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A computer scientist's reconstruction of quantum theory*

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Abstract
The rather unintuitive nature of quantum theory has led numerous people to develop sets of (physically motivated) principles that can be used to derive quantum mechanics from the ground up, in order to better understand where the structure of quantum systems comes from. From a computer scientist's perspective we would like to study quantum theory in a way that allows interesting transformations and compositions of systems and that also includes infinite-dimensional datatypes. Here we present such a compositional reconstruction of quantum theory that includes infinite-dimensional systems. This reconstruction is noteworthy for three reasons: it is only one of a few that includes no restrictions on the dimension of a system; it allows for both classical, quantum, and mixed systems; and it makes no *a priori* reference to the structure of the real (or complex) numbers. This last point is possible because we frame our results in the language of category theory, specifically the categorical framework of effectus theory.

Keywords: effectus, reconstruction of quantum theory, effect algebra, sequential product, Jordan algebra, von Neumann algebra, JBW-algebra

(Some figures may appear in colour only in the online journal)

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Contents

1. Introduction

Quantum theory is famously unintuitive. Furthermore, it is not *a priori* clear why its mathematical machinery (complex Hilbert spaces, bounded operators, tensor products) should lead to a correct description of nature. This has led numerous people throughout the last hundred years to try and reconstruct quantum theory from first principles. The idea here being that if one can find a set of reasonable assumptions that are only satisfied by quantum theory and not by any other hypothetical physical theory then one has a better grasp on understanding why this mathematics describe nature so well.

There are many such *reconstructions of quantum theory*. Much early work was based on the orthomodular lattices of von Neumann's *quantum logic* [7]. These approaches focused on the *sharp* observables (projections) of a quantum system and mostly considered an infinite-dimensional system in isolation, i.e. one which is not composable with other systems (see [20] for a review). In contrast, much of the work on reconstructions in the last two decades has instead focused on finite-dimensional systems that can interact with each other and be combined into composite systems. Most of these results take an *operational* approach, which entails that they fundamentally presuppose the nature of classical probability theory in order to describe classical interactions such as measurement and probabilistic mixtures of processes [6, 21]. This requires the *a priori* usage of real numbers and convex sets in their frameworks. The principles themselves in these approaches come in many different guises: some are based on information processing properties [4, 9, 17, 24], others on properties of entanglement [8, 53], properties of pure processes [58, 64, 66], or on any other of a multitude of properties [37, 38, 42, 50, 52, 55, 67]. Each of these approaches sheds new light on how quantum theory is 'special' among a large selection of hypothetical physical theories.

From a computer scientist's point of view, the compositional nature of many modern works that shed light on the interactions of systems is preferable over the older work that dealt with systems in isolation. However, the restriction to finite dimension is less desirable, as many natural datatypes to describe programs require infinite-dimensional systems. Indeed, in order to describe, say, the natural numbers type in a quantum programming language we require an infinite-dimensional algebra [11, 13, 56]. In addition, most reconstructions of quantum theory employ principles that are only satisfied by quantum systems, but not mixed classical–quantum systems. This prevents the inclusion of systems needed to describe quantum programming languages that have a 'quantum data/classical control' architecture [28, 59].

One then wonders whether there is a reconstruction of quantum theory that includes infinite-dimensional systems and also allows for mixed classical–quantum systems, i.e. where the target of reconstruction is a category of operator algebras. In addition it would be desirable if we could sidestep the *a priori* usage of real numbers and convex sets and instead work in a more abstract categorical setting. To phrase this question more concretely:

> Are there nice assumptions on a category such that any such category must be a category of quantum types with quantum programs between them?

In this paper we present a reconstruction with these three desirable properties. Firstly, our assumptions hold for infinite-dimensional types, in contrast to the axioms of the vast majority of the existing reconstructions. Secondly, our axioms include classical types and mixed quantum–classical types, whereas most reconstructions restrict to purely-quantum types. Finally, our axioms do not presuppose the real or complex numbers. As far as we are aware, this is the first reconstruction with all these desirable properties together. Additionally, the core of our reconstruction does not assume a symmetric monoidal structure. We only need the presence of a tensor product for the final step.

Our axioms split roughly into three groups. The first group specifies our basic framework: the category is an *effectus* [12, 14], a basic type of structure that has very minimal assumptions while still allowing us to speak of states and predicates. We make the additional familiar assumption that the predicate spaces are directed-complete (i.e. that they form a *dcpo*).

The second group deals with additional categorical structure, *filters* and *comprehensions* [15], that imposes the well-behavedness of certain filters and pure maps. We believe these assumptions and the structure they imply might be of interest in their own right, and so we give a name to the sort of effectus with these properties: a $\diamond$-*effectus* (pronounced ‘diamond-effectus’).

Finally, in the third group, we require some more operationally motivated axioms which state the well-behavedness of the operation of *sequential measurement* (to wit, we require our predicate spaces to form *sequential effect algebras* (SEA) [29]).

Our assumptions are satisfied by the category of von Neumann algebras with normal positive linear contractions in the opposite direction (representing quantum theory) and also by the category of complete Boolean algebras (representing deterministic classical logic). Our main result is a rough converse to this. More formally, our reconstruction proceeds in three steps. First, we show that a category satisfying our assumptions embeds into the product of the category of complete Boolean algebras and the category of directed-complete *JB-algebras* [36]. JB-algebras are a type of infinite-dimensional Jordan algebra that are closely related to C^*-algebras. The reason our category embeds into a product category of Boolean algebras and JB-algebras is because the scalars in the category are ‘spatial’ and can be probabilistic in one part of the space and sharp in another part. By restricting to categories with ‘irreducible’ scalars we show that it either embeds into the category of Boolean algebras *or* the category of *JBW-algebras*, a particularly well-behaved type of JB-algebra that is closely related to von Neumann algebras. Hence, when assuming the scalars are irreducible we get a dichotomy: either the category is describing classical deterministic logic or it is describing a theory of quantum systems. Finally, we impose additional symmetric monoidal structure on the category, so that we can form composite systems. This forces each JBW-algebra to be a *JW-algebra*, a Jordan algebra that embeds into a von Neumann algebra.

While systems in our category correspond to either Boolean algebras or JBW-algebras, our assumptions do not force any of the JBW-algebras to be ‘quantum-like’. For instance, the category of associative JBW-algebras (or equivalently, commutative von Neumann algebras) and normal positive linear contractions satisfies all our assumptions. However, these algebras are all classical in the sense that they correspond to measurable spaces. We see the possibility of fully classical examples as a strength of our approach, as it means our assumptions capture those properties that are shared between classical, quantum, and quantum–classical systems, without restricting to some subset of these systems *a priori*. The existence of quantum systems can be forced on the category by assuming any of a multitude of assumptions that are only satisfied by quantum systems. For instance, we could assume that each map can be dilated [76, section 3.7.1], similar to the requirement of the existence of purifications in [9, 58, 64].

1.1. Related work

This reconstruction essentially combines two previous reconstructions by one of the authors [66, 67]. The first of these [66] also used the effectus framework and used assumptions related to pure maps. The second [67] used assumptions based on sequential measurement. Both of these reconstructions relied on the convex structure imposed by the real numbers and were restricted to finite dimension. In this paper we combine the assumptions of these reconstructions. This allows us to remove these restrictions on dimension and convexity.

Some other reconstructions that are similar in that they are framed in the language of category theory are that of Tull [64] and Selby *et al* [58]. These are both inspired by the Oxford school of categorical quantum mechanics and as such deal with symmetric monoidal categories, dagger structures, and compact closure (i.e. *cups and cap*s, also known as *map-state duality* or the *Choi–Jamiołkowski isomorphism*). Tull’s reconstruction is almost entirely categorical, retrieving a category of matrices over a particular type of ring. To retrieve quantum theory one then only has to impose that the ring in question is the complex numbers.

The assumptions of the reconstruction are essentially those of the Pavia reconstruction [9], but then translated into the language of category theory. The reconstruction of Selby *et al* [58] imposes a more standard GPT framework at the start of the reconstruction, but the assumptions themselves are all clearly motivated from a categorical viewpoint. These two reconstructions are inherently restricted to finite dimension as compact closure is a core property of them, although it is conceivable that there is a way around that by using non-standard analysis [26]. Another significant difference is that these reconstructions rely on the Oxford school of categorical quantum mechanics, whilst our reconstruction is more closely aligned to 'standard' category theory in the sense that many of our assumptions can be framed in terms of universal properties.

One selling point of our work is that we do not need to assume the structure of the real numbers *a priori*. Some other ways to get the correct set of scalars are known as well. A seminal result is that of Solèr [61], who showed that if an infinite-dimensional generalised Hilbert space over some division ring is orthomodular, then the ring in question must be the real numbers, complex numbers or the quaternions. Another approach is given by the work of Heunen [39] and Vicary [70]. They both derive (related) sets of conditions under which the scalars of a suitable dagger-category embed into the complex numbers, and in Heunen's case, under which the category itself embeds into the category of complex Hilbert spaces. Whereas we work in the setting of effectuses and impose an order-theoretic condition, directed completeness, they work in dagger categories and impose a cardinality condition, that the number of scalars is at most equal to the continuum. A drawback of their results is that the scalars only *embed* into the complex numbers. For instance, the field of rational numbers is allowed in their results and so is the (non-Archimedean) field of rational functions. This embedding generally does not preserve the ordering of the elements. Recently, Heunen and Kornell improved upon the result by Heunen and found a set of categorical conditions that force a category to be equivalent to the category of real or complex Hilbert spaces (containing both finite- and infinite-dimensional spaces) [41]. Their result uses Solèr's theorem to show the ring of scalars is the field of real or complex numbers. Their axioms are categorically natural and based on the theory of dagger monoidal categories with dagger biproducts.

While most (modern) reconstructions focus on finite-dimensional systems, there are some exceptions. A particularly relevant one is the work of Alfsen and Shultz [1, 2]. They find geometric conditions for when a convex set is isomorphic to the state space of a quantum system or, more generally, a Jordan operator algebra. Our proof works essentially by showing that our spaces satisfy (something similar to) the conditions they find. A number of reconstructions of infinite-dimensional quantum theory rely heavily on the work of Alfsen and Shultz, for instance [33, 51, 54]. Especially this last one resembles our work in that they also assume a completeness condition for the order on predicates, and that they assume the existence of filters, although some other assumptions of [54] do not have a clear motivation.

1.2. Structure of the paper

We recall all the definitions and some known results we will need in section 2. In particular, we recall the basic definitions of effectus theory (section 2.1), the notion of directed completeness (section 2.2), order unit spaces (OUS) (section 2.3), Jordan operator algebras (section 2.4), and SEAs (section 2.5). We state our main results formally in section 3. Before proving these results, we will first see some consequences of having well-behaved filters and comprehensions in section 4 and in section 5 we show how some of our assumptions, in particular directed completeness, conspire to force an effectus to split into a sharp part and a convex part, which forms the backbone of our reconstruction. Then in section 6 we prove our main result: that an effectus satisfying our assumptions embeds into the product category of Boolean algebras and JB-algebras. We finish our reconstruction by also considering a tensor product in section 7 which allows us to prove that an effectus satisfying our assumptions must embed into the category of JW-algebras. We end the paper with some concluding remarks in section 8.

2. Preliminaries

The assumptions of our reconstruction and the steps in our proof rely on definitions from several somewhat disparate fields, namely effectus theory, SEAs and Jordan operator algebras. In this section we will recall all these concepts.

2.1. Effectus theory

The basic assumption of our reconstruction is that our category **C** is an *effectus* [12, 14]. This is a weak structure that allows for a basic notion of state and predicate. The requirement that a category be an effectus should be compared to the requirement that a set be a topological space: one rarely considers just an arbitrary topological space as it has so little structure. The strength of topological spaces though, is that they allow for the definition of many important notions on top of it. Similarly, an effectus on its own has little structure, but allows for the definition of many interesting notions. An effectus can be defined in two ways: either axiomatising a category of total maps or of partial maps. Though we will not use it in the rest of the paper, we will give the definition of the total form first as it has the cleanest definition. Although seemingly obscure at first, many categories with a coproduct that behaves as a probabilistic disjunction are effectuses.

Definition 1. A category **C** is an **effectus in total form** iff

(a) **C** has finite coproducts (hence an initial object 0) and a final object 1;

(b) All diagrams of the following form[4] are pullbacks

$$\begin{array}{ccc} X+Y & \xrightarrow{\text{id}+!} & X+1 \\ {\scriptstyle !+\text{id}}\downarrow & & \downarrow{\scriptstyle !+\text{id}} \\ 1+Y & \xrightarrow[\text{id}+!]{} & 1+1 \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{!} & 1 \\ {\scriptstyle \kappa_1}\downarrow & & \downarrow{\scriptstyle \kappa_1} \\ X+Y & \xrightarrow[!+!]{} & 1+1 \end{array} \tag{1}$$

(c) And the following two arrows are jointly monic.

$$1+1+1 \underset{[\kappa_2,\kappa_1,\kappa_2]}{\overset{[\kappa_1,\kappa_2,\kappa_2]}{\rightrightarrows}} 1+1$$

A **partial map** from $X \to Y$ is an arrow $X \to Y + 1$; a **state** on X is an arrow $1 \to X$; a **predicate** on X is an arrow $X \to 1 + 1$. The partial maps of an effectus can be composed in the obvious way, and hence we get a category Par(**C**) of partial maps (formally, $\_ + 1$ is the maybe monad on **C** and Par(**C**) is its Kleisli category).

Example 2. We just give a few examples. For a more comprehensive list, see [14].

[4] We write κ_i for coproduct coprojections; square brackets $[f, g]$ for coproduct cotupling; $h + k = [\kappa_1 \circ h, \kappa_2 \circ k]$ and ! for the unique maps associated to either the final object 1 or initial object 0.

(a) The category of sets and functions is an effectus in total form. The states of a set A correspond to the elements of A and the predicates correspond to the subsets of A.
(b) The category of sets and probabilistic functions[5] is an effectus in total form. The states on a set A correspond to probability distributions on A and predicates are maps $A \to [0, 1]$.
(c) The opposite category of (finite-dimensional) C^*-algebras with positive unital linear maps forms an effectus in total form. States of an algebra $\mathfrak{A}$ are positive unital linear maps $\omega : \mathfrak{A} \to \mathbb{C}$, and the predicates correspond to elements of $[0, 1]_{\mathfrak{A}}$.

Remark 3. In this last example we used the opposite category of C^*-algebras. This is because C^*-algebras are spaces of observables (and hence predicates), while effectuses are defined in terms of states. Using the language of physicists we would say that effectuses are in the Schrödinger picture, while C^*-algebras are in the Heisenberg picture. In this paper we will often see the necessity of working with an opposite category for this reason.

For our purposes it will be more convenient to work with the category of partial maps of an effectus. That category can be axiomatised on its own as an *effectus in partial form*, but that requires some preparation. To start, that category comes with a partial addition on the maps.

Definition 4. A **partial commutative monoid** (**PCM**) is a set M with an element $0 \in X$ and a *partial* binary operation $\varovee : M \times M \rightharpoonup M$ such that for all $x, y, z \in M$

- $(x \varovee y) \varovee z = x \varovee (y \varovee z)$ (associativity),
- $x \varovee y = y \varovee x$ (commutativity), *and*
- $0 \varovee x = x$ (unitality).

Here '$=$' is taken to be a *Kleene equality*.[6] We write $x \perp y$ to denote $x \varovee y$ is defined. A function $f : M \to N$ between PCMs is **additive** if $f(0) = 0$ and $f(x) \varovee f(y) = f(x \varovee y)$ for all $x \perp y$ in M. The Cartesian product $M \times N$ of two PCMs is again a PCM in the obvious way. A map $g : M \times N \to L$ is **biadditive** if its restrictions $g(x, -)$, $g(-, y)$ for arbitrary x and y are additive. We say a category is **enriched over PCMs** if each homset is a PCM and the composition maps are biadditive.

A category enriched over PCMs has a partial addition operation defined on its morphisms that interacts suitably with composition. This acts as an abstraction and generalisation of the *coarse-graining* operation present in, for instance, generalised probabilistic theories [6]. When the sum of two morphisms f and g is defined, it means that there is a sense in which f and g can coexist as different branches of a probabilistic process. The sum morphism $f \varovee g$ then corresponds to their coarse-graining where we forget which of the two processes actually happened. In an effectus we also have coproducts to model the probabilistic disjunction of systems, and the coarse-graining operation interacts suitably with these coproducts.

Definition 5. A category $\mathbf{C}$ with zero morphisms $0 : A \to B$ (such as when it is enriched over PCMs) has for each coproduct $\coprod_{j\in J} A_j$ **partial projections** $\rhd_i : \coprod_{j\in J} A_j \to A_i$ characterized by $\rhd_i \circ \kappa_i = \mathrm{id}$ and $\rhd_i \circ \kappa_k = 0$ for $k \neq i$. A family $(f_j : B \to A_j)_{j\in J}$ of morphisms in $\mathbf{C}$ is **compatible** if there exists an $f : B \to \coprod_{j\in J} A_j$ such that $\rhd_j \circ f = f_j$ for each $j \in J$.

A **finitely partially additive category** (finPAC) [10] is a category with finite coproducts that is enriched over PCMs so that the coproduct and PCM operations interact suitably:

[5] I.e. the Kleisli category of the finite distribution monad (but note that the Kleisli category of the Giry monad on measurable spaces is also an example of an effectus).
[6] Kleene equality: if either side is defined, then so is the other, and they are equal. Hence an equation like $x \varovee y = z$ is taken to mean both that $x \varovee y$ is defined, as well as that we have the equality $x \varovee y = z$.

- **Compatible sum axiom**: compatible pairs of morphisms $f, g : A \to B$ are summable in $\mathbf{C}(A, B)$.
- **Untying axiom**: if $f, g : A \to B$ are summable, then $\kappa_1 \circ f, \kappa_2 \circ g : A \to B + B$ are summable too.

An effectus in partial form is a finPAC, but we need a bit more: that the predicates, the homsets $\mathbf{C}(A, I)$, are a special kind of PCM.

Definition 6. An **effect algebra** [25] is a PCM $(E, \varovee, 0)$ with a 'top' element $1 \in E$ such that for each $x \in E$,

- There is a unique $x^{\perp} \in E$ (called the orthosupplement) satisfying $x \varovee x^{\perp} = 1$, and
- $x \perp 1$ implies $x = 0$.

For $x, y \in E$ we write $x \leqslant y$ whenever there is a $z \in E$ with $x \varovee z = y$.. This turns E into a poset with minimum 0 and maximum $1 = 0^{\perp}$. The map $x \mapsto x^{\perp}$ is an order anti-isomorphism. Furthermore $x \perp y$ if and only if $x \leqslant y^{\perp}$. We write **EA** for the category of effect algebras and additive maps. Note that additive maps automatically preserve the order (i.e. are monotone).

Example 7. Let $(B, 0, 1, \wedge, \vee, (\ )^{\perp})$ be an orthomodular lattice. Then B is an effect algebra with the partial addition defined by $x \perp y \Leftrightarrow x \wedge y = 0$ and in that case $x \varovee y = x \vee y$.. The orthosupplement $(\ )^{\perp}$ is given by the orthocomplement itself. The lattice order coincides with the effect algebra order (defined above). See e.g. [75, proposition 27].

Example 8. For a unital C^*-algebra $\mathfrak{A}$, the set of effects $[0, 1]_{\mathfrak{A}}$ is an effect algebra. This is the motivating example.

Definition 9. An **effectus in partial form** is a finPAC $\mathbf{C}$ with a distinguished **unit** object $I \in \mathbf{C}$ satisfying the following conditions.

- The PCM $\mathbf{C}(A, I)$ is an effect algebra for all A. We write $\mathbf{1}_A$ and $\mathbf{0}_A = 0_{AI}$ for the top and bottom of $\mathbf{C}(A, I)$.
- $\mathbf{1}_B \circ f = \mathbf{0}_A$ implies $f = 0_{AB}$ for all $f : A \to B$.
- $\mathbf{1}_B \circ f \perp \mathbf{1}_B \circ g$ implies $f \perp g$ for all $f, g : A \to B$.

We call a map $f : A \to B$ **total** when $\mathbf{1}_B \circ f = \mathbf{1}_A$.

Viewing an effectus (in partial form) as an abstraction of a generalised probabilistic theory, we can give an interpretation to these axioms. That the predicates form an effect algebra means, first, that we have a *deterministic* predicate $\mathbf{1}_A$ for every system A so that the processes in the theory are non-signalling [18], and second, that for every predicate p we have its *negation* $p^{\perp}$. The existence of negations in non-signalling GPTs is usually a consequence of the ability to coarse-grain measurements. The second and third axioms can be interpreted as a weak form of *operational equivalence*, stating that maps are zero, respectively summable, when they are zero, respectively summable, on every predicate [62].

One might object that the axioms of an effectus in partial form seem arbitrary. However, it turns out that they correspond exactly to effectuses in total form, which as we saw in definition 1, have a much more clear categorical definition.

Remark 10. Let $\mathbf{C}$ be an effectus in total form. Then $\mathrm{Par}(\mathbf{C})$ is an effectus in partial form. Conversely, for an effectus in partial form $\mathbf{D}$, the category of **total maps** $\mathrm{Tot}(\mathbf{D})$ is an effectus in total form. This is, in fact, a two-categorical equivalence between the category of effectuses in total form and the category of effectuses in partial form [12].

Example 11. Adapting example 2 to the partial case we see that the category of sets and partial functions is an effectus in partial form. So is the Kleisli category of the subdistribution monad and the opposite category of C^*-algebras with contractive positive linear maps. The category $\mathbf{EA}^{\mathrm{op}}$ is also an effectus in partial form.

For the remainder of the paper we will work solely with effectuses in partial form and simply refer to them as effectuses. For clarity, let us translate some of the important notions: in an effectus (in partial form),

- A **predicate** is a map $A \to I$,
- A **state** is a total map $I \to A$, and
- A **scalar** is a map $I \to I$.

Definition 12. For any object A in an effectus we write $\mathrm{Pred}(A)$ for the effect algebra of predicates on A. For a morphism $f : A \to B$ we write $\mathrm{Pred}(f) : \mathrm{Pred}(B) \to \mathrm{Pred}(A)$ for the map defined by $\mathrm{Pred}(f)(p) := p \circ f$.

It is clear that Pred is a functor from $\mathbf{C}$ to $\mathbf{EA}^{\mathrm{op}}$. The image $\mathrm{Pred}(\mathbf{C})$ is an effectus, and it is equivalent to $\mathbf{C}$ iff Pred is faithful, which is equivalent to the following.

Definition 13 (cf [16]). We say an effectus $\mathbf{C}$ is **separated by predicates** if for a pair of morphisms $f, g : A \to B$ we have $f = g$ when $p \circ f = p \circ g$ for all $p \in \mathrm{Pred}(B)$.

Separation by predicates is analogous to the condition of *local tomography* in the setting of generalised probabilistic theories [6]. We also have a dual definition, which asks the same, but for states.

Definition 14. Let $\mathbf{C}$ be an effectus. We say it is **separated by states** when for all pairs of morphisms $f, g : A \to B$ we have $f = g$ iff $f \circ \omega = g \circ \omega$ for all states $\omega : I \to A$.

Just as with predicates, we can also construct a 'state functor', which goes into a category of *abstract convex sets* [16, 46, 76], but we will not need this in this paper.

These definitions of separation allow for infinite-dimensional systems. In the literature on generalised probabilistic theories it is however common to only consider finite-dimensional ones. We can enforce finite dimensionality by requiring that a finite number of predicates must suffice to separate maps.

Definition 15. We say an effectus has **finite tomography** when for each object A there is a finite set of predicates $p_1, \ldots, p_k$ such that for any pair of morphisms $f, g : B \to A$ we have $f = g$ iff $p_i \circ f = p_i \circ g$ for all i.

The operational reasoning behind this assumption is that we can physically only probe a state transformation through the application of a predicate, and we can only ever do this a finite number of times. Hence, finite tomography says that there is such a finite set of probings that is sufficient to fully determine a transformation. In section 5.4 we will see that in the setting we care about, that an effectus with finite tomography is a product category of Boolean algebras and finite-dimensional generalised probabilistic theories.

2.1.1. Effect monoids. The set of scalars $\mathbf{C}(I, I)$ in an effectus has a rich structure: as a set of predicates on I it is an effect algebra, but it also has a multiplication that comes from the composition of scalars. Its structure is axiomatised as follows.

Definition 16. An **effect monoid**[7] [44] is an effect algebra $(M, \ovee, 0, ^{\perp}, \cdot)$ with an additional (total) binary operation $\cdot$, such that the following conditions hold for all $a, b, c \in M$.

- Unit: $a \cdot 1 = a = 1 \cdot a$.
- Distributivity: if $b \perp c$, then $a \cdot b \perp a \cdot c$, $b \cdot a \perp c \cdot a$,

$$a \cdot (b \ovee c) = (a \cdot b) \ovee (a \cdot c), \quad \text{and} \quad (b \ovee c) \cdot a = (b \cdot a) \ovee (c \cdot a).$$

  Or, in other words: the operation $\cdot$ is bi-additive.
- Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

We call an element p of M **idempotent** whenever $p^2 := p \cdot p = p$.

Example 17. Any Boolean algebra $(B, 0, 1, \wedge, \vee, (\ )^{\perp})$, being an orthomodular lattice, is an effect algebra by example 7, and, moreover, a commutative effect monoid with multiplication defined by $x \cdot y = x \wedge y$. In particular, every element is idempotent.

Example 18. In any effectus, the set of scalars is an effect monoid with $s \cdot t := s \circ t$.

Example 19. Let X be a compact Hausdorff space and denote its space of continuous functions into the complex numbers by $C(X) := \{f : X \to \mathbb{C}, f \text{ continuous}\}$. This is a commutative unital C^*-algebra (and conversely by the Gel'fand theorem, any commutative C^*-algebra with unit is of this form). Its unit interval $[0, 1]_{C(X)} = C(X, [0, 1])$ consisting of continuous functions $f : X \to [0, 1]$ is a commutative effect monoid.

Remark 20. A physical or logical theory which has probabilities of the form $[0, 1]_{C(X)}$ can be seen as a theory with a natural notion of space, where probabilities are allowed to vary continuously over the space X. This is explored in for instance reference [23].

Example 21. Given two effect algebras/monoids E_1 and E_2 we define their **direct sum** $E_1 \oplus E_2$ as the Cartesian product with pointwise operations. This is again an effect algebra/monoid. Effect algebras/monoids that cannot be written as a non-trivial direct sum we call **irreducible**.

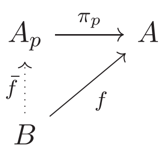
Example 22. Let M be an effect monoid and let $p \in M$ be some idempotent. Define $pM := \{p \cdot a;\ a \in M\}$. This is an effect monoid with $(p \cdot a)^{\perp} := p \cdot a^{\perp}$ and all other operations inherited from M. The map $a \mapsto (p \cdot a, p^{\perp} \cdot a)$ is an isomorphism $M \cong pM \oplus p^{\perp}M$ [71]. In particular, an effect monoid is irreducible iff it has no non-trivial idempotents.

2.1.2. Filters and comprehensions. So far we have discussed the general structure of an effectus, which is present in a large class of examples. Now we will look at additional structure that is more specialised.

[7] The category of effect algebras has an algebraic tensor product that makes the category symmetric monoidal [45]. The monoids in the category of effect algebras resulting from this tensor product are the effect monoids, hence the name.

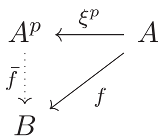
We will require the existence of certain universal maps into and out of subsystems, which can be motivated operationally [66] as filters and arise categorically as adjunctions [15] (see remark 27). Additionally, filters and comprehensions are closely related to the categorical notion of (co)kernels (see remark 26) and hence to the notion of *ideal compressions* of [9] (see [63, section 4.4.2] for details).

Definition 23. Let $p : A \to I$ be a predicate in an effectus. A **comprehension** for p consists of an object A_p and a map $\pi_p : A_p \to A$ such that $\mathbf{1}_A \circ \pi_p = p \circ \pi_p$ that is **final** with this property: whenever $f : B \to A$ is such that $\mathbf{1}_A \circ f = p \circ f$ then there is a unique $\bar{f} : B \to A_p$ with $\pi_p \circ \bar{f} = f$. That is:



We say an effectus **has comprehensions** when every predicate has a comprehension.

Definition 24. Let $p : A \to I$ be a predicate in an effectus. A **filter**[8] for p is an object A^p and map $\xi^p : A \to A^p$ such that $\mathbf{1} \circ \xi^p \leqslant p$ which is **initial** for this property: for any map $f : A \to B$ which satisfies $\mathbf{1} \circ f \leqslant p$ there is a unique $\bar{f} : A^p \to B$ with $\bar{f} \circ \xi^p = f$. That is:



We say an effectus **has filters** when every predicate has a filter.

The reason we call these maps filters is because applying a filter ξ^p corresponds in our categories of interest to the 'post-selection' of the predicate p, so that after application we have 'filtered' the state to ensure p is true.

Note that as filters and comprehensions are defined by a universal properties, that they are unique up to unique isomorphism.

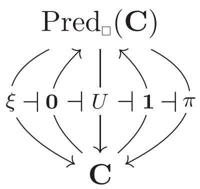
Example 25. In [15] many examples of categories with filters and comprehension are given. Here we will restrict ourselves to discussing them for the 'quantum' example of the C^*-algebra $B(\mathcal{H})$ of bounded operators on a Hilbert space with positive linear contractions in opposite direction between them. Let $p \in B(\mathcal{H})$ be an effect, i.e. $0 \leqslant p \leqslant 1$. Denote by P the largest projection (idempotent effect) below p, i.e. P projects to the eigenspace of p of eigenvalue 1. Denote this space by $\mathcal{K} \subseteq \mathcal{H}$. Then the **standard comprehension** of p is the map $\pi_p : B(\mathcal{H}) \to B(\mathcal{K})$ given by $\pi_p(B) = PBP$. Now let $\mathcal{K}' \subseteq \mathcal{H}$ be $\mathcal{K}' = (\ker p)^\perp$, i.e. the closure of the eigenspaces of p of non-zero eigenvalue. Then p's **standard filter** is the map $\xi^p : B(\mathcal{K}') \to B(\mathcal{H})$ given by $\xi^p(q) = \sqrt{p}q\sqrt{p}$.

[8] A filter for p is exactly the same thing as what is called a **quotient** for $p^\perp$ in many other papers on effectuses [14]. In those papers ξ_p correspond to our $\xi^{p^\perp}$.

Remark 26. It can be shown that an effectus has comprehensions iff it has *kernels* [76, section 200]. An effectus has *cokernels* iff all maps have an *image* and every *sharp* effect has a filter [76, section 205]. We will give definitions of the image and sharpness in the next subsection, but for now let us note that we can hence interpret filters as 'fuzzy cokernels'.

Filters and comprehensions have a different categorical characterisation due to Jacobs.

Remark 27. Let $\mathbf{C}$ be an effectus. Let $\mathrm{Pred}_\Box(\mathbf{C})$ denote its **Grothendieck category** which has as objects pairs $(A \in \mathbf{C}, p \in \mathrm{Pred}(A))$ and morphisms $f : (A, p) \to (B, q)$ given by $f : A \to B$ satisfying $p \leqslant (q^\perp \circ f)^\perp$. There is an obvious forgetful functor $U : \mathrm{Pred}_\Box(\mathbf{C}) \to \mathbf{C}$. Conversely there are two canonical ways to embed $\mathbf{C}$ into $\mathrm{Pred}_\Box(\mathbf{C})$, namely by mapping an object A to $(A, \mathbf{0})$ and by mapping A to $(A, \mathbf{1})$. These two embeddings turn out to be left and right adjoint to the forgetful functor [15]:



The $\mathbf{0}$ embedding has a left adjoint iff $\mathbf{C}$ has filters[9], and the $\mathbf{1}$ embedding has a right adjoint iff $\mathbf{C}$ has comprehensions [12, chapter 5].

We note a number of properties of filters and comprehensions that we will use without further reference.

Proposition 28 ([14]). *Let $\mathbf{C}$ be an effectus which has filters and comprehensions.*

- *Every filter is epic, every comprehension is monic.*
- *If ξ is a filter for a, then $\mathbf{1} \circ \xi = a$.*
- *Comprehensions are total: $\mathbf{1} \circ \pi = \mathbf{1}$.*

By the second point of this proposition, we can extract the predicate of a filter by applying $\mathbf{1} \circ (\ )$. Analogously, to get the predicate of a comprehension we ask for its image [14].

Definition 29. Let $f : A \to B$ be a morphism in an effectus. The **image** of f, when it exists, is the smallest predicate $p : B \to I$ such that $p \circ f = \mathbf{1} \circ f$, i.e. if $q : B \to I$ is also such that $q \circ f = \mathbf{1} \circ f$, then $p \leqslant q$. We denote the image of f by $\mathrm{im}\, f$. We say an effectus **has images** when all the maps have an image.

Note that for a von Neumann algebra, the image of a positive map is also known as its *carrier* [73, section 63].

2.1.3. Pure maps. Filters and comprehensions allow us to define an abstract notion of *pure map*. In pure finite-dimensional quantum theory there is a well-established notion of pure map: Kraus rank-1, i.e. a map of the form $T \mapsto A^* T A$ for some operator A. These correspond to the evolution of a system that does not include measurement (but may include set-up or loss of

[9] This is the reason that filters for p are referred to as *quotients* for $p^\perp$ in the effectus literature.

knowledge). For infinite-dimensional systems, there are many proposals, but we argue for the following.

Definition 30 ([76, definition 201II]). Let $f : A \to B$ be a map in an effectus. We say it is **pure** when $f = \pi \circ \xi$ for some filter ξ and comprehension π.

Example 31. In the category of von Neumann algebras with normal completely positive contractive linear maps in the opposite direction, a map $f : B(\mathcal{H}) \to B(\mathcal{K})$ is pure iff it is Kraus rank-1, as desired. In fact, every map factors (in a Stinespring-like fashion) as a pure map after a normal $*$-homomorphism [74] and a map is pure iff this $*$-homomorphism is surjective.

Note that the ordering of the filter and comprehension $\pi \circ \xi$ is important. This raises a question whether pure maps are closed under composition, i.e. whether the pure maps form a subcategory. Let us note first that some compositions always result in a pure map again.

Lemma 32. *Let* $\mathbf{C}$ *be an effectus with filters. Then the following are true.*

- *A composition of filters is again a filter.*
- *If* $\mathbf{C}$ *also has images and compatible comprehensions, then a composition of comprehensions is again a comprehension.*

Proof. Let ξ^p and ξ^q be filters for p respectively q. We claim that $\xi^p \circ \xi^q$ is a filter for $p \circ \xi_q$. To this end we let ξ be a filter for $p \circ \xi^q$. Then there is a unique g such that $\xi^p \circ \xi^q = g \circ \xi$, which we need to show is an isomorphism. As $1 \circ \xi = p \circ \xi^q \leqslant 1 \circ \xi^q = q$ we have $\xi = h_1 \circ \xi^q$ for a unique h_1. Because $1 \circ h_1 \circ \xi^q = 1 \circ \xi = p \circ \xi^q$ we have $1 \circ h_1 = p$ because ξ^q is epic and hence $h_1 = h_2 \circ \xi^p$. Then

$$g \circ h_2 \circ \xi^p \circ \xi^q \;=\; g \circ \xi \;=\; \xi^p \circ \xi^q \quad \text{and} \quad h_2 \circ g \circ \xi \;=\; h_2 \circ \xi^p \circ \xi^q \;=\; h_1 \circ \xi^q \;=\; \xi$$

so that because $\xi^p \circ \xi^q$ and ξ are epic we have $g \circ h_2 = \mathrm{id}$ and $h_2 \circ g = \mathrm{id}$.

Now suppose $\mathbf{C}$ has images and compatible comprehensions and let π_p and π_q be comprehensions for sharp predicates p respectively q. We will show that $\pi_p \circ \pi_q$ is a comprehension for $\mathrm{im}(\pi_p \circ \pi_q)$. To this end let f be any map with $\mathrm{im}\,\pi_p \circ \pi_q \circ f = 1 \circ f$. As $\mathrm{im}\,\pi_p \circ \pi_q \leqslant \mathrm{im}\,\pi_p = p$ we also have $p \circ f = 1 \circ f$ and hence $f = \pi_p \circ g_1$ for a unique g_1. Let ξ^p be a filter for p such that $\xi^p \circ \pi_p = \mathrm{id}$. Then $q \circ \xi^p \circ \pi_p \circ \pi_q = q \circ \pi_q = 1 \circ \pi_q = 1 \circ \pi_p \circ \pi_q$ and hence $q \circ \xi^p \geqslant \mathrm{im}\,\pi_p \circ \pi_q$ so that

$$q \circ g_1 \;=\; q \circ \xi^p \circ \pi_p \circ g_1 \;=\; q \circ \xi^p \circ f \;\geqslant\; \mathrm{im}\;\pi_p \circ \pi_q \circ f \;=\; 1 \circ f \;=\; 1 \circ g_1.$$

Hence there is a unique g_2 such that $g_1 = \pi_q \circ g_2$ which gives $f = \pi_p \circ g_1 = (\pi_p \circ \pi_q) \circ g_2$. Uniqueness of g_2 with the property that $f = (\pi_p \circ \pi_q) \circ g_2$ follows because $\pi_p \circ \pi_q$ is monic. □

Hence, the question whether a composition of pure maps is again pure is reduced to the question whether a composition $\xi \circ \pi$ 'in the wrong order' can be written as $\pi' \circ \xi'$ for some different comprehension π' and filter ξ'. This is true in our main examples from quantum theory (indeed the composition of two Kraus rank-1 maps is again Kraus rank-1). For an effectus it is something that needs to be imposed additionally, i.e. by demanding that the pure maps form a subcategory.

In fact, inspired by quantum theory, for our reconstruction we will require that the pure maps form a *dagger category* where for each pure map $f : A \to B$ we have a pure map $f^\dagger : B \to A$ such that $(f^\dagger)^\dagger = f$ and $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$. Recall that in a dagger category we say f is **†-adjoint** to g when $f^\dagger = g$ and we say f is an **†-isometry** when $f^\dagger \circ f = \mathrm{id}$. Finally, we say f is **†-positive** when $f = g \circ g^\dagger$ for some g.

2.1.4. Sharp predicates and diamond-adjointness. The dagger structure on pure maps has to imposed 'externally', but suitable effectuses with filters and comprehensions have interesting dualities of pure maps themselves already. In order to see this we need to introduce a couple of concepts.

Definition 33 ([66]). Let $p : A \to I$ be a predicate. We call p **sharp** when there is some morphism $f : B \to A$ such that $\mathrm{im}\, f = p$. We define $\mathrm{SPred}(A)$ to be the poset of sharp predicates of A.

When we have images and comprehensions, we can find for each predicate the largest sharp predicate beneath it.

Definition 34. The **floor** of p is defined as $\lfloor p \rfloor := \mathrm{im}\,\pi$, where π is any comprehension for p. The **ceiling** is defined as the De Morgan dual: $\lceil p \rceil := \lfloor p^\perp \rfloor^\perp$.

The floor of a predicate is well-defined due to the following lemma.

Lemma 35. *Let f and g be composable maps and suppose* $\mathrm{im}\, f \circ g$ *and* $\mathrm{im}\, f$ *exist. Then* $\mathrm{im}(f \circ g) \leqslant \mathrm{im}\, f$. *Furthermore, if g is an isomorphism, then* $\mathrm{im}(f \circ g) = \mathrm{im}\, f$.

Proof. We of course have $1 \circ (f \circ g) = (1 \circ f) \circ g = (\mathrm{im}\, f \circ f) \circ g = \mathrm{im}\, f \circ (f \circ g)$, and hence $\mathrm{im}\, f \leqslant \mathrm{im}\, f \circ g$.

If g is an isomorphism, then we furthermore have $\mathrm{im}\, f = \mathrm{im}(f \circ g) \circ g^{-1} \leqslant \mathrm{im}\, f \circ g \leqslant \mathrm{im}\, f$, and hence $\mathrm{im}\, f = \mathrm{im}\, f \circ g$. □

Indeed, for any two comprehensions π, π' for the same predicate, there exists an isomorphism α with $\pi = \pi' \circ \alpha$ so that $\mathrm{im}\,\pi \leqslant \mathrm{im}\,\pi' \leqslant \mathrm{im}\,\pi$ by lemma 35. For an effect $A \in B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ (viewed in the effectus of positive contractive maps between C^*-algebras in the opposite direction), the floor is the projection on $\mathcal{H}$ corresponding to the closure of the eigenvalue 1 subspace of A, while the ceiling is the projection onto the support of A, i.e. $\mathrm{Ker}(A)^\perp$.

While the floor of a predicate is always sharp by definition, in the general setting of effectuses it is not necessarily the case that the ceiling of a predicate is always sharp. This needs to be imposed additionally, and will give us the definition of a ⋄-effectus.

Definition 36. An effectus is a **⋄-effectus** (pronounced 'diamond-effectus') when it has images, filters, comprehensions and if a predicate p is sharp iff $p^\perp$ is sharp.

The reason we call it a ⋄-effectus, is because of the 'possibilistic' structure that is present in such an effectus.

Definition 37. Let A and B be objects in a ⋄-effectus. For any $f : A \to B$ we define

$$f^\diamond : \mathrm{SPred}(B) \to \mathrm{SPred}(A) \quad \text{and} \quad f_\diamond : \mathrm{SPred}(A) \to \mathrm{SPred}(B)$$

by $f^{\diamond}(p) := \lceil p \circ f \rceil$ and $f_{\diamond}(p) := \mathrm{im}(f \circ \pi_p)$, where π_p is any comprehension for p.

These maps forget the exact probabilities involved in f (indeed, for instance $(\frac{1}{2}f)^{\diamond} = f^{\diamond}$), and only remember what is possible.

The importance of this $\diamond$-structure is that it allows us to have a notion of adjointness between arbitrary processes, somewhat like a 'dagger':

Definition 38. We say maps $f : A \to B$ and $g : B \to A$ in a $\diamond$-effectus are **$\diamond$-adjoint** when $f^{\diamond} = g_{\diamond}$. An endomap $f : A \to A$ is **$\diamond$-self-adjoint** when $f^{\diamond} = f_{\diamond}$.

As mentioned in the previous section, we will want our pure maps to form a dagger-category. We want this dagger to interact nicely with the $\diamond$-structure, requiring that if $f^{\dagger} = g$, that then $f_{\diamond} = g^{\diamond}$, i.e. that f and g are $\diamond$-adjoint when they are $\dagger$-adjoint.

Remark 39. In general, a map does not have a unique $\diamond$-adjoint. A trivial reason is that $f^{\diamond} = (\frac{1}{2}f)^{\diamond}$. Interestingly, even if $f^{\diamond} = g^{\diamond}$ and $\mathbf{1} \circ f = \mathbf{1} \circ g$, then it does not have to be the case that $f = g$. If the previous is true (with fixed f for arbitrary g), then we say f is **rigid**. Rigidity for maps between von Neumann algebras is studied in [73, section 102].

Example 40. Let $\mathfrak{A}$ be a von Neumann algebra and let $a \in \mathfrak{A}$. Then the conjugation endomaps $b \mapsto a^{*}ba$ and $b \mapsto aba^{*}$ on $\mathfrak{A}$ are $\diamond$-adjoint and so are the standard comprehension and standard filter of a given projection (see example 25) [73, section 101].

There is much more to be said about $\diamond$-effectuses. We will study them in more detail in section 4.

2.1.5. Monoidal effectuses. In most works dealing with GPTs, the notion of a composite system is important. To talk about composite systems in a category we need monoidal structure: a tensor product. Effectuses do not need to have monoidal structure, and our main result does also not require the existence of a tensor product. However, to make the final jump in our reconstruction from general JBW-algebras to von Neumann algebras, we will require a tensor product. So let us give a definition of a monoidal effectus.

Definition 41. We say an effectus is **monoidal** when it has a symmetric monoidal structure $(\otimes, I)$ such that

- The tensor unit I is also the designated unit object of the effectus,
- The tensor product is 'biadditive', i.e. for any morphisms f, g, h with $f \perp g$ we have $(f \ovee g) \otimes h = (f \otimes h) \ovee (g \otimes h)$ and $0 \otimes h = 0$,
- And the tensor product preserves $\mathbf{1}$—that is: $\mathbf{1}_A \otimes \mathbf{1}_B = \mathbf{1}_{A\otimes B}$.

Let $\lambda_A : I \otimes A \to A$ denote the natural isomorphism for the tensor unit and let $s, t : I \to I$ be some scalars. Then for any morphism $f : A \to B$ we can define the map $s \cdot f$ as the composition $s \cdot f := \lambda_B \circ (s \otimes f) \circ \lambda_A^{-1}$. This gives us a scalar multiplication on morphisms in a monoidal effectus. Let us note the following straightforwardly verifiable facts.

Lemma 42. *Let* $\mathbf{C}$ *be a monoidal effectus, and let* $s, t : I \to I$ *be scalars. Then the following holds.*

- *Scalar multiplication respects composition: for any* $f : A \to B$ *and* $g : B \to C$ *we have* $g \circ (s \cdot f) = s \cdot (g \circ f) = (s \cdot g) \circ f$.
- *Scalar multiplication respects addition: for any* $f \perp g : A \to B$ *we have* $(s \ovee t) \cdot f = s \cdot f \ovee t \cdot f$ *and* $s \cdot (f \ovee g) = s \cdot f \ovee s \cdot g$..
- *For any predicate* $p : A \to I$ *we have* $s \cdot p = s \circ p$. *In particular* $s \cdot t = s \circ t$ *so that* $s \cdot (t \cdot f) = (s \cdot t) \cdot f = (s \circ t) \cdot f$.

2.2. Directed completeness

We will require the predicates to form a *dcpo*: a directed-complete poset. This requirement turns out to be surprisingly strong.

Definition 43. We say an effectus **C** is **directed complete** when all the predicate spaces Pred(A) are directed complete.[10] If in addition these suprema are preserved by all maps (i.e. the maps are *Scott continuous*) then we say **C** is **normal**.

Example 44. The category of sets is a normal effectus, as the predicate spaces are all complete Boolean algebras. The category of finite-dimensional C^*-algebras and positive linear contractions (in the opposite direction) is normal. This is not the case when including infinite-dimensional algebras. However, the category of von Neumann algebras with (normal) positive linear contractions is a (normal) directed-complete effectus.[11]

The scalars in a directed-complete effectus form a directed-complete effect monoid. In contrast to arbitrary effect monoids, the directed-complete ones are well-understood [71].

Example 45. Any complete Boolean algebra is a directed-complete effect monoid.

Example 46. Let X be an **extremally-disconnected** compact Hausdorff space, i.e. where the closure of every open set is open. Then $C(X, [0, 1])$ is a directed-complete effect monoid.

Theorem 47 ([71]). *Let* M *be a directed-complete effect monoid. Then there exists a complete Boolean algebra* B *and an extremally-disconnected compact Hausdorff space* X *such that* $M \cong B \oplus C(X, [0, 1])$.

Corollary 48. *If* **C** *is a directed-complete effectus with trivial object* I, *then there is an extremally-disconnected compact Hausdorff space* X *and a complete Boolean algebra* B *such that* $\mathrm{Pred}(I) \cong B \oplus C(X, [0, 1])$.

The characterisation result of theorem 47 has a corollary for irreducible effect monoids, also proven in [71].

[10] An effect algebra E is said to be directed complete, when every upwards-directed subset $U \subseteq E$ (i.e. where for every $x, y \in U$ there exists $z \geqslant x, y$ in U) has a supremum. As $(\ )^\perp$ is an order anti-automorphism this upwards-directed completeness is equivalent to downwards-directed completeness.

[11] In fact, a C^*-algebra is a von Neumann algebra iff its unit interval is directed-complete and it is separated by its normal states.

Theorem 49. *Let M be an irreducible directed-complete effect monoid. Then M is isomorphic (as an effect monoid) to* $\{0\}$, $\{0, 1\}$ *or* $[0, 1]$.

Hence, in a directed-complete effectus with irreducible scalars we have three possibilities for the scalars. These three different possibilities were analysed in [16]. If $\mathrm{Pred}(I) \cong \{0\}$ the entire category is equivalent to the trivial one-object category, so we can safely ignore this possibility. If $\mathrm{Pred}(I) \cong \{0, 1\}$, then we do not have any immediate useful consequences, however if we assume the effectus is separated by states or separated by predicates, then this implies a lot of structure, namely that all the predicate spaces are *orthoalgebras*. We will look at these in the next section, together with their counterpart, *convex* effect algebras that arise when $\mathrm{Pred}(I) \cong [0, 1]$.

2.3. Orthoalgebras, convexity and order unit spaces

Definition 50. Let E be an effect algebra. It is an **orthoalgebra** when 0 is the only self-summable element; i.e. when for every a with $a \perp a$, we have $a = 0$. We denote the full subcategory of **EA** consisting of the orthoalgebras by **OA**.

Examples of orthoalgebras include orthomodular lattices and Boolean algebras.

Proposition 51 ([16]). *Let* $\mathbf{C}$ *be an effectus separated by states where* $\mathrm{Pred}(I) \cong \{0, 1\}$. *Then every predicate space is an orthoalgebra.*

Definition 52. An effect algebra is **convex** [31] when there is a map $\cdot : [0, 1] \times E \to E$, where $[0, 1]$ is the regular unit interval, obeying the following axioms for all $x, y \in E$ and $\lambda,\ \mu \in [0, 1]$:

- $\lambda \cdot (\mu \cdot x) = (\lambda\mu) \cdot x$.
- If $\lambda + \mu \leqslant 1$, then $\lambda \cdot x \perp \mu \cdot x$ and $\lambda \cdot x \ovee \mu \cdot x = (\lambda + \mu) \cdot x$.
- $1 \cdot x = x$.
- $\lambda \cdot (x \ovee y) \ = \ \lambda \cdot x \ovee \lambda \cdot y$.

Denote by $\mathbf{EA}_c$, respectively $\mathbf{DCEA}_c$, the subcategory of **EA** consisting of (directed-complete) convex effect algebras and morphisms that preserve the convex action.

Example 53. Let V be an ordered real vector space (such as the space of self-adjoint elements of a C^*-algebra). Then any interval $[0, u]_V$ where $u \geqslant 0$ is a convex effect algebra with the obvious action of the real unit interval. Conversely, for any convex effect algebra E, we can find an ordered real vector space V and $u \in V$ such that E is isomorphic as a convex effect algebra to $[0, u]_V$ [32]. This is, in fact, an equivalance of categories [47].

Ordered real vector spaces play a central role in *generalised probabilistic theories* [6] that are often used in operational reconstructions of quantum theory, cf [27, 62]. For directed-complete convex effect algebras, this equivalence restricts to a more specific type of vector space.

Definition 54. An **order unit space** (OUS) $(V, \leqslant, 1)$ is an ordered vector space $(V, \leqslant)$ with a designated **order unit** 1 such that the induced semi-norm defined by $\|v\| := \inf\{\lambda \in$

$\mathbb{R}\ ;\ -\lambda 1 \leqslant v \leqslant \lambda 1\}$ is a norm and where the positive cone of V is closed in its topology. A **Banach** OUS is furthermore complete in this norm. We say an OUS is **directed complete** when its unit interval is.[12] Denote by **DCOUS** the category of directed-complete OUSs and positive linear contractions. Note that any directed-complete OUS is Banach [77, lemma 1.1].

Proposition 55 (cf [16, proposition 55]). *The equivalence between ordered vector spaces and convex effect algebras restricts to an equivalence* $\mathbf{DCOUS} \cong \mathbf{DCEA}_c$.

Proposition 56 ([16]). *Let* $\mathbf{C}$ *be an effectus with* $\mathrm{Pred}(I) \cong [0, 1]$. *Then all predicate spaces are convex effect algebras with the convex action given by* $\lambda \cdot p := \lambda \circ p$ *for* $\lambda \in \mathrm{Pred}(I)$.

This last result in particular implies that if $\mathbf{C}$ is additionally directed complete that all the predicate spaces are the unit intervals of directed-complete OUSs, and thus that the predicate functor gives a functor from $\mathbf{C}$ to $\mathbf{DCOUS}^{\mathrm{op}}$.

2.4. Jordan operator algebras

Our reconstruction of quantum theory will show that our category embeds into a category of *Jordan algebras*. These are a type of algebras originally introduced as a generalisation of a quantum system [48], but were quickly found to be very close to regular quantum systems. Indeed, the type of infinite-dimensional Jordan algebras we consider here, JBW-algebras, can be shown to embed into a von Neumann algebra up to a so-called ‘exceptional ideal’, and hence we do not lose much by working with Jordan algebras instead of C^*-algebras.

In this section we introduce the necessary concepts related to JBW-algebras. First, let us introduce the ‘Jordan version’ of a C^*-algebra.

Definition 57 ([36, proposition 3.1.6]). A **JB-algebra** $(A, *, 1, \leqslant)$ is a Banach OUS equipped with a binary operation $*\colon A \times A \to A$ satisfying for all $a, b, c \in A$:

- Commutativity: $a * b = b * a$.
- Unit: $a * 1 = 1 * a = a$.
- The **Jordan identity**: $(a * b) * (a * a) = a * (b * (a * a))$.
- If $-1 \leqslant a \leqslant 1$, then $0 \leqslant a * a \leqslant 1$.

Example 58. Let $\mathfrak{A}$ be a unital C^*-algebra. Let $\mathfrak{A}_{\mathrm{sa}}$ denote the space of self-adjoint elements and write $a * b := \frac{1}{2}(ab + ba)$ for the **special Jordan product**. Then $(\mathfrak{A}_{\mathrm{sa}}, *, 1, \leqslant)$ where $\leqslant$ is the standard order of a C^*-algebra is a JB-algebra. Its norm is the regular C^*-norm.

Example 59. A finite-dimensional JB-algebra A is a **Euclidean Jordan algebra** (EJA): a Jordan algebra equipped with an inner product $\langle \cdot, \cdot \rangle : A \times A \to \mathbb{R}$ such that $\langle a * b, c \rangle = \langle b, a * c \rangle$ for all $a, b, c \in A$ (and vice versa any EJA is a JB-algebra). The EJAs have been fully classified [48]: they are direct sums of simple EJAs, and these are either matrix algebras over the real, complex or quaternionic fields, a type of algebra known as a *spin factor*, or the

[12] This is equivalent to the whole OUS being *bounded* directed complete, i.e. where every bounded directed subset has a supremum.

so-called *exceptional* algebra of 3×3 Hermitian matrices over the octonions. Except for this last one, each of these algebras can be embedded into a C^*-algebra.

Remark 60. In reference [66] a reconstruction of finite-dimensional quantum theory is given that can be stated using our language (up to some details) as follows: let $\mathbf{C}$ be a state-separated $\diamond$-effectus with finite tomography where $\mathrm{Pred}(I) \cong [0, 1]$ such that

- The pure maps form a dagger category,
- A comprehension of a sharp predicate is †-adjoint to a filter of the same predicate,
- And comprehensions are †-isometries.

Then $\mathrm{Pred}(\mathbf{C})$ embeds into the category of EJAs (and positive contractive linear maps). If additionally $\mathbf{C}$ is symmetric monoidal in a suitably compatible way, then $\mathrm{Pred}(\mathbf{C})$ embeds into the category of finite-dimensional C^*-algebras.

It will be the goal of this paper to find a version of this result that works in infinite dimension. To do this we need the right generalisation of EJAs to infinite dimension. This turns out not to be JB-algebras, but *JBW-algebras*. These are a class of JB-algebras that have more structure. They relate to JB-algebras in an analogous manner to how von Neumann algebras (i.e. W^*-algebras) relate to C^*-algebras, hence the 'W' in 'JBW'.

Definition 61. Let A be an OUS (such as a JB-algebra). A **state** of A is a positive unital linear map $\omega : A \to \mathbb{R}$. We say a state (or more generally any positive linear map) is **normal** when it preserves suprema of directed sets: $\omega(\bigvee S) = \bigvee_{s\in S} \omega(s)$ for any directed S. We say A has a **separating set** of normal states when for any two $a, b \in A, a \neq b$ we can find a normal state ω such that $\omega(a) \neq \omega(b)$.

Definition 62. A JB-algebra A is a **JBW-algebra** when it is directed complete and has a separating set of normal states. We denote by $\mathbf{JBW}_{\mathrm{pc}}$ the category of JBW-algebras with positive linear contractions, and by $\mathbf{JBW}_{\mathrm{npc}}$ for the wide subcategory of normal positive linear contractions.

Example 63. Let $\mathfrak{A}$ be a **von Neumann algebra**, i.e. a C^*-algebra that is directed complete and has a separating set of normal states [49]. Then its space of self-adjoint elements $\mathfrak{A}_{\mathrm{sa}}$ equipped with the special Jordan product is a JBW-algebra.

JBW-algebras are very close to the more familiar von Neumann algebras. Indeed, a large class of JBW-algebras comes from von Neumann algebras:

Definition 64. A JBW-algebra A is a **JW-algebra** when it is Jordan-isomorphic to an ultraweakly-closed subset of the self-adjoint elements of a von Neumann algebra.

The counterpart to such 'well-behaved' algebras are the *exceptional* Jordan algebras.

Definition 65. Let A be a JB-algebra. We call A **purely exceptional** when any Jordan homomorphism $\phi : A \to \mathfrak{A}_{\mathrm{sa}}$ into a C^*-algebra $\mathfrak{A}$ is necessarily zero.

Theorem 66 ([36, theorem 7.2.7]). *Let A be a JBW-algebra. Then there is a unique decomposition $A = A_{\text{sp}} \oplus A_{\text{ex}}$ where A_{sp} is a JW-algebra and A_{ex} is a purely exceptional JBW-algebra.*

Interestingly, purely exceptional JBW-algebras only come in one type. To state this result, we need some more definitions.

Definition 67. Let X be a Stonean space (i.e. an extremally disconnected compact Hausdorff space). We call X **hyperstonean** when $C(X,\mathbb{R}) := \{f : X \to \mathbb{R} \text{ continuous}\}$ is separated by normal states.

Note that a compact Hausdorff space X is hyperstonean if and only if $C(X,\mathbb{R})$ is an associative JBW-algebra (or equivalently when $C(X,\mathbb{C})$ is a commutative von Neumann algebra).

Example 68 ([60]). Let X be a hyperstonean space and let $E = M_3(\mathbb{O})_{\text{sa}}$ denote the exceptional Albert algebra of 3×3 self-adjoint matrices of octonions $\mathbb{O}$ equipped with the standard Jordan product. Denote by $C(X,E)$ the set of continuous functions $f : X \to E$. Then $C(X,E)$ is a purely exceptional JBW-algebra with the Jordan product given pointwise by $(f * g)(x) = f(x) * g(x)$.

Theorem 69 ([60]). *Let A be a purely exceptional JBW-algebra. Then there exists a hyperstonean space X, such that $A \cong C(X, M_3(\mathbb{O}))$.*

Combining theorems 66 and 69 we see that any JBW-algebra splits up into a part that embeds into a von Neumann algebra and a part that is characterised by a hyperstonean space.

All of the above are standard results regarding Jordan operator algebras, and JBW-algebras in particular. In the PhD thesis of one of the authors, the properties of JBW-algebras with regards to effectus theory were studied [69, chapter 5]. Therein it was shown that $\mathbf{JBW}^{\text{op}}_{\text{npc}}$ is a $\diamond$-effectus [69, theorem 4.6.1], and that its pure maps form a †-category [69, theorem 4.6.27]. In fact, all of the properties outlined in remark 60 (apart from finite tomography of course) are satisfied by $\mathbf{JBW}^{\text{op}}_{\text{npc}}$ [69, proposition 4.6.30]. The dagger structure was found by first considering the **$\diamond$-positive** maps of JBW-algebras. We say a map f is $\diamond$-positive when $f = g \circ g$ for g a $\diamond$-self-adjoint map. As shown in [69, theorem 4.6.17], for pure $\diamond$-positive maps f and g, if $f(1) = g(1)$, then $f = g$. There is hence for every predicate a *unique* $\diamond$-positive map. We will impose a similar uniqueness property in our reconstruction.

Ideally we would now be able to directly generalise the result of remark 60 by just dropping the requirement on finite tomography and showing that the resulting categories embed in $\mathbf{JBW}_{\text{npc}}$. We have however not been able to do so and our proof requires some additional assumptions.

2.5. Sequential products

Our reconstruction relies heavily on the categorical structures outlined in section 2.1. However, we will also need some assumptions that are of a more operational nature. In particular, we consider the operation of 'measuring' a predicate. This will take the form of a self-map $\text{asrt}_p : A \to A$ for each predicate $p \in \text{Pred}(A)$ that 'asserts' that p is true. For an effect $p \in B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ this map is of the form $\text{asrt}_p(q) = \sqrt{p}q\sqrt{p}$. When given a set of assert maps for each predicate we denote $p \,\&\, q := q \circ \text{asrt}_p$ for the *sequential product* that can be interpreted as 'observe p and then observe q' [30]. We will require $\&$ to satisfy a number of assumptions that will make Pred(A) into a SEA [29]. Before we give the definition, let us motivate some of these conditions.

The sequential product $p \ \& \ q$ of two effects p and q represents the sequential measurement of first p and then q. An important difference between classical and quantum systems is that in a classical system we can measure without disturbance, and hence the order of measurement is not important: $p \ \& \ q = q \ \& \ p$ for all predicates p and q. In a quantum system this is generally not the case, and the order of measurement is important (indeed, this is essentially Heisenberg uncertainty.) However, what is interesting in quantum theory is that some measurements are *compatible*, meaning that the order of measurement for those measurements is not important. We will use the symbol $p|q$ to denote that $p \ \& \ q = q \ \& \ p$.

Definition 70. A **sequential effect algebra** (SEA) [29] E is an effect algebra with an additional (total) binary operation $\&$, called the **sequential product**, satisfying the axioms listed below, where $a, b, c \in E$. Elements a and b are said to **commute**, written $a|b$, whenever $a \ \& \ b = b \ \& \ a$.

(a) $a \ \& \ (b \ovee c) = a \ \& \ b \ovee a \ \& \ c$ whenever $b \perp c$.
(b) $1 \ \& \ a = a$.
(c) $a \ \& \ b = 0 \Rightarrow b \ \& \ a = 0$.
(d) If $a|b$, then $a|b^{\perp}$ and $a \ \& \ (b \ \& \ c) = (a \ \& \ b) \ \& \ c$ for all c.
(e) If $c|a$ and $c|b$ then also $c|a \ \& \ b$ and if furthermore $a \perp b$, then $c \mid a \ovee b$.

A SEA E is called **normal** when E is directed complete, and

(f) Given directed $S \subseteq E$ we have $a \ \& \ \bigvee S = \bigvee_{s \in S} a \ \& \ s$, and $a| \bigvee S$ when $a|s$ for all $s \in S$.

Normal SEAs were studied in [72], where they were shown to have many desirable properties. Let us note some of these properties here for later reference. Note that we call an effect p idempotent when $p \ \& \ p = p$.

Lemma 71 ([72]). *Let E be a normal SEA and let $a, b \in E$ be arbitrary. Then the following are true*

(*a*) *There is a smallest idempotent effect above a, which we denote by $\lceil a \rceil$.*
(*b*) *There is a largest idempotent effect beneath a, which we denote by $\lfloor a \rfloor$.*
(*c*) *If $b \ \& \ a = a$, then $b \geqslant \lceil a \rceil$.*
(*d*) *The idempotents form a complete lattice.*

Remark 72. In [72] it is also shown that normal SEAs satisfy a spectral theorem, and that in particular every effect can be written as a supremum and norm-limit of a sequence of *simple* effects, effects that are finite linear combinations of idempotent effects. This implies that the sharp effects span a norm-dense set of effects. These properties will sometimes implicitly be used in our reconstruction, in particular in proving lemma 121.

The unit interval of a JBW-algebra is an example of a normal SEA. This is defined in terms of the **quadratic product**. Let A be a JBW-algebra and $a, b \in A$ arbitrary. Then we define $Q_a : A \to A$ as the map $Q_a b = 2a * (a * b) - a^2 * b$. While this might look arbitrary, when A is a JW-algebra this boils down to $Q_a b = aba$ using the product in the underlying von Neumann algebra. It is then perhaps not too surprising that the operation $a \,\&\, b := Q_{\sqrt{a}} b$ on the unit interval of a JBW-algebra defines a normal sequential product [68].

What is perhaps more surprising is that there is also a converse to this. As it will inform the structure of our reconstruction proof, let us now recall two properties introduced in [67] that force a convex normal SEA to have a Jordan algebra structure. Analogously to the definition for OUSs, we call a map $\omega : E \to [0, 1]$ for a convex SEA E a state when ω is linear (i.e. additive and preserves the scalar multiplication) and $\omega(1) = 1$.

Definition 73. We say the sequential product of a convex SEA E is **compressible** when for all idempotent effects $p \in E$ the following implication holds for all states $\omega : E \to [0, 1]$: if $\omega(p) = 1$, then $\omega(p \,\&\, a) = \omega(a)$ for all $a \in E$.

What this property says is that if an effect p already holds with certainty on a state ω, then measuring p does not affect the probabilities of other effects holding in the state ω.[13] The second property is a weaker version of the *fundamental identity of quadratic Jordan algebras* [57].

Definition 74. We say the sequential product of a SEA E is **quadratic** when for any two idempotents $p, q \in E$ we have $q \,\&\, (p \,\&\, q) = (q \,\&\, p)^2$.

Theorem 75 ([67, theorem 4]). *Let E be a convex normal SEA and suppose its sequential product is compressible and quadratic. Then E is order-isomorphic to the unit interval of a directed-complete JB-algebra.*[14]

3. Summary of main results

Now that we have laid out all the definitions and known results we will need, we can state the type of effectus we will need for our reconstruction.

Definition 76. A **sequential** effectus is a normal effectus (cf definition 43) separated by states satisfying the following.

(a) The effectus has filters and comprehensions.
(b) Comprehensions have images.
(c) The pure maps form a dagger-category.
(d) Every pure map f is $\diamond$-adjoint to $f^\dagger$.
(e) For every predicate $p \in \mathrm{Pred}(A)$ there is a unique $\dagger$-positive pure map $\mathrm{asrt}_p : A \to A$ satisfying $\mathbf{1}_A \circ \mathrm{asrt}_p = p$ called the **assert map** of p.

[13] The work of Alfsen and Shultz uses the notion of a *compression*, which is a special type of an idempotent map [1]. The multiplication maps $L_p(a) = p \,\&\, a$ of convex SEAs always satisfy three of the four conditions of being a compression. The fourth condition, namely the implication $\omega \circ L_{p^\perp} = 0 \Rightarrow \omega \circ L_p = \omega$, is satisfied iff the SEA is compressible, hence the name.

[14] As far as the authors are aware, there is no known example of a convex sequential effect algebra that is not compressible, nor one that is not quadratic. Hence, it might be that these properties hold for all convex SEAs and thus that the conditions in this theorem can be simplified.

(f) For every object A, the operation &: Pred(A) × Pred(A) → Pred(A) given by $p \;\&\; q := q \circ \mathrm{asrt}_p$ is a normal sequential product, making Pred(A) into a normal SEA.

Remark 77. Points (a)–(d) are variations on the properties outlined in remark 60. Point (e) is a new assumption. We remark that the uniqueness condition of point (e) can be framed as the implication $\mathbf{1} \circ f^\dagger \circ f = \mathbf{1} \circ g^\dagger \circ g \Rightarrow f^\dagger \circ f = g^\dagger \circ g$ for any pure f and g. In this sense it is similar to the *CPM axiom* [19] for an *environment structure* (i.e. choice of pure maps) which states that $\mathbf{1} \circ f = \mathbf{1} \circ g \Leftrightarrow f^\dagger \circ f = g^\dagger \circ g$. We will see that for sharp predicates this definition of an assert map reduces to that of definition 92.

Remark 78. The opposite category of JBW-algebras $\mathbf{JBW}_{\mathrm{npc}}^{\mathrm{op}}$ is a sequential effectus. This follows from the results of [69, chapter 4]. In particular, points (a) and (b) follows from [69, theorem 4.6.1], (c) and (d) from [69, theorem 4.6.29], (e) from [69, theorem 4.6.17] and (f) from [69, theorem 4.7.18]

We will prove three theorems about sequential effectuses of increasing specificity. First, for a general sequential effectus:

Theorem. *Let* $\mathbf{C}$ *be a sequential effectus. Then* $\mathbf{C}$ *is equivalent to a product of effectuses* $\mathbf{C} \cong \mathbf{C}_1 \times \mathbf{C}_2$ *where the predicate spaces of* $\mathbf{C}_1$ *are complete Boolean algebras and those of* $\mathbf{C}_2$ *are directed-complete JB-algebras. In particular, letting* **CBA** *denote the category of complete Boolean algebras and monotone maps, we have predicate functors* $\mathrm{Pred} : \mathbf{C}_1 \to \mathbf{CBA}^{\mathrm{op}}$ *and* $\mathrm{Pred} : \mathbf{C}_2 \to \mathbf{JB}_{\mathrm{npc}}^{\mathrm{op}}$ *that are faithful iff* $\mathbf{C}$ *is separated by predicates.*

Hence, in a sequential effectus, the predicate spaces either correspond to Boolean algebras (and hence are classical) or they are (directed complete) JB-algebras, and so can represent quantum systems. When we restrict our scalars to be irreducible, the predicate spaces in fact correspond to JBW-algebras.

Theorem. *Let* $\mathbf{C}$ *be a sequential effectus with irreducible scalars. Then all predicate spaces are either complete Boolean algebras or they are all the unit interval of a JBW-algebra. Furthermore, there is a functor* $\mathrm{Pred} : \mathbf{C} \to \mathbf{D}^{\mathrm{op}}$ *where* $\mathbf{D}$ *is either* **CBA** *of* $\mathbf{JBW}_{\mathrm{npc}}$, *and this functor is faithful iff* $\mathbf{C}$ *is separated by predicates.*

Remark 79. In the first theorem we talk about directed-complete JB-algebras, while in the second we talk about JBW-algebras, i.e. those directed-complete JB-algebras that are separated by normal states. The difference comes from the fact that in the first case our scalars can satisfy $\mathrm{Pred}(I) \cong [0, 1]_{C(X)}$ where X is an arbitrary Stonean space, and hence $C(X)$ does not have to be a JBW-algebra. This might seem surprising as in definition 76 we require separation of states and that all maps are normal. However, this talks about states as maps $I \to A$ in the effectus, while the condition to be a JBW-algebra requires normal states as maps $V_A \to \mathbb{R}$, where V_A is the OUS corresponding to A. For instance, to satisfy the condition of separation by states on I we only need the map $\mathrm{id} : I \to I$. It is unclear which, categorically nice, condition we could require that forces systems to be separated by normal states in the correct sense, without restricting the scalars to the irreducible ones.

Now, by theorem 66 we know that any JBW-algebra splits up into a direct sum of a JW-algebra (which is part of a von Neumann algebra), and a purely exceptional algebra. By going to the monoidal setting, the purely exceptional algebras become 'disallowed' as they do not have nice tensor products. Let us denote by $\mathbf{JW}_{\text{npc}}$ the full subcategory of $\mathbf{JBW}_{\text{npc}}$ consisting of the JW-algebras.

Definition 80. A **monoidal sequential effectus** is a sequential effectus that is monoidal and such that

- The tensor product of two pure maps is pure,
- For pure f and g we have $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$.

Theorem. *Let* $\mathbf{C}$ *be a monoidal sequential effectus with irreducible scalars not equal to* $\{0, 1\}$. *Then there is a functor* $F : \mathbf{C} \to \mathbf{JW}_{\text{npc}}^{\text{op}}$ *satisfying* $F(\text{Pred}(A)) \cong [0, 1]_{F(A)}$. *This functor is faithful if and only if* $\mathbf{C}$ *is separated by predicates.*

The other option for irreducible scalars is that they are equal to the Booleans $\{0, 1\}$, which results in predicate spaces being complete Boolean algebras. So we either get deterministic classical systems, or probabilistic quantum systems in the form of (certain subspaces) of von Neumann algebras.

A couple of remarks about this result are in order.

Remark 81. None of our assumptions requires the predicate spaces to correspond to a noncommutative algebra. The category could be entirely classical in the sense that all the underlying von Neumann algebras are commutative. We could impose any of a number of additional assumptions that would require there to be truly quantum systems, such as the existence of dilations, which are explored in the context of effectus theory in [76, section 3.7.1].

Remark 82. While the category of JBW-algebras $\mathbf{JBW}_{\text{npc}}^{\text{op}}$ does satisfy all our (non-monoidal) assumptions (see remark 78), the category of JW-algebras $\mathbf{JW}_{\text{npc}}^{\text{op}}$ does *not* satisfy our monoidal assumptions, for the simple reason that it doesn't have a well-behaved tensor product. In [34] it is shown that the 'obvious' choice of tensor product only works for some JW-algebras and that this would preclude the existence of some types of systems, such as quaternionic ones. Instead a different tensor product can be defined, which does work for all JW-algebras, but this differs from the standard tensor product in odd ways, such as setting the tensor product of a qubit system $M_2(\mathbb{C})_{\text{sa}}$ with itself to $M_4(\mathbb{C})_{\text{sa}} \oplus M_4(\mathbb{C})_{\text{sa}}$. This tensor does not satisfy our assumptions, since it can map pure predicates to non-pure predicates. The functor into $\mathbf{JBW}_{\text{npc}}^{\text{op}}$ we find is hence not monoidal. However, this does not preclude the possibility of our category actually embedding into a subcategory of $\mathbf{JBW}_{\text{npc}}^{\text{op}}$ that *does* have a monoidal structure for which our functor is monoidal. We suspect that the types of subcategories of $\mathbf{JW}_{\text{npc}}$ that satisfy our assumptions can contain either *only* real algebras *or* complex ones, in which case the obvious choice of tensor product would be the correct one and the functor is monoidal. In fact, if our category contains an object isomorphic to $M_2(\mathbb{C})_{\text{sa}}$ (i.e. a qubit), then using [34, theorem 5.5], all objects must be isomorphic to the self-adjoint part of a (complex) C^*-algebra, so that we would get back the standard quantum-theoretical framework.

Before we are able to prove these results, there are a couple of results about more general effectuses we need to prove. We give a schematic overview of the proof and the different intermediate results and structures in figure 1.

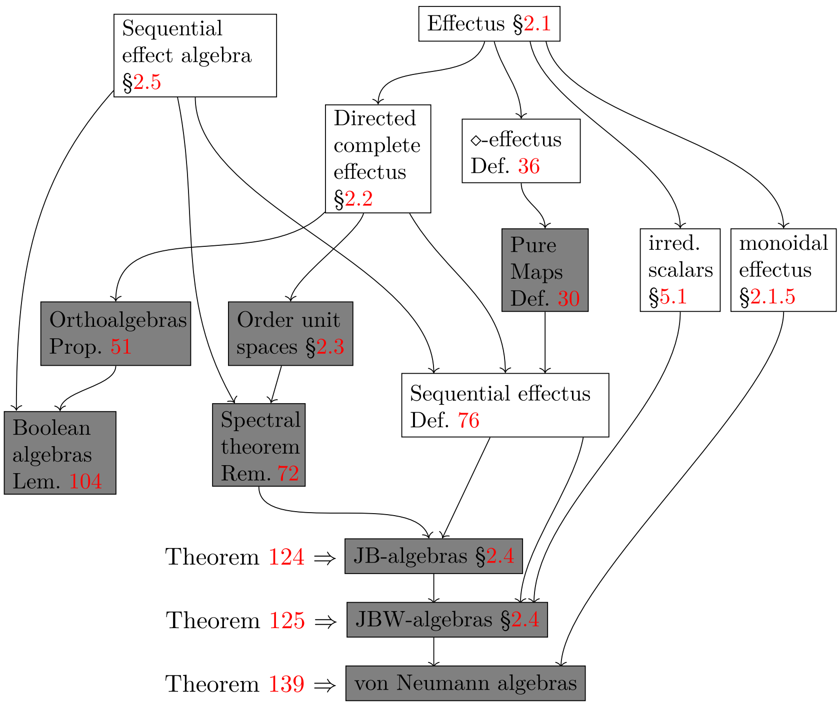


Figure 1. Structure of the proof. White boxes represent assumptions, while grey boxes represent derived concepts and structures. An arrow from A to B denote that the concept or proof for B depends on A.

In section 4 we study $\diamond$-effectuses without any further structure. We show that the $f_\diamond$ and $f^\diamond$ maps are related in interesting ways (proposition 84), and that this leads to the sharp predicates forming an orthomodular lattice (proposition 88). In fact, for any $\diamond$-effectus $\mathbf{C}$ there is a functor $\mathbf{C} \to \mathbf{OMLatGal}$ into the category $\mathbf{OMLatGal}$ of orthomodular lattices and Galois connections between them (proposition 90).

Then in section 5 we study various conditions under which an effectus is equivalent to a product of effectuses. Particularly relevant to our reconstruction is proposition 103 which states that a state-separated directed-complete effectuses satisfying some conditions that follows from our reconstruction assumptions splits up into an effectus where the scalars are a Boolean algebra and an effectus where the scalars are convex.

With these results in hand we can prove our main (non-monoidal) reconstruction results in section 6 and the monoidal reconstruction in section 7.

4. Properties of filters and comprehensions

In this section we will take a closer look at the properties of filters, comprehensions, images and $\diamond$-adjointness, which form the backbone of our reconstruction.

First, let us establish some basic properties of the floor and ceiling operations (definition 34).

Proposition 83. *In an effectus with images and comprehensions, the following are true for any predicates $q \leqslant p$ and composable map f.*

(a) $\lfloor p \rfloor \leqslant p \leqslant \lceil p \rceil$.	(d) $\lceil p \circ f \rceil = \lceil \lceil p \rceil \circ f \rceil$.
(b) $\lfloor \lfloor p \rfloor \rfloor = \lfloor p \rfloor$.	(e) $\lceil p \rceil \circ f = 0 \Leftrightarrow p \circ f = 0$.
(c) $\lfloor q \rfloor \leqslant \lfloor p \rfloor$ and $\lceil q \rceil \leqslant \lceil p \rceil$.	(f) p is sharp if and only if $\lfloor p \rfloor = p$.

Proof. Let $p, q : A \to I$ be predicates, and let $\pi_p : A_p \to A$ be a comprehension for p, and $\pi_q : A_q \to A$ a comprehension for q.

(a) Of course $1 \circ \pi_p = p \circ \pi_p$, and hence $\lfloor p \rfloor := \operatorname{im} \pi_p \leqslant p$. Hence also $\lfloor p^\perp \rfloor \leqslant p^\perp$, and thus $\lceil p \rceil := \lfloor p^\perp \rfloor^\perp \geqslant (p^\perp)^\perp = p$.

(b) We will show that π_p is a comprehension for $\lfloor p \rfloor$, and hence $\pi_{\lfloor p \rfloor} = \pi_p \circ \Theta$ for some isomorphism Θ. The result then follows using lemma 35, because $\lfloor \lfloor p \rfloor \rfloor := \operatorname{im} \pi_{\lfloor p \rfloor} = \operatorname{im} \pi_p \circ \Theta = \operatorname{im} \pi_p = \lfloor p \rfloor$.

Note first that $\lfloor p \rfloor \circ \pi_p = \operatorname{im} \pi_p \circ \pi_p = 1 \circ \pi_p$. Now let $f : B \to A$ be some map with $\lfloor p \rfloor \circ f = 1 \circ f$. As $\lfloor p \rfloor \leqslant p$, we then also have $p \circ f = 1 \circ f$, and hence by the universal property of π_p there is a unique $\overline{f}$ with $f = \pi_p \circ \overline{f}$. Hence, π_p is also a comprehension for $\lfloor p \rfloor$.

(c) We have $1 \circ \pi_q = q \circ \pi_q \leqslant p \circ \pi_q \leqslant 1 \circ \pi_q$, and hence $p \circ \pi_q = 1 \circ \pi_q$. Hence $\pi_q = \pi_p \circ \overline{\pi_q}$ for a unique $\overline{\pi_q}$. With lemma 35 we calculate $\lfloor q \rfloor := \operatorname{im} \pi_q = \operatorname{im} \pi_p \circ \overline{\pi_q} \leqslant \operatorname{im} \pi_p = \lfloor p \rfloor$. To show $\lceil q \rceil \leqslant \lceil p \rceil$, we note that as $q \leqslant p$, we have $p^\perp \leqslant q^\perp$ and hence $\lfloor p^\perp \rfloor \leqslant \lfloor q^\perp \rfloor$. Then: $\lceil q \rceil := \lfloor q^\perp \rfloor^\perp \leqslant \lfloor p^\perp \rfloor^\perp = \lceil p \rceil$.

(d) First note that since $\lceil p \rceil \circ f \geqslant p \circ f$, we have by point (c): $\lceil \lceil p \rceil \circ f \rceil \geqslant \lceil p \circ f \rceil$. It remains to show the inequality in the other direction.

Because $p \circ (f \circ \pi_{(p \circ f)^\perp}) = 0$, there must be an h with $f \circ \pi_{(p \circ f)^\perp} = \pi_{p^\perp} \circ h$. By point (b) there must be some isomorphism Θ such that $\pi_{p^\perp} = \pi_{\lfloor p^\perp \rfloor} \circ \Theta = \pi_{\lceil p \rceil^\perp} \circ \Theta$. We then calculate:

$$\lceil p \rceil \circ f \circ \pi_{(p \circ f)^\perp} \;=\; \lceil p \rceil \circ \pi_{p^\perp} \circ h \;=\; \lceil p \rceil \circ \pi_{\lceil p \rceil^\perp} \circ \Theta \circ h \;=\; 0.$$

Hence $\lceil p \rceil \circ f \leqslant \operatorname{im} \pi_{(p \circ f)^\perp}{}^\perp = \lfloor (p \circ f)^\perp \rfloor^\perp = \lceil p \circ f \rceil$. Using points (c) and (b): $\lceil \lceil p \rceil \circ f \rceil \leqslant \lceil \lceil p \circ f \rceil \rceil = \lceil p \circ f \rceil$.

(e) Of course if $\lceil p \rceil \circ f = 0$, then $p \circ f \leqslant \lceil p \rceil \circ f = 0$. For the other direction, we remark that $\lceil 0 \rceil = \lfloor 1 \rfloor^\perp = 1^\perp = 0$, so that by the previous point: $0 = \lceil 0 \rceil = \lceil p \circ f \rceil = \lceil \lceil p \rceil \circ f \rceil$, and hence $\lceil p \rceil \circ f \leqslant \lceil \lceil p \rceil \circ f \rceil = 0$.

(f) If $\lfloor p \rfloor = p$, then $p = \operatorname{im} \pi_p$, and hence p is sharp. Now suppose p is sharp, and hence is the image of some map f: $p = \operatorname{im} f$. Then by the universal property of π_p, there is some $\overline{f}$ such that $f = \pi_p \circ \overline{f}$. We then calculate using lemma 35 $p = \operatorname{im} f = \operatorname{im} \pi_p \circ \overline{f} \leqslant \operatorname{im} \pi_p = \lfloor p \rfloor$. As $\lfloor p \rfloor \leqslant p$ by point (a), we are done. □

In the setting where we have filters, comprehensions and images, the floor of a predicate is already well-behaved, but the ceiling is not necessarily. In particular, it is not necessarily the case that $\lceil p \rceil$ is sharp. To ensure this we must require that p is sharp iff $p^\perp$ is sharp. This lead us to the notion of a $\diamond$-effectus (definition 36) and the special types of maps we can define in them that we repeat here for clarity.

Definition. An effectus is a **$\diamond$-effectus** (pronounced 'diamond-effectus') when it has images, filters, comprehensions and if a predicate p is sharp iff $p^\perp$ is sharp.

Recall that we write SPred(A) for the set of sharp predicates of A.

Definition. Let A and B be objects in a $\diamond$-effectus. For any $f : A \to B$ we define

$$f^\diamond : \mathrm{SPred}(B) \to \mathrm{SPred}(A) \quad \text{and} \quad f_\diamond : \mathrm{SPred}(A) \to \mathrm{SPred}(B)$$

by $f^\diamond(p) := \lceil p \circ f \rceil$ and $f_\diamond(p) := \mathrm{im}(f \circ \pi_p)$, where π_p is any comprehension for p.

For the following proposition it will be useful to introduce a third such map: $f^\Box(p) := (f^\diamond(p^\perp))^\perp$.

Proposition 84. *Let $f : A \to B$ and $g : B \to C$ be maps in a $\diamond$-effectus and let $p \in \mathrm{SPred}(B)$ and $q \in \mathrm{SPred}(A)$. Then the following are true.*

(a) $f^\diamond$ and $f^\Box$ are monotone.	(e) $f_\diamond \circ f^\Box \circ f_\diamond = f_\diamond$.
(b) $f^\diamond(p) \leqslant q^\perp \Leftrightarrow f_\diamond(q) \leqslant p^\perp$.	(f) $(\mathrm{id})^\diamond = (\mathrm{id})_\diamond = (\mathrm{id})^\Box = \mathrm{id}$.
(c) $f_\diamond$ and $f^\Box$ form a Galois connection.	(g) $(f \circ g)^\diamond = g^\diamond \circ f^\diamond$, $(f \circ g)^\Box = g^\Box \circ f^\Box$.
(d) $f_\diamond$ is monotone.	(h) $(f \circ g)_\diamond = f_\diamond \circ g_\diamond$.

Proof.

(a) Suppose $p \leqslant q$. Then $p \circ f \leqslant q \circ f$ and hence $f^\diamond(p) = \lceil p \circ f \rceil \leqslant \lceil q \circ f \rceil = f^\diamond(q)$. Also $q^\perp \leqslant p^\perp$ and hence $\lceil q^\perp \circ f \rceil \leqslant \lceil p^\perp \circ f \rceil$. Taking complements again gives $f^\Box(p) \leqslant f^\Box(q)$.

(b) Suppose $f^\diamond(p) \leqslant q^\perp$. Then $p \circ f \leqslant \lceil p \circ f \rceil = f^\diamond(p) \leqslant q^\perp = \mathrm{im}\ \pi_q{}^\perp$ and hence $p \circ f \circ \pi_q = 0$ so that $p \leqslant \mathrm{im}\ f \circ \pi_q{}^\perp$. But then $f_\diamond(q) = \mathrm{im}\, f \circ \pi_q \leqslant p^\perp$.

(c) Suppose $f_\diamond(q) \leqslant p$. We need to show $q \leqslant f^\Box(p)$. The previous point gives $f_\diamond(q) \leqslant p$ iff $f^\diamond(q^\perp) \leqslant p^\perp$ and hence $p \leqslant f^\diamond(q^\perp)^\perp = : f^\Box(q)$.

(d) As $f_\diamond(q) \leqslant f_\diamond(q)$ the previous point gives $q \leqslant f^\Box(f_\diamond(q))$. Now suppose $p \leqslant q$. Then $p \leqslant q \leqslant f^\Box(f_\diamond(q))$ so that again by the previous point $f_\diamond(p) \leqslant f_\diamond(q)$.

(e) We have $f_\diamond(f^\Box(p)) \leqslant p$. Letting $p := f_\diamond(q)$ we get $f_\diamond f^\Box f_\diamond(q) \leqslant f_\diamond(q)$. We also have $q \leqslant f^\Box f_\diamond(q)$ and hence applying the monotone $f_\diamond$ to both sides gives the other inequality.

(f) For sharp p we have $\lceil p \rceil = p = \mathrm{im}\, \pi_p$. The statements then follow easily.

(g) By proposition 83: $(f \circ g)^\diamond(p) = \lceil p \circ f \circ g \rceil = \lceil \lceil p \circ f \rceil \circ g \rceil = g^\diamond(f^\diamond(p))$. Furthermore $(f \circ g)^\Box(p) = (f \circ g)^\diamond(p^\perp)^\perp = g^\diamond(f^\diamond(p^\perp))^\perp = g^\diamond(f^\Box(p)^\perp)^\perp = g^\Box(f^\Box(p))$.

(h) $(f \circ g)_\diamond$ is left Galois adjoint to $(f \circ g)^\Box$. As Galois adjoints are unique it suffices to show that $f_\diamond \circ g_\diamond$ is also left Galois adjoint to $(f \circ g)^\Box$. We calculate:

$$f_\diamond(g_\diamond(p)) \leqslant q \Leftrightarrow g_\diamond(p) \leqslant f^\Box(q) \Leftrightarrow p \leqslant g^\Box(f^\Box(q)) = (f \circ g)^\Box(q).$$

$\Box$

In our reconstruction we need the concept of $\diamond$-adjointness (definition 38): f and g are $\diamond$-adjoint when $f^\diamond = g_\diamond$.

Lemma 85. *$\diamond$-adjointness is a symmetric relation: $f^\diamond = g_\diamond$ iff $g^\diamond = f_\diamond$.*

Proof. Suppose $f^\diamond = g_\diamond$. Using proposition 84(b) twice, we calculate:

$$f_\diamond(p) \leqslant q^\perp \quad\Leftrightarrow\quad f^\diamond(q) \leqslant p^\perp \quad\Leftrightarrow\quad g_\diamond(q) \leqslant p^\perp \quad\Leftrightarrow\quad g^\diamond(p) \leqslant q^\perp.$$

Then take $q := f_\diamond(p)^\perp$ and $q := g^\diamond(p)^\perp$ to get inequalities in both directions that proves $f_\diamond(p) = g^\diamond(p)$. □

The above results are needed for our main result. We will now prove some additional properties of $\diamond$-effectuses that have no further bearing on our reconstruction, but might be of independent interest.

First, sharp predicates in a $\diamond$-effectus are also 'sharp' in a more standard sense.

Lemma 86. *Let p be a sharp predicate in a $\diamond$-effectus. Then p is also **ortho-sharp**, namely $p \wedge p^\perp = \mathbf{0}$.*

Proof. Let q be any predicate on the same object as p. Suppose $q \leqslant p$ and $q \leqslant p^\perp$. We need to show that $q = \mathbf{0}$. Note $\lceil q \rceil \leqslant \lceil p \rceil = \lfloor p^\perp \rfloor^\perp = (p^\perp)^\perp = p$ (using proposition 83(f)), and similarly $\lceil q \rceil \leqslant p^\perp$. Let $\pi_{\lceil q \rceil}$ and π_p be comprehensions for $\lceil q \rceil$, respectively p. Then by the universal property of π_p there is a unique f such that $\pi_{\lceil q \rceil} = \pi_p \circ f$. We then calculate

$$\mathbf{1} \circ \pi_{\lceil q \rceil} \;=\; \lceil q \rceil \circ \pi_{\lceil q \rceil} \;=\; \lceil q \rceil \circ \pi_p \circ f \;\leqslant\; p^\perp \circ \pi_p \circ f \;=\; \mathbf{0}.$$

Hence $\pi_{\lceil q \rceil} = 0$ so that $q \leqslant \lceil q \rceil = \mathrm{im}\,\pi_{\lceil q \rceil} = \mathbf{0}$ as desired. □

Lemma 87. *Let $p, q \in \mathrm{SPred}(A)$ be two sharp predicates in a $\diamond$-effectus. Then the following holds.*

- *The supremum $p \vee q$ in $\mathrm{Pred}(A)$ exists and is sharp.*
- *The sharp predicates $\mathit{SPred}(A)$ form an **ortholattice**: a lattice with orthocomplement satisfying $p \wedge p^\perp = \mathbf{0}$ and $p \vee p^\perp = \mathbf{1}$.*

Proof. Let $p, q \in \mathrm{SPred}(A)$. We claim that $p \wedge q = (\pi_p)_\diamond(\pi_p^\Box(q))$. First of all, as $q \leqslant q$, we have $\pi_p^\Box(q) \leqslant \pi_p^\Box(q)$ and thus $(\pi_p)_\diamond(\pi_p^\Box(q)) \leqslant q$. Second, $(\pi_p)_\diamond(\pi_p^\Box(q)) \leqslant (\pi_p)_\diamond(1) = \mathrm{im}\ \pi_p = p$, so it is indeed a lower bound. Now let r be any sharp element with $r \leqslant p$ and $r \leqslant q$. Then as in the previous point $\pi_r = \pi_p \circ h$ for some map h. Now using that $(\pi_p)_\diamond = (\pi_p)_\diamond \circ (\pi_p)^\Box \circ (\pi_p)_\diamond$; cf proposition 84(e):

$$(\pi_r)_\diamond \;=\; (\pi_p)_\diamond \circ h_\diamond \;=\; (\pi_p)_\diamond \circ (\pi_p)^\Box \circ (\pi_p)_\diamond \circ h_\diamond \;=\; (\pi_p)_\diamond \circ (\pi_p)^\Box \circ (\pi_r)_\diamond$$

and thus $r = (\pi_r)_\diamond(1) = (\pi_p)_\diamond \circ (\pi_p)^\Box(r) \leqslant (\pi_p)_\diamond \circ (\pi_p)^\Box(q)$. Now for a general $a \leqslant p, q$, we will also have $\lceil a \rceil \leqslant p, q$ and hence $a \leqslant \lceil a \rceil \leqslant p \wedge q$.

Now to show $\mathrm{SPred}(A)$ forms an ortholattice, first note that $(p^\perp \vee q^\perp)^\perp = p \wedge q$ so that is indeed a lattice. By lemma 86 we have $p \wedge p^\perp = \mathbf{0}$ and hence $p \vee p^\perp = (p^\perp \wedge p)^\perp = \mathbf{0}^\perp = \mathbf{1}$. □

Proposition 88. *Let A be an object in a $\diamond$-effectus. Then SPred(A) is a sub-effect-algebra of* Pred(A)*. Furthermore, SPred(A) is an orthomodular lattice.*

Proof. SPred(A) contains $\mathbf{0}$, $\mathbf{1}$ and is closed under the complement so it remains to show that it is closed under sums. Let $p, q \in$ SPred(A) and suppose p and q are summable. We claim that $p \ovee q = p \vee q$ so that $p \ovee q$ is indeed sharp. We have $p \leqslant q^\perp$ and hence $p \wedge q \leqslant q^\perp \wedge q = \mathbf{0}$. That $p \ovee q = p \vee q$ then follows from the identity $a \ovee b = (a \wedge b) \ovee (a \vee b)$ that holds in effect algebras [76, proposition 177].

Now any ortholattice that is also an effect algebra is an orthomodular lattice [22, proposition 1.5.8]. □

Definition 89 ([43]). We define **OMLatGal** to be the category where the objects are orthomodular lattices and the morphisms are Galois connections $f : A \rightleftarrows B : g$.

Proposition 90. *Let* $\mathbf{C}$ *be a $\diamond$-effectus. The assignment $A \mapsto$* SPred(A) *and $f \mapsto (f_\diamond, f^\square)$ gives a functor* $\mathbf{C} \to \mathbf{OMLatGal}$.

Proof. SPred(A) is an orthomodular lattice by proposition 88. That $f_\diamond$ and $f^\square$ form a Galois connection is proposition 84(c), and that the assignment is functorial is given by point (g) of that same proposition. □

4.1. Assert maps

So far we have not required any type of coherence between filters and comprehensions. For the next results we however do need to know a bit more about their interplay.

Definition 91. Let $\mathbf{C}$ be an effectus with filters and comprehensions. We say the filters and comprehensions are **compatible** when for every comprehension π_p of a sharp predicate p there exists a filter ξ^p of p such that $\xi^p \circ \pi_p = \mathrm{id}$.

In the setting we care about, there will be a dagger structure on some of the maps in an effectus which in particular will entail that $\pi_p^\dagger$ is a filter with $\pi_p^\dagger \circ \pi_p = \mathrm{id}$, so that they are indeed compatible. In an effectus with compatible filters and comprehensions we can for every sharp predicate define a special type of idempotent map that acts as a measurement for this predicate.

Definition 92. Let $\mathbf{C}$ be an effectus with compatible filters and comprehensions, and let $p : A \to I$ be a sharp predicate. We define the **assert map** $\mathrm{asrt}_p : A \to A$ for p to be $\mathrm{asrt}_p := \pi_p \circ \xi^p$ where π_p and ξ^p are a compatible pair of a comprehension and filter for p.

Note first that assert maps are uniquely defined, since if π_p' and $(\xi^p)'$ are another compatible pair, then $\pi_p' = \pi_p \circ \Theta_1$ and $(\xi^p)' = \Theta_2 \circ \xi^p$ for some isomorphisms Θ_1 and Θ_2 and furthermore $\mathrm{id} = (\xi^p)' \circ \pi_p' = \Theta_2 \circ \xi^p \circ \pi_p \circ \Theta_1 = \Theta_2 \circ \Theta_1$ so that $\Theta_2 = \Theta_1^{-1}$ and hence $\pi_p' \circ (\xi^p)' = \pi_p \circ \Theta_1 \circ \Theta_1^{-1} \circ \xi^p = \pi_p \circ \xi^p = \mathrm{asrt}_p$. Additionally, since the filter and comprehension are compatible we get $\mathrm{asrt}_p \circ \mathrm{asrt}_p = \mathrm{asrt}_p$.

Recall that in definition 76 we also referred to assert maps for predicates. These turn out to coincide with the assert maps we have defined here (when the predicate is sharp).

Example 93. Let $B(\mathcal{H})$ be an object in the opposite category of C^*-algebras. A sharp predicate then corresponds to a projector $P : \mathcal{H} \to \mathcal{H}$. The assert map for P is then given by $\mathrm{asrt}_P(A) = PAP$.

Note that we have $1 \circ \mathrm{asrt}_p = \mathrm{im}\,\mathrm{asrt}_p = p$.

Lemma 94. *Let* $\mathbf{C}$ *be an effectus with images and compatible filters and comprehensions. Let* $p \in \mathrm{Pred}(A)$ *be a sharp predicate and let* $f : B \to A$ *and* $g : A \to B$ *be morphisms in the effectus. The following are true:*

(*a*) $\mathrm{im}\, f \leqslant p \Leftrightarrow \mathrm{asrt}_p \circ f = f$.

(*b*) $1 \circ g \leqslant p \Leftrightarrow g \circ \mathrm{asrt}_p = g$.

Proof. For the first point: if $\mathrm{asrt}_p \circ f = f$, then $\mathrm{im}\, f = \mathrm{im}\,\mathrm{asrt}_p \circ f \leqslant \mathrm{im}\,\mathrm{asrt}_p = p$. Conversely, if $\mathrm{im}\, f \leqslant p$, then $p \circ f = 1 \circ f$ so that by the universal property of π_p we have $f = \pi_p \circ \overline{f}$ for some $\overline{f}$. Now $\overline{f} = \mathrm{id} \circ \overline{f} = \xi^p \circ \pi_p \circ \overline{f} = \xi^p \circ f$ so that $f = \pi_p \circ \overline{f} = \pi_p \circ \xi^p \circ f = \mathrm{asrt}_p \circ f$.

For the second point: suppose $g \circ \mathrm{asrt}_p = g$. Then $1 \circ g = (1 \circ g) \circ \mathrm{asrt}_p \leqslant 1 \circ \mathrm{asrt}_p = p$. Conversely, if $1 \circ g \leqslant p$, then by the universal property of ξ^p we have $g = \overline{g} \circ \xi^p$ for some $\overline{g}$. Now $\overline{g} = \overline{g} \circ \mathrm{id} = \overline{g} \circ \xi^p \circ \pi_p = g \circ \pi_p$ so that $g = \overline{g} \circ \xi^p = g \circ \pi_p \circ \xi^p = g \circ \mathrm{asrt}_p$. □

5. Decomposing into sharp and convex systems

Our first step in reconstructing quantum theory is showing how we can retrieve convexity and real probabilities from the abstract framework of effectus theory. This relies mostly on showing how properties of the scalars of the effectus lift to the entirety of the category. We already saw examples of that in section 2.3, where if the scalars were $\{0, 1\}$ the predicate spaces were orthoalgebras, while if they were $[0, 1]$, then the predicate spaces would be convex effect algebras. Here we will generalise these results.

5.1. Decomposing an effectus

If the effect monoid of scalars of an effectus is reducible, then we can lift this to the level of the category. This is because scalars in an effectus have an action on the predicate spaces through composition: given a scalar $s : I \to I$ and a predicate $a : A \to I$, we construct the scaled predicate $s \cdot a := s \circ a$.

Proposition 95. *Let* $\mathbf{C}$ *be an effectus with a non-trivial idempotent scalar* s*, i.e.* $s \notin \{0, 1\}$ *and* $s^2 = s$*. Then* $\mathrm{Pred}(\mathbf{C})$ *embeds non-trivially into a product of categories.*

Proof. We sketch a proof. Let $A \in \mathbf{C}$. For every $p \in \mathrm{Pred}(A)$ we have $p = \mathbf{1}_I \circ p = (s \ovee s^\perp) \circ p = s \circ p \ovee s^\perp \circ p \equiv p_1 \ovee p_2$ and hence we can write $\mathrm{Pred}(A) \cong A_1 \oplus A_2 \equiv s \cdot \mathrm{Pred}(A) \oplus s^\perp \cdot \mathrm{Pred}(A)$. Let $f : A \to B$ be a morphism in $\mathbf{C}$. Under the functor Pred this becomes $\mathrm{Pred}(f) : \mathrm{Pred}(B) \to \mathrm{Pred}(A)$. Using the idempotence of s we note that

$$\begin{aligned}\mathrm{Pred}(f)(p) &= \mathrm{Pred}(f)(s \circ p \mathbin{\varovee} s^{\perp} \circ p)\\ &= \mathrm{Pred}(f)(s \circ s \circ p \mathbin{\varovee} s^{\perp} \circ s^{\perp} \circ p)\\ &= s \circ \mathrm{Pred}(f)(s \circ p) \mathbin{\varovee} s^{\perp} \circ \mathrm{Pred}(f)(s^{\perp} \circ p)\end{aligned}$$

and hence $\mathrm{Pred}(f) = (f_1, 0) \mathbin{\varovee} (0, f_2)$ where $f_i : \mathrm{Pred}(B_i) \to \mathrm{Pred}(A_i)$. We then have a faithful functor $\mathrm{Pred}(\mathbf{C}) \to \mathbf{C}_1 \times \mathbf{C}_2$ where $\mathbf{C}_1$ has objects $s \cdot \mathrm{Pred}(A)$ and morphisms $f : s \cdot \mathrm{Pred}(A) \to s \cdot \mathrm{Pred}(B)$ in **EA** (or rather the subcategory consisting of effect modules over $s \cdot \mathrm{Pred}(I)$). We define $\mathbf{C}_2$ analogously, but with s and $s^{\perp}$ interchanged. □

Generally it will not be the case that Pred(**C**) is equivalent to a product of categories in the above situation. However, this is the case if the effectus has some more structure. First, let us recall the concept of the *Karoubi envelope*, which allows us to speak of subsystems in a general category, as given by idempotent maps.

Definition 96. Let **C** be any category. We say a morphism $t : A \to A$ in **C** is **idempotent** when $t \circ t = t$. We define the **Karoubi envelope** of **C** to be the category Split(**C**) which has as objects idempotents $t : A \to A$ in **C**, and has morphisms $f : (t : A \to A) \to (s : B \to B)$ corresponding to a morphism $f : A \to B$ in **C** satisfying $s \circ f \circ t = f$.

Note that **C** embeds fully and faithfully into Split(**C**) via $A \mapsto \mathrm{id}_A$ and $f \mapsto f$. Intuitively we can think of an object $t : A \to A$ as the subobject of A where t holds. We call this category Split(**C**) as it makes every idempotent *split*, meaning that for an idempotent $t : A \to A$ we can find a pair of maps $\xi^t : A \leftrightarrows A_t : \pi_t$ so that $\pi_t \circ \xi^t = t$ and $\xi^t \circ \pi_t = \mathrm{id}_{A_t}$.

Proposition 97. *Let* **C** *be an effectus in partial form. Then* Split(**C**) *is also an effectus in partial form. Furthermore:*

- *If **C** is separated by predicates, then so is* Split(**C**).
- *If **C** is separated by states, then so is* Split(**C**).
- *If **C** is monoidal, then so is* Split(**C**).

Proof. We just give a sketch of the necessary constructions to prove this. The trivial object of Split(**C**) is id_I. The predicates of an object $t : A \to A$ in Split(**C**) then correspond to maps $p : A \to I$ satisfying $p \circ t = p$. Note that always $p \leqslant \mathbf{1}_A \circ t$, but that the converse is not necessarily true. The truth element is $\mathbf{1}_t := \mathbf{1}_A \circ t$ and falsity is just $\mathbf{0}_t := \mathbf{0}_A$. We set $p^{\perp_t} := p^{\perp} \circ t$. It is straightforward to verify that Pred(t) is an effect algebra. We define the coproduct as $t + s : A + B \to A + B$ in the obvious way. The coproduct maps then become $\kappa_1' := (t + s) \circ \kappa_1 \circ t$ and similarly for κ_2'. All the axioms of an effectus are now easily checked.

Now suppose **C** is separated by predicates. Let $f, g : t \to s$ in Split(**C**) where $s : B \to B$ and suppose $p' \circ f = p' \circ g$ for all $p' : s \to \mathrm{id}_I$. We need to show that $f = g$. Let $p : B \to I$ be an arbitrary predicate on B. Then $p \circ f = p \circ (s \circ f \circ t) = (p \circ s) \circ (s \circ f \circ t) = (p \circ s) \circ f$. Now since $p \circ s$ is a predicate on s, we have $(p \circ s) \circ f = (p \circ s) \circ g$. Doing the argument in reverse we then get $p \circ f = p \circ g$. Now predicate separation in **C** gives $f = g$ as desired. The proof for preservation of separation by states works analogously.

If $\mathbf{C}$ is monoidal, we define the monoidal structure in Split($\mathbf{C}$) to be the same as in $\mathbf{C}$, with the monoidal unit being id_I. We do need to modify the coherence isomorphisms though. For instance, for an object $t : A \to A$ in Split($\mathbf{C}$) we set $\lambda_t : t \otimes \mathrm{id}_I \to t$ to be $\lambda_t := t \circ \lambda_A \circ (t \otimes \mathrm{id}_I)$. Using the naturality of λ_A it is then straightforward to verify that this satisfies the correct equations. The additional equations required for a monoidal effectus are also easily checked. □

Proposition 98. *Let* $\mathbf{C}$ *be a monoidal effectus with a non-trivial idempotent scalar s. Then* Split($\mathbf{C}$) $\cong \mathbf{C}_s \times \mathbf{C}_{s^\perp}$ *for some non-trivial effectuses* $\mathbf{C}_s$ *and* $\mathbf{C}_{s^\perp}$.

Proof. Without loss of generality we may assume that $\mathbf{C}$ has split idempotents (since otherwise we could take Split($\mathbf{C}$) instead).

Define $\mathbf{C}_s$ to be the category with the same objects as $\mathbf{C}$, but with morphisms restricted to those of the form $s \cdot f$ for $f : A \to B$ in $\mathbf{C}$ and with identity $s \cdot \mathrm{id}_X$ for an object A. It is easy to see that the coproduct and enrichment on $\mathbf{C}$ restricts to $\mathbf{C}_s$ and hence $\mathbf{C}_s$ is a finPAC. It has the same distinguished object I, but for every object A the maximum element of the predicate space $\mathbf{C}_s(A, I)$ is $s \cdot \mathbf{1}_A$, and the orthosupplement for p is $s \cdot p^\perp$. Hence $\mathbf{C}_s$ is an effectus as well. The same holds for $\mathbf{C}_{s^\perp}$ with the obvious definition.

We will show $\mathbf{C} \cong \mathbf{C}_s \times \mathbf{C}_{s^\perp}$. First we need a few definitions. For every object A in $\mathbf{C}$, pick a splitting $\xi^s : A \leftrightarrows A_s : \pi^s$ of the idempotent $s \cdot \mathrm{id}_A$. For any morphism $f : A \to B$, we define $f_s : A_s \to B_s$ as $f_s := \xi^s \circ f \circ \pi_s$. Note $(\mathrm{id}_A)_s = \mathrm{id}_{A_s}$ and $(f \circ g)_s = f_s \circ g_s$. Furthermore $s \cdot \xi^s = \xi^s \circ s \cdot \mathrm{id} = \xi^s \circ \pi_s \circ \xi^s = \xi^s$ and so $s \cdot \mathrm{id}_{A_s} = \pi^s \circ (s \cdot \xi^s) = \mathrm{id}_{A_s}$ and $s^\perp \cdot \mathrm{id}_{A_s} = 0$. Define $A_{s^\perp}$, $\pi_{s^\perp}$, $\xi^{s^\perp}$ and $(\ )_{s^\perp}$ similarly.

Now, we can define the functors $F : \mathbf{C}_s \times \mathbf{C}_{s^\perp} \leftrightarrows \mathbf{C} : G$ by

$$F(A,B) = A_s + B_{s^\perp} \qquad F(f,g) = f_s + g_{s^\perp} \qquad G(A) = (A_s, A_{s^\perp}) \qquad G(f) = (f_s, f_{s^\perp}).$$

To show that these functors form an equivalence of categories, first observe that $F(G(A)) = A_s + A_{s^\perp}$. We have $\xi^s \circ \pi_s = \mathrm{id}$ so that $\mathbf{1} \circ \pi_s = \mathbf{1}$, and hence $\mathbf{1} \circ \xi^s = s \cdot \mathbf{1}$ and similarly $\mathbf{1} \circ \xi^{s^\perp} = s^\perp \cdot \mathbf{1}$. As these are summable predicates, the sum map $\langle \xi^s, \xi^{s^\perp} \rangle := (\kappa_1 \circ \xi^s) \mathbin{\otimes\!\!\!\!\vee} (\kappa_2 \circ \xi^{s^\perp}) : A \to A_s + A_{s^\perp}$ exists, see [76, 181_{VII}]. Note that this map is an isomorphism with inverse $[\pi_s, \pi_{s^\perp}]$. From this it follows that $\alpha := [\pi_s, \pi_{s^\perp}]$ is a natural isomorphism $FG \Rightarrow \mathrm{id}$.

For the other direction of the equivalence, first consider the maps

$$(A_s + B_{s^\perp})_s \underset{\pi_s}{\overset{\xi^s}{\leftrightarrows}} A_s + B_{s^\perp} \underset{\rhd_1}{\overset{\kappa_1}{\leftrightarrows}} A_s \underset{\pi_s}{\overset{\xi^s}{\leftrightarrows}} A. \tag{2}$$

Note $\xi^s \circ \kappa_1 \circ \xi^s \circ \pi_s \circ \rhd_1 \circ \pi_s = \xi^s \circ (\mathrm{id}_{A_s} + 0_{B_{s^\perp}}) \circ \pi_s$. Remember $s \cdot \mathrm{id}_{B_{s^\perp}} = 0$, hence

$$s \cdot (\xi^s \circ \kappa_1 \circ \xi^s \circ \pi_s \circ \rhd_1 \circ \pi_s) = \xi^s \circ (s \cdot \mathrm{id}_{A_s} + s \cdot \mathrm{id}_{B_{s^\perp}}) \circ \pi_s = s \cdot \mathrm{id}.$$

We can also calculate $s \cdot (\pi_s \circ \rhd_1 \circ \pi_s \circ \xi^s \circ \kappa_1 \circ \xi^s) = s \cdot \mathrm{id}$ so that the maps in (2) are each others inverse in $\mathbf{C}_s$. We can do a similar calculation for the map

$$\pi_{s^\perp} \circ \rhd_2 \circ \pi_{s^\perp} : (A_s + B_{s^\perp})_{s^\perp} \to B$$

to show it is an isomorphism in $\mathbf{C}_{s^\perp}$. Hence, the map $\beta = (\pi_s \circ \rhd_1 \circ \pi_s, \pi_{s^\perp} \circ \rhd_2 \circ \pi_{s^\perp})$ gives a natural isomorphism $GF \Rightarrow \mathrm{id}$. □

Note that this result does not in fact require all the idempotents to split, just the ones that correspond to scalar multiplication by an idempotent scalar. We can use this fact to get a different variant of this result.

Proposition 99. *Let* $\mathbf{C}$ *be a monoidal effectus which has images and compatible filters and comprehensions and let s be a non-trivial idempotent scalar. Then* $\mathbf{C} \cong \mathbf{C}_s \times \mathbf{C}_{s^\perp}$ *for some non-trivial effectuses* $\mathbf{C}_s$ *and* $\mathbf{C}_{s^\perp}$.

Proof. We show that the idempotent maps $s \cdot \mathrm{id}_A$ split for every object A. The rest of the proof then proceeds as it does in proposition 98.

Note first that $s \cdot \mathbf{1}_A$ is sharp as it is the image of $s \cdot \mathrm{id}_A$. Hence, there is a comprehension $\pi_{s\cdot\mathbf{1}_A}$ and compatible filter $\xi^{s\cdot\mathbf{1}_A}$ and we can define $\mathrm{asrt}_{s\cdot\mathbf{1}_A} := \pi_{s\cdot\mathbf{1}_A} \circ \xi^{s\cdot\mathbf{1}_A}$. We have $\mathbf{1} \circ (s^\perp \cdot \mathrm{asrt}_{s\cdot\mathbf{1}_A}) = (s^\perp \cdot \mathbf{1}) \circ \mathrm{asrt}_{s\cdot\mathbf{1}_A} = \mathbf{0}$ so that $s^\perp \cdot \mathrm{asrt}_{s\cdot\mathbf{1}_A} = 0$. Hence, $s \cdot \mathrm{asrt}_{s\cdot\mathbf{1}_A} = \mathrm{asrt}_{s\cdot\mathbf{1}_A}$. Then using lemma 94 we get $s \cdot \mathrm{id}_A = \mathrm{asrt}_{s\cdot\mathbf{1}_A} \circ (s \cdot \mathrm{id}_A) = s \cdot \mathrm{asrt}_{s\cdot\mathbf{1}_A} = \mathrm{asrt}_{s\cdot\mathbf{1}_A}$. So $s \cdot \mathrm{id}_A$ splits via $\pi_{s\cdot\mathbf{1}_A} \circ \xi^{s\cdot\mathbf{1}_A}$. □

Even without monoidal structure we can get a similar result, but then we have to require separation by either predicates or states.

Proposition 100. *Let* $\mathbf{C}$ *be an effectus which is separated by states or predicates and which has images and compatible filters and comprehensions and let s be a non-trivial idempotent scalar. Then* $\mathbf{C} \cong \mathbf{C}_s \times \mathbf{C}_{s^\perp}$ *for some non-trivial effectuses* $\mathbf{C}_s$ *and* $\mathbf{C}_{s^\perp}$.

Proof. This follows along the same lines as the proof of proposition 98, but instead of the maps $s \cdot \mathrm{id}$ we consider the maps $\mathrm{asrt}_{s\circ\mathbf{1}} = \pi_{s\circ\mathbf{1}} \circ \xi^{s\circ\mathbf{1}}$. We let the category $\mathbf{C}_s$ have the same objects as $\mathbf{C}$ but with morphisms $f : A \to B$ satisfying $\mathrm{im}\, f \leqslant s \circ \mathbf{1}$ and $\mathbf{1} \circ f \leqslant s \circ \mathbf{1}$. The identity on an object A in $\mathbf{C}_s$ is $\mathrm{asrt}_{s\circ\mathbf{1}}$. We define $\mathbf{C}_{s^\perp}$ analogously. As in proposition 98 we define $f_s := \xi^{s\circ\mathbf{1}} \circ f \circ \pi_{s\circ\mathbf{1}}$, and we get functors $F : \mathbf{C}_s \times \mathbf{C}_{s^\perp} \leftrightarrows \mathbf{C} : G$. Showing that these form an equivalence of categories follows entirely analogously, except for a complication with establishing that the right-inverse of $\alpha := [\pi_{s\circ\mathbf{1}}, \pi_{s^\perp\circ\mathbf{1}}]$ is $\langle \xi^{s\circ\mathbf{1}}, \xi^{s^\perp\circ\mathbf{1}} \rangle$, which requires predicate separation to be proven. We first calculate $[\pi_{s\circ\mathbf{1}}, \pi_{s^\perp\circ\mathbf{1}}] \circ \langle \xi^{s\circ\mathbf{1}}, \xi^{s^\perp\circ\mathbf{1}} \rangle = \mathrm{asrt}_{s\circ\mathbf{1}} \ovee \mathrm{asrt}_{s^\perp\circ\mathbf{1}}$. Now assume $\mathbf{C}$ is separated by predicates. For any predicate p we have $p \circ \mathrm{asrt}_{s\circ\mathbf{1}} = (s \circ p) \circ \mathrm{asrt}_{s\circ\mathbf{1}} \ovee (s^\perp \circ p) \circ \mathrm{asrt}_{s\circ\mathbf{1}} = s \circ p$ (by lemma 94), so that $p \circ (\mathrm{asrt}_{s\circ\mathbf{1}} \ovee \mathrm{asrt}_{s^\perp\circ\mathbf{1}}) = s \circ p \ovee s^\perp \circ p = p$. As also $p \circ \mathrm{id} = p$ we get by predicate separation $[\pi_{s\circ\mathbf{1}}, \pi_{s^\perp\circ\mathbf{1}}] \circ \langle \xi^{s\circ\mathbf{1}}, \xi^{s^\perp\circ\mathbf{1}} \rangle = \mathrm{id}$. If instead $\mathbf{C}$ is separated by states, we can do a similar argument, but with the extra complication that we need s to commute with all scalars. This is however true for any idempotent element in an effect monoid [71, lemma 20]. □

These results could perhaps be generalised so that every effectus can be presented as a presheaf from the Boolean algebra of idempotent scalars to the category of effectuses, akin to the results in reference [3], but as this is not necessary for our results we leave this as future work.

5.2. Decomposing a directed-complete effectus

Recall that in a directed-complete effectus, the effect monoid of scalars splits up into a Boolean algebra and a convex effect algebra. We hence get the following.

Proposition 101. *Let $\mathbf{C}$ be a directed-complete effectus that is separated by states and let $A \in \mathbf{C}$. Then* $\mathrm{Pred}(A) \cong E_1 \oplus E_2$ *where E_1 is an orthoalgebra and E_2 is convex* (*see definitions* 50 *and* 52).

Proof. Write $M = \mathrm{Pred}(I)$ for the scalars of $\mathbf{C}$. By corollary 48, we know $M \cong M_1 \oplus M_2$ where M_1 is a complete Boolean algebra and $M_2 \cong C(X, [0, 1])$. Let s be the idempotent that projects onto the Boolean part, i.e. $s \equiv (1, 0) \in M_1 \oplus M_2$.

Each predicate $p \in E := \mathrm{Pred}(A)$ splits as $s \circ p \owedge s^{\perp} \circ p$. Write $E_1 \equiv sE \equiv \{s \circ p;\ p \in E\}$ and $E_2 \equiv s^{\perp}E$. As s is an idempotent, sE is an effect algebra itself. In fact $\mathrm{Pred}(A) \cong E_1 \oplus E_2$. We claim that E_1 is an orthoalgebra and that E_2 is convex. The latter statement is easily seen as M_2 is convex, and hence we can define a convex action on E_2 via $\lambda \cdot p := (\lambda \cdot s_1) \circ p$.

To show that E_1 is an orthoalgebra we need to prove that whenever $p_1 \owedge p_1$ is defined for some $p_1 \in E_1$ that then $p_1 = 0$. So suppose $p_1 \owedge p_1$ is defined. Then for any state $\omega : I \to A$, the scalar $(p_1 \owedge p_1) \circ \omega = p_1 \circ \omega \owedge p_1 \circ \omega$ is defined. As $p_1 \in E_1$ we have $s_1 \circ p_1 = p_1$, and hence we have $p_1 \circ \omega = s_1 \circ (p_1 \circ \omega) \in M_1$. As $p_1 \circ \omega \perp p_1 \circ \omega$ and M_1 is a Boolean algebra, and so in particular an orthoalgebra, we must then have $p_1 \circ \omega = 0$. Now, by separation of states we see that $p_1 = 0$, as desired. □

Hence, in a directed-complete effectus with separating states, each predicate space splits up into a sharp part, given by an orthoalgebra, and a probabilistic part, given by a convex effect algebra. By using the equivalence of convex effect algebras with vector spaces this extends to the following.

Proposition 102. *Let $\mathbf{C}$ be a directed-complete effectus that is separated by states.* Pred *gives a functor* $\mathrm{Pred} : \mathbf{C} \to \mathbf{OA}^{\mathrm{op}} \times \mathbf{DCEA}_c^{\mathrm{op}} \cong \mathbf{OA}^{\mathrm{op}} \times \mathbf{DCOUS}^{\mathrm{op}}$, *where $\mathbf{OA}$ is the category of orthoalgebras and $\mathbf{DCOUS}$ is the category of directed-complete OUSs.*

For this result this factorization is merely 'internal' to the predicate spaces, and not necessarily reflected in the structure of the objects of $\mathbf{C}$, as the axioms of a plain effectus do not force 'predicate sub-spaces' to correspond to objects in the category. However, when we have suitable filters and comprehensions we get a stronger result.

Proposition 103. *Let $\mathbf{C}$ be a directed-complete effectus that is separated by states and which has images and compatible filters and comprehensions. Then $\mathbf{C}$ is equivalent to a product of effectuses $\mathbf{C} \cong \mathbf{C}_1 \times \mathbf{C}_2$ and* Pred *gives functors* $\mathrm{Pred} : \mathbf{C}_1 \to \mathbf{OA}^{\mathrm{op}}$ *and* $\mathrm{Pred} : \mathbf{C}_2 \to \mathbf{DCEA}_c^{\mathrm{op}} \cong \mathbf{DCOUS}^{\mathrm{op}}$, *where $\mathbf{OA}$ is the category of orthoalgebras and $\mathbf{DCOUS}$ is the category of directed-complete OUSs.*

Proof. Let s be the idempotent scalar that splits the scalars of $\mathbf{C}$ into a Boolean algebra and a convex set. Then apply proposition 100 to get $\mathbf{C} \cong \mathbf{C}_1 \times \mathbf{C}_2$ where the scalars of $\mathbf{C}_1$ are a Boolean algebra and those of $\mathbf{C}_2$ are convex. Then proposition 101 shows that the predicate spaces of $\mathbf{C}_1$ are orthoalgebras and that of $\mathbf{C}_2$ are convex effect algebras. To go from directed-complete convex effect algebras to OUSs, apply proposition 55. □

5.3. Decomposing normal sequential effect algebras

In our reconstruction we will look at effectuses whose predicate spaces are normal SEAs (definition 70). By the previous results, if the effectus has state separation we can consider normal SEAs that are either orthoalgebras or are convex. In the former case, the situation simplifies even further.

Lemma 104. *Let E be an orthoalgebra that is also a SEA. Then E is a Boolean algebra with* $a \ \& \ b = a \wedge b$.

Proof. For any $a \in E$ we note that $a \ \& \ a^{\perp}$ is summable with itself as $1 = 1 \ \& \ 1 = (a \ovee a^{\perp}) \ \& \ (a \ovee a^{\perp}) \geq 2(a \ \& \ a^{\perp})$. Since E is an orthoalgebra we must then have $a \ \& \ a^{\perp} = 0$ so that a is an idempotent (as $a = a \ \& \ 1 = a \ \& \ (a \ovee a^{\perp}) = a^2 \ovee a \ \& \ a^{\perp} = a^2$)). But as every element is then an idempotent, E must be a Boolean algebra [72, proposition 45]. □

Proposition 105. *Let* **C** *be a directed-complete effectus with separating states, and let* $A \in \mathbf{C}$. *If* Pred(A) *is a normal SEA, then* Pred(A) $\cong A_b \oplus A_c$ *where* A_b *is a complete Boolean algebra and* A_c *is a convex effect algebra, or more specifically, the unit interval of a directed-complete OUS.*[15]

Proof. By proposition 101 we see that Pred(A) $\cong A_b \oplus A_c$ where A_b is an orthoalgebra and A_c is convex. As A_b is a principal downset and $p \ \& \ q \leqslant p$, we see that A_b is also a SEA and so it must then be a Boolean algebra by lemma 104. □

5.4. Decomposing finite-dimensional effectuses

Let us briefly demonstrate how we can talk about finite-dimensional systems in effectus theory as the literature on generalised probabilistic theories often restricts to working with finite-dimensional spaces.

Proposition 106. *Let* **C** *be a directed-complete effectus which has finite tomography. Then there exists a finite set A and* $n \in \mathbb{N}$ *such that* Pred(I) $\cong \mathcal{P}(A) \oplus [0,1]^n$, *where* $\mathcal{P}(A)$ *denotes the powerset of A.*

Proof. By finite tomography we know that there is a finite collection $p_1, \ldots, p_k \in$ Pred(I) such that $a, b \in$ Pred(I) are equal iff $p_i \cdot a = p_i \cdot b$ for all i. As **C** is a directed complete effectus we know that Pred(I) $\cong B \oplus C(X, [0,1])$ for some complete Boolean algebra B and extremally-disconnected compact Hausdorff space X. Let s denote the idempotent scalar splitting these two parts. Then $\{s \cdot p_i\}$ is a set of elements of B that suffice to separate the elements of B. This is only possible if B is finite, hence $\mathcal{P}(A) \cong B$ for some finite set A.

Similarly, the set $\{s^{\perp} \cdot p_i\}$ separates $C(X, [0,1])$ and in fact $C(X)$ itself. This is only possible if $C(X)$ is finite dimensional, viz if X is finite (indeed, the $s^{\perp} \cdot p_i$ can only distinguish elements that lie in their span, and hence $C(X)$ can have dimension at most k). The only finite compact Hausdorff spaces are discrete and hence $C(X) \cong \mathbb{R}^n$ for some n. □

[15] In [72] it was shown that any normal SEA splits up into three parts: a Boolean algebra, a convex effect algebra and a type of effect algebra called 'purely almost convex'. The previous proposition shows that the 'purely almost-convex' case does not occur in the context of an effectus. Hence, this pathological type of SEA is prevented from existing in the more compositional setting of an effectus.

As $\mathcal{P}(A)$ is equivalent to a product of the trivial Boolean algebra $\{0, 1\}$ and $[0, 1]^n$ is a product of the 'trivial' convex effect algebra [0, 1], the above result means that, by iterated application of proposition 95, for a directed complete effectus with finite tomography $\mathbf{C}$, the category Pred($\mathbf{C}$) embeds into a product of categories $B_1 \times \cdots \times B_k \times C_1 \times \cdots \times C_k$ where each B_i is a category with 'scalars' $\{0, 1\}$ and each C_i is a category with 'scalars' [0, 1]. If the effectus additionally has images, compatible filters and comprehensions then we can apply proposition 100 to see that the effectus is equivalent to a product of effectuses where the scalars are either $\{0, 1\}$ or [0, 1]. For each C_i we can also show that the predicate spaces correspond to finite-dimensional vector spaces using finite tomography, and hence we have retrieved the standard framework of generalised probabilistic theories.

6. The reconstruction

We now have all we need to start proving our main results. Let us repeat here for ease of reference the definition of a sequential effectus.

Definition (Restatement of definition 76). A **sequential** effectus is a normal effectus (cf definition 43) separated by states satisfying the following.

(a) The effectus has filters and comprehensions.
(b) Comprehensions have images.
(c) The pure maps form a dagger category.
(d) Every pure map f is $\diamond$-adjoint to $f^\dagger$.
(e) For every predicate $p \in \text{Pred}(A)$ there is a unique †-positive pure map $\text{asrt}_p : A \to A$ satisfying $\mathbf{1}_A \circ \text{asrt}_p = p$ called the **assert map** of p.
(f) For every object A, the operation $\& : \text{Pred}(A) \times \text{Pred}(A) \to \text{Pred}(A)$ given by $p \;\&\; q := q \circ \text{asrt}_p$ is a normal sequential product, making Pred(A) into a normal SEA.

For the remainder of the section, we will let $\mathbf{C}$ be a sequential effectus, and we let A denote an arbitrary system in $\mathbf{C}$. By assumption Pred(A) is a normal SEA. By proposition 105, Pred(A) is then a direct sum of a complete Boolean algebra and a convex SEA. We aim to use a version of theorem 75 to show the convex part is the unit interval of a JB-algebra. This means that we need to show that the sequential product given by the assert maps is compressible and quadratic (up to some technical modifications that will become apparant later on).

6.1. Sequential effectuses are $\diamond$-effectuses

Our definition refers to $\diamond$-adjointness, but this concept is only well-behaved when the effectus is a $\diamond$-effectus. So let us start by showing that a sequential effectus is indeed a $\diamond$-effectus. This means we still need to show that images of all maps exist and that p is sharp iff $p^\perp$ is sharp. For any predicate p we will write $p^2 := p \;\&\; p := p \circ \text{asrt}_p$. Following definition 70 we call predicates with $p^2 = p$ **idempotent**.

Proposition 107. *Let $p \in \text{Pred}(A)$ be any predicate. Then the image of asrt_p exists and is equal to $\lceil p \rceil$, where $\lceil p \rceil$ is both as in lemma 71 as well as in definition 34. In particular, $\lceil p \rceil$ is sharp, p is sharp iff $p^\perp$ is sharp, and p is sharp iff it is idempotent.*

Proof. Let $q \in \mathrm{Pred}(A)$ be a predicate with $\mathbf{1} \circ \mathrm{asrt}_p = q \circ \mathrm{asrt}_p$. I.e. $p = q \,\&\, p$. By lemma 71(c) we then have $q \geqslant \lceil p \rceil$. As $\lceil p \rceil \,\&\, p = p$ we see that indeed $\mathrm{im}\, \mathrm{asrt}_p = \lceil p \rceil$. In particular the ceiling given in lemma 71(c) is a sharp predicate. As asrt_p is †-self-adjoint, and thus $\diamond$-self-adjoint we also calculate $\mathrm{im}\ \mathrm{asrt}_p = (\mathrm{asrt}_p)_\diamond(\mathbf{1}) = \mathrm{asrt}_p^\diamond(\mathbf{1}) = \lceil \mathbf{1} \circ \mathrm{asrt}_p \rceil = \lceil p \rceil$, where here $\lceil p \rceil$ is as in definition 34, so that the two possible definitions of the ceiling coincide, and in particular $\lceil p \rceil$ is sharp. Now suppose p is sharp, so that $\lfloor p \rfloor = p$. Then $p^\perp = \lfloor p \rfloor^\perp = \lceil p^\perp \rceil$ is sharp. So indeed p sharp iff $p^\perp$ sharp.

Suppose again that p is sharp. Then $\lceil p \rceil = p$, and hence $\mathrm{im}\, \mathrm{asrt}_p = p$ so that $p^2 := p \circ \mathrm{asrt}_p = \mathbf{1} \circ \mathrm{asrt}_p = p$. So p is idempotent. Conversely, if $p^2 = p$, then $p \geqslant \mathrm{im}\, \mathrm{asrt}_p = \lceil p \rceil \geqslant p$ so that $p = \lceil p \rceil$ and p is sharp. □

Proposition 108. *All morphisms have images.*

Proof. Let f be a morphism in our category. From lemma 71(d) we know that the idempotents of a normal SEA form a complete lattice. Hence, we can define $\mathrm{im}\, f := \bigwedge\{p | p^2 = p, p \circ f = \mathbf{1} \circ f\}$. By normality of f we have $\mathrm{im}\, f \circ f = \mathbf{1} \circ f$. Now, if $q \circ f = \mathbf{1} \circ f$, then also $\lfloor q \rfloor \circ f = \mathbf{1} \circ f$ (follows from proposition 83(e)). As $\lfloor q \rfloor$ is sharp, it is idempotent, and hence $\mathrm{im}\, f \leqslant \lfloor q \rfloor \leqslant q$ as desired. □

Corollary 109. *A sequential effectus is a $\diamond$-effectus.*

6.2. The sequential product is compressible

Now we will venture to prove that the sequential product is compressible (cf definition 73). To do this we need to know more about the relationship between the assert maps, filters, comprehensions and the dagger.

Lemma 110. *For all predicates $p \in \mathrm{Pred}(A)$ we have $\mathrm{asrt}_p^2 = \mathrm{asrt}_{p^2}$.*

Proof. By definition asrt_p is †-positive, and so in particular is †-self-adjoint. Hence, $\mathrm{asrt}_p^2 = \mathrm{asrt}_p^\dagger \circ \mathrm{asrt}_p$ is also †-positive. We have $\mathbf{1} \circ \mathrm{asrt}_p^2 = p \circ \mathrm{asrt}_p = p^2 = \mathbf{1} \circ \mathrm{asrt}_{p^2}$ so that by the uniqueness of †-positive maps: $\mathrm{asrt}_p^2 = \mathrm{asrt}_{p^2}$. □

Lemma 111. *Comprehensions and filters are compatible (cf definition 91), i.e. for any comprehension π_p of a sharp predicate p there exists a filter ξ^p of p such that $\xi^p \circ \pi_p = \mathrm{id}$. Furthermore, $\pi_p \circ \xi^p = \mathrm{asrt}_p$.*

Proof. Let p be a sharp predicate, which is hence idempotent. As asrt_p is pure we have $\mathrm{asrt}_p = \pi \circ \xi$ for some comprehension π and filter ξ. Now,

$$\pi \circ \xi \;=\; \mathrm{asrt}_p \;=\; \mathrm{asrt}_{p^2} \;=\; \mathrm{asrt}_p \circ \mathrm{asrt}_p \;=\; \pi \circ \xi \circ \pi \circ \xi$$

so that $\xi \circ \pi = \mathrm{id}$ (as filters are epic and comprehensions are monic). As $\mathbf{1} \circ \pi = \mathbf{1}$ we calculate $\mathbf{1} \circ \xi = \mathbf{1} \circ \pi \circ \xi = \mathbf{1} \circ \mathrm{asrt}_p = p$ so that ξ is a filter for p. Furthermore, $\mathrm{im}\, \pi = p$

as $\mathbf{1} \circ \pi = \mathbf{1} \circ \pi \circ \xi \circ \pi = \mathbf{1} \circ \mathrm{asrt}_p \circ \pi = p \circ \pi$ and if $q \circ \pi = \mathbf{1} \circ \pi$ then $p = \mathbf{1} \circ \mathrm{asrt}_p = \mathbf{1} \circ \pi \circ \xi = q \circ \pi \circ \xi = q \circ \mathrm{asrt}_p = q \,\&\, p$, so that $p \geqslant \lceil q \rceil \geqslant q$ by lemma 71(c). Hence π is a comprehension for p. Now let π' be another comprehension for p. Then $\pi' = \pi \circ \Theta$ for some isomorphism Θ. Define $\xi' := \Theta^{-1} \circ \xi$. Then $\pi' \circ \xi' = \pi \circ \xi = \mathrm{asrt}_p$ and $\xi' \circ \pi' = \mathrm{id}$. □

Note that as a consequence of this lemma, the assert map for a sharp predicate coincides with the definition of an assert map in definition 92.

Proposition 112. *Let p be a sharp predicate. Then $\pi_p^\dagger = \xi^p$, where π_p and ξ^p form a pair of a comprehension and a filter of p with $\xi^p \circ \pi_p = \mathrm{id}$ and $\pi_p \circ \xi^p = \mathrm{asrt}_p$.*

Proof. Let π_p be a comprehension of a sharp predicate p, and let ξ^p be a filter of p such that $\xi^p \circ \pi_p = \mathrm{id}$ which exists by lemma 111. By definition 76, π_p is $\diamond$-adjoint to $\pi_p^\dagger$. Hence $\lceil \mathbf{1} \circ \pi_p^\dagger \rceil = (\pi_p^\dagger)^\diamond(\mathbf{1}) = (\pi_p)_\diamond(\mathbf{1}) = \mathrm{im}\,\pi_p = p$. Similarly, we calculate $\mathrm{im}\,(\xi^p)^\dagger = p$. By the universal property of filters respectively comprehensions there are then unique maps h and g such that $\pi_p^\dagger = h \circ \xi^p$ and $(\xi^p)^\dagger = \pi_p \circ g$. Using $\xi^p \circ \pi_p = \mathrm{id}$ twice we calculate $\mathrm{id} = \mathrm{id}^\dagger = \pi_p^\dagger \circ (\xi^p)^\dagger = h \circ \xi^p \circ \pi_p \circ g = h \circ g$. As a result $\mathbf{1} = \mathbf{1} \circ \mathrm{id} = \mathbf{1} \circ h \circ g \leqslant \mathbf{1} \circ g$ so that g is unital, and hence $\mathbf{1} \circ (\xi^p)^\dagger = \mathbf{1} \circ \pi_p \circ g = \mathbf{1}$.

By uniqueness of $\dagger$-positive maps we have $(\xi^p)^\dagger \circ \xi^p = \mathrm{asrt}_{\mathbf{1} \circ (\xi^p)^\dagger \circ \xi^p} = \mathrm{asrt}_{\mathbf{1} \circ \xi^p} = \mathrm{asrt}_p = \pi_p \circ \xi^p$. Because ξ^p is epic we conclude that indeed $(\xi^p)^\dagger = \pi_p$. □

As a consequence of this proposition we note that $\pi_p^\dagger \circ \pi_p = \mathrm{id}$ and $\pi_p \circ \pi_p^\dagger = \mathrm{asrt}_p$ for any sharp p. This makes the comprehensions into *dagger-kernels* [40], and furthermore, we have now recovered the conditions specified in remark 60. We also have the following corollary.

Corollary 113. *Let Θ be an isomorphism. Then $\Theta^\dagger = \Theta^{-1}$.*

Proof. Θ is a filter for $\mathbf{1}$, which is sharp, and Θ^{-1} is a comprehension for $\mathbf{1}$. As $\Theta^{-1} \circ \Theta = \mathrm{id}$ and $\Theta \circ \Theta^{-1} = \mathrm{id} = \mathrm{asrt}_{\mathbf{1}}$, they satisfy the conditions of the previous proposition. □

Proposition 114. *The sequential product is compressible (cf definition 73).*

Proof. Let p be sharp and let ω be a state such that $p \circ \omega = 1$. We need to show that $\mathrm{asrt}_p \circ \omega = \omega$ as then $(p \,\&\, a) \circ \omega := a \circ \mathrm{asrt}_p \circ \omega = a \circ \omega$ as desired. But as $p \circ \omega = 1$ implies $\mathrm{im}\,\omega \leqslant p$, this follows immediately from lemma 94(a). □

6.3. The sequential product is quadratic

That the sequential product is also quadratic (cf definition 74) requires a bit more work.

Lemma 115. *Every pure map f factors as $f = \pi_{\mathrm{im}\, f} \circ \Theta \circ \xi^{\lceil \mathbf{1} \circ f \rceil} \circ \mathrm{asrt}_{\mathbf{1} \circ f}$ where Θ is an isomorphism.*

Proof. As f is pure, it is by definition of the form $f = \pi \circ \xi$ for some comprehension π and filter ξ. It is then straightforward to show that in fact $f = \pi_{\mathrm{im}\ f} \circ \Theta \circ \xi^{\mathbf{1}\circ f}$, where Θ is an isomorphism.

It hence remains to show that $\xi^{\lceil \mathbf{1}\circ f\rceil} \circ \mathrm{asrt}_{\mathbf{1}\circ f}$ is a filter for $\mathbf{1} \circ f$. Write $p := \mathbf{1} \circ f$. Note first of all that

$$\mathbf{1} \circ (\xi^{\lceil p\rceil} \circ \mathrm{asrt}_p) \;=\; \lceil p\rceil \circ \mathrm{asrt}_p \;=\; \mathbf{1} \circ \mathrm{asrt}_{\lceil p\rceil} \circ \mathrm{asrt}_p \;\overset{94.d}{=}\; \mathbf{1} \circ \mathrm{asrt}_p \;=\; p$$

so that it remains to show that $\xi^{\lceil p\rceil} \circ \mathrm{asrt}_p$ is a filter. We see that

$$\begin{aligned}
\mathrm{im}\,(\xi^{\lceil p\rceil} \circ \mathrm{asrt}_p) &= (\xi^{\lceil p\rceil} \circ \mathrm{asrt}_p)_\diamond(\mathbf{1}) \\
&= (\xi^{\lceil p\rceil})_\diamond((\mathrm{asrt}_p)_\diamond(\mathbf{1})) \\
&= (\xi^{\lceil p\rceil})_\diamond(\lceil p\rceil) \\
&= \mathrm{im}\,(\xi^{\lceil p\rceil} \circ \pi_{\lceil p\rceil}) \\
&= \mathrm{im}\ \mathrm{id} \\
&= \mathbf{1}.
\end{aligned}$$

Being a composition of pure maps, $\xi^{\lceil p\rceil} \circ \mathrm{asrt}_p$ is a pure map itself, and hence is equal to $\pi \circ \xi$ for some comprehension π and filter ξ. Now we calculate $\mathbf{1} = \mathrm{im}(\xi^{\lceil p\rceil} \circ \mathrm{asrt}_p) = \mathrm{im}(\pi \circ \xi) \leqslant \mathrm{im}\,\pi$ so that $\mathrm{im}\,\pi = \mathbf{1}$ and hence π is an isomorphism. We conclude that $\xi^{\lceil p\rceil} \circ \mathrm{asrt}_p$ is a filter. $\square$

Proposition 116. *Let p and q be arbitrary predicates on the same object. Then*

$$\mathrm{asrt}^2_{p\,\&\,q} \;=\; \mathrm{asrt}_p \circ \mathrm{asrt}^2_q \circ \mathrm{asrt}_p.$$

Proof. First we note that for any assert map $(\mathrm{asrt}_p)^\diamond = (\mathrm{asrt}_p)_\diamond$, as assert maps are †-self-adjoint. Now we calculate $\mathbf{1} \circ \mathrm{asrt}_q \circ \mathrm{asrt}_p = p \ \&\ q$ and

$$\begin{aligned}
\mathrm{im}\ (\mathrm{asrt}_q \circ \mathrm{asrt}_p) &= (\mathrm{asrt}_q \circ \mathrm{asrt}_p)_\diamond(\mathbf{1}) \\
&= (\mathrm{asrt}_q)_\diamond \circ (\mathrm{asrt}_p)_\diamond(\mathbf{1}) \\
&= (\mathrm{asrt}_q)_\diamond(\lceil p\rceil) \\
&= (\mathrm{asrt}_q)^\diamond(\lceil p\rceil) \\
&= \lceil \lceil p\rceil \circ \mathrm{asrt}_q\rceil \\
&= \lceil p \circ \mathrm{asrt}_q\rceil,
\end{aligned}$$

where the last step follows from proposition 83(d). Write $p \circ \mathrm{asrt}_q = q \ \&\ p$ and use lemma 115 to get

$$\mathrm{asrt}_q \circ \mathrm{asrt}_p \;=\; \pi_{\lceil q\,\&\,p\rceil} \circ \Theta \circ \xi^{\lceil p\,\&\,q\rceil} \circ \mathrm{asrt}_{p\,\&\,q}$$

for some isomorphism Θ. Applying the dagger to both sides and using proposition 112 and corollary 113 gives us:

$$\begin{aligned}\mathrm{asrt}_p \circ \mathrm{asrt}_q &= (\mathrm{asrt}_q \circ \mathrm{asrt}_p)^\dagger = \mathrm{asrt}_{p\,\&\,q} \circ (\xi^{\lceil p\,\&\,q\rceil})^\dagger \circ \Theta^\dagger \circ \pi^\dagger_{\lceil q\,\&\,p\rceil}\\ &= \mathrm{asrt}_{p\,\&\,q} \circ \pi_{\lceil p\,\&\,q\rceil} \circ \Theta^{-1} \circ \xi^{\lceil q\,\&\,p\rceil}.\end{aligned}$$

Finally, we calculate:

$$\begin{aligned}\mathrm{asrt}_p \circ \mathrm{asrt}_q^2 \circ \mathrm{asrt}_p &= \mathrm{asrt}_{p\,\&\,q} \circ \pi_{\lceil p\,\&\,q\rceil} \circ \Theta^{-1} \circ \xi^{\lceil q\,\&\,p\rceil} \circ \pi_{\lceil q\,\&\,p\rceil} \circ \Theta \circ \xi^{\lceil p\,\&\,q\rceil} \circ \mathrm{asrt}_{p\,\&\,q}\\ &= \mathrm{asrt}_{p\,\&\,q} \circ \pi_{\lceil p\,\&\,q\rceil} \circ \xi^{\lceil p\,\&\,q\rceil} \circ \mathrm{asrt}_{p\,\&\,q}\\ &= \mathrm{asrt}_{p\,\&\,q} \circ \mathrm{asrt}_{\lceil p\,\&\,q\rceil} \circ \mathrm{asrt}_{p\,\&\,q}\\ &= \mathrm{asrt}_{p\,\&\,q} \circ \mathrm{asrt}_{p\,\&\,q}\\ &= \mathrm{asrt}_{(p\,\&\,q)^2}.\end{aligned}$$

And hence we are done. □

Now we can conclude that the sequential product in Pred(A) is quadratic (definition 74).

Corollary 117. *Let p and q be sharp predicates. Then $(p \ \& \ q)^2 = p \ \& \ (q \ \& \ p)$.*

Proof. Just plug **1** into the expression of the previous proposition and use $\mathrm{asrt}_q^2 = \mathrm{asrt}_q$ for sharp q. □

Remark 118. An interesting question to ask is whether the dagger structure we impose on the pure maps is structure or a property, i.e. whether it is unique given the other assumptions we impose on it. For a sequential effectus with irreducible scalars this turns out to be the case (although we only know a very indirect proof). For a sequential effectus with general scalars this problem is still open, but the dagger for most maps is indeed fixed: lemma 115 shows that each pure map decomposes into an assert map, filter for a sharp predicate, isomorphism, and comprehension for a sharp predicate, so that it suffices to consider the uniqueness of the dagger for these four classes of maps. Using proposition 112 and corollary 113 it is relatively straightforward to show the dagger is uniquely defined for isomorphisms and filters and comprehensions for sharp predicates. This leaves the question as to whether the assert maps (for non-sharp predicates) are independent of the chosen dagger. We require the assert maps to lead to a normal sequential product. Uniqueness of these products is analysed in [65]. In particular, by applying [65, theorem V.19] we can show that our assert maps are unique when the predicate is *simple* (a finite linear combination of sharp predicates) and *invertible*. These predicates form a norm-dense set of the predicates. Without any further conditions it is not clear whether the assert maps are unique for non-simple predicates, and hence whether the dagger is indeed uniquely defined. However, this is the case when the scalars are irreducible. Then our predicate spaces are JBW-algebras (theorem 125). Our assert maps turn out to be $\diamond$-*positive*, meaning that they are a composition of $\diamond$-self-adjoint maps, and in [69, theorem 4.6.17] it is shown that $\diamond$-positivity uniquely determines a map by its action on the unit in a JBW-algebra, so that in this setting the assert maps are uniquely determined. This proof is very indirect, so it would be nice to find a more straightforward way to prove the uniqueness of the dagger that relies on the abstract conditions of our category and not on the peculiarities of JBW-algebras.

6.4. The predicate spaces are JB-algebras

By assumption our effectus $\mathbf{C}$ has filters and comprehensions and is separated by states. By proposition 108 it has images, and by lemma 111 the filters and comprehensions are compatible. Hence, proposition 103 applies and $\mathbf{C} \cong \mathbf{C}_1 \times \mathbf{C}_2$ where $\mathbf{C}_1$ has Boolean scalars and $\mathbf{C}_2$ has convex scalars. The predicate spaces of $\mathbf{C}_1$ are then complete Boolean algebras, so we are done with that part. We may then focus on $\mathbf{C}_2$. So without loss of generality assume that $\mathbf{C}$ has convex scalars, so that all predicate spaces are convex normal SEAs. Let A again denote an object of $\mathbf{C}$ and let V_A denote the OUS such that $\mathrm{Pred}(A) \cong [0,1]_{V_A}$.

By the previous results the sequential product is compressible and quadratic, so it looks like we can now use theorem 75 to finish our proof that the predicate spaces are JB-algebras. However, a subtle issue now arises. The notion of being compressible as defined in definition 73 refers to states $V_A \to \mathbb{R}$, while our notion of state internal to the effectus is a map $V_A \to C(X)$ where $C(X) =: V_I$ for the trivial object I. Furthermore, definition 73 requires the property to hold for *all* states, while here we have only shown it to hold for states internal to the category. However, it is still possible to get to the conclusion of theorem 75. To do so, we have to delve into the details behind theorem 75.

The crucial part of the proof of theorem 75 is to show that the operators $D_p := \mathrm{asrt}_p - \mathrm{asrt}_{p^\perp}$ for sharp predicates $p \in \mathrm{Pred}(A)$, when viewed as acting on the OUS V_A, are *order derivations*.

Definition 119. Let W be an OUS, and let $\delta : W \to W$ be a bounded linear map. We call δ an **order derivation** when $e^{t\delta} := \sum_{n=0}^{\infty} \frac{(t\delta)^n}{n!}$ is an order isomorphism for all $t \in \mathbb{R}$.

A useful way to prove a map is an order derivation is to use the following proposition.

Proposition 120 ([2, proposition 1.108]). *Let W be a Banach OUS, and let $\delta : W \to W$ be a bounded linear map. Then δ is an order derivation if and only if for all $a \in W^+$ and states $\omega : W \to \mathbb{R}$ the following implication holds:*

$$\omega(a) = 0 \quad \Rightarrow \quad \omega(\delta(a)) = 0.$$

We are interested in taking $\delta := D_p := \mathrm{asrt}_p - \mathrm{asrt}_{p^\perp}$ for some sharp predicate p. The condition $\omega(a) = 0 \Rightarrow \omega(\delta(a)) = 0$ then becomes equivalent to the implication $\omega(a) = 0 \Rightarrow \omega(p \,\&\, a) = \omega(p^\perp \,\&\, a)$. In [67] it is shown that this implication follows when the SEA is compressive and quadratic. We have shown these conditions, but only for the states *internal* to the effectus. So we have the following.

Lemma 121. *Let $p \in \mathrm{Pred}(A)$ be sharp and $a \in \mathrm{Pred}(A)$ arbitrary. Let $\omega : I \to A$ be any state on A. Then $a \circ \omega = 0 \Rightarrow (p \,\&\, a) \circ \omega = (p^\perp \,\&\, a) \circ \omega$.*

Proof. This follows in exactly the same way as for compressive quadratic SEAs as shown in proposition 46 of [67]. □

As our internal states do not necessarily correspond to the type of states mentioned in proposition 120, we need to rework the proof of that statement to make it apply in our situation. In particular, we need to modify the proof of theorem 1.106 in [2] on which proposition 120 depends.

Proposition 122. *Let $p \in \mathrm{Pred}(A)$ be a sharp predicate. Then $D_p := \mathrm{asrt}_p - \mathrm{asrt}_{p^\perp}$ is an order derivation on V_A.*

Proof. To mimic the notation of theorem 1.106 of [2], write $\delta := D_p$. We need to show that $e^{t\delta} \geqslant 0$ for all $t \in \mathbb{R}_{>0}$ (the result for $t < 0$ follows by repeating the argument with $-\delta$). To do this, it suffices to show that $(1 - \lambda\delta)^{-1} \geqslant 0$ for all $\lambda < \frac{1}{2}\|\delta\|^{-1}$, as we can then calculate

$$e^{t\delta} \;=\; (e^{-t\delta})^{-1} \;=\; \left(\lim_n (1 - t/n\,\delta)^n\right)^{-1} \;=\; \lim_n \left((1 - t/n\,\delta)^{-1}\right)^n$$

which then indeed is positive as $t/n < \frac{1}{2}\|\delta\|^{-1}$ for sufficiently large n.

To prove $(1 - \lambda\delta)^{-1} \geqslant 0$, we can use equation (1.82) of [2], which says this is the case precisely when $1 - \lambda\delta$ maps non-positive elements to non-positive elements.

Hence, let $y \in V_A$ with $y \notin V_A^+$. We will show that $(1 - \lambda\delta)y \notin V_A^+$ when $\lambda < \frac{1}{2}\|\delta\|$. We do this by finding a state ω such that $\omega((1 - \lambda\delta)y) < 0$. As we have $\omega(a) \geqslant 0$ for all $a \geqslant 0$ this establishes that $(1 - \lambda\delta)y$ is indeed not positive.

Using the spectral theorem of convex normal SEAs write $y = y^+ - y^-$ for $y^+, y^- \geqslant 0$ and $y^+ \;\&\; y^- = 0$. Let $\alpha = \| y^- \|$ be the 'absolute value of the minimal eigenvalue of y'. Again using the spectral theorem we can then find an idempotent effect $p \neq 0$ which projects onto the part where y is 'very negative', i.e. $p \,\&\, y < -\frac{\alpha}{2}p$. By separation of states there is a state $\omega : I \to A$ such that $p \circ \omega \neq 0$. We may assume that $\mathrm{im}\,\omega \leqslant p$, as otherwise we can simply take $\omega' := \mathrm{asrt}_p \circ \omega$. Note that we can equivalently view ω as a positive linear map $\omega : V_A \to C(X)$ where $C(X)$ corresponds to the predicate space of I.

Now, note that $\omega(y) = \omega(p \,\&\, y) \leqslant \omega(-\frac{\alpha}{2}p) = -\frac{\alpha}{2}\omega(1)$. Let $y' \in V_A$ be such that $\|y - y'\| < \frac{\alpha}{2}$. In particular, this means that for any state σ we have $\sigma(y - y') < \frac{\alpha}{2}\sigma(1)$. Then we also have

$$\omega(y') \;=\; \omega(y' - y + y) \;=\; \omega(y' - y) + \omega(y) \;<\; \frac{\alpha}{2}\omega(1) - \frac{\alpha}{2}\omega(1) \;=\; 0.$$

As a result, for any $z \in V_A$ with $\| z \| < 1$ we see that $\omega(y + \frac{\alpha}{2}z) < 0$ so that $\omega(z) < -\frac{2}{\alpha}\omega(y)$. As this holds for all z with $\| z \| < 1$ we get $\|\omega\| \leqslant -\frac{2}{\alpha}\omega(y)$.

Set $x := y^+$. As $x \;\&\; y^- = 0$, we have $\omega(x) = 0$ so that by lemma 121 we have $\omega(\delta x) = 0$. Note furthermore that $\| y - x \| = \| y^- \| = \alpha$.

Let $\lambda \in \mathbb{R}_{>0}$. We calculate:

$$\begin{aligned}
\omega((1 - \lambda\delta)y) &= \omega(y) - \lambda\omega(\delta y) \\
&= \omega(y) - \lambda\omega(\delta(y - x)) \\
&\leqslant \omega(y) + \lambda\|\omega\|\,\|\delta\|\,\|y - x\| \\
&\leqslant \omega(y) - \frac{2\lambda}{\alpha}\omega(y)\|\delta\|\alpha \\
&= (1 - 2\lambda\|\delta\|)\omega(y).
\end{aligned}$$

Hence, if $2\lambda \| \delta \| < 1$ we see $\omega((1 - \lambda\delta)y) < 0$. As ω is positive, this means that $(1 - \lambda\delta)y \notin V_A^+$ when $\lambda < \frac{1}{2}\|\delta\|^{-1}$. As y was an arbitrary non-positive element, we indeed conclude that $1 - \lambda\delta$ carries $V_A \backslash V_A^+$ into itself. □

Proposition 123. *The space V_A is a JB-algebra.*

Proof. This can be shown by invoking theorem 9.48 of [1] (which itself uses theorem 9.43 of [1]). We can use this theorem 9.48 because of proposition 122. We give a brief sketch of the proof.

Let $p \in \mathrm{Pred}(A)$ be sharp. We claim that the operator $T_p := \frac{1}{2}(\mathrm{id} + D_p) := \frac{1}{2}(\mathrm{id} + \mathrm{asrt}_p - \mathrm{asrt}_{p^\perp})$ acts as the Jordan product operator of p, i.e. that we can define $p * a := T_p a$. For an element $a = \sum_i \lambda_i p_i \in V_A$ we then define its Jordan product operator by linearity as $T_a := \sum_i \lambda_i T_{p_i}$. Each element of V_A can be written as the norm limit of elements of the form $\sum_i \lambda_i p_i$, and hence we get a Jordan product for all elements by continuity.

The crucial point we need to check is commutativity of the Jordan product. For this it suffices to check commutativity on the sharp predicates: $p * q = q * p$. This translates to $T_p q = T_q p$. Because $T_p 1 = p$ and $T_q 1 = q$ we can write our desired identity as $[T_p, T_q]1 = 0$, where $[f, g] := f \circ g - g \circ f$ is the standard commutator bracket. The statement $[T_p, T_q]1 = 0$ is easily seen to be equivalent to $[D_p, D_q]1 = 0$. We have shown that D_p and D_q are order derivations (proposition 122). The commutator of two order derivations is again an order derivation [2, proposition 1.114]. These facts are combined together with some algebra in theorem 9.48 of [1] to show that indeed $[D_p, D_q]1 = 0$.

As mentioned above, for elements $a := \sum_i \lambda_i p_i$ and $b := \sum_j \mu_j q_j$ in V_A we define $a *** b := T_a b$ where $T_a := \sum_i \lambda_i T_{p_i}$. That this is well-defined follows from the commutativity of the T_{p_i} and T_{q_j}. We can copy the argument of theorem 9.43 of [1] to show it is continuous in the norm, so that it extends to all elements of V_A. Similarly, we can follow theorem 9.43 of [1] to show that * satisfies the Jordan identity and the final implication $-1 \leqslant a \leqslant 1 \Rightarrow 0 \leqslant a * a \leqslant 1$ of definition 57. □

6.5. Proof of the main theorems

We can now prove our main reconstruction results. First, the statement for sequential effectuses with arbitrary scalars.

Theorem 124. *Let* $\mathbf{C}$ *be a sequential effectus. Then* $\mathbf{C}$ *is equivalent to a product of effectuses* $\mathbf{C} \cong \mathbf{C}_1 \times \mathbf{C}_2$ *where the predicate spaces of* $\mathbf{C}_1$ *are complete Boolean algebras and those of* $\mathbf{C}_2$ *are directed-complete JB-algebras. In particular, letting* $\mathbf{CBA}$ *denote the category of complete Boolean algebras and monotone maps, we have predicate functors* $\mathrm{Pred} : \mathbf{C}_1 \to \mathbf{CBA}^{\mathrm{op}}$ *and* $\mathrm{Pred} : \mathbf{C}_2 \to \mathbf{JB}^{\mathrm{op}}_{\mathrm{npc}}$ *that are faithful iff* $\mathbf{C}$ *is separated by predicates.*

Proof. By assumption our effectus $\mathbf{C}$ has filters and comprehensions and is separated by states. By proposition 108 it has images, and by lemma 111 the filters and comprehensions are compatible. Hence, proposition 103 applies and $\mathbf{C} \cong \mathbf{C}_1 \times \mathbf{C}_2$ where the predicate spaces of $\mathbf{C}_1$ are orthoalgebras and those of $\mathbf{C}_2$ are convex. By assumption the predicate spaces are normal SEAs, so that by lemma 104 the predicate spaces of $\mathbf{C}_1$ are complete Boolean algebras. That the predicate spaces of $\mathbf{C}_2$ are unit intervals of JB-algebras is given by proposition 123. The theorem now follows easily. □

Unfortunately, we do not know any way in which we can restrict the directed-complete JB-algebras to JBW-algebras in this setting with arbitrary scalars. The fact that we require all maps to be normal and that the effectus is separated by states is not enough to give 'separation by normal states' in the sense required for JBW-algebras (cf definition 61). However, when we restrict the scalars to irreducible scalars, then the concept of state for the effectus corresponds to that for JB-algebras, and hence we do get JBW-algebras.

Theorem 125. *Let* $\mathbf{C}$ *be a sequential effectus with irreducible scalars. Then all predicate spaces are either complete Boolean algebras or they are all the unit interval of a JBW-algebra. Furthermore, there is a functor* $\mathrm{Pred} : \mathbf{C} \to \mathbf{D}^{\mathrm{op}}$ *where* $\mathbf{D}$ *is either* $\mathbf{CBA}$ *of* $\mathbf{JBW}_{\mathrm{npc}}$*, and this functor is faithful iff* $\mathbf{C}$ *is separated by predicates.*

Proof. Let $\mathbf{C}$ be a sequential effectus with irreducible scalars. Then $\mathrm{Pred}(I) = \{0\}$, or $\mathrm{Pred}(I) = \{0, 1\}$ or $\mathrm{Pred}(I) = [0, 1]$. In the first case, $\mathbf{C}$ is equivalent to the trivial single-object category, and hence the theorem is trivially true. If $\mathrm{Pred}(I) = \{0, 1\}$, then the predicate spaces are orthoalgebras, and as they are also normal SEAs, they are complete Boolean algebras, and hence we get $\mathrm{Pred} : \mathbf{C} \to \mathbf{CBA}^{\mathrm{op}}$ as required. Finally, if $\mathrm{Pred}(I) = [0, 1]$ then all the predicate spaces are convex, so that the previous results give $\mathrm{Pred} : \mathbf{C} \to \mathbf{JB}^{\mathrm{op}}_{\mathrm{npc}}$. Let $A \in \mathbf{C}$ be an object, then the predicate space $\mathrm{Pred}(A)$ is separated by normal states $\omega : I \to A$. These correspond to normal states $\omega^* : V_A \to \mathbb{R}$ on the JB-algebra V_A. As V_A is then directed-complete and separated by normal states, it is a JBW-algebra. Hence, the predicate functor restricts to $\mathrm{Pred} : \mathbf{C} \to \mathbf{JBW}^{\mathrm{op}}_{\mathrm{npc}}$. □

With these theorems we see that each object in a sequential effectus splits up into an object whose predicates form a Boolean algebra and an object whose predicates correspond to the unit interval of a JB-algebra, a model for a quantum system. If the scalars of the effectus are irreducible then the predicate functor restricts to either Boolean algebras or JBW-algebras, meaning that any such category either models deterministic classical logic, or probabilistic quantum logic.

We can however still get a bit closer to quantum theory, and see that each predicate space embeds into a von Neumann algebra.

7. Reconstruction for monoidal effectuses

One could argue that JBW-algebras are not a proper model for a quantum system since they also include the purely exceptional algebras (cf theorems 66 and 69). However, as we will work towards now, when the sequential effectus is monoidal in a suitable way then the only allowed JBW-algebras for the predicate spaces are JW-algebras. This reflects the result in finite dimension, shown in many different contexts, that a distinguishing factor between Jordan algebras and C^*-algebras is that the former do not allow well-behaved tensor products [5, 35, 58, 66, 67].

Definition (Restatement of definition 80). A **monoidal sequential effectus** is a sequential effectus that is monoidal and such that

- The tensor product of two pure maps is pure,
- For pure f and g we have $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$.

For this section let us assume that we are working with a monoidal sequential effectus with irreducible scalars. Then by theorem 125 we may assume all our predicate spaces are either Boolean algebras or JBW-algebras. We are interested in the latter case, so let us assume the scalars are $[0, 1]$ instead of $\{0, 1\}$ so that our predicate spaces are JBW-algebras. In this section we will show that the monoidal structure forces our predicate spaces to be JW-algebras, which boils down to showing that the predicate spaces cannot contain exceptional subalgebras. In this section we will need to use some more structure present in JBW-algebras than before. We will introduce the necessary concepts when needed.

For the remainder of this section let $\mathbf{C}$ denote a monoidal sequential effectus with scalars $[0, 1]$. Let A and B denote objects in $\mathbf{C}$. By previous results we have JBW-algebras V_A and V_B such that $\mathrm{Pred}(A) \cong [0, 1]_{V_A}$ and $\mathrm{Pred}(B) \cong [0, 1]_{V_B}$. Additionally, the tensor product $A \otimes B$ has an associated JBW-algebra $V_{A\otimes B}$. Recall that the definition of a monoidal effectus (definition 41) gives us $\mathbf{1}_A \otimes \mathbf{1}_B = \mathbf{1}_{A\otimes B}$ and $(a \ovee b) \otimes c = (a \otimes c) \ovee (b \otimes c)$. The scalar action of $[0, 1]$ on the predicates also works nicely with the tensor product (see lemma 42). As a result we get a bilinear positive unital map $V_A \times V_B \to V_{A\otimes B}$, which we will also denote by $\otimes$.

Our first step is getting a better handle on the interaction of the Jordan product with the tensor product. First we recall from section 2.5 that a JBW-algebra has a sequential product operation given by the quadratic product as $a \,\&\, b = Q_{\sqrt{a}} b$. As is shown in [69, theorem 4.6.17], the map $Q_{\sqrt{a}}$ is the unique $\diamond$-positive map on a JBW-algebra satisfying $Q_{\sqrt{a}} 1 = a$. As asrt_a is a $\diamond$-positive map on the JBW-algebra V_A satisfying $\mathbf{1} \circ \mathrm{asrt}_a = a$ we must then have $\mathrm{asrt}_a = Q_{\sqrt{a}}$ on V_A.

Proposition 126. *Let* $a \in \mathrm{Pred}(A)$ *and* $b \in \mathrm{Pred}(B)$. *Then* $\mathrm{asrt}_{a\otimes b} = \mathrm{asrt}_a \otimes \mathrm{asrt}_b$. *In particular* $\mathrm{asrt}_{a\otimes \mathbf{1}} = \mathrm{asrt}_a \otimes \mathrm{id}$ *and* $a^2 \otimes b^2 = (a \otimes b)^2$.

Proof. Note $\mathbf{1} \circ (\mathrm{asrt}_a \otimes \mathrm{asrt}_b) = (\mathbf{1} \otimes \mathbf{1}) \circ (\mathrm{asrt}_a \otimes \mathrm{asrt}_b) = (\mathbf{1} \circ \mathrm{asrt}_a) \otimes (\mathbf{1} \circ \mathrm{asrt}_b) = a \otimes b$, so by uniqueness of $\dagger$-positive maps, it remains to show that $\mathrm{asrt}_a \otimes \mathrm{asrt}_b$ is $\dagger$-positive. But this follows because $(\mathrm{asrt}_{\sqrt{a}} \otimes \mathrm{asrt}_{\sqrt{b}})^2 = \mathrm{asrt}^2_{\sqrt{a}} \otimes \mathrm{asrt}^2_{\sqrt{b}} = \mathrm{asrt}_a \otimes \mathrm{asrt}_b$.

For $\mathrm{asrt}_{a\otimes \mathbf{1}} = \mathrm{asrt}_a \otimes \mathrm{id}$ we simply note that $\mathrm{asrt}_{\mathbf{1}} = \mathrm{id}$, and for $a^2 \otimes b^2 = (a \otimes b)^2$ we calculate $a^2 \otimes b^2 = (a \circ \mathrm{asrt}_a) \otimes (b \circ \mathrm{asrt}_b) = (a \otimes b) \circ (\mathrm{asrt}_a \otimes \mathrm{asrt}_b) = (a \otimes b) \circ \mathrm{asrt}_{a\otimes b} = (a \otimes b)^2$. □

Corollary 127. *Let* $p \in \mathrm{Pred}(A)$ *and* $q \in \mathrm{Pred}(B)$ *be sharp predicates. Then* $p \otimes q$ *is sharp.*

Denote by $*$ the Jordan product on V_A and write $T_a(b) = a * b$ for the Jordan product map of a (and similarly for V_B). T_a is not a positive map and hence cannot be part of the effectus. However, if p is sharp (i.e. idempotent) then $T_p = \frac{1}{2}(\mathrm{id} + Q_p - Q_{p^\perp})$ so that it is a linear combination of maps that do lie in the effectus, so that we can still speak of tensor products of these maps. Note that additionally $Q_p = \mathrm{asrt}_{p^2} = \mathrm{asrt}_p$.

Proposition 128. *Let* $a \in V_A$ *be arbitrary and* $\mathbf{1} \in V_B$, *then* $T_{a\otimes \mathbf{1}} = T_a \otimes \mathrm{id}$. *Similarly, for* $\mathbf{1} \in V_A$ *and* $b \in V_B$ *we have* $T_{\mathbf{1}\otimes b} = \mathrm{id} \otimes T_b$.

Proof. We only show the first equation, as the second follows analogously. We prove the result for $a = p$ sharp. By the norm-continuity and linearity of the Jordan product in the first argument, this is sufficient as the sharp elements span a dense set.

Note first that $(p \otimes \mathbf{1})^\perp = p^\perp \otimes \mathbf{1}$ and $\mathrm{id}_{A\otimes B} = \mathrm{id}_A \otimes \mathrm{id}_B$. We then calculate:

$$\begin{aligned} T_{p\otimes \mathbf{1}} &= \frac{1}{2}\big(\mathrm{id} \otimes \mathrm{id} + \mathrm{asrt}_{p\otimes \mathbf{1}} - \mathrm{asrt}_{(p\otimes \mathbf{1})^\perp}\big) = \frac{1}{2}\big(\mathrm{id} \otimes \mathrm{id} + \mathrm{asrt}_p \otimes \mathrm{id} - \mathrm{asrt}_{p^\perp} \otimes \mathrm{id}\big) \\ &= \left(\frac{1}{2}(\mathrm{id} + \mathrm{asrt}_p - \mathrm{asrt}_{p^\perp})\right) \otimes \mathrm{id} = T_p \otimes \mathrm{id}. \end{aligned}$$

□

The Jordan product is of course commutative, however, this does not mean that the Jordan product maps T_a all commute. When $T_aT_b = T_bT_a$ we say that a and b **operator commute**.

Corollary 129. *For all $a \in V_A$ and $b \in V_B$, $a \otimes \mathbf{1}$ and $\mathbf{1} \otimes b$ operator commute.*

Proposition 130. *The maps $a \mapsto a \otimes \mathbf{1}$ and $b \mapsto \mathbf{1} \otimes b$ are normal injective Jordan homomorphisms.*

Proof. We only show this for $a \mapsto a \otimes \mathbf{1}$ as the other one follows analogously. That it is a Jordan homomorphism, i.e. $(a_1 \otimes 1) * (a_2 \otimes 1) = (a_1 * a_2) \otimes 1$, follows immediately from proposition 128.

To show it is injective suppose $a \otimes \mathbf{1} = a' \otimes \mathbf{1}$. Let ω be any state on the first system, and ω' any state on the second system. Then $\omega(a) = \omega(a)\omega'(\mathbf{1}) = (\omega \otimes \omega')(a \otimes \mathbf{1}) = (\omega \otimes \omega')(a' \otimes \mathbf{1}) = \omega(a')$. Since states separate the predicates, we then necessarily have $a = a'$.

Now as it is an injective unital Jordan homomorphism, the restriction to their domain is an order-isomorphism, and hence the maps must be normal. □

We now wish to prove that the quadratic product 'commutes' with the tensor product for arbitrary elements of the JBW-algebras: $Q_{a\otimes b} = Q_a \otimes Q_b$ (equality here is understood as equality as linear maps on $V_{A\otimes B}$). Proposition 126 shows this result for effects a and b. To extend this result for arbitrary a and b we need a concept related to the quadratic product, known as the **triple product**. For $a, b, c \in V$ for V a Jordan algebra we define $Q_{a,b}c = (a * b) * c + (c * b) * a - (a * c) * b$. To motivate this, when V is a JW-algebra, this gives $Q_{a,b}c = \frac{1}{2}(acb + bca)$ where the product is in the underlying von Neumann algebra. The triple product is related to the quadratic product via $Q_a = Q_{a,a}$. Note that $Q_{a,b} = Q_{b,a}$ and that it is bilinear in its two arguments: $Q_{a_1+a_2,b} = Q_{a_1,b} + Q_{a_2,b}$. In particular, $Q_{a_1+a_2} = Q_{a_1+a_2,a_1+a_2} = Q_{a_1,a_1} + Q_{a_2,a_2} + 2Q_{a_1,a_2}$.

Proposition 131. *Let $a \in V_A$ and $b \in V_B$ be arbitrary. Then $Q_{a\otimes b} = Q_a \otimes Q_b$.*

Proof. First suppose $a \in [0,1]_{V_A}$ and $b \in [0,1]_{V_A}$. Then $Q_a = \mathrm{asrt}_{a^2}$ and $Q_b = \mathrm{asrt}_{b^2}$, so that by proposition 126 $Q_{a\otimes b} = \mathrm{asrt}_{(a\otimes b)^2} = \mathrm{asrt}^2_{a\otimes b} = (\mathrm{asrt}_a \otimes \mathrm{asrt}_b)^2 = \mathrm{asrt}_{a^2} \otimes \mathrm{asrt}_{b^2} = Q_a \otimes Q_b$. Now for an arbitrary positive a we have $Q_a = Q_{\|a\|a/\|a\|} = \| a \|^2 Q_{a/\|a\|}$, and hence the desired result also follows when $a \geqslant 0$ and $b \geqslant 0$.

Now suppose $a = a_1 + a_2$ where $a_1, a_2 \geqslant 0$. Then $Q_a = Q_{a_1} + Q_{a_2} + 2Q_{a_1,a_2}$. We expand $Q_{a\otimes b}$ in two different ways. First we see that $Q_{a\otimes b} = Q_{a_1\otimes b+a_2\otimes b} = Q_{a_1\otimes b} + Q_{a_2\otimes b} + 2Q_{a_1\otimes b,a_2\otimes b} = Q_{a_1} \otimes Q_b + Q_{a_2} \otimes Q_b + 2Q_{a_1\otimes b,a_2\otimes b}$. Secondly, $Q_{a\otimes b} = Q_a \otimes Q_b = Q_{a_1} \otimes Q_b + Q_{a_2} \otimes Q_b + 2Q_{a_1,a_2} \otimes Q_b$. Comparing terms in both of these decompositions of $Q_{a\otimes b}$ we see that necessarily $Q_{a_1\otimes b,a_2\otimes b} = Q_{a_1,a_2} \otimes Q_b$.

We can use this equation, and do a similar trick, but starting with $Q_{a_1\otimes b,a_2\otimes b}$ where $b = b_1 + b_2$ to give us the equation $2Q_{a_1,a_2} \otimes Q_{b_1,b_2} = Q_{a_1\otimes b_1,a_2\otimes b_2} + Q_{a_1\otimes b_2,a_2\otimes b_1}$.

Finally, suppose a and b are arbitrary. Write $a = a^+ - a^-$ and $b = b^+ - b^-$ where $a^+, a^-, b^+, b^- \geqslant 0$. Now if we expand both the expression $Q_{(a^+-a^-)\otimes(b^+-b^-)}$ and $Q_{a^+-a^-} \otimes Q_{b^+-b^-}$ as much as possible using linearity and apply the previous rewrite rules, it is easily verified that these two expression are indeed equal. □

In order to proceed we need to use the concept of *universal von Neumann algebras*.

Theorem 132 ([36, theorem 7.1.9]). *Let V be a JBW-algebra. Then there exists an (up to isomorphism) unique von Neumann algebra $W^*(V)$ and a normal Jordan homomorphism $\psi : V \to W^*(V)_{\text{sa}}$ such that $\psi(V)$ generates $W^*(V)$ as a von Neumann algebra and if $\mathfrak{B}$ is a von Neumann algebra with a normal Jordan homomorphism $\phi : V \to \mathfrak{B}_{\text{sa}}$, then there is a unique normal $*$-homomorphism $\hat{\phi} : W^*(V) \to \mathfrak{B}$ such that $\hat{\phi} \circ \psi = \phi$.*

Corollary 133. *A JBW-algebra V is a JW-algebra if and only if $\psi : V \to W^*(V)_{\text{sa}}$ is injective.*

Proof. If ψ is injective, then V is of course a JW-algebra. Conversely, if V is a JW-algebra, then there must be an injective normal Jordan homomorphism $\phi : V \to \mathfrak{B}_{\text{sa}}$ for some von Neumann algebra $\mathfrak{B}$, and hence by the universal property of $W^*(V)$, $\hat{\phi} \circ \psi = \phi$, which shows that ψ must be injective. □

Definition 134. Let V be a JBW-algebra. We call $s \in V$ a **symmetry** when $s^2 = 1$. Two idempotents $p, q \in V$ are **exchangeable by a symmetry** if there exists a symmetry s such that $Q_s p = q$.

Lemma 135 ([1, lemma 4.4]). *Let V be a JBW-algebra where the identity is the sum of at least 4 idempotents that are mutually exchangeable by a symmetry. Then V is a JW-algebra.*

Lemma 136. *Let $V \neq \{0\}$ be a purely exceptional JBW-algebra. Then the identity of V is the sum of three orthogonal non-zero idempotents exchangeable by a symmetry.*

Proof. By theorem 69 we can write $V = C(X, E)$ where $E = M_3(\mathbb{O})_{\text{sa}}$ for some hyperstonean space X. As X is a type I_3 JBW-factor there exist orthogonal non-zero idempotents $q_1, q_2, q_3 \in E$ mutually exchangeable by a symmetry such that $q_1 + q_2 + q_3 = 1_E$ [36, theorem 2.8.3]. Let $s_{ij} \in E$ for $i, j \in \{1, 2, 3\}$ be symmetries so that $Q_{s_{ij}} q_i = q_j$. Define then $f_i : X \to E$ as the constant function $f_i(x) = q_i$, and similarly $g_{ij} : X \to E$ by $g_{ij}(x) = s_{ij}$. Then indeed for every $x \in X : (Q_{g_{ij}} f_i)(x) = Q_{s_{ij}} q_i = q_j = f_j(x)$. □

Lemma 137. *Let $p_1, q_1 \in V_A$ be idempotents exchangeable by a symmetry $s_1 \in V_A$, and let $p_2, q_2 \in V_B$ be idempotents exchangeable by a symmetry $s_2 \in V_B$. Then $p_1 \otimes p_2$ and $q_1 \otimes q_2$ are idempotents exchangeable by $s_1 \otimes s_2$.*

Proof. That $p_1 \otimes p_2$ and $q_1 \otimes q_2$ are idempotents follows by corollary 127. That $s_1 \otimes s_2$ is a symmetry follows by proposition 131, because $(s_1 \otimes s_2)^2 = Q_{s_1 \otimes s_2} 1 = (Q_{s_1} \otimes Q_{s_2})(1 \otimes 1) = s_1^2 \otimes s_2^2 = 1 \otimes 1 = 1$. By the same proposition: $Q_{s_1 \otimes s_2}(p_1 \otimes p_2) = (Q_{s_1} \otimes Q_{s_2})(p_1 \otimes p_2) = (Q_{s_1} p_1) \otimes (Q_{s_2} p_2) = q_1 \otimes q_2$. □

Proposition 138. *V_A is a JW-algebra.*

Proof. Since V_A is a JBW-algebra we can write $V_A = V_1 \oplus V_2$ where V_1 is a JW-algebra and V_2 is purely exceptional (theorem 66). We need to show that $V_2 = \{0\}$. Towards contradiction, suppose that $V_2 \neq \{0\}$.

Let $p \in V_A$ be the central idempotent corresponding to V_2. Then we have an object A_p in $\mathbf{C}$, by considering the comprehension π_p, so that $\mathrm{Pred}(A_p) \cong [0,1]_{V_2}$. Let q_1, q_2, q_3 be a set of idempotents in A_p exchangeable by symmetries s_{ij} for $i, j \in \{1, 2, 3\}$, which exists by lemma 136. Consider the system $A_p \otimes A_p$. By lemma 137 $s_{ik} \otimes s_{jl}$ is a symmetry for all $i, k, j, l \in \{1, 2, 3\}$. This set of symmetries makes all nine idempotents $\{q_i \otimes q_j;\ i, j \in \{1, 2, 3\}\}$ in $A_p \otimes A_p$ mutually exchangeable by a symmetry.

Hence, by lemma 135, $V_{A_p \otimes A_p}$ must be a JW-algebra. So then $V_{A_p \otimes A_p}$ embeds into $W^*(V_{A_p \otimes A_p})$ via an injective Jordan homomorphism (corollary 133). But we also have an injective Jordan homomorphism from V_2 to $V_{A_p \otimes A_p}$ given by $a \mapsto a \otimes 1$ (proposition 130). Hence, V_2 embeds into $W^*(V_{A_p \otimes A_p})$. This contradicts the fact that V_2 is purely exceptional, so that we indeed must have had $V_2 = \{0\}$. □

Let us denote by $\mathbf{JW}_{\mathrm{npc}}$ the full subcategory of $\mathbf{JBW}_{\mathrm{npc}}$ consisting of the JW-algebras. Combining what we have seen before we then get the following theorem.

Theorem 139. *Let* $\mathbf{C}$ *be a monoidal sequential effectus with irreducible scalars not equal to* $\{0, 1\}$. *Then there is a functor* $F : \mathbf{C} \to \mathbf{JW}_{\mathrm{npc}}^{\mathrm{op}}$ *satisfying* $F(\mathrm{Pred}(A)) \cong [0, 1]_{F(A)}$. *This functor is faithful if and only if* $\mathbf{C}$ *is separated by predicates.*

Recall that the other option for the irreducible scalars is that they are equal to the Booleans $\{0, 1\}$, which results in predicate spaces being complete Boolean algebras. So we either get deterministic classical systems, or probabilistic quantum systems in the form of (subspaces) of von Neumann algebras.

Remark 140. Not all JW-algebras are allowed in our setting, but analysing which ones precisely has proven difficult. By adapting the arguments of [66] we can show that no finite-dimensional quaternionic systems are allowed, but it is unclear how to adapt the argument for infinite-dimensional quaternionic systems. We also conjecture that no 'true' spin-factors (those that aren't isomorphic to a matrix algebra) are allowed, which means that our JW-algebras restrict to the *universally reversible* ones, for which a characterisation of the universal von Neumann algebra is known (cf theorem 6.2.5 and proposition 7.3.3 of [36]).

8. Conclusion and discussion

We have shown that an effectus with directed-complete predicate spaces and suitable additional structure embeds into a category of Boolean algebras and a category of JB-algebras. Requiring the scalars to be irreducible allows us to restrict to JBW-algebras, and imposing a tensor product restricts us further to JW-algebras, which are subspaces von Neumann algebras. This demonstrates that quantum theory, including both infinite-dimensional systems as well as mixed quantum–classical systems, can be reconstructed from abstract categorical grounds without even *a priori* referring to the structure of real numbers or convex sets. While there have been other categorical approaches to reconstructing quantum theory [58, 64], they had to insert the real numbers at some point to get standard quantum theory, making our reconstruction the first to be fully categorical.

There are a number of open questions related to our results. The first is about the interpretation of our axioms. While most have an operational interpretation, some others are merely of mathematical interest, such as the requirements on ⋄-adjointness or uniqueness of †-positive maps. It would be interesting to find a 'meaning' for these axioms, or failing that, a derivation of these properties from other more operational assumptions. We also suspect that similar results to ours should be attainable with fewer assumptions. The assumptions in definition 76 are in any case not minimal, as some of the axioms of a sequential product follow 'for free'. The ones that require explicit inclusion are related to the commutativity of certain assert maps. Proving these follow from the other assumptions in the abstract setting seems difficult however, as in [68] these commutativity conditions are derived for JBW-algebras using the Fuglede–Putnam–Rosenblum theorem, so an abstract proof would circumvent the use of this rather powerful theorem. The finite-dimensional reconstruction of [66] outlined in remark 60 does not require any of the assumptions related to sequential measurement. As the only examples we have of categories satisfying the assumptions of remark 60 are the categories of Jordan algebras we consider here, it might be that we can do our infinite-dimensional reconstruction without referring to properties of sequential measurement at all.

Another open question is how we can get get the systems to correspond to JBW-algebras without restricting the scalars to be irreducible.

Finally, there are several open questions related to our monoidal reconstruction. As we pointed out in remark 140, we suspect that there are several restrictions on the types of JW-algebras allowed in categories satisfying our assumptions. In several other finite-dimensional reconstructions they first show that their systems are Jordan algebras, and then use the tensor product to find that only real and complex systems are allowed, by using a dimension-counting argument [4, 58, 66, 67]. This technique obviously does not work in infinite dimension, but there might be more sophisticated cardinality arguments that allow us to do the same. Alternatively, if we add some assumption ensuring there is a system with an atomic projection, then we might be able to adapt [34, theorem 5.5], in order to show that every system must correspond to a real or complex C^*-algebra. These questions are all related to determining which subcategories of $\mathbf{JW}^{\mathrm{op}}_{\mathrm{npc}}$ satisfy our assumptions. This problem seems hard, as the categories could for instance only contain 'continuous' systems that have no atomic projections, apart from the trivial system $\mathbb{R}$ that has to be included. We could simplify this classification by requiring at least some type I factor to exist in the category, which would make it much easier to analyse.

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Data availability statement

No new data were created or analysed in this study.

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