A Remark on the High Energy Limit of Scattering Cross-Sections.

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Pomeranchuk[1] has shown that under rather plausible assumptions concerning the very high energy dependence of total cross-section - the main of which being that they behave most as constants at infinity - the difference of the cross-sections for a particle and its charge conjugate on the same target tends to vanish at infinity. More explicitly : if $\sigma^+(E)$ and $\sigma^-(E)$ are the total cross-sections for a particle and its charge conjugate ($\pi^+ - \pi^-$, proton-antiproton, $K^+ - K^-$, $K^0 - \bar{K}^0$, etc.) at total energy E of the incoming particle in the laboratory system on a specific target, and if

$$\lim_{E \to \infty} \sigma^+(E) = \sigma^+(\infty) \qquad \lim_{E \to \infty} \sigma^-(E) = \sigma^-(\infty) \tag{1}$$

$\sigma^+(\infty)$ and $\sigma^-(\infty)$ being finite constants (or zero), then Pomeranchuk shows that

$$\sigma^+(\infty) = \sigma^-(\infty) \tag{2}$$

Hypothesis (1) is justified in P. by making essential use of the finite range of the forces between elementary particles there, studying expressions characteristic of potential scattering, it is shown with plausible arguments that

$$\lim_{E \to \infty} \frac{A^{\pm}(E)}{E} = \text{constant (eventually zero)} \tag{3}$$

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where $A^{\pm}(E)$ is the forward elastic scattering amplitude for the corresponding process averaged over possible spins. By virtue of the optical theorem; the condition (3) on the imaginary part of $A^{\pm}(E)$ is equivalent to (1), for the real part, $D^{\pm}(E)$, (3) implies

$$\lim_{E \to \infty} \frac{D^{\pm}(E)}{E} = \text{constant} \qquad \text{(or zero)} \tag{4}$$

Both conditions (1) and (4) (deriving from assumption (3) [2]) have been used in P. in order to prove (2). We want to stress that (3), even if quite reasonable from a physical point of view, has not been proved so far by using only general principles of field theory. We believe however, that having not any rigorous suggestion it is worthwhile to draw conclusions from conditions that seem quite plausible and are dictated by physical intuition drawn from general potential scattering formalism.

The property (2) is of importance for the application of dispersion relations : in fact, in many cases where the unsubtracted philosophy is by no means acceptable, it is hoped that the unsubtracted dispersion relations for the difference of particle and antiparticle amplitudes can converge because of the vanishing of $\sigma^{+} - \sigma^{-}$ at very high energies. Such dispersion relations, subject to that hope, were used and analyzed for several processes and proved to give meaningful results. This is the case for π-N S-wave scattering [3] (for the combination $\alpha_1 - \alpha_3$) and for K-N scattering [4]. In order that those applications make any sense, it is necessary and sufficient, however, that integrals of the type

$$\int^{t} \frac{\sigma^{-}(E) - \sigma^{+}(E)}{E} \, dE \tag{5}$$

converge for $t \to \infty$. Even if (2) makes more plausible the convergence of (5), it is by no means sufficient : to ensure it, it would be necessary to know how $\sigma^{-} - \sigma^{+}$ goes to zero at very high energies.

We note that if

$$\lim_{E \to \infty} (\sigma^{-}(E) - \sigma^{+}(E)) \to \left(\frac{1}{\log E}\right) \qquad (6)$$

(behaviour which in some cases has been interpreted to be Pomeranchuk's prediction), then (5) would diverge as log log t. In P., however, no discussion is done on how the limit (2) is reached.

We want to show that using the same starting point as P., something can be said on how $\sigma^{-} - \sigma^{+}$ go to 0; in fact we show in this note that integral (5) is indeed convergent.

Let us start from the substracted dispersion relation whose convergence is guaranteed by (1) :

$$D^{+}(E) = \tfrac{1}{2}(1+\tfrac{E}{M})\, D^{+}(M) + \tfrac{1}{2}(1-\tfrac{E}{M})\, D^{-}(M) + \sum_b \frac{A_b p^2}{E+E_b} + \frac{p^2}{\pi}\int_{E_o}^{\infty}\left(\frac{\sigma^{+}(E')}{(E'-E)} + \frac{\sigma^{-}(E')}{(E'+E)}\right)\frac{dE'}{p'} \qquad (7)$$

where $p = (E^2 + M^2)^{\frac{1}{2}}$, M being the mass of the particle in question and $\sum_b A_b p^2/E+E_b$ the contribution of possible poles (bound states at energies E_b) A_b being constants. The lowest limit of integration E_o depends on the process in question (possible existence of unphysical regions). We could start from the dispersion relation for $D^{-}(E)$ without changing the conclusions we shall reach.

Let us define

$$f(E) = \sigma^{-}(E) - \sigma^{+}(E) \qquad (8)$$

then (7) can be rewritten

$$\frac{D^+(E)}{E} = \tfrac{1}{2}\left(\frac{1}{E} + \frac{1}{M}\right) D^+(M) + \tfrac{1}{2}\left(\frac{1}{E} - \frac{1}{M}\right) D^-(M) + \sum_b \frac{A_b(E-M^2/E)}{E+E_b} +$$

$$+ \left(E - \frac{M^2}{E}\right) \frac{2}{\pi} \int_{E_o}^{\infty} \frac{E'}{p'} \frac{\sigma^+(E)}{E'^2-E^2} dE' + \left(E - \frac{M^2}{E}\right) \frac{1}{\pi} \int_{E_o}^{\infty} \frac{f(E')}{E'(E'+E)} dE' \tag{9}$$

The condition (4) is clearly satisfied by the terms containing D(M) and the bound state contributions in the r.h.s. of (9). This is also the case for the integral over σ^+ : if in fact we subdivide the integration at an energy $\mathcal{E}$ sufficiently high so that for $E' > \mathcal{E}$, $\sigma^+(E')$ can be considered a constant, its contribution for $E \gg \mathcal{E}$ is essentially given by

$$\frac{2}{\pi} \frac{1}{E} \int_{E_o}^{\mathcal{E}} \frac{E'}{p'} \sigma^+(E')\, dE' - \frac{2}{\pi} \frac{\mathcal{E}}{E} \sigma^+(\infty) \tag{10}$$

Then, if we call

$$G(E) = \int_{E_o}^{\infty} \frac{\varphi(E')dE'}{(E'+E)}, \quad \text{with} \quad \varphi(E) = \frac{f(E)}{p} \tag{11}$$

condition (4) implies that

$$\lim_{E \to \infty} E\, G(E) \simeq \text{constant} \tag{12}$$

On the other hand the following theorem on Stieltjes transforms holds:[5)]

Theorem 1

Let G(E) be the (convergent) integral defined by (11) ($E_o > 0$) and let

$$\lim_{E \to \infty} \frac{1}{E} \int_{E_o}^{E} E' \varphi(E') dE' = 0 \tag{13}$$

then

$$\lim_{E \to \infty} \left\{ EG(E) - \int_{E_o}^{E} \varphi(E')dE' \right\} = 0 \qquad (14)$$

The condition (13) is trivially satisfied in our case because of (2) (although it is less restrictive then (2) since it implies only the existence of limits (1) and (2) in the mean). From (14) and (12) the convergence of the integral

$$\int_{E_o}^{\infty} \varphi(E')dE' = \int_{E_o}^{\infty} \frac{\sigma^+ - \sigma^-}{p'} dE' \qquad (15)$$

follows, q.e.d. [6)]

As a consequence, we see that the limiting behaviour (6) is by no means compatible with the constancy of cross-sections at high energies, i.e. the difference $\sigma^-(E) - \sigma^+(E)$ must go to zero faster than $\frac{1}{\log E}$.

We want to note, finally, that an immediate consequence of our conclusion is that

$$\lim_{E \to \infty} \Delta(E) = 0 \qquad (16)$$

where

$$\Delta(E) = \frac{D^-(E) - D^+(E)}{E} \qquad (17)$$

By using the unsubstracted dispersion relation for $D^-(E) - D^+(E)$ whose validity we have just discussed, we can write

$$\Delta(E) = \sum_b \frac{B_b}{E^2 - E_b^2} + \frac{2}{\pi} \int_{E_o}^{\infty} \frac{p' f(E')}{E'^2 - E^2} \, dE' \qquad (18)$$

where B_b are the constant residua of the bound state contributions. Changing the integration variable to E'^2, the integral of the r.h.s. of (18) is reduced to the form of a Hilbert transform. Its convergence and the possibility of inverting the Hilbert transform [7] are ensured by the exclusion of (15). This means that the expression

$$\int^{\infty} \frac{\Delta(E')}{E'^2 - E^2} \, dE'^2$$

must converge for all values of E (and go to zero for $E \to \infty$) from which (16) follows.

An empirical discussion on the limit value for $\Delta(E)$ was given by Goldberger et al. [2] in connection with the possibility of using (18) as a sum rule. Here instead we have shown, from rather general theoretical grounds, that (18) is valid without substraction constants and that, as a consequence, the limit (16) holds.

R E F E R E N C E S

1) I. Ia. Pomeranchuk, J.Exptl.Theoret.Phys. (U.S.S.R.) 34, 725 (1958); J.E.T.P. 7,499 (1958), hereafter referred to as P.

2) We note that the requirement (3) is stronger than the more obvious one that $A^{\pm}(E)/E$ stay bounded at infinity. However, both our conclusion and that of Pomeranchuk depend on the supposed existence of the limits.

3) M.L. Goldberger, H. Miyazawa and R. Oehme, Phys.Rev. 99, 986 (1955).

See also : A. Salam, Proceedings of CERN Symposium (1956), Vol.II, p.179.

4) P.T. Matthews and A. Salam, Phys.Rev. 110, 569 (1958);

R.H. Dalitz, Proceedings of the 1958 Annual Conference on High Energy Physics at CERN, p.192.;

E. Galzenati and B. Vitale, Phys.Rev. 113, 1635 (1959);

D. Amati, Phys.Rev. 113, 1692 (1959).

5) This is a slight modification of a theorem by Hardy and Littlewood Proc. London Math. Soc. (2) 30, 34 (1930). Compare also with D.V. Widder - "The Laplace Transform" - Princeton University Press 1946, Lemma 5, p. 193.

6) (14) shows that if $D^{\pm}(E)/E$ satisfies only the weaker condition to be bounded at infinity, then the existence of the integral (5) does not necessarily follow. The following example : $\sigma^{+} - \sigma^{-} \sim \frac{\sin(\ln E)}{\ln E}$ ($\rightarrow 0$ for $E \rightarrow \infty$), $EG(E) \sim \sin(\ln E)$, gives an oscillating behaviour of $\int_{E_0}^{E} \varphi(E')dE'$ for $E \rightarrow \infty$. On the other hand if the limit (4) exists, then the integral (15) converges also in the more general case of bounded σ^{+} and σ^{-} without the requirement of the actual existence of a limit for $\sigma^{+} - \sigma^{-}$ at $E \rightarrow \infty$. This last fact is the consequence of another theorem of Hardy and Littlewood (loc. cit.). A trivial example : $\sigma^{+} - \sigma^{-} \sim \sin E$.

7) E.C. Titchmarsh - "Introduction to the Theory of Fourier Integrals" - Oxford Press 1948, p. 724.