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Finsler–modified Randers cosmological models in Einstein field theory

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Abstract In this research article, we have considered the Finsler–Modified Randers Cosmological model (FMRCM) with Cosmological constant Λ for generalized Finsler–Randers space time, to investigate the solutions of this model in Einstein theory of cosmology in different variations of energy conditions. Also, we have analyzed the role of cosmological constant Λ in various scenarios, exploring its impact on both accelerating and decelerating phases of cosmic expansion.

1 Introduction

Riemannian geometry is a particular case of Finslerian geometry [1–3]. A Finsler space is fundamentally defined by a generating function $\mathcal{F}(x, y)$ on the tangent bundle $\mathcal{TM}$ of a manifold $\mathcal{M}$ where $\mathcal{F}$ is positively homogeneous of degree one in y. In particular, the Finsler–Randers space-time is examined in [4]. Finsler geometry was introduced by Paul Finsler in 1918. Earlier, in 1854, B. Riemann formulated the Riemannian metric $ds^2 = g_{ij}dx^i dx^j$, which defines the distance between two points x and $x + y$. The generating function $\mathcal{F}(x, y)$ satisfies the following properties:

(i) $\mathcal{F}$ is continuous on $(\mathcal{TM})$ and smooth on $(\tilde{\mathcal{TM}}) = \mathcal{TM}/\{0\}$, namely the tangent bundle minus the null set $\{(x, y) \in \mathcal{TM}/\mathcal{F}(x, y) = 0\}$.

(ii) $\mathcal{F}$ is positively homogeneous of first degree on its second argument
$\mathcal{F}(x, ky) = k\mathcal{F}(x, y)$ for every $k > 0$.

(iii) For each $x \in \mathcal{M}$ the fundamental metric tensor $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}$ is non singular, with $i, j = 0, 1, 2, 3, 4, ...(n-1)$.

The pair $(\mathcal{M}, \mathcal{F})$ is called a Finsler manifold and the symmetric bilinear form $g = g_{ij}(x, y)dx^i.dx^j$ is called the Finsler metric tensor of the Finsler manifold $(\mathcal{M}, \mathcal{F})$. Sometimes, a function $\mathcal{F}$ satisfying the above conditions is said to be regular Finsler metric.

According to the Big-Bang theory, the entire universe was once in the form of a single point with extremely high energy, density and pressure. After explosion this energy converted into mass. This mass leads to the formation of physical structure of the universe. Thereafter geometry was used to study it.

Now a days, the open problem is that to calculate the exact value of this cosmological constant Λ [5]. In general, by extending the Riemannian metric we can establish a Finslerian geometric structure on a manifold, leading the way for generalized gravitational field theories. There have been rapid developments in the applications of Finsler geometry in the context of fundamental relativity (FR), particularly in general relativity, astrophysics, and cosmology. Many researchers have studied these topics, [1,6–22]. In this context, Stavrinos, Koretsis, and Stathakopoulos [5] suggested that the Finsler–Randers field equations yield a Hubble parameter containing an extra geometrical term, which may serve as a potential candidate for dark energy.

Spatially homogeneous cosmological models enable the study of distorted and rotating universes, broadening the scope of cosmological research. These models help to estimate the effects of anisotropy on primordial element production and the observed anisotropies in the cosmic microwave background radiation (CMBR) [12]. In addition to the observational insights provided by Hawking and Ellis [14], sev-

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eral theoretical considerations have motivated the study of anisotropic cosmologies.

G. Randers, in his research on asymmetric metrics in four-dimensional spacetime within general relativity [21], proposed the Finsler–Randers cosmological model. A broad class of anisotropic cosmological models exists and has been widely studied in cosmology [19]. Some theoretical arguments support the presence of an anisotropic universe [17]. Additionally, anisotropic cosmological models offer an alternative to the assumption of specific initial conditions in standard FRW models. The universe may also exhibit an irregular expansion mechanism. Therefore, studying cosmological models in which early-stage anisotropies diminish over time could provide valuable insights into cosmic evolution [13].

Stavrinos et al. [5] investigated a Friedmann-like Robertson–Walker model in a generalized metric spacetime with weak anisotropy. More recently, Basilakos and P. C. Stavrinos [8] explored the cosmological equivalence between Finsler–Randers spacetime and the Dvali–Gabadadze–Porrati (DGP) gravity model. Building upon these studies. We aim to analyze the evolution of the universe within the framework of Finsler–Randers cosmology. Stavrinos et al. [23] developed a cosmological model based on the dynamics of a varying vacuum in Finsler–Randers cosmology, incorporating a cosmological fluid source described as an ideal fluid. Additionally, they investigated the connection between Finsler–Randers theory and General Relativity, exploring the conditions under which the former approaches the latter. In this research paper, we study the Friedman-like Robertson–Walker model and the cosmological constant Λ within the framework of generalized Finsler–Randers cosmology. Furthermore, we investigate both the accelerating and decelerating expansion of the universe.

1.1 A brief introduction of osculating Riemannian metric

In this subsection, we introduce the osculating Riemannian metric associated with a Finsler metric $(\mathcal{M}, \mathcal{F})$. As mentioned already, the fundamental geometrical objects of Finsler geometry are defined on the total space $\mathcal{TM}$ of the tangent bundle $\pi_{\mathcal{M}} : \mathcal{TM} \to \mathcal{M}$ regarded as $2n$-dimensional differentiable manifold with the canonical coordinates (x, y) [24]. Here $x = (x^i)$ and $y = (y^i)$ are obviously independent variables. For instance $g_{ij} : \mathcal{M}/0 \to \mathcal{R}$, where 0 denotes the zero section of the tangent bundle. On the other hand, since $\pi_{\mathcal{M}} : \mathcal{TM} \to \mathcal{M}$ is a fiber bundle, we can consider a local section $Y : U \to \mathcal{T}U$,where $U \subset \mathcal{M}$ is an open neighborhood on $\mathcal{M}$ such that $Y(x) \neq 0$, for all $x \in U$. We have $\pi_{\mathcal{M}}(Y(x)) = x$ on U. If we fix such a local section Y of $\mathcal{M} : \mathcal{TM} \to \mathcal{M}$, all geometrical objects defined on the manifold $\mathcal{TM}$ can be pulled back to $\mathcal{M}$, for instance $\bar{g}_{ij}oY$ is a function on U, hence we can define

$$\bar{g_{ij}}(x) := \bar{g}_{ij}(x, y)|_{y=Y(x)}, x \in U. \tag{1.1}$$

The pair $(U, \bar{g}_{ij})$ is a Riemannian manifold and this $\bar{g}_{ij}$ is called the Y -osculating Riemannian metric associated to $(\mathcal{M}, \mathcal{F})$. The Christoffel symbols of the first kind associated with the osculating Riemannian metric (1.1) are defined as follows

$$\bar{\gamma}_{ijk}(x) := \frac{1}{2}\left(\frac{\partial}{\partial x^j}[\bar{g_{ij}}(x, Y(x))] + \frac{\partial}{\partial x^k}[\bar{g_{ij}}(x, Y(x))] - \frac{\partial}{\partial x^i}[\bar{g_{ij}}(x, Y(x))]\right) \tag{1.2}$$

and by using the derivative law of composed functions we get

$$\bar{\gamma}_{ijk}(x) = \bar{\gamma}_{ijk}(x, y)|_{y=Y(x)} + 2\left(\bar{C}_{ijl}\frac{\partial Y^l}{\partial x^k} + \bar{C}_{ijl}\frac{\partial Y^l}{\partial x^j} - \bar{C}_{jkl}\frac{\partial Y^l}{\partial x^i}\right)|_{y=Y(x)}. \tag{1.3}$$

If Y is a non-vanishing global section of $\mathcal{TM}$, i.e. $Y(x) \neq 0$, for all $x \in \mathcal{M}$, then we can define the osculating Riemannian manifold $(\mathcal{M}, g_{ij})$. However, observe that the existence of globally non-vanishing sections of $\mathcal{TM}$ depends on the topology of $\mathcal{M}$. For instance in the case of a 2- dimensional sphere, such sections do not exist. It is known that all non-compact manifolds admit non-vanishing global vector fields. Compact manifolds admit non-vanishing global vector fields if and only if the Euler characteristic vanishes [24]. We will always assume that non-vanishing global vector fields exist on our differential manifold $\mathcal{M}$. With the assumption above, in the case of an (α, β)-metric, let us consider the vector field $Y = \mathcal{A}$ having the components $\mathcal{A}_i = a_{ij}\mathcal{A}^j$. The vector field A being globally non-vanishing on $\mathcal{M}$ is equivalent with the fact that β has no zero points. With these notations, we consider the $\mathcal{A}$-osculating Riemannian manifold $(\mathcal{M}, \bar{g}_{ij})$, where $\bar{g}_{ij} = \bar{g}_{ij}(x, \mathcal{A})$. By denoting by a the length of $\mathcal{A}$ with respect to α, we have $\bar{a}^2 = \mathcal{A}_i\mathcal{A}^i = \alpha^2(x, \mathcal{A})$, $Y_i(x, \mathcal{A}) = \mathcal{A}_i$ and the $\mathcal{A}$-Riemannian metric takes the form

$$\bar{g}_{ij}(x) = \frac{\mathcal{L}_\alpha}{\bar{a}}|_{y=\mathcal{A}(x)}a_{ij} + \left(\frac{\mathcal{L}_{\alpha\alpha}}{\bar{a}^2} + 2\frac{\mathcal{L}_{\alpha\beta}}{\bar{a}} + \mathcal{L}_{\beta\beta} - \frac{\mathcal{L}_\alpha}{\bar{a}^3}\right)|_{y=\mathcal{A}(x)}\mathcal{A}_i\mathcal{A}_j. \tag{1.4}$$

Furthermore, we have $\beta(x, \mathcal{A}) = a^2$, $p_i(x, \mathcal{A}) = 0$ i.e. $\bar{C}_{ijk} = 0$ (see for instance [25, p. 8]).

On the other hand, in case of $Y = \mathcal{A}$, (see for instance [25, p. 7])

$\mathcal{F}^i_{jk}(x, y)|_{y=\mathcal{A}(x)} = \bar{\gamma}^i_{jk}(x, y)|_{y=\mathcal{A}(x)}$, and furthermre, from (1.3) we get $\bar{\gamma}_{ijk}(x) = \bar{\gamma}_{ijk}(x, y)|_{y=\mathcal{A}(x)}$.

Hence we obtain the fundamental result that for a Finsler space with (α, β)-metric the linear $\mathcal{A}$-connection associated with the Cartan connection is the Levi-Civita connection of the Barthal connection is the Levi-Civita connection of the $\mathcal{A}$-Riemannian space. The curvature tensor of an affine connection with local coefficients $(\Gamma^i_{ij}(x)$ is given by

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}. \tag{1.5}$$

Also, let $\bar{\gamma}^i_{jk}$ are Levi-Civita coefficients, we obtain the expressions for curvature tensor

$$\bar{R}^i_{jkl} = \frac{\partial \gamma^i_{jl}}{\partial x^k} - \frac{\partial \gamma^i_{jk}}{\partial x^l} + \gamma^m_{jl}\gamma^i_{mk} - \gamma^m_{jk}\gamma^i_{ml}. \tag{1.6}$$

and

$$\bar{R}_{jl} = \sum_i \left[\frac{\partial \gamma^i_{jl}}{\partial x^i} - \frac{\partial \gamma^i_{ji}}{\partial x^l} + \sum_m (\gamma^m_{jl}\gamma^k_{mk} - \gamma^m_{jk}\gamma^k_{ml}) \right], \tag{1.7}$$

respectively, where $i, j, k, l, m \in \{0, 1, 2, 3\}$.

The contractions of the curvature tensor lead to the generalized Ricci tensor and Ricci scalar, respectively given by

$$\bar{R}_{jl} = \bar{R}^i_{jil}, \bar{R}^j_l = \bar{g}^{jm}\bar{R}_{ml}, \tag{1.8}$$

and

$$\bar{R} = \bar{R}^i_i. \tag{1.9}$$

2 Einstein theory of generalized Finsler–Randers cosmological model with cosmological constant

The energy conditions in general relativity facilitate the derivation of powerful and general theorems concerning the behavior of strong gravitational fields and cosmological geometries. However, in this section, we focus on the generalized Finsler–Randers cosmological model with a cosmological constant and examine the corresponding energy conditions for this framework.

Riemannian geometry represents a special case of Finslerian geometry [1–3]. Fundamentally, a Finsler space is defined by a generating function $\mathcal{F}(x, y)$ on the tangent bundle $\mathcal{TM}$ of a manifold $\mathcal{M}$. The function $\mathcal{F}$ is positively homogeneous of degree one in y. Specifically, the case of a Finsler–Randers space-time is discussed in [4].

$$\mathcal{F}(x, y) = \sqrt{a_{ij}(x)y^i y^j} + b_i y^i,$$

but we have generalized Finsler Randers space-time such as

$$\mathcal{F}(x, y) = \sqrt{g_{ij}(x)y^i y^j} + Ab_i y^i, \tag{2.1}$$

where $A = \begin{cases} const., \\ \frac{q}{mc^2} \end{cases}$, a_{ij} are component of a Riemannian metric and $b_i = (b_0, 0, 0, 0)$ is weak primordial vector field with $|b|1$. The Finslerian metric tensor g_{ij} is constructed by Hessian

$$g_{ij} = \frac{1}{2}\frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}. \tag{2.2}$$

The zero order Friedmann equation is given by [26]

$$\frac{3k}{a^2} = 8\pi G\rho_0 - 3\mathcal{H}^2 + \Lambda,$$

where k is the curvature parametre of universe, $\mathcal{H}$ is the Hubble parameter and Λ is the cosmological constant. The generalized Finsler–Randers field equation with cosmological constant is given by [27]

$$\mathcal{R}_{ij} - \frac{1}{2}g_{ij}\mathcal{T} = -\frac{8\pi G}{c^4}\mathcal{T}_{ij} + \frac{\Lambda}{8\pi G}g_{ij}, \tag{2.3}$$

where $\mathcal{R}_{ij}$ is Ricci tensor, $\mathcal{T}_{ij}$ is the energy momentum tensor, Λ is the Cosmological constant [26] and $\mathcal{T}$ is the trace of energy momentum tensor and the term $-(\Lambda/8\pi G)g_{ij}$ represents the contribution of the cosmological constant to the field equations. Modelling the expanding universe as Finslerian perfect fluid [23] that induces rediation and matter with four velocity u_i for comoving observers, we have

$$\mathcal{T}_{ij} = -\left(P - \frac{\Lambda}{12\pi G}\right)g_{ij} + \left(\rho + P + \frac{\Lambda}{12\pi G}\right)u_i u_j, \tag{2.4}$$

where ρ and P are the total energy density and pressure with cosmological constant Λ of the cosmic fluid respectively. Thus, the energy momentum tensor becomes

$$\mathcal{T}_{ij} = diag\left[\left(\rho - \frac{\Lambda}{6\pi G}\right), -\left(P - \frac{\Lambda}{12\pi G}\right)g_{11}, -\left(P - \frac{\Lambda}{12\pi G}\right)g_{22}, -\left(P - \frac{\Lambda}{12\pi G}\right)g_{33}\right]. \tag{2.5}$$

In view of [7,10,16], we apply the weak, dominant and strong energy conditions within the framework of Finslerian cosmology for our models. In the locally Minkowski frame, these conditions are expressed as $T^0_0 = (\rho - \frac{\Lambda}{6\pi G})$, $T^1_1 = T^2_2 = T^3_3 = -(P - \frac{\Lambda}{12\pi G})$. Obviously the root of matrix equation is

$$|\mathcal{T}_{ij} - rg_{ij}| = diag\left[\left(\left(\rho - \frac{\Lambda}{6\pi G}\right) - r\right), \left(r + \left(P - \frac{\Lambda}{12\pi G}\right)\right), \left(r + \left(P - \frac{\Lambda}{12\pi G}\right)\right), \left(r + \left(P - \frac{\Lambda}{12\pi G}\right)\right)\right]. \quad (2.6)$$

It gives the eigen values r for the energy momentum tesor as $r_0 = (\rho - \frac{\Lambda}{6\pi G})$ and $r_1 = r_2 = r_3 = -(P - \frac{\Lambda}{12\pi G})$. Also, we put $\Lambda = 3K$ for $\Lambda > 0$ and $\Lambda = -3K$ for $\Lambda < 0$ where $K > 0$ be any real number [5].

We assume the energy conditions for Finsler modified Randers cosmological model are as follows:

(i) Null energy condition with cosmological constant (NECCC);

(a)
$$\rho + P \geq 0, \Lambda < 0 \quad (2.7)$$

(b)
$$\rho + P \geq 0, \Lambda > 0. \quad (2.8)$$

(ii) Weak energy condition with cosmological constant (WECCC);

(a)
$$r_0 \geq 0 \Rightarrow \rho \geq 0, r_0 - r_i \geq 0 \Rightarrow \rho + P \geq 0, \Lambda > 0. \quad (2.9)$$

(b)
$$r_0 \geq 0 \Rightarrow \rho \geq 0, r_0 - r_i \geq 0 \Rightarrow \rho + P \geq 0, \Lambda < 0. \quad (2.10)$$

(iii) Strong energy condition with cosmological constant (SECCC);

(a)
$$r_0 - \sum r_i \geq 0 \Rightarrow \rho + 3P \geq 0 and \rho + P \geq 0, \Lambda > 0. \quad (2.11)$$

(b)
$$r_0 - \sum r_i \geq 0 \Rightarrow \rho + 3P \geq 0 and \rho + P \geq 0, \Lambda < 0. \quad (2.12)$$

(iii) Dominant energy condition with cosmological constant (DECCC);

(a)
$$r_0 \geq 0 \Rightarrow \rho \geq 0, -r_0 \leq -r_i \leq r_0 \Rightarrow \rho \pm P \geq 0, \Lambda > 0. \quad (2.13)$$

(b)
$$r_0 \geq 0 \Rightarrow \rho \geq 0, -r_0 \leq -r_i \leq r_0 \Rightarrow \rho \pm P \geq 0, \Lambda < 0. \quad (2.14)$$

Note: Among all these energy conditions, the first condition indicates the accelerating expansion of the universe, while the second condition suggests its deceleration.. FRW metric is

$$a_{ij} = diag\left(1, -\frac{a^2}{1 - kr^2}, -a^2r^2, -a^2r^2sin^2\theta\right) \quad (2.15)$$

where a is a function of t only and k is the curvature parameter having the values -1, 0, +1 for open, flat and closed models respectively. The non-zero components of the Ricci tensors are

$$\mathcal{R}_{00} = 3\left(\frac{\ddot{a}}{a} - \frac{3\dot{a}}{4a}\dot{u_0}\right), \quad (2.16)$$

and

$$\mathcal{R}_{ii} = -\left(\frac{\ddot{a}a + 2\dot{a}^2 + 2k + \frac{11}{4}a\dot{a}\dot{u_0}}{\Delta_{ii}}\right), \quad (2.17)$$

where $\Delta_{11} = 1 - kr^2$, $\Delta_{22} = r^2$ and $\Delta_{33} = r^2sin^2\theta$. From gravitational FR field equation (2.1) for comoving observers, then the FRW Einstein field equation with a cosmological constant can be given in the following form [5]

$$\frac{\ddot{a}}{a} + \frac{3\dot{a}}{4a}\dot{u_0} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}, \quad (2.18)$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} + \frac{11}{4}\frac{\dot{a}}{a}\dot{u_0} = 4\pi G(\rho - P) + \Lambda, \quad (2.19)$$

From (2.18) and (2.19), we get

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}z_t = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}. \quad (2.20)$$

where over the dot denotes the derivatives with respect to the cosmic time t and $z_t = b_0 < 0$ and z_t are defined as $z_t = \dot{u_0}$ [5,23]. Using the Hubble parameter, Eq. (2.20) becomes

$$\mathcal{H}^2 + \mathcal{H}z_t + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}. \quad (2.21)$$

Equation (2.21) represents the modified Friedmann equation, where the additional term $\mathcal{H}z_t - \frac{\Lambda}{3}$ influences the dynamics of the universe. If we set $b_0 = 0$ or $(C_{000} = 0)$, it follows that $z_t = -\frac{\Lambda}{3} = 0$. In this case, the field equations (2.19) and (2.20) reduce to the standard Einstein equations, whose solution corresponds to the usual Friedmann equation. Next, we analyze the solutions that give rise to two distinct physically relevant cosmological scenarios, which are of interest in describing both the decelerating and accelerating phases of the universe.

Case 1: de Sitter solution with cosmological constant

In cosmology, it is well established that the current epoch of accelerated cosmic expansion can be effectively approximated by this model. This solution describes an exponential growth of the scale factor, leading to a constant Hubble parameter.

From [6], we use the scalar factor as: $a = ce^{\sigma t}$, where c and σ are constants. For $\sigma^2 > 0$, it gives an accelerating universe. Using this scalar factor, the Hubble parameter becomes

$$\mathcal{H}(t) = \frac{\dot{a}}{a} = \frac{\sigma c e^{\sigma t}}{ce^{\sigma t}} = \sigma \tag{2.22}$$

Using energy conditions and equation (2.21), we obtain the energy density as

$$\rho = \frac{3}{8\pi G}\left(\sigma^2 + \sigma z_t + \frac{k}{c^2 e^{2\sigma t}} + \frac{\Lambda}{3}\right). \tag{2.23}$$

From Eq. (2.18), we get the pressure as

$$P = -\frac{1}{8\pi G}\left[3\sigma^2 + \frac{5\sigma z_t}{2} + \frac{k}{c^2 e^{2\sigma t}} - \frac{\Lambda}{3}\right]. \tag{2.24}$$

From Eqs. (2.23) and (2.24), we obtain

$$\rho + P = -\frac{k}{4\pi G c^2 c^2 e^{2\sigma t}} + \frac{\sigma z_t}{16\pi G} + \frac{k}{2\pi G c^2 e^{2\sigma t}} + \frac{\Lambda}{12\pi G}. \tag{2.25}$$

and

$$\rho - P = -\frac{3\sigma^2}{4\pi G} + \frac{11\sigma z_t}{16\pi G} + \frac{k}{2\pi G c^2 e^{2\sigma t}} - \frac{\Lambda}{12\pi G}. \tag{2.26}$$

Again, from above equations, we haave

$$\rho + 3P = -\frac{3\sigma^2}{4\pi G} - \frac{9\sigma z_t}{16\pi G} + \frac{\Lambda}{4\pi G}. \tag{2.27}$$

Notice that if $z_t = \frac{\Lambda}{3} = 0$, Eq. (2.21) reduces to the standard Friedmann equation. The above observations suggest that the universe is anisotropic at an early stage and becomes isotropic at later times. For physically viable choices where $z_t < 0$, two different cases arise:

(a) $z_t = -e^{-t}$
and
(b) $z_t = -t^{-n}$.

where n is a positive constant. (a) If $z_t = -e^{-t}$: Putiing this value of z_t in Eqs. (2.23) and (2.24), the value of pressure and density are obtained as

$$\rho = \frac{3}{8\pi G}\left(\sigma^2 - \sigma e^{-t} + \frac{k}{c^2 e^{2\sigma t}}\right) + \frac{\Lambda}{8\pi G}. \tag{2.28}$$

and

$$P = \frac{3\sigma^2}{8\pi G} - \frac{5\sigma e^{-t}}{16\pi G} - \frac{k}{8\pi G c^2 e^{2\sigma t}} + \frac{\Lambda}{24\pi G}. \tag{2.29}$$

From Eqs. (2.28) and (2.29), we obtained

$$\rho + P = -\frac{\sigma e^{-t}}{16\pi G} + \frac{k}{2\pi G c^2 e^{2\sigma t}} + \frac{\Lambda}{6\pi G}. \tag{2.30}$$

and

$$\rho - P = \frac{3\sigma^2}{4\pi G} - \frac{11\sigma e^{-t}}{16\pi G} + \frac{k}{2\pi G c^2 e^{2\sigma t}} + \frac{\Lambda}{12\pi G}. \tag{2.31}$$

Also, the condition $\rho + 3P$ is

$$\rho + 3P = -\frac{3\sigma^2}{4\pi G} + \frac{9\sigma e^{-t}}{16\pi G} + \frac{\Lambda}{4\pi G}. \tag{2.32}$$

Now we discuss the energy conditions with cosmological constant, from equation (2.30) the null energy condition with cosmological constant (NECCC) is satisfied if

$$\rho + P \geq 0 \Rightarrow -\frac{k}{4\pi G c^2 c^2 e^{2\sigma t}} + \frac{\sigma z_t}{16\pi G} + \frac{k}{2\pi G c^2 e^{2\sigma t}} + \frac{\Lambda}{12} \geq 0$$
$$\Rightarrow c^2 \leq \frac{24k}{(3\sigma e^{(2\sigma-1)t} - 8\Lambda e^{2\sigma t})} = \mathcal{B}_1$$

from Eqs. (2.28) and (2.30) the weak energy condition with cosmological constant (WECCC) is satisfied if

$$c^2 \leq min\{\frac{k}{e^{2\sigma t}(3\sigma e^{-t} - 3\sigma^2 - \Lambda)}, \frac{24k}{(3\sigma e^{(2\sigma-1)t} - 8\Lambda e^{2\sigma t})}\} = \mathcal{B}_2$$

The strong energy condition is satisfied if

$$\sigma \in [\frac{9e^{-t} - \sqrt{81e^{2t} - 192\Lambda}}{24}, \frac{9e^{-t} + \sqrt{81e^{2t} - 192\Lambda}}{24}] = \mathcal{B}_3$$

The dominant energy condition with cosmological constant (DECCC) is satisfied if

$$c^2 \leq \{\frac{k}{e^{2\sigma t}(3\sigma e^{-t} - 3\sigma^2 - \Lambda)}, \frac{24k}{(3\sigma e^{(2\sigma-1)t} - 8\Lambda e^{2\sigma t})}, \frac{24k}{(33\sigma e^{-t} - 36\sigma^2 - 4\Lambda)e^{2\sigma t}}\} = \mathcal{B}_4$$

From these observations, we find that for any value of t, the Null Energy Condition with Cosmological Constant (NECCC), the Weak Energy Condition with Cosmological Constant (WECCC), and the Dominant Energy Condition with Cosmological Constant (DECCC) are satisfied if $c^2 \leq min\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_4\}$, On the other hand, the Strong Energy Condition with Cosmological Constant (SECCC) is satisfied in this model if $\sigma \in \left[\frac{9e^{-t} - \sqrt{81e^{2t} - 192\Lambda}}{24}, \frac{9e^{-t} + \sqrt{81e^{2t} - 192\Lambda}}{24}\right] = \mathcal{B}_3$.

Furthermore, we observe that for large cosmic time t, the NECCC, WECCC, and DECCC continue to hold, whereas the SECCC is violated. This violation is responsible for the current accelerated expansion of the universe.

Now, we will discuss the effect of Λ in both conditions

(a): $\Lambda > 0$ **i.e.** $\Lambda = 3K$, then we have

From Eq. (2.30) the null energy condition with cosmological constant (NECCC) is satisfied if

$\Rightarrow c^2 \leq \frac{24k}{(3\sigma e^{(2\sigma-1)t} - 24Ke^{2\sigma t})} = \mathcal{B}_1$

from Eqs. (2.28) and (2.30) the weak energy condition with cosmological constant (WECCC) is satisfied if

$c^2 \leq min\left\{\frac{k}{e^{2\sigma t}(3\sigma e^{-t} - 3\sigma^2 - 3K)}, \frac{24k}{(3\sigma e^{(2\sigma-1)t} - 24Ke^{2\sigma t})}\right\} = \mathcal{B}_2$

The strong energy condition is satisfied if

$\sigma \in [\frac{9e^{-t} - \sqrt{81e^{2t} + 576K}}{24}, \frac{9e^{-t} + \sqrt{81e^{2t} + 576K}}{24}] = \mathcal{B}_3.$

The dominant energy condition with cosmological constant (DECCC) is satisfied if

$c^2 \leq \{\frac{k}{e^{2\sigma t}(3\sigma e^{-t} - 3\sigma^2 - 3K)}, \frac{24k}{(3\sigma e^{(2\sigma-1)t} - 24Ke^{2\sigma t})}, \frac{24k}{(33\sigma e^{-t} - 36\sigma^2 - 12K)e^{2\sigma t}}\} = \mathcal{B}_4$

From these observations, we find that for any value of t, the Null Energy Condition with Cosmological Constant (NECCC), the Weak Energy Condition with Cosmological Constant (WECCC), and the Dominant Energy Condition with Cosmological Constant (DECCC) are satisfied if $c^2 \leq min\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_4\}$, On the other hand, the Strong Energy Condition with Cosmological Constant (SECCC) is satisfied in this model if $\sigma \in \left[\frac{9e^{-t} - \sqrt{81e^{2t} - 576K}}{24}, \frac{9e^{-t} + \sqrt{81e^{2t} - 576K}}{24}\right] = \mathcal{B}_3$.

Furthermore, we observe that for large cosmic time t, the NECCC, WECCC, and DECCC continue to hold, whereas the SECCC is violated. This violation is responsible for the current accelerated expansion of the universe.

(b): $\Lambda < 0$ **i.e.** $\Lambda = -3K$, then we have

From equation (2.30) the null energy condition with cosmological constant (NECCC) is satisfied if

$\Rightarrow c^2 \leq \frac{24k}{(3\sigma e^{(2\sigma-1)t} + 24Ke^{2\sigma t})} = \mathcal{B}_1$

From Eqs. (2.28) and (2.30) the weak energy condition with cosmological constant (WECCC) is satisfied if

$c^2 \leq min\left\{\frac{k}{e^{2\sigma t}(3\sigma e^{-t} - 3\sigma^2 + 3K)}, \frac{24k}{(3\sigma e^{(2\sigma-1)t} + 24Ke^{2\sigma t})}\right\} = \mathcal{B}_2$

The strong energy condition is satisfied if

$\sigma \in \left[\frac{9e^{-t} - \sqrt{81e^{2t} + 576K}}{24}, \frac{9e^{-t} + \sqrt{81e^{2t} + 576K}}{24}\right] = \mathcal{B}_3$

The dominant energy condition with cosmological constant (DECCC) is satisfied if

$c^2 \leq \left\{\frac{k}{e^{2\sigma t}(3\sigma e^{-t} - 3\sigma^2 + 3K)}, \frac{24k}{(3\sigma e^{(2\sigma-1)t} + 24Ke^{2\sigma t})}, \frac{24k}{(33\sigma e^{-t} - 36\sigma^2 + 12K)e^{2\sigma t}}\right\} = \mathcal{B}_4$

From these observations, we find that for any value of t, the Null Energy Condition with Cosmological Constant (NECCC), the Weak Energy Condition with Cosmological Constant (WECCC), and the Dominant Energy Condition with Cosmological Constant (DECCC) are satisfied if $c^2 \leq min\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_4\}$, on the other hand, the Strong Energy Condition with Cosmological Constant (SECCC) is satisfied in this model if $\sigma \in [\frac{9e^{-t} - \sqrt{81e^{2t} + 576K}}{24}, \frac{9e^{-t} + \sqrt{81e^{2t} + 576K}}{24}] = \mathcal{B}_3$.

However, we also observed that for large cosmic time t, SECCC, WECCC, DECCC and SECCC are satisfied, so for the condition $\Lambda = -3K$, there is no information about expanding of universe.

(b) If $z_t = -t^{-n}$:

Putiing this value of z_t in equations (2.23) and (2.24), the value of pressure and density are obtained as

$$\rho = \frac{3}{8\pi G}\left(\sigma^2 - \sigma t^{-n} + \frac{k}{c^2 e^{2\sigma t}}\right) + \frac{\Lambda}{8\pi G}. \tag{2.33}$$

and

$$P = \frac{3\sigma^2}{8\pi G} - \frac{5\sigma t^{-n}}{16\pi G} - \frac{k}{8\pi G c^2 e^{2\sigma t}} + \frac{\Lambda}{24\pi G}. \tag{2.34}$$

From Eqs. (2.33) and (2.34), we obtained

$$\rho + P = -\frac{\sigma t^{-n}}{16\pi G} + \frac{k}{2\pi G c^2 e^{2\sigma t}} + \frac{\Lambda}{6\pi G}. \tag{2.35}$$

and

$$\rho - P = \frac{3\sigma^2}{4\pi G} - \frac{11\sigma t^{-n}}{16\pi G} + \frac{k}{2\pi G c^2 e^{2\sigma t}} + \frac{\Lambda}{12\pi G}. \tag{2.36}$$

Also, the condition $\rho + 3P$ is

$$\rho + 3P = -\frac{3\sigma^2}{4\pi G} + \frac{9\sigma t^{-n}}{16\pi G} + \frac{\Lambda}{4\pi G}. \tag{2.37}$$

Now we discuss the energy conditions with cosmological constant, from equation (2.35) the null energy condition with cosmological constant (NECCC) is satisfied if

$\rho + P \geq 0 \Rightarrow -\frac{\sigma t^{-n}}{16\pi G} + \frac{k}{4\pi G c^2 e^{2\sigma t}} + \frac{\Lambda}{6\pi G} \geq 0$

$\Rightarrow c^2 \leq \frac{24k}{e^{2\sigma t}(3\sigma t^{-n} - 8\Lambda)} = \mathcal{A}_1$

from Eqs. (2.33) and (2.35) the weak energy condition with cosmological constant (WECCC) is satisfied if

$c^2 \leq min\left\{\frac{k}{e^{2\sigma t}(3\sigma t^{-n} - 3\sigma^2 - \Lambda)}, \frac{24k}{(3\sigma e^{2\sigma t} t^{-n} - 8\Lambda e^{2\sigma t})}\right\} = \mathcal{A}_2$

The strong energy condition is satisfied if

$\sigma \in \left[\frac{9t^{-n} - \sqrt{81t^{2n} + 192\Lambda}}{24}, \frac{9t^{-n} + \sqrt{81t^{2n} + 192\Lambda}}{24}\right] = \mathcal{A}_3$

The dominant energy condition with cosmological constant (DECCC) is satisfied if

$c^2 \leq \left\{ \frac{k}{e^{2\sigma t}(3\sigma t^{-n} - 3\sigma^2 - \Lambda)}, \frac{24k}{(3\sigma e^{(2\sigma t} t^{-n} - 8\Lambda e^{2\sigma t})}, \frac{24k}{(33\sigma t^{-n} - 36\sigma^2 - 4\Lambda)e^{2\sigma t}} \right\} = \mathcal{A}_4$

From these observations, we find that for any value of t, the Null Energy Condition with Cosmological Constant (NECCC), the Weak Energy Condition with Cosmological Constant (WECCC), and the Dominant Energy Condition with Cosmological Constant (DECCC) are satisfied if $c^2 \leq min\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4\}$, on the other hand, the Strong Energy Condition with Cosmological Constant (SECCC) is satisfied in this model if$\sigma \in \left[\frac{9t^{-n} - \sqrt{81t^{2n} + 192\Lambda}}{24}, \frac{9t^{-n} + \sqrt{81t^{2n} + 192\Lambda}}{24} \right] = \mathcal{A}_3$.

Furthermore, we observe that for large cosmic time t, the NECCC, WECCC, and DECCC continue to hold, whereas the SECCC is violated. This violation is responsible for the current accelerated expansion of the universe.

Now, we will discuss the effect of Λ in both conditions

(a): $\Lambda > 0$ **i.e.** $\Lambda = 3K$, then we have

From Eq. (2.35) the null energy condition with cosmological constant (NECCC) is satisfied if

$c^2 \leq \frac{24k}{e^{2\sigma t}(3\sigma t^{-n} - 24K)} = \mathcal{A}_1$

from equations (2.33) and (2.35) the weak energy condition with cosmological constant (WECCC) is satisfied if

$c^2 \leq min\left\{ \frac{k}{e^{2\sigma t}(3\sigma t^{-n} - 3\sigma^2 - 3K)}, \frac{24k}{(3\sigma e^{2\sigma t} t^{-n} - 24K e^{2\sigma t})} \right\} = \mathcal{A}_2$

The strong energy condition is satisfied if

$\sigma \in \left[\frac{9t^{-n} - \sqrt{81t^{2n} + 576K}}{24}, \frac{9t^{-n} + \sqrt{81t^{2n} + 576K}}{24} \right] = \mathcal{A}_3$

The dominant energy condition with cosmological constant (DECCC) is satisfied if

$c^2 \leq \left\{ \frac{k}{e^{2\sigma t}(3\sigma t^{-n} - 3\sigma^2 - 3K)}, \frac{24k}{(3\sigma e^{(2\sigma t} t^{-n} - 24K e^{2\sigma t})}, \frac{24k}{(33\sigma t^{-n} - 36\sigma^2 - 12K)e^{2\sigma t}} \right\} = \mathcal{A}_4$

From these observations that any value of t, NECCC, WECCC and DECCC are satisfied in this case if $c^2 \leq min\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4\}$, whereas SECCC is satisfied in this model if $\sigma \in \left[\frac{9t^{-n} - \sqrt{81t^{2n} + 576K}}{24}, \frac{9t^{-n} + \sqrt{81t^{2n} + 576K}}{24} \right] = \mathcal{A}_3$.

However, we also observed that for large cosmic time t, SECCC, WECCC and DECCC are satisfied, whereas SECCC is violated, which is responsible for current accelerated expansion of universe.

(b): $\Lambda < 0$ **i.e.** $\Lambda = -3K$, then we have

From Eq. (2.35) the null energy condition with cosmological constant (NECCC) is satisfied if

$c^2 \leq \frac{24k}{e^{2\sigma t}(3\sigma t^{-n} + 24K)} = \mathcal{A}_1$

from Eqs. (2.33) and (2.35) the weak energy condition with cosmological constant (WECCC) is satisfied if

$c^2 \leq min\left\{ \frac{k}{e^{2\sigma t}(3\sigma t^{-n} - 3\sigma^2 + 3K)}, \frac{24k}{(3\sigma e^{2\sigma t} t^{-n} + 24K e^{2\sigma t})} \right\} = \mathcal{A}_2$

The strong energy condition is satisfied if

$\sigma \in \left[\frac{9t^{-n} - \sqrt{81t^{2n} - 576K}}{24}, \frac{9t^{-n} + \sqrt{81t^{2n} - 576K}}{24} \right] = \mathcal{A}_3$

The dominant energy condition with cosmological constant (DECCC) is satisfied if

$c^2 \leq \left\{ \frac{k}{e^{2\sigma t}(3\sigma t^{-n} - 3\sigma^2 + 3K)}, \frac{24k}{(3\sigma e^{(2\sigma t} t^{-n} + 24K e^{2\sigma t})}, \frac{24k}{(33\sigma t^{-n} - 36\sigma^2 + 12K)e^{2\sigma t}} \right\} = \mathcal{A}_4$

From these observations, we find that for any value of t, the Null Energy Condition with Cosmological Constant (NECCC), the Weak Energy Condition with Cosmological Constant (WECCC), and the Dominant Energy Condition with Cosmological Constant (DECCC) are satisfied if $c^2 \leq min\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4\}$, on the other hand, the Strong Energy Condition with Cosmological Constant (SECCC) is satisfied in this model if $\sigma \in [\frac{9t^{-n} - \sqrt{81t^{2n} - 576K}}{24}, \frac{9t^{-n} + \sqrt{81t^{2n} - 576K}}{24}] = \mathcal{A}_3$.

However, we also observed that for large cosmic time t, SECCC, WECCC, DECCC and SECCC are satisfied, so for the condition $\Lambda = -3K$, there is no information about expanding of universe.

Case-2: Power Law solution with Cosmological constant Power-law solutions are fundamental in standard cosmology, as they help describe the evolution of various cosmological phases, including radiation-dominated, matter-dominated, and dark energy-dominated eras. Consider a universe where the scale factor follows a power-law form [20], given by $a = ct^{\omega}$, where c and ω are constants. When $\omega > 1$ the universe undergoes accelerated expansion.

now, the Hubble parameter becomes

$$\mathcal{H}(t) = \frac{\dot{a}}{a} = \frac{\omega c t^{\omega - 1}}{c t^{\omega}} = \frac{\omega}{t}, \tag{2.38}$$

Using this value of Hubble parameter in Eq. (2.21), we have

$$\rho = \frac{3}{8\pi G} \left(\frac{\omega^2}{t^2} + \frac{\omega}{t} z_t + \frac{k}{c^2 t^{2\omega t}} + \frac{\Lambda}{3} \right). \tag{2.39}$$

With the scalar factor and from Eq. (2.18), we get the pressure as

$$P = \frac{-3\omega^2 + 2\omega}{8\pi G t^2} - \frac{5\omega}{16\pi G t} z_t - \frac{k}{8\pi G c^2 t^{2\omega}} - \frac{\Lambda}{24\pi G}. \tag{2.40}$$

From Eqs. (2.39) and (2.40), we have

$$\rho + P = \frac{2\omega}{8\pi G t^2} + \frac{\omega}{16\pi G t} z_t + \frac{k}{4\pi G c^2 t^{2\omega}} + \frac{\Lambda}{12\pi G}. \tag{2.41}$$

and

$$\rho - P = \frac{\omega(3\omega-1)}{4\pi G t^2} + \frac{11\omega}{16\pi G t} z_t + \frac{k}{2\pi G c^2 t^{2\omega}} + \frac{\Lambda}{6\pi G}. \tag{2.42}$$

Also, the value of $\rho + 3P$ is

$$\rho + 3P = \frac{3\omega(2-\omega)}{8\pi G t^2} - \frac{9\omega}{16\pi G t} z_t + \frac{\Lambda}{4\pi G}. \tag{2.43}$$

I this case also, we discuss, as same in case-1 two different scinario:

(i) When $z_t = -e^{-t}$

Putiing the value of z_t in Eqs. (2.39) and (2.40), we can determine the value of the pressure and density respectively as

$$\rho = \frac{3}{8\pi G}\left(\frac{\omega^2}{t^2} - \frac{\omega}{t}e^{-t} + \frac{k}{c^2 t^{2\omega t}} + \frac{\Lambda}{3}\right). \tag{2.44}$$

With the scalar factor and from Eq. (2.18), we get the pressure as

$$P = \frac{-3\omega^2 + 2\omega}{8\pi G t^2} + \frac{5\omega}{16\pi G t}e^{-t} - \frac{k}{8\pi G c^2 t^{2\omega}} - \frac{\Lambda}{24\pi G}. \tag{2.45}$$

From Eqs. (2.44) and (2.45), we obatin

$$\rho + P = \frac{2\omega}{8\pi G t^2} + \frac{\omega}{16\pi G t}e^{-t} + \frac{k}{4\pi G c^2 t^{2\omega}} + \frac{\Lambda}{12\pi G}. \tag{2.46}$$

and

$$\rho - P = \frac{\omega(3\omega-1)}{4\pi G t^2} - \frac{11\omega}{16\pi G t}e^{-t} + \frac{k}{2\pi G c^2 t^{2\omega}} + \frac{\Lambda}{6\pi G}. \tag{2.47}$$

Also, the value of $\rho + 3P$ is

$$\rho + 3P = \frac{3\omega(2-\omega)}{8\pi G t^2} + \frac{9\omega}{16\pi G t}e^{-t} + \frac{\Lambda}{4\pi G}. \tag{2.48}$$

From Eq. (2.46), The null energy conditions with cosmological constant (NECCC) satisfied if

$$\rho + P \geq \frac{2\omega}{8\pi G t^2} + \frac{\omega}{16\pi G t}e^{-t} + \frac{k}{4\pi G c^2 t^{2\omega}} + \frac{\Lambda}{12\pi G} \geq 0$$

$$\Rightarrow \frac{2\omega}{8\pi G t^2} + \frac{\omega}{16\pi G t}e^{-t} + \frac{k}{4\pi G c^2 t^{2\omega}} + \frac{\Lambda}{12\pi G} \geq 0$$

$$\Rightarrow c^2 \leq \frac{12k}{(12\omega t^{2(\omega-1)} + 3\omega e^{-t} t^{2\omega-1} + 4\Lambda t^{2\omega})} = \mathcal{Q}_1.$$

The weak energy condition with cosmological condition (WECCC) is satisfied if

$$c^2 \leq min\left\{\frac{3k}{3\omega e^{-t} t^{(2\omega-1)} - 3\omega^2 t^{(2\omega-2)} - \Lambda t^{2\omega}}, \frac{12k}{(12\omega t^{2(\omega-1)} + 3\omega e^{-t} t^{2\omega-1} + 4\Lambda t^{2\omega})}\right\} = \mathcal{Q}_2.$$

Also, The dominant energy condition with cosmological constant (DECCC) is satisfied if

$$c^2 \leq min\left\{\frac{3k}{3\omega e^{-t} t^{(2\omega-1)} - 3\omega^2 t^{(2\omega-2)} - \Lambda t^{2\omega}}, \frac{12k}{(12\omega t^{2(\omega-1)} + 3\omega e^{-t} t^{2\omega-1} + 4\Lambda t^{2\omega})}, \frac{24k}{(12\omega t^{2(\omega-1)} - 36\omega^2 t^{2(\omega-1)} + 33\omega t^{2\omega+1} e^{-t} + 8\Lambda t^{2\omega})}\right\} = \mathcal{Q}_3.$$

and the strong energy condition is satisfied if

$$\omega \in \left[\frac{(12e^t - 9t) - \sqrt{(12e^t-9t)^2 + 96e^{2t}t^2\Lambda}}{12e^t}, \frac{(12e^t - 9t) + \sqrt{(12e^t-9t)^2 + 96e^{2t}t^2\Lambda}}{12e^t}\right],$$

From these observations, we find that for any value of t, the Null Energy Condition with Cosmological Constant (NECCC), the Weak Energy Condition with Cosmological Constant (WECCC), and the Dominant Energy Condition with Cosmological Constant (DECCC) are satisfied if $c^2 \leq min\{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3\}$, on the other hand, the Strong Energy Condition with Cosmological Constant (SECCC) is satisfied in this model if $\omega \in \left[\frac{(12e^t - 9t) - \sqrt{(12e^t-9t)^2 + 96e^{2t}t^2\Lambda}}{12e^t}, \frac{(12e^t - 9t) + \sqrt{(12e^t-9t)^2 + 96e^{2t}t^2\Lambda}}{12e^t}\right] = \mathcal{Q}_4$.

Furthermore, we observe that for large cosmic time t, the NECCC, WECCC, and DECCC continue to hold, whereas the SECCC is violated. This violation is responsible for the current accelerated expansion of the universe.

Now, we will discuss the effect of Λ in both conditions

(a): $\Lambda > 0$ **i.e.** $\Lambda = 3K$, then we have

From Eq. (2.46), The null energy conditions with cosmological constant (NECCC) satisfied if

$$c^2 \leq \frac{12k}{(12\omega t^{2(\omega-1)} + 3\omega e^{-t} t^{2\omega-1} + 12K t^{2\omega})} = \mathcal{Q}_1.$$

The weak energy condition with cosmological condition (WECCC) is satisfied if

$$c^2 \leq min\left\{\frac{3k}{3\omega e^{-t} t^{(2\omega-1)} - 3\omega^2 t^{(2\omega-2)} - 3K t^{2\omega}}, \frac{12k}{(12\omega t^{2(\omega-1)} + 3\omega e^{-t} t^{2\omega-1} + 12K t^{2\omega})}\right\} = \mathcal{Q}_2.$$

Also, The dominant energy condition with cosmological constant (DECCC) is satisfied if

$$c^2 \leq min\left\{\frac{3k}{3\omega e^{-t} t^{(2\omega-1)} - 3\omega^2 t^{(2\omega-2)} - 3K t^{2\omega}}, \frac{12k}{(12\omega t^{2(\omega-1)} + 3\omega e^{-t} t^{2\omega-1} + 12K t^{2\omega})}, \frac{24k}{(12\omega t^{2(\omega-1)} - 36\omega^2 t^{2(\omega-1)} + 33\omega t^{2\omega+1} e^{-t} + 24K t^{2\omega})}\right\} = \mathcal{Q}_3.$$

and the strong energy condition is satisfied if

$$\omega \in \left[\frac{(12e^t - 9t) - \sqrt{(12e^t-9t)^2 + 288Ke^{2t}t^2}}{12e^t}, \frac{(12e^t - 9t) + \sqrt{(12e^t-9t)^2 + 288Ke^{2t}t^2}}{12e^t}\right],$$

From these observations, we find that for any value of t, the Null Energy Condition with Cosmological Constant

(NECCC), the Weak Energy Condition with Cosmological Constant (WECCC), and the Dominant Energy Condition with Cosmological Constant (DECCC) are satisfied if $c^2 \leq min\{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3\}$, on the other hand, the Strong Energy Condition with Cosmological Constant (SECCC) is satisfied in this model if $\omega \in \Big[\frac{(12e^t-9t)-\sqrt{(12e^t-9t)^2+288Ke^{2t}t^2}}{12e^t}, \frac{(12e^t-9t)+\sqrt{(12e^t-9t)^2+288Ke^{2t}t^2}}{12e^t}\Big] = \mathcal{Q}_4$.
Furthermore, we observe that for large cosmic time t, the NECCC, WECCC, and DECCC continue to hold, whereas the SECCC is violated. This violation is responsible for the current accelerated expansion of the universe.

(b): $\Lambda < 0$ **i.e.** $\Lambda = -3K$, then we have

From Eq. (2.46), The null energy conditions with cosmological constant (NECCC) satisfied if

$c^2 \leq \frac{12k}{(12\omega t^{2(\omega-1)}+3\omega e^{-t}t^{2\omega-1}-12Kt^{2\omega})} = \mathcal{Q}_1$.

The weak energy condition with cosmological condition (WECCC) is satisfied if

$$c^2 \leq min\Big\{\frac{3k}{3\omega e^{-t}t^{(2\omega-1)}-3\omega^2 t^{(2\omega-2)}+3Kt^{2\omega}}, \frac{12k}{(12\omega t^{2(\omega-1)}+3\omega e^{-t}t^{2\omega-1}-12Kt^{2\omega})}\Big\} = \mathcal{Q}_2.$$

Also, The dominant energy condition with cosmological constant (DECCC) is satisfied if

$$c^2 \leq min\Big\{\frac{3k}{3\omega e^{-t}t^{(2\omega-1)}-3\omega^2 t^{(2\omega-2)}+3Kt^{2\omega}}, \frac{12k}{(12\omega t^{2(\omega-1)}+3\omega e^{-t}t^{2\omega-1}-12Kt^{2\omega})}, \frac{24k}{(12\omega t^{2(\omega-1)}-36\omega^2 t^{2(\omega-1)}+33\omega t^{2\omega+1}e^{-t}-24Kt^{2\omega})}\Big\} = \mathcal{Q}_3.$$

and the strong energy condition is satisfied if

$$\omega \in \Big[\frac{(12e^t-9t)-\sqrt{(12e^t-9t)^2-288Ke^{2t}t^2}}{12e^t}, \frac{(12e^t-9t)+\sqrt{(12e^t-9t)^2-288Ke^{2t}t^2}}{12e^t}\Big],$$

From these observations, we find that for any value of t, the Null Energy Condition with Cosmological Constant (NECCC), the Weak Energy Condition with Cosmological Constant (WECCC), and the Dominant Energy Condition with Cosmological Constant (DECCC) are satisfied if $c^2 \leq min\{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3\}$, on the other hand, the Strong Energy Condition with Cosmological Constant (SECCC) is satisfied in this model if $\omega \in \Big[\frac{(12e^t-9t)-\sqrt{(12e^t-9t)^2-288Ke^{2t}t^2}}{12e^t}, \frac{(12e^t-9t)+\sqrt{(12e^t-9t)^2-288Ke^{2t}t^2}}{12e^t}\Big] = \mathcal{Q}_4$.
However, we also observed that for large cosmic time t, SECCC, WECCC, DECCC and SECCC are satisfied, so for the condition $\Lambda = -3K$, there is no information about expanding of universe.

(ii) When $z_t = -t^{-n}$
Putiing the value of z_t in equations (2.39) and (2.40), we can determine the value of the pressure and density respectively as

$$\rho = \frac{3}{8\pi G}\left(\frac{\omega^2}{t^2} - \frac{\omega}{t^{n+1}} + \frac{k}{c^2 t^{2\omega t}} + \frac{\Lambda}{3}\right). \tag{2.49}$$

With the scalar factor and from Eq. (2.18), we get the pressure as

$$P = \frac{-3\omega^2+2\omega}{8\pi G t^2} + \frac{5\omega}{16\pi G t^{n+1}} - \frac{k}{8\pi G c^2 t^{2\omega}} - \frac{\Lambda}{24\pi G}. \tag{2.50}$$

From Eqs. (2.44) and (2.45), we obatin

$$\rho+P = \frac{2\omega}{8\pi G t^2} + \frac{\omega}{16\pi G t^{n+1}} + \frac{k}{4\pi G c^2 t^{2\omega}} + \frac{\Lambda}{12\pi G}. \tag{2.51}$$

and

$$\rho - P = \frac{\omega(3\omega-1)}{4\pi G t^2} - \frac{11\omega}{16\pi G t^{n+1}} + \frac{k}{2\pi G c^2 t^{2\omega}} + \frac{\Lambda}{6\pi G}. \tag{2.52}$$

Also, the value of $\rho + 3P$ is

$$\rho + 3P = \frac{3\omega(2-\omega)}{8\pi G t^2} + \frac{9\omega}{16\pi G t^{n+1}} + \frac{\Lambda}{4\pi G}. \tag{2.53}$$

From Eq. (2.46), The null energy conditions with cosmological constant (NECCC) satisfied if

$$\rho + P \geq \frac{2\omega}{8\pi G t^2} + \frac{\omega}{16\pi G t^{n+1}} + \frac{k}{4\pi G c^2 t^{2\omega}} + \frac{\Lambda}{12\pi G} \geq 0$$
$$\Rightarrow \frac{2\omega}{8\pi G t^2} + \frac{\omega}{16\pi G t^{n+1}} + \frac{k}{4\pi G c^2 t^{2\omega}} + \frac{\Lambda}{12\pi G} \geq 0$$
$$\Rightarrow c^2 \leq \frac{12k}{(-12\omega t^{2(\omega-2)}-3\omega t^{2\omega-n-1}-4\Lambda t^{2\omega})} = \mathcal{S}_1.$$

The weak energy condition with cosmological condition (WECCC) is satisfied if

$$c^2 \leq min\Big\{\frac{3k}{3\omega t^{(2\omega-n-1)}-3\omega^2 t^{(2\omega-2)}-\Lambda t^{2\omega}}, \frac{12k}{(-12\omega t^{2(\omega-2)}-3\omega t^{2\omega-n-1}-4\Lambda t^{2\omega})}\Big\} = \mathcal{S}_2.$$

Also, The dominant energy condition with cosmological constant (DECCC) is satisfied if

$$c^2 \leq min\Big\{\frac{12k}{(-12\omega t^{2(\omega-2)}-3\omega t^{2\omega-n-1}-4\Lambda t^{2\omega})}, \frac{12k}{(-12\omega t^{2(\omega-2)}-3\omega t^{2\omega-n-1}-4\Lambda t^{2\omega})}, \frac{24k}{(12(3\omega^2-\omega)t^{2(\omega-2)}-33\omega t^{2(\omega-n-1)}-8\Lambda t^{2\omega})}\Big\} = \mathcal{S}_3.$$

and the strong energy condition is satisfied if

$$\omega \in \Big[\frac{(12t^{-2}+9t^{-n-1})-\sqrt{(12t^{-2}+9t^{-n-1})^2+96t^{-2}\Lambda}}{12t^{-2}}, \frac{(12t^{-2}+9t^{-n-1})+\sqrt{(12t^{-2}+9t^{-n-1})^2+96t^{-2}\Lambda}}{12t^{-2}}\Big],$$

From these observations, we find that for any value of t, the Null Energy Condition with Cosmological Constant

(NECCC), the Weak Energy Condition with Cosmological Constant (WECCC), and the Dominant Energy Condition with Cosmological Constant (DECCC) are satisfied if $c^2 \leq min\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\}$, on the other hand, the Strong Energy Condition with Cosmological Constant (SECCC) is satisfied in this model if $\omega \in \left[\frac{(12t^{-2}+9t^{-n-1})-\sqrt{(12t^{-2}+9t^{-n-1})^2+96t^{-2}\Lambda}}{12t^{-2}}, \frac{(12t^{-2}+9t^{-n-1})+\sqrt{(12t^{-2}+9t^{-n-1})^2+96t^{-2}\Lambda}}{12t^{-2}}\right]$.
Furthermore, we observe that for large cosmic time t, the NECCC, WECCC, and DECCC continue to hold, whereas the SECCC is violated. This violation is responsible for the current accelerated expansion of the universe.

Now, we will discuss the effect of Λ in both conditions

(a): $\Lambda > 0$ **i.e.** $\Lambda = 3K$, then we have

From Eq. (2.46), The null energy conditions with cosmological constant (NECCC) satisfied if

$$c^2 \leq \frac{12k}{(-12\omega t^{2(\omega-2)} - 3\omega t^{2\omega-n-1} - 12Kt^{2\omega})} = \mathcal{S}_1.$$

The weak energy condition with cosmological condition (WECCC) is satisfied if

$$c^2 \leq min\left\{\frac{3k}{3\omega t^{(2\omega-n-1)} - 3\omega^2 t^{(2\omega-2)} - 3Kt^{2\omega}}, \frac{12k}{(-12\omega t^{2(\omega-2)} - 3\omega t^{2\omega-n-1} - 12Kt^{2\omega})}\right\} = \mathcal{S}_2.$$

Also, The dominant energy condition with cosmological constant (DECCC) is satisfied if

$$c^2 \leq min\left\{\frac{12k}{(-12\omega t^{2(\omega-2)} - 3\omega t^{2\omega-n-1} - 12Kt^{2\omega})}, \frac{12k}{(-12\omega t^{2(\omega-2)} - 3\omega t^{2\omega-n-1} - 12Kt^{2\omega})}, \frac{24k}{(12(3\omega^2-\omega)t^{2(\omega-2)} - 33\omega t^{2(\omega-n-1)} - 24Kt^{2\omega})}\right\} = \mathcal{S}_3.$$

and the strong energy condition is satisfied if

$$\omega \in \left[\frac{(12t^{-2}+9t^{-n-1})-\sqrt{(12t^{-2}+9t^{-n-1})^2+288Kt^{-2}}}{12t^{-2}}, \frac{(12t^{-2}+9t^{-n-1})+\sqrt{(12t^{-2}+9t^{-n-1})^2+288Kt^{-2}}}{12t^{-2}}\right],$$

From these observations, we find that for any value of t, the Null Energy Condition with Cosmological Constant (NECCC), the Weak Energy Condition with Cosmological Constant (WECCC), and the Dominant Energy Condition with Cosmological Constant (DECCC) are satisfied if $c^2 \leq min\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\}$, on the other hand, the Strong Energy Condition with Cosmological Constant (SECCC) is satisfied in this model if $\omega \in \left[\frac{(12t^{-2}+9t^{-n-1})-\sqrt{(12t^{-2}+9t^{-n-1})^2+288Kt^{-2}}}{12t^{-2}}, \frac{(12t^{-2}+9t^{-n-1})+\sqrt{(12t^{-2}+9t^{-n-1})^2+288Kt^{-2}}}{12t^{-2}}\right]$.
Furthermore, we observe that for large cosmic time t, the NECCC, WECCC, and DECCC continue to hold, whereas the SECCC is violated. This violation is responsible for the current accelerated expansion of the universe.

(b): $\Lambda < 0$ **i.e.** $\Lambda = -3K$, then we have

From Eq. (2.46), The null energy conditions with cosmological constant (NECCC) satisfied if

$$c^2 \leq \frac{12k}{(-12\omega t^{2(\omega-2)} - 3\omega t^{2\omega-n-1} + 12Kt^{2\omega})} = \mathcal{S}_1.$$

The weak energy condition with cosmological condition (WECCC) is satisfied if

$$c^2 \leq min\left\{\frac{3k}{3\omega t^{(2\omega-n-1)} - 3\omega^2 t^{(2\omega-2)} + 3Kt^{2\omega}}, \frac{12k}{(-12\omega t^{2(\omega-2)} - 3\omega t^{2\omega-n-1} + 12Kt^{2\omega})}\right\} = \mathcal{S}_2.$$

Also, The dominant energy condition with cosmological constant (DECCC) is satisfied if

$$c^2 \leq min\left\{\frac{12k}{(-12\omega t^{2(\omega-2)} - 3\omega t^{2\omega-n-1} + 12Kt^{2\omega})}, \frac{12k}{(-12\omega t^{2(\omega-2)} - 3\omega t^{2\omega-n-1} + 12Kt^{2\omega})}, \frac{24k}{(12(3\omega^2-\omega)t^{2(\omega-2)} - 33\omega t^{2(\omega-n-1)} + 24Kt^{2\omega})}\right\} = \mathcal{S}_3.$$

and the strong energy condition is satisfied if

$$\omega \in \left[\frac{(12t^{-2}+9t^{-n-1})-\sqrt{(12t^{-2}+9t^{-n-1})^2-288Kt^{-2}}}{12t^{-2}}, \frac{(12t^{-2}+9t^{-n-1})+\sqrt{(12t^{-2}+9t^{-n-1})^2-288Kt^{-2}}}{12t^{-2}}\right],$$

From these observations, we find that for any value of t, the Null Energy Condition with Cosmological Constant (NECCC), the Weak Energy Condition with Cosmological Constant (WECCC), and the Dominant Energy Condition with Cosmological Constant (DECCC) are satisfied if $c^2 \leq min\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\}$, on the other hand, the Strong Energy Condition with Cosmological Constant (SECCC) is satisfied in this model if $\omega \in \left[\frac{(12t^{-2}+9t^{-n-1})-\sqrt{(12t^{-2}+9t^{-n-1})^2-288Kt^{-2}}}{12t^{-2}}, \frac{(12t^{-2}+9t^{-n-1})+\sqrt{(12t^{-2}+9t^{-n-1})^2-288Kt^{-2}}}{12t^{-2}}\right]$.
However, we also observed that for large cosmic time t, SECCC, WECCC, DECCC and SECCC are satisfied, so for the condition $\Lambda = -3K$, there is no information about expanding of universe.

3 Conclsion

Stavrinos et al. [23] developed the cosmological scenario of the dynamics varying vacuum Finsler–Randers cosmology with a cosmological fluid source described by an ideal fluid. Also, explored the limit of General Relativity provided by the Finsler Randers theory. In this research paper, we investigate generalized Finsler–Randers (FR) cosmological models within the framework of Einstein-modified theories of cosmology. We have explore the behavior of the model in Einstein's theory by considering the physical variables $z_t = -e^{-t}$ and $z_t = -t^{-n}$, obtaining solutions in the presence of the cosmological constant Λ. Furthermore, we analyze the energy conditions, including the null energy condition with the cosmological constant (NECCC), the weak energy condition with the cosmological constant (WECCC), the strong energy condition with the cosmological constant (SECCC), and the dominant energy condition with the cosmological constant (DECCC). Our study determines the conditions under which the generalized FR cosmological model remains physically stable in Einstein's cosmology. Finally, we have presented all energy conditions under two distinct cases of the cosmological constant $\Lambda = 3K$ and $\Lambda = -3K$ respectively, corresponding to both the accelerating and decelerating phases of the universe.

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