

Article

Darboux Associated Curves of a Null Curve on Pseudo-Riemannian Space Forms

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Abstract: In this work, the Darboux associated curves of a null curve on pseudo-Riemannian space forms, i.e., de-Sitter space, hyperbolic space and a light-like cone in Minkowski 3-space are defined. The relationships of such partner curves are revealed including the relationship of their Frenet frames and the curvatures. Furthermore, the Darboux associated curves of k-type null helices are characterized and the conclusion that a null curve and its self-associated curve share the same Darboux associated curve is obtained.

Keywords: null curve; Darboux associated curve; pseudo-Riemannian space form

1. Introduction

The geometry in Minkowski space is very important and interesting in both mathematics and physics. It is well-known that there exist three kinds of curves, i.e., space-like curve, time-like curve and light-like (null) curve in Minkowski space. Many topics in classical differential geometry of Riemannian manifolds can be extended into those of Lorentz–Minkowski manifolds. However, the geometry of null curves has no Riemannian analogs because one can not define the arc length parameter of null curves in a natural way due to the norm of the light-like vector vanishing everywhere.

Bejancu [1] studied the properties of general null curves in Minkowski space. In 1998, Nersessian and Romos [2,3] have shown the importance of null curves in physics and mathematics by showing that there exists a geometric particle model associated with null curves in Minkowski space. Inoguchi and Lee [4] proved the existence of a canonical representation of null curves in Minkowski 3-space. All these works proved that it is possible to study the null curves if the appropriate method can be applied. Wang and Pei [5] defined the Darboux (rotation) vector of a null curve which describes the direction of rotation axis of a Cartan frame in Minkowski 3-space. Nesovic et al. [6] defined a k-type null Cartan slant helices lying on a time-like surface according to their Darboux frame. One of the authors and Kim [7] defined the structure function of null curves and studied the directional associated curves of a null curve in Minkowski 3-space. Making use of the structure function of null curves, most of works about null curves have been pushed forward greatly, such as the generalized null scrolls that are characterized via the structure function of null curves [8].

In this paper, the Darboux associated curves of a null curve on three pseudo-Riemannian space forms are defined and studied. In Section 2, some fundamental facts of null curves, space forms and the Darboux vector of a null curve are reviewed. In Section 3, the relationships between a null curve and its Darboux associated curve on three pseudo-Riemannian space forms, i.e., de-Sitter space, hyperbolic space and light-like cone are discussed respectively. Particularly, the Darboux associated curves of k-type null helices are characterized and some typical examples are given. Last but not least, the relationship of two null curves which share the same Darboux associated curves is found.

Throughout this paper, all geometric objects under consideration are smooth and regular unless otherwise stated.

2. Preliminaries

Let \mathbb{E}_1^3 be the Minkowski 3-space with natural Lorentzian metric

$$\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 - dx_3^2$$

in terms of the natural coordinate system (x_1, x_2, x_3) . A vector v in \mathbb{E}_1^3 is said to be space-like, time-like and light-like (null) if $\langle v, v \rangle > 0$ or $v = 0$, $\langle v, v \rangle < 0$ and $\langle v, v \rangle = 0$ ($v \neq 0$), respectively. The norm of a vector v is defined by $\|v\| = \sqrt{|\langle v, v \rangle|}$.

An arbitrary curve r in \mathbb{E}_1^3 is space-like, time-like or light-like (null) if its tangent vector r' is space-like, time-like or light-like (null), correspondingly. For null curves, we have

Proposition 1. [7] Let $r(s)$ be a null curve parameterized by the null arc length s (i.e., $\|r''(s)\| = 1$) in \mathbb{E}_1^3 . Then there exists a unique Cartan frame $\{r'(s) = T(s), N(s), B(s)\}$ such that

$$\begin{cases} T'(s) = N(s), \\ N'(s) = \kappa(s)T(s) - B(s), \\ B'(s) = -\kappa(s)N(s), \end{cases} \quad (1)$$

where $\langle T(s), T(s) \rangle = \langle B(s), B(s) \rangle = \langle T(s), N(s) \rangle = \langle B(s), N(s) \rangle = 0$, $\langle T(s), B(s) \rangle = \langle N(s), N(s) \rangle = 1$.

In the sequel, $T(s)$, $N(s)$ and $B(s)$ is called the tangent, principal normal and binormal vector field of $r(s)$, respectively. From Equation (1), it is easy to know that $\kappa(s) = -\frac{1}{2}\langle r'''(s), r'''(s) \rangle$. The function $\kappa(s)$ is called the null curvature of $r(s)$ which is an invariant under Lorentzian transformations [4]. Hence, a null curve is only determined by its null curvature.

In [7], the authors introduced the structure function and the representation formula of a null curve. Namely,

Proposition 2. [7] Let $r = r(s) : I \rightarrow \mathbb{E}_1^3$ be a null curve. Then r can be written as

$$r(s) = \int \frac{f}{2f'}(f - f^{-1}, 2, f + f^{-1})ds, \quad (f' = \frac{df}{ds}), \quad (2)$$

where f is called the structure function of r . And the null curvature of r can be expressed by

$$\kappa(s) = \frac{1}{2}[(\log f')']^2 - (\log f')''. \quad (3)$$

For a null curve $r(s)$ with Frenet frame $\{T(s), N(s), B(s)\}$ and null curvature $\kappa(s)$, the Darboux (rotation) vector field $D(s)$ along $r(s)$ is defined as $D(s) = \kappa(s)T(s) + B(s)$ (see details in [5]).

Definition 1. [9] Let p be a fixed point in \mathbb{E}_1^3 and $r > 0$ be a constant. Then the pseudo-Riemannian space forms, i.e., the de-Sitter space $\mathbb{S}_1^2(p, r)$, the hyperbolic space $\mathbb{H}_0^2(p, r)$ and the light-like cone $\mathbb{Q}_1^2(p)$ are defined as

$$\mathbb{M}^2(\varepsilon_0) = \{x \in \mathbb{E}_1^3 : \langle x - p, x - p \rangle = \varepsilon_0 r^2\} = \begin{cases} \mathbb{S}_1^2(p, r) \mid \varepsilon_0 = 1; \\ \mathbb{H}_0^2(p, r) \mid \varepsilon_0 = -1; \\ \mathbb{Q}_1^2(p) \mid \varepsilon_0 = 0. \end{cases}$$

The point p is called the center of $\mathbb{S}_1^2(p, r)$, $\mathbb{H}_0^2(p, r)$ and $\mathbb{Q}_1^2(p)$. When p is the origin and $r = 1$, we simply denote them by \mathbb{S}_1^2 , \mathbb{H}^2 and \mathbb{Q}_1^2 .

Proposition 3. [9] Let $r = r(s) : I \rightarrow \mathbb{S}_1^2$ be a curve with Frenet frame $\{\alpha, \beta, r = \gamma\}$. Then

$$\begin{cases} \alpha'(s) = \kappa(s)\beta(s), \\ \beta'(s) = \kappa(s)\alpha(s) - \varepsilon\gamma(s), \\ \gamma'(s) = \beta(s), \end{cases} \quad (4)$$

where $\langle \gamma, \gamma \rangle = 1$, $\langle \beta, \beta \rangle = -\langle \alpha, \alpha \rangle = \varepsilon = \pm 1$.

Proposition 4. [9] Let $r = r(s) : I \rightarrow \mathbb{H}^2$ be a curve with Frenet frame $\{\alpha, \beta, r = \gamma\}$. Then

$$\begin{cases} \alpha'(s) = \kappa(s)\beta(s), \\ \beta'(s) = -\kappa(s)\alpha(s) + \gamma(s), \\ \gamma'(s) = \beta(s), \end{cases} \quad (5)$$

where $\langle \gamma, \gamma \rangle = -1$, $\langle \beta, \beta \rangle = \langle \alpha, \alpha \rangle = 1$.

Proposition 5. [9] Let $r = r(s) : I \rightarrow \mathbb{Q}_1^2$ be a curve with Frenet frame $\{\alpha, \beta, r = \gamma\}$. Then

$$\begin{cases} \alpha'(s) = -\kappa(s)\beta(s), \\ \beta'(s) = \kappa(s)\gamma(s) - \alpha(s), \\ \gamma'(s) = \beta(s), \end{cases} \quad (6)$$

where $\langle \gamma, \gamma \rangle = 0$, $\langle \beta, \beta \rangle = \langle \alpha, \gamma \rangle = 1$.

Definition 2. Let $r = r(s) : I \rightarrow \mathbb{E}_1^3$ be a null curve with Darboux (rotation) vector field $D(s)$. Then the curve $\bar{r}(s) : I \rightarrow \mathbb{M}^2(\varepsilon_0) \subset \mathbb{E}_1^3$ defined as

$$\bar{r}(s) = \frac{D(s)}{\|D(s)\|} = \frac{\kappa(s)T(s) + B(s)}{\sqrt{|2\kappa|}}$$

is called the Darboux associated curve of $r(s)$ on $\mathbb{M}^2(\varepsilon_0)$.

Remark 1. Obviously, $D(s) = D$ is space-like (respectively time-like or light-like) if and only if $\kappa > 0$ (respectively $\kappa < 0$ or $\kappa = 0$) because $\langle D, D \rangle = 2\kappa$. If D is space-like (respectively time-like), then $\bar{r}(s) \in \mathbb{S}_1^2$ (respectively \mathbb{H}^2). If D is light-like, then the Darboux associated curve of $r(s)$ degenerates as $\bar{r}(s) = B(s) \in \mathbb{Q}_1^2$.

Remark 2. In particular, $D(s)$ and $\bar{r}(s)$ is constant since $D'(s) = \kappa'(s)T(s) = 0$ when the null curvature $\kappa(s)$ is constant.

3. Main Results

In this section, we study the Darboux associated curve of a null curve on de-Sitter space \mathbb{S}_1^2 , hyperbolic space \mathbb{H}^2 and light-like cone \mathbb{Q}_1^2 , respectively.

3.1. Darboux Associated Curve of a Null Curve on De-Sitter Space

Theorem 1. Let $r = r(s) : I \rightarrow \mathbb{E}_1^3$ be a null curve with Frenet frame $\{T, N, B\}$ and null curvature $\kappa(s) > 0$, $\bar{r}(s)$ its Darboux associated curve on \mathbb{S}_1^2 with Frenet frame $\{\alpha, \beta, \gamma\}$ and curvature $\bar{\kappa}(s)$. Then the Frenet frames of $r(s)$ and $\bar{r}(s)$ satisfy

$$\alpha = \pm N, \quad \beta = \pm \frac{\kappa T - B}{\sqrt{2\kappa}}, \quad \gamma = \bar{r} = \frac{\kappa T + B}{\sqrt{2\kappa}};$$

the curvature $\bar{\kappa}(s)$ of $\bar{r}(s)$ can be given by

$$\bar{\kappa} = \frac{\pm\sqrt{2\kappa}}{(\log\sqrt{2\kappa})'} = \frac{\pm\|D\|}{(\log\|D\|)'}.$$
 (7)

Proof of Theorem 1. When $\kappa > 0$, i.e., $\bar{r}(s) \in \mathbb{S}_1^2$ for all s , we have

$$\gamma = \bar{r} = \frac{D}{\|D\|} = \frac{\kappa T + B}{\sqrt{2\kappa}}.$$
 (8)

Differentiating Equation (8) by using Equations (4) and (1), it gives

$$\beta \frac{d\bar{s}}{ds} = \frac{\kappa'}{2\sqrt{2\kappa}} T - \frac{\kappa'}{2\kappa\sqrt{2\kappa}} B,$$
 (9)

where \bar{s} is the arc length of \bar{r} .

Taking scalar product on both sides of Equation (9), we obtain

$$\varepsilon \left(\frac{d\bar{s}}{ds} \right)^2 = -\left(\frac{\kappa'}{2\kappa} \right)^2.$$
 (10)

Equation (10) implies $\varepsilon = -1$, i.e., $\langle \beta, \beta \rangle = -\langle \alpha, \alpha \rangle = -1$ and $\left(\frac{d\bar{s}}{ds} \right)^2 = \left(\frac{\kappa'}{2\kappa} \right)^2$. Hence, Equation (9) can be rewritten by

$$\beta = \pm \frac{\kappa T - B}{\sqrt{2\kappa}}.$$
 (11)

Differentiating Equation (11), we get

$$\begin{aligned} \bar{\kappa}\alpha + \gamma &= \frac{\kappa T + B}{\sqrt{2\kappa}} + \frac{2\kappa\sqrt{2\kappa}}{\kappa'} N \\ &= \gamma + \frac{2\kappa\sqrt{2\kappa}}{\kappa'} N. \end{aligned}$$
 (12)

From Equation (12), we can find easily

$$\bar{\kappa}\alpha = \frac{2\kappa\sqrt{2\kappa}}{\kappa'} N.$$
 (13)

Taking scalar product on both sides of Equation (13), we obtain $\alpha = \pm N$ and

$$\bar{\kappa}^2 = \frac{8\kappa^3}{\kappa'^2}.$$

Thus, the curvature $\bar{\kappa}(s)$ of $\bar{r}(s)$ is given by Equation (7). \square

3.2. Darboux Associated Curve of a Null Curve on Hyperbolic Space

Theorem 2. Let $r = r(s) : I \rightarrow \mathbb{E}_1^3$ be a null curve with Frenet frame $\{T, N, B\}$ and null curvature $\kappa(s) < 0$, $\bar{r}(s)$ its Darboux associated curve on \mathbb{H}^2 with Frenet frame $\{\alpha, \beta, \gamma\}$ and curvature $\bar{\kappa}(s)$. Then the Frenet frames of $r(s)$ and $\bar{r}(s)$ satisfy

$$\alpha = \pm N, \quad \beta = \pm \frac{\kappa T - B}{\sqrt{-2\kappa}}, \quad \gamma = \frac{\kappa T + B}{\sqrt{-2\kappa}}$$

the curvature $\bar{\kappa}(s)$ of $\bar{r}(s)$ can be given by

$$\bar{\kappa} = \frac{\pm\sqrt{-2\kappa}}{(\log\sqrt{-2\kappa})'} = \frac{\pm\|D\|}{(\log\|D\|)'}.$$

Proof of Theorem 2. When $\kappa < 0$, i.e., $\bar{r}(s) \in \mathbb{H}^2$ for all s , by similar calculations in Theorem 1, we can find the conclusions easily. \square

3.3. Darboux Associated Curve of a Null Curve on Light-like Cone

Theorem 3. The Darboux associated curve of a null curve on light-like cone degenerates as a point on \mathbb{Q}_1^2 .

Proof of Theorem 3. When $\kappa = 0$, i.e., $\bar{r}(s) = B(s) \in \mathbb{Q}_1^2$ for all s . From Remark 2, the conclusion is straightforward. \square

4. Darboux Associated Curves of K-Type Null Helices on Pseudo-Riemannian Space Forms

First of all, let us review the definition of k-type null helix in \mathbb{E}_1^3 and the relevant results.

Definition 3. [10] Let $r(s) : I \rightarrow \mathbb{E}_1^3$ be a null curve with Frenet frame $\{T, N, B\}$. If there exists a nonzero constant vector field V such that $\langle T, V \rangle \neq 0$ (respectively $\langle N, V \rangle \neq 0, \langle B, V \rangle \neq 0$) is a constant for all $s \in I$, then $r(s)$ is said to be a k-type ($k=1,2,3$) null helix and V is called the axis of $r(s)$.

Remark 3. If the vector T , N or B in the Frenet frame of $r(s)$ is constant, then every fixed direction V satisfies the above definition. Throughout this paper, we assume this situation never happens.

Theorem 4. [11] Let $r = r(s) : I \rightarrow \mathbb{E}_1^3$ be a null curve. Then

1. $r(s)$ is a 1-type null helix or 3-type null helix if and only if its null curvature $\kappa(s)$ is a constant;
2. $r(s)$ is a 2-type null helix if and only if its null curvature $\kappa(s) = \frac{a}{2(s+b)^2}$, ($a \in \mathbb{R} - \{0\}$, $b \in \mathbb{R}$).

Remark 4. From Theorem 4, a 1-type or 3-type null helix has constant null curvature, so their Darboux associated curves are not considered here by Remark 2.

Theorem 5. [8] Let $r = r(s) : I \rightarrow \mathbb{E}_1^3$ be a 2-type null helix with null curvature $\kappa(s) = \frac{a}{2(s+b)^2}$, ($a \in \mathbb{R} - \{0\}$, $b \in \mathbb{R}$). Then $r(s)$ and its structure function $f(s)$ can be represented as

1. when $a = c^2 - 1$ and $c \neq 0, \pm 1$, we have

$$\begin{cases} f(s) = s^c \text{ or } f(s) = s^{-c}, \\ r(s) = \frac{1}{2c} \left(\frac{s^{c+2}}{c+2} - \frac{s^{2-c}}{2-c}, s^2, \frac{s^{c+2}}{c+2} + \frac{s^{2-c}}{2-c} \right); \end{cases}$$

2. when $a = -1$ and $c \neq 0$, we have

$$\begin{cases} f(s) = \frac{c}{\log s} \text{ or } f(s) = \frac{\log s}{c}, \\ r(s) = \frac{s^2}{4} \left(c - \frac{\log^2 s}{c} + \frac{\log s}{c} - \frac{1}{2c}, 2\log s - 1, c + \frac{\log^2 s}{c} - \frac{\log s}{c} + \frac{1}{2c} \right); \end{cases}$$

3. when $a = -c^2 - 1$ and $c \neq 0$, we have

$$\begin{cases} f(s) = \frac{2}{c} \tan(\frac{c}{2} \log s) \text{ or } f(s) = -\frac{2}{c} \tan^{-1}(\frac{c}{2} \log s), \\ r(s) = (\frac{s^2}{8c^2} (4 - c^2 - 2c \sin(c \log s) - 4 \cos(c \log s)), \frac{s^2}{4+c^2} (\frac{2}{c} \sin(c \log s) - \cos(c \log s)), \\ \frac{s^2}{8c^2} (4 + c^2 - \frac{2c(4 - c^2)}{4 + c^2} \sin(c \log s) - \frac{4(4 - c^2)}{4 + c^2} \cos(c \log s))). \end{cases}$$

Theorem 6. Let $r = r(s) : I \rightarrow \mathbb{E}_1^3$ be a 2-type null helix with null curvature $\kappa(s) = \frac{a}{2(s+b)^2}$, ($a \in \mathbb{R} - \{0\}$, $b \in \mathbb{R}$). Then the Darboux vector field $D(s)$ of $r(s)$ can be written as

1. when $a = c^2 - 1$ and $c \neq 0, \pm 1$, we have

$$D(s) = \frac{1}{2c} ((-1 - c)s^{c-1} + (1 - c)s^{-c-1}, \frac{2(c^2 - 1)}{s}, (-1 - c)s^{c-1} - (1 - c)s^{-c-1}).$$

2. when $a = -1$ and $c \neq 0$, we have

$$D(s) = (-\frac{c}{2s} + \frac{\log^2 s}{2cs} + \frac{1 + \log s}{cs}, \frac{-1 - \log s}{s}, -\frac{c}{2s} - \frac{\log^2 s}{2cs} - \frac{1 + \log s}{cs}).$$

3. when $a = -c^2 - 1$ and $c \neq 0$, we have

$$\begin{aligned} D(s) = & \frac{1}{2cs} (\frac{(c^2 + 4)[\cos(c \log s) - c \sin(c \log s)] + (c^2 + 1)(c^2 - 4)}{2c}, \\ & - 2 \sin(c \log s) - 2c \cos(c \log s), \frac{(4 - c^2)[\cos(c \log s) - c \sin(c \log s)] - (c^2 + 1)(c^2 + 4)}{2c}). \end{aligned}$$

Proof of Theorem 6. From Proposition 1 and Theorem 5, through direct calculations, the Frenet frame of $r(s)$ can be expressed as follows:

1. when $a = c^2 - 1$ and $c \neq 0, \pm 1$, we have

$$\begin{cases} T = \frac{s}{2c} (s^c - s^{-c}, 2, s^c + s^{-c}), \\ N = \frac{1}{2c} ((c + 1)s^c + (c - 1)s^{-c}, 2, (c + 1)s^c - (c - 1)s^{-c}), \\ B = \frac{1}{4c} (-(c + 1)^2 s^{c-1} + (c - 1)^2 s^{-c-1}, \frac{2(c^2 - 1)}{s}, -(c + 1)^2 s^{c-1} - (c - 1)^2 s^{-c-1}). \end{cases}$$

Therefore, the Darboux vector $D(s) = \kappa(s)T(s) + B(s)$ of $r(s)$ can be written by

$$D(s) = \frac{1}{2c} ((-1 - c)s^{c-1} + (1 - c)s^{-c-1}, \frac{2(c^2 - 1)}{s}, (-1 - c)s^{c-1} - (1 - c)s^{-c-1}).$$

2. when $a = -1$ and $c \neq 0$, we have

$$\begin{cases} T = \frac{s \log s}{2} (\frac{c}{\log s} - \frac{\log s}{c}, 2, \frac{c}{\log s} + \frac{\log s}{c}), \\ N = \frac{1}{2} (c - \frac{\log^2 s}{c} - \frac{2 \log s}{c}, 2 \log s + 2, c + \frac{\log^2 s}{c} + \frac{2 \log s}{c}), \\ B = (\frac{\log^2 s}{4cs} + \frac{1 + \log s}{cs} - \frac{c}{4s}, -\frac{2 + \log s}{2s}, -\frac{\log^2 s}{4cs} - \frac{1 + \log s}{cs} - \frac{c}{4s}). \end{cases}$$

Therefore, the Darboux vector $D(s) = \kappa(s)T(s) + B(s)$ of $r(s)$ can be written by

$$D(s) = \left(-\frac{c}{2s} + \frac{\log^2 s}{2cs} + \frac{1 + \log s}{cs}, \frac{-1 - \log s}{s}, -\frac{c}{2s} - \frac{\log^2 s}{2cs} - \frac{1 + \log s}{cs} \right).$$

3. when $a = -c^2 - 1$ and $c \neq 0$, we have

$$\left\{ \begin{array}{l} T = \frac{s}{c} \left(\frac{4-c^2}{4c} - \frac{4+c^2}{4c} \cos(c \log s), \sin(c \log s), \frac{4+c^2}{4c} - \frac{4-c^2}{4c} \cos(c \log s) \right), \\ N = \left(\frac{4-c^2}{4c^2} - \frac{4+c^2}{4c^2} (\cos(c \log s) - c \sin(c \log s)), \frac{\sin(c \log s) + c \cos(c \log s)}{c}, \right. \\ \left. \frac{4+c^2}{4c^2} - \frac{4-c^2}{4c^2} (\cos(c \log s) - c \sin(c \log s)) \right), \\ B = \left(\frac{(c^2+4)[(1-c^2)\cos(c \log s) - 2c \sin(c \log s)] + (c^2+1)(c^2-4)}{8c^2s}, \right. \\ \left. \frac{(c^2-1)\sin(c \log s) - 2c \cos(c \log s)}{2cs}, \right. \\ \left. \frac{(4-c^2)[(1-c^2)\cos(c \log s) - 2c \sin(c \log s)] - (c^2+1)(c^2+4)}{8c^2s} \right). \end{array} \right.$$

Therefore, the Darboux vector $D(s) = \kappa(s)T(s) + B(s)$ of $r(s)$ can be written by

$$D(s) = \frac{1}{2cs} \left(\frac{(c^2+4)[\cos(c \log s) - c \sin(c \log s)] + (c^2+1)(c^2-4)}{2c}, \right. \\ \left. - 2 \sin(c \log s) - 2c \cos(c \log s), \frac{(4-c^2)[\cos(c \log s) - c \sin(c \log s)] - (c^2+1)(c^2+4)}{2c} \right).$$

□

Theorem 7. Let $r = r(s) : I \rightarrow \mathbb{E}_1^3$ be a 2-type null helix with null curvature $\kappa(s) = \frac{a}{2(s+b)^2}$, $(a \in \mathbb{R} - \{0\}, b \in \mathbb{R})$. Then the Darboux associated curve $\bar{r}(s)$ of $r(s)$ can be expressed as

1. when $a = c^2 - 1$ and $c \neq 0, \pm 1$, we have

$$\bar{r}(s) = \frac{1}{2c\sqrt{c^2-1}}((-1-c)s^c + (1-c)s^{-c}, 2(c^2-1), (-1-c)s^c - (1-c)s^{-c}).$$

2. when $a = -1$ and $c \neq 0$, we have

$$\bar{r}(s) = \left(-\frac{c}{2} + \frac{\log^2 s}{2c} + \frac{1 + \log s}{c}, -1 - \log s, -\frac{c}{2} - \frac{\log^2 s}{2c} - \frac{1 + \log s}{c} \right).$$

3. when $a = -c^2 - 1$ and $c \neq 0$, we have

$$\bar{r}(s) = \frac{1}{2c\sqrt{c^2+1}} \left(\frac{(c^2+4)[\cos(c \log s) - c \sin(c \log s)] + (c^2+1)(c^2-4)}{2c}, \right. \\ \left. - 2 \sin(c \log s) - 2c \cos(c \log s), \frac{(4-c^2)[\cos(c \log s) - c \sin(c \log s)] - (c^2+1)(c^2+4)}{2c} \right).$$

Proof of Theorem 7. By Definition 2 and Theorem 6, the Darboux associated curve $\bar{r}(s)$ of $r(s)$ can be obtained easily. □

From Theorem 1, Theorem 2 and Theorem 4, we have the following result.

Corollary 1. Let $r = r(s) : I \rightarrow \mathbb{E}_1^3$ be a null curve. Then $r(s)$ is a 2-type null helix if and only if its Darboux associated curve $\bar{r}(s)$ has nonzero constant curvature.

Proof of Corollary 1. Let $r(s)$ be a 2-type null helix in \mathbb{E}_1^3 . From Theorem 4, the null curvature is $\kappa(s) = \frac{a}{2(s+b)^2}$, ($a \in \mathbb{R} - \{0\}$, $b \in \mathbb{R}$). By Theorems 1 and 2, the curvature $\bar{\kappa}$ of its Darboux associated curve $\bar{r}(s)$ is given by

$$\bar{\kappa} = \pm \sqrt{|a|}.$$

Thus $\bar{r}(s)$ has nonzero constant curvature. Conversely, if $\bar{\kappa} = c$ is a nonzero constant then, from Theorems 1 and 2, we know

$$\frac{\pm \sqrt{|2\kappa|}}{(\log \sqrt{|2\kappa|})'} = c.$$

Solving the above differential equation, we get

$$\kappa(s) = \frac{a}{2s^2}, (a = \pm c^2).$$

By Theorem 4, $r(s)$ is a 2-type null helix. \square

Example 1. Let $r : I \rightarrow \mathbb{E}_1^3$ be a 2-type null helix with $\kappa(s) = 4s^{-2}$. By Theorem 5, $r(s)$ can be expressed by

$$r(s) = \frac{1}{6} \left(\frac{1}{5}s^5 + s^{-1}, s^2, \frac{1}{5}s^5 - s^{-1} \right).$$

From Theorem 7, the Darboux associated curve $\bar{r}(s)$ can be obtained as

$$\bar{r}(s) = -\frac{1}{6\sqrt{2}} (2s^3 + s^{-3}, -8, 2s^3 - s^{-3}),$$

which satisfies $\langle \bar{r}, \bar{r} \rangle = 1$, i.e., $\bar{r}(s) \in \mathbb{S}_1^2$. (see Figure 1 and 2)

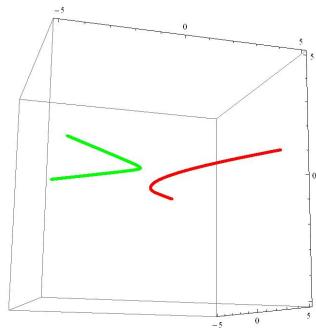


Figure 1. The positions of r (red) and \bar{r} (green).

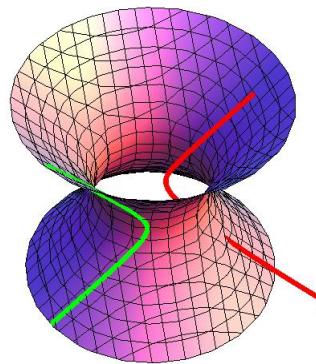


Figure 2. r (red) and \bar{r} (green) on \mathbb{S}^2_1 .

Example 2. Let $r : I \rightarrow \mathbb{E}^3_1$ be a 2-type null helix with $\kappa(s) = -\frac{1}{2}s^{-2}$. By Theorem 5, $r(s)$ can be expressed by

$$r(s) = \frac{1}{4}s^2(\log^2 s - \log s - \frac{1}{2}, 2\log s - 1, -\log^2 s + \log s - \frac{3}{2}).$$

From Theorem 7, the Darboux associated curve $\bar{r}(s)$ can be obtained as

$$\bar{r}(s) = -(\frac{1}{2}\log^2 s + \log s + \frac{1}{2}, \log s + 1, -\frac{1}{2}\log^2 s - \log s - \frac{3}{2}),$$

which satisfies $\langle \bar{r}, \bar{r} \rangle = -1$, i.e., $\bar{r}(s) \in \mathbb{H}^2$. (see Figure 3 and 4)

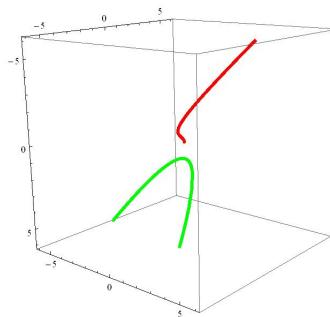


Figure 3. The positions of r (red) and \bar{r} (green).

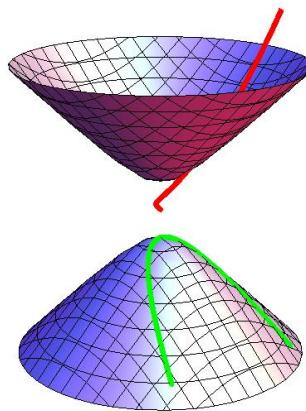


Figure 4. r (red) and \bar{r} (green) on \mathbb{H}^2 .

Example 3. Let $r : I \rightarrow \mathbb{E}_1^3$ be a 2-type null helix with $\kappa(s) = -s^{-2}$. By Theorem 5, $r(s)$ can be expressed by

$$r(s) = \frac{s^2}{40} (15 - 10 \sin(\log s) - 20 \cos(\log s), 16 \sin(\log s) - 8 \cos(\log s), 25 - 6 \sin(\log s) - 12 \cos(\log s)).$$

From Theorem 7, the Darboux associated curve $\bar{r}(s)$ can be written as

$$\bar{r}(s) = \frac{\sqrt{2}}{8} (-6 - 5 \sin(\log s) + 5 \cos(\log s), -4 \sin(\log s) - 4 \cos(\log s), -10 - 3 \sin(\log s) + 3 \cos(\log s)),$$

which satisfies $\langle \bar{r}, \bar{r} \rangle = -1$, i.e., $\bar{r}(s) \in \mathbb{H}^2$. (see Figure 5 and 6)

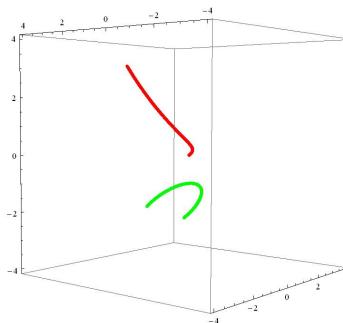


Figure 5. The positions of r (red) and \bar{r} (green).

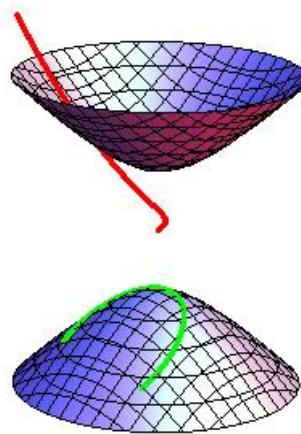


Figure 6. r (red) and \bar{r} (green) on \mathbb{H}^2 .

Last but not least, we study the null curves which share the same Darboux associated curves on pseudo-Riemannian space forms.

Theorem 8. Let $r_i : I \rightarrow \mathbb{E}_1^3$ be null curves with non-constant null curvatures κ_i ($i = 1, 2$). If the Darboux associated curves \bar{r}_i of r_i satisfy $\bar{r}_1 = \bar{r}_2$, then the null curvatures κ_i of r_i satisfy $\kappa_1 = \kappa_2$ or $\kappa_1 = \frac{1}{\kappa_2}$.

Proof of Theorem 8. (Case 1) Assume that $\kappa_i > 0$ ($i = 1, 2$), i.e., $r_i(s) \in \mathbb{S}_1^2$ for all s . Since $\bar{r}_1 = \bar{r}_2$, from Theorem 1, we have

$$\frac{\kappa_1 T_1 + B_1}{\sqrt{2\kappa_1}} = \frac{\kappa_2 T_2 + B_2}{\sqrt{2\kappa_2}}. \quad (14)$$

Differentiating Equation (14), we have

$$\frac{\kappa'_1(\kappa_1 T_1 - B_1)}{\kappa_1 \sqrt{2\kappa_1}} = \frac{\kappa'_2(\kappa_2 T_2 - B_2)}{\kappa_2 \sqrt{2\kappa_2}}. \quad (15)$$

Taking scalar product on both sides of Equation (15), we obtain

$$(\frac{\kappa'_1}{\kappa_1})^2 = (\frac{\kappa'_2}{\kappa_2})^2. \quad (16)$$

From Equation (16), we have

$$[(\log \kappa_1)']^2 = [(\log \kappa_2)']^2,$$

which implies

$$(\log \kappa_1)' = \pm (\log \kappa_2)'.$$

Then, we have $\kappa_1 = \kappa_2$ or $\kappa_1 = \frac{1}{\kappa_2}$.

(Case 2) Assume that $\kappa_i < 0$ ($i = 1, 2$), i.e., $r_i(s) \in \mathbb{H}^2$ for all s . By the similar process as in (Case 1), the same result can be achieved. \square

In 2015, one of the authors and Kim defined the self-associated curves of a null curve as follows:

Definition 4. [7] Let $r = r(s) : I \rightarrow \mathbb{E}_1^3$ be a null curve with Frenet frame $\{T, N, B\}$. The null curve defined by $\tilde{r}(\tilde{s}) = \int B(\tilde{s})d\tilde{s}$ is called the self-associated curve of r , where \tilde{s} is the null arc length parameter of \tilde{r} .

Theorem 9. [7] Let $r = r(s) : I \rightarrow \mathbb{E}_1^3$ be a null curve framed by $\{T, N, B\}$ and \tilde{r} its self-associated curve with Frenet frame $\{\tilde{T}, \tilde{N}, \tilde{B}\}$. Then

1. the Frenet frame of \tilde{r} is related by $\tilde{T} = B$, $\tilde{N} = \pm N$, $\tilde{B} = T$;
2. the null curvature $\tilde{\kappa}$ of \tilde{r} satisfy $\tilde{\kappa} = \frac{1}{\kappa}$.

Corollary 2. Let $r = r(s) : I \rightarrow \mathbb{E}_1^3$ be a null curve and \tilde{r} its self-associated curve. Then they share the same Darboux associated curve on pseudo-Riemannian space forms.

Proof of Corollary 2. From Definition 2 and Theorem 9, the Darboux associated curve \tilde{r} of r can be written by

$$\tilde{r} = \frac{\tilde{\kappa} \tilde{T} + \tilde{B}}{\sqrt{|2\tilde{\kappa}|}} = \frac{\kappa T + B}{\sqrt{|2\kappa|}} = \bar{r}, \quad (17)$$

where \bar{r} is the Darboux associated curve of r . Then the result is achieved easily. \square

Remark 5. Even the result in Corollary 2 shows that two null curves whose null curvature are reciprocal have the same Darboux associated curves, but it does not mean the opposite of conclusion in Theorem 8 holds.

Example 4. Let $r = r(s) : I \rightarrow \mathbb{E}_1^3$ be a 2-type null helix with $\kappa(s) = 4s^{-2}$. $r(s)$ and $\tilde{r}(s)$ have been expressed in Example 1. From Definition 4, through direct calculations, the self-associated curve $\tilde{r}(s)$ of $r(s)$ can be expressed by

$$\tilde{r}(s) = \left(-\frac{16s}{3} - \frac{4}{15s^5}, -\frac{8}{3s^2}, -\frac{16s}{3} + \frac{4}{15s^5} \right)$$

which shares the same Darboux associated curve $\bar{r}(s)$ with $r(s)$. (see Figure 7 and 8)

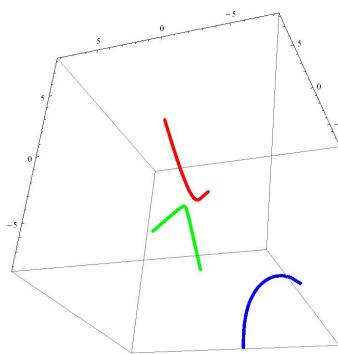


Figure 7. The positions of r (red), \tilde{r} (blue) and \bar{r} (green).

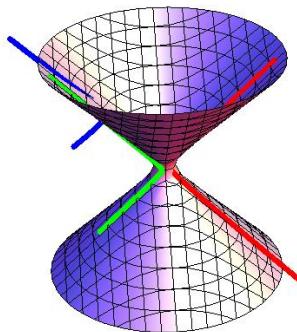


Figure 8. r (red), \tilde{r} (blue) and \bar{r} (green) on \mathbb{S}^2_1 .

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