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EFFECT OF CALORIMETER RESOLUTION ON HIGH-P_T PARTICLE DISTRIBUTIONS

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ABSTRACT

For a particular parametrization of high transverse momentum particle distributions and calorimeter resolutions, analytic formulae are derived for folding and unfolding the the distributions. The effect of the tail of the resolution is also studied.

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INTRODUCTION

Let us assume that we measure a quantity x which has a frequency distribution f(x) by using a measuring instrument with resolution $D(x_0,x)$. The resolution $D(x_0,x)$ is the differential probability dP/dx_0, where x_0 is the apparent value of the quantity x. The apparent (observed) frequency distribution will be given by the following convolution integral in the proper domain $x_1 \leqslant x \leqslant x_2$,

$$f_0(x_0) = \int_{x_1}^{x_2} f(x)\ D(x_0,x)\ dx$$

$$\text{with} \quad \int_{x_1}^{x_2} D(x_0,x)\ dx_0 = 1 \tag{1}$$

In practice one measures $f_0(x_0)$ and $D(x_0,x)$ and wants to determine f(x). This is a Fredholm integral equation of the first kind. It could be solved, in general, by numerical methods.[1,2]
The inverse problem is of interest too, namely by knowing $D(x_0,x)$ and f(x) to determine $f_0(x_0)$.

We will concentrate on cases where the parametrization of the frequency distribution and the resolution are such that the above problem can be solved analytically. For a different parametrization leading to analytic solution see ref. 3.

We will assume that $f_0(x_0)$, or f(x), and $D(x_0,x)$ are precisely known wherever it is necessary. To determine the effects of errors on $f_0(x\ )$, or f(x), and $D(x_0,x)$ one could specify the errors on the parameter of the above functions and determine the error of the final result in a straightforward way.

Parametrizations for high transverse momentum distribution calculations

In the case where a high transverse momentum (high-P_T) spectrum of final particle(s) in an elementary particle interaction is measured with a calorimeter, one has the definitions, $x=P_T$, $x_0=P_{T0}$ and $f(x)=dN/dP_T$, $f_0(x_0)=dN/dP_{T0}$. Where P_T, P_{T0} are transverse momenta, N is number of events. First we deal with the problem of the production of single particles with high-P_T. Their distribution $f(x)=dN/dx(=dN/dP_T)$ can be parametrized for small ranges of P_T by the equation,

$$\frac{dN}{dx} = A \cdot e^{-B\,x} \qquad (2)$$

A) Let us first take a resolution which is in general a Gaussian without tails,

$$D(x_0, x) = D_1(x_0, x) = \frac{1}{A_1}\left(e^{-\frac{(x_0-x)^2}{2\cdot s^2}} - e^{-\frac{g^2}{2}}\right) \quad x_1 < x < x_2$$

$$D(x_0, x) = 0 \qquad x < x_1 \;,\quad x_2 < x$$

$$x_{1,2} = x_0 + \frac{gc}{2}\left(g \cdot c \pm \sqrt{g^2 \cdot c^2 + 4\, x_0}\right) \qquad (3)$$

$$s = c \cdot \sqrt{x}$$

$$A_1 = \sqrt{2\pi} \cdot c \cdot \sqrt{x} \left(\mathrm{erf}\left(\frac{g}{\sqrt{2}}\right) - \frac{e^{-\frac{g^2}{2}} \cdot 2 \cdot g}{\sqrt{2 \cdot \pi}}\right)$$

g is a constant. Its meaning becomes obvious if one puts $x_0 = x - gs$ and $x_0 = x + gs$ into $D_1(x_0, x)$, then $D_1(x_0, x)$ becomes zero so that g is a measure of where the Gaussian tails are cut.
x_1, x_2 are obtained by requiring $D_1(x_0, x)$ 0.
This $D(x_0, x)$ resolution becomes Gaussian if $g = \infty$.

The choice of the above form was made in order to examine the effects of Gaussian tails.

The variance s_D^2 of $D(x_0, x)$ is given by,

$$s_D^2 = s^2 \cdot \left[\mathrm{erf}\left(\frac{g}{\sqrt{2}}\right) - \frac{2g}{\sqrt{2\pi}} e^{-\frac{g^2}{2}} - \frac{2g^3}{3} e^{-\frac{g^2}{2}} \frac{1}{\sqrt{2\pi}\, A_2}\right] \qquad (4)$$

$$A_2 = \mathrm{erf}\left(\frac{g}{\sqrt{2}}\right) - e^{-\frac{g^2}{2}} \frac{2g}{\sqrt{2\pi}}$$

The Full Width at Half Maximum, $FWHM_D$, for $D(x_0,x)$ is given by,

$$FWHM_D = 2\cdot s\cdot\sqrt{-2\cdot \ln\left[\tfrac{1}{2}\left(1+e^{-g^2/2}\right)\right]} \qquad (5)$$

For large values of g, $s_D \approx s$.

The convolution integral equ. (1) with the parametrizations of equations (2) and (3) becomes,

$$f_0(x_0) = \frac{2\cdot A}{\sqrt{2\pi}\,C A_2}\left[e^{\frac{x_0}{c^2}}\int_{u_1}^{u_2} du\left(e^{-a^2u^2-\frac{b^2}{u^2}} - e^{-\frac{g^2}{2}-Bu^2}\right)\right]$$

$$u_1=\sqrt{x_1}\ ,\quad u_2=\sqrt{x_2} \qquad (6)$$

$$a=\sqrt{B+\frac{1}{2c^2}}$$

$$b=\frac{x_0}{\sqrt{2}\cdot c}$$

It is easy using reference 4a to estimate this integral up to error functions. The result is,

$$f_0(x_0) = \frac{A\cdot e^{\frac{x_0}{c^2}}}{\sqrt{2}\cdot c\cdot A_2}\cdot\frac{1}{2a}\cdot$$

$$\cdot\left\{e^{2ab}\left[\mathrm{erf}\left(a\cdot u_2+\frac{b}{u_2}\right)-\mathrm{erf}\left(au_1+\frac{b}{u_1}\right)\right]+e^{-2ab}\left[\mathrm{erf}\left(au_2-\frac{b}{u_2}\right)-\mathrm{erf}\left(au_1-\frac{b}{u_1}\right)\right]\right\} - \qquad (7)$$

$$-\ \frac{A}{cA_2}\cdot\frac{e^{-g^2/2}}{\sqrt{2B}}\left[\left(\mathrm{erf}(\sqrt{B}u_2)-\mathrm{erf}(\sqrt{B}u_1)\right]\right.$$

For $g\to\infty$ and $u_1=0$, $u_2\to\infty$ ($A_2\to 1$) one gets for the Gaussian case, with s varying with x as $s= c\cdot\sqrt{x}$,

$$f_0(x_0) = \frac{A}{\sqrt{1+2Bc^2}}\cdot e^{-\left(\sqrt{1+2Bc^2}-1\right)\cdot\frac{1}{c^2}\cdot x_0} \qquad (8)$$

This result could be obtained directly, without going through (7), using reference 4b and initially taking $x_1=0$, $x_2=\infty$.

If we write $f_0(x_0)=A_0\cdot e^{-B_0\cdot x_0}$ then from (8) we get,

$$A_0=A/\sqrt{1+2Bc^2}$$

$$B_0=\left(\sqrt{1+2Bc^2}-1\right)/c^2 \qquad (9)$$

These are the folding relations for the case of Gaussian resolution for the distribution $f(x)=A\cdot e^{-B\cdot x}$. We observe that the apparent distribution is still exponential with a different decay constant, B_0. By solving equations (9) for A and B one gets the unfolding relations,

$$A=(1+B_0\cdot c^2)\cdot A_0$$
$$B=B_0+\frac{B_0^2\cdot c^2}{2} \tag{10}$$

A useful relation which gives the value of $x(=x_m)$ where the integrand of the integral in equ. (6) becomes maximum, for the case of the Gaussian resolution function, is,

$$x_m=\tfrac{1}{2}\left(-c^2+\sqrt{c^4+4(1+2Bc^2)x_0^2}\right)/(1+2Bc^2) \tag{11}$$

B) We consider now the case where equ. (2) is still valid but the resolution is given by,

$$D(x_0,x)=\frac{1}{\sqrt{2\pi}\cdot s}\cdot e^{-\frac{(x_0-x)^2}{2\cdot s^2}} \qquad s=\text{constant, independent of } x \tag{12}$$

For $x_1 < x < x_2$ one easily obtains for the convolution integral of equ. (1),

$$f_0(x_0)=\frac{A}{2}e^{\frac{B^2\cdot s^2}{2}}\left[\mathrm{erf}\left(\frac{x_2}{\sqrt{2}\cdot s}+\frac{B\cdot s}{\sqrt{2}}-\frac{x_0}{\sqrt{2}\cdot s}\right)-\mathrm{erf}\left(\frac{x_1}{\sqrt{2}\cdot s}+\frac{B\cdot s}{\sqrt{2}}-\frac{x_0}{\sqrt{2}\cdot s}\right)\right]\cdot e^{-B\cdot x_0} \tag{13}$$

If $x_2\rightarrow\infty$ then,

$$f_0(x_0)=\frac{A}{2}e^{\frac{B^2\cdot s^2}{2}}\left[1-\mathrm{erf}\left(\frac{x_1}{\sqrt{2}\cdot s}+\frac{B\cdot s}{\sqrt{2}}-\frac{x_0}{\sqrt{2}\cdot s}\right)\right]\cdot e^{-B\cdot x_0} \tag{14}$$

For $x_0 \gg \frac{x_1}{\sqrt{2}\cdot s}+\frac{B\cdot s}{\sqrt{2}}$ we get,

$$f_0(x_0)=A\cdot e^{\frac{B^2\cdot s^2}{2}}\cdot e^{-B\cdot x_0} \tag{15}$$

In this case the distribution is just multiplied by a factor without change in its decay constant. We will not make use of this result because we think the resolutions of the calorimeters, so far reported, are better approximated with the formulae used in case A.

Comments on the resolution function

Let us concentrate on the Gaussian case of A. The constant c def[illegible]d by $s=c\cdot\sqrt{x}$, is measured by calibrating the calorimeter with particles of known momentum (we ignore any differences between momentum and energy of the measured particles). One then obtains a relation $s_P=c_P\sqrt{P}$ for the momenta instead of the transverse momenta, assuming that the calorimeter response behaves in this way. If the calorimeter is placed such that the measured particle has a polar angle θ with respect to the beam direction, then obviously $P_T=P\cdot\sin\theta$ and $s= c_P\cdot\sqrt{\sin\theta}\cdot\sqrt{P_T}=c\cdot\sqrt{P_T}=c\cdot\sqrt{x}$.

A problem occurs when $x_0\,(P_{T0})$ is small enough that the resolution function has significant values even for $x<0$ and the question arises with the normalization condition of equ. (1). In fact, for small x the resolution is not as simple as our parametrization indicates. One could imagine one of the following, either apply our formulae for large values of x such that there is no problem or imagine that the side of the resolution function near x=0 is modified such that the normalization condition of equ. (1) holds. This will not affect our results.

Jet measurements with a calorimeter

When measuring a jet of many particles determining c becomes a lit[illegible] more involved. One approach is to use $c=c_P\cdot\sin\theta$ where θ is the jet vector polar angle and c_P is an average of the hadronic and electromagnetic contributions to the jet properly weighted. Notice that c_P for the electromagnetic contribution is much smaller than the corresponding c_P of the hadronic component of the jet. A better approach to dealing with the problem is disribed below.

We can write for the momentum of the jet, $P_J=\sum_i P_i\cdot\cos a_i$, where P_i are the momenta of the fragments of the jet (we assume they are massless) and a_i are their angles with respect to the jet direction. For Gaussian resolutions one could write,

$$s_J^2=\sum_i s_i^2\cdot\cos^2 a_i$$

where $s_i^2= c_{Pi}^2\cdot P_i$ (16)

then $s_J^2=\sum_i c_{Pi}^2\cdot P_i\cdot\cos^2 a_i=c_J^2\cdot P_J$

c_{Pi} could be assumed to vary with the location on the face of the calorimeter where the particles hit. c_{Pi} depend also on the type of particle entering the calorimeter. Photons and electrons have smaller c_{Pi} than pions. On an event by event basis one could calculate c_J and then calculate c from $c=c_J\cdot\sqrt{\sin\theta_J}$, θ_J is the polar angle of the jet.

Although slightly different methods could be implemented which give results which differ slightly, the above could be chosen for simplicity. It is clear that c will, in general, vary from event to event. If c is plotted for a high-P_T spectrum one will obtain a distribution with finite width. The correct way to solve the unfolding problem is to take events having similar values of c and unfold them together. Then add the results of the different c intervals. The approach for folding a jet spectrum is quite similar. A cruder approximation would be to take as c the average value of c and do only one unfolding (folding

It is easy to see from equ. (16) that s_J for a jet could be smaller or larger than the respective s_i for the same value of momentum depending on how s_i varies with momentum.

Applications, results, conclusions

An attempt to use equ. (7) with algebraic approximations for the error function from reference 4c proved not to be very useful due to inaccuracies in the approximations. In order to compare the analytical result of equ. (8) with Gaussian tails in the resolution, to the case where Gaussian tails do not exist, we calculated the integral of equ. (6) for different cuts (g) of the Gaussian tail, by using the 20-point Gaussian integration from reference 4d.

A typical result is displayed in fig. 1. B is taken to have typical values consistent with the results of references 5,6 and 7. c is typical for calorimeters.[8] The conclusion from fig. 1 is that as long as the ratio of the apparent (observed) rate of events to the real rate is approximately below 3 the two methods (with Gaussian tail or without) give results within 10% of each other. The resolution with the Gaussian tail gives higher rates. Notice that we consider for the two cases that at each momentum the variances are the same (s_D, s).

We fitted the result for $pp\rightarrow\pi^+ + X$ from reference 5 with two exponentials ($dN/dP_T \approx \sigma_{inv}\cdot P_T$). The fit was done graphically and had deviations of the order of 5%. Then we took c_P to be 0.919 $(GeV/c)^{\frac{1}{2}}$ and $\sin\theta=0.057$

folded the fit with equ. (8). Fig. 2 indicates that the ratio of observ events to real events is approximately 2 at $P_T = 7$GeV/c. If one consi rs reference 9 and scales the distributions to fixed target experiments with an approximately 400 GeV beam, our exponential parametrization for the higher P_T points of reference 5 also seems to be resonable beyond P_T of 7 GeV/c.

Fig. 3 shows different resolutions with the same $FWHM_D$ or the same s_D ,obtained by adding two Gaussian resolutions with proper weighting, thus simulating the case of worse than Gaussian tails but still keeping s_D or $FWHM_D$ resonable. The results are displayed in fig. 4. It is clear that the effect is worse than that for single Gaussian tails.

Table I gives some characteristics of the various resolutions used to emphasize the fact that small changes in the tail have large effects as displayed in fig. 4.

We comment on the parametrizations of reference 3 and ours, and apparent differences in the two results. Reference 3, which also derives analytical formula, uses as parametrization for $f(x)=A/P^n$ and the same parametrization as this present work for s_P ($s_P = c_P\sqrt{P}$). The conclusion of reference 3 is that as P (or P_T) increases the ratio of apparent events to real events decreases. In our case where we selected parametrization of the form $A \cdot e^{-B \cdot x}$ the oposite is true. In reality neither parametrization is correct to fit the data for a large range of P_T. To illuminate this fact we give the following numbers. The local power n of a local $1/P_T^n$ fit to the 400 GeV $pp \to \pi^+ + X$ data (fit to invariant cross section multiplied by P_T) of reference 5 for P_T=2. GeV/c is 8.4, while for P_T=6. GeV/c it is 11.2. We mention that the values of B for the exponentials that fit the low and high P_T part of the same spectrum are 3.8 $(GeV/c)^{-1}$ and 2.4 $(GeV/c)^{-1}$ respectively. The correct approach is to fit the global P_T data (as we did for the result of fig. 2). One probably should be careful for the contributions of $P \approx 0$ to the convolution integral of reference 3.

In conclusion we remark that a) for resonable calorimeter resolution of Gaussian type with s varying as $c_P \cdot \sqrt{P}$ our analytic formula of equ. (8) is useful, and b) one should be careful not to have large tails in a calorimeter to be used for measuring high-P_T spectra, because they worsen the spectrum (of particles) due to a contamination of hig P_T events from events of lower P_T.

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FIGURE CAPTIONS

Figure 1: R (apparent rate)/(real rate) versus P_T.
The continuous line is the result of folding a distribution of the form $\exp(-3.0\ P_T)$ with a Gaussian resolution with $s_P = 0.8 \cdot \sqrt{P}$ and $\sin\theta = 0.07$.
The dotted line is the result of folding the same distribution with a Gaussian resolution without tails (g=0.6) and the same standard deviation dependence and same $\sin\theta$ as above. The numbers on the top of the figure show the ratios of R's of the two cases at each P_T.

Figure 2: R (apparent rate)/(real rate) versus P_T obtained by fitting the result of reference 5 with two exponentials and folding them with a Gaussian resolution ($\sin\theta = 0.057$, $c_P = 0.845\,(\mathrm{GeV/c})^{1/2}$.

Figure 3: Resolution function D versus P_T for x_0 (P_{T0}) 5 GeV/c, $\sin\theta = 0.07$.

a) Line - - - is the case of a single Gaussian with $c_P = 0.85\ (\mathrm{GeV/c})^{1/2}$.

b) Line —— is the sum of two Gaussians with $c_P = 0.85\,(\mathrm{GeV/c})^{1/2}$ and $2.55\,(\mathrm{GeV/c})^{1/2}$ with weights 0.9 and 0.1 respectively.

c) Line —— - - is one Gaussian with c_P^2 the average of the squares of the two Gaussians of case b), $c_P = (0.9 \cdot (0.85)^2 + 0.1 \cdot (2.55)^2\ )^{1/2} = 1.14\,(\mathrm{GeV/c})^{1/2}$.

d) Line —— - is the sum of two Gaussians with $c_P = 0.85\,(\mathrm{GeV/c})^{1/2}$ and $3.4\,(\mathrm{GeV/c})^{1/2}$ with weights 0.9 and 0.1 respectively.

Figure 4: R (apparent rate)/(real rate) versus P_T for the different resolutions of figure 3. $B = 2.5\ (\mathrm{GeV/c})^{-1}$.
Line - - - is for case a) of figure 3. Line —— is case c), line —— - is case b) and line —— - - is case d).

TABLE I

	c_P $(GeV/c)^{1/2}$	FWHM GeV/c	P_T/P_{To}	A
a)	0.85	1.20	1.31	0.991
b)	1.14	1.20	1.48	0.971
c)	1.14	1.60	1.41	0.991
d)	1.31	1.20	1.57	0.964

The letters a),b),c) and d) refer to the cases of resolutions of figure 3. P_T/P_{To} is the ratio of the P_T where the resolution function has a value equal to 1/100 its peak value, to P_{To} wich is the P_T of the peak (5 GeV/c). The value of P_T the larger than P_{To} has being chosen. A is the "area" of the resolution function between P_{To} -FWHM and P_{To}+FWHM.

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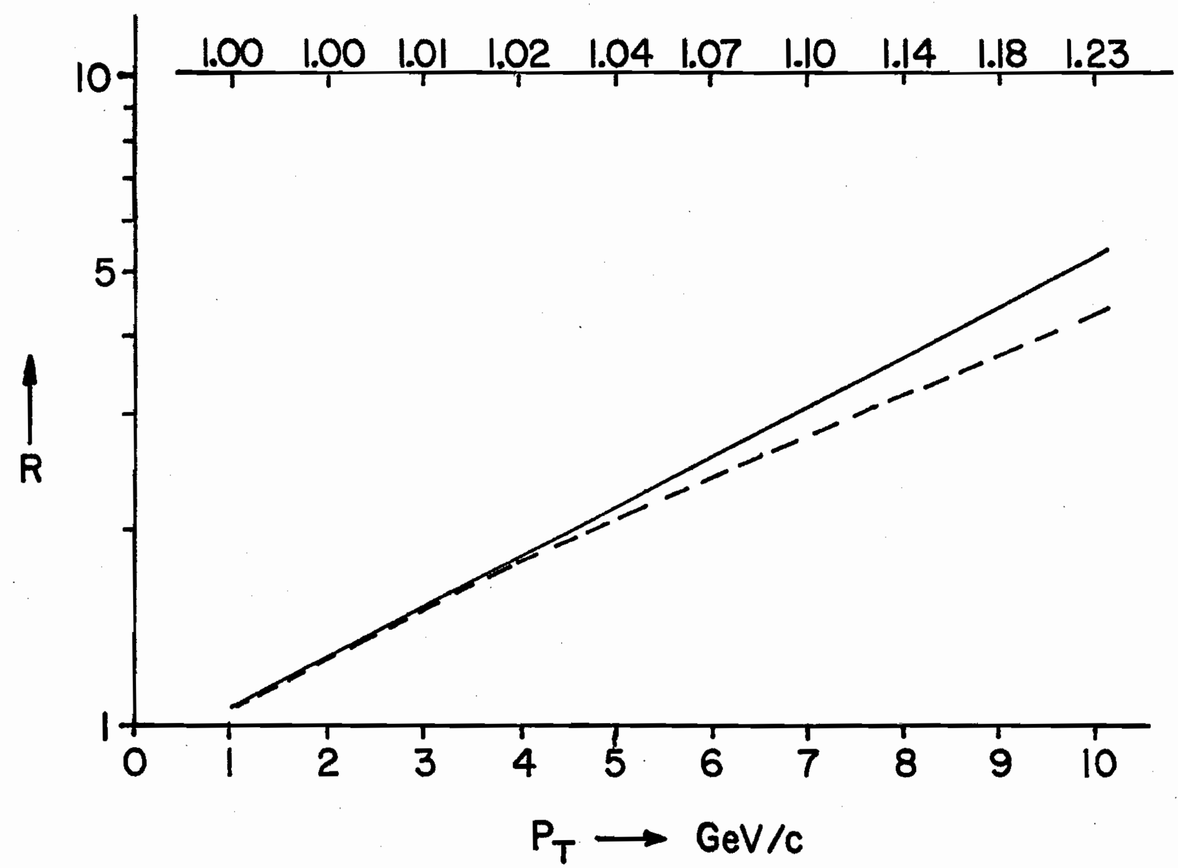
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10
5
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0 1 2 3 4 5 6 7 8 9 10
P_T GeV/c

FIGURE 1

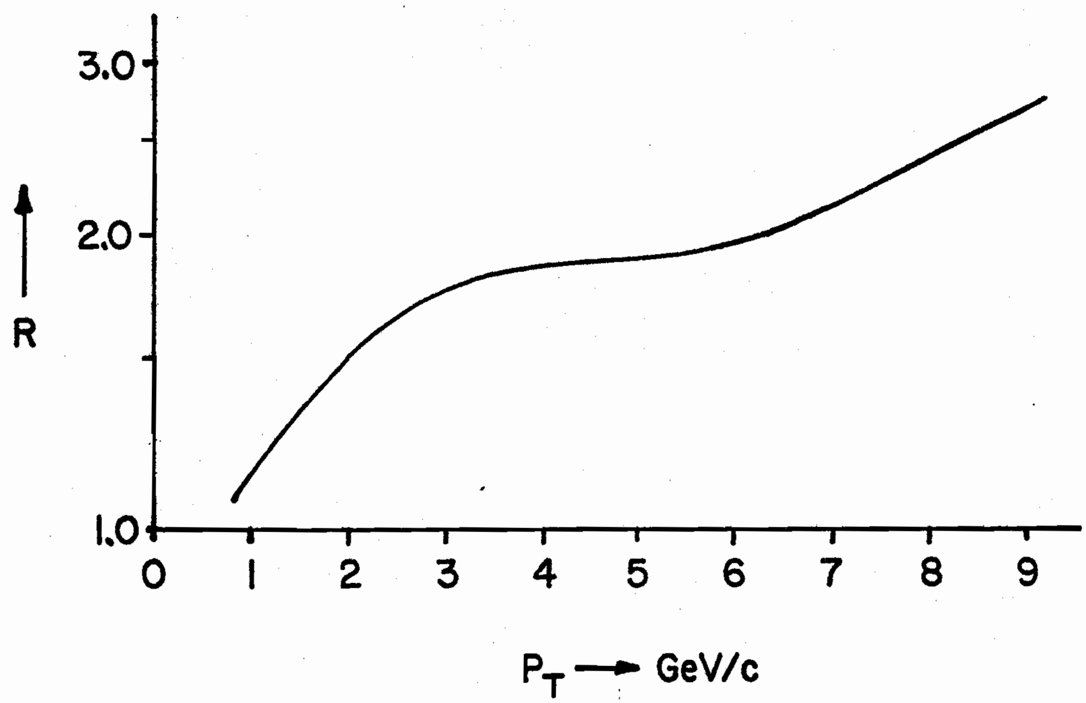

FIGURE 2

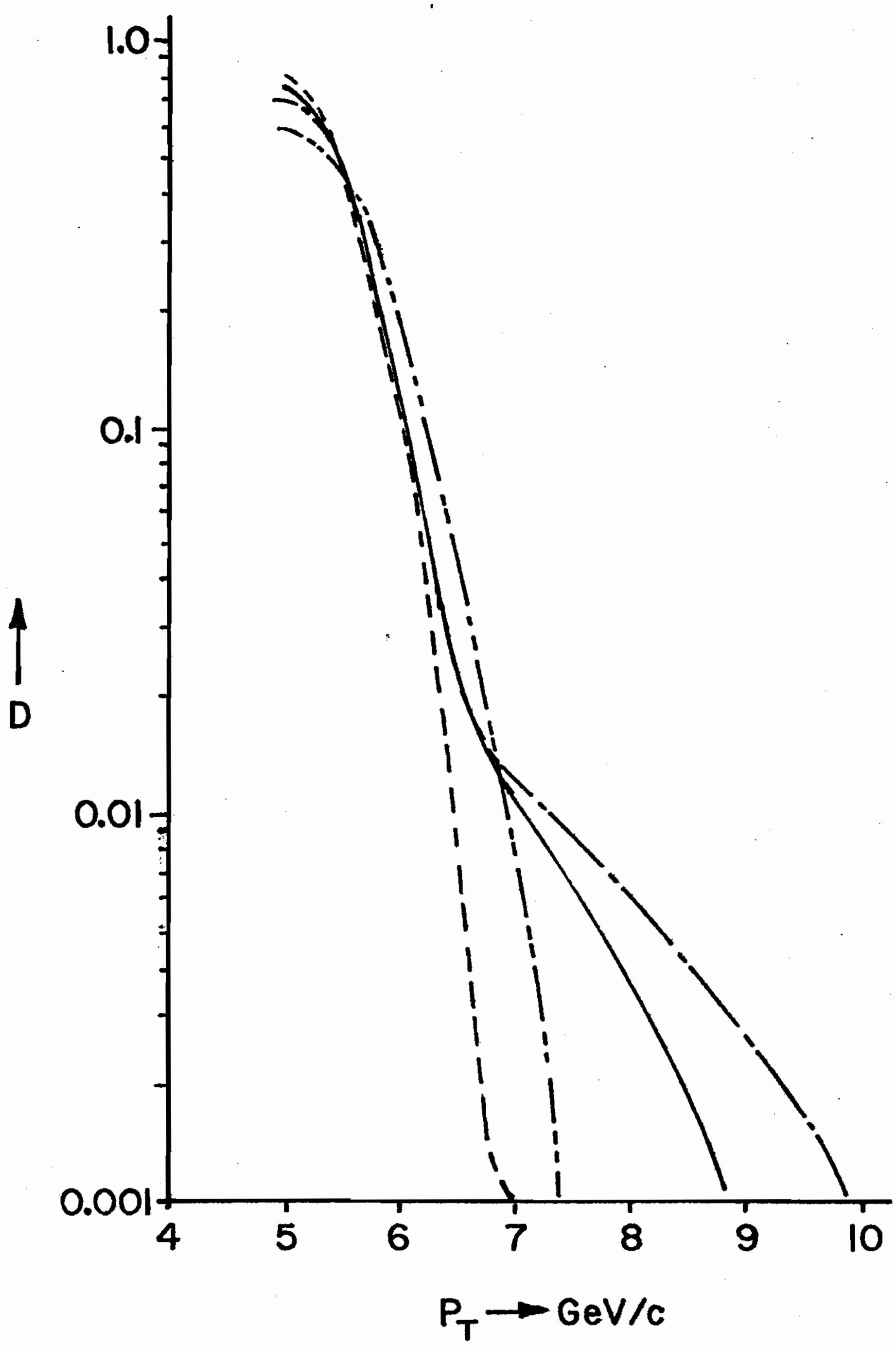

FIGURE 3

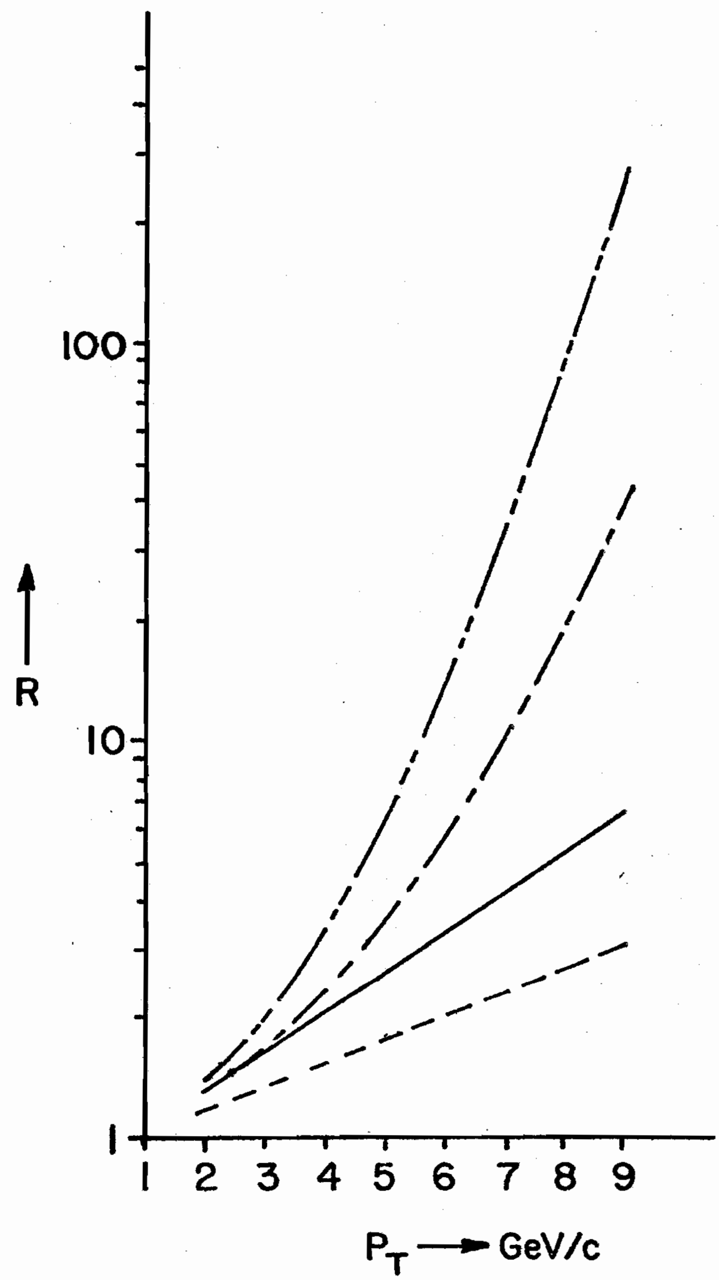

FIGURE 4