Supersymmetry of de-Sitter Solutions in String Theory

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Scientific abstract

In the first part of the Thesis, we investigate some background material which is utilized in the second part, where the original research is presented. We present the proof of how the supercovariant derivative acting on a spinor is Spin-gauge covariant. We introduce the spinorial geometry techniques used to analyze the Killing spinor equations (KSEs), and give an example in the case of the gaugino KSE of certain warped product solutions of $D = 10$ heterotic supergravity. We describe the isometries of de Sitter space and we give the ansatz for warped product dS_4 solutions. Then, we briefly prove some classical no-go theorems for warped product de Sitter solutions.

In the second part of the Thesis, the necessary and sufficient conditions for warped product dS_4 solutions in $D = 11$ supergravity to preserve the minimal $N = 8$ supersymmetry are determined. We find, on integrating the KSE along the dS_4 directions, that the necessary and sufficient conditions for supersymmetry are encoded in a single gravitino-type equation, which is satisfied by a spinor ψ_+ whose components depend only on the co-ordinates of the internal space. The spinor ψ_+ is associated with two possible stabilizer groups, $SU(3)$ or G_2. We derive explicitly the Spin(7) gauge transformations which are used to write ψ_+ in simple canonical forms with stabilizer subgroups SU(3) or G_2. We then solve the linear system obtained from the KSEs. In particular, we show that the linear system implies there are no solutions for which the stabilizer of ψ_+ is G_2. For the case of SU(3) stabilizer subgroup, the KSEs determine all components of the 4-form flux in terms of the geometry of the internal manifold, and we present the geometric conditions and the components of the flux, written in a SU(3) covariant fashion.

Contents

CHAPTER 1

INTRODUCTION

1.1. String Theory and Supersymmetry

Exact solutions of Einstein gravity coupled to matter, (M, g, T), consist of an n-dimensional spacetime M equipped with a metric g and an energy-momentum tensor T of some specified form of matter, which may be required to satisfy some appropriate energy condition, as well as requirements relating to the absence of closed timelike curves. This set of data is required to satisfy the Einstein field equations

$$R_{AB} - \frac{1}{2} R g_{AB} + \Lambda g_{AB} = 8\pi T_{AB} \ , \tag{1.1}$$

where R_{AB}, R and Λ are the Ricci tensor, Ricci scalar and the cosmological constant, respectively. Due to the nonlinearity of the PDEs encoded in (1.1) it is difficult to find exact solutions, apart from cases in which the geometry is assumed to have a high degree of symmetry. There are many different possible choices for energy momentum tensor; for empty space one takes $T_{AB} = 0$, other possibilities correspond to the energy momentum tensors of perfect fluids, or of electromagnetic fields. One aim of this thesis is to explore, using supersymmetry, techniques for constructing solutions to the Einstein field equations of $D = 11$ supergravity, in the particular context of geometries associated to the region near to the horizon of supersymmetric black holes. We will discuss in further detail the formalism of $D = 11$ supergravity, supersymmetry, and near horizon geometries, in later chapters.

One particularly desired goal in physics is that of the construction of grand unified theories, which enable the relationship between (apparently) distinct fundamental laws of nature to be more fully understood, by unifying them into a single theory valid at higher energy scales. This also provides insight into the the so-called hierarchy problem. A hierarchy problem occurs when fundamental energy scales in nature are vastly different, such as the electroweak scale $m_{\mathrm{EW}} \sim 10^3$ GeV and the Planck scale $M_{\mathrm{Pl}} \sim 10^{18}$ GeV. The understanding of the hierarchy problem has been one of the greatest driving forces behind the construction of theories beyond the Standard Model. The vast difference between the weak and the Planck scales could itself be explained by some spontaneous symmetry breaking. On the other hand, the physics responsible for making a sensible quantum theory of gravity is revealed only at the Planck scale. One might therefore expect that there could be a hypothetical unified theory, which would fully describe the four interactions existing in nature: the gravitational, weak nuclear, strong nuclear, and electromagnetic forces. However, the mechanism by which such a unified theory gives rise to such a huge difference in energy scales is still undetermined.

By the early 1970s, it was shown that the Standard Model is a promising theory to describe the weak nuclear, the strong nuclear, and electromagnetic forces using a quantum field theory framework, although omitting gravity. The gravitational interaction is explained by Einstein's theory of General Relativity. In spite of considerable efforts, the unification of gravity with the standard model has yet to be fully realised. There are a number of significant obstacles to such a unification. Firstly, the union of gravity with quantum theory produces a non-renormalizable quantum field theory due to the General Relativity power-counting failure. In addition, at the classical level of General Relativity, the theory breaks down when certain types of singularities occur, such as in black holes. One might hope that a quantum theory of gravity, appropriately unified with the Standard Model, would produce some mechanism for dealing with such singularities - currently this is unclear.

A number of potential candidates for such a quantum theory of gravity have been postulated. In this thesis, we shall be concerned with aspects of String Theory. This has been particularly successful in the sense that String Theory produces the Einstein equations via the vanishing of a certain beta function which is associated with a quantum field theory defined on the string worldsheet. Moreover, String Theory has provided significant insights and new techniques for understanding the microscopic origin of black hole entropy in terms of branes. Notably, Strominger and Vafa [1] examined the extreme Reisser-Nordström $D = 5$ black hole solution from the perspective of the microstates associated with the D-branes. Such extended objects arise in String Theory, and can be used to construct black hole geometries. The black hole entropy was counted in terms of D-brane states, and it was found that this result agreed with the classic entropy obtained from the black hole horizon area in the limit of large charges and spins. Physically, this limit corresponds to taking a sufficiently large collection of such D-branes, which then form the black hole.

Other significant developments are the construction of gauge-gravity dualities, originating in the AdS/CFT correspondence which established a duality between String Theory on $AdS_5 \times S^5$ and a conformal field theory ($\mathcal{N} = 4$ super-Yang-Mills) defined on the boundary [2]. In addition, some more spacetime attempts have been made to find gravitational duals of quantum field theories of relevance to condensed matter physics; although these are rather more speculative in nature. Other mathematical constructions, such as mirror symmetry, have been motivated by String Theory. As such String Theory is understood to remain the most promising candidate for a unified theory capable of combining the standard model with gravity.

There are, however, a number of problems with the consistent formulation of String Theory, when attempting to describe our 4-dimensional universe. The primary issue is that such a formulation requires that spacetime must be more than 4-dimensional. This necessitates embedding our 4-dimensional universe into a 10-dimensional spacetime, with the additional dimensions being compact. Typically, it is assumed that the extra dimensions are appropriately "small", and one obtains an effective 4-dimensional theory via dimensional reduction (the Kaluza-Klein mechanism [3]) [1]. A secondary, though not insignificant problem which is associated with this is the choice of the compact internal space. In the absence of fluxes, this is required to be a 6-dimensional Calabi-Yau manifold, which is both Ricci-flat and Kähler. These conditions are quite restrictive, and it was originally hoped that the number of such manifolds might be relatively small, enabling some mechanism to be constructed which would produce a "natural" choice of Calabi-Yau manifold. However, it is now known that the family of Calabi-Yau manifolds is very large indeed, which significantly complicates our attempts to understand how a particular Calabi-Yau manifold,

[1] Other approaches do however exist in which the extra dimensions are "large"; in this case it is necessary to construct a mechanism, typically involving appropriately chosen configurations of branes, whereby gravity is effectively restricted to act in four dimensions in a manner which is consistent with observations.

whose geometry gives rise to the Standard Model on dimensional reduction, can be found. Moreover, the question of how such a Calabi-Yau manifold is effectively "selected" out of the multitude of other possibilities is also unresolved.

There are five a-priori different types of supersymmetric String Theories: Type IIA, Type IIB, type Type I, and Heterotic (with gauge group $E_8 \times E_8$, or $SO(32)$) [4]. Initially, this was also understood to be problematic, as it was unclear which of these theories should be considered to be the candidate unified theory of quantum gravity. However, it is now appreciated that there is an extensive web of dualities relating these theories. It is further proposed that all such perturbative D=10 string theories arise as limits from a strongly coupled non-perturbative theory, "M-theory", whose low energy limit is $D = 11$ supergravity. At the level of supergravity, there are mechanisms, via dimensional reduction and T-duality, for obtaining $D = 10$ supergravity theories from $D = 11$ supergravity.

Supersymmetry is a key ingredient in the construction of String Theory; it resolves a number of divergences in a consistent fashion and also imposes conditions on the dimensionality of spacetime. The transformations associated with supersymmetry relate fields of different spins and statistics. Supersymmetry is a spacetime symmetry mapping bosons into fermions, and vice versa. Bosons follow Bose-Einstein statistics and have integer-valued spin, whereas fermions follow Fermi-Dirac statistics and have half-integer-valued spin. In a theory in which supersymmetry is realized, each particle has an associated superpartner particle. Generically, supersymmetry is required in String Theory in order to ensure no physical tachyons appear at vacuum level. Moreover, for solutions in theories which preserve some residual supersymmetry, there is some control in the size of quantum corrections, with implications for the hierarchy problem.

In order to understand supersymmetry, it is necessary to consider spinors whose components depend on the spacetime co-ordinates of the manifold. Locally, in a specific chart, such spinors can be defined with respect to the associated local co-ordinate system. However, there are nontrivial topological requirements for a manifold to consistently globally admit spinors. To see how this arises, we note that spinors lie within a vector space Δ_c, which corresponds to a Spin representation. Explicitly, Δ_c corresponds to a vector space of complexified poly-forms, which is introduced explicitly in the context of spinorial geometry techniques in Chapters 2 and 5.

There is a close link between the Lorentz group $SO(n-1,1)$ and $Spin(n-1,1)$. In particular, there exists a 2:1 homomorphism $\varphi : Spin(n-1,1) \to SO(n-1,1)$ given by $\varphi(\tilde{\Lambda}) = \Lambda$ where

$$\tilde{\Lambda}^{-1}\Gamma^M\tilde{\Lambda} = \Lambda^M{}_N\Gamma^N \ , \tag{1.2}$$

where M, N are local frame indices, and Γ_M are gamma matrices which satisfy

$$\Gamma_M\Gamma_N + \Gamma_N\Gamma_M = 2\eta_{MN}\ \mathrm{id}\ . \tag{1.3}$$

The detailed relationship between Λ and $\tilde{\Lambda}$, in terms of generators, is calculated in Chapter 2.1. It is however, straightforward to see that the map φ is 2:1, as $\varphi(\tilde{\Lambda}) = \varphi(-\tilde{\Lambda})$.

Given a manifold M, we can cover M with co-ordinate patches $O_{(\alpha)}$ such that locally, on each $O_{(\alpha)}$, there exists an orthonormal frame $\{\mathbf{e}^M_{(\alpha)} : M = 1, ..., n\}$ with respect to which the metric is

$$ds^2 = \eta_{MN}\mathbf{e}^M_{(\alpha)}\mathbf{e}^N_{(\alpha)}\ . \tag{1.4}$$

Moreover, on each $O_{(\alpha)}$, we can define a spinor $\psi_{(\alpha)} \in \Delta_c$. We wish to consistently "patch together" such locally defined spinors on the overlap regions $O_{(\alpha\beta)} = O_{(\alpha)} \cap O_{(\beta)}$, analogously to how the components of vector fields are related by appropriate Jacobian transformations on co-ordinate patch overlaps. We begin, however, by considering the relationship between $\{\mathbf{e}^M_{(\alpha)}\}$ and $\{\mathbf{e}^M_{(\beta)}\}$ on $O_{(\alpha\beta)}$. This naturally leads to a relationship between $\psi_{(\alpha)}$ and $\psi_{(\beta)}$, on using the 2:1 correspondence φ between $Spin(n-1,1)$ and $SO(n-1,1)$. Firstly, in order to preserve the orthonormality of the frame, there must exist $\Lambda_{(\alpha\beta)} \in SO(n-1,1)$ such that

$$\mathbf{e}^M_{(\alpha)} = \Lambda_{(\alpha\beta)}{}^M{}_N\mathbf{e}^N_{(\beta)}\ . \tag{1.5}$$

In turn, this implies that an identity relation of the $\Lambda_{(\alpha\beta)}$ on the triple overlap regions must be satisfied. Indeed, in the triple intersection regions $O_{(\alpha\beta\gamma)} = O_{(\alpha)} \cap O_{(\beta)} \cap O_{(\gamma)}$, one obtains

$$\Lambda_{(\alpha\beta)}\Lambda_{(\beta\gamma)}\Lambda_{(\gamma\alpha)} = \mathrm{id}\,. \tag{1.6}$$

There is also a corresponding identity for the associated $Spin(n-1,1)$ transformations $\tilde{\Lambda}_{(\alpha\beta)}$. To see this, note that on $O_{(\alpha\beta)}$ the spinors $\psi_{(\alpha)}$ and $\psi_{(\beta)}$ are related by

$$\psi_{(\alpha)} = \tilde{\Lambda}_{(\alpha\beta)}\psi_{(\beta)}\ , \tag{1.7}$$

where $\varphi(\tilde{\Lambda}_{(\alpha\beta)}) = \Lambda_{(\alpha\beta)}$. Once more, this condition implies that in the triple intersection regions $O_{(\alpha\beta\gamma)}$,

$$\tilde{\Lambda}_{(\alpha\beta)}\tilde{\Lambda}_{(\beta\gamma)}\tilde{\Lambda}_{(\gamma\alpha)} = \text{id}\,. \tag{1.8}$$

However, as the correspondence φ is 2:1, one may freely replace $\tilde{\Lambda}_{(\alpha\beta)}$ in the above expression with $-\tilde{\Lambda}_{(\alpha\beta)}$. There is therefore, a priori, a sign ambiguity in (1.8). If it is the case that one can choose consistently the $\tilde{\Lambda}$ such that (1.8) holds for all triple intersections, then M is said to admit a spin structure, and M is a spin manifold. Most manifolds are not spin manifolds, so the requirement of spin structure is an important topological restriction when we consider supersymmetric supergravity solutions from a global perspective. However, we remark that in terms of the class of de-Sitter supergravity solutions which we analyse in Chapter 3 onwards, such solutions are typically not globally well defined, as a consequence of certain no-go theorems described in Chapter 3.4. Hence, for these types of solutions, we do not necessarily have a globally well-defined spin structure, and the analysis of supersymmetry is considered locally.

String Theory is not only a theory of strings, but contains extended objects, D-branes, which as we have mentioned, as having provided key insight into black hole entropy. Geometrically, D-branes correspond to hypersurfaces on which strings may end. In terms of M-theory, it is known that $D = 11$ supergravity contains solitonic membranes, $M2$-branes [5], and $M5$-branes [6], which play an important role in the dynamics of the theory and provide a large family of supersymmetric solutions. Both of these solitons preserve 1/2 of the supersymmetry and are known as 1/2-BPS solutions. It is important to understand the spectrum of BPS solutions in M-theory associated with intersecting M-branes, because these can be used to obtain black holes in $D = 4$ and $D = 5$ via appropriate dimensional reduction. More general black objects, such as $D = 5$ black rings, can also be obtained in a similar fashion. Harmonic superpositions of M-branes describe classes of supersymmetric configurations of 2 or 3 orthogonally intersecting $M2$-branes and $M5$-branes of $D = 11$ supergravity [7, 8].

The main feature of supersymmetric p-brane solutions of supergravity theories is that they are expressed in terms of harmonic functions depending only on the transverse spatial coordinates. There exist some universal rules to obtain stable supersymmetric solutions via brane intersections:

(i) a configuration of k orthogonally intersecting branes preserve at least $1/2^k$ of the maximal amount of supersymmetry,

(ii) p-branes of the same type can intersect only over a $(p-2)$-brane,

(iii) a $M2$-brane can intersect a $M5$-brane over a string.

(iv) A fundamental string may end on a D-brane.

These rules, when applied to intersecting M-branes in $D = 11$, are consistent with the corresponding rules in $D = 10$, on making appropriate dimensional reductions. The metric of the corresponding intersecting brane configurations is diagonal, and the components depend on various powers of the harmonic functions associated to each type of brane. The harmonic functions depend only on those directions which are transverse to all of the branes in the configuration [1].

The possible supersymmetric M-brane configurations that preserve 1/4 of supersymmetry are $M2 \perp M2$, $M5 \perp M5$, and $M5 \perp M2$. To illustrate the $M2 \perp M2$ intersection schematically in terms of worldvolume "X" and transverse "$-$" directions, we can consider the following:

Direction	0	1	2	3	4	5	6	7	8	9	♯
M2	X	X	X	-	-	-	-	-	-	-	-
M2	X	-	-	X	X	-	-	-	-	-	-

The first $M2$-brane has worldvolume directions 0, 1, 2 and transverse directions 3, 4, 5, 6, 7, 8, 9, ♯; the second $M2$-brane has worldvolume directions 0, 3, 4 and transverse directions 1, 2, 6, 7, 8, 9, ♯. The configuration preserves 8 supersymmetries. Corresponding diagrams for the $M5 \perp M5$ and the $M2 \perp M5$ intersections are as follows:

Direction	0	1	2	3	4	5	6	7	8	9	♯
M5	X	X	X	X	X	X	-	-	-	-	-
M5	X	X	X	X	-	-	X	X	-	-	-

[1]For partially smeared brane configurations, or for branes intersecting at angles, these rules for the formulation of the metric are modified in such a way that the harmonic functions depend on more of the co-ordinates, in a more complicated fashion; see e.g. [9].

and

Direction	0	1	2	3	4	5	6	7	8	9	♯
M5	X	X	X	X	X	X	-	-	-	-	-
M2	X	X	-	-	-	-	X	-	-	-	-

The possible supersymmetric configurations that preserve 1/8 of supersymmetry are $M2 \perp M2 \perp M2$, $M5 \perp M2 \perp M2$, $M5 \perp M5 \perp M2$, and $M5 \perp M5 \perp M5$. Moreover, there are 1/16 supersymmetric configurations with four intersecting M-branes, i.e. $M2 \perp M2 \perp M2 \perp M2$, $M2 \perp M2 \perp M2 \perp M5$, $M5 \perp M5 \perp M2 \perp M2$, and $M5 \perp M5 \perp M5 \perp M2$. These four-intersecting configurations are not asymptotically flat because the dimension of the overall transverse space is $d < 3$, with the exception of $M5 \perp M5 \perp M2 \perp M2$ for which $d = 3$, as can be seen from the following diagram:

Direction	0	1	2	3	4	5	6	7	8	9	♯
M5	X	X	X	X	X	X	-	-	-	-	-
M5	X	X	X	X	-	-	X	X	-	-	-
M2	X	-	-	-	X	-	X	-	-	-	-
M2	X	-	-	-	-	X	-	X	-	-	-

This solution can be dimensionally reduced along the 1, 2, 3, 4, 5, 6, 7 directions to produce a dyonic $D = 4$ black hole solution (associated with the directions 0, 8, 9, ♯) as described in [10]. Another physically interesting intersecting brane configuration in terms of black holes is the 3-charge $D = 5$ black hole solution [11] can be obtained from the dimensional reduction of the $M2 \perp M2 \perp M2$ geometry:

Direction	0	1	2	3	4	5	6	7	8	9	♯
M2	X	X	X	-	-	-	-	-	-	-	-
M2	X	-	-	X	X	-	-	-	-	-	-
M2	X	-	-	-	-	X	X	-	-	-	-

The $D = 11$ solution is reduced along the 1, 2, 3, 4, 5, 6 directions. The resulting electrically charged black hole geometry corresponds to the resulting metric along the 0, 7, 8, 9, ♯ directions, and is a solution of $N = 2$, $D = 5$ supergravity coupled to vector multiplets, which preserves 4 supersymmetries. This can be further generalized by intersections with additional $M5$-branes [12, 13] as illustrated in the table:

Direction	0	1	2	3	4	5	6	7	8	9	♯
M2	X	X	X	-	-	-	-	-	-	-	-
M2	X	-	-	X	X	-	-	-	-	-	-
M2	X	-	-	-	-	X	X	-	-	-	-
M5	X	-	-	X	X	X	X	X	-	-	-
M5	X	X	X	-	-	X	X	X	-	-	-
M5	X	X	X	X	X	-	-	X	-	-	-

Although the M2-M2 brane, and M5-M5 brane intersections are consistent with the rules described above, the M2-M5 brane intersections are not. This configuration therefore constitutes a special modification of the types of M-brane intersections described previously. It can also be verified that the addition of the M5-branes with such orientations does not break any more of the supersymmetry - the solution preserves the same amount of supersymmetry as the $M2 \perp M2 \perp M2$ geometry. The M2-branes carry conserved electric charges, whereas the M5-brane magnetic charges are not conserved - instead they generate a non-zero magnetic dipole moment. Moreover, the metric associated with this solution is not diagonal, and the components are not given in terms of harmonic functions as for the other cases. When reduced along the 1, 2, 3, 4, 5, 6 directions the resulting $D = 5$ solution corresponds to a 3-charge supersymmetric black ring, whose event horizon has topology $S^1 \times S^2$. This is in contrast to the S^3 event horizon topology obtained from the dimensional reduction of the $M2 \perp M2 \perp M2$ geometry. The existence of such black rings implies that there is no black hole uniqueness in 5 dimensions, as the asymptotic charges do not uniquely specify the solution.

More generally, there has been consistent progress made in constructing systematic classifications of supersymmetric solutions in supergravity theories. Such classifications have played a crucial role in the construction of novel black hole solutions [14, 15] and supersymmetric black ring solutions [16]. The classifications have also been used to find more exotic composite black objects such as "black Saturn" solutions [17, 18] which consist of a black hole with S^3 horizon topology, surrounded by an arbitrary number of black rings, each of which has $S^1 \times S^2$ horizon topology. The first systematic investigation of the classification of supersymmetric solutions was undertaken by Tod in [19] in which he analyzed all possible forms of the metric which admit a supercovariantly constant spinor in the minimal ungauged $N = 2$ $D = 4$ supergravity. The necessary and sufficient conditions for a geometry to be supersymmetric were determined. This classification was then generalized to other $D = 4$ supergravity theories including dilaton and axion scalar fields in [20]. However, the analysis was performed using two component spinor notation and as such was specific to $D = 4$ theories. Following on from this, the first systematic classification of supersymmetric solutions in $D = 5$ minimal ungauged supergravity was constructed in [21]. The method initially used to construct such $D > 4$ classifications used via spinor bilinears [21–23]. This was utilized to fully classify all supersymmetric solutions in minimal $N = 2, D = 5$ supergravity, and was later applied to obtain the necessary and sufficient conditions for supersymmetric solutions in $D = 11$ supergravity to preserve the minimal ($N = 1$) supersymmetry. This method is based on the insight that a pair of spinors can be associated to various k-form spinor bilinears. Fierz identities impose algebraic conditions on the form bilinears, and the Killing spinor equations (KSE) also impose conditions on the covariant derivatives of the spinor form bilinears. Such algebraic and differential conditions are then used to find conditions on the geometry and the fluxes of the theory.

The main limitation of using the spinor bilinears method for classifying supersymmetric solutions is that it is difficult to classify supersymmetric solutions which preserve more than the minimal amount of supersymmetry in any given theory, especially in higher dimensional $D = 10$ and $D = 11$ supergravities, for which solutions may preserve as many as 32 supersymmetries. A key reason for this is that applying the Fierz identities to obtain meaningful algebraic conditions on the multitude of possible spinor form bilinears is computationally prohibitive. In order to address this limitation, the *spinorial geometry* approach to classifying supersymmetric solutions was developed. This was first proposed by Gillard, Gran and Papadopoulos [24]. The spinorial geometry method consists of expressing spinors in a particular representation in terms of multi-differential forms. These can then be appropriately explicitly simplified into certain canonical forms, utilizing $Spin$-gauge transformations. On computing spinor bilinears explicitly, using such canonical forms, the algebraic conditions on the bilinears can be obtained directly without the need for extensive use of Fierz identities. Moreover, the different components of the Killing spinor equations can be explicitly determined. This produces a linear system of equations which can be solved to provide conditions on the spacetime geometry, as well as determining certain components of the supergravity fluxes in terms of the geometry. Further details of the spinorial geometry method, including some simple examples of its application, will be presented in the following chapters. It will later be utilized to determine a classification of supersymmetric warped product dS_4 geometries in $D = 11$ supergravity.

1.2. de Sitter Space and String Theory

De Sitter geometry is of particular interest in terms of string cosmology and also in the context of the holographic principle. De Sitter spacetime plays a central role in the understanding of our present universe. From the work of [25–27] it has been observed that our universe is asymptotically dS_4, corresponding to a very small positive cosmological constant. However, the observed value of the cosmological constant differs by many orders of magnitude from the vacuum energy density value predicted by quantum field theory [28, 29]. Moreover, in the context of string cosmology there are also difficulties in obtaining de Sitter space via compactification from higher dimensions. In particular, there are no go-theorems proving that smooth warped de Sitter solutions with compact, without boundary, internal manifold cannot be found in ten- and eleven-dimensional supergravity [30, 31, 2]. Issues relating to quantum gravity in de Sitter space have been investigated in [32].

In terms of holography, the AdS/CFT correspondence relates string theory in Anti-de Sitter (AdS) space to conformal field theories (CFT) defined on an appropriate boundary [33]. This has been particularly useful in developing a deeper understanding of the microscopic nature of the entropy-area law [34, 35]. In spite of the considerable insights produced via the holographic principle, there are still many open issues in this area. Building from the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence proposed by Brown and Henneaux in [36], the relation between quantum gravity on de Sitter space and conformal field theory on a sphere, the so-called dS/CFT correspondence, was considered in [37–39]. However, our understanding of the conjectured dS/CFT correspondence is less complete than for the case of AdS/CFT for a number of reasons. Firstly, in contrast to AdS, there is a lack of de Sitter space solutions in string theory (or in any quantum gravity theory) in which the conjecture can be tested. Also, there are subtle issues with defining the dual CFT on the past and future spheres $\mathcal{I}^{\pm}$, relating to the causal structure of dS space. Nevertheless, the macroscopic entropy-area law applies to a very wide class of black holes, including asymptotically flat, asymptotically AdS, and also asymptotically dS cases. The universality of this law provides strong motivation for understanding de Sitter holography.

Motivated by this, it is of particular interest to systematically understand the different types of de Sitter solutions which are possible in $D = 10$ and $D = 11$ supergravity. Such a classification may provide interesting new applications of the dS/CFT correspondence. As it is possible to embed dS_n inside both $\mathbb{R}^{1,n}$ and AdS_{n+1} as a warped product geometry [40], it follows that the maximally supersymmetric $AdS_7 \times S^4$ solution, as well as $\mathbb{R}^{1,10}$, can both be regarded as examples of warped product dS_4 geometries. However, as we shall establish here, there is a much larger class of supersymmetric warped product dS_4 solutions in $D = 11$ supergravity than these two very special solutions, and this is also somewhat in contrast to the results of recent analysis of supersymmetric warped product dS_n geometries for $5 \leqslant n \leqslant 10$.

In terms of $D = 11$ supergravity, there has been recent progress in the classification of supersymmetric warped product dS_n geometries for $5 \leqslant n \leqslant 10$ [41]. There are a number of different possibilities:

- For $7 \leqslant n \leqslant 10$, the geometry is the maximally supersymmetic $\mathbb{R}^{1,10}$ solution with vanishing 4-form flux.
- For warped product dS_6 solutions, the solution is either the maximally supersymmetric $AdS_7 \times S^4$ solution, or $\mathbb{R}^{1,6} \times N$ where N is a hyper-Kähler 4-manifold.
- The warped product dS_5 solutions are all examples of generalized M5-brane solutions for which the transverse space is $\mathbb{R} \times N$, where N is a hyper-Kähler 4-manifold.

It is clear from this list that the possible warped product dS_n geometries for $5 \leqslant n \leqslant 10$ is very highly constrained. In addition, a similar recent analysis of warped product dS_n solutions in heterotic supergravity [42], including first order α' corrections, has also produced a rather restricted class of such solutions. In this case, for $n \geqslant 3$, the geometry is $\mathbb{R}^{1,n} \times M_{9-n}$, where M_{9-n} is a $(9 - n)$-dimensional manifold. The dilaton depends only on the co-ordinates of M_{9-n}, and all p-form fields have components only along the M_{9-n} directions. The heterotic warped product dS_2 solutions are the direct product $AdS_3 \times M_7$ solutions which have been classified in [43]. Compared to these types of solutions, the conditions on supersymmetric warped product dS_4 solutions in $D = 11$ supergravity are rather weaker.

Motivated by these results, in this thesis we obtain the necessary and sufficient conditions for warped product dS_4 solutions in $D = 11$ supergravity to preserve the minimal $N = 8$ supersymmetry. We find, on integrating the Killing spinor equations along the dS_4 directions, that all of the necessary and sufficient conditions for supersymmetry are encoded in a single gravitino-type equation, which is satisfied by a spinor ψ_+ whose components depend only on the co-ordinates of the internal space. We analyse the solutions of this equation using spinorial geometry techniques. This technique was introduced in [24] and consists of writing the Killing spinors in terms of multi-differential forms and, utilizing the gauge-covariance of the KSE, gauge transformations are then used to write the spinors in one of several simple canonical forms. The main outcome of this approach is a linear system which imposes conditions on the spin connection and the fluxes of the theory. This in turn can be used to obtain conditions on the geometry which are necessary and sufficient for supersymmetry. These techniques have been applied to classify a wide variety of supergravity solutions [44].

In the case of warped product dS_4 solutions, we state explicitly the Spin(7) gauge transformations which are used to write the spinor ψ_+ in canonical forms with stabilizer subgroups SU(3) and G_2. We then solve the linear system obtained from the Killing spinor equations. In particular, we show that the linear system implies that there are no Killing spinors for which the stabilizer of ψ_+ is G_2. For the case of SU(3) stabilizer subgroup, the Killing spinor equations determine all components of the 4-form flux in terms of the geometry of the internal manifold, and we present the geometric conditions and the components of the flux, written in a SU(3) covariant fashion. On considering these conditions, we note that the warped product dS_4 geometries are manifestly less restricted in terms of the geometric structure and the 4-form flux in comparison to the warped product dS_n solutions for $5 \leqslant n \leqslant 10$. Our analysis does not utilize the global techniques developed for the investigation of supersymmetric black holes [45]; we consider only local properties of the Killing spinor equations. This avoids the no-go theorems which exclude warped product dS_n solutions when the warp factor and 4-form flux are smooth, and the internal manifold is smooth and compact without boundary.

1.3. Plan of the Thesis

The plan for the remainder of the Thesis is as follows:

(1) In chapter 2 we present the proof of how the supercovariant derivatives acting on a spinor transforms in a covariant fashion when a Spin-gauge transformation acts on the spinor. Then, we briefly give an introduction to the spinorial geometry techniques used to analyze the Killing spinor equations.

(2) In chapter 3 we outline in some details properties of de Sitter geometries. In the first part, we investigate the isometries of de Sitter space, and prove that the only differential forms on de Sitter space for which the Lie derivatives with respect to all of the isometries vanish are constant functions, and constant multiples of the volume form. In the second part, we summarize the bosonic field equations, Bianchi identities, and Killing spinor equations of $D = 11$ supergravity. In the third part we describe the ansatz for the warped product dS_4 solutions. Finally in the fourth part, we briefly describe some classical no-go theorems which forbid the existence of warped product de Sitter solutions for which the internal manifold is smooth, and compact without boundary, and the warp factor is smooth.

(3) In chapter 4 we derive several integrability conditions from the Killing spinor equations, and we demonstrate how some of these integrability conditions can be derived from others. We also explicitly integrate up the Killing spinor equations along the dS_4 directions, and show how the Killing spinor equations reduce to a single gravitino-type equation for a spinor ψ_+ which depends only on the internal manifold co-ordinates. We also prove that the supersymmetric dS_4 warped product solutions preserve $N = 8n$ supersymmetries for $n = 1, 2, 3, 4$.

(4) In chapter 5 we utilize spinorial geometry techniques, and prove that the spinor ψ_+ can be written in one of several particularly simple canonical forms, on applying appropriate Spin(7) gauge transformations. Furthermore, depending on the type of canonical form, we prove that such a spinor has stabilizer subgroup which is either SU(3) or G_2; in the SU(3) case we also consider several possible special sub-cases.

(5) In chapter 6 we present the SU(3) covariant conditions on the flux and geometry, obtained from the gravitino-type equation in the case for which the spinor ψ_+ has

stabilizer subgroup SU(3). We also prove that there are no supersymmetric warped product dS_4 solutions for which the stabilizer subgroup of ψ_+ is G_2.

(6) In chapter 7 we present our conclusions and discuss some possible future work on the classification of warped product dS_4 solutions in $D = 11$ supergravity with enhanced supersymmetry.

The Thesis also contains a number of Appendices, which provide supporting material to the above chapters:

(a) Appendix A contains some general conventions.

(b) Appendix B summarizes some key details of spinorial geometry. The description of the Clifford algebra representation utilized in the spinorial geometry techniques is given. This representation is used in the analysis of the Killing spinor equations in chapters 5 and 6.

(c) Appendix C consists of a number of gamma matrix identities that are utilized in the analysis of the integrability conditions in chapter 4.

(d) Appendix D consists of a detailed description of the derivation of the values of the constants appearing in equation (4.23) in chapter 4.

(e) Appendix E states the linear system of equations in the spin connection and the gauge field strength components which are obtained from the gravitino type equation (4.37) in the case when the stabilizer group of the spinor is $SU(3)$.

(f) Appendix F contains some relations which are used to covariantize the solutions of the gravitino type equation (4.37) in terms of various $SU(3)$ covariant forms.

The original research in this Thesis, corresponding to [46], consists of chapter 3.3, and all of chapters 4, 5, 6, 7; together with Appendices C, D, E, and F.

CHAPTER 2

PROPERTIES OF KILLING SPINORS

In this chapter we shall summarize some properties of the Killing spinor equation which will later be utilized to solve the Killing spinor equations of $D = 11$ supergravity for warped product dS_4 solutions. In particular, we begin in the first part of this chapter by considering the Levi-Civita term in the KSE. Such a term is present in all supercovariant derivatives which appear in supergravity theories. We shall illustrate how the Levi-Civita connection acting on a spinor ϵ transforms in a covariant fashion when a Spin-gauge transformation acts on the spinor. In addition to the Levi-Civita connection term, there is also a theory dependent flux term in the supercovariant derivative; the nature of this term depends on the type of supergravity theory under consideration. However, in all cases it is straightforward to prove that this algebraic term transforms covariantly. We shall therefore concentrate on the properties of the Levi-Civita term in the supercovariant derivative.

Having described the transformation properties of the supercovariant derivative, we shall briefly introduce the spinorial geometry technique which will be used to analyse the KSE. This utilizes an explicit representation of the Clifford algebra, which acts on spinors which are certain types of multi-differential forms defined on an auxiliary space. We shall illustrate this method with an explicit example from $D = 6$ gauge theory, to demonstrate the key aspects of this approach for solving the KSE.

2.1. Gauge Covariance of Killing Spinor Equation

A key ingredient of the spinorial geometry method is the Spin$(D-1,1)$ gauge covariance of the Killing Spinor equations (KSEs). The KSEs are the vanishing conditions of the supersymmetry variations of the fields. These are evaluated in the background where all fermions vanish, which in turn implies that the supersymmetry variations of the bosons are trivially satisfied. The KSE associated to the gravitino is the vanishing of the supercovariant derivative, whereas the KSEs associated to the remaining fermions are all algebraical equations.

In order to analyse the gauge covariance of KSE, capital latin letters such as A, B denote D-dimensional frame indices and $\Omega_{A,BC}$ denotes the spin-connection. A supercovariant derivative is defined as

$$\mathcal{D}_A = \nabla_A + \sigma_A(e,F)$$
$$\nabla_A \equiv \partial_A + \frac{1}{4}\Omega_{A,CD}\Gamma^{CD} \quad \text{where} \tag{2.1}$$

∇_A denotes the Levi-Civita connection, and $\sigma_A(e,F)$ is a Clifford algebra element which depends on the spacetime coframe e and the fluxes F. The expression of $\sigma_A(e,F)$ in terms of the fields is theory dependent. In this section we present the proof of the gauge covariance of the Levi-Civita part of the supercovariant derivative.

Let M be a spacetime of dimension D and let ϵ be a spinor satisfying a gravitino KSE of some theory. Under Spin$(D-1,1)$, ϵ transforms as

$$\epsilon' = e^X\epsilon$$
$$X \equiv f_{AB}\Gamma^{AB} \quad f_{AB} = -f_{BA} \quad . \tag{2.2}$$

Here X is an arbitrary real linear combination of generators of $Spin(D-1,1)$. The parameter f_{AB} is real antisymmetric factor and depends on the spacetime co-ordinates, $f_{AB} = f_{AB}(x)$ and $\Gamma_{AB} \equiv \frac{1}{2}[\Gamma_A, \Gamma_B]$, where Γ_A is a gamma matrix. By using Eq. (2.2), we can rewrite the Levi-Civita covariant derivative as follows

$$\nabla_A\epsilon = (\partial_A + \frac{1}{4}\Omega_{A,BC}\Gamma^{BC})\epsilon$$
$$= e^{-X}(\partial_A + e^X\partial_A e^{-X} + \frac{1}{4}\Omega_{A,BC}\tilde{\Gamma}^{BC})\epsilon' \ , \tag{2.3}$$

where $\tilde{\Gamma}^{AB} \equiv e^X\Gamma^{AB}e^{-X}$. In particular, $\tilde{\Gamma}^{AB}$ and Γ^{AB} correspond to two equivalent representations of the Clifford algebra. We shall show that $\tilde{\Gamma}^A$ and Γ^A are connected by local

Lorentz SO$(D-1,1)$ transformation. To do this, we define an auxiliary function $\Theta(\tau)$ depending on an arbitrary parameter τ as follows

$$\Theta^C(\tau) \equiv e^{\tau X}\Gamma^C e^{-\tau X} \quad . \tag{2.4}$$

On differentiating this expression with respect to τ and by exploiting the commutation relation

$$\left[\Gamma^A, \Gamma^{BC}\right] = 4\eta^{A[B}\Gamma^{C]}\,, \tag{2.5}$$

it follows that

$$\frac{\mathrm{d}\Theta^C}{\mathrm{d}\tau} = -4f^C{}_A\Theta^A\,. \tag{2.6}$$

Moreover, $\Theta^C(0) = \Gamma^C$. It follows that

$$\Theta^C(\tau) = \left(e^{-4\tau f}\right)^C{}_A\Gamma^A \tag{2.7}$$

and therefore

$$\tilde{\Gamma}^C = \Theta^C(1) = \left(e^{-4f}\right)^C{}_A\Gamma^A \tag{2.8}$$

Now, we shall show that $\left(e^{-4f}\right)^M{}_A \in \mathrm{SO}(D-1,1)$.

To do so, we define another auxiliary function U depending on parameter λ as follows

$$U_{CB}(\lambda) \equiv \left(e^{-4\lambda f}\right)^A{}_C\,\eta_{AD}\left(e^{-4\lambda f}\right)^D{}_B\ . \tag{2.9}$$

By taking the derivative of $U_{CB}(\lambda)$ with respect to λ, we find

$$\begin{aligned}\frac{\mathrm{d}U_{CB}}{\mathrm{d}\lambda} &= -4f^A{}_N\left(e^{-4\lambda f}\right)^N{}_C\eta_{AD}\left(e^{-4\lambda f}\right)^D{}_B - 4\left(e^{-4\lambda f}\right)^A{}_C\eta_{AD}\left(e^{-4\lambda f}\right)^N{}_B f^D{}_N \\ &= -4\left(e^{-4\lambda f}\right)^N{}_C\big(f_{DN}+f_{ND}\big)\left(e^{-4\lambda f}\right)^D{}_B \\ &= 0\,,\end{aligned} \tag{2.10}$$

where we have made use of the fact that

$$\left(e^{-4f\lambda}\right)^A{}_C f^C{}_B = f^A{}_C\left(e^{-4f\lambda}\right)^C{}_B\ . \tag{2.11}$$

Hence U_{CB} is constant, therefore

$$U_{CB}(\lambda) = U_{CB}(0) = \eta_{CB} \quad . \tag{2.12}$$

By setting $\lambda = 1$ in (2.9) and comparing with (2.12), we find that

$$\left(e^{-4f\delta}\right)^{A}{}_{C}\,\eta_{AD}\left(e^{-4f\delta}\right)^{D}{}_{B} = \eta_{CB}\ , \tag{2.13}$$

and hence $\left(e^{-4f}\right)^{M}{}_{A} \in \mathrm{SO}(D-1,1)$.

We can rewrite Eq. (2.3) by using Eq.(2.8) as follows

$$\nabla_A\epsilon = e^{-X}\left(\partial_A + e^{X}\partial_A e^{-X} + \frac{1}{4}\Omega_{A,BC}\left(e^{-4f\tau}\right)^{B}{}_{D}\left(e^{-4f\tau}\right)^{C}{}_{E}\Gamma^{DE}\right)\epsilon' \quad . \tag{2.14}$$

The last step of the KSE covariance proof is to perform a local Lorentz $\mathrm{SO}(D-1,1)$ transformation of the type

$$\hat{\partial}_A = \left(e^{-4f}\right)^{B}{}_{A}\partial_B$$
$$\hat{e}^A = \left(e^{4f}\right)^{A}{}_{B}e^B\ . \tag{2.15}$$

With this frame choice, we find that Eq.(2.14) can be rewritten as

$$\nabla_A\epsilon = e^{-X}\left(e^{4f}\right)^{B}{}_{A}\left(\hat{\partial}_B + \frac{1}{4}\hat{\Omega}_{B,CD}\Gamma^{CD}\right)\epsilon' \quad , \tag{2.16}$$

that is gauge covariant with respect to the $\mathrm{Spin}(D-1,1)$, up to a Local Lorentz transformation. Having established the covariance of the Levi-Civita part of the supercovariant derivative with respect to $\mathrm{Spin}(D-1,1)$ gauge transformations, we can then utilize such transformations to make the process of solving the Killing spinor equations more straightforward. This will involve using spinorial geometry techniques.

2.2. Spinorial Geometry Techniques

Spinorial geometry techniques were first introduced in [24] in the context of $D = 11$ supergravity. These methods have also been used to classify supersymmetric techniques in numerous supergravity theories [44, 47, 48]. There are a number of key steps in applying the spinorial geometry method for the analysis of KSEs.

(i) The spinors of the theory in question are certain types of multi-differential forms, defined on an appropriate auxiliary space. The multi-form components depend on the spacetime co-ordinates.

(ii) A representation for the Clifford algebra is chosen for which the gamma matrices typically act as creation or annihilation operators acting on the space of spinors.

(iii) For the analysis of solutions preserving the minimal proportion of supersymmetry, a Spin-gauge transformation is utilized in order to choose a gauge in which the spinor takes one of several particularly simple canonical forms. The nature of these canonical forms depends on the theory in question.

(iv) The KSE are then evaluated explicitly, working in the gauge for which the spinor is in one of the simple canonical forms. This produces a linear system involving the spin connection, and various components of fluxes.

(v) This linear system is then solved explicitly to obtain conditions on some (though not necessarily all) of the flux components, as well as conditions on the spin connection (i.e. conditions on the geometry).

(vi) These conditions are then rewritten in a manifestly gauge-invariant fashion in terms of various gauge-invariant spacetime spinor form bilinears

We remark that by utilizing an optimal choice of gauge, as described in (iii), the components of the gauge-invariant spacetime spinor form bilinears can be computed explicitly in the gauge for which the spinor is in a simple canonical form. Consequently, the components of the spacetime spinor form bilinears are particularly simple as well, and it is also straightforward to directly see the different types of algebraic conditions which the spacetime spinor form bilinears satisfy. This obviates the need to make use of Fierz identities, which simplifies the analysis significantly. In addition, the spinorial geometry approach has also been used to analyse supersymmetric solutions which preserve more than the minimal amount of supersymmetry [49–52]; such calculations are prohibitively difficult to undertake using other methods.

Having described general aspects of the spinorial geometry method, it will be useful to consider a simple explicit application, in the context of a gaugino type equation on a 6-dimensional manifold. This is an algebraic condition, and such a condition can be obtained in the context of certain warped product solutions of $D = 10$ heterotic supergravity, in the case for which the internal space M_6 is 6-dimensional [44, 53, 54]. The reduced gaugino KSE on M_6 is

$$F_{AB}\Gamma^{AB}\epsilon = 0 \quad A, B = 1, \dots, 6 \quad , \tag{2.17}$$

where F_{AB} represents the non-abelian flux of the theory (e.g. heterotic supergravity) and $\Gamma_{AB} \equiv \frac{1}{2}[\Gamma_A, \Gamma_B]$.

In Appendix B, we explain how the Dirac spinor representation can be constructed in terms of differential forms, following [55,56]. In this representation, the Dirac spinors in 6 dimensions consist of the complex span of differential forms on $\mathbb{R}^3$, i.e, an arbitrary Dirac spinor can be written as a complex linear combination of

$$\{1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\} \tag{2.18}$$

where $\{e_1, e_2, e_3\}$ are a basis of 1-forms on $\mathbb{R}^3$, and $e_{12} = e_1 \wedge e_2$, $e_{123} = e_1 \wedge e_2 \wedge e_3$. In even dimensions, the Dirac representation is reducible into Weyl chiral Δ_4^+ and Weyl anti-chiral Δ_4^- representations, $\Delta_8 = \Delta_4^+ \oplus \Delta_4^-$. These are determined by the corresponding projections

$$\mathcal{P}_\pm = \frac{1}{2}(\mathbb{I} \mp i\Gamma_{123456}) \tag{2.19}$$

which also commute with $F_{AB}\Gamma^{AB}$. Hence, without loss of generality, we may assume that the spinor ϵ appearing in (2.17) is chiral. Such a Weyl chiral spinor can be written as an even-degree multi-form $\epsilon \in \Lambda_{\text{even}}(\mathbb{C}^3)$:

$$\epsilon = \alpha 1 + \beta_1 e_{12} + \beta_2 e_{13} + \beta_3 e_{23} \quad , \tag{2.20}$$

where α, β_i with $i = 1, 2, 3$ are complex functions which in general depend on the co-ordinates of M_6. Now, we write the Clifford algebra in oscillator basis as follows

$$\Gamma_\alpha = \sqrt{2}\, e_\alpha \wedge \qquad \Gamma_{\bar\alpha} = \sqrt{2}\, i_{e_\alpha} \quad , \tag{2.21}$$

where $\alpha = 1, 2, 3$ and $\bar\alpha = \bar 1, \bar 2, \bar 3$. We remark that the matrices (2.21) satisfy the Clifford algebra in complex basis for $\mathbb{R}^6$.

To proceed further, we shall now simplify further the spinor ϵ given in (2.20). In particular, we shall apply certain Spin(6) gauge transformations as follows:

(a) We can set $\beta_1 = 0$ by a SU(2) gauge trasformation generated by

$$\frac{1}{2}(\Gamma_{12} + \Gamma_{\bar 1\bar 2}), \quad \frac{i}{2}(\Gamma_{12} - \Gamma_{\bar 1\bar 2}), \quad \frac{i}{2}(\Gamma_{1\bar 1} + \Gamma_{2\bar 2}) \tag{2.22}$$

which acts transitively on $\text{span}_{\mathbb{C}}\{1, e_{12}\}$, leaving invariant $\text{span}_{\mathbb{C}}\{e_{23}, e_{13}\}$.

(b) We can set $\beta_2 = 0$ by a SU(2) gauge trasformation generated by

$$\frac{1}{2}(\Gamma_{13} + \Gamma_{\bar 1\bar 3}), \quad \frac{i}{2}(\Gamma_{13} - \Gamma_{\bar 1\bar 3}), \quad \frac{i}{2}(\Gamma_{1\bar 1} + \Gamma_{3\bar 3}) \tag{2.23}$$

which acts transitively on $\text{span}_{\mathbb{C}}\{1, e_{13}\}$, leaving invariant $\text{span}_{\mathbb{C}}\{e_{23}, e_{12}\}$.

(c) We can set $\beta_3 = 0$ by a SU(2) gauge transformation generated by

$$\frac{1}{2}(\Gamma_{23} + \Gamma_{\bar{2}\bar{3}}), \quad \frac{i}{2}(\Gamma_{23} - \Gamma_{\bar{2}\bar{3}}), \quad \frac{i}{2}(\Gamma_{2\bar{2}} + \Gamma_{3\bar{3}}) \tag{2.24}$$

which acts transitively on $\text{span}_{\mathbb{C}}\{1, e_{23}\}$, leaving invariant $\text{span}_{\mathbb{C}}\{e_{13}, e_{12}\}$.

After applying these gauge transformations, the spinor ϵ can be taken to be $\epsilon = f1$ where f is a real function. The gaugino KSE (2.17) in this canonical gauge becomes

$$(F_{\bar{\alpha}\bar{\beta}}\Gamma^{\bar{\alpha}\bar{\beta}} + 2F_{\alpha\bar{\beta}}\delta^{\alpha\bar{\beta}})1 = 0 \quad , \tag{2.25}$$

or equivalently,

$$(2F^{\alpha\beta}e_{\alpha\beta} + 2F_{\alpha\bar{\beta}}\delta^{\alpha\bar{\beta}})1 = 0 \quad . \tag{2.26}$$

Considering the linear independence the spinor basis elements, Eq. (2.25) implies

$$F_{\bar{\alpha}\bar{\beta}} = 0 \qquad F_{\alpha}{}^{\alpha} = 0 \quad . \tag{2.27}$$

Hence, in the language of (almost) complex geometry, F is (1,1) traceless real 2-form.

Finally, we wish to covariantize these results by defining a 2-form bilinear ω as follows

$$\omega \equiv \frac{i}{2}\langle 1, \Gamma_{MN}1\rangle \mathbf{e}^M \wedge \mathbf{e}^N = -i\delta_{\alpha\bar{\beta}}\,\mathbf{e}^{\alpha} \wedge \mathbf{e}^{\bar{\beta}} \quad . \tag{2.28}$$

where $\langle , \rangle$ is the Hermitian inner product defined in (B.3), $\mathbf{e}^M$, $M = 1, \ldots, 6$, denotes a real frame on M_6, and $\mathbf{e}^{\bar{\alpha}} = (\mathbf{e}^{\alpha})^*$ for $\alpha = 1, 2, 3$. We remark that ω is an almost hermitian form on M_6, which is associated with an almost complex structure I, given by $\omega_{AB} = \delta_{AC}I^C{}_B$. Then, the results of (2.17) can be rewritten as

$$F_{AB} = F_{CD}I^C{}_A I^D{}_B \ , \quad F^A{}_B I^B{}_A = 0 \quad . \tag{2.29}$$

This example illustrates the key principles relating to spinorial geometry, and applications to solving Killing spinor equations. An alternative approach would be to utilize Fierz identities; in lower dimensions such an approach is tractable. However, when solving the KSE associated with warped product de Sitter solutions in D=11 supergravity, extracting algebraic conditions on the spinor bilinears via Fierz identities is significantly more complicated when compared to using spinorial geometry techniques. Hence, to undertake the $D = 11$ analysis described in this thesis, we shall utilize spinorial geometry.

CHAPTER 3

PROPERTIES OF dS GEOMETRIES

In this chapter, we shall consider in some detail properties of de Sitter geometries. We shall begin by investigating the isometries of de Sitter space, which is the maximally symmetric Lorentzian manifold, whose Riemann curvature satisfies

$$R_{ABCD} = \frac{R}{n(n-1)}(g_{AC}g_{BD} - g_{AD}g_{BC}) \tag{3.1}$$

with $R > 0$. Any metric whose curvature satisfies such an identity must be locally isometric to de Sitter space. In the first part of this section, we shall present the proof that for dS_n, the only differential forms for which the Lie derivative with respect to all dS_n isometries vanishes are constant functions, and constant multiples of the volume form of dS_n. Such properties of isometries of dS_n will be utilized in the rest of this chapter, in the context of considering warped-product dS_4 solutions in $D = 11$ supergravity. To do this, in the second part of this section, we summarize some key properties of $D = 11$ supergravity, including the bosonic field equations, and the $D = 11$ supercovariant derivative. Then, in the third part of this section, we describe in further detail the properties of warped-product dS_4 solutions in $D = 11$ supergravity, focusing on the reduction of the Einstein and gauge field equations, and the Bianchi identities, to the 7-dimensional internal manifold. These equations are obtained by assuming that the Lie derivative of the 4-form F with respect to all of the isometries of dS_4 vanishes. Finally, in the fourth part of this chapter, we shall briefly describe some classical no-go theorems which forbid the existence of warped product de Sitter solutions for which the internal manifold is smooth, and compact without boundary, and the warp factor is smooth.

3.1. Isometries of de Sitter Space

In this section we shall show that, on dS_n, the only differential forms whose Lie derivatives with respect to all of the isometries of dS_n vanish are constant functions, and constant multiples of the volume form of dS_n. Hence, for the case of dS_4 solutions, it follows that the 4-form flux F must be the sum of a constant multiple of the volume form of dS_4 and a 4-form on the internal space M_7.

In order to demonstrate this, we first must determine the isometries of dS_n. It will be convenient to adopt the following choice of co-ordinates for dS_n

$$ds^2 = \frac{1}{\mathcal{R}^2}\eta_{\mu\nu}dx^\mu dx^\nu \quad , \quad \mathcal{R} = 1 + \frac{1}{4}Kx^\alpha\eta_{\alpha\beta}x^\beta \quad \alpha,\beta = 1,\dots,n \tag{3.2}$$

In the case of dS_n, K is taken to be a positive constant, which is proportional to the scalar curvature. Anti-de-Sitter space and flat space correspond to taking $K < 0$ and $K = 0$ respectively. We shall consider the case of de Sitter space, with $K > 0$, henceforth; however the analysis of isometries in this section also holds for $K < 0$ and $K = 0$ as well. Let V be a vector field

$$V = V^\mu\frac{\partial}{\partial x^\mu} \quad . \tag{3.3}$$

V is an isometry of the metric of dS_n if and only if V satisfies the Killing equation:

$$\mathcal{L}_V g_{\mu\nu} = 0 \quad \Rightarrow \quad \nabla_{(\mu}V_{\nu)} = 0 \tag{3.4}$$

where

$$\mathcal{L}_V g_{\mu\nu} = V^\rho\partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu V^\rho + g_{\mu\rho}\partial_\nu V^\rho \quad . \tag{3.5}$$

In terms of the co-ordinates we have chosen for dS_n, this condition is equivalent to

$$\eta_{\alpha\nu}\partial_\mu V^\alpha + \eta_{\alpha\mu}\partial_\nu V^\alpha - \frac{K}{\mathcal{R}}\eta_{\mu\nu}x^\alpha\eta_{\alpha\beta}V^\beta = 0 \tag{3.6}$$

It is straightforward to show that the following vector fields are Killing vectors:

$$U^\sigma = x^{\tilde{\alpha}}\eta^{\sigma\tilde{\beta}} - x^{\tilde{\beta}}\eta^{\sigma\tilde{\alpha}} \quad , \tag{3.7}$$

and

$$W^\sigma = \left(1 - \frac{K}{4}x^\alpha\eta_{\alpha\beta}x^\beta\right)\eta^{\sigma\tilde{\lambda}} + \frac{K}{2}x^\sigma x^{\tilde{\lambda}} \quad , \tag{3.8}$$

where $\tilde{\alpha}, \tilde{\beta}, \tilde{\lambda}$ are fixed indices. There are $\frac{1}{2}n(n-1)$ linearly independent U^σ Killing vectors, and there are n linearly independent W^σ Killing vectors. Consequently, this set of vector fields contains $\frac{1}{2}n(n+1)$ linearly independent Killing vectors, which is the maximum possible number of Killing vectors for a n-dimensional manifold. Hence, this set of Killing vectors is a basis for the set of Killing vectors of dS_n.

Now we shall consider a p-form ω such that $\mathcal{L}_U\omega = \mathcal{L}_W\omega = 0$ for all possible U and W Killing vectors. We first take the case of $p = 0$, and $\omega = f$ for a function f. In this case, $\mathcal{L}_U\omega = \mathcal{L}_W\omega = 0$ implies that

$$x^\alpha \partial^\beta f - x^\beta \partial^\alpha f = 0 \tag{3.9}$$

and

$$\left(1 - \frac{K}{4}|x|^2\right)\partial^\lambda f + \frac{K}{2}x^\sigma x^\lambda \partial_\sigma f = 0 \tag{3.10}$$

where $\partial^\alpha f = \eta^{\alpha\nu}\partial_\nu f$ and $|x|^2 = \eta_{\nu\tau}x^\nu x^\tau$. In particular, (3.10) implies (3.9), and so it suffices to consider (3.10). On contracting (3.10) with x_λ, where $x_\lambda = \eta_{\lambda\rho}x^\rho$, one finds

$$x^\rho \partial_\rho f = 0 \tag{3.11}$$

and on substituting this condition into (3.10) one then obtains $\partial_\mu f = 0$, so f must be constant. Next, consider the case for which ω is a p-form for $1 \leqslant p \leqslant n$. The condition $\mathcal{L}_U\omega = 0$ implies that

$$x^\alpha \partial^\beta \omega_{\nu_1\ldots\nu_p} - x^\beta \partial^\alpha \omega_{\nu_1\ldots\nu_p} + p\delta^\alpha_{[\nu_1}\omega^\beta{}_{\nu_2\ldots\nu_p]} - p\delta^\beta_{[\nu_1}\omega^\alpha{}_{\nu_2\ldots\nu_p]} = 0 \tag{3.12}$$

and the condition $\mathcal{L}_W\omega = 0$ implies that

$$\begin{aligned}\left(1 - \frac{K}{4}|x|^2\right)\partial^\lambda \omega_{\nu_1\ldots\nu_p} + \frac{K}{2}x^\sigma x^\lambda \partial_\sigma \omega_{\nu_1\ldots\nu_p} - \frac{K}{2}p x_{[\nu_1}\omega^\lambda{}_{\nu_2\ldots\nu_p]} \\ + \frac{K}{2}p x^\lambda \omega_{\nu_1\ldots\nu_p} + \frac{K}{2}p x_\sigma \delta^\lambda_{[\nu_1}\omega^\sigma{}_{\nu_2\ldots\nu_p]} = 0\end{aligned} \tag{3.13}$$

Utilizing (3.12), we eliminate $p\delta^\lambda_{[\nu_1}\omega^\sigma{}_{\nu_2\ldots\nu_p]}$ from the final term in (3.13) to obtain

$$\left(1 + \frac{K}{4}|x|^2\right)\partial^\lambda \omega_{\nu_1\ldots\nu_p} = -\frac{K}{2}p x^\lambda \omega_{\nu_1\ldots\nu_p} \tag{3.14}$$

On substituting this expression for $\partial^\lambda \omega_{\nu_1\ldots\nu_p}$, for $\lambda = \alpha, \beta$ into (3.12) we then find

$$p\delta^\alpha_{[\nu_1}\omega^\beta{}_{\nu_2\ldots\nu_p]} - p\delta^\beta_{[\nu_1}\omega^\alpha{}_{\nu_2\ldots\nu_p]} = 0 \tag{3.15}$$

On contracting this expression over α, ν_1 one then finds

$$(n-p)\omega^{\beta}{}_{\nu_2 \ldots \nu_p} = 0 \tag{3.16}$$

Hence, if $p \neq n$, then $\omega = 0$. If $p = n$, then we write

$$\omega_{\nu_1 \ldots \nu_n} = \frac{h}{(1 + \frac{K}{4}|x|^2)^n} \epsilon_{\nu_1 \ldots \nu_n} \tag{3.17}$$

for a function h, where $\epsilon_{\nu_1 \ldots \nu_n}$ is the alternating symbol in n dimensions. Substituting this expression back into (3.14) one finds that $\partial_\lambda h = 0$, so h is constant. It follows in this case that ω must be a constant multiple of the volume form on dS_n.

Hence, we have shown that the only differential forms whose Lie derivatives with respect to all of the isometries of dS_n vanish are constant functions, and constant multiples of the volume form of dS_n. In the case of warped product dS_4 solutions in $D = 11$ supergravity, this result will enable a particular simple decomposition of the 4-form field strength F.

3.2. D=11 Supergravity

In this section, we describe some key properties of $D = 11$ supergravity, and state the conditions which supersymmetric bosonic solutions of this theory must satisfy. Such solutions are called supersymmetric solutions. This theory was first constructed in [57]. The bosonic fields of $D = 11$ supergravity consist of a metric g, and 3-form gauge potential A, with 4-form field strength $F = \mathrm{d}A$. In addition, there is a fermionic Majorana gravitino field, ψ. The action, including fermionic terms, is given by

$$\begin{aligned} S &= \frac{1}{2\kappa^2} \int \Bigg(R - \frac{1}{48} F_{ABCD} F^{ABCD} \\ &- \bar{\psi}_A \Gamma^{ABC} D_B \left(\frac{1}{2}(\Omega + \check{\Omega}) \right) \psi_C + \frac{1}{192} \bar{\psi}_E \Gamma^{ABCDEF} \psi_F \big(F_{ABCD} + \check{F}_{ABCD} \big) + \\ &+ \frac{1}{16} \bar{\psi}_C \Gamma^{AB} \psi_D \big(F_{AB}{}^{CD} + \check{F}_{AB}{}^{CD} \big) \Bigg) \mathrm{dvol}_{11} + S_{CS} \end{aligned} \tag{3.18}$$

where

$$S_{\mathrm{CS}} = \frac{1}{12\kappa^2} \int F \wedge F \wedge A \quad , \tag{3.19}$$

is a Chern-Simons term, $A, B, C \ldots$ are $D = 11$ frame indices, and

- κ^2 is proportional to the gravitational constant;

- dvol_{11} is the $D = 11$ volume form
- $\check{F}_{ABCD} = 4\partial_{[A}A_{BCD]} - \frac{3}{2}\bar{\psi}_{[A}\Gamma_{BC}\psi_{D]}$
- Ω is the the spin-connection, $\Omega_{MAB} = \Omega^0_{MAB} + K_{MAB}$, where Ω^0_{MAB} is the spin-connection with vanishing torsion and K_{MAB} is the cotorsion term;
- $K_{MAB} = -\frac{1}{4}(\bar{\psi}_M\Gamma_B\psi_A - \bar{\psi}_A\Gamma_M\psi_B + \bar{\psi}_B\Gamma_A\psi_M) + \frac{1}{8}\bar{\psi}_N\Gamma^{NL}{}_{MAB}\psi_L$;
- $\check{\Omega}_{MAB} = \Omega^0_{MAB} - \frac{1}{4}(\bar{\psi}_M\Gamma_B\psi_A - \bar{\psi}_A\Gamma_M\psi_B + \bar{\psi}_B\Gamma_A\psi_M)$;
- $D_M\left(\frac{1}{2}(\Omega + \check{\Omega})\right)$ is the covariant derivative with connection given by $\frac{1}{2}(\Omega + \check{\Omega})$;

The supersymmetric variations of the bosonic and fermionic fields are

$$(\delta \mathrm{e}^A)_B = \frac{1}{2}\bar{\epsilon}\Gamma^A\psi_B \tag{3.20}$$

$$(\delta A)_{MNP} = \frac{3}{4}\bar{\epsilon}\Gamma_{[MN}\psi_{P]} \tag{3.21}$$

$$(\delta\psi)_A = D_A(\check{\Omega})\epsilon - \frac{1}{288}\left(\Gamma^{MNPQ}{}_A - 8\Gamma^{NPQ}\delta^M_A\right)\check{F}_{MNPQ}\epsilon \tag{3.22}$$

where e^A is the vielbein, and ϵ is a Majorana spinor.

Bosonic supersymmetric solutions of $D = 11$ supergravity are those for which the fermions, and the variation of the gravitino, vanish. Requiring that the gravitino variation should vanish imposes the Killing spinor equations (KSE) of $D = 11$ supergravity:

$$\mathcal{D}_A\epsilon = 0\ , \tag{3.23}$$

where the supercovariant derivative $\mathcal{D}_A$ is defined as

$$\mathcal{D}_M \equiv \nabla_M - \frac{1}{288}\left(\Gamma_M{}^{A_1A_2A_3A_4} - 8\delta^{A_1}_M\Gamma^{A_2A_3A_4}\right)F_{A_1A_2A_3A_3A_4}\ . \tag{3.24}$$

On setting the fermionic fields to zero in the action (3.18), one also obtains the bosonic field equations. The Einstein equation is

$$R_{AB} - \frac{1}{2}Rg_{AB} - \frac{1}{12}F_{AB_1B_2B_3}F_B{}^{B_1B_2B_3} + \frac{1}{96}g_{AB}F^2 = 0 \tag{3.25}$$

and the gauge field equation is

$$\mathrm{d} * F - \frac{1}{2}F \wedge F = 0\ . \tag{3.26}$$

The 4-form F is also required to satisfy the Bianchi identity

$$dF = 0 \tag{3.27}$$

Having introduced the conditions which supersymmetric solutions of $D = 11$ supergravity must satisfy, we shall next consider the particular case of warped product dS_4 solutions in $D = 11$ supergravity. We shall state how the metric and 4-form F decompose for such solutions, assuming that the Lie derivative of F with respect to all of the dS_4 isometries vanishes; we will also reduce all of the bosonic field equations and Bianchi identities to field equations on the 7-dimensional internal manifold.

3.3. Warped Product dS_4 in $D = 11$ Supergravity

In order to analyse supersymmetric warped product dS_4 solutions, we shall split the $D = 11$ spacetime in a 4+7 fashion $ds^2 = dS_4 \times_w M_7$, where $\times_w$ denotes a warped product of dS_4 with an internal manifold M_7. In terms of the $D = 11$ frame, capital latin letters such as A,B denote $D = 11$ frame indices. These D=11 frame indices are split in a 4+7 fashion as follows: we use greek letters for dS_4 frame directions, and latin letters from the middle of the alphabet and onwards for M_7. Latin letters from the beginning of the alphabet denote M_7 spacetime indices. M_7 is equipped with local co-ordinates y^a, whereas dS_4 is equipped with local co-ordinates x^μ. For further details about the conventions used are set out in Appendix A.

The warped dS_4 product metric g is

$$ds^2 = A^2 ds^2_{dS_4} + ds^2_{M_7} = \eta_{\mu\nu}\mathbf{e}^\mu\mathbf{e}^\nu + \delta_{ij}\mathbf{e}^i\mathbf{e}^j \ , \tag{3.28}$$

where the vielbein frame is defined as

$$\begin{cases} \mathbf{e}^\mu & \equiv \ \frac{A}{\mathcal{R}} dx^\mu \\ \mathbf{e}^i & \equiv \ e^i_b dy^b \end{cases} \tag{3.29}$$

with

$$\mathcal{R}(x) = \left(1 + \frac{1}{4}K x_\nu x^\nu\right) \ , \qquad x_\nu \equiv x^\alpha \eta_{\alpha\nu} \ . \tag{3.30}$$

The conformal factor A and the vielbein e^j_a depend only on y^a co-ordinates. The scalar K is constant and greater than zero.

We require that the Lie derivative of F with respect to all of the isometries of dS_4 must vanish. Under such an assumption, from the analysis of the previous section, F must decompose as follows:

$$F = c\,\mathrm{dvol}(dS_4) + X \ , \tag{3.31}$$

where c is a constant due to the Bianchi identity and X is a closed 4-form on M_7 depending only on y^a co-ordinates. The gauge field equation (3.26) is equivalent to

$$d(A^4 \star_7 X) = cX \ . \tag{3.32}$$

It will be convenient to state the non-vanishing components of the spin-connection, and curvature components. The non-vanishing spin-connection components are

$$\begin{aligned} \Omega_{\mu,\nu\rho} &= \frac{K}{A} x_{[\nu} \eta_{\rho]\mu} \\ \Omega_{\mu,i\nu} &= -\frac{\nabla_i A}{A} \eta_{\mu\nu} \\ \Omega_{ijk} &= \Omega_{ijk}(M_7) \ , \end{aligned} \tag{3.33}$$

where on the LHS Greek indices are frame indices on dS_4, and on the RHS they are co-ordinate indices on dS_4. ∇_i denotes the Levi-Civita connection on M_7.

The non-vanishing Riemann tensor components are

$$\begin{aligned} R_{\mu\nu\alpha\beta} &= (\eta_{\mu\alpha}\eta_{\beta\nu} - \eta_{\nu\alpha}\eta_{\beta\mu}) \left(\frac{K}{A^2} - \frac{\nabla_i A \nabla^i A}{A^2} \right) \\ R_{i\alpha j\beta} &= -\frac{1}{A} \nabla_i \nabla_j A \, \eta_{\alpha\beta} \\ R_{ijkl} &= R_{ijkl}(M_7) \end{aligned} \tag{3.34}$$

where on the LHS Greek indices are frame indices on dS_4, and on the RHS they are co-ordinate indices on dS_4. The Ricci curvature tensor components are

$$\begin{aligned} R_{\mu\nu} &= \eta_{\mu\nu} \left(3A^{-2}K - A^{-1}\nabla_i\nabla^i A - 3A^{-2}\nabla_i A \nabla^i A \right) \\ R_{\mu i} &= 0 \\ R_{ij} &= -4A^{-1}\nabla_i\nabla_j A + R_{ij}(M_7) \end{aligned} \tag{3.35}$$

where on the LHS Greek indices are frame indices on dS_4, and on the RHS they are co-ordinate indices on dS_4.

The $(\mu\nu)$-component of the Einstein equations of motion (3.25), imply that

$$3KA^{-1} - \nabla_i\nabla^i A - 3A^{-1}\nabla_i A\nabla^i A + \frac{1}{3}c^2 A^{-7} + \frac{A}{144}X^2 = 0 \; . \qquad (3.36)$$

From the (ij)-component of the Einstein equation of motion (3.25) and the third equation in (3.35), one finds

$$R_{ij}(M_7) = 4A^{-1}\nabla_i\nabla_j A + \frac{1}{12}X_{ia_1a_2a_3}X_j{}^{a_1a_2a_3} + \frac{1}{6}c^2A^{-8}\delta_{ij} - \frac{1}{144}X^2\delta_{ij} \; . \qquad (3.37)$$

On taking the trace of (3.37), and using (3.32) and (3.36), we obtain

$$R(M_7) - 8A^{-1}\nabla_i\nabla^i A - 12A^{-2}\nabla_i A\nabla^i A + 12A^{-2}K + \frac{1}{6}c^2A^{-8} - \frac{1}{144}X^2 = 0 \; . \quad (3.38)$$

3.4. No-Go Theorem for Regular de Sitter solutions

Having performed this reduction, we briefly revisit the topic of global properties of warped product dS_4 solutions. There are some particularly important AdS geometries which arise in the context of maximally supersymmetric solutions in $D = 11$ supergravity [58]. It is known that such solutions correspond to flat space $\mathbb{R}^{10,1}$, a maximally supersymmetric plane wave solution, and two direct product AdS geometries:

- $\mathrm{AdS}_7(-7R) \times \mathrm{S}^4(8R)$ and $F = \sqrt{6R}\mathrm{dvol}(\mathrm{S}^4)$, where $\mathrm{AdS}_7(-7R)$ is 7-dimensional AdS spacetime with scalar curvature $-7R$, $\mathrm{S}^4(8R)$ is 4-dimensional sphere with scalar curvature $8R$, and $R > 0$ is the constant scalar curvature of the overall $D = 11$ geometry.

- $\mathrm{AdS}_4(8R) \times S^7(-7R)$ and $F = \sqrt{-6R}\mathrm{dvol}(\mathrm{AdS}_4)$, where $\mathrm{AdS}_4(8R)$ is 4-dimensional AdS spacetime with scalar curvature $8R$, $\mathrm{S}^7(-7R)$ is 7-dimensional sphere with scalar curvature $8R$, and $R < 0$ is the constant scalar curvature of the overall $D = 11$ geometry.

Those two solutions can be interpreted as the near-horizon limits of the $M5$ and $M2$ brane solutions respectively [6]. As we shall be interested later on in the classification of warped product dS_4 geometries in $D = 11$ supergravity, we note that the case of $\mathrm{AdS}_7 \times S^4$ provides an explicit, and moreover maximally supersymmetric, example of such a solution. In this case, the metric can be written as a warped product $\mathrm{dS}_4 \times_w M_7$.

To demonstrate this, we take the M5-brane solution in eleven-dimensional supergravity [6],

$$ds^2 = U^{-1/3}(-dt^2 + \sum_{n=1}^{5}(dx_n)^2) + U^{2/3}\sum_{n=1}^{5}(dy_n)^2 \tag{3.39}$$

where Q is a positive constant corresponding to the $M5$-brane charge,

$$U \equiv 1 + \frac{Q}{r^3} \quad , \quad r \equiv \sqrt{\sum_{n=1}^{5} y_n^2} \quad . \tag{3.40}$$

After taking the near-horizon limit and applying an appropriate change of co-ordinates, the near-horizon limit of Eq. (3.39) becomes:

$$\begin{aligned} ds^2 &= A^2\, ds^2(dS_4) + \frac{6}{L}d\theta_1^2 + \frac{6}{L}\cosh\theta_1{}^2 d\theta_2^2 + \frac{6}{L}\cosh\theta_1{}^2\cosh\theta_2{}^2 d\theta_3^2 \\ &\quad + \frac{d\phi^i\delta_{ij}d\phi^j}{\left(1+\frac{L}{6}\phi^k\delta_{kl}\phi^l\right)^2} \ , \end{aligned} \tag{3.41}$$

where $\mu = 0, 1, 2, 3,\ i = 1, 2, 3, 4,\ L = \frac{3}{2}Q^{-2/3}$, the warp factor is

$$A = \cosh\theta_1 \cosh\theta_2 \sinh\theta_3 \ , \tag{3.42}$$

and the metric

$$ds^2(dS_4) = \frac{dx^\mu \eta_{\mu\nu} dx^\nu}{\left(1+\frac{L}{24}x^\alpha\eta_{\alpha\beta}x^\beta\right)^2} \tag{3.43}$$

is the metric of dS_4. The first line in (3.41) corresponds to the metric of AdS_7 with curvature $R = -7L$, and the portion of the metric on the second line is the metric on S^4 with curvature $R = 8L$. The metric on the internal space M_7 is obtained from the metric in (3.41) by excluding the contribution from the dS_4 metric.

It is useful to consider the global properties of this warped product geometry, in terms of the internal space M_7 and the warp factor A found for this solution. If the co-ordinate θ_1 is periodically identified, then the metric components, and the warp factor, are not smooth functions of θ_1 on making a complete revolution in θ_1. Alternatively, if θ_1 is not bounded, then the warp factor A is also unbounded on M_7. Hence, although the $AdS_7 \times S^4$ geometry is smooth, when decomposed as a warped product dS_4 geometry with an internal manifold M_7, one cannot construct such a decomposition for which the M_7 is smooth and compact without boundary, and the warp factor A is a smooth function on M_7.

Having considered the specific case of $AdS_7 \times S^4$, written as a warped product dS_4 solution, we shall now consider investigate a more general type of no-go theorem for warped product dS solutions [30, 59, 60]. In particular, suppose (for a contradiction) one assumes that the internal manifold M_7 is smooth and compact without boundary, and moreover that the warp factor A is smooth. We first establish that A cannot have any zeroes on M_7 unless $c = 0$. We shall do this also by an argument via contradiction; let us suppose that there is a zero for A on M_7. If $c \neq 0$, then (3.36) implies

$$\frac{1}{3}c^2 A^{-6} = -3K + A\nabla_i\nabla^i A + 3\nabla_i A\nabla^i A - \frac{A^2}{144}X^2 \ . \tag{3.44}$$

If there is a zero for A on M_7 then there exists a sequence of points on M_7 approaching this zero for which the LHS becomes unbounded. However, the RHS of this equation is a smooth function on M_7 and hence must be bounded. Hence, there is a contradiction, and therefore one must have $c = 0$.

Consider then (3.36) ; if A has any zeroes on M_7 then $c = 0$, and (3.36) is equivalent to

$$\frac{1}{4}\nabla_i\nabla^i(A^4) = 3KA^2 + \frac{A^4}{144}X^2 \tag{3.45}$$

However, on integrating this equation over M_7 one obtains a contradiction, as the contribution from the LHS is zero, whereas the contribution from the RHS is positive. Alternatively, if A does not admit any zeroes on M_7 then c need not vanish, and (3.36) is equivalent to

$$\frac{1}{4}\nabla_i\nabla^i(A^4) = 3KA^2 + \frac{A^4}{144}X^2 + \frac{1}{3}c^2A^{-4} \tag{3.46}$$

All of the terms on the RHS of this expression are smooth, and so we may integrate all terms in the above equation over M_7. Once more, there is a contradiction, because the contribution from the LHS is zero, whereas the contribution from the RHS is positive.

This no-go theorem implies that in our investigation of warped product dS_4 geometries we cannot utilize the same type of global analysis, such as establishing Lichnerowicz type theorems which were used to obtain quite strong conditions on warped product AdS solutions [61], as well as proving the non-existence of warped product AdS_6 solutions in $D = 11$ supergravity [62]. Instead, our analysis of the KSEs of the warped product dS_4 geometries will be purely local in nature.

CHAPTER 4

INTEGRATING THE KSE FOR dS_4

In this chapter, we shall begin the analysis of the KSE for warped product dS_4 solutions in $D = 11$ supergravity. The content of this chapter, as well as in chapters 5, 6, 7 (and supporting Appendices) constitutes original research presented in [46]. We proceed by computing the integrability conditions associated with the existence of a non-zero Killing spinor ϵ which is covariantly constant with respect to the supercovariant derivative of $D = 11$ supergravity,

$$\mathcal{D}_A\epsilon = 0 \ , \qquad (4.1)$$

where capital latin letters such as A, B denote $D = 11$ frame indices. These results will be particularly useful when we explicitly integrate up the KSE along the dS_4 directions in the next section. We remark that the $D = 11$ frame indices are split in a 4+7 fashion as follows: we use greek letters for dS_4 frame directions, and latin letters from the middle of the alphabet and onwards for M_7 frame directions.

4.1. Integrability Conditions from the KSE

From the equation $\mathcal{D}_M\epsilon = 0$, where

$$\mathcal{D}_M \equiv \nabla_M - \frac{1}{288}\left(\Gamma_M{}^{A_1A_2A_3A_4} - 8\delta_M^{A_1}\Gamma^{A_2A_3A_4}\right)F_{A_1A_2A_3A_4} \ , \qquad (4.2)$$

we find

$$\frac{\partial}{\partial x^\mu}\epsilon = \frac{1}{\mathcal{R}}\left(-\frac{1}{4}Kx^\alpha\Gamma_{\alpha\mu} + \frac{1}{2}\nabla_k A\Gamma^k\Gamma_\mu + \frac{A}{288}\Gamma_\mu \not{X} - \frac{c}{6}A^{-3}\Gamma_\mu\tilde{\Gamma}^4\right)\epsilon \qquad (4.3)$$

and

$$\frac{\partial}{\partial y^a}\epsilon = e^j_a\left(\frac{1}{288}\not{\Gamma X}_j + \frac{c}{12}A^{-4}\Gamma_j\tilde{\Gamma}^4 - \frac{1}{36}\not{X}_j - \frac{1}{4}\Omega_{j,lm}\Gamma^{lm}\right)\epsilon \ , \tag{4.4}$$

where

$$\tilde{\Gamma}^4 \equiv \Gamma^0\Gamma^1\Gamma^2\Gamma^3 \ . \tag{4.5}$$

In the chosen notation, Appendix A,

$$\not{X}_C \equiv X_{CA_1\ldots A_3}\Gamma^{A_1\ldots A_3} \ , \quad \text{and} \quad \not{\Gamma X}_C \equiv \Gamma_{CA_1\ldots A_4}X^{A_1\ldots A_4} \quad . \tag{4.6}$$

We remark that (4.4) is equivalent to

$$\nabla_i\epsilon = \left(\frac{1}{288}\not{\Gamma X}_i + \frac{c}{12}A^{-4}\Gamma_i\tilde{\Gamma}^4 - \frac{1}{36}\not{X}_i\right)\epsilon \ , \tag{4.7}$$

where ∇_i denotes the Levi-Civita connection on M_7.

We use these expressions to derive several integrability conditions. First, from the integrability condition on dS_4 spacetime

$$\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\right)\epsilon = 0 \ , \tag{4.8}$$

we get

$$\left(|\nabla A|^2 - K - \frac{c^2}{9}A^{-6} - \frac{A^2}{(144)^2}\not{X}^2 + \frac{2}{3}cA^{-3}\nabla_i A\Gamma^i\tilde{\Gamma}^4 - \frac{1}{18}A\nabla_i A\not{X}^i\right)\epsilon = 0 \ . \tag{4.9}$$

On the other hand, from the integrability condition with one direction on dS_4 and the other on M_7, i.e.

$$\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^a} - \frac{\partial}{\partial y^a}\frac{\partial}{\partial x^\mu}\right)\epsilon = 0 \tag{4.10}$$

we get

$$\begin{aligned}\Big(& - \frac{1}{2}\nabla_i\nabla_k A\Gamma^k + \frac{A}{288}\nabla_i\not{X} + \frac{5}{6}\frac{A}{288}\left(\Gamma_{[il_1l_2}{}^{j_3j_4}\,\delta^{j_2}_{l_3}\,\delta^{j_1}_{l_4]}X_{j_1j_2j_3j_4}\,X^{l_1l_2l_3l_4}\right) \\ & + \frac{5}{6}\frac{A}{288}\left(\Gamma_{[i}X_{l_1l_2l_3l_4]}\,X^{l_1l_2l_3l_4}\right) + \frac{c}{864}A^{-3}(10\not{X}_i - \Gamma_i\not{X})\,\tilde{\Gamma}^4 \\ & + \frac{A}{144}\Gamma_{l_1}{}^{j_3j_4}\,X_i{}^{l_1j_1j_2}\,X_{j_1j_2j_3j_4} + \frac{c}{2}A^{-4}\nabla_i A\,\tilde{\Gamma}^4 + \frac{1}{72}\nabla_k A\Gamma_{l_1l_2l_3i}\,X^{l_1l_2l_3k} \\ & - \frac{c}{12}A^{-4}\nabla_k A\Gamma^k{}_i\,\tilde{\Gamma}^4 + \frac{1}{12}\nabla_k A\Gamma^{mn}\,X_{imn}{}^k\Big)\epsilon = 0 \ .\end{aligned} \tag{4.11}$$

The integrability conditions (4.9) and (4.11) are, however, not independent; (4.9) is implied by (4.11). To see this, contract (4.11) with Γ^i, and using equation of motion (3.36) and the Bianchi Identity $\mathrm{d}F = 0$, we are able to derive the integrability condition (4.9).

So far, we have analyzed the integrability conditions involving the dS_4 part of the covariant derivative (4.3). The integrability condition on M_7 given by

$$[\nabla_i, \nabla_j]\,\epsilon = \frac{1}{4} R_{ijmn}\Gamma^{mn}\epsilon \ , \tag{4.12}$$

is

$$\begin{aligned}
\frac{1}{4}R_{ijmn}\Gamma^{mn}\epsilon \quad = \quad & \Bigg[\frac{1}{288}\left(\nabla_i(\slashed{\Gamma X}_j) - \nabla_j(\slashed{\Gamma X}_i)\right) - \frac{c}{3}A^{-5}\left(\nabla_i A\Gamma_j - \nabla_j A\Gamma_i\right)\tilde{\Gamma}^4 \\
- \ & \frac{1}{36}\left(\nabla_i \slashed{X}_j - \nabla_j \slashed{X}_i\right) + \frac{1}{288^2}\left(\slashed{\Gamma X}_j\slashed{\Gamma X}_i - \slashed{\Gamma X}_i\slashed{\Gamma X}_j\right) \\
+ \ & \frac{1}{36^2}\left(\slashed{X}_j\slashed{X}_i - \slashed{X}_i\slashed{X}_j\right) + \frac{1}{288}\frac{c}{12}A^{-4}\left(\slashed{\Gamma X}_j\Gamma_i - \Gamma_i\slashed{\Gamma X}_j\right)\tilde{\Gamma}^4 \\
+ \ & \frac{1}{288}\frac{c}{12}A^{-4}\left(\Gamma_j\slashed{\Gamma X}_i - \slashed{\Gamma X}_i\Gamma_j\right)\tilde{\Gamma}^4 + \frac{1}{288}\frac{1}{36}\left(\slashed{\Gamma X}_i\slashed{X}_j - \slashed{X}_j\slashed{\Gamma X}_i\right) \\
+ \ & \frac{1}{288}\frac{1}{36}\left(\slashed{X}_i\slashed{\Gamma X}_j - \slashed{\Gamma X}_j\slashed{X}_i\right) + \frac{c^2}{72}A^{-8}\Gamma_{ij} \\
+ \ & \frac{c}{432}A^{-4}\left(\slashed{X}_i\Gamma_j - \Gamma_j\slashed{X}_i\right)\tilde{\Gamma}^4 + \frac{c}{432}A^{-4}\left(\Gamma_i\slashed{X}_j - \slashed{X}_j\Gamma_i\right)\tilde{\Gamma}^4\Bigg]\epsilon \ .
\end{aligned} \tag{4.13}$$

In fact, (4.11) is implied by (4.13). To see this, contract (4.13) with Γ^j and use the Einstein equation (3.37), the Bianchi identity, $R_{l[ijk]} = 0$, and the condition $dX = 0$, as well as the gauge field equations (3.32). In particular:

- The condition $\mathrm{d}X = 0$ is used to derive:

$$4\Gamma^{l a_1 a_2 a_3}\nabla_l X_{k a_1 a_2 a_3} = \nabla_k \slashed{X} \ . \tag{4.14}$$

- The gauge field equation (3.32) in components is

$$16\nabla^i A\slashed{X}_i + 4A\nabla^i\slashed{X}_i + cA^{-3}\slashed{X}\tilde{\Gamma}^4 = 0 \ . \tag{4.15}$$

- We obtain two equations by multiplying (4.15) by Γ_i on the left and on the right,

$$4A\Gamma_i\nabla_j\slashed{X}^j + cA^{-3}\Gamma_i\slashed{X}\,\tilde{\Gamma}^4 + 16\nabla_j A\Gamma_i\slashed{X}^j = 0 \ , \tag{4.16}$$

$$4A\nabla_j\slashed{X}^j\Gamma_i + cA^{-3}\slashed{X}\,\tilde{\Gamma}^4\Gamma_i + 16\nabla_j A\slashed{X}^j\Gamma_i = 0 \ . \tag{4.17}$$

- Subtracting (4.16) and (4.17) implies that

$$4A(\Gamma_i \nabla_j \not{X}^j - \nabla_j \not{X}^j \Gamma_i) + cA^{-3}(\Gamma_i \not{X} - \not{X}\Gamma_i)\,\tilde{\Gamma}^4 + 16\nabla_j A(\Gamma_i \not{X}^j - \not{X}^j \Gamma_i) = 0\ , \tag{4.18}$$

it follows that

$$A\Gamma_{ia_1a_2a_3}\nabla_k X^{ka_1a_2a_3} = -cA^{-3}\not{X}_i\tilde{\Gamma}^4 - 4\nabla_k A\Gamma_{ia_1a_2a_3}X^{ka_1a_2a_3}\ , \tag{4.19}$$

where the terms in the brackets in (4.18) have been listed in (C.1) and (C.14).

- Adding (4.16) and (4.17) implies that

$$4A(\Gamma_i \nabla_j \not{X}^j + \nabla_j \not{X}^j \Gamma_i) + cA^{-3}(\Gamma_i \not{X} + \not{X}\Gamma_i)\,\tilde{\Gamma}^4 + 16\nabla_j A(\Gamma_i \not{X}^j + \not{X}^j \Gamma_i) = 0\ , \tag{4.20}$$

and from this condition, it follows that

$$A\Gamma^{ab}\nabla^k X_{kiab} = -\frac{1}{12}cA^{-3}\Gamma_i\not{X}\tilde{\Gamma}^4 + \frac{1}{3}cA^{-3}\not{X}_i\tilde{\Gamma}^4 - 4\nabla^k A\Gamma^{ab}X_{kiab}\ , \tag{4.21}$$

where the terms in the brackets in (4.20) have been listed in (C.6) and (C.9).

Hence, it follows that the integrability conditions (4.9) and (4.11) are both implied by (4.13), which is derived from the integrability condition of (4.7).

4.2. Integration of KSE

In this section, we will explicitly integrate the KSE along the dS_4 directions. In this analysis, we shall show that the KSE reduce to a single gravitino-type KSE acting on a spinor ψ which is independent of the dS_4 co-ordinates. To begin, we shall define a spinor Φ, as follows:

$$\Phi \equiv \frac{A}{288}\not{X}\epsilon - \frac{1}{2}\nabla_k A\Gamma^k\epsilon + acA^{-3}\tilde{\Gamma}^4\epsilon\ , \tag{4.22}$$

where a is a constant to be fixed. We have chosen the relative coefficients between $\not{X}$ and $d\!\!\!/A$ in (4.22) motivated by the first two terms in (4.11). We shall show that one can choose the constant a, as well as other constants k_1, k_2, q_1, q_2, q_3, q_4, q_5 such that

$$\nabla_i\Phi + k_1[\text{Eq. (4.11)}] + k_2 A^{-1}\Gamma^i[\text{Eq. (4.9)}]$$
$$+q_1 \not{\Gamma}\not{X}_i\Phi + q_2\not{X}_i\Phi + q_3 c A^{-4}\Gamma_i\tilde{\Gamma}^4\Phi + q_4 A^{-1}\nabla_k A\Gamma_i\Gamma^k\Phi + q_5 A^{-1}\nabla_i A\Phi = 0\ . \tag{4.23}$$

Details of this calculation are presented in Appendix D. One finds that

$$k_1 = -1 \quad a = -\frac{1}{6} \quad q_1 = \frac{1}{288} \quad q_2 = -\frac{1}{36} \quad q_3 = -\frac{1}{12} \quad k_2 = q_4 = q_5 = 0\ . \tag{4.24}$$

Given this choice of constants, the spinor Φ is

$$\Phi = \left(\frac{A}{288}\not{X} - \frac{1}{2}\nabla_k A\Gamma^k - \frac{c}{6}A^{-3}\tilde{\Gamma}^4\right)\epsilon\ , \tag{4.25}$$

which satisfies the following equations

$$\frac{\partial}{\partial x^\mu}\Phi = \frac{1}{\mathcal{R}}\left[-\frac{1}{4}Kx^\alpha\Gamma_{\alpha\mu} + \Gamma_\mu\left(\frac{1}{2}\nabla_k A\Gamma^k + \frac{A}{288}\not{X} + \frac{c}{6}A^{-3}\tilde{\Gamma}^4\right)\right]\Phi \tag{4.26}$$

$$\frac{\partial}{\partial y^a}\Phi = e^j_a\left(-\frac{1}{288}\not{\Gamma}\not{X}_j + \frac{c}{12}A^{-4}\Gamma_j\tilde{\Gamma}^4 + \frac{1}{36}\not{X}_j - \frac{1}{4}\Omega_{j,lm}\Gamma^{lm}\right)\Phi\quad . \tag{4.27}$$

These equations are similar, but not identical, to the original Killing spinor equations for ϵ (4.3)-(4.4). The differences are in terms of certain signs appearing in (4.26)-(4.27), which are flipped with respect to (4.3)-(4.4) - in (4.26) the second and the fourth term with respect to (4.3) and in (4.27) the first and the third term with respect to (4.4).

Equations (4.26) and (4.27), will be particularly useful in the process of integrating up the KSE along the dS_4 directions. By using (4.9), (4.26) becomes

$$\frac{\partial}{\partial x^\mu}\Phi = -\frac{K}{4\mathcal{R}}x^\alpha\Gamma_{\alpha\mu}\Phi - \frac{K}{4\mathcal{R}}\Gamma_\mu\epsilon\ . \tag{4.28}$$

By using the definition of Φ (4.25), one can rewrite $\frac{\partial}{\partial x^\mu}\epsilon$ as

$$\frac{\partial}{\partial x^\mu}\epsilon = \frac{1}{\mathcal{R}}\left(-\frac{1}{4}Kx^\alpha\Gamma_{\alpha\mu}\epsilon + \Gamma_\mu\Phi\right)\ . \tag{4.29}$$

Applying a second derivative $\frac{\partial}{\partial x^\nu}$ to (4.29), using (4.28) and finally exploiting (4.29) to cancel $\mathcal{R}^{-1}\Gamma_\mu\Phi$ terms, one gets a second order differential equation for ϵ, namely

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\epsilon + \frac{K}{4\mathcal{R}}(x_\mu\frac{\partial}{\partial x^\nu}\epsilon + x_\nu\frac{\partial}{\partial x^\mu}\epsilon) - \frac{K^2}{16\mathcal{R}^2}x_\mu x_\nu\epsilon + \frac{K}{4\mathcal{R}}\eta_{\mu\nu}\epsilon = 0\ . \tag{4.30}$$

On defining η by

$$\epsilon = \mathcal{R}^{-\frac{1}{2}}\eta \ , \tag{4.31}$$

it is straightforward to see that (4.30) is equivalent to

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\eta = 0 \ , \tag{4.32}$$

and hence this equation can be integrated to find

$$\eta = \psi + x^\lambda \tau_\lambda \ , \tag{4.33}$$

where ψ, τ_λ with $\lambda = 0, 1, 2, 3$ are Majorana spinors which do not depend on the x_μ co-ordinates.

Given this expression for ϵ, i.e.

$$\epsilon = \mathcal{R}^{-\frac{1}{2}}(\psi + x^\lambda \tau_\lambda) \ , \tag{4.34}$$

we substitute it into the KSEs (4.3) and (4.4). As the spinors ψ, τ_λ are independent of the dS_4 co-ordinates, on expanding (4.3) and (4.4) order-by-order in x_α, we find various conditions.

In particular, from the KSE along the dS_4 directions (4.3), the vanishing of $x-$independent terms imply that the Majorana spinors τ_μ are given in terms of ψ, as follows:

$$\tau_\mu = \Gamma_\mu\left(\frac{A}{288}\not{X} - \frac{1}{2}\nabla_k A\Gamma^k - \frac{c}{6}A^{-3}\tilde{\Gamma}^4\right)\psi \ . \tag{4.35}$$

The vanishing of the terms that are linear in x_μ in (4.3) imply

$$\left(|\nabla A|^2 - K - \frac{c^2}{9}A^{-6} - \frac{A^2}{(144)^2}\not{X}^2 + \frac{2}{3}cA^{-3}\nabla_i A\Gamma^i\tilde{\Gamma}^4 - \frac{1}{18}A\nabla_i A\not{X}^i\right)\psi = 0 \ , \tag{4.36}$$

and we remark that this condition is equivalent to the integrability condition (4.9), but with ϵ replaced with ψ. The terms in (4.3) which are quadratic in x_μ vanish identically; this then exhausts the content of (4.3).

Next we consider the KSE along the seven-dimensional internal directions, (4.4). Again, we substitute in (4.34) and expand order-by-order in dS_4 co-ordinates. The vanishing of $x-$independent terms gives

$$\nabla_i\psi = \left(\frac{1}{288}\Gamma\not{X}_i + \frac{c}{12}A^{-4}\Gamma_i\tilde{\Gamma}^4 - \frac{1}{36}\not{X}_i\right)\psi \ . \tag{4.37}$$

The above equation (4.37) implies that ψ satisfies a gravitino KSE along the internal directions, which is identical to the condition (4.37) but with ϵ replaced with ψ.

From the terms in (4.4) which are linear in x_μ we obtain

$$\begin{aligned}
&\Big[\frac{A}{288}\nabla_i \not{X} - \frac{1}{2}\nabla_i\nabla_k A\Gamma^k + \frac{A}{1728}\Gamma_i X^2 + \frac{A}{864}\Gamma_{j_1j_2j_3}{}^{l_1l_2}X_{ij_4l_1l_2}X^{j_1j_2j_3j_4} \\
-&\ \frac{A}{432}\Gamma_{j_1}X_{ij_2j_3j_4}X^{j_1j_2j_3j_4} - \frac{A}{576}\Gamma_{ij_1j_2}{}^{l_1l_2}X_{j_3j_4l_1l_2}X^{j_1j_2j_3j_4} \\
-&\ \frac{1}{864}cA^{-3}\Gamma_i\not{X}\tilde{\Gamma}^4 + \frac{5}{432}cA^{-3}\not{X}_i\tilde{\Gamma}^4 + \frac{A}{144}\Gamma^m{}_{ab}X_{impq}X^{pqab} \\
+&\ \frac{1}{72}\nabla_k A\Gamma_{ij_1j_2j_3}X^{kj_1j_2j_3} + \frac{1}{12}\nabla_k A\Gamma^{ab}X_i{}^k{}_{ab} \\
-&\ \frac{c}{12}A^{-4}\nabla_k A\Gamma^k\Gamma_i\tilde{\Gamma}^4 + \frac{7}{12}cA^{-4}\nabla_i A\tilde{\Gamma}^4\Big]\psi = 0
\end{aligned} \tag{4.38}$$

which is identical to the integrability condition (4.11), with ϵ replaced by ψ. This then exhausts the content of (4.4).

Hence, we have shown that the spinor ϵ is given by

$$\epsilon = \mathcal{R}^{-\frac{1}{2}}(\psi + x^\lambda \tau_\lambda) \ , \tag{4.39}$$

where

$$\tau_\lambda = \Gamma_\lambda \left(\frac{A}{288}\not{X} - \frac{1}{2}\nabla_k A\Gamma^k - \frac{c}{6}A^{-3}\tilde{\Gamma}^4 \right)\psi \ . \tag{4.40}$$

The Majorana spinor ψ is independent of the dS_4 co-ordinates, and satisfies (4.37). Furthermore, ψ must also satisfy the algebraic conditions (4.38) and (4.36). However, as we have shown in the previous section, the integrability conditions of (4.37), together with the bosonic field equations and Bianchi identities, imply that (4.38) holds. Furthermore, we have shown that (4.38) also implies (4.36). Hence, the necessary and sufficient conditions for supersymmetry are encoded in (4.37).

4.3. Counting the supersymmetries

Having determined that the necessary and sufficient conditions for supersymmetry are given by (4.37), we shall now count the number of solutions to this equation. In particular, if ψ satisfies (4.37), then so does $\Gamma_{\mu\nu}\psi$. We choose a null basis for the Majorana representation of Spin(10,1) and take the dS_4 frame directions to correspond with the $+, -, 1, \bar{1}$ directions, see Appendix B. The frame directions associated with the internal manifold M_7 correspond to the $2, 3, 4, \bar{2}, \bar{3}, \bar{4}, \#$ directions.

With these conventions for the de Sitter and internal manifold frames, we define light-cone projection operators as

$$P_\pm \equiv \frac{1}{2}\left(\mathbb{I} \pm \Gamma_{+-}\right) \ , \tag{4.41}$$

where $\Gamma_{+-} = \frac{1}{2}[\Gamma_+, \Gamma_-]$. As the projection operator $P_\pm$ commutes with the supercovariant derivative (4.37), we then decompose the spinor ψ using the lightcone projectors and we define $\psi_\pm$ to be

$$\psi_\pm \equiv P_\pm \psi \quad \Rightarrow \quad \Gamma_\pm \psi_\pm = 0 \ . \tag{4.42}$$

Without loss of generality, utilizing these projection operators, any supersymmetric solution must admit a positive chirality solution ψ_+ to (4.37). Given such a ψ_+ spinor, we can then define

$$\begin{aligned} \tilde{\psi}_+ &\equiv i\Gamma_{1\bar{1}}\psi_+ \\ \psi_- &\equiv \Gamma_-(\Gamma_1 + \Gamma_{\bar{1}})\psi_+ \\ \tilde{\psi}_- &\equiv i\Gamma_-(\Gamma_1 - \Gamma_{\bar{1}})\psi_+ \ . \end{aligned} \tag{4.43}$$

$\tilde{\psi}_+$ is an additional positive chirality solution to (4.37), and $\{\psi_-, \tilde{\psi}_-\}$ are two negative chirality solutions to (4.37). $\{\psi_+, \tilde{\psi}_+, \psi_-, \tilde{\psi}_-\}$ are linearly independent, as by construction they are mutually orthogonal with respect to the Dirac inner product $\langle \cdot\ ,\ \cdot \rangle$.

It would therefore appear, a priori, that the number of supersymmetries is $4n$. However, there are, in fact further additional spinors. To see this, note that (4.27) implies that

$$\check{\psi}_+ \equiv (\Gamma_1 + \Gamma_{\bar{1}})\left(\frac{A}{288}\not{X} - \frac{1}{2}\nabla_k A\Gamma^k - \frac{c}{6}A^{-3}\tilde{\Gamma}^4\right)\psi_+ \ , \tag{4.44}$$

is also a positive chirality solution of (4.37). Furthermore, it can be shown that $\{\psi_+, \tilde{\psi}_+, \check{\psi}\}$ are linearly independent. To see this, suppose that

$$\check{\psi}_+ = c_1\psi_+ + ic_2\Gamma_{1\bar{1}}\psi_+ \ , \tag{4.45}$$

for real constants c_1, c_2. Acting on both sides of this condition with the operator $(\Gamma_1 + \Gamma_{\bar{1}})\left(\frac{A}{288}\not{X} - \frac{1}{2}\nabla_k A\Gamma^k - \frac{c}{6}A^{-3}\tilde{\Gamma}^4\right)$, and utilizing the integrability condition (4.36) to simplify the LHS, we find

$$-\frac{K}{2}\psi = (c_1^2 + c_2^2)\psi \ , \tag{4.46}$$

where we have also used (4.45) to simplify the RHS. It is clear that this admits no solution, as $K > 0$. Hence, we find that we can construct four linearly independent positive chirality spinors which solve (4.37), corresponding to $\{\psi_+, \tilde{\psi}_+, \check{\psi}_+, \tilde{\check{\psi}}_+\}$, where $\tilde{\check{\psi}}_+ \equiv i\Gamma_{1\bar{1}}\check{\psi}$. There are also four corresponding negative chirality spinors given by $\{\psi_-, \tilde{\psi}_-, \check{\psi}_-, \tilde{\check{\psi}}_-\}$, where

$$\check{\psi}_- = \Gamma_-(\Gamma_1 + \Gamma_{\bar{1}})\check{\psi}_+ \ , \qquad \tilde{\check{\psi}}_- = i\Gamma_{1\bar{1}}\check{\psi}_- \ . \tag{4.47}$$

Hence we have constructed 8 linearly independent solutions to (4.37),

$$\{\psi_+, \tilde{\psi}_+, \check{\psi}_+, \tilde{\check{\psi}}_+, \psi_-, \tilde{\psi}_-, \check{\psi}_-, \tilde{\check{\psi}}_-\} \tag{4.48}$$

and it follows that the number of supersymmetries for warped product dS_4 solutions is $8n$, $n = 1, 2, 3, 4$.

We remark that the existence of the additional spinors $\check{\psi}_\pm$, $\tilde{\check{\psi}}_\pm$ is somewhat analogous to results found in the analysis of near-horizon geometries of supersymmetric extremal black holes [45] and also for warped product AdS solutions [61]. In these cases, given a Killing spinor, one also finds that additional Killing spinors can be generated by the action of certain algebraic operators constructed out of the fluxes of the theory. For the case of near-horizon geometries, and for warped product AdS_2 solutions, the construction of such operators relies on global properties of the geometries via generalized Lichnerowicz type theorems. However, for warped product AdS_n ($n \geqslant 3$) solutions, one can also construct the additional Killing spinors algebraically using purely local constructions somewhat analogous to the de-Sitter analysis.

CHAPTER 5

SPINORIAL GEOMETRY TECHNIQUES FOR dS$_4$ KSEs

In this chapter we shall use Spin(7) gauge transformations to bring the spinor ψ_+ to one of several simple canonical forms. We will describe the gauge transformations used to do this explicitly.

We consider the 32-dimensional space of Majorana spinors Δ_{32} (see Appendix B). The most general form of a positive chirality Majorana spinor $\psi_+ \in \Delta_{32}$ can be expressed by using (B.7), i.e.

$$
\begin{aligned}
\psi_+ &= w1 + \bar{w}e_{1234} + \lambda^1 e_1 + \bar{\lambda}^1 e_{234} + \lambda^j e_j - \frac{1}{3!}(*\bar{\lambda})^{l_1 l_2 l_3} e_{l_1 l_2 l_3} \\
&+ \Omega^q e_{1q} - \frac{1}{2!}\bar{\Omega}^q \epsilon_q{}^{mn} e_{mn} \ ,
\end{aligned}
\tag{5.1}
$$

with $l, q, m, m = 2, 3, 4$. As the action of SU(N) on $\mathbb{C}P^{N-1}$ is transitive, one can apply a SU(3) gauge transformation in the 2,3,4 directions to set, without loss of generality, $\Omega^3 = \Omega^4 = 0$,[1] i.e.

$$
\psi_+ = w1 + \bar{w}e_{1234} + \lambda^1 e_1 + \bar{\lambda}^1 e_{234} + \lambda^j e_j - \frac{1}{3!}(*\bar{\lambda})^{l_1 l_2 l_3} e_{l_1 l_2 l_3} + \Omega e_{12} - \bar{\Omega} e_{34} \ . \tag{5.2}
$$

To proceed further, we define T^1, T^2, T^3 as

$$
T_1 \equiv \frac{1}{2}(\Gamma_{34} + \Gamma_{\bar{3}\bar{4}}) \quad T_2 \equiv \frac{i}{2}(\Gamma_{34} - \Gamma_{\bar{3}\bar{4}}) \quad T_3 \equiv T_1 T_2 = \frac{i}{2}(\Gamma_{3\bar{3}} + \Gamma_{4\bar{4}}) \ . \tag{5.3}
$$

[1]Generally, the complex value Ω^2 can be set to be real with the same SU(3) transformation used to set $\Omega^3 = \Omega^4 = 0$. It does not happen in this specific case due to the fact that Ω^2 will be promoted to be complex value in the next gauge transformation.

It is straightforward to verify that T^i with $i = 1, 2, 3$, which satisfy the algebra of the imaginary unit quaternions, preserve the span of the following basis elements

$$\begin{aligned} v_1 \equiv &\quad (1 + e_{1234}) \qquad v_2 \equiv i(1 - e_{1234}) \\ v_3 \equiv &\quad (e_{12} - e_{34}) \qquad v_4 \equiv i(e_{12} + e_{34}) \end{aligned} \tag{5.4}$$

and we remark that the Spin(7) gauge transformation generated by the T_i is of the form $p^4\mathrm{id} + p^i T_i$ where $(p^1, p^2, p^3, p^4) \in S^3$.

Then one can carry out an SO(2) gauge transformation generated by T_3 to set $w \in \mathbb{R}$. So far, the spinor ψ_+ can be written as

$$\psi_+ = w(1 + e_{1234}) + \lambda^1 e_1 + \bar{\lambda}^1 e_{234} + \lambda^j e_j - \frac{1}{3!}(*\bar{\lambda})^{l_1 l_2 l_3} e_{l_1 l_2 l_3} + \Omega e_{12} - \bar{\Omega} e_{34} \ . \tag{5.5}$$

An SU(3) gauge transformation generated by $i(\Gamma_{2\bar{2}} - \frac{1}{2}\Gamma_{3\bar{3}} - \frac{1}{2}\Gamma_{4\bar{4}})$, which leaves $\{1, e_{1234}\}$ invariant, is then used to set $\Omega \in \mathbb{R}$, so

$$\psi_+ = w(1 + e_{1234}) + \lambda^1 e_1 + \bar{\lambda}^1 e_{234} + \lambda^j e_j - \frac{1}{3!}(*\bar{\lambda})^{l_1 l_2 l_3} e_{l_1 l_2 l_3} + \Omega(e_{12} - e_{34}) \ . \tag{5.6}$$

We next exploit an SO(2) transformation generated by T_1, acting on v_1 and v_3 to put $\Omega = 0$. Then, we make a further SU(3) gauge transformation along the $2, 3, 4$ directions to set $\lambda^3 = \lambda^4 = 0$ with $\lambda^2 \in \mathbb{R}$, i.e.

$$\psi_+ = w(1 + e_{1234}) + \lambda^1 e_1 + \bar{\lambda}^1 e_{234} + \lambda^2(e_2 - e_{134}) \ . \tag{5.7}$$

In order to simplify further the spinor ψ_+, we shall introduce additional Spin(7) generators L_1, L_2, L_3 given by

$$L_1 \equiv \frac{1}{\sqrt{2}}\Gamma_\sharp(\Gamma_2 + \Gamma_{\bar{2}}), \quad L_2 \equiv \frac{i}{\sqrt{2}}\Gamma_\sharp(\Gamma_2 - \Gamma_{\bar{2}}), \quad L_3 \equiv L_1 L_2 = i\Gamma_{2\bar{2}} \ . \tag{5.8}$$

The L_j also satisfy the algebra of the imaginary unit quaternions, and commute with the T_i, and the Spin(7) gauge transformation generated by the L_j is of the form $q^4\mathrm{id} + q^j L_j$ where $(q^1, q^2, q^3, q^4) \in S^3$. We shall then consider a generic gauge transformation generated by the T_i and L_j of the acting on the spinor (5.7) of the form

$$\left(p^4\mathrm{id} + p^i T_i\right)\left(q^4\mathrm{id} + q^j L_j\right)\psi_+ \ . \tag{5.9}$$

We set $q^2 = q^3 = 0$ and $q^1 = \sin\sigma$, $q^4 = \cos\sigma$, such that

$$w\lambda^2 \cos 2\sigma + \frac{1}{2}\sin 2\sigma(\omega^2 - (\lambda^2)^2 - |\lambda^1|^2) = 0 \tag{5.10}$$

and

$$
\begin{aligned}
p^1 &= \ell \mathrm{Re}(\lambda^1)\sin\sigma, \quad p^2 = -\ell \mathrm{Im}(\lambda^1)\sin\sigma, \\
p^3 &= 0, \quad\quad\quad\quad\quad p^4 = \ell(\omega\cos\sigma - \lambda^2\sin\sigma)
\end{aligned}
\tag{5.11}
$$

where the constant ℓ is chosen such that $(p^1, p^2, p^3, p^4) \in S^3$. With this choice of parameters, the gauge transformation given in (5.9) can be used to set $\lambda^2 = 0$ in (5.7), so the simplest canonical form for the spinor ψ_+ is given by

$$
\psi_+ = w(1 + e_{1234}) + \lambda e_1 + \bar{\lambda} e_{234} \qquad w \in \mathbb{R},\ \lambda \in \mathbb{C}\ . \tag{5.12}
$$

5.1. Stabilizer Group of ψ_+

It is useful to consider the stabilizer subgroup of Spin(7) which leaves ψ_+ invariant. In particular, we must determine the generators $f^{ij}\Gamma_{ij}$, where $f^{ij} \in \mathbb{R}$ are antisymmetric in i, j, and satisfy

$$
f^{ij}\Gamma_{ij}\psi_+ = 0 \qquad i, j = \#, \alpha, \bar{\alpha}\ . \tag{5.13}
$$

The conditions obtained from (5.13) are

$$
\begin{aligned}
2wf^{\alpha\beta} &= \sqrt{2}\bar{\lambda} f^{\#\bar{\rho}} \epsilon_{\bar{\rho}}{}^{\alpha\beta} \\
2\lambda f^{\alpha\beta} &= \sqrt{2} w f^{\#\bar{\rho}} \epsilon_{\bar{\rho}}{}^{\alpha\beta} \\
f^{\alpha}{}_{\alpha} &= 0\ .
\end{aligned}
\tag{5.14}
$$

Depending on w and λ, there are two possible different stabilizer subgroups:

(a) if $w^2 - |\lambda|^2 \neq 0$ then (5.14) implies that $f_{\alpha\beta} = 0$, $f_{\sharp\alpha} = 0$ and $f^{\alpha}{}_{\alpha} = 0$, that is $f \in \mathfrak{su}(3)$, hence the stabilizer is $SU(3)$. The stabilizer subgroup is generated by $\Gamma_{\alpha\bar{\beta}}$ for $\alpha \neq \beta$, together with $i(\Gamma_{1\bar{1}} - \Gamma_{2\bar{2}})$ and $i(\Gamma_{1\bar{1}} - \Gamma_{3\bar{3}})$.

(b) if $w^2 - |\lambda|^2 = 0$, then (5.14) implies that $f \in \mathfrak{g}_2$. In particular, the spinor ψ_+ has a 3-form bilinear φ which is the canonical G_2 invariant 3-form given in (6.53). The stabilizer group is generated by the eight $SU(3)$ generators listed above, together with the additional 6 generators

$$
\begin{aligned}
&\{2\Gamma_{\sharp\rho} + 2\Gamma_{\sharp\bar{\rho}} + \frac{1}{\sqrt{2}}\big(e^{i\zeta}\epsilon_{\rho}{}^{\bar{\alpha}\bar{\beta}}\Gamma_{\bar{\alpha}\bar{\beta}} + e^{-i\zeta}\epsilon_{\bar{\rho}}{}^{\alpha\beta}\Gamma_{\alpha\beta}\big), \\
&2i\Gamma_{\sharp\rho} - 2i\Gamma_{\sharp\bar{\rho}} + \frac{i}{\sqrt{2}}\big(e^{i\zeta}\epsilon_{\rho}{}^{\bar{\alpha}\bar{\beta}}\Gamma_{\bar{\alpha}\bar{\beta}} - e^{-i\zeta}\epsilon_{\bar{\rho}}{}^{\alpha\beta}\Gamma_{\alpha\beta}\big)\}
\end{aligned}
\tag{5.15}
$$

for $\rho = 1, 2, 3$, and we have set $\lambda = e^{i\zeta} w$ for real ζ.

In the $SU(3)$ stabilized case it is particularly useful to consider the complex $SU(3)$ invariant spinor bilinear scalar $\langle \psi_+, \Gamma_{\bar{1}} \psi_+ \rangle = 2\sqrt{2} w \lambda$. There are various different cases, corresponding to whether this scalar vanishes, or it does not vanish:

(i) $w \neq 0,\ \lambda = 0$,

(ii) $\lambda \neq 0,\ w = 0$,

(iii) $w \neq 0$, $\lambda \neq 0$.

In fact, it is straightforward to see that the spinors associated with cases (i) and (ii) above are related by a $Pin(7)$ transformation. To see this, consider the spinor from case (ii),

$$\psi_+ = \lambda \mathrm{e}_1 + \bar{\lambda} \mathrm{e}_{234} \ . \tag{5.16}$$

The Spin(7) gauge transformation generated by L_3 produces a $SO(2)$ which acts transitively on $\{e_1 + e_{234}, i(e_1 - e_{234})\}$, and hence without loss of generality we can set $\psi_+ = \lambda(e_1 + e_{234})$ for $\lambda \in \mathbb{R}$. Next, note that

$$\Gamma_{234}(e_1 + e_{234}) = -(1 + e_{1234}) \ . \tag{5.17}$$

It therefore follows that the spinor ψ_+ in case (ii) is Spin(7) gauge-equivalent to a spinor which in turn is Pin-equivalent, with respect to $\Gamma_{234} \in Pin(7)$, to the spinor in case (i). The effect of the Γ_{234} transformation is to flip holomorphic with anti-holomorphic directions and to reflect along the # direction, namely

$$\alpha \ \to \ \bar{\alpha} \qquad , \qquad \# \ \to \ -\# \ . \tag{5.18}$$

It is therefore sufficient to consider spinors ψ_+ corresponding to the G_2 stabilizer case, and the two $SU(3)$ stabilizer cases $(i), (iii)$. Having determined the stabilizers associated with these three canonical types of spinors, we next proceed to obtain a linear system of equations by substituting these expressions for ψ_+ into (4.37). The linear system consists of relations between the flux and spin-connection, which when covariantized with respect to the appropriate stabilizer group, give rise to conditions on the flux X and the geometry of the internal manifold M_7. In the following sections, we shall present the covariant solution of the linear system for each of the stabilizer subgroups.

CHAPTER 6

SUMMARY OF SOLUTIONS OF THE LINEAR SYSTEM

In this chapter, we shall solve the linear system obtained from the Killing spinor equations. In particular, we shall show that the linear system implies that there are no Killing spinors for which the stabilizer of ψ_+ is G_2. For the case of a SU(3) stabilizer subgroup, the Killing spinor equations determine all components of the 4-form flux in terms of the geometry of the internal manifold, and we determine the geometric conditions and the components of the flux, written in a SU(3) covariant fashion.

6.1. SU(3) Invariant Spinor

In this section, we solve the KSEs (4.37) when the stabilizer of ψ_+ is $SU(3)$, corresponding to

$$\psi_+ = w(1 + e_{1234}) + \lambda e_1 + \bar{\lambda} e_{234} \qquad w \in \mathbb{R},\ \lambda \in \mathbb{C} \ , \tag{6.1}$$

for $w^2 - |\lambda|^2 \neq 0$. We begin by considering the case for which both w and λ are non-vanishing. Furthermore, we will write $\lambda = \rho e^{i\theta}$, where $\rho > 0$ and $\theta \in [0, 2\pi[$ are two real spacetime functions. The associated linear system and the components of the flux are presented in Appendix E. The linear system is initially expressed non-covariantly in terms of SU(3)-components of the spin-connection and the fluxes, but, it can be rewritten in SU(3)-covariant form by using the SU(3) gauge invariant bilinears. In Appendix F we set out the main relations which are used to write the relations in a manifestly $SU(3)$ covariant fashion, in terms of the following $SU(3)$ invariant bilinears:

$$\xi \equiv e^{\#} \ , \qquad \omega \equiv -i\delta_{\alpha\bar{\beta}} e^{\alpha} \wedge e^{\bar{\beta}} \ , \qquad \chi \equiv \frac{1}{3!}\epsilon_{\alpha\beta\gamma} e^{\alpha} \wedge e^{\beta} \wedge e^{\gamma} \ . \tag{6.2}$$

The above forms are obtained from the following $SU(3)$-invariant spinor bilinears:

$$\langle \psi_+, \Gamma_a \psi_+ \rangle e^a = -2(w^2 - |\lambda|^2)\xi \tag{6.3}$$

$$\frac{1}{2!}\langle \psi_+, \Gamma_{ab}\tilde{\Gamma}^4 \psi_+ \rangle e^a \wedge e^b = -2(w^2 - |\lambda|^2)\omega \tag{6.4}$$

$$\begin{aligned} \frac{1}{3!}\langle \psi_+, (\Gamma_1 + \Gamma_{\bar{1}})\Gamma_{abc} \psi_+ \rangle e^a \wedge e^b \wedge e^c &= 2iw(\bar{\lambda} - \lambda)\xi \wedge \omega \\ &+ 4(w^2 - \lambda^2)\chi + 4(w^2 - \bar{\lambda}^2)\bar{\chi} \end{aligned} \tag{6.5}$$

where $a, b, c = \alpha, \bar{\alpha}, \#$. There are also several $Spin(7)$ invariant scalar bilinears, such as

$$\begin{aligned} \langle \psi_+, \psi_+ \rangle &= 2(w^2 + |\lambda|^2) \\ \langle \psi_+, (\Gamma_1 + \Gamma_{\bar{1}})\psi_+ \rangle &= 2\sqrt{2}w(\lambda + \bar{\lambda}) \\ \langle \psi_+, i(\Gamma_1 - \Gamma_{\bar{1}})\psi_+ \rangle &= 2\sqrt{2}iw(-\lambda + \bar{\lambda}) \ . \end{aligned} \tag{6.6}$$

When we later covariantize the conditions on the flux and the geometry, there are various polynomials in w^2 and $|\lambda|^2$ which can be rewritten in a manifestly gauge-invariant way in terms of these gauge-invariant spinor bilinears. In terms of spinor bilinears, a complete set of gauge invariant spinor bilinears completely encodes the algebraic properties of the spinor, modulo appropriate gauge transformations. This was utilized in the initial classification of supergravity solutions [19–21] prior to the development of spinorial geometry techniques. We remark that (6.3), (6.4), (6.5) and (6.6) do not constitute a full set of spinor bilinears.

In constructing the solution to the linear system (E.2)-(E.13), it is convenient to make use of the two Lee forms built from χ, and ω, which are

$$Z_i \equiv (\nabla^j \chi_{jkl})\bar{\chi}^{kl}{}_i \ , \qquad W_i \equiv (\nabla^j \omega_{jk})\omega^k{}_i \ . \tag{6.7}$$

Furthermore, we use $\mathcal{L}_\xi \Upsilon$ to denote the Lie derivative of Υ along the vector field which is dual with respect to the metric on M_7 to the 1-form ξ.

After some computation, the SU(3)-covariant conditions involving the warp factor A and the geometry of M_7 are as follows (here $i, j, k = \#, \alpha, \bar{\alpha}$ are frame indices on M_7):

$$\nabla^i \xi_i = -\frac{6\xi^j}{(w^2 - |\lambda|^2)^2}\Big[w(2w^2 + 3|\lambda|^2)(\mathrm{d}w)_j + (3w^2 + 2|\lambda|^2)\mathfrak{Re}(\lambda \mathrm{d}\bar{\lambda})_j\Big] \tag{6.8}$$

$$\nabla^i \xi^j \omega_{ij} = -cA^{-4} - 6w^2 \frac{\xi^k \mathfrak{Im}(\lambda \mathrm{d}\bar{\lambda})_k}{w^4 - |\lambda|^4} \tag{6.9}$$

$$\begin{aligned} \chi_{ijk}(\mathcal{L}_\xi \bar{\chi})^{ijk} &= -6\xi^l\Big[\bar{\lambda}\frac{(2w^4 + 2|\lambda|^4 + 11w^2|\lambda|^2)}{(w^2 + |\lambda|^2)(w^2 - |\lambda|^2)^2}(\mathrm{d}\lambda)_l \\ &+ \lambda\frac{(7w^4 + 4|\lambda|^4 + 4w^2|\lambda|^2)}{(w^2 + |\lambda|^2)(w^2 - |\lambda|^2)^2}(\mathrm{d}\bar{\lambda})_l \\ &+ 3w\frac{(2w^2 + 3|\lambda|^2)}{(w^2 - |\lambda|^2)^2}(\mathrm{d}w)_l\Big] - 4icA^{-4} \end{aligned} \tag{6.10}$$

$$\begin{aligned} i\chi_{ijk}(\mathrm{d}\omega)^{ijk} &= 9\sqrt{2}\xi^l\Big[\bar{\lambda}\frac{(9w^2 + |\lambda|^2)}{(w^2 - |\lambda|^2)^2}(\mathrm{d}w)_l + w\frac{(w^2 + 4|\lambda|^2)}{(w^2 - |\lambda|^2)^2}(\mathrm{d}\bar{\lambda})_l \\ &+ 5\frac{\bar{\lambda}^2 w}{(w^2 - |\lambda|^2)^2}(\mathrm{d}\lambda)_l\Big] \end{aligned} \tag{6.11}$$

$$(\mathrm{d}\xi)^{ij}\chi_{ijk} = \frac{\sqrt{2}}{(w^2 - |\lambda|^2)}\left[w\frac{\bar{\lambda}}{\lambda}(\mathrm{d}\lambda)_k - w(\mathrm{d}\bar{\lambda})_k\right] \tag{6.12}$$

$$Z = \frac{4}{w^2 - |\lambda|^2}\left[w^2 \mathrm{d}\log\rho + w^2 i_{\mathrm{d}\theta} w - \frac{|\lambda|^2}{w}\mathrm{d}w + \xi\left(\frac{|\lambda|^2}{w}(\mathrm{d}w)_j - w^2(\mathrm{d}\log\rho)_j\right)\xi^j\right] \tag{6.13}$$

$$\begin{aligned} W &= -\frac{1}{3}\frac{1}{w^2 - |\lambda|^2}\Big[\frac{1}{w}(w^2 - 4|\lambda|^2)\left(\mathrm{d}w - \xi(\mathrm{d}w)_j\xi^j\right) \\ &+ (5w^2 + 4|\lambda|^2)\left(\mathrm{d}\log\rho + i_{\mathrm{d}\theta}w - \xi(\mathrm{d}\log\rho)_j\xi^j\right)\Big] \end{aligned} \tag{6.14}$$

$$\mathcal{L}_\xi \xi = \frac{1}{3}\Big[\frac{1}{w}(7w^2+2|\lambda|^2)\left(\mathrm{d}w - \xi(\mathrm{d}w)_j\xi^j\right) + (w^2+2|\lambda|^2)\left(\xi(\mathrm{d}\log\rho)_j\xi^j - \mathrm{d}\log\rho - i_{\mathrm{d}\theta}w\right)\Big] \quad (6.15)$$

$$\chi_{ml[i}(\mathrm{d}\xi)^{mn}\bar{\chi}_{j]n}{}^{l} = \frac{1}{3}cA^{-4}\omega_{ij}\frac{(w^2+|\lambda|^2)}{(w^2-|\lambda|^2)} + 2\omega_{ij}\frac{w^2}{(w^4-|\lambda|^4)}\xi^k\mathfrak{Im}(\lambda\mathrm{d}\bar{\lambda})_k \quad (6.16)$$

$$\begin{aligned} &8|\lambda|^2(5w^2+|\lambda|^2)\mathrm{d}w + 4w\mathrm{d}\rho^2(w^2+5|\lambda|^2) + 8w\rho^2\, i_{\mathrm{d}\theta}\omega(w^2-|\lambda|^2) \\ &-\xi^i\left[8|\lambda|^2(5w^2+|\lambda|^2)\mathrm{d}w + 4w\mathrm{d}\rho^2(w^2+5|\lambda|^2) + 8w\rho^2\, i_{\mathrm{d}\theta}\omega(w^2-|\lambda|^2)\right]_i = 0 \ . \end{aligned} \quad (6.17)$$

We also obtain a $SU(3)$ invariant expression for the flux X. In general, any real 4-form on M_7 can be written as

$$X = \mathrm{e}^\# \wedge Y + \omega\wedge\sigma + \beta\wedge\chi + \bar{\beta}\wedge\bar{\chi} + X^{\mathrm{TT}} \quad (6.18)$$

where

- σ is a real two-form;
- β is a complex one-form, and $\bar{\beta}$ is its complex conjugate;
- Y is real 3-form;
- X^{TT} is the traceless (2,2)-part of the flux.

We remark that X^{TT} is the only part of the flux that is not fixed by the linear system. However, a traceless $(2,2)$ 4-form in 6 dimensions vanishes identically. To see this, note that X^{TT} is dual (in 6 dimensions) to a (1,1) 2-form R, $R = *_6 X^{\mathrm{TT}}$. Furthermore, by definition

$$\begin{aligned} R_{\alpha\bar{\beta}} &= \frac{1}{4!}\varepsilon_{\alpha\bar{\beta}}{}^{b_1b_2b_3b_4}X^{\mathrm{TT}}_{b_1b_2b_3b_4} = \frac{1}{4}\varepsilon_{\alpha\bar{\beta}}{}^{\mu_1\mu_2\bar{\nu}_1\bar{\nu}_2}X^{\mathrm{TT}}_{\mu_1\mu_2\bar{\nu}_1\bar{\nu}_2} \\ &= \frac{i}{4}\epsilon_\alpha{}^{\bar{\nu}_1\bar{\nu}_2}\varepsilon_{\bar{\beta}}{}^{\mu_1\mu_2}X^{\mathrm{TT}}_{\mu_1\mu_2\bar{\nu}_1\bar{\nu}_2} = \frac{i3!}{4}\delta^{\beta\mu_1\mu_2}_{\alpha\nu_1\nu_2}X^{\mathrm{TT}}_{\mu_1\mu_2}{}^{\nu_1\nu_2} = 0 \end{aligned} \quad (6.19)$$

as the contribution from trace terms in the final term vanishes. Hence R vanishes identically, and so $X^{\mathrm{TT}} = 0$.

It follows that the flux can be written as

$$X = e^{\#} \wedge Y + \omega \wedge \sigma + \beta \wedge \chi + \bar{\beta} \wedge \bar{\chi} \tag{6.20}$$

where all of these terms are fixed by the Killing spinor equations. In particular, the components of the real 2-form σ and of the complex 1-form β are given by

$$\begin{aligned} \sigma_{ij} &= -\frac{(w^2-|\lambda|^2)}{(w^2+|\lambda|^2)}\left(\mathcal{L}_\xi \omega\right)_{ij} \\ &- 2\omega_{ij}\frac{\xi^k}{(w^2-|\lambda|^2)(w^2+|\lambda|^2)}\left[w(w^2+2|\lambda|^2)(\mathrm{d}w)_k + (2w^2+|\lambda|^2)\mathfrak{Re}(\lambda \mathrm{d}\bar{\lambda})_k\right] \end{aligned} \tag{6.21}$$

$$\begin{aligned} \beta_i &= -3\sqrt{2}\frac{\lambda w}{(w^2-|\lambda|^2)}\left[\left(\mathcal{L}_\xi \xi\right)_i + i\left(\mathcal{L}_\xi \xi\right)_j \omega^j{}_i\right] + \frac{3}{2}\frac{(w^2+|\lambda|^2)}{(w^2-|\lambda|^2)}(\mathrm{d}\xi)^{kj}\bar{\chi}_{kji} \\ &+ i\frac{(w^4+4w^2|\lambda|^2+|\lambda|^4)}{(w^2-|\lambda|^2)(w^2+|\lambda|^2)}\left(\mathcal{L}_\xi \omega\right)^{kj}\bar{\chi}_{kji}\ . \end{aligned} \tag{6.22}$$

The real 3-form Y has components

$$Y_{ijk} = q\omega_{[i}{}^l(\mathrm{d}\omega)_{jk]l} + (\omega \wedge V)_{ijk} + h\chi_{ijk} + \bar{h}\bar{\chi}_{ijk} \tag{6.23}$$

where q and h are functions, and V is a real one-form, given by:

$$q = -3\frac{(w^2-|\lambda|^2)}{(w^2+|\lambda|^2)} \tag{6.24}$$

$$V_i = \frac{(w^2-|\lambda|^2)}{(w^2+|\lambda|^2)}\omega_i{}^j(\mathcal{L}_\xi \xi)_j \tag{6.25}$$

$$\begin{aligned} h &= \frac{3\sqrt{2}}{2}\frac{\xi^k}{(w^2-|\lambda|^2)}\left[\lambda(\mathrm{d}w)_k + \frac{w^3}{(w^2+|\lambda|^2)}(\mathrm{d}\lambda)_k + \frac{w\lambda^2}{(w^2+|\lambda|^2)}(\mathrm{d}\bar{\lambda})_k\right] \\ &- q\frac{i}{6}\bar{\chi}^{ijk}(\mathrm{d}\omega)_{ijk}\ . \end{aligned} \tag{6.26}$$

6.1.1. SU(3) Invariant Spinor with $\lambda = 0$, $w \neq 0$

Next, we consider the special case of the $SU(3)$ invariant spinor

$$\psi_+ = w(1 + e_{1234}) \qquad w \in \mathbb{R}, \quad w \neq 0\ . \tag{6.27}$$

The $SU(3)$ covariant geometric conditions involving the warp factor and the geometry of M_7 which are obtained from the linear system are:

$$(\mathrm{d}\omega)^{(3,0)} = (\mathrm{d}\omega)^{(0,3)} = 0 \tag{6.28}$$

$$\mathrm{d}(w^4\xi) = -\frac{c}{3}A^{-4}\omega w^4 \tag{6.29}$$

$$\bar{\chi}^{ijk}\left(\mathcal{L}_\xi\omega\right)_{jk} = 0 \tag{6.30}$$

$$Z_i = -20(w^{-1}\mathrm{d}w)_i + 20\xi_i\xi^k(w^{-1}\mathrm{d}w)_k \tag{6.31}$$

$$W_i = 8(w^{-1}\mathrm{d}w)_i - 8\xi_i\xi^k(w^{-1}\mathrm{d}w)_k \tag{6.32}$$

$$\mathfrak{Im}\left(\bar{\chi}^{ijk}(\mathcal{L}_\xi\chi)_{ijk}\right) - 4cA^{-4} = 0 \tag{6.33}$$

$$\nabla^i\xi_i = -12\xi^k w^{-1}(\mathrm{d}w)_k \tag{6.34}$$

$$\nabla^i\xi^j\omega_{ij} = -cA^{-4}\ . \tag{6.35}$$

The flux X can be expressed as

$$X = \mathrm{e}^{\#}\wedge Y + \omega\wedge\sigma \tag{6.36}$$

with

$$\sigma = -w^{-2}\mathcal{L}_\xi(w^2\omega) \tag{6.37}$$

and

$$Y_{ijk} = -3\omega_{[i}{}^{l}(\mathrm{d}\omega)_{jk]l} + (\omega\wedge V)_{ijk} \tag{6.38}$$

where

$$V_i = \omega_i{}^{j}(\mathcal{L}_\xi\xi)_j\ . \tag{6.39}$$

6.2. G_2 Invariant Spinor

In this section, we shall consider the case when the stabilizer of ψ_+ is G_2, corresponding to the case

$$\psi_+ = w(1 + e_{1234}) + \lambda e_1 + \bar{\lambda} e_{234} \qquad w \in \mathbb{R},\ \lambda \in \mathbb{C} \ , \tag{6.40}$$

with $w^2 = |\lambda|^2$. We shall show that this orbit admits no solutions to the Killing spinor equations and the bosonic field equations. To establish this result, we set $\lambda \equiv e^{i\zeta} w$, where ζ is a real function. The geometric conditions we obtained by solving the linear system are:

$$\mathrm{d}w = \mathrm{d}\zeta = 0 \tag{6.41}$$

$$\Omega_{\mu,\alpha}{}^{\alpha} = 0 \tag{6.42}$$

$$\Omega_{\#,\alpha}{}^{\alpha} = i\frac{c}{6}A^{-4} \tag{6.43}$$

$$\Omega_{\#,\alpha\beta}\epsilon^{\alpha\beta\gamma} = \sqrt{2}\, e^{i\zeta}\, \Omega_{\#,\#}{}^{\gamma} \tag{6.44}$$

$$\Omega_{\bar{\mu},\alpha\beta}\epsilon^{\alpha\beta\gamma} = \sqrt{2}\, e^{i\zeta}\, \Omega_{\bar{\mu},\#}{}^{\gamma} \tag{6.45}$$

$$2\Omega_{\bar{\mu}}{}^{\gamma\rho} - \sqrt{2}\, e^{-i\zeta}\epsilon^{\gamma\rho\alpha}\Omega_{\bar{\mu},\#\alpha} + i\sqrt{2}\,\frac{c}{6}e^{-i\zeta}A^{-4}\epsilon_{\bar{\mu}}{}^{\gamma\rho} = 0 \ . \tag{6.46}$$

Furthermore, we find that all of the components of the flux X vanish,

$$X = 0 \ . \tag{6.47}$$

As $X = 0$, the integrability condition $\Gamma^j\,[\nabla_i, \nabla_j]\,\psi_+$ from (4.11) implies that

$$\Big[-\frac{1}{2}\nabla_i\nabla_k A\Gamma^k + \frac{c}{2}A^{-4}\nabla_i A\,\tilde{\Gamma}^4 - \frac{c}{12}A^{-4}\nabla_k A\Gamma^k{}_i\,\tilde{\Gamma}^4\Big]\psi_+ = 0 \ . \tag{6.48}$$

Multiplying (6.48) by Γ_l, we find

$$\Big[-\frac{1}{2}\nabla_i\nabla_k A(\delta_l^k + \Gamma_l{}^k) + \frac{c}{2}A^{-4}\nabla_i A\Gamma_l\,\tilde{\Gamma}^4 \\ -\frac{c}{12}A^{-4}\nabla_k A(\Gamma_l{}^k{}_i + \delta_l^k\Gamma_i - \delta_{li}\Gamma^k)\,\tilde{\Gamma}^4\Big]\psi_+ = 0 \ . \tag{6.49}$$

We next take the inner product of (6.49) with ψ_+, noting that the anti-hermitian terms vanish identically as ψ_+ is Majorana. The hermitian part gives

$$\langle\psi_+, \Big[-\frac{1}{2}\nabla_i\nabla_l A - \frac{c}{12}A^{-4}\nabla_k A\Gamma_l{}^k{}_i\tilde{\Gamma}^4\Big]\psi_+\rangle = 0 \ . \tag{6.50}$$

The symmetric part of (6.50) then gives

$$\nabla_i\nabla_l A||\psi_+||^2 = 0 \quad \Rightarrow \quad \nabla_i\nabla_l A = 0 \ , \tag{6.51}$$

and the antisymmetric part of (6.50) implies

$$-\frac{c}{12}A^{-4}\nabla_k A\langle\psi_+, \Gamma_l{}^k{}_i\tilde{\Gamma}^4\psi_+\rangle = 0 \ . \tag{6.52}$$

The 3-form spinor bilinear in (6.52) is proportional to the G_2-invariant 3-form φ given by

$$\varphi = e^{\#}\wedge\omega - 2i\sqrt{2}\,\mathfrak{Im}(e^{i\theta}\chi) \ , \tag{6.53}$$

$$\varphi_{\#\alpha\bar{\beta}} \equiv -i\delta_{\alpha\bar{\beta}} \qquad\qquad \varphi_{\alpha\beta\gamma} = -i\sqrt{2}\,e^{i\theta}\epsilon_{\alpha\beta\gamma} = (\varphi_{\bar{\alpha}\bar{\beta}\bar{\gamma}})^* \ . \tag{6.54}$$

Hence (6.52) implies that

$$\varphi_{lik}\nabla^k A = 0 \tag{6.55}$$

which in turn implies that $dA = 0$, so A is constant. However, from the Einstein field equation (3.36) we obtain

$$3KA^{-1} - \nabla_i\nabla^i A - 3A^{-1}\nabla_i A\nabla^i A + \frac{1}{3}c^2A^{-7} + \frac{A}{144}X^2 = 0 \ . \tag{6.56}$$

It is clear that this equation admits no solution in the case for which A is constant and $X = 0$, as the LHS is strictly positive. Therefore, we conclude that there are no supersymmetric warped product dS_4 solutions for which the spinor ψ_+ is G_2 invariant.

It follows that all warped product dS_4 must lie within the SU(3) cases. We have previously considered the maximally supersymmetric solution $AdS_7\times S^4$ as the near-horizon M5-brane limit, and showed that it is a warped product dS_4 solution. In fact, the half-supersymmetric M5-brane geometry is also another example of a warped product dS_4 solution. To see this, note that one can write $\mathbb{R}^{1,4}$ as a warped product dS_4 solution as follows:

$$ds^2(\mathbb{R}^{1,4}) = \frac{z^2}{\mathcal{R}^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{K}dz^2 \tag{6.57}$$

where $\mathcal{R}$ is given by (3.2). It is straightforward to show that the Riemann curvature tensor of (6.57) vanishes, so the metric is locally isometric to that on $\mathbb{R}^{1,4}$. On embedding $\mathbb{R}^{1,4}$ into the $\mathbb{R}^{1,5}$ which appears in the M5-brane metric, it follows that the half-supersymmetric M5-brane also is warped product dS_4 solution. Furthermore, one can also write AdS_6 as a warped product $\mathbb{R}^{1,4}$ geometry, and consequently also as a warped product dS_4 solution. Consequently, the warped product AdS_6 solutions found in [61] also provide examples of warped product dS_4 solutions preserving $N = 16$ supersymmetry. We remark that although a non-existence theorem for warped product AdS_6 solutions in $D = 11$ supergravity was established in [62], this theorem assumes a smooth warp factor and a smooth and compact without boundary internal space. For warped product dS_4 solutions we do not assume such global properties, as we wish to evade the no-go theorems. Hence warped product AdS_6 solutions, which are also warped product dS_4 solutions, can arise in this context.

CHAPTER 7

CONCLUSIONS AND FUTURE WORK

We have obtained the necessary and sufficient conditions for warped product $dS_4 \times_w M_7$ solutions in $D = 11$ supergravity to preserve the minimal $N = 8$ supersymmetry. To do this, we first integrated explicitly the gravitino equation along the dS_4 directions. This reduces the conditions imposed by supersymmetry to a gravitino-type equation on M_7 acting on a Majorana spinor ψ_+, whose components depend only on the co-ordinates of M_7. Using spinorial geometry techniques, the spinor ψ_+ was then simplified to two possible canonical forms by Spin(7) gauge transformations. These two canonical forms have stabilizer subgroups corresponding to G_2 and $SU(3)$. In the G_2 case, we show that there is no solution to the Killing spinor equations. For the $SU(3)$ case we have determined the 4-form flux in terms of $SU(3)$ invariant geometric structures on M_7 (6.20)-(6.26) (6.36)-(6.39), as well as determining all of the conditions imposed on the geometry of M_7 (6.8)-(6.17) (6.28)-(6.35). In particular, the geometric conditions, a priori, appear rather weak, though in the case of the $SU(3)$ invariant spinor with $\lambda = 0$, the conditions on the $SU(3)$ structure simplify somewhat, to imply for example that $d\omega$ has vanishing $(3, 0)$ and $(0, 3)$ components.

Having obtained these conditions for the $N = 8$ solutions, it would be interesting to further investigate the resulting (local) conditions on the geometry. It would be useful to determine if they could be used, for example, to construct some useful set of co-ordinates for M_7. It would also be interesting to consider the $N = 16$ case, as well as the $N = 24$ and $N = 32$ cases. In particular, for the latter two cases of $N = 24$ and $N = 32$ supersymmetry, it is possible to find further conditions on such solutions utilizing the homogeneity theorem analysis constructed in [63]. To proceed with this, suppose that there we have N linearly independent solutions $\{\psi^r : r = 1, \dots, N\}$ for $N = 24$ or $N = 32$ to the gravitino equation (4.37). We then consider the integrability condition (4.36), which implies

$$\left(|\nabla A|^2 - K - \frac{c^2}{9} A^{-6} - \frac{A^2}{(144)^2} \not{X}^2 + \frac{2}{3} c A^{-3} \nabla_i A \Gamma^i \tilde{\Gamma}^4 - \frac{1}{18} A \nabla_i A \not{X}^i \right) \psi^s = 0 \ . \quad (7.1)$$

This implies that

$$\begin{aligned} \langle \psi^r, \Gamma_0 \tilde{\Gamma}^4 \Big(& |\nabla A|^2 - K - \frac{c^2}{9} A^{-6} - \frac{A^2}{(144)^2} \not{X}^2 \\ & + \frac{2}{3} c A^{-3} \nabla_i A \Gamma^i \tilde{\Gamma}^4 - \frac{1}{18} A \nabla_i A \not{X}^i \Big) \psi^s \rangle = 0 \end{aligned} \quad (7.2)$$

and hence

$$c \langle \psi^r, \Gamma_0 \Gamma^i \psi^s \rangle \nabla_i A = 0 \ . \quad (7.3)$$

On defining vector fields $\Theta_i^{rs} = \langle \psi^r, \Gamma_0 \Gamma_i \psi^s \rangle$, this implies

$$c \mathcal{L}_{\Theta^{rs}} A = 0 \ . \quad (7.4)$$

For $N = 24$ and $N = 32$ solutions, it follows from the homogeneity theorem analysis of [63] that the Θ^{rs} span pointwise the tangent space of M_7, and hence

$$c dA = 0 \ . \quad (7.5)$$

If $c \neq 0$, then this implies that $dA = 0$. However, (6.56) implies that there are no solutions for which A is constant. Hence, for $N = 24$ or $N = 32$ solutions, we must take $c = 0$.

This determines all possible $N = 32$ warped product dS_4 solutions. From [58], where all maximally supersymmetric solutions in $D = 11$ supergravity were determined, the maximally supersymmetric solutions are $\mathbb{R}^{1,10}$ with $F = 0$; $AdS_4 \times S^7$ with 4-form F proportional to the volume form of AdS_4, $AdS_7 \times S^4$, with 4-form F proportional to the volume form of S^4, and a maximally supersymmetric plane wave solution which has $F \neq 0$, but $F^2 = 0$. In terms of possible $N = 32$ warped product dS_4 solutions, the condition $c = 0$ implies that $F^2 \geqslant 0$ with equality if and only if $F = 0$. Hence we exclude $AdS_4 \times S^7$ and the maximally supersymmetric plane wave as $N = 32$ warped product dS_4 solutions, because the $AdS_4 \times S^7$ solution has $F^2 < 0$, and the maximally supersymmetric plane wave solution has $F \neq 0$, but $F^2 = 0$. It follows that the $N = 32$ warped product dS_4 solutions are $\mathbb{R}^{1,10}$ and $AdS_7 \times S^4$. In particular, it is possible to explicitly write both $\mathbb{R}^{1,4}$ and AdS_7 as warped product dS_4 geometries, as in (6.57) and also [40]. It would be interesting to further understand the possible $N = 16$ and $N = 24$ warped product dS_4 solutions, though the homogeneity theorem does not apply to the $N = 16$ solutions.

APPENDICES

APPENDIX A

CONVENTIONS

We use the mostly plus sign signature $\eta = \mathrm{diag}(-,+,\ldots,+)$. The gamma matrices satisfy

$$\{\Gamma_A, \Gamma_B\} = 2g_{AB} \ . \tag{A.1}$$

In these conventions, we take

$$\Gamma_{0123456789\#} = \mathbb{I} \ , \tag{A.2}$$

and consequently the following duality relation holds

$$\Gamma_{A_1\ldots A_p} = (-1)^{\frac{(p+1)(p-2)}{2}} \frac{1}{(11-p)!} \varepsilon_{A_1\ldots A_p}{}^{A_{p+1}\ldots A_{11}} \Gamma_{A_{p+1}\ldots A_{11}} \ , \tag{A.3}$$

where

$$\varepsilon_{0123456789\#} = +1 \ . \tag{A.4}$$

The Hodge star of a p-form ω is defined by

$$*\omega_{A_1\ldots A_{11-p}} = \frac{1}{p!} \varepsilon_{A_1\ldots A_{11-p}}{}^{B_1\ldots B_p} \omega_{B_1\ldots B_p} \ . \tag{A.5}$$

For every k-form ω, one can define a Clifford algebra element $\not{\omega}$ given by

$$\not{\omega} \equiv \omega_{A_1\ldots A_k} \Gamma^{A_1\ldots A_k} \ . \tag{A.6}$$

In addition, one can define

$$\not{\omega}_C \equiv \omega_{CA_1\ldots A_k} \Gamma^{A_1\ldots A_k} \ , \quad \text{and} \quad \not{\Gamma}\not{\omega}_C \equiv \Gamma_{CA_1\ldots A_k} \omega^{A_1\ldots A_k} \ . \tag{A.7}$$

APPENDIX B

SPINORS FROM FORMS

The Majorana representation of $Spin(10,1)$ can be constructed from the $Spin(9,1)$ spinor representations and then adding the tenth gamma matrix $\Gamma_\#$. This construction is derived in an explicit representation, in terms of differential forms, in [47,48], see also [55,56]. We take the space U of 1-forms on $\mathbb{R}^5$, with basis $\{e_1,\dots,e_5\}$. The space of Dirac spinors, $\Delta_c = \Lambda^*(U\otimes\mathbb{C})$, is identified with the complexified space of multi-forms constructed from this basis. Δ_c is equipped with a canonical Euclidean Hermitian inner product $\langle\ \cdot\ ,\ \cdot\ \rangle$

We then take the following representation for the gammma matrices:

$$\begin{aligned}
\Gamma_0\eta &= -e_5\wedge\eta + i_{e_5}\eta \quad \Gamma_5\eta = e_5\wedge\eta + i_{e_5}\eta \\
\Gamma_i\eta &= e_i\wedge\eta + i_{e_i}\eta \quad i=1,\dots,4 \\
\Gamma_{i+5}\eta &= i(e_i\wedge\eta - i_{e_i}\eta)
\end{aligned} \tag{B.1}$$

where $\eta\in\Delta_c$ and i_{e_i} is the inner derivative along the direction e_i. The tenth gamma matrix can be chosen as

$$\Gamma_\# = -\Gamma_{0123456789}\ . \tag{B.2}$$

One can verify that $\Gamma_\#^2 = \mathbb{I}$. The gamma matrices satisfy the Clifford Algebra, namely $\Gamma_A\Gamma_B + \Gamma_B\Gamma_A = 2\eta_{AB}\mathbb{I}$. The Hermitian inner product, acting only on 1-forms, is defined by

$$\langle z^a e_a, w^b e_b\rangle = (z^a)^*\eta_{ab}w^b\ , \tag{B.3}$$

and is then extended to the complexified space of multi-forms, Δ_c.

The gamma matrices are chosen such that Γ_0 is skew-hermitian and Γ_i, $i = 1, \ldots, 9$ are hermitian with respect to $\langle \cdot , \cdot \rangle$. The $Spin(10, 1)$ invariant Dirac inner product is defined as

$$D(\eta, \theta) = \langle \Gamma_0 \eta, \theta \rangle \ . \tag{B.4}$$

In eleven dimensions a spinor can be Majorana; the reality condition is

$$\eta^* = \Gamma_{6789} \eta \ , \tag{B.5}$$

where $C = \Gamma_{6789}$ is the charge conjugation matrix, and $C*$ commutes with the gamma matrices, i.e. $C * \Gamma_A = \Gamma_A C*$. The Dirac representation of $Spin$(10,1) admits an oscillator basis as

$$\begin{aligned} \Gamma_- &= \tfrac{1}{\sqrt{2}} (\Gamma_5 - \Gamma_0) = \sqrt{2}\, e_5 \wedge \qquad \Gamma_\alpha = \tfrac{1}{\sqrt{2}} (\Gamma_\alpha - i\Gamma_{\alpha+5}) = \sqrt{2}\, e_\alpha \wedge \\ \Gamma_+ &= \tfrac{1}{\sqrt{2}} (\Gamma_5 + \Gamma_0) = \sqrt{2}\, i_{e_5} \qquad \Gamma_{\bar\alpha} = \tfrac{1}{\sqrt{2}} (\Gamma_\alpha + i\Gamma_{\alpha+5}) = \sqrt{2}\, i_{e_\alpha} \end{aligned} \tag{B.6}$$

and $\Gamma_\sharp$ defined as in (B.2). In this oscillator basis, the gamma matrices satisfy the Clifford Algebra, $\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2\eta_{AB} \mathbb{I}$, with non-vanishing components are $\eta_{+-} = \eta_{\sharp\sharp} = 1$, $\eta_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$.

We note that $(\Gamma_+)^\dagger = \Gamma_-$ and $(\Gamma_\alpha)^\dagger = \Gamma_{\bar\alpha}$; $(\Gamma^+, \Gamma^{\bar\alpha})$ act as creation operators on the Clifford vacuum represented by the 0-degree form 1, where $\Gamma^A = \eta^{AB} \Gamma_B$. A general spinor ϵ can be written as

$$\epsilon = \sum_{k=0}^{5} \frac{1}{k!} \phi_{\bar a_1 \ldots \bar a_k} \Gamma^{\bar a_1 \ldots \bar a_k} 1 \quad , \quad \bar a = +, \bar\alpha \ . \tag{B.7}$$

APPENDIX C

USEFUL RELATIONS

In this appendix, some expressions used to compute Eqs. (4.9), (4.11), (4.38) are shown.

$$\begin{aligned} \Gamma_i \not{X} - \not{X}\Gamma_i &= 8\not{X}_i \,, && \text{(C.1)} \\ \not{X}_i\not{X} + \not{X}\not{X}_i &= -72\Gamma_j{}^{k_1k_2}X_i{}^{jl_1l_2}X_{l_1l_2k_1k_2} \,, && \text{(C.2)} \\ \not{\Gamma}\not{X}_i &= \Gamma_i\not{X} - 4\not{X}_i \,, && \text{(C.3)} \\ \not{X}^2 &= -72\,\Gamma^{l_1l_2k_1k_2}X_{l_1l_2j_1j_2}X^{j_1j_2}{}_{k_1k_2} + 24X^2 \,, && \text{(C.4)} \end{aligned}$$

$$\begin{aligned} \not{X}\not{\Gamma}\not{X}_i + \not{\Gamma}\not{X}_i\not{X} &= 240\left(\Gamma_{[i}X_{j_1j_2j_3j_4]}X^{j_1j_2j_3j_4} - \Gamma_{[ij_1j_2}{}^{l_1l_2}X_{j_3j_4]l_1l_2}X^{j_1j_2j_3j_4}\right) \\ &= 48\,\Gamma_iX^2 - 192\,\Gamma_{j_1}X_{ij_2j_3j_4}X^{j_1j_2j_3j_4} - 144\,\Gamma_{ij_1j_2}{}^{l_1l_2}X_{j_3j_4l_1l_2}X^{j_1j_2j_3j_4} \\ &+ 96\,\Gamma_{j_1j_2j_3}{}^{l_1l_2}X_{ij_4l_1l_2}X^{j_1j_2j_3j_4} \,, && \text{(C.5)} \end{aligned}$$

$$\begin{aligned} \Gamma_k\not{X}_i + \not{X}_i\Gamma_k &= 6\,\Gamma^{l_1l_2}X_{ikl_1l_2} \,, && \text{(C.6)} \\ \Gamma^k\not{\Gamma}\not{X}_i + \not{\Gamma}\not{X}_i\Gamma^k &= 10\Gamma_{[j_1j_2j_3j_4}\delta^k_{i]}X^{j_1j_2j_3j_4} = 2\delta^k_i\not{X} - 8\,\Gamma_{j_1j_2j_3i}X^{j_1j_2j_3k} \,, && \text{(C.7)} \\ \not{X}\Gamma^{lm} - \Gamma^{lm}\not{X} &= 16\,\Gamma_{j_1j_2j_3}{}^{[m}\delta^{l]}_{j_4}X^{j_1j_2j_3j_4} \,, && \text{(C.8)} \\ \not{X}\Gamma_j + \Gamma_j\not{X} &= 2\not{\Gamma}\not{X}_j \,, && \text{(C.9)} \end{aligned}$$

$$
\begin{aligned}
\not{\Gamma X}_l \not{\Gamma X}_k - \not{\Gamma X}_k \not{\Gamma X}_l \;&=\; 32\Gamma_{i_1 i_2 i_3}{}^{j_1 j_2 j_3} X_k{}^{i_1 i_2 i_3} X_{l j_1 j_2 j_3} - 96\Gamma_l{}^{i_1 i_2 j_2 j_3 j_3} X_{k j_1 i_1 i_2} X^{j_1}{}_{j_2 j_3 j_3} \\
&+\; 144\Gamma_{kl}{}^{i_1 i_2 j_2 j_3} X_{i_1 i_2 j_1 j_3} X^{j_1 j_3}{}_{j_2 j_3} - 96\Gamma_{k i_1 i_2 i_3}{}^{j_2 j_3} X^{i_1 i_2 i_3 j_1} X_{l j_1 j_2 j_3} \\
&-\; 48\Gamma_{kl} X^2 + 192\Gamma_l{}^{j_3} X_k{}^{j_1 j_2 j_3} X_{j_1 j_2 j_3 j_3} - 576\Gamma_{i_1}{}^{j_3} X^{k\, i_1 j_1 j_2} X_{l j_3 j_1 j_2} \\
&+\; 192\Gamma_{k i_1} X^{i_1 j_1 j_2 j_3} X_{l j_1 j_2 j_3} \;, \qquad \text{(C.10)}
\end{aligned}
$$

$$
\begin{aligned}
\not{\Gamma X}_l \Gamma_k - \Gamma_k \not{\Gamma X}_l \;&=\; -2\Gamma_{k l i_1 i_2 i_3 i_4} X^{i_1 i_2 i_3 i_4} \;, && \text{(C.11)} \\
\not{\Gamma X}_k \not{X}_l - \not{X}_l \not{\Gamma X}_k \;&=\; -72\Gamma_k{}^{i_1 i_2 j_3} X_{i_1 i_2 j_1 j_2} X_{l j_3}{}^{j_1 j_2} + 48\Gamma_{i_1 i_2 i_3 j_3} X^{i_1 i_2 i_3 j_1} X_{kl}{}^{j_3}{}_{j_1} \;, && \text{(C.12)} \\
\not{\Gamma X}_k \Gamma^{mn} - \Gamma^{mn} \not{\Gamma X}_k \;&=\; 4\not{\Gamma X}^{[n} \delta_k^{m]} + 16\Gamma_k{}^{[n}{}_{i_1 i_2 i_3} X^{m] i_1 i_2 i_3} \;, && \text{(C.13)} \\
\Gamma_k \not{X}_l - \not{X}_l \Gamma_k \;&=\; 2\Gamma_{k i_1 i_2 i_3} X_l{}^{i_1 i_2 i_3} \;, && \text{(C.14)} \\
\Gamma^{mn} \not{X}_k - \not{X}_k \Gamma^{mn} \;&=\; 12\Gamma^{[m}{}_{i_1 i_2} X_k{}^{n] i_1 i_2} \;, && \text{(C.15)} \\
\not{X}_l \not{X}_k - \not{X}_k \not{X}_l \;&=\; 2\Gamma^{i_1 i_2 i_3 j_1 j_2 j_3} X_{l i_1 i_2 i_3} X_{k j_1 j_2 j_3} - 36\Gamma_{i_1}{}^{j_3} X_l{}^{i_1 j_1 j_2} X_{k j_3 j_1 j_2} \;. && \text{(C.16)}
\end{aligned}
$$

APPENDIX D

DERIVATION OF EQUATION (4.23)

Given the spinor Φ defined in (4.22), we consider (4.23). In particular we begin by examining the following terms:

$$\nabla_i \Phi + k_1 [\text{Eq. } (4.11)] \ , \tag{D.1}$$

where k_1, k_2 are some constants to be determined. To begin with, note that the terms which are linear in X are:

$$\begin{aligned} & (1+k_1)\frac{A}{288}\nabla_i \not{X} + \frac{1}{576}\nabla_i A \not{X} + \frac{(12a+1-4k_1)}{12}\frac{1}{288} c A^{-3}\Gamma_i \not{X}\tilde{\Gamma}^4 \\ + & \frac{(5k_1-1-18a)}{432} c A^{-3}\not{X}_i \tilde{\Gamma}^4 - \frac{1}{576}\nabla^k A \Gamma_{ki}\not{X} + \frac{1}{48}\nabla_k A \Gamma^{kj_1j_2j_3} X_{ij_1j_2j_3} \\ + & \frac{(3+4k_1)}{48}\nabla^k A \Gamma^{ab} X_{ikab} + \frac{k_1}{72}\nabla_k A \Gamma_{ij_1j_2j_3} X^{kj_1j_2j_3} \ . \end{aligned} \tag{D.2}$$

In order to set to zero the term involving $\nabla_i \not{X}$, we set $k_1 = -1$. Having done so, we then consider imposing the condition

$$\begin{aligned} & \nabla_i \Phi - [\text{Eq. } (4.11)] + k_2 A^{-1}\Gamma^i [\text{Eq. } (4.9)] + q_1 \not{\Gamma X}_i \Phi + q_2 \not{X}_i \Phi \\ & + q_3 c A^{-4}\Gamma_i \tilde{\Gamma}^4 \Phi + q_4 A^{-1}\nabla_k A \Gamma_i \Gamma^k \Phi + q_5 A^{-1}\nabla_i A \Phi = 0 \ , \end{aligned} \tag{D.3}$$

and compute all of the terms on the LHS, choosing the constants $a, k_2, q_1, q_2, q_3, q_4, q_5$ so that the identity above holds. The terms involving the quadratic contribution of X are

$$\begin{aligned} & A\Big[\frac{(4k_2+1-288q_1)}{1152}\Gamma^{l_1 l_2}{}_{ij_1j_2}X_{j_3j_4l_1l_2}X^{j_1j_2j_3j_4} \\ + & \frac{(288q_1-72q_2-3)}{1728}\Gamma_{j_1j_2j_3}{}^{l_1l_2}X_{ij_4l_1l_2}X^{j_1j_2j_3j_4} \\ + & \frac{(8k_2-288q_1-72q_2-1)}{576}\Gamma^{abm}X_{impq}X^{pq}{}_{ab} - \frac{(1+4k_2-288q_1)}{12}\frac{1}{288}\Gamma_i X^2 \\ + & \frac{(3-288q_1+72q_2)}{864}\Gamma_{j_1}X_{ij_2j_3j_4}X^{j_1j_2j_3j_4}\Big] . \end{aligned} \tag{D.4}$$

The terms involving the linear contribution of X are

$$\begin{aligned} & \frac{(1-288q_1+2q_4+2q_5)}{576}\nabla_i A \not{X} + \frac{(12a+5+12q_3+3456aq_1)}{12}\frac{1}{288}cA^{-3}\Gamma_i\not{X}\tilde{\Gamma}^4 \\ + & \frac{(288q_1-2q_4-1)}{576}\nabla^k A\Gamma_{ki}\not{X} + \frac{(1+24q_2-96q_1)}{48}\nabla_k A\Gamma^{kj_1j_2j_3}X_{ij_1j_2j_3} \\ + & \frac{(8k_2-1-288q_1-72q_2)}{48}\nabla^k A\Gamma^{ab}X_{ikab} - \frac{(1+4k_2-288q_1)}{72}\nabla_k A\Gamma_{ij_1j_2j_3}X^{kj_1j_2j_3} \\ - & \frac{(6+18a+1728aq_1-432q_2a)}{432}cA^{-3}\not{X}_i\tilde{\Gamma}^4 \quad . \end{aligned} \tag{D.5}$$

The terms involving no contribution of X are

$$\begin{aligned} & \frac{1}{24}(1-16k_2-24aq_4+12q_3)cA^{-4}\nabla^k A\Gamma_k\Gamma_i\tilde{\Gamma}^4 \\ + & \frac{1}{12}(16k_2-7-36a-12q_3+24aq_4+12aq_5)cA^{-4}\nabla_i A\tilde{\Gamma}^4 \\ + & (k_2-\frac{1}{2}q_4)A^{-1}|\nabla A|^2\Gamma_i - k_2A^{-1}K\Gamma_i - \frac{1}{2}q_5A^{-1}\nabla_k A\nabla_i A\Gamma^k \\ - & \frac{1}{36}(3a+4k_2+36q_3a)c^2A^{-7}\Gamma_i . \end{aligned} \tag{D.6}$$

By requiring that all terms in the above expressions should vanish, we are able to determine the constant values, that are

$$a=-\frac{1}{6} \quad q_1=\frac{1}{288} \quad q_2=-\frac{1}{36} \quad q_3=-\frac{1}{12} \quad k_2=q_4=q_5=0 . \tag{D.7}$$

APPENDIX E

KSE LINEAR SYSTEM - SU(3) STABILIZER

The linear system associated to the KSEs (4.37), with the spinor given by

$$\psi_+ = w(1 + e_{1234}) + \lambda e_1 + \bar{\lambda} e_{234} \qquad w \in \mathbb{R},\ \lambda \in \mathbb{C} \tag{E.1}$$

is as follows:

$$\partial_\# w + \frac{w}{2}\Omega_{\#,\alpha}{}^{\alpha} + \frac{1}{24} w X_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} - i\frac{c}{12}A^{-4}w - \frac{\sqrt{2}}{3}\bar{\lambda}X_{\#234} = 0 \tag{E.2}$$

$$\partial_\# \lambda + \frac{\lambda}{2}\Omega_{\#,\alpha}{}^{\alpha} - \frac{1}{24}\lambda X_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} + \frac{\sqrt{2}}{3} w X_{\#234} - i\frac{c}{12}A^{-4}\lambda = 0 \tag{E.3}$$

$$w\Omega_{\#,\alpha\beta}\epsilon^{\gamma\alpha\beta} - \sqrt{2}\lambda\Omega_{\#,\#}{}^{\gamma} + \frac{\sqrt{2}}{3}\lambda X_{\#\alpha}{}^{\alpha\gamma} + \frac{w}{3}X^{\gamma}{}_{234} = 0 \tag{E.4}$$

$$\bar{\lambda}\epsilon^{\gamma\alpha\beta}\Omega_{\#\alpha\beta} - \sqrt{2}w\Omega_{\#,\#}{}^{\gamma} - \frac{w\sqrt{2}}{3}X_{\#\alpha}{}^{\alpha\gamma} - \frac{\bar{\lambda}}{3}X^{\gamma}{}_{234} = 0 \tag{E.5}$$

$$\partial_\mu w + \frac{1}{2}w\Omega_{\mu,\alpha}{}^{\alpha} - \frac{1}{4}wX_{\mu\#\alpha}{}^{\alpha} = 0 \tag{E.6}$$

$$\partial_\mu \lambda + \frac{1}{2}\lambda\Omega_{\mu,\alpha}{}^{\alpha} + \frac{1}{4}\lambda X_{\mu\#\alpha}{}^{\alpha} = 0 \tag{E.7}$$

$$\partial_\mu w - \frac{w}{2}\Omega_{\mu,\alpha}{}^{\alpha} - \frac{w}{12}X_{\#\mu\alpha}{}^{\alpha} - \frac{\sqrt{2}}{3}\lambda X_{\mu\bar{2}\bar{3}\bar{4}} = 0 \tag{E.8}$$

$$\partial_\mu \bar\lambda - \frac{\bar\lambda}{2}\Omega_{\mu,\alpha}{}^{\alpha} + \frac{1}{12}\bar\lambda X_{\#\mu\alpha}{}^{\alpha} + \frac{\sqrt{2}}{3} w X_{\mu\bar2\bar3\bar4} = 0 \tag{E.9}$$

$$w\Omega_{\bar\mu,\alpha\beta}\varepsilon^{\alpha\beta\gamma} - \sqrt{2}\lambda\Omega_{\bar\mu,\#}{}^{\gamma} - \frac{w}{2}\varepsilon^{\gamma\alpha\beta}X_{\bar\mu\#\alpha\beta} - \frac{w}{3}\varepsilon_{\bar\mu}{}^{\gamma\rho}X_{\#\rho\alpha}{}^{\alpha} + \frac{\sqrt{2}}{6}\lambda\varepsilon_{\bar\mu}{}^{\gamma\rho}X_{\rho\bar2\bar3\bar4} = 0 \tag{E.10}$$

$$\begin{aligned} w\Omega_{\bar\mu,}{}^{\gamma\rho} - \frac{\sqrt{2}}{2}\bar\lambda\varepsilon^{\gamma\rho\alpha}\Omega_{\bar\mu,\#\alpha} + \varepsilon_{\bar\mu}{}^{\gamma\rho}\Big[\frac{w}{6}X_{\#\bar2\bar3\bar4} - \frac{\bar\lambda\sqrt{2}}{24}X_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} \\ + i\frac{c\sqrt{2}}{12}\bar\lambda A^{-4}\Big] - \frac{\sqrt{2}}{4}\bar\lambda\varepsilon^{\gamma\rho\alpha}X_{\bar\mu\alpha\beta}{}^{\beta} = 0 \end{aligned} \tag{E.11}$$

$$\bar\lambda\Omega_{\bar\mu,\alpha\beta}\varepsilon^{\alpha\beta\gamma} - \sqrt{2}w\Omega_{\bar\mu,\#}{}^{\gamma} + \frac{\bar\lambda}{2}\varepsilon^{\gamma\alpha\beta}X_{\bar\mu\#\alpha\beta} + \frac{\bar\lambda}{3}\varepsilon_{\bar\mu}{}^{\gamma\rho}X_{\#\rho\alpha}{}^{\alpha} - \frac{\sqrt{2}}{6}w\varepsilon_{\bar\mu}{}^{\gamma\rho}X_{\rho\bar2\bar3\bar4} = 0 \tag{E.12}$$

$$\begin{aligned} \lambda\Omega_{\bar\mu,}{}^{\gamma\rho} - \frac{\sqrt{2}}{2}w\varepsilon^{\gamma\rho\alpha}\Omega_{\bar\mu,\#\alpha} + \varepsilon_{\bar\mu}{}^{\gamma\rho}\Big[\frac{\sqrt{2}w}{24}X_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} \\ + i\frac{c\sqrt{2}}{12}wA^{-4} - \frac{\lambda}{6}X_{\#\bar2\bar3\bar4}\Big] + \frac{\sqrt{2}}{4}w\varepsilon^{\gamma\rho\alpha}X_{\bar\mu\alpha\beta}{}^{\beta} = 0\ . \end{aligned} \tag{E.13}$$

E.1. Solution for $\lambda \neq 0$, $w \neq 0$

From the linear system (E.2)-(E.13), we find that the components of the flux are given by the following expressions

$$X_{\#\alpha}{}^{\alpha\gamma} = \frac{3}{w^2 - |\lambda|^2}\left[\sqrt{2}w\bar\lambda\varepsilon^{\gamma\alpha\beta}\Omega_{\#,\alpha\beta} - (w^2 + |\lambda|^2)\Omega_{\#,\#}{}^{\gamma}\right] \tag{E.14}$$

$$X_{\bar\gamma 234} = \frac{3}{w^2 - |\lambda|^2}\left[2\sqrt{2}\lambda w\Omega_{\#,\#}{}^{\gamma} - (w^2 + |\lambda|^2)\Omega_{\#,\alpha\beta}\varepsilon^{\gamma\alpha\beta}\right] \tag{E.15}$$

$$X_{\#234} = \frac{1}{w^2 - |\lambda|^2}\left[-\frac{3}{\sqrt{2}}\lambda\partial_\# w - \frac{3}{\sqrt{2}}w\partial_\#\lambda - \frac{3}{\sqrt{2}}w\lambda\Omega_{\#,\alpha}{}^{\alpha} + i\frac{c}{2\sqrt{2}}A^{-4}\lambda w\right] \tag{E.16}$$

$$\begin{aligned} X_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} &= \frac{1}{w^2 - |\lambda|^2}\big[-24w\partial_\# w - 24\bar\lambda\partial_\#\lambda \\ &- 12(w^2 + |\lambda|^2)\Omega_{\#,\alpha}{}^{\alpha} + 2icA^{-4}(w^2 + |\lambda|^2)\big] \end{aligned} \tag{E.17}$$

$$X_{\bar{\mu}\#\alpha\beta} = 2\Omega_{\bar{\mu},\alpha\beta} - \sqrt{2}\frac{\lambda}{w}\varepsilon_{\gamma\alpha\beta}\Omega_{\bar{\mu},\#}{}^{\gamma} + \delta_{\bar{\mu}[\alpha}\left(2\Omega_{\#,\#\beta]} - \sqrt{2}\frac{\lambda}{w}\varepsilon_{\beta]\delta\sigma}\Omega_{\#}{}^{\delta\sigma}\right) \tag{E.18}$$

$$\begin{aligned} X_{\bar{\mu}\alpha\beta}{}^{\beta} &= \sqrt{2}\frac{w}{\bar{\lambda}}\varepsilon_{\alpha\gamma\rho}\Omega_{\bar{\mu},}{}^{\gamma\rho} - 2\Omega_{\bar{\mu},\#\alpha} + \frac{\delta_{\bar{\mu}\alpha}}{w^2 - |\lambda|^2}\Big(3w\partial_{\#}w - \frac{w^2}{\bar{\lambda}}\partial_{\#}\bar{\lambda} \\ &+ 4\bar{\lambda}\partial_{\#}\lambda + \Omega_{\#,\alpha}{}^{\alpha}(3w^2 + 2|\lambda|^2) - i\frac{c}{6}A^{-4}(4|\lambda|^2 + w^2)\Big) . \end{aligned} \tag{E.19}$$

From the linear system (E.2)-(E.13), we also find the following geometric conditions:

$$\begin{aligned} & 4(w^2 - |\lambda|^2)\partial_{\mu}w + 2w(w^2 - |\lambda|^2)\Omega_{\mu,\alpha}{}^{\alpha} \\ - \; & 3\sqrt{2}w^2\lambda\varepsilon_{\mu\alpha\beta}\Omega_{\#,}{}^{\alpha\beta} + 3w(w^2 + |\lambda|^2)\Omega_{\#,\#\mu} = 0 \end{aligned} \tag{E.20}$$

$$\begin{aligned} & 4(w^2 - |\lambda|^2)\partial_{\mu}\lambda + 2\lambda(w^2 - |\lambda|^2)\Omega_{\mu,\alpha}{}^{\alpha} \\ + \; & 3\sqrt{2}w\lambda^2\varepsilon_{\mu\alpha\beta}\Omega_{\#,}{}^{\alpha\beta} - 3\lambda(w^2 + |\lambda|^2)\Omega_{\#,\#\mu} = 0 \end{aligned} \tag{E.21}$$

$$\begin{aligned} & 4(w^2 - |\lambda|^2)\partial_{\mu}w - 2w(w^2 - |\lambda|^2)\Omega_{\mu,\alpha}{}^{\alpha} \\ + \; & \sqrt{2}\lambda(5w^2 + 4|\lambda|^2)\varepsilon_{\mu\alpha\beta}\Omega_{\#,}{}^{\alpha\beta} - w(w^2 + 17|\lambda|^2)\Omega_{\#,\#\mu} = 0 \end{aligned} \tag{E.22}$$

$$\begin{aligned} & 4(w^2 - |\lambda|^2)\partial_{\mu}\bar{\lambda} - 2\bar{\lambda}(w^2 - |\lambda|^2)\Omega_{\mu,\alpha}{}^{\alpha} \\ - \; & \sqrt{2}w(4w^2 + 5|\lambda|^2)\varepsilon_{\mu\alpha\beta}\Omega_{\#,}{}^{\alpha\beta} + \bar{\lambda}(17w^2 + |\lambda|^2)\Omega_{\#,\#\mu} = 0 \end{aligned} \tag{E.23}$$

$$\begin{aligned} & 2\bar{\lambda}w\varepsilon^{\gamma\alpha\beta}\Omega_{\bar{\mu},\alpha\beta} - \sqrt{2}(w^2 + |\lambda|^2)\Omega_{\bar{\mu},\#}{}^{\gamma} \\ - \; & 2\bar{\lambda}w\varepsilon^{\gamma\beta}{}_{\bar{\mu}}\Omega_{\#,\#\beta} + \sqrt{2}(w^2 + |\lambda|^2)\Omega_{\#,\bar{\mu}}{}^{\gamma} = 0 \end{aligned} \tag{E.24}$$

$$\begin{aligned} & 8(|\lambda|^2 + w^2)\Omega_{\bar{\mu}}{}^{\gamma\rho} - 8\sqrt{2}w\bar{\lambda}\varepsilon^{\gamma\rho\alpha}\Omega_{\bar{\mu},\#\alpha} \\ + \; & \sqrt{2}\varepsilon_{\bar{\mu}}{}^{\gamma\rho}\Big[2w\bar{\lambda}\Omega_{\#,\alpha}{}^{\alpha} - 2w\partial_{\#}\bar{\lambda} - 2\bar{\lambda}\partial_{\#}w + icA^{-4}\bar{\lambda}w\Big] = 0 \end{aligned} \tag{E.25}$$

$$6(\bar{\lambda}\partial_{\#}\lambda - \lambda\partial_{\#}\bar{\lambda}) + 6(w^2 + |\lambda|^2)\Omega_{\#,\alpha}{}^{\alpha} - icA^{-4}(w^2 + |\lambda|^2) = 0 \tag{E.26}$$

$$
\begin{aligned}
& 2w(w^2 - |\lambda|^2)\Omega^{\alpha}{}_{,\alpha\gamma} + \sqrt{2}\lambda(w^2 - |\lambda|^2)\Omega^{\alpha}{}_{,\#}{}^{\beta}\varepsilon_{\alpha\beta\gamma} \\
- \quad & w\Omega_{\#,\#\gamma}(w^2 + 5|\lambda|^2) + \sqrt{2}\lambda(2w^2 + |\lambda|^2)\varepsilon_{\gamma\alpha\beta}\Omega_{\#,}{}^{\alpha\beta} = 0
\end{aligned}
\tag{E.27}
$$

$$
\begin{aligned}
& 2\sqrt{2}w(w^2 - |\lambda|^2)\Omega^{\alpha\beta\gamma}\varepsilon_{\alpha\beta\gamma} - 4\bar{\lambda}(w^2 - |\lambda|^2)\Omega^{\alpha}{}_{,\#\alpha} - 30w\bar{\lambda}\partial_{\#}w \\
- \quad & 6w^2\partial_{\#}\bar{\lambda} - 24\bar{\lambda}^2\partial_{\#}\lambda - 6\bar{\lambda}(w^2 + 2|\lambda|^2)\Omega_{\#,\alpha}{}^{\alpha} + 3icA^{-4}\bar{\lambda}w^2 = 0
\end{aligned}
\tag{E.28}
$$

$$
\begin{aligned}
& (w^2 - |\lambda|^2)^2\,(\Omega_{\mu,\#\bar{\alpha}} - \Omega_{\bar{\alpha},\#\mu}) + \delta_{\mu\bar{\alpha}}\big[(3w^2 + 2|\lambda|^2)(\lambda\partial_{\#}\bar{\lambda} - \bar{\lambda}\partial_{\#}\lambda) \\
- \quad & 2(w^4 + 3w^2|\lambda|^2 + |\lambda|^4)\Omega_{\#,\alpha}{}^{\alpha} + i\frac{c}{3}A^{-4}(2w^4 + w^2|\lambda|^2 + 2|\lambda|^4)\big] = 0\ .
\end{aligned}
\tag{E.29}
$$

E.2. Solution for $\boldsymbol{\lambda} = 0$, $\boldsymbol{w} \neq 0$

We next present the components of the flux and the geometric conditions associated to the KSEs (4.37), with the spinor given by

$$\psi_+ = w(1 + e_{1234}) \qquad w \in \mathbb{R}\ . \tag{E.30}$$

We find that the components of the flux are given by the following expressions

$$X_{\#234} = X_{\bar{\mu}234} = 0 \tag{E.31}$$

$$X_{\#\alpha}{}^{\alpha\gamma} = -3\Omega_{\#,\#}{}^{\gamma} \tag{E.32}$$

$$X_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} = -24w^{-1}\partial_{\#}w \tag{E.33}$$

$$X_{\bar{\mu}\#\alpha\beta} = 2\Omega_{\bar{\mu},\alpha\beta} + 2\delta_{\bar{\mu}[\alpha}\Omega_{\#,\#\beta]} \tag{E.34}$$

$$X_{\bar{\mu}\nu\beta}{}^{\beta} = (\Omega_{\bar{\mu},\#\nu} + \Omega_{\nu,\#\bar{\mu}}) + 4\delta_{\bar{\mu}\nu}w^{-1}\partial_{\#}w\ . \tag{E.35}$$

Furthermore, we find that the geometric conditions are given by the following expressions

$$\Omega_{\#,\alpha\beta}\varepsilon^{\gamma\alpha\beta} = 0 \tag{E.36}$$

$$\Omega_{\#,\#\mu} = -4w^{-1}\partial_\mu w \tag{E.37}$$

$$\Omega_{\mu,\alpha}{}^{\alpha} = 4w^{-1}\partial_\mu w \tag{E.38}$$

$$\Omega_{\mu,\alpha\beta} = 0 \tag{E.39}$$

$$\Omega_{\mu,\#\beta} = 0 \tag{E.40}$$

$$\Omega^{\alpha}{}_{\alpha\beta} = -2\frac{\partial_\beta w}{w} \tag{E.41}$$

$$\Omega^{\alpha}{}_{\#\alpha} - 6\frac{\partial_\# w}{w} - \frac{i}{2}cA^{-4} = 0 \tag{E.42}$$

$$(\Omega_{\bar{\alpha},\#\beta} - \Omega_{\beta,\#\bar{\alpha}}) - i\frac{c}{3}A^{-4}\delta_{\bar{\alpha}\beta} = 0 \tag{E.43}$$

$$\Omega_{\#,\alpha}{}^{\alpha} = i\frac{c}{6}A^{-4} \; . \tag{E.44}$$

APPENDIX F

COVARIANT RELATIONS

In this Appendix, we present the main relations used to covariantize the linear system. These expressions relate spin connection terms to SU(3)-covariant terms involving the $SU(3)$ invariant 1-forms ξ, ω and χ, their Lie derivatives with respect to ξ, and also the Lee forms W and Z:

$$\nabla_{\#}\xi_{\mu} = (\mathcal{L}_{\xi}\xi)_{\mu} = -\Omega_{\#,\#\mu} \tag{F.1}$$

$$\nabla_{\bar{\alpha}}\xi_{\beta} = -\Omega_{\bar{\alpha},\#\beta} \tag{F.2}$$

$$\nabla_{\#}\omega_{\alpha\beta} = 2i\Omega_{\#,\alpha\beta} \tag{F.3}$$

$$W_{\alpha} = -\Omega_{\#,\#\alpha} - 2\Omega^{\beta}{}_{\beta\alpha} \tag{F.4}$$

$$W_{\bar{\alpha}} = -\Omega_{\#,\#\bar{\alpha}} - 2\Omega^{\bar{\beta}}{}_{\bar{\beta}\bar{\alpha}} \tag{F.5}$$

$$Z_{\bar{\rho}} = 2\Omega_{\#,\#\bar{\rho}} + 2\Omega_{\bar{\rho},\gamma}{}^{\gamma} + 2\Omega^{\bar{\gamma}}{}_{\bar{\gamma}\bar{\rho}} \tag{F.6}$$

$$Z_{\rho} = 2\Omega_{\#,\#\rho} + 2\Omega_{\rho,\bar{\gamma}}{}^{\bar{\gamma}} + 2\Omega^{\gamma}{}_{\gamma\rho} \tag{F.7}$$

$$(\mathcal{L}_{\xi}\omega)_{\alpha\beta} = 2i\Omega_{\#,\alpha\beta} + i(\mathrm{d}\xi)_{\alpha\beta} \tag{F.8}$$

$$(\mathcal{L}_\xi\omega)_{\bar\alpha\bar\beta} = -2i\Omega_{\#,\bar\alpha\bar\beta} - i(\mathrm{d}\xi)_{\bar\alpha\bar\beta} \tag{F.9}$$

$$(\mathcal{L}_\xi\omega)_{\alpha\bar\beta} = i(\Omega_{\bar\beta,\#\alpha} + \Omega_{\alpha,\#\bar\beta}) \tag{F.10}$$

$$(\mathcal{L}_\xi\chi)_{\alpha\beta\gamma} = 3\Omega_{\#,[\gamma}{}^{\lambda}\varepsilon_{\alpha\beta]\lambda} - 3\Omega_{[\gamma,\#}{}^{\lambda}\varepsilon_{\alpha\beta]\lambda} = (\Omega_{\#,\lambda}{}^{\lambda} - \Omega_{\lambda,\#}{}^{\lambda})\varepsilon_{\alpha\beta\gamma} \tag{F.11}$$

$$(\mathcal{L}_\xi\chi)_{\alpha\beta\bar\gamma} = (\Omega_{\#,\bar\gamma}{}^{\lambda} - \Omega_{\bar\gamma,\#}{}^{\lambda})\varepsilon_{\lambda\alpha\beta} \tag{F.12}$$

$$\mathcal{L}_\xi\bar\chi = (\mathcal{L}_\xi\chi)^* \ . \tag{F.13}$$

The spin-connection components are rewritten in terms of those covariant quantities as

$$\Omega_{\mu,\alpha}{}^{\alpha} = -\frac{1}{2}\left((\mathcal{L}_\xi\xi)_\mu + W_\mu + Z_\mu\right) \tag{F.14}$$

$$\Omega_{\#,\#\mu} = -\nabla_\#\xi_\mu = -i\nabla_\#\omega_{\#\mu} = -(\mathcal{L}_\xi\xi)_\mu \tag{F.15}$$

$$\Omega_{\#,\alpha\beta} = -\frac{i}{2}\nabla_\#\omega_{\alpha\beta} = -\frac{1}{2}\left(i(\mathcal{L}_\xi\omega)_{\alpha\beta} + (\mathrm{d}\xi)_{\alpha\beta}\right) \tag{F.16}$$

$$\Omega_{\bar\alpha,\beta\gamma} = -\frac{i}{2}(\mathrm{d}\omega)_{\bar\alpha\beta\gamma} \tag{F.17}$$

$$\Omega_{\bar\alpha,\#\bar\beta} = \frac{1}{2}\left(i(\mathcal{L}_\xi\omega)_{\bar\alpha\bar\beta} - (\mathrm{d}\xi)_{\bar\alpha\bar\beta} - \varepsilon_{\bar\beta}{}^{\gamma\rho}(\mathcal{L}_\xi\chi)_{\gamma\rho\bar\alpha}\right) \tag{F.18}$$

$$\Omega_{\bar\alpha\bar\beta\bar\gamma} = \frac{i}{2}\nabla_{\bar\alpha}\omega_{\bar\beta\bar\gamma} \tag{F.19}$$

$$\varepsilon_{\alpha\beta\gamma}\nabla^\alpha\omega^{\beta\gamma} = \frac{1}{3}\varepsilon_{\alpha\beta\gamma}(\mathrm{d}\omega)^{\alpha\beta\gamma} \tag{F.20}$$

$$\Omega_{\bar\alpha,\#\beta} = -\nabla_{\bar\alpha}\xi_\beta = -i\nabla_{\bar\alpha}\omega_{\#\beta} \tag{F.21}$$

$$\Omega_{\#,\rho}{}^{\rho} = -\frac{1}{6}\varepsilon_{\alpha\beta\gamma}(\mathcal{L}_\xi\bar\chi)^{\alpha\beta\gamma} + \nabla^\lambda\xi_\lambda \tag{F.22}$$

$$\Omega^{\beta}{}_{\beta\alpha} = \frac{1}{2}\left((\mathcal{L}_{\xi}\xi)_{\alpha} - W_{\alpha}\right) \tag{F.23}$$

$$\Omega^{[\alpha}{}_{\#}{}^{\beta]} = -\frac{1}{2}(\mathrm{d}\xi)^{\alpha\beta} \tag{F.24}$$

$$\Omega^{\alpha}{}_{\#\alpha} = -\nabla^{\alpha}\xi_{\alpha} = -i\nabla^{\alpha}\omega_{\#\alpha}\ . \tag{F.25}$$

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