}_q(\mathfrak{sl}(2))$ in dimension $l+1$ are q^l and $-\,q^l$. So, the matrices of the irreducible representations, ρ_+ and ρ_- in dimension $l+1$ are

$$\begin{aligned}
&\rho_+(K) = \mathrm{diag}[q^l, q^{l-2}, \ldots, q^{-\,l}], && \rho_-(K) = -\rho_+(K),\\
&\rho_+(X_+) = \rho_l(E), && \rho_-(X_+) = -\,\rho_+(X_+), \qquad (79)\\
&\rho_+(X_-) = \rho_l(F), && \rho_-(X_-) = \rho_+(X_-).
\end{aligned}$$

where $\rho_l(E), \rho_l(E)$, are matrices from (76).

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