

**The All-Loop
Finite-Size Next-To-Next-To-Leading-Order
One-Cut
Large-Mode-Number, Fixed-Winding-Number or Weak-Coupling
Solution to the Quantum String Bethe Ansatz Equations
in the Non-Compact Rank-One $SL(2)$ Sector**

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I. INTRODUCTION

The first section of this dissertation is devoted to introducing the reader into the realms of the AdS/CFT correspondence, and in particular, the powerful concept of integrability in it. There are many reviews of the topic available, such as [3–20].

A. The 't Hooft's Limit and String/Gauge Dualities

1. String and Gauge Theories

Ever since the milestone of the establishing of the Standard Model as a quantum-field-theoretical description of the unified three forces of nature (electromagnetic, weak and strong interactions), perhaps the greatest quest of physics has been to attach the fourth force, gravity, to the unified picture. For more than three decades, this ultimate goal has been pursued within an intrinsically novel approach, by many considered to be very promising, although not lacking its fierce opponents (even arguing for its complete lack of connection with the real world) — a theory of 2-dimensional objects, “strings” [21]. String theory itself has been initially designed to describe strong interactions (“dual resonance model”). This “old string theory” has been fueled by the appearance of approximately linear Regge trajectories in observations of the πN scattering, where a certain duality between amplitudes in the s - and t -channels has been noticed [22]. Veneziano in 1968 [23] derived an expression for a 4-point amplitude which is manifestly s - t crossing-symmetric and yields a linear Regge trajectory,

$$A(s, t) \sim \frac{\Gamma(-\alpha(s)) \Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))}, \quad \alpha(s) = \alpha(0) + \alpha' s. \quad (1)$$

Its physical interpretation was presented in 1970 independently by Nambu, Nielsen and Susskind [24] in terms of an infinite number of harmonic oscillators, therefore, a string. A meson $q\bar{q}$ is described as an open string with a quark q and an anti-quark \bar{q} on its opposite ends, and excitations of this string correspond to meson's states; the string's “Nambu-Goto action” reads

$$S_{\text{Nambu-Goto}} = -T \int \int d\tau d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}, \quad T = \frac{1}{2\pi\alpha'}, \quad (2)$$

where T is the string's tension, related above to the Regge slope α' . But the string techniques of handling strong interactions quickly encountered serious difficulties; for example, they implied wrong meson masses, included tachyons, and required 26 dimensions (10 for the supersymmetric counterpart) to live. Finally, it was quantum chromodynamics (QCD) that succeeded as a model for strong interactions, and string methods have been abandoned (except for describing long-distance behavior of $q\bar{q}$ pairs, bound with a linear potential due to formation of so-called flux tubes of strong color field, for example in the setting of the “Lund string model” [25]). There opened, however, and even greater opportunity for string theory; since the closed string spectrum contains the graviton, the hope of quantizing gravity, so far beyond reach, started to materialize [26]. In this approach, neither gauge fields nor gravity are fundamental any more, but follow from more basic strings as their low-lying excitations.

2. The 't Hooft's Limit

It was 't Hooft in 1974 [27] who first suggested that this fundamental string theory may actually be equivalent (“dual”) to a gauge theory. In this way, oddly, neither description would be supreme, but one would proceed from the other. In particular, amazingly, a strongly-coupled gauge theory might find a dual description in terms of a string theory which happens to be tractable. And this duality, importantly, is not an effective approximate string picture of QCD flux tubes, but an exact correspondence of the two theories. Let us briefly revisit his idea. Consider a $(3+1)$ -dimensional Yang-Mills theory with the non-Abelian gauge group $SU(N)$, for example one sketched by the following Lagrangian with 3- and 4-point vertices,

$$S = \frac{1}{g_{\text{YM}}^2} \int d^4x \mathcal{L}, \quad \mathcal{L} = \text{Tr}(\partial_\mu \Phi_i \partial^\mu \Phi_i) + c^{ijk} \text{Tr}(\Phi_i \Phi_j \Phi_k) + d^{ijkl} \text{Tr}(\Phi_i \Phi_j \Phi_k \Phi_l), \quad (3)$$

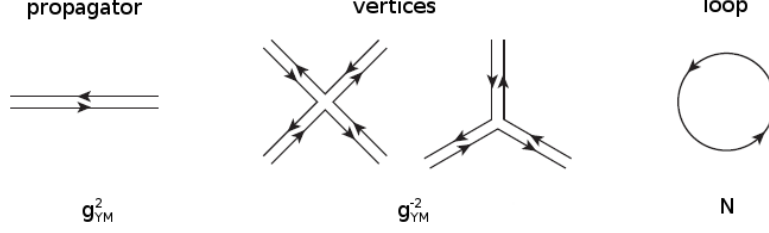


FIG. 1: The 't Hooft double-line notation, and factors for the Lagrangian (3). The opposite arrows describe the fundamental and anti-fundamental representations (4).

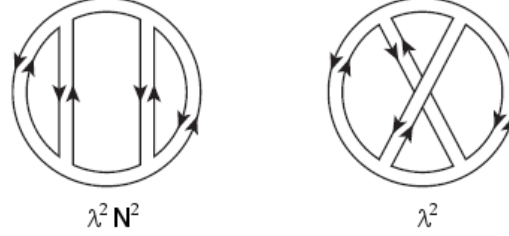


FIG. 2: Examples of a genus-zero (sphere; planar) and genus-one (torus; non-planar) Feynman diagram. The latter is suppressed as compared to the former in the 't Hooft limit (6).

where each $(\Phi_i)_{AB}$ is an $N \times N$ Hermitian and traceless matrix field in the adjoint representation. The 't Hooft's method will work for any gauge theory with adjoint fields, and is based on the decomposition into the fundamental and anti-fundamental representations,

$$\text{adjoint} = N \otimes \bar{N} - 1, \quad (4)$$

which allows to pictorially encode the two color indices as double lines in Feynman diagrams, creating so-called “fat diagrams.” The basic graphs stemming from (3), and the associated factors of g_{YM} and N , are shown in figure 1. In the 't Hooft's double-line notation, Feynman graphs start resembling discretized 2-dimensional surfaces, or in other words, any diagram can be drawn without crossing a line on a surface of a certain genus g , thus providing its polygonisation. More precisely, a graph with E propagators (edges), V vertices and F loops (faces) can be drawn without crossings on a surface with Euler characteristics $\chi = -E + V + F$, *i.e.*, genus $\chi = 2 - 2g$. Collecting the data from figure 1, such a diagram would come with the factor of

$$(g_{YM}^2)^E (g_{YM}^{-2})^V N^F = \lambda^{E-V} N^\chi, \quad \text{where} \quad \lambda \equiv g_{YM}^2 N = \text{the 't Hooft coupling constant}, \quad (5)$$

see figure 2.

Now, the 't Hooft's breakthrough idea was to extend the analysis of QCD, which has the gauge group $SU(3)$, to an arbitrary number of colors N , which then is considered a free parameter of the theory, and take the limit

$$g_{YM} \rightarrow 0, \quad N \rightarrow \infty, \quad \text{such that} \quad \lambda = \text{fixed} \quad (\text{'t Hooft}). \quad (6)$$

In QCD, one investigates perturbative phenomena, such as deep inelastic scattering, which have high energies and are described by a small coupling g_{YM} . In addition, there are non-perturbative phenomena, such as quark confinement, which is a low-energy physics, when the coupling is large. The 't Hooft limit (6) has this advantage that it still incorporates the non-perturbative regime, while considerably simplifying calculations.

Consequently, the theory (its partition function Z) can be investigated order by order in $1/N$, which then can subsequently be expanded in small λ (quantum loops). Thanks to (5), at each order of $1/N$, the graphs can be drawn on a surface of a given genus, *i.e.*, the large- N expansion is in fact an expansion in the topology of 2-dimensional surfaces,

$$Z(N, \lambda) = \sum_{g \geq 0} N^{2-2g} \sum_{l \geq 0} \lambda^l z_{g,l} = N^2(\text{sphere}) + (\text{torus}) + \frac{1}{N^2}(\text{torus with two holes}) + \dots \quad (7)$$

Now, this gauge-theory series resembles a closed string perturbation theory expansion of a hypothetical “QCD string” in the string coupling constant $g_s \sim 1/N$, and then in the string tension $1/\alpha' \sim \sqrt{\lambda}$, which was the first indication that the two theories may be much closer to each other than imagined. Furthermore, large N means a weakly-coupled string theory. Sadly, it was not at all clear how to build such a string for a given field theory. Using the gauge-side insight, the corresponding string would be described through the summing of all the planar graphs, which, however, poses an enormous challenge (the “large- N master equation” problem). A possible string/gauge duality was, therefore, firmly believed to exist, although nobody could explicitly construct one. Once found, one would come the full circle back to the original attempts of explaining strong interactions in terms of string theory.

B. The $(3+1)$ -Dimensional $SU(N)$ $\mathcal{N} = 4$ Supersymmetric Yang–Mills Theory

In this subsection, we introduce the “gauge side” of the so-called “AdS/CFT correspondence,” the first and best-investigated realization of the ’t Hooft’s idea (to which we will come in subsection ID) — namely, the $(3+1)$ -dimensional pure $SU(N)$ $\mathcal{N} = 4$ supersymmetric Yang–Mills theory [28].

1. The Supersymmetry Algebra and Its Representations

Let us start from recalling the basic ingredients of the supersymmetry (SUSY) graded Lie algebra in the flat $(3+1)$ -dimensional spacetime with the metric $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$. This symmetry between bosons and fermions is generated by “supercharges,” which are left and right Weyl spinors of the Lorentz group $SO(1, 3)$, commuting with spacetime translations P_μ ,

$$Q_\alpha^a \quad (\text{left}), \quad \bar{Q}_{\dot{\alpha}a} = (Q_\alpha^a)^\dagger \quad (\text{right}), \quad \text{where} \quad \alpha, \dot{\alpha} = 1, 2, \quad a = 1, 2, \dots, \mathcal{N}. \quad (8)$$

Here \mathcal{N} denotes the number of independent SUSYs. The supercharges submit to the anti-commutation relations,

$$\{Q_\alpha^a, \bar{Q}_{\dot{\beta}b}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \delta_b^a, \quad \{Q_\alpha^a, Q_\beta^b\} = 2\epsilon_{\alpha\beta} Z^{ab}, \quad (9)$$

where, also for further reference, $\sigma_\mu \equiv (1, \sigma_i)$, $\bar{\sigma}_\mu \equiv (1, -\sigma_i)$, $\sigma_{\mu\nu} \equiv \frac{1}{4}\sigma_{[\mu}\bar{\sigma}_{\nu]}$, $\bar{\sigma}_{\mu\nu} \equiv \frac{1}{4}\bar{\sigma}_{[\mu}\sigma_{\nu]}$, with $\sigma_{1,2,3}$ being the standard Pauli matrices; moreover, the anti-symmetric and commuting with everything generators Z^{ab} are called the “central charges.” Observe that the above algebra of supercharges displays the invariance w.r.t. rotations in the index a which form the group

$$SU(\mathcal{N}) \quad (\text{the R-symmetry}). \quad (10)$$

The prime question is to find representations of this SUSY algebra. The representations describing massless particles are derived most easily by switching to a frame of reference in which the particle’s 4-momentum $P^\mu = (E, 0, 0, E)$, with $E > 0$. Then the first relation (9) reduces to

$$\{Q_\alpha^a, (Q_\beta^b)^\dagger\} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix} \delta_b^a, \quad (11)$$

which can be quickly translated, upon exploiting the assumption that the norms in our Hilbert space are always non-negative (unitary representation), into the following statements: $Q_2^a = 0$, $Z^{ab} = 0$, while Q_1^a and $(Q_1^a)^\dagger$ act as respectively the lowering and raising operators for helicity by $1/2$, therefore forming a $2^\mathcal{N}$ -dimensional representation of the Clifford algebra associated with the group $SO(2\mathcal{N})$; the states of this representation are derived from the highest-helicity state through the lowering operators Q_1^a with all possible a ’s. For example, in the case of $\mathcal{N} = 4$ and a CPT-invariant spectrum (*i.e.*, symmetric w.r.t. the change of sign of helicity) of massless particles with helicities ≤ 1 , there are respectively 1, 4, 6, 4, 1 states with helicities 1, $1/2$, 0, $-1/2$, -1 .

The massive representations are conveniently computed in the rest frame of the particle, *i.e.*, $P^\mu = (M, 0, 0, 0)$, with $M > 0$. It is useful to further simplify the resulting (9) by virtue of the R-symmetry; namely, we diagonalize the central charges Z^{ab} into 2×2 blocks, $Z = \text{diag}(\epsilon Z_1, \dots, \epsilon Z_{\lfloor \mathcal{N}/2 \rfloor}, \mathbb{I})$, where \mathbb{I} appears for \mathcal{N} odd, and $\epsilon^{12} = -\epsilon^{21} = 1$. Consequently, we split the index a into $\hat{a} = 1, 2$ running inside the blocks, and $\check{a} = 1, 2, \dots, \lfloor \mathcal{N}/2 \rfloor$ counting the blocks; we define then $\mathcal{Q}_{\alpha}^{\check{a}\pm} \equiv \frac{1}{2}(Q_{\alpha}^{(1,\hat{a})} \pm (\sigma^0)_{\alpha\beta}(Q_{\beta}^{(2,\hat{a})})^\dagger)$, which yields the only non-zero anti-commutation relations

$$\left\{ \mathcal{Q}_{\alpha}^{\check{a}\pm}, \left(\mathcal{Q}_{\beta}^{\check{b}\pm} \right)^\dagger \right\} = \delta_b^{\check{a}} \delta_{\alpha}^{\beta} (M \pm Z_{\check{a}}). \quad (12)$$

An immediate implication of (12) and the unitarity is that the r.h.s. must not be negative,

$$M \geq |Z_{\check{a}}|, \quad \text{for all} \quad \check{a} = 1, 2, \dots, \lfloor \mathcal{N}/2 \rfloor; \quad (13)$$

this is the so-called “Bogomol’nyi–Prasad–Sommerfeld (BPS) bound.” If for some $\check{a} = 1, 2, \dots, \check{a}_0$ it is saturated, then one of every pair of the corresponding supercharges $\mathcal{Q}_{\alpha}^{\check{a}\pm}$ vanishes, and the SUSY representation is shortened — it is then the $2^{2(\mathcal{N}-\check{a}_0)}$ -dimensional representation of the Clifford algebra associated with the group $SO(4(\mathcal{N}-\check{a}_0))$;

it is called “ $1/2^{\tilde{a}_0}$ BPS.” An extremely important property of BPS states is that their scaling dimensions (36) do not receive quantum corrections, *i.e.*, are not renormalized,

$$\Delta = \Delta_{\text{bare}}, \quad \text{for operators in BPS multiplets;} \quad (14)$$

such operators are termed “protected.”

2. The Field Content, Lagrangian, Observables

After this recollection, let us proceed to the interesting part, the $\mathcal{N} = 4$ SYM theory. If we restrict ourselves to the physically relevant situation of particles carrying spins ≤ 1 , the fields spanning any SUSY representation will be spin-1 vectors, spin-1/2 Weyl fermions, and spin-0 scalars. For $\mathcal{N} = 4$, the only possible multiplet is the “gauge multiplet” consisting of

$$\text{the gluon } A_\mu, \quad \text{the Weyl gluinos } \lambda_\alpha^a, \bar{\lambda}_{\dot{\alpha}}^{\bar{a}}, \quad \text{the real scalars } \Phi_i. \quad (15)$$

Here, $\mu = 0, 1, 2, 3$ denotes a Lorentz vector index; $i = 1, 2, \dots, 6$ an R-symmetry vector; $\alpha, \dot{\alpha} = 1, 2$ a Lorentz spinor; $a, \bar{a} = 1, 2, 3, 4$ an R-symmetry spinor. The Weyl spinors can be encoded as a 16-component 10-dimensional Majorana–Weyl spinor Ψ_κ , $\kappa = 1, 2, \dots, 16$. All of these fields are in the adjoint representation of the gauge group, *i.e.*,

$$X = \sum_{p=1}^{N^2-1} X^p T^p, \quad (16)$$

where X stands for any of the canonical fields, and $(T^p)_{AB}$ are the $N \times N$ Hermitian generators of $\text{SU}(N)$.

The Lagrangian is uniquely determined by the symmetries of the theory, and reads,

$$S = \frac{1}{2g_{\text{YM}}^2} \int d^4x \mathcal{L}(A_\mu, \Psi, \Phi_i), \quad \mathcal{L} = \text{Tr} \left(-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi_i D^\mu \Phi_i - \sum_{i < j} [\Phi_i, \Phi_j]^2 + i \bar{\Psi} \Gamma^\mu D_\mu \Psi - \bar{\Psi} \Gamma_i [\Phi_i, \Psi] \right), \quad (17)$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and $D_\mu \equiv \partial_\mu - i[A_\mu, \cdot]$, and (Γ^μ, Γ_i) are the 10-dimensional 16×16 Dirac matrices. There are two free parameters of the theory, the Yang–Mills coupling constant g_{YM} and the rank of the gauge group N ; the instanton angle θ_I will be of no importance in our considerations, and its contribution to (17) has been disregarded.

The observables of the theory are the correlation functions of gauge-invariant local composite operators [29] which arise in the following way: By “letters,” we mean the components of the gauge multiplet (14) at a given point x , or rather, their gauge-covariant versions, $\{D_\mu \cdot|_x, \Psi(x), \bar{\Psi}(x), \Phi_i(x)\}$. From them, we form “words,” which are their products; they will have definite scaling dimensions Δ (see below). The gauge invariance is imposed by creating “sentences,” *i.e.*, traces of such products (single-trace operators) or products of several such traces (multi-trace operators), for example,

$$\mathcal{O} = \text{Tr}(D_\mu \Phi_i D_\nu \lambda_\alpha^A), \quad \mathcal{O} = \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \text{Tr}(\lambda_\alpha^A \bar{\lambda}_{\dot{\alpha}}^{\bar{A}}), \quad \text{etc.} \quad (18)$$

The correlation functions of composite operators at different spacetime points are defined as usual,

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \dots \mathcal{O}_M(x_M) \rangle = \frac{\delta^M}{\delta J_1(x_1) \delta J_2(x_2) \dots \delta J_M(x_M)} Z, \quad (19)$$

where the partition function is the standard path integral with sources $J_i(x)$,

$$Z[J_1(x), J_2(x), \dots, J_M(x)] = \int [dA][d\lambda][d\bar{\lambda}][d\Phi] e^{i \int d^4x (\mathcal{L} + \sum_{i=1}^M J_i(x) \mathcal{O}_i(x))}. \quad (20)$$

3. The Renormalization Group and Anomalous Dimension

The critical property of the theory is its “conformal invariance,” which we will now address. In order to go there, let us briefly review the renormalization group (RG) method. In computing correlators in a generic QFT, we encounter divergences, which we then subtract systematically at any order in perturbation theory. This can be done in many ways, as with infinity, we can also subtract an arbitrary finite piece, which results in many renormalization schemes. In the process of renormalization, there certainly must appear a mass scale μ , called the “renormalization scale,” which is arbitrary and remains in the finite correlation function. Therefore, any physical quantity will depend on both the renormalization scheme and scale, with its different values connected by finite renormalizations, which can be shown to form a group. Any physical quantity must then be invariant under transformations from this group, which is expressed through the “renormalization group equations” (RGE). Let us sketch how they arise in the $D = 6$ -dimensional ϕ^3 theory, in the minimal subtraction (MS) scheme. The renormalized coupling constant, mass, and M -point truncated connected correlation function in the momentum space ($p = (p_1, p_2, \dots, p_M)$) are related to their bare values through

$$g_{\text{ren.}}(\mu) = Z_g(\mu)^{-1} g_{\text{bare}}, \quad m_{\text{ren.}}(\mu) = Z_m(\mu)^{-1} m_{\text{bare}},$$

$$G_{\text{ren.}}^{\text{trunc.,conn.,}M\text{-point}}(p, g_{\text{ren.}}, m_{\text{ren.}}, \mu) = Z_3(\mu)^{M/2} G_{\text{bare}}^{\text{trunc.,conn.,}M\text{-point}}(p, g_{\text{bare}}, m_{\text{bare}}). \quad (21)$$

The bare objects do not depend on μ . We start from the coupling constant, and use dimensional regularization,

$$g_{\text{bare}} = g_0 \mu_0^\epsilon, \quad g_{\text{ren.}} = g \mu^\epsilon, \quad (22)$$

where $\epsilon = (6 - D)/2$, and μ_0 and μ are the renormalization scales for respectively the bare and the renormalized coupling constants. The two first relations in (21) can now be rewritten with help of (22) as two differential equations,

$$\mu \frac{dg}{d\mu} \Big|_{g_{\text{bare}}, m_{\text{bare}}=\text{fixed}} = \beta, \quad \mu \frac{dm}{d\mu} \Big|_{g_{\text{bare}}, m_{\text{bare}}=\text{fixed}} = -m\gamma_m, \quad (23)$$

where for short $m \equiv m_{\text{ren.}}$, we keep the bare values fixed, and

$$\beta(g, m, \mu) \equiv -\epsilon g - \frac{\mu}{Z_g} \frac{dZ_g}{d\mu} g, \quad \gamma_m(g, m, \mu) \equiv \frac{\mu}{Z_m^{1/2}} \frac{dZ_m^{1/2}}{d\mu}. \quad (24)$$

These “RG functions” are finite, as divergences in Z_g , Z_m cancel out; the first one is called the “Gell-Mann-Low beta-function”; in the MS scheme, they can be shown to depend only on g , which conveniently decouples both RGEs (23). The RGE for the correlation function (the “’t Hooft–Weinberg equation”) can be derived from the condition that its unrenormalized value does not depend on μ , which yields

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} - \gamma_m m \frac{\partial}{\partial m} - M\gamma \right) G_{\text{ren.}}^{\text{trunc.,conn.,}M\text{-point}} = 0, \quad (25)$$

where the RG function corresponding to the renormalization of the correlation function is called the “anomalous dimension,” whose name we will soon explain,

$$\gamma(g, m, \mu) \equiv \frac{\mu}{2Z_3} \frac{dZ_3}{d\mu} \Big|_{g_{\text{bare}}, m_{\text{bare}}=\text{fixed}}. \quad (26)$$

Again, in the MS scheme, it depends only on g . These equations (23), (25) guarantee that the renormalized theory will lead to physical predictions independent of the arbitrary renormalization scale μ .

Let us show the solution to the ’t Hooft–Weinberg equation (25). To find it, we introduce a dimensionless scale Λ for the momentum by replacing $p \rightarrow \Lambda p$. Moreover, we notice, by naively counting mass dimensions, that

$$G_{\text{ren.}}^{\text{trunc.,conn.,}M\text{-point}}(\Lambda p, g, m, \mu) = \mu^{6-M\Delta_{\text{bare}}} \Gamma\left(\frac{\Lambda p}{\mu}, g, \frac{m}{\mu}\right), \quad (27)$$

where $\Delta_{\text{bare}} = (D - 2)/2 = 2$ is the bare (classical, tree-level) mass dimension of the field $\phi(x)$ in the ϕ_6^3 theory, and the object Γ is dimensionless. Further, we define the “running coupling constant” \bar{g} and the “running mass” \bar{m} as

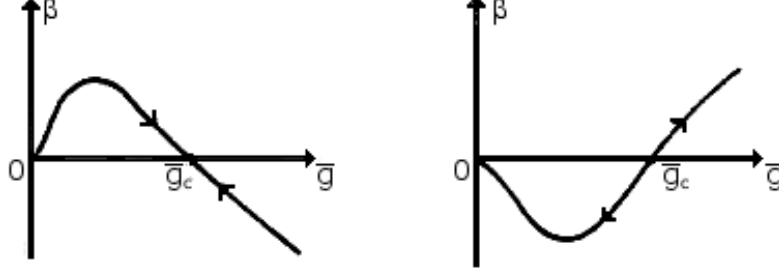


FIG. 3: The two generic scenarios for the dependence of the beta-function β on the running coupling constant $\bar{g}(t)$. The arrows represent the direction in which $\bar{g}(t)$ moves as t grows to infinity, *i.e.*, approaches the UV limit. The value \bar{g}_c at which $\beta = 0$ is called, respectively, the UV or IR fixed point.

these renormalized quantities at the scale μ/Λ and not μ , which according to (23) and in the variable $t \equiv -\log(\Lambda)$, obey the RGEs,

$$\frac{d\bar{g}}{dt} = \beta(\bar{g}), \quad \frac{1}{\bar{m}} \frac{d\bar{m}}{dt} = -1 - \gamma_m(\bar{g}). \quad (28)$$

In these new variables, and with aid of (27), the 't Hooft–Weinberg equation becomes

$$\left(\frac{d}{dt} + \omega_M(\bar{g}) \right) G_{\text{ren.}}^{\text{trunc.,conn.,}M\text{-point}}(e^{-t}p, \bar{g}, \bar{m}, \mu) = 0, \quad \text{where} \quad \omega_M(\bar{g}) \equiv 6 - M\Delta_{\text{bare}} - M\gamma(\bar{g}). \quad (29)$$

Equations (28) and (29) in the integrated form read

$$t = \int_{\bar{g}}^{\bar{g}(t)} \frac{d\bar{g}'}{\beta(\bar{g}')}, \quad \bar{m}(t) = m \exp \left(-t - \int_0^t dt' \gamma_m(\bar{g}(t')) \right), \quad (30)$$

$$G_{\text{ren.}}^{\text{trunc.,conn.,}M\text{-point}}(e^t p, g, m, \mu) = G_{\text{ren.}}^{\text{trunc.,conn.,}M\text{-point}}(p, \bar{g}(t), \bar{m}(t), \mu) \exp \left((6 - M\Delta_{\text{bare}})t - M \int_0^t dt' \gamma(\bar{g}(t')) \right). \quad (31)$$

Let us first discuss the running of the coupling constant, which is governed by the beta-function. Generically, there are two possible scenarios, see figure 3: there must be $\beta(0) = 0$, but there can also be an additional zero,

$$\beta(\bar{g}_c) = 0. \quad (32)$$

Then, for $t \rightarrow \pm\infty$ (which is respectively called the “UV” or “IR limit”), respectively for the left or right plot in figure 3, $\bar{g}(t) \rightarrow \bar{g}_c$, which is called the “UV/IR fixed point.” How does this affect the dependence on the momentum scale Λ of the correlation functions? For simplicity, assume that there is no mass in the theory, $m = 0$. First, remark that if there were no divergences in the theory, *i.e.*, no renormalization were necessary, then all the RG functions would be zero, and the Feynman amplitudes would display the behavior implied by the naive mass–dimension counting,

$$G_{\text{ren.}}^{\text{trunc.,conn.,}M\text{-point}}(e^t p, g, m, \mu) = (e^t)^{6-M\Delta_{\text{bare}}} G_{\text{ren.}}^{\text{trunc.,conn.,}M\text{-point}}(p, g, m e^{-t}, \mu) \quad (\text{a finite theory}). \quad (33)$$

If divergences are present, this is modified by the running of the parameters. Let us see what happens if we consider the theory at a fixed point. Then, $\int_0^t dt' \gamma(\bar{g}(t')) \rightarrow \gamma(\bar{g}_c)t$ as $t \rightarrow \pm\infty$, and (31) yields

$$G_{\text{ren.}}^{\text{trunc.,conn.,}M\text{-point}}(e^t p, g, 0, \mu) = (e^t)^{6-M\Delta_{\text{bare}}-M\gamma(\bar{g}_c)} G_{\text{ren.}}^{\text{trunc.,conn.,}M\text{-point}}(p, \bar{g}_c, 0, \mu) \quad (\text{at a fixed point}), \quad (34)$$

hence, the true mass dimension is modified from the bare one Δ_{bare} by the quantity $\gamma(\bar{g}_c)$; this justifies its name, “anomalous dimension.” (We have assumed here $\bar{g}_c > 0$. If it is zero, then $\gamma(0) = 0$ does not modify the bare mass dimension. If moreover it is an UV fixed point, the theory is called “asymptotically free,” such as QCD.) A crucial insight is that a theory at a RG fixed point is scale-invariant, *i.e.*, invariant w.r.t. “dilatations” of the spacetime,

$$\mathcal{D} : x_\mu \rightarrow \Lambda x_\mu, \quad (35)$$

because $\beta = 0$ implies that the theory does not depend on the RG scale μ . Accordingly, a canonical field operator $\phi(x)$ transforms as

$$\phi(x) \rightarrow \Lambda^\Delta \phi(\Lambda x), \quad (36)$$

where Δ is called the field’s “scaling dimension.” If we are at the classical level, or if the theory is free, this is identical to the bare scaling dimension, obtained by a naive counting. But if we have an interacting field at the quantum level, renormalization effects will change the bare scaling dimension by the anomalous part,

$$\Delta = \Delta_{\text{bare}} + \Delta_{\text{an}}, \quad (37)$$

where $\Delta_{\text{an}} \equiv \gamma(\bar{g}_c)$. As mentioned before (14), an extremely important exception in supersymmetric theories is constituted by operators in shortened BPS multiplets, which do not receive the anomalous correction from renormalization effects.

For $\mathcal{N} = 4$ SYM, the situation is even stronger: Contrary to a non-supersymmetric Yang–Mills, which is scale-invariant classically, but at the quantum level the anomaly breaks the invariance under dilatations, in $\mathcal{N} = 4$ SYM, there are no UV divergences in the correlation functions of the canonical fields $\{A_\mu, \Psi, \Phi_i\}$, hence, the RG beta-function vanishes to all orders in perturbation theory [30, 31],

$$\beta(g_{\text{YM}}) = 0 \quad (\mathcal{N} = 4 \text{ SYM}), \quad (38)$$

which implies that the scale invariance is retained even quantum-mechanically, *i.e.*, that the coupling g_{YM} does not run with the RG scale μ , and that all the fields are massless. This is a consequence of a large number of SUSYs. We will now see even more — that this is then a UV-finite conformal field theory (CFT), with the superconformal group $\text{SU}(2, 2|4)$ (48) as a symmetry even at the quantum level. Remark also that even though the canonical fields do not undergo renormalization, the composite “sentences” (18) in general will; this is related to UV divergences associated with dealing with products of operators at the same spacetime point (see the discussion around (169)). Finally, as is clear from (34), $\beta = 0$ does not imply the absence of anomalous dimensions.

4. The Superconformal Group

QFTs which possess scale invariance are usually also symmetric w.r.t. a larger “conformal group,” *i.e.*, transformations that modify the metric only up to a local scale factor,

$$dx_\mu dx^\mu \rightarrow \Omega(x)^2 dx_\mu dx^\mu. \quad (39)$$

We may understand it as follows: Scale invariance implies, according to the Noether’s theorem, that the “dilatation current” $j_{(\mathfrak{D})}^\mu = T^{\mu\nu} x_\nu$ is conserved, where $T^{\mu\nu}$ is the energy–momentum tensor; this is equivalent to its tracelessness, $T^\mu_\mu = 0$. If now we consider an arbitrary current $j_{(\xi)}^\mu = T^{\mu\nu} \xi_\nu$, the tracelessness can be checked to lead to the following statement: this current will be conserved provided ξ_ν satisfies the “conformal Killing equation,”

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{1}{2} \partial_\rho \xi^\rho \eta_{\mu\nu}. \quad (40)$$

In a flat Minkowski spacetime with $D > 2$ dimensions, the most general solution, called the “conformal Killing vector,” is four-fold,

$$\begin{aligned} \xi^\mu &= a^\mu & (\text{translations}), & & \xi^\mu &= \omega^{\mu\nu} x_\nu & (\text{Lorentz}), \\ \xi^\mu &= \Lambda x^\mu & (\text{dilatations}), & & \xi^\mu &= 2(bx)x^\mu - x^2 b^\mu & (\text{special conformal}). \end{aligned} \quad (41)$$

It is determined by $(D+1)(D+2)/2$ parameters Λ , a^μ , b^μ , $\omega^{\mu\nu} = -\omega^{\nu\mu}$. (For $D = 2$, there are infinitely many parameters.) These four sorts of transformations form together a group. It is generated by the D generators of spacetime translations P_μ , and $D(D-1)/2$ generators of special Lorentz transformations $L_{\mu\nu}$, which comprise the Poincaré group, and moreover by the one generator of dilatations \mathfrak{D} , and D generators of the “special conformal transformations” K_μ . These generators obey the commutation relations

$$i[L_{\mu\nu}, L_{\rho\sigma}] = \eta_{\mu\rho} L_{\nu\sigma} + \eta_{\nu\sigma} L_{\mu\rho} - \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\sigma} L_{\nu\rho},$$

$$i[L_{\mu\nu}, P_\rho] = \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu, \quad i[L_{\mu\nu}, K_\rho] = \eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu, \quad i[L_{\mu\nu}, \mathfrak{D}] = 0, \quad (42)$$

$$i[P_\mu, K_\nu] = 2(\eta_{\mu\nu}\mathfrak{D} - L_{\mu\nu}), \quad i[P_\mu, \mathfrak{D}] = -P_\mu, \quad i[K_\mu, \mathfrak{D}] = K_\mu,$$

which can be concisely rewritten as

$$i[M_{mn}, M_{m'n'}] = \eta_{mm'}M_{nn'} + \eta_{nn'}M_{mm'} - \eta_{mn'}M_{nm'} - \eta_{nm'}M_{mn'}, \quad (43)$$

where the indices run over $0, 1, \dots, D+1$, the extended metric is $\eta_{mn} \equiv \text{diag}(-, +, +, \dots, +, -)$, and where we define the anti-symmetric symbols

$$M_{mn} \equiv \begin{pmatrix} L_{\mu\nu} & \frac{1}{2}(K_\mu - P_\mu) & \frac{1}{2}(K_\mu + P_\mu) \\ -\frac{1}{2}(K_\mu - P_\mu) & 0 & \mathfrak{D} \\ -\frac{1}{2}(K_\mu + P_\mu) & -\mathfrak{D} & 0 \end{pmatrix}. \quad (44)$$

These are precisely the structure relations of the group $\text{SO}(2, D)$, which in our case of $D = 4$ can also be written as the covering $\text{SU}(2, 2)$.

Let us moreover mention that $\mathcal{N} = 4$ SYM is conjectured to display a certain global discrete symmetry. This theory is invariant under the shift in the instanton angle, $\theta_I \rightarrow \theta_I + 2\pi$, but is supposed to submit to an even wider symmetry. Namely, forming the complex coupling constant,

$$\tau_{\text{YM}} \equiv \frac{\theta_I}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2}, \quad (45)$$

in which language the above shift is realized as $\tau_{\text{YM}} \rightarrow \tau_{\text{YM}} + 1$ — the “Montonen–Olive S–duality conjecture” claims that also $\tau_{\text{YM}} \rightarrow -1/\tau_{\text{YM}}$ is a quantum symmetry. The full invariance group then reads

$$\text{SL}(2, \mathbb{Z}) : \tau_{\text{YM}} \rightarrow \frac{a\tau_{\text{YM}} + b}{c\tau_{\text{YM}} + d}, \quad \text{where} \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \quad (\text{S–duality}). \quad (46)$$

In addition to the above conformal symmetry, and also obviously the R–symmetry $\text{SO}(6) \sim \text{SU}(4)$ — the theory is invariant under the $\mathcal{N} = 4$ Poincaré SUSY, generated by the 16 supercharges $Q_\alpha^a, \bar{Q}_{\dot{\alpha}a}$. Not only this, but it is noticed that these supercharges and the special conformal generators K_μ do not commute; both being symmetries, their commutator must be one, too, which yields 16 “conformal supercharges” $S_{\alpha a}, \bar{S}_{\dot{\alpha}}^a$. The non-zero (anti-)commutators read

$$\begin{aligned} i[L_{\mu\nu}, Q_\alpha^a] &= (\sigma_{\mu\nu})_{\alpha\beta} \epsilon^{\beta\gamma} Q_\gamma^a, & i[L_{\mu\nu}, \bar{Q}_{\dot{\alpha}a}] &= (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} \epsilon^{\dot{\beta}\dot{\gamma}} \bar{Q}_{\dot{\gamma}a}, \\ i[L_{\mu\nu}, S_{\alpha a}] &= (\sigma_{\mu\nu})_{\alpha\beta} \epsilon^{\beta\gamma} S_{\gamma a}, & i[L_{\mu\nu}, \bar{S}_{\dot{\alpha}}^a] &= (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} \epsilon^{\dot{\beta}\dot{\gamma}} \bar{S}_{\dot{\gamma}}^a, \\ i[P_\mu, S_{\alpha a}] &= i(\sigma_\mu)_{\alpha\dot{\beta}} \epsilon^{\dot{\beta}\dot{\gamma}} \bar{Q}_{\dot{\gamma}a}, & i[P_\mu, \bar{S}_{\dot{\alpha}}^a] &= i(\bar{\sigma}_\mu)_{\dot{\beta}\alpha} \epsilon^{\beta\gamma} Q_\gamma^a, \\ i[K_\mu, Q_\alpha^a] &= i(\sigma_\mu)_{\alpha\dot{\beta}} \epsilon^{\dot{\beta}\dot{\gamma}} \bar{S}_{\dot{\gamma}}^a, & i[K_\mu, \bar{Q}_{\dot{\alpha}a}] &= i(\bar{\sigma}_\mu)_{\dot{\beta}\alpha} \epsilon^{\beta\gamma} S_{\gamma a}, \\ i[\mathfrak{D}, Q_\alpha^a] &= \frac{1}{2} Q_\alpha^a, & i[\mathfrak{D}, \bar{Q}_{\dot{\alpha}a}] &= \frac{1}{2} \bar{Q}_{\dot{\alpha}a}, \\ i[\mathfrak{D}, S_{\alpha a}] &= -\frac{1}{2} S_{\alpha a}, & i[\mathfrak{D}, \bar{S}_{\dot{\alpha}}^a] &= -\frac{1}{2} \bar{S}_{\dot{\alpha}}^a, \\ \{Q_\alpha^a, \bar{Q}_{\dot{\beta}b}\} &= (\sigma_\mu)_{\alpha\dot{\beta}} \delta_b^a P_\mu, & \{S_{\alpha a}, \bar{S}_{\dot{\beta}}^b\} &= (\sigma_\mu)_{\alpha\dot{\beta}} \delta_a^b K_\mu. \end{aligned} \quad (47)$$

Together with the bosonic counterpart, therefore, the full global continuous symmetry group is

$$\text{SU}(2, 2|4). \quad (48)$$

5. Conformal and Superconformal Primary Operators

Having described the symmetries of $\mathcal{N} = 4$ SYM, let us move to the structure of “conformal” and “superconformal primary operators” present in any CFT. First, notice that representations of the conformal group are spanned by the

eigenstates of the dilatation operator \mathfrak{D} with an eigenvalue $-i\Delta$. From (42) and (47), we observe that the generators K_μ and $S_{\alpha a}$ lower the scaling dimension by 1 and 1/2, respectively, while on the other hand, P_μ and $Q_{\alpha a}$ raise it by 1 and 1/2. They are all symmetry generators, hence, they do not lead out of the states of the theory. Thus, they can be treated as annihilation and creation operators, respectively, to build a representation. Since in a unitary CFT, negative scaling dimensions are forbidden, applying either of K_μ and $S_{\alpha a}$ to a given operator of a definite dimension must at some point yield 0. Therefore, we single out the last operator in such a series, *i.e.*, the operator of the lowest dimension in a given conformal/superconformal multiplet,

$$[K, \mathcal{O}]_\pm = 0 \quad (\text{conformal primary}), \quad [S, \mathcal{O}]_\pm = 0 \quad (\text{superconformal primary}), \quad \text{where} \quad \mathcal{O} \neq 0. \quad (49)$$

The latter is always also the former, but not conversely. The other operators in a multiplet are termed “conformal/superconformal descendants,” *i.e.*, \mathcal{O} is a descendant of \mathcal{O}' if

$$\mathcal{O} = [P, \mathcal{O}']_\pm \quad (\text{conformal descendant}), \quad \mathcal{O} = [Q, \mathcal{O}']_\pm \quad (\text{superconformal descendant}). \quad (50)$$

It can be proven that in $\mathcal{N} = 4$ SYM, the superconformal primaries are only the gauge-invariant operators built from the scalars Φ_i in such a way that the R-symmetry indices i are symmetrized, for example,

$$\sum_{i=1}^6 \text{Tr}(\Phi_i \Phi_i) \quad (\text{the Konishi operator}). \quad (51)$$

The conformal primaries and their descendants are very important also because they are the operators with definite scaling dimensions. Their 2-point correlation functions are constrained by conformal invariance to be diagonal and of the form

$$\langle \mathcal{O}_\alpha(x) \mathcal{O}_\beta(y) \rangle = \frac{\delta_{\alpha\beta}}{(x-y)^{2\Delta(\mathcal{O}_\alpha)}}. \quad (52)$$

More generally, operators of the theory transforming under unitary irreducible representations of the full symmetry group $\text{SU}(2, 2|4)$ (48) are labeled by the eigenvalues of the Cartan generators of the bosonic subgroup,

$$\left(\underbrace{\Delta}_{\text{SO}(1,1)}, \underbrace{S_1, S_2}_{\text{SO}(1,3)}, \underbrace{J_1, J_2, J_3}_{\text{SO}(6)} \right), \quad (53)$$

where Δ is the scaling dimension, S_1, S_2 the conformal spins, and J_1, J_2, J_3 the R-charges.

Finally, in order to practically proceed with anomalous dimensions, one often resorts to perturbation theory,

$$\Delta_{\text{an}} = \sum_{l \geq 1} \lambda^l \sum_{g \geq 0} N^{-2g} \Delta_{l,g}, \quad (54)$$

compare (7). This non-trivial procedure is avoided for operators in the BPS multiplets, introduced above, whose scaling dimensions are protected against renormalization (14), such as the half-BPS (or “chiral”) operators, which for a single trace can be argued to acquire the form

$$\mathcal{O}_{(J)}^{\text{half-BPS}}(x) = \psi_{i_1 i_2 \dots i_J} \text{Tr}(\Phi_{i_1}(x) \Phi_{i_2}(x) \dots \Phi_{i_J}(x)), \quad (55)$$

where $\psi_{i_1 i_2 \dots i_J}$ is a rank- J symmetric traceless tensor. For them,

$$\Delta(\mathcal{O}_{(J)}^{\text{half-BPS}}) = J. \quad (56)$$

C. The Type IIB Superstring Theory in the $\text{AdS}_5 \times S^5$ Background

Let us now proceed to the opposite “string side” of the AdS/CFT correspondence, which is the announced type IIB superstring theory in the curved target spacetime $\text{AdS}_5 \times S^5$.

1. Coordinate Charts in Anti-de Sitter Space

Let us start from succinctly describing the geometry of the $\text{AdS}_5 \times S^5$ manifold. The (Minkowskian) d -dimensional “anti-de Sitter space” [32] (of radius L), AdS_d , together with the “de Sitter space” dS_d , are the closest cousins of the usual flat spacetime. They all are solutions to the vacuum Einstein equations with the cosmological constant Λ ,

$$R_{mn} - \frac{1}{2}Rg_{mn} = \frac{1}{2}\Lambda g_{mn}. \quad (57)$$

Since the Ricci tensor is proportional to the metric, $R_{mn} = \frac{\Lambda}{2-d}g_{mn}$, they are Einstein spaces. Furthermore, they are all maximally symmetric, *i.e.*, admit a kinematical symmetry with $d(d+1)/2$ generators, just as the Poincaré group, which can be stated as $R_{mnm'n'} = \frac{1}{d(d-1)}R(g_{mm'}g_{nn'} - g_{mn'}g_{nm'})$. They have also a constant curvature, respectively, negative, positive and zero; the Ricci scalar reads $R = \frac{d\Lambda}{2-d}$, and the radius is related to it as $L^2 = (d-1)(d-2)/|\Lambda|$.

It is easiest to define AdS_d in the (Cartesian) “embedding coordinates” (y^0, y^1, \dots, y^d) of $\mathbb{R}^{2,d-1}$ with the metric $\text{diag}(-, +, +, \dots, +, -)$ as the connected hyperboloid with the isometry $\text{SO}(2, d-1)$ given through the equation

$$-(y^0)^2 + \sum_{i=1}^{d-1} (y^i)^2 - (y^d)^2 = -L^2. \quad (58)$$

For the d -dimensional sphere (having isometry $\text{SO}(d+1)$) of the same radius L , the equation obeyed by $(x^1, x^2, \dots, x^{d+1})$ of \mathbb{R}^{d+1} reads

$$\sum_{i=1}^{d+1} (x^i)^2 = +L^2. \quad (59)$$

Note: Since $\mathbb{R}^{2,d-1}$ has two temporal dimensions, AdS_d will include closed time-like curves; they can be eliminated by passing to the “universal covering space,” denoted by CAdS_d , which is commonly understood as the proper definition of AdS_d ; this is done by extending the AdS time coordinate $t \in [-\pi, \pi]$ (see below (60)) to $t \in \mathbb{R}$.

These constrained embedding coordinates are conveniently traded for the “global coordinates,” which, as the name indicates, parameterize the entire manifold; for AdS_d , they read

$$y^0 = L \cosh(\rho) \cos(t), \quad y^i = L \sinh(\rho) \Omega^i, \quad y^d = L \cosh(\rho) \sin(t), \quad (60)$$

where $\sum_{i=1}^{d-1} (\Omega^i)^2 = 1$, and their range is $\rho \geq 0$ and $t \in [-\pi, \pi]$, the latter then extended to $t \in \mathbb{R}$. Let us, for further reference, explicitly print this change of variables for $d = 5$, and for both the anti-de Sitter and the sphere,

$$\eta_1 \equiv y^1 + iy^2 = L \sinh(\rho) \cosh(\bar{\psi}) e^{i\varphi_1}, \quad \eta_2 \equiv y^3 + iy^4 = L \sinh(\rho) \sinh(\bar{\psi}) e^{i\varphi_2}, \quad \eta_0 \equiv y^0 + iy^5 = L \cosh(\rho) e^{it}, \quad (61)$$

$$\xi_1 \equiv x^1 + ix^2 = L \sin(\gamma) \cos(\psi) e^{i\phi_1}, \quad \xi_2 \equiv x^3 + ix^4 = L \sin(\gamma) \sin(\psi) e^{i\phi_2}, \quad \xi_3 \equiv x^5 + ix^6 = L \cos(\gamma) e^{i\phi_3}, \quad (62)$$

where

$$\rho \geq 0, \quad t \in \mathbb{R}, \quad \bar{\psi} \in [0, \pi], \quad \varphi_1 \in [0, \pi], \quad \varphi_2 \in [0, 2\pi], \quad (63)$$

$$\gamma \in [0, \pi], \quad \phi_3 \in [0, \pi], \quad \psi \in [0, \pi], \quad \phi_1 \in [0, \pi], \quad \phi_2 \in [0, 2\pi]. \quad (64)$$

The metric is induced from the metric of the ambient space, is non-degenerate and with a Lorentzian/Euclidean signature, and in the global variables acquires the form

$$ds_{\text{AdS}_5}^2 = L^2 \left(d\rho^2 - \cosh^2(\rho) dt^2 + \sinh^2(\rho) \underbrace{(d\bar{\psi}^2 + \cos^2(\bar{\psi}) d\varphi_1^2 + \sin^2(\bar{\psi}) d\varphi_2^2)}_{d\Omega_3^2} \right), \quad (65)$$

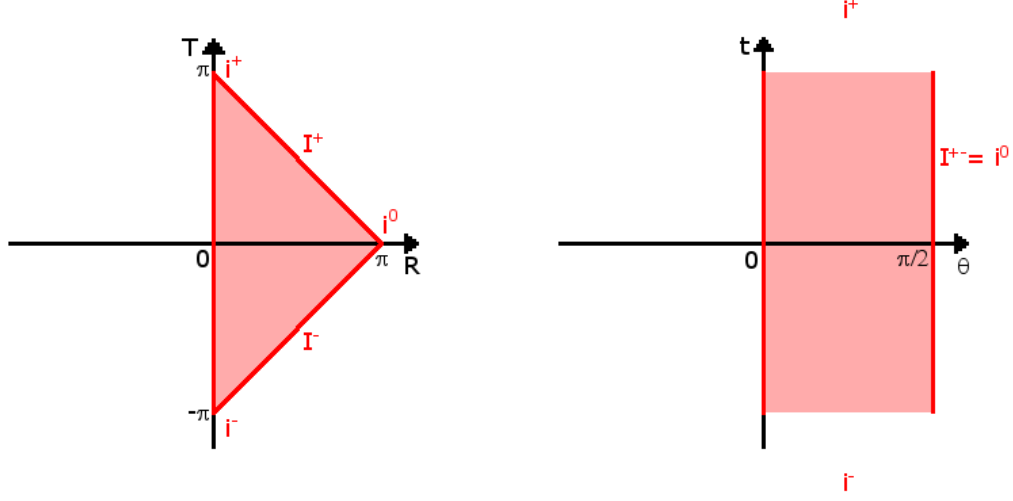


FIG. 4: LEFT: The Penrose diagram for the flat Minkowski spacetime in $d > 2$ dimensions. i^\pm at $(R, T) = (0, \pm\pi)$ represent the future/past time-like infinities, *i.e.*, where future/past-directed time-like geodesics end; i^0 at $(R, T) = (\pi, 0)$ is the space-like infinity. The diagram is the triangle bound by the vertical line $R = 0$ corresponding to $r = 0$, and by the future/past null infinities \mathcal{I}^\pm at $R \pm T = \pi$.

RIGHT: The Penrose diagram for CAdS_d ($d > 2$). It is the infinite strip bound by the vertical lines $\theta = 0$, corresponding to $\rho = 0$, and $\theta = \pi/2$, corresponding to the null and space-like infinities.

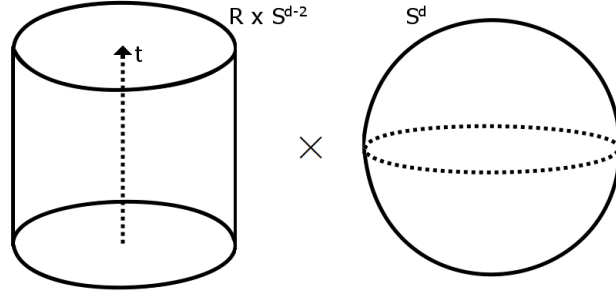


FIG. 5: A sketch of the global $\text{AdS}_d \times S^d$.

$$ds_{S^5}^2 = L^2 \left(d\gamma^2 + \cos^2(\gamma) d\phi_3^2 + \sin^2(\gamma) \underbrace{(d\psi^2 + \cos^2(\psi) d\phi_1^2 + \sin^2(\psi) d\phi_2^2)}_{d\Omega_3^2} \right), \quad (66)$$

while for a general number of dimensions d , the only difference is in the unit sphere metric, $d\Omega_{d-2}^2$ instead of $d\Omega_3^2$.

Various other useful sets of coordinates on AdS_d are possible, too. First, we describe the chart needed to draw the Penrose diagram of this spacetime, *i.e.*, a representation which properly captures the spacetime's topological and causal structure, and moreover, has a metric which is locally conformally equivalent to the spacetime's metric, but all the infinities are brought to be within finite distances. For example, for the standard flat Minkowski spacetime ($d > 2$), with $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2$, one first transforms the coordinates to the light-cone ones, $u_\pm \equiv t \pm r$, then one makes them finite by $\tilde{u}_\pm \equiv \arctan(u_\pm)$, to finally return to time- and space-like variables by $T \equiv \tilde{u}_+ + \tilde{u}_-$, $R \equiv \tilde{u}_+ - \tilde{u}_-$, with $T \in [-\pi, \pi]$, $R \in [0, \pi]$. The resulting metric is conformally equivalent to $ds_{\text{Einstein static}}^2 = -dT^2 + dR^2 + \sin^2(R) d\Omega_{d-2}^2$, which describes $\mathbb{R} \times S^{d-1}$, the “Einstein static universe.” The Penrose diagram is obtained in the chart (R, T) , and is a triangle, see figure 4, left. This procedure of “conformal compactification” can be repeated for AdS_d ($d > 2$). Here, it is enough to trade $\rho \geq 0$ for $\theta \in [0, \pi/2]$ through $\tan(\theta) \equiv \sinh(\rho)$ to get

$$ds_{\text{AdS}_d}^2 = \frac{L^2}{\cos^2(\theta)} (d\theta^2 - dt^2 + \sin^2(\theta) d\Omega_{d-2}^2). \quad (67)$$

This means that the pertinent Penrose diagram is an infinite strip, see figure 4, right, rotated additionally around the t axis. (For $d = 2$, one has $\theta \in [-\pi/2, \pi/2]$ instead, and the strip is twice broader.)

Yet another important frame is provided by the so-called ‘‘Poincaré coordinates.’’ We first define the light-cone variables, $u_{\pm} \equiv (y^0 \pm y^{d-1})/L^2$, and then the space-like $\tilde{y}^i \equiv y^i/(Lu_-)$, $i = 1, 2, \dots, d-2$, and the time-like $\tau \equiv y^d/(Lu_+)$; this yields

$$ds_{\text{AdS}_d}^2 = L^2 \left(\frac{1}{u_-^2} du_-^2 + u_-^2 \left(-d\tau^2 + \sum_{i=1}^{d-2} d(\tilde{y}^i)^2 \right) \right). \quad (68)$$

But notice that this is singular at $u_- = 0$, and therefore, our space is divided in two by the hyperplane $y^0 = y^{d-1}$; one chart of the Poincaré coordinates, with $u_- > 0$, covers one half of AdS_d , while the chart with $u_- < 0$ the other half. (The remaining variables run over the entire \mathbb{R} .) By the transformation $z \equiv 1/u_-$, (68) becomes

$$ds_{\text{AdS}_d}^2 = \frac{L^2}{z^2} \left(dz^2 - d\tau^2 + \sum_{i=1}^{d-2} d(\tilde{y}^i)^2 \right), \quad (69)$$

where we choose the chart with $z > 0$ (called the ‘‘Poincaré patch (wedge)’’), which conformally is a half of a flat Minkowski spacetime (and thus, in particular, its Penrose diagram in appropriate coordinates will be a triangle).

Without going into details, let us state that AdS_d in the Poincaré patch has a ‘‘conformal boundary,’’ in the sense of the boundary of its Penrose diagram, and it lies at $z = 0$ ($u_- \rightarrow \infty$). This boundary is the conformal compactification of the $(d-1)$ -dimensional Minkowski spacetime, and the isometry $\text{SO}(2, d-1)$ acts on it as the conformal group. For the global AdS_d , the conformal boundary is $\mathbb{R} \times S^{d-2}$, see figure 5.

2. The Metsaev–Tseytlin Action

It is not known how to construct type IIB superstring theory in this background in the Ramond–Neveu–Schwarz formulation; see however [33]. Its action in the Green–Schwarz formulation was proposed by Metsaev and Tseytlin in 1998 [34], see also [35]: it is described by a non-linear sigma-model with the isometry $\text{SO}(2, 4) \times \text{SO}(6)$, whose bosonic part (the fermionic one will be irrelevant for our discussion) reads,

$$S_{\text{IIB strings in AdS}_5 \times S^5}|_{\text{bosonic}} = \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{+\infty} d\tau \int_0^{2\pi} d\sigma (\mathcal{L}_{\text{AdS}_5} + \mathcal{L}_{S^5}), \quad (70)$$

where

$$\mathcal{L}_{\text{AdS}_5} = -\frac{1}{2} \eta^{\alpha\beta} \eta^{mn} \partial_{\alpha} Y_m \partial_{\beta} Y_n + \frac{1}{2} \tilde{\Lambda} (\eta^{mn} Y_m Y_n + 1), \quad (71)$$

$$\mathcal{L}_{S^5} = -\frac{1}{2} \eta^{\alpha\beta} \eta^{ij} \partial_{\alpha} X_i \partial_{\beta} X_j + \frac{1}{2} \Lambda (\delta^{ij} X_i X_j - 1), \quad (72)$$

where the target space metrics $\eta^{mn} = \text{diag}(-, +, +, +, +, -)$ and $\eta^{ij} = \text{diag}(+, +, +, +, +, +)$, respectively, while the world-sheet metric $\eta^{\alpha\beta} = \text{diag}(-, +)$. We remark the Lagrange multipliers designed to impose the sigma model constraints. Furthermore, the closed-string periodicity $\sigma \sim \sigma + 2\pi$ and the Virasoro constraints

$$\frac{\delta}{\delta \eta^{\alpha\beta}} S|_{\text{bosonic}} = 0, \quad i.e., \quad \begin{cases} \eta^{mn} (\partial_{\tau} Y_m \partial_{\tau} Y_n + \partial_{\sigma} Y_m \partial_{\sigma} Y_n) + \partial_{\tau} X_i \partial_{\tau} X_i + \partial_{\sigma} X_i \partial_{\sigma} X_i = 0 \\ \eta^{mn} \partial_{\tau} Y_m \partial_{\sigma} Y_n + \partial_{\tau} X_i \partial_{\sigma} X_i = 0 \end{cases} \quad (73)$$

must be required. We have already made use in (70) of the string tension value implied by the AdS/CFT correspondence, $T = L^2/(2\pi\alpha') = \sqrt{\lambda}/(2\pi)$ (116), where λ is the 't Hooft coupling constant (5). An additional complication is the presence, among other background fields, of the R–R self-dual field strength \tilde{F}_5 (95).

The global isometry of the product manifold (for $d = 5$), $\text{SO}(2, 4) \times \text{SO}(6)$, is realized as shifts in the global coordinates (61), (62). Crucially, this symmetry is precisely equal to the bosonic subgroup of the invariance group of $\mathcal{N} = 4$ SYM (48). Moreover, if one takes into account also the SUSY present in the setup, one finds that type IIB superstrings in our background display invariance w.r.t. the Lie supergroup $\text{SU}(2, 2|4)$, which exactly matches the

$\mathcal{N} = 4$ SYM counterpart; this is the first clue of a relationship between the two theories. The Noether global charges follow easily from the action (70),

$$S_{mn} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (Y_m \partial_\tau Y_n - Y_n \partial_\tau Y_m), \quad J_{ij} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (X_i \partial_\tau X_j - X_j \partial_\tau X_i). \quad (74)$$

In the AdS/CFT setting, on the string side, the first set describes the isometry of AdS_5 , while the second one of S^5 ; on the gauge side, they correspond to the conformal symmetry and the R-symmetry, respectively. Comparing the charges S_{mn} with the generators M_{mn} (44) of the conformal symmetry of $\mathcal{N} = 4$ SYM, we find that they should be related as

$$S_{\mu\nu} \leftrightarrow L_{\mu\nu}, \quad S_{\mu 4} \leftrightarrow \frac{1}{2} (K_\mu - P_\mu), \quad S_{\mu 5} \leftrightarrow \frac{1}{2} (K_\mu + P_\mu), \quad S_{45} \leftrightarrow \mathfrak{D}. \quad (75)$$

Out of (74), there can be selected 6 Cartan charges,

$$E \equiv S_{50}, \quad S_1 \equiv S_{12}, \quad S_2 \equiv S_{34}, \quad J_1 \equiv J_{12}, \quad J_2 \equiv J_{34}, \quad J_3 \equiv J_{56}, \quad (76)$$

with the following interpretation: E corresponds to shifts in the variable t (the anti-de Sitter time), $S_{1,2}$ in $\varphi_{1,2}$, and $J_{1,2,3}$ in $\phi_{1,2,3}$. Via AdS/CFT, they are mapped to the 6 Cartan charges of the bosonic symmetry of $\mathcal{N} = 4$ SYM (53). Notice in particular that

$$E \leftrightarrow -\frac{1}{2} (K_0 + P_0). \quad (77)$$

Due to the reasons explained in paragraph ID 8, AdS/CFT relates a string theory in the entire space $\text{AdS}_5 \times S^5$ to a gauge theory in its conformal boundary, $\mathbb{R} \times S^3$ (see figures 4, 5). However, while deriving the Maldacena's duality, a relationship is found between a string theory in a part of $\text{AdS}_5 \times S^5$, namely, its Poincaré patch, and a gauge theory living in its conformal boundary, \mathbb{R}^4 . A certain conformal transformation is needed to map the two, and this operation changes the operator $-\frac{1}{2}(K_0 + P_0)$ into the dilatation operator \mathfrak{D} , so that string energies E correspond to scaling dimensions Δ (126).

Let us finally mention that rescaled versions of (76) will be useful,

$$\mathcal{E} \equiv \frac{E}{\sqrt{\lambda}}, \quad \mathcal{S}_{1,2} \equiv \frac{S_{1,2}}{\sqrt{\lambda}}, \quad \mathcal{J}_{1,2,3} \equiv \frac{J_{1,2,3}}{\sqrt{\lambda}}. \quad (78)$$

They will stay finite in the BMN limit (139).

D. The Maldacena's Conjecture

1. The AdS/CFT Correspondence

The first concrete realization of the 't Hooft's idea of a duality between gauge and string theories was discovered by Maldacena in 1997 [36–38], and identifies the maximally supersymmetric ($\mathcal{N} = 4$) Yang–Mills theory with the gauge group $SU(N)$ in $(3 + 1)$ dimensions — with the type IIB superstring theory in the $(9 + 1)$ –dimensional background of the anti-de Sitter space times the sphere, $AdS_5 \times S^5$. It had been preceded by various speculations of how to construct such a duality, for example by exploiting an analogy between the large- N loop equation and the string Schrödinger equation, in the work of Polyakov from 1981 [39]; it was the first time when this hypothetical “QCD string” was conjectured to live not in 4 dimensions of our physical spacetime, but rather in a novel background having 5 non-compact dimensions. This idea has indeed proven to be the right track through the Maldacena's conjecture and subsequent developments [40] — the string theories corresponding to several 4-dimensional large- N conformal gauge theories with the $SO(2, 4)$ symmetry have been discovered to occupy target spaces with 5 non-compact and 5 compact dimensions of the general form

$$AdS_5 \times X^5, \quad (79)$$

where X^5 is an Einstein manifold (*i.e.*, having $R_{ij} \sim g_{ij}$) with a positive curvature.

Let us begin with sketching the picture of the “AdS/CFT correspondence,” in order to eventually move to a more detailed discussion. It deals with a flat $(9 + 1)$ –dimensional spacetime $\mathbb{R}^{1,9}$, and a stack of N coinciding Dirichlet's D3-branes at $x^4 = x^5 = \dots = x^9 = 0$ in it. A Dp-brane has a striking feature that it gives rise, through the spectrum of massless excitations of open strings attached to it, to the $(p + 1)$ –dimensional maximally supersymmetric $U(1)$ gauge theory living in its world-volume. Consequently, N such branes placed one on each other lead to the gauge group $SU(N)$ [41], and this is how the “gauge side” of the correspondence is recovered. On the other hand, if N is large, the stack of Dp-branes back-reacts on the geometry of the spacetime, being a heavy object in a theory with gravity. It will then be described by some metric and also other background fields, in particular, the Ramond–Ramond $(p + 1)$ –form potential. This will mean type IIB superstring theory with an R–R charged p –brane background on the “string side” of the correspondence. The equivalence of these two points of view on the brane system constitutes the celebrated Maldacena's duality.

2. Ten-Dimensional Supergravity

We will now set the stage for the string side. We will be working with various supergravity (SUGRA) theories in the flat Minkowski spacetime with D dimensions. For the SUSY generators Q_α^a , $\alpha = 1, 2, \dots, \dim_{\mathbb{C}} S$, $a = 1, 2, \dots, \mathcal{N}$, we will now adapt the Dirac spinor representation $S = S_{\text{Dirac}}$, given by the standard Clifford–Dirac matrices Γ_μ ($J_{\mu\nu} = \frac{i}{4}[\Gamma_\mu, \Gamma_\nu]$ and $\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}$); its complex dimension is $\dim_{\mathbb{C}} S_{\text{Dirac}} = 2^{\lfloor D/2 \rfloor}$. The SUSY algebra looks analogous to (9), but with the Pauli matrices σ_μ replaced by Γ_μ . In order to find massless representations of this algebra, we proceed as described around (11), and also this time half of the supercharges vanish, while the other half serve as raising/lowering operators of helicity by $1/2$. Moreover, we restrict the multiplets to include only particles of spins ≤ 2 ; the reason is that it has been shown that massless interacting particles of spin > 2 cannot be causal, while we want to include a spin-2 massless graviton. This means that there may be at most 8 raising supercharges,

$$\mathcal{N} \dim_{\mathbb{R}} S \leq 32. \quad (80)$$

We will consider the largest spacetime dimension for which this inequality is saturated, which is $D = 11$ and $\mathcal{N} = 1$. Then, there are 32 Majorana supercharges. There exists a unique SUGRA theory with these parameters [42]. It consists of the following fields,

$$\text{graviton (symmetric traceless rank-2) } G_{\mu\nu}, \quad \text{anti-symmetric rank-3 } A_{\mu\nu\rho}, \quad \text{Majorana gravitino } \phi_{\mu\alpha}, \quad (81)$$

which are governed by the action (in the Einstein frame, as there is no dilaton here) whose bosonic piece reads

$$S_{D=11, \mathcal{N}=1 \text{ SUGRA}}|_{\text{bosonic}} = \frac{1}{2\kappa_{11}^2} \int \left(\sqrt{G} \left(R_G - \frac{1}{2} |F_4|^2 \right) - \frac{1}{6} A_3 \wedge F_4 \wedge F_4 \right), \quad (82)$$

where $G \equiv -\text{Det}G_{\mu\nu}$, $|F_p|^2 \equiv \frac{1}{p!}G^{\mu_1\nu_1}\dots G^{\mu_p\nu_p}\bar{F}_{\mu_1\dots\mu_p}F^{\nu_1\dots\nu_p}$, the bar denoting complex conjugation, $A_3 \equiv (1/3!)A_{\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho$, and $F_4 \equiv dA_3$, while $\kappa_{11} = \sqrt{8\pi G_{11}}$ is the gravitational constant in 11 dimensions. This SUGRA theory will be of importance to us only indirectly — it gives rise to a theory of a lower dimensionality, namely $D = 10$, through the standard “Kaluza–Klein dimensional reduction procedure.” We use it in the following way: one coordinate y of \mathbb{R}^D is compactified into a circle of radius R , while the remaining $(D - 1)$ ones $x^{\bar{\mu}}$ remain untouched; then, $R \rightarrow 0$ is taken, which removes one dimension. This leaves a certain imprint on the fields of the theory. For example, a gauge field $A_\mu(x^\nu)$ transforming under the fundamental representation of $\text{SO}(1, D - 1)$ and obeying periodic b.c. on the circle — changes into a scalar $A_y(x^{\bar{\nu}})$ and a vector $A_{\bar{\mu}}(x^{\bar{\nu}})$ of $\text{SO}(1, D - 2)$. More generally, a field transforming under some tensor representation T of $\text{SO}(1, D - 1)$ and having periodic b.c. on the circle — will change into fields corresponding to representations whose direct sum is the restriction of T to the subgroup $\text{SO}(1, D - 2)$. The same is true for any spinor representation S . Fields satisfying any non-periodic b.c. decouple. Now, this method is used to reduce the above $D = 11$, $\mathcal{N} = 1$ SUGRA to either type IIA or type IIB $D = 10$, $\mathcal{N} = 2$ SUGRA. This choice of parameters again saturates the bound (80), in which case there are 16 Majorana–Weyl supercharges. The field content for type IIB version of this theory, which is our goal, reads

$$\begin{aligned} &\text{graviton } G_{\mu\nu}, \quad \text{axion–dilaton } C + i\Phi, \\ &\text{antisymmetric rank-2 tensor } B_{\mu\nu} + iA_{2\mu\nu}, \quad \text{antisymmetric self-dual rank-4 tensor } A_{4\mu\nu\rho\sigma}^+, \\ &\text{Majorana–Weyl gravitinos } \psi_{\mu\alpha}^I, \quad \text{Majorana–Weyl dilatinos } \lambda_\alpha^I, \end{aligned} \quad (83)$$

where $I = 1, 2$, and the plus in A_4^+ indicates self-duality. The gravitinos have the same chirality, as the dilatinos do. The action is derived from (82) by dimensional reduction. A complication is the presence of the self-dual field strength coming from A_4^+ , and one considers an action for both dualities, only subsequently imposing the self-duality condition as an additional equation of motion. We have [43],

$$\begin{aligned} S_{D=10, \mathcal{N}=2 \text{ type IIB SUGRA}}|_{\text{bosonic}} &= \frac{1}{4\kappa_{10}^2} \int \left(\sqrt{G} e^{-2\Phi} \left(2R_G + 8\partial_\mu \Phi \partial^\mu \Phi - |H_3|^2 \right) - \right. \\ &\quad \left. - \sqrt{G} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2 \right) + A_4^+ \wedge H_3 \wedge F_3 \right), \end{aligned} \quad (84)$$

where $F_1 \equiv dC$, $H_3 \equiv dB$, $F_3 \equiv dA_2$, $F_5 \equiv dA_4^+$, $\tilde{F}_3 \equiv F_3 - CH_3$, $\tilde{F}_5 \equiv F_5 - \frac{1}{2}A_2 \wedge H_3 + \frac{1}{2}B \wedge F_3$, and finally we must impose the self-duality $\star \tilde{F}_5 = \tilde{F}_5$. It is customary to rewrite this action in the Einstein’s frame through the Weyl transformation $G_{E\mu\nu} \equiv e^{-\Phi/2} G_{\mu\nu}$,

$$\begin{aligned} S_{D=10, \mathcal{N}=2 \text{ type IIB SUGRA}}|_{\text{bosonic}} &= \\ &= \frac{1}{4\kappa_{10}^2} \int \left(\sqrt{G_E} \left(2R_{G_E} - \frac{\partial_\mu \bar{\tau} \partial^\mu \tau}{(\text{Im}\tau)^2} - \frac{1}{2} |F_1|^2 - |G_3|^2 - \frac{1}{2} |\tilde{F}_5|^2 \right) + iA_4^+ \wedge \bar{G}_3 \wedge G_3 \right), \end{aligned} \quad (85)$$

where $\tau \equiv C + i e^{-\Phi}$ and $G_3 \equiv (F_3 - \tau H_3)/\sqrt{\text{Im}\tau}$.

3. Black Holes and Black Branes

We will leave our introductory discussion of $D = 10$, $\mathcal{N} = 2$ type IIB SUGRA at this point, and proceed to brane solutions in it. To get to branes, let us begin with the “Schwarzschild black hole” solution to the Einstein’s equation without matter in $(3 + 1)$ dimensions ($G = 1$), which is their earliest predecessor,

$$ds_{\text{Schwarzschild}}^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\Omega_2^2. \quad (86)$$

It is static and spherically symmetric; in fact, it is the most general solution with spherical symmetry and no sources according to the Birkhoff’s theorem. It is, however, generated by a point mass M at $r = 0$, as may be verified by

going to the Newtonian limit, but this source is inside the event horizon. At $r = r_H \equiv 2M$, which is called an “event horizon,” this metric has a singularity; the hypersurface $r = r_H$ is null; to reach it from the outside or to escape from its inside takes an infinite time (the quantum-mechanical Hawking’s thermal radiation does escape, though), which may be checked by considering a radial ray of light ($ds^2 = 0$, $d\theta = d\phi = 0$) around the horizon, $t \sim \log(|r - r_H|)$. This is, however, merely a superficial singularity, as there may be found another frame in which the metric is regular, as can be inferred from the finiteness of the Ricci scalar R at the horizon; the first such set of coordinates, which moreover covers the entire space, has been found by Kruskal and Szekeres.

The Schwarzschild solution can be generalized to include, in addition to the mass M , an electric charge Q sitting at $r = 0$, by considering an Einstein–Maxwell system with the energy–momentum tensor $T_{\mu\nu} = \frac{1}{4\pi}(g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma})$, which yields the so-called “Reissner–Nordström black hole,”

$$ds_{\text{Reissner–Nordström}}^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{1}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}dr^2 + r^2 d\Omega_2^2, \quad E_t(r) = \frac{Q}{r^2}. \quad (87)$$

This time, there appear two horizons (“inner” and “outer”), at $r = r_{H\pm} \equiv M \pm \sqrt{M^2 - Q^2}$. Notice that if there were $M < Q$, there would be no horizon, and so, the singularity at $r = 0$ would be naked; this is believed to be forbidden in a physical theory by the Penrose’s cosmic censorship rule. Hence, it is required to choose

$$M \geq Q. \quad (88)$$

A very potent situation occurs when this inequality is saturated, $M = Q$; we then obtain an “extremal black hole.” It has a number of interesting and generic features: First, its two horizons coincide, and the metric (after $R \equiv r - Q$) acquires the form,

$$ds_{\text{extremal Reissner–Nordström}}^2 = -\frac{1}{H(R)^2}dt^2 + H(R)^2(dR^2 + R^2 d\Omega_2^2), \quad \text{where} \quad H(R) \equiv 1 + \frac{Q}{R}; \quad (89)$$

the extremal solution, therefore, is defined through $H(r)$, which is a harmonic function in 3 dimensions,

$$\Delta_{(3)}H(r) \sim Q\delta^{(3)}(r). \quad (90)$$

Such a function encodes all the properties of an extremal black hole, for example its horizon, Hawking temperature, Bekenstein–Hawking entropy, *etc.* Second, this extremal black hole behaves as a half-BPS object with the bound (88). We will not elaborate on this more except saying that it can be understood by thinking of our Einstein–Maxwell system as a bosonic part of the short gravity supermultiplet of $(3 + 1)$ -dimensional $\mathcal{N} = 2$ SUGRA. Third, the extremal black hole interpolates between its two limits: for $r \gg r_H$, the metric reduces to the flat Minkowski space, while for $r \approx r_H$ (the “near-horizon limit”), to

$$ds_{\text{AdS}_2 \times S^2}^2 = -\frac{R^2}{Q^2}dt^2 + \frac{Q^2}{R^2}dR^2 + Q^2 d\Omega_2^2, \quad (91)$$

which (upon $R \rightarrow Q^2 R$) we recognize as the metric of $\text{AdS}_2 \times S^2$ in the Poincaré coordinates (68), and which is called the “Bertotti–Robinson geometry.” The spatial part is degenerated into an infinitely long tube of the geometry of $\mathbb{R} \times S^2$, called a “throat.” As solutions in $\mathcal{N} = 2$ SUGRA, both these limits are maximally supersymmetric (they have 8 Killing spinors), while the extremal Reissner–Nordström black hole, which breaks half of the SUSYs (it has 4 Killing spinors), spatially interpolates between them; therefore, it behaves like a soliton, which interpolates between two vacua. These are generic features of extremal black holes, also in other backgrounds, for multi-center solutions, with additional angular momentum, *etc.*

Let us now return to SUGRA theories. Generalizations of charged black holes to a SUGRA setting are found as follows: If we have a theory with an antisymmetric rank- $(p + 1)$ tensor $A_{\mu_1 \dots \mu_{p+1}}$, written as the $(p + 1)$ -form $A_{p+1} \equiv (1/(p + 1)!)A_{\mu_1 \dots \mu_{p+1}}dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}}$, it is natural to couple it with $(p + 1)$ -dimensional (p spatial dimensions) surfaces Σ_{p+1} through the action

$$S_{p+1} = T_{p+1} \int_{\Sigma_{p+1}} A_{p+1}, \quad (92)$$

which is diff-invariant and invariant under Abelian rank- p gauge transformations, $A_{p+1} \rightarrow A_{p+1} + d\rho_p$. Such classical (and static) solutions to the SUGRA equations with an A_{p+1} charge are called “ p -branes.” Let us note that each

p -brane has its “magnetic dual” which is a $(D - p - 4)$ -brane coupled to $A_{D-p-3}^{\text{magn.}}$ such that $dA_{D-p-3}^{\text{magn.}} \equiv \star dA_{p+1}$. We introduce also a bit of terminology: a “D p -brane” is a p -brane which is a source to an antisymmetric tensor field from the R–R sector. We will be interested in the extremal case; they will also be described by harmonic functions $((D - p - 1)$ -dimensional), break half of the SUSYs, and interpolate between two solutions of higher SUSY, just as discussed above for the Reissner–Nordström extremal black hole. Before we show these properties, let us remark that for the physical number of dimensions, $D = 4$, there are no other extremal branes which are localized in space than black holes, *i.e.*, 0-branes. Indeed, a 1-brane, called a “cosmic string,” is given by the harmonic function $H(x_1, x_2) = \log(|x_1 + ix_2|)$, while a 2-brane, called a “domain wall,” by $H(x) = 1 + a|x|$; both functions increase as we move away from the source, thus affecting the entire space. But in $D > 4$, we can have localized extremal p -branes. In order to find what brane solutions are allowed in a given SUGRA theory, we must review its field content w.r.t. antisymmetric tensors. For example, in our $D = 10$ $\mathcal{N} = 2$ type IIB SUGRA, there will be (83) four possibilities plus their four magnetic duals,

τ	D(−1)	D7
$B_{\mu\nu}$	F1	NS5
$A_{2\mu\nu}$	D1	D5
$A_{4\mu\nu\rho\sigma}^+$	D3	D3.

(93)

The only NS–NS field here, $B_{\mu\nu}$, gives rise to a 1-brane F1, which is just the fundamental string; $\tau = C + ie^{-\Phi}$ yields the D(−1)-instanton, which we will not discuss in more detail. The most interesting for us will be the self-dual field $A_{4\mu\nu\rho\sigma}^+$ producing a D3-brane, which is self-dual w.r.t. electric–magnetic duality. To derive the metrics of these branes, we notice that a brane has the Poincaré symmetry $\mathbb{R}^{p+1} \times \text{SO}(1, p)$ in its world-volume, and also is rotationally symmetric $\text{SO}(D - p - 1)$ in its transverse space. This implies that the brane’s metric should be a rescaling of the Minkowski metric in the longitudinal coordinates plus a rescaling of the Euclidean metric in the transverse coordinates; substituting such a form into the SUGRA equations, we find [44],

$$ds_{p\text{-brane in } D=10 \text{ } \mathcal{N}=2 \text{ type II SUGRA}}^2 = H_p(r)^{-1/2} \left(-f_p(r) dt^2 + \sum_{i=1}^p (dx^i)^2 \right) + H_p(r)^{1/2} (f_p(r)^{-1} dr^2 + r^2 d\Omega_{8-p}^2) \quad (94)$$

(this is the so-called “string metric,” related to the usual Einstein metric through $ds_E^2 = e^{-\Phi/2} ds^2$), while some other background fields, the dilaton and antisymmetric rank- $(p + 1)$ tensor,

$$e^\Phi = H_p(r)^{\frac{3-p}{4}}, \quad A_{p+1} = \frac{1}{g_s} (H_p(r)^{-1} - 1) dx^0 \wedge dx^1 \wedge \dots \wedge dx^p, \quad (95)$$

where the most general $(9 - p)$ -dimensional harmonic function H_p of the transverse coordinates, respecting the $\text{SO}(9 - p)$ invariance, and yielding a metric with the Minkowski limit at $r \rightarrow \infty$, is derived along with $f_p(r)$ to be

$$H_p(r) = 1 + \frac{L^{7-p}}{r^{7-p}}, \quad f_p(r) = 1 + \frac{r_H^{7-p}}{r^{7-p}}, \quad (96)$$

where L is some length, and $r = r_H$ is the position of the p -brane’s horizon. Since the only dimensionful parameter in the theory is the string length $l_s = \sqrt{\alpha'}$, L must be proportional to it, and in fact, for D p -branes,

$$L^{7-p} = \alpha_p r_p^{7-p}, \quad \text{where} \quad \alpha_p = \sqrt{1 + \left(\frac{r_H^{7-p}}{2r_p^{7-p}} \right)^2} - \frac{r_H^{7-p}}{2r_p^{7-p}}, \quad r_p^{7-p} = g_s N \left(\sqrt{\alpha'} \right)^{7-p} (4\pi)^{\frac{5-p}{2}} \Gamma\left(\frac{7-p}{2}\right), \quad (97)$$

where we have also introduced N if we have many coincident p -branes. L is chosen in such a manner that the flux of the self-dual R–R field strength through the 5-sphere reads

$$\int_{S^5} \tilde{F}_5 = N. \quad (98)$$

Notice the dependence on the (dynamically generated) closed string coupling constant, $g_s \equiv e^\phi$, where $\phi \equiv \langle \Phi \rangle$ is the VEV of the dilaton field. Let us now restrict our interest to only the extremal limit, when the horizon and the singularity at $r = 0$ merge, *i.e.*, $r_H = 0$; if $r_H \ll L$, the solution is called “near-extremal.” They can be shown to preserve precisely 16 out of 32 SUSYs, thus being half-BPS.

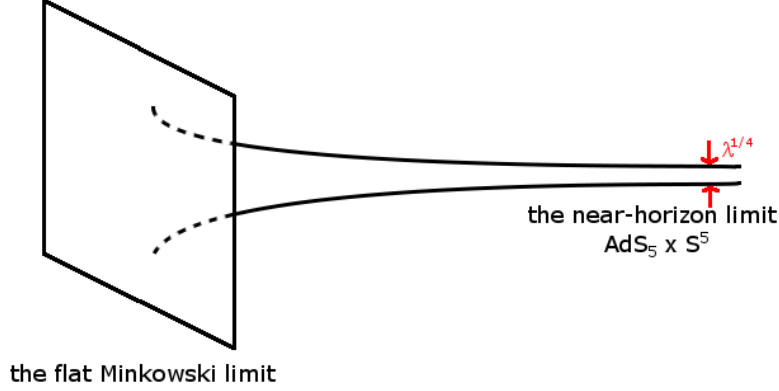


FIG. 6: The two limits of the extremal D3-brane metric (99): the flat $(9+1)$ -dimensional Minkowski spacetime far from the brane ($r \gg L$), and the near-horizon or throat limit ($r \ll L$), yielding the $\text{AdS}_5 \times S^5$ geometry. The size of the latter, in string units, is unexpectedly given by the 't Hooft coupling constant, $\lambda^{1/4}$ (106).

The most interesting case is $p = 3$, *i.e.*, a stack of N coincident extremal D3-branes in $D = 10$ $\mathcal{N} = 2$ type IIB SUGRA, and for a number of reasons: First, its world-volume possesses the $(3+1)$ -dimensional Poincaré invariance. Second, the axion C and dilaton Φ fields are constant. For all $p \neq 3$, the dilaton is not constant, and explodes at $r = 0$ (95); consequently, the string coupling g_s depends on the position in the space, which hardens the analysis; also, the instanton angle $\theta_1 = 2\pi\langle C \rangle$ is not constant. (The other fields are given by $B_{\mu\nu} = A_{2\mu\nu} = 0$, $F_{5\mu_1\mu_2\mu_3\mu_4\mu_5}^+ = \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}\partial^{\mu_6}H_3$.) Third, its metric,

$$ds_{\text{extremal D3-brane}}^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \left(1 + \frac{L^4}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2), \quad (99)$$

with

$$L^4 = 4\pi g_s N (\alpha')^2, \quad (100)$$

is [45] non-singular at the horizon $r = 0$, despite the appearances, and in fact, for $r \rightarrow 0$, and in the variable $z \equiv L^2/r$, it tends to the metric of $\text{AdS}_5 \times S^5$ (with equal radii of curvature L of the two components) in the Poincaré coordinates (69),

$$ds_{\text{AdS}_5 \times S^5}^2 = \frac{L^2}{z^2} (-dt^2 + dz^2 + d\vec{x}^2) + L^2 d\Omega_5^2, \quad (101)$$

analogously to the Reissner–Nordström case (91). On the contrary, for all $p \neq 3$, the horizon is singular, of zero area. Similarly as before, thus, the extremal D3-brane interpolates between two maximally symmetric spaces, the flat $(9+1)$ -dimensional Minkowski spacetime for $r \gg L$, and $\text{AdS}_5 \times S^5$ near the horizon $r \ll L$, see figure 6. Let us elaborate more on this finiteness: It may be verified that all possible curvature measures of the geometry (99) are finite for $r \approx 0$; they are actually small if L is much larger than the string scale ($L \gg l_s$), as for example

$$R_{mnm'n'} = -\frac{1}{L^2} (g_{mm'}g_{nn'} - g_{mn'}g_{nm'}) \quad (\text{AdS}_5), \quad R_{iji'j'} = +\frac{1}{L^2} (g_{ii'}g_{jj'} - g_{ij'}g_{ji'}) \quad (S^5) \quad (102)$$

(see the discussion below (57)), *i.e.*, the small-curvature SUGRA approximation of type IIB superstring theory works well then. Since $(L/l_s)^4 = 4\pi g_s N$ (100), we infer that if $g_s N \ll 1$, the SUGRA approximation of the full superstring solution breaks down, even though the string perturbation theory is valid due to $g_s \ll 1$; on the other hand, if $g_s N \gg 1$, the SUGRA solution is a good approximation, and moreover it is possible to remain within the string perturbative regime of $g_s \ll 1$ provided $N \gg 1$ sufficiently. The fourth important advantage of $p = 3$ is that superstring theories defined in the extremal D3-brane background depend, in particular, on the complex constant (the “modular parameter”)

$$\tau_s = \langle C \rangle + i e^{-\langle \Phi \rangle} = \frac{\theta_1}{2\pi} + \frac{i}{g_s}. \quad (103)$$

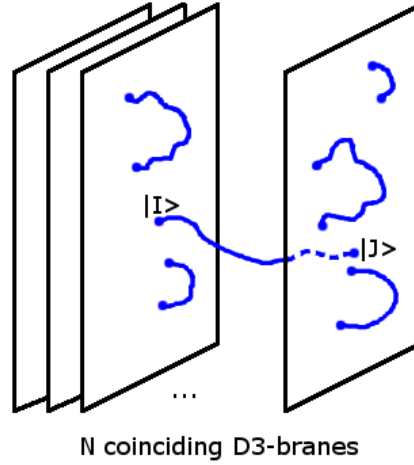


FIG. 7: D-branes, open strings ending on them, and Chan-Patton factors.

Now, type IIB SUGRA is invariant w.r.t. $SU(1, 1) \sim SL(2, \mathbb{R})$, which acts transitively on τ ; for type IIB superstring theory, this must be restricted to the subgroup $SL(2, \mathbb{Z})$ [46] due to the identification $\theta_1 \sim \theta_1 + 2\pi$ ($\tau_s \sim \tau_s + 1$). Hence, there is the $SL(2, \mathbb{Z})$ symmetry of superstring theories in the background of the extremal D3-brane; it will correspond to the Montonen–Olive S-duality of $\mathcal{N} = 4$ SYM (46).

4. Black Branes as D-Branes

Let us now switch the point of view on the above stack of N coinciding extremal D3-branes, in order to see how they may give rise to the gauge side of the AdS/CFT correspondence. We have introduced above Dp -branes as certain solutions to SUGRA equations, but they are expected to extend to solutions of full superstring theory (of which SUGRA is the low-energy $\alpha' \rightarrow 0$ limit), *i.e.*, receive α' -corrections to the background fields. Let us see what happens to Dp -branes if we treat them as string theory solutions, in the perturbative regime of $g_s \ll 1$ (in which string theory is well-defined through the strings' world-sheets' genus expansion). Then, according to (94), (96), the Dp -brane metric becomes flat everywhere except for the $(p+1)$ -dimensional hyperplane Σ_{p+1} , where the metric explodes. In other words, a Dp -brane becomes, for weak string coupling, a localized topological defect (a “wall”) of a flat spacetime. This means [47] that strings propagate in a flat spacetime, except when they touch a D-brane — then they open, and boundary conditions are imposed on the string's endpoints: the Neumann b.c. (*i.e.*, a free motion with the speed of light; $\partial_\sigma X_\mu = 0$, for $\sigma = 0, l$) for the $(p+1)$ tangential coordinates, and the Dirichlet b.c. (*i.e.*, fixed; $X_\mu = \text{const}$, for $\sigma = 0, l$) for the $(D-p-1)$ coordinates transverse to the brane. However, D-branes are much more than just boundary conditions; Polchinski in 1995 [48] showed in his breakthrough work that they also are dynamical half-BPS objects carrying an elementary unit of charge w.r.t. antisymmetric rank- $(p+1)$ R-R tensor of type II superstring theory, *i.e.*, in fact

$$Dp\text{-branes (b.c. for the string's endpoints)} = \text{extremal } p\text{-branes (solutions to SUGRA)}. \quad (104)$$

It is a standard calculation in string theory and a fascinating observation to find that the spectrum of an open string ending on a single Dp -brane includes a massless $U(1)$ gauge field living in the brane's world-volume. (The other fields include $(9-p)$ massless Goldstone scalars, as well as fermions, completing the SUSY.) In the low-energy $\alpha' \rightarrow 0$ limit, only these massless fields remain. Moreover, since the brane is a half-BPS object, and so, it preserves 16 out of the total 32 supercharges, forming $\mathcal{N} = 1$ 11-dimensional spinor, or equivalently, $\mathcal{N} = 8$ 4-dimensional spinors — we obtain $\mathcal{N} = 4$ (*i.e.*, maximally supersymmetric) $U(1)$ gauge theory in a flat $(p+1)$ -dimensional Minkowski spacetime. If we take N such parallel branes, each endpoint of an open string may be placed on any of them; accordingly, endpoints are labeled by so-called “Chan-Patton factors” $|I\rangle$, with $I = 1, 2, \dots, N$ enumerating the D-branes, see figure 7, and so, there are N^2 possible string configurations (type II strings are oriented). If the branes

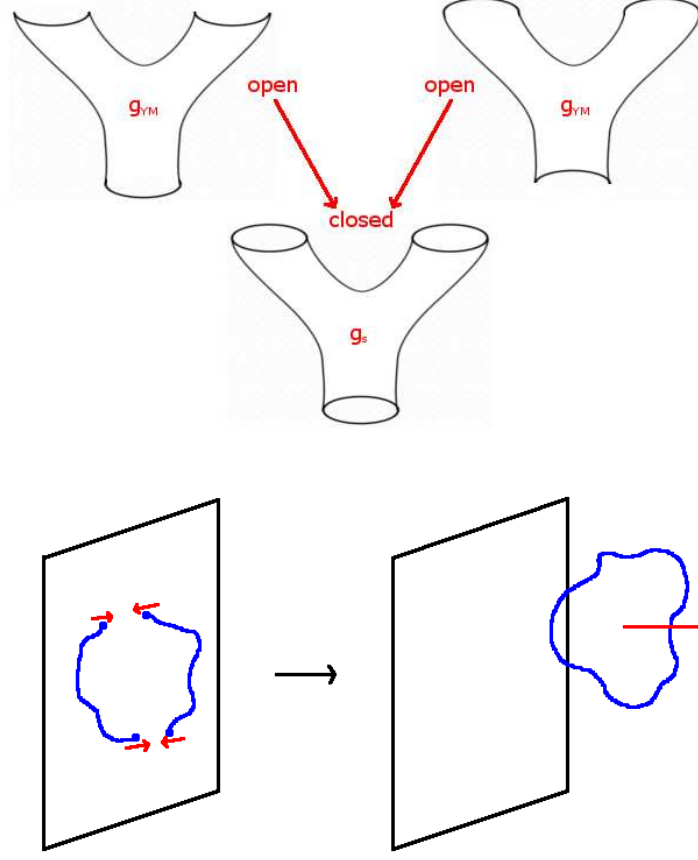


FIG. 8: UP: A heuristic derivation of (105). Two splitting vertices of open strings can be glued together to form a closed string splitting vertex (a “pair of pants”).

DOWN: An explanation of Hawking radiation in terms of open strings living on a D-brane. Two open strings collide with their endpoints, create a closed string, which eventually detaches from the D-brane, and radiates away.

are separated from each other, the resulting theory has the low-energy excitations including a $U(1)^N$ gauge field. But if we take all the separations to zero, the gauge group is enhanced to $U(N)$. Actually, it is $SU(N)$, as the factor $U(1) = U(N)/SU(N)$ decouples from most of equations as corresponding to the center of mass of the stack of branes. Furthermore, one can find a relationship between the gauge coupling constant g_{YM} and the closed string coupling constant g_s ,

$$g_{YM}^2 = 4\pi g_s (2\pi l_s)^{p-3}. \quad (105)$$

A pictorial argument in favor of this formula is given in figure 8, up: the gauge coupling constant g_{YM} is essentially the open string coupling constant g_s^{open} [26], which is equal to $\sqrt{g_s}$, as two open string splitting vertices can be joined along their boundaries to form a closed string splitting vertex. This relation can also be systematically derived from the D-brane’s Born-Infeld action, $S_{\text{Born-Infeld}} = -T_p \text{Tr} \int d^{p+1}x \sqrt{-\text{Det}(G_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}$. To summarize, setting also $p = 3$: a string theory on a stack of N coinciding D3-branes in the flat $(9+1)$ -dimensional spacetime possesses, as its low-lying excitations (the only ones for $\alpha' \rightarrow 0$), $(3+1)$ -dimensional Poincaré-invariant $\mathcal{N} = 4$ SYM theory with the gauge group $SU(N)$, with the coupling constant $g_{YM}^2 = 4\pi g_s$. Note that this is in agreement with the AdS/CFT-implied equivalence of the $SL(2, \mathbb{Z})$ symmetries on both sides (45), (103), $\tau_{YM} = \tau_s$. An immediate consequence is that the size of the throat L (100) in string units reads

$$\frac{L}{\sqrt{\alpha'}} = \lambda^{1/4}, \quad (106)$$

i.e., is expressed through the ’t Hooft coupling (5). This is the second hint (the first one was the equality of the symmetry groups, $SO(2,4) \times SO(6)$) for the AdS/CFT correspondence: λ , originally appearing in the string-like

topological expansion (7) of large-color (6) gauge theories — has now been reproduced in a gravitational setting! Finally, let us reiterate the discussion from below (102): the small-curvature SUGRA regime is when $\lambda \gg 1$, which precisely is the strong-coupling regime on the gauge side; conversely, for $\lambda \ll 1$, perturbative methods can be exploited in $\mathcal{N} = 4$ SYM, but on the opposite side, we find a genuinely quantum string theory in a highly curved background. This means that AdS/CFT is a “strong/weak duality,” which is both an unprecedented advantage (as one can access a strongly-coupled theory via a weakly-coupled one) and a great hindrance (as a proof of the conjecture seems very inaccessible) at the same time.

5. The Hawking’s Radiation

We have described above two viewpoints on the system of coinciding D-branes. But is there any relationship between the open string theory on the stack of D3-branes, yielding $(3 + 1)$ -dimensional $\mathcal{N} = 4$ SYM, and the SUGRA (or, more broadly, closed string) theory of the $(9 + 1)$ -dimensional spacetime curved by this stack? The third indication of such a connection comes from the following argument: If we consider a near-extremal p -brane, it will have a horizon at some $r = r_H \approx 0$ of a finite size, and this horizon will emit the Hawking’s thermal radiation; this radiation is described precisely by the closed string theory in the spacetime curved by the stack. On the other hand, the Hawking radiation can also be understood in the open string theory language as the unitary process in which two open strings attached to the stack collide with their endpoints, thus forming a closed string, which then detaches from the stack and radiates into the outer space, see figure 8, down. A calculation of entropies for both cases has first been made in [49]. In the gravitational setting, the method of [50] has been used: One considers the Euclidean continuation of the p -brane solution, and the Euclidean time is periodic with circumference $\beta = 1/T = \partial S_{\text{BH}}/\partial E$, where $S_{\text{BH}} = 2\pi A_H/\kappa^2$ is the Bekenstein–Hawking entropy, A_H the horizon’s area, E the excess energy of the brane above its extremal value. Now, it is proven that T must acquire such a value that the geometry has no conical singularity at the horizon. For the solution (94), $p = 3$, in the near-extremal limit and close to the horizon, it is shown that $\beta = \pi L^2/r_H$. The area of the horizon is then derived from the metric, and reads $A_H = (r_H/L)^3 V_3 L^4 \Omega_5$, where V_3 is the volume of the D3-brane, and Ω_5 of a unit 5-sphere; this leads to the Bekenstein–Hawking entropy,

$$S_{\text{BH}} = \frac{\pi^2}{2} V_3 N^2 T^3. \quad (107)$$

On the other hand, the entropy of the $\mathcal{N} = 4$ SYM gauge supermultiplet is investigated. The theory is assumed to be free, and standard techniques of statistical mechanics are used to find

$$S_{\text{free } \mathcal{N} = 4 \text{ SYM}} = \frac{2\pi^2}{3} V_3 N^2 T^3. \quad (108)$$

The two results amazingly coincide up to a numerical factor; in particular, the gravitational computation reproduces the N^2 scaling (obvious on the gauge side, as there are $\sim N^2$ degrees of freedom), and the T^3 scaling (which is required by CFT arguments). The numerical-factor discrepancy is caused by the fact that the two calculations have been performed under two different assumptions: the former requires the SUGRA limit, $\lambda \gg 1$, while the latter, the weak-gauge-coupling limit $\lambda \ll 1$. Hence, (107) and (108) are really two sides (large- λ and small- λ) of the same coin, and there should be

$$S = \frac{2\pi^2}{3} V_3 N^2 T^3 f(\lambda), \quad (109)$$

where an attack on the function $f(\lambda)$ can be carried either in a weakly-coupled gauge theory (which then gives a prediction for a strongly-coupled string theory), or in quantum-string α' -corrections to SUGRA (thus, saying something about a strongly-coupled gauge theory); one respectively obtains [51, 52],

$$f(\lambda) = 1 - \frac{3}{2\pi^2} \lambda + \frac{3 + \sqrt{2}}{\pi^3} \lambda^{3/2} + \dots \quad (\lambda \ll 1), \quad f(\lambda) = \frac{3}{4} + \frac{45}{32} \zeta(3) \lambda^{-3/2} + \dots \quad (\lambda \gg 1). \quad (110)$$

This is a generic situation in AdS/CFT problems, which are of strong/weak nature. Let us also mention that this type of links between black holes and D-branes was pioneered in 1996 by Strominger and Vafa [53], although for a 5-dimensional SUGRA and a more complicated system of intersecting D1- and D5-branes, see also [54], and constituted an important step toward AdS/CFT.

6. The Maldacena's Limit

So, having introduced above some background knowledge and initial clues, let us finally come to establishing the Maldacena's correspondence; we will say what theories really stand on both sides, what the limits of applicability are, and how their observables relate. Maldacena's seminal paper [36] contained a critical observation that a very intriguing limit (the “decoupling limit”) is to send the string length $l_s = \sqrt{\alpha'}$ to zero,

$$\alpha' \rightarrow 0, \quad g_s, N = \text{fixed} \quad (\text{Maldacena/decoupling}). \quad (111)$$

(Technically, one cannot take such a limit of a dimensionful quantity. As we will see, it properly means to consider energy scales such that massive string excitations decouple, and only a SUGRA remains.)

Let us first understand what happens to our system treated as a SUGRA solution. Start from recalling that the extremal black 3-brane metric (99) interpolates between two regions: for $r \gg L$, it tends to a flat Minkowski spacetime, while for $r \ll L$ (the near-horizon/throat limit), to $\text{AdS}_5 \times S^5$ (101) (or more precisely, its Poincaré patch (69)). Notice that from the point of view of $\text{AdS}_5 \times S^5$ (obtained from the brane through $r \rightarrow 0$, *i.e.*, $z \rightarrow \infty$), the original brane lives at $r \rightarrow \infty$, *i.e.*, $z \rightarrow 0$; this, as mentioned in paragraph IC 1, is a part of the boundary of the anti-de Sitter space, which is also a $(3+1)$ -dimensional Minkowski spacetime. Now, the idea is that we zoom into the near-horizon region; this region is then magnified in a singular way by taking (111) in the following fashion,

$$\alpha' \rightarrow 0, \quad r \rightarrow 0, \quad \text{such that} \quad U \equiv \frac{r}{\alpha'} = \text{fixed}. \quad (112)$$

Indeed, in the limit (112), in the harmonic function $H_3(r)$ (96), (106) we can forget about the 1, and get $H_3(r) \approx L^4/r^4 = \lambda/(U^4(\alpha')^2)$; then, the metric becomes

$$ds_{\text{AdS}_5 \times S^5}^2 = \underbrace{\alpha' \sqrt{\lambda}}_{L^2} \underbrace{\left(\frac{U^2}{\lambda} (-dt^2 + d\vec{x}^2) + \frac{dU^2}{U^2} + d\Omega_5^2 \right)}_{ds_{\text{AdS}_5 \times S^5, L=1}^2 \text{ (radius } L=1)}. \quad (113)$$

By removing the overall factor of α' , this small piece of the spacetime close to the horizon of the 3-brane is blown up. Consequently, the non-linear sigma-model describing the string theory in the 3-brane's background (we forget about the other background fields),

$$S_{3\text{-brane sigma-model}} = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{\gamma} \gamma^{\alpha\beta} G_{MN}(x) \partial_\alpha X^M \partial_\beta X^N, \quad G_{MN}(x) dx^M dx^N = ds_{\text{extremal D3-brane}}^2 \quad (114)$$

(the metric is given by (99); $M, N = 0, 1, \dots, 9$), possesses a well-defined smooth limit

$$S_{\text{throat sigma-model}} = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \sqrt{\gamma} \gamma^{\alpha\beta} \tilde{G}_{MN}(x) \partial_\alpha X^M \partial_\beta X^N, \quad \tilde{G}_{MN}(x) dx^M dx^N = ds_{\text{AdS}_5 \times S^5, L=1}^2. \quad (115)$$

Notice that the resulting metric is that of $\text{AdS}_5 \times S^5$ with both radii of curvature $L = 1$, and that the singular string tension in (114) has absorbed the α' from (113) to become finite and determined by the 't Hooft coupling,

$$T_{3\text{-brane sigma-model}} = \frac{1}{2\pi\alpha'} \quad \text{changes into} \quad T \equiv T_{\text{throat sigma-model}} = \frac{1}{2\pi\alpha'} \cdot \alpha' \sqrt{\lambda} = \frac{\sqrt{\lambda}}{2\pi}; \quad (116)$$

compare also (68). To summarize, we can say that the part of the 3-brane, and its contribution to the string dynamics, which survives the Maldacena's limit (111) is the throat, while the asymptotically flat region decouples.

Let us also mention that yet another way to perceive this decoupling of the two limits, $r \ll L$ and $r \gg L$, under (111), is to consider absorption cross-sections σ of massless particles (closed string excitations, for example, the dilaton) incident from the flat region to the throat region (see the Klebanov's paper in [49]). Such a wave partially penetrates the throat, and partially reflects back. The massless scalar wave equation for the 3-brane metric (99) is reduced to an equation for the radial part,

$$\left(\frac{d^2}{d\tilde{r}^2} - \frac{15}{4\tilde{r}^2} + 1 + \frac{\omega^4 L^4}{\tilde{r}^4} \right) \phi(\tilde{r}) = 0, \quad \text{where} \quad \tilde{r} \equiv \omega r, \quad (117)$$

and $\omega = p^0$ is the energy of the wave, assumed to be small, $\omega \ll 1/L$. A SUGRA computation then yields

$$\sigma = \frac{\pi^4}{8} \omega^3 L^8 \ll L^5 \sim (\alpha')^{5/2}, \quad (118)$$

which means that for low energies of the particles, and in the limit (111), it is hard for them to be absorbed by the throat. Therefore, the Maldacena's limit puts a barrier between the two regions, thus decoupling them from each other. (This cross-section can alternatively be calculated within the D-brane picture, just as for the entropy, using the low-energy world-volume action for our system of D-branes, coupled to massless bulk scalars. The leading-order result happens to precisely reproduce (118), which is another hint in favor of the AdS/CFT correspondence. Subleading string- and gauge-theory perturbative corrections may be attacked, as in (110).)

Finally, let us support the necessary scaling (112) by the following argumentation: Consider an arbitrary excited string state at some point of the throat, $r \ll L$, having energy E_r ; this energy in string units, $E_r \sqrt{\alpha'}$, we keep fixed, in order to get a meaningful theory in the throat. The same energy, but measured at some point of the asymptotical region $r \gg L$ (at infinity), E_∞ , is red-shifted due to the factor $H_3(r)^{-1/2}$ in front of $-dt^2$ (94), $E_\infty = H_3(r)^{-1/4} E_r$. Since $H_3(r) \approx L^4/r^4 = \lambda(\alpha')^2/r^4$, for $r \ll L$, this relation is $E_\infty = \lambda^{-1/4}(r/\alpha')(E_r \sqrt{\alpha'})$. This energy measured at infinity we must also keep fixed; we have seen that $r \rightarrow \infty$ describes the boundary of $\text{AdS}_5 \times S^5$, and we will soon see that it is in this boundary that the pertinent gauge theory lives; hence, this energy is really measured in gauge theory, and thus, should stay fixed. To retain these two energies finite, there must, therefore, be $U = r/\alpha' = \text{fixed}$. Remark also that U represents an energy scale in the gauge theory, as E_∞/U remains finite.

We have, thus, argued that the limit (111), investigated from the gravitational point of view, produces two decoupled theories: • type IIB superstring theory in the maximally-symmetric $\text{AdS}_5 \times S^5$ (of radii L given by (106)) background of the throat, in the presence of the R-R 5-form flux of integer magnitude N (98), described by the non-linear $\text{AdS}_5 \times S^5$ (scaled up to $L = 1$) sigma-model whose coupling α' has been effectively replaced by $1/\sqrt{\lambda}$; and • free type IIB SUGRA in the bulk of the spacetime.

The same limit (111) may, on the other hand, be apprehended using the viewpoint of the $\mathcal{N} = 4$ SYM theory living on our stack of N coinciding D3-branes. To this end, we consider type IIB superstring theory in the flat $(9+1)$ -dimensional spacetime with the D-brane stack in it. There are three ingredients of this theory: • First, the “brane modes,” *i.e.*, open strings with the endpoints attached to the D-branes, which we have explained to be described by the pure $\text{SU}(N)$ $\mathcal{N} = 4$ SYM theory in the $(3+1)$ dimensions of the stack's world-volume, up to higher-derivative terms of order $\mathcal{O}(\alpha')$. • Second, the “bulk modes,” *i.e.*, closed type IIB strings in the bulk of the spacetime, behaving as the SUGRA (83), (84) coupled to massive string modes, out of which only the SUGRA remains at $\alpha' \rightarrow 0$. • Third, interactions of these two sectors, which must be proportional to the Newton's constant, *i.e.*, to $g_s(\alpha')^2$, and which, consequently, vanish in the Maldacena's low-energy limit (111), thus leaving us with two decoupled theories. Finally, comparing the first theories in the two decoupled pairs of theories — we arrive at the AdS/CFT correspondence.

7. Versions of the AdS/CFT Correspondence

Let us now discuss various versions of this duality. As it has been formulated above, it requires that the SUGRA approximation of string theory holds, *i.e.*, that the radius of curvature of the background is large in string units, $L \gg l_s$, which means (106) $\lambda \gg 1$, and that quantum-string corrections are small, $g_s \ll 1$. Astonishingly, on the opposite side, we find the planar $\mathcal{N} = 4$ SYM in its strong-coupling regime (the perturbative regime is exactly opposite, $\lambda \ll 1$),

$$\begin{aligned} (\text{classical type IIB SUGRA in } \text{AdS}_5 \times S^5) & \Leftrightarrow (\text{strongly-coupled planar } \text{SU}(N) \mathcal{N} = 4 \text{ SYM}) \\ g_s \rightarrow 0, N \rightarrow \infty, \text{ such that } \lambda = \text{fixed, and then } \lambda \rightarrow \infty. \end{aligned} \quad (119)$$

Therefore, AdS/CFT allows us to access this highly non-perturbative sector of a planar gauge theory through classical SUGRA calculations! Then, one could consider going away from the SUGRA limit by taking into account subleading corrections: on the string side, in small α' , *i.e.*, on the gauge side, in small $1/\sqrt{\lambda}$. If all these corrections agree, a stronger version of AdS/CFT holds:

$$\begin{aligned} (\text{classical type IIB superstring theory in } \text{AdS}_5 \times S^5) & \Leftrightarrow (\text{planar } \text{SU}(N) \mathcal{N} = 4 \text{ SYM}) \\ g_s \rightarrow 0, N \rightarrow \infty, \text{ such that } \lambda = \text{fixed.} \end{aligned} \quad (120)$$

This is the “’t Hooft-limit version” of the duality. Indeed, on the gauge side, we take the strict ’t Hooft limit (6), which singles out only planar fat Feynman graphs. On the string side, it translates into sending the string coupling g_s

to zero, which singles out only tree-level string graphs, *i.e.*, strings are classical. Hence, again astonishingly, classical string computations lead to results in the full quantum planar $\mathcal{N} = 4$ SYM! Once again: to have a 't Hooft gauge theory with finite coupling λ , it is mandatory to go beyond the original SUGRA formulation, and work with genuine string theory. Further, one could again try to move away from the limit (120) by considering subleading corrections: on the string side, in small g_s , *i.e.*, on the gauge side, in small $1/N$. This relates the string-loop expansion in terms of closed-string world-sheets with the 't Hooft's genus expansion in terms of discretized surfaces (7), and is the most advanced fulfilment of the 't Hooft's program. If all these corrections coincide, then the strongest version of the duality is true,

$$\begin{aligned} \text{(full quantum type IIB superstring theory in } \text{AdS}_5 \times S^5) \quad & \Leftrightarrow \quad \text{(SU}(N) \mathcal{N} = 4 \text{ SYM)} \\ & g_s, N \text{ finite.} \end{aligned} \quad (121)$$

This one is actually believed to be valid. In this work, however, we will need to assume only (120).

8. String Energies and Scaling Dimensions

We will not elaborate much on the broad subject of how to relate the observables (*i.e.*, representations of the common invariance group $\text{SU}(2, 2|4)$) of the two theories. First of all, recall that the original D-brane system, where the $\mathcal{N} = 4$ SYM lives, is from the point of view of the throat's $\text{AdS}_5 \times S^5$ geometry its boundary. Since the gauge theory is placed on the 4-dimensional conformal boundary of the 5-dimensional anti-de Sitter space where the string theory lives, AdS/CFT happens also to be the first concrete and most successful realization of the 't Hooft–Susskind “holographic principle” [55], according to which all the information in a space is encoded in the boundary of its volume. There is one subtlety we must comment on regarding the boundary: We have obtained above the throat geometry in the Poincaré chart (101), which does not cover the entire space. The boundary of the Poincaré wedge can then be showed to be conformally related to \mathbb{R}^4 with the Minkowski signature, and there the gauge theory is defined. But the anti-de Sitter space is larger, and one should appropriately modify AdS/CFT so to include the whole space. This is done by a conformal transformation in a Wick-rotated Poincaré metric. Namely, recall first that the conformal boundary of the whole AdS_5 lies in the global coordinates (65) at $\rho \rightarrow \infty$, in the frame (67) at $\theta = \pi/2$, and its metric is

$$ds_{\text{boundary of the global AdS}_5}^2 = \underbrace{\frac{L^2}{\cos^2(\theta)}}_{\rightarrow \infty} (-dt^2 + d\Omega_3^2); \quad (122)$$

the infinite conformal factor is then removed, and the conformal boundary is obtained to be $\mathbb{R} \times S^3$, as is also clear from the Penrose diagram, see figure 4. (Observe that such a rescaling is not unique. Therefore, for it to make sense in the context of AdS/CFT, the boundary gauge theory needs to be scale-invariant.) To make a connection with the conformal boundary of the Poincaré wedge, which lies at $z = 0$ (69),

$$ds_{\text{boundary of the Poincaré patch of AdS}_5}^2 = \underbrace{\frac{L^2}{z^2}}_{\rightarrow \infty} (-d\tau^2 + d\vec{y}^2), \quad (123)$$

we should continue both metrics (122), (123) to the Euclidean signature, and perform the conformal transformation $t = \log(\tau)$, to find that they are equivalent. Hence, we may rephrase the statement of AdS/CFT as follows,

$$\text{(type IIB superstring theory in the global } \text{AdS}_5 \times S^5) \quad \Leftrightarrow \quad \text{(SU}(N) \mathcal{N} = 4 \text{ SYM in } \mathbb{R} \times S^3). \quad (124)$$

The corresponding relation on the gauge side is in the form of the well-known CFT's “state-operator map” (“radial quantization”),

$$\text{quantum states in the } \mathbb{R} \times S^3 \mathcal{N} = 4 \text{ SYM} \quad \rightarrow \quad \text{local operators in the } \mathbb{R}^4 \mathcal{N} = 4 \text{ SYM}. \quad (125)$$

One extremely important implication of this mapping is that the Cartan operator generating translations in the global anti-de Sitter time t , whose eigenvalue is the string energy E , should be matched with the Cartan operator generating spacetime rescalings in \mathbb{R}^4 of $\mathcal{N} = 4$ SYM, whose eigenvalue is the scaling dimension Δ ; consequently, AdS/CFT predicts that

$$E(\lambda; S_1, S_2, J_1, J_2, J_3, \{n_t\}) = \Delta(\lambda; S_1, S_2, J_1, J_2, J_3, \{n_t\}) \quad (126)$$

(where we underline that this is an equality of two functions of λ , over its entire range, as well as the relevant Cartan charges (53), (76), and possibly higher conserved charges, analogs of the flat-space oscillation numbers), *i.e.*, in the version we will need it,

$$\left(\begin{array}{c} \text{the spectrum of energies} \\ \text{of non-interacting type IIB superstrings} \\ \text{in the AdS}_5 \times S^5 \text{ background} \end{array} \right) = \left(\begin{array}{c} \text{the spectrum of scaling dimensions} \\ \text{of gauge-invariant single-trace local operators} \\ \text{in planar } \mathcal{N} = 4 \text{ SYM in } \mathbb{R}^4 \end{array} \right). \quad (127)$$

The prime way of testing AdS/CFT will then be to calculate string theory's E and gauge theory's Δ , and to compare the two. This is, however, fundamentally hindered by the strong/weak nature of the duality, discussed above — the perturbative regimes of both theories generically lie opposite to each other. For example, in the 't Hooft-limit version of AdS/CFT (120), one approaches E under the assumption of $\lambda \gg 1$, *i.e.*, through the standard inverse-tension string-theory perturbative series,

$$E = \sqrt{\lambda} \mathcal{E}_0 + \mathcal{E}_1 + \frac{1}{\sqrt{\lambda}} \mathcal{E}_2 + \dots, \quad (128)$$

while Δ is computed in the planar-gauge-theory perturbative sector of $\lambda \ll 1$,

$$\Delta = \Delta_0 + \lambda \Delta_1 + \lambda^2 \Delta_2 + \dots; \quad (129)$$

compare (110). Formula (126) means that these should be strong- λ and weak- λ expansions of one and the same function. A notable exception is given by BPS operators (*e.g.*, chiral primaries and their descendants), whose scaling dimensions are protected and do not receive quantum corrections (14) (and are determined by the R-symmetry), and also, much more non-trivially, by a certain novel double-scaling limit (“BMN”), introduced in the next subsection, in which we move a “little” from BPS (“near-BPS”), and the two theories develop an overlapping perturbative region, thus permitting direct checks; this will then, remarkably, be the first tractable instance of a string/gauge duality. It is eventually taken even farther from BPS (“far-from-BPS”), see subsection I F.

E. Large Angular Momenta and the PP-Wave AdS/CFT

In the previous subsection, we have given an introduction and a heuristic argument in favor of the AdS/CFT correspondence, as well as the relation between the parameters of both theories (105), (106), and the prediction for the equality between the energies of non-interacting strings and the scaling dimensions of gauge-invariant single-trace local operators (127). Due to the strong/weak nature of AdS/CFT, any quantitative test of $E = \Delta$ beyond the BPS sector seems quite hopeless. The idea by Berenstein, Maldacena and Nastase from 2002 [56] was designed to define a new regime where this critical difficulty could be overcome.

1. The Penrose's Limit

The BMN prescription on the string side is based upon taking a certain limit such that the geometry of $\text{AdS}_5 \times S^5$ simplifies enough to allow string quantization. It is founded upon the observation that if in any spacetime one considers a null geodesic, the spacetime close to it becomes a so-called “pp-wave.” Let us begin with defining this notion. A pp-wave is a solution in any theory including gravity representing a gravitational plane-parallel-wave; in a flat background, its most general metric reads

$$ds_{\text{pp-wave}}^2 = -4dx^+ dx^- + K(x^+, x^i) (dx^+)^2 + \sum_{I=2}^D (dx^I)^2, \quad (130)$$

where $x^\pm \equiv (x^0 \pm x^1)/2$ are the light-cone coordinates, x^I the transverse ones, while K is a harmonic function. A pp-wave can be found in our $D = 10$ -dimensional type IIB SUGRA (83)–(85); its harmonic function obeys

$$\partial_I \partial^I K = -32|\omega|^2, \quad \text{being of the general form} \quad K = \sum_{I,J=2}^{10} K_{IJ} x^I x^J, \quad (131)$$

where $\omega = \omega_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$ is a certain 4-form subject to $d\omega = d \star \omega = 0$, and $|\omega|^2 \equiv \omega_{\mu\nu\rho\sigma} \omega^{\mu\nu\rho\sigma}$. Moreover, the self-dual R-R 5-form field strength is given by

$$\tilde{F}_5 = dx^+ \wedge (\omega + \star \omega). \quad (132)$$

This general solution may be shown to preserve 1/2 of the SUSYs (*i.e.*, 16). However, it has been noticed [57–59] that there exists a particular value of K which yields the solution to be maximally supersymmetric; it is then a highly distinguished background for type IIB superstring theory, as there exist only two other backgrounds sharing this property, namely, the flat spacetime and $\text{AdS}_5 \times S^5$. This special value reads

$$K = -\mu^2 \sum_{I=2}^{10} (x^I)^2, \quad \text{for which the non-zero components of the 5-form,} \quad \tilde{F}_{+1234} = \tilde{F}_{+5678} = 4\mu, \quad (133)$$

where μ is a constant (of mass dimension 1; see below).

As announced, a pp-wave metric was discovered by Penrose in 1976 [60] to appear as a geometry seen by a particle moving along a null geodesic of any spacetime. Let us specify this theorem in our setup. To this end, consider a relativistic point particle (a degenerated string) moving along a great circle of S^5 with the speed of light; in the global coordinates (61), (62),

$$\begin{aligned} \rho = 0, & \quad t = \kappa\tau, & \bar{\psi} = 0, & \quad \varphi_1 = 0, & \quad \varphi_2 = 0, \\ \gamma = \frac{\pi}{2}, & \quad \phi_3 = \kappa\tau, & \psi = 0, & \quad \phi_1 = 0, & \quad \phi_2 = 0. \end{aligned} \quad (134)$$

where κ is a constant. It can easily be shown to fulfill the equations of motion stemming from the action (70), and the Virasoro constraints (73). The conserved charges (76) are derived to be

$$E = \sqrt{\lambda}\kappa, \quad S_1 = S_2 = J_1 = J_2 = 0, \quad J_3 = \sqrt{\lambda}\kappa, \quad (135)$$

i.e., in particular, $E = J$. In order to zoom into the neighborhood of this trajectory, the following procedure (the “Penrose limit”) is executed: the light-cone coordinates $\tilde{x}^\pm \equiv (t \pm \phi_3)/2$ are introduced; the variables are rescaled as

$$x^+ \equiv \frac{\tilde{x}^+}{\mu}, \quad x^- \equiv \mu L^2 \tilde{x}^-, \quad r \equiv L\rho, \quad y \equiv L\gamma, \quad (136)$$

where recall that L is the common radius of AdS_5 and S^5 , and remark that μ is used to establish the proper length dimension; finally, the limit $L \rightarrow \infty$ is taken. Substituting this to the $\text{AdS}_5 \times S^5$ metric (65), (66), we find that the prefactor L^2 cancels out, and the resulting metric is finite and precisely equal to the maximally-supersymmetric plane-wave one (130), (133) (plus corrections $\mathcal{O}(1/L^2)$). Hence, we have discovered that the Penrose limit of the maximally-supersymmetric space $\text{AdS}_5 \times S^5$ is the maximally-supersymmetric pp-wave.

2. Free Quantum Strings in the PP-Wave

Now, let us understand what this Penrose procedure means for AdS/CFT. Consider the symmetry generators of translations in the AdS time t and rotations in the angle ϕ_3 (*i.e.*, in the plane x^5, x^6), *i.e.*, $E \leftrightarrow i\partial_t$ and $J \leftrightarrow -i\partial_{\phi_3}$. Via AdS/CFT, they will correspond to the scaling dimension Δ and the R-charge J of the U(1) subgroup of $\text{SO}(6)$ that rotates the scalars Φ_5, Φ_6 . The momenta conjugate to the rescaled light-cone coordinates read then

$$2p^+ = i\partial_{x^-} = \frac{1}{\mu L^2} i(\partial_t - \partial_{\phi_3}) \leftrightarrow \frac{E + J}{\mu L^2}, \quad H_{\text{light-cone}} = 2p^- = \mu i(\partial_t + \partial_{\phi_3}) \leftrightarrow \mu(E - J). \quad (137)$$

If we want to build a string theory in the background of the pp-wave, these light-cone momenta of physical string states need to be kept finite while taking the Penrose limit $L \rightarrow \infty$. To ensure this, one surprisingly finds that there must be

$$E \sim J \sim L^2 \rightarrow \infty, \quad (138)$$

or taking into account the expression for the radius L (100),

$$\begin{aligned} g_s = \text{fixed}, \quad N \rightarrow \infty \quad (i.e., \lambda \rightarrow \infty), \quad \text{such that} \quad J \sim \sqrt{N}, \\ i.e., \quad \lambda' \equiv \frac{\lambda}{J^2} = (\text{the BMN coupling}) = \text{fixed}, \quad (BMN). \\ \text{and} \quad E - J = \text{fixed} \end{aligned} \quad (139)$$

This is the celebrated “BMN limit.” Its breakthrough meaning and a hallmark is that we not only take the ’t Hooft coupling λ to be large (which simplifies the string side, but at the same time takes us into the strongly-coupled non-perturbative sector of the gauge theory), but we also combine it with a large angular momentum in S^5 (R-charge) J in the described fashion. Then, there emerges a new effective coupling λ' , which can be small even when λ is large, thereby providing an expansion parameter on the gauge side. This gives a hope that the two theories may develop an overlapping perturbative regime.

What is very remarkable is that our string theory in the pp-wave background supported by the R-R 5-form (130), (133) can be exactly quantized, in sharp contradistinction to the one in the $\text{AdS}_5 \times S^5$ background. The pertinent Green-Schwarz action was calculated in 2002 by Metsaev [61], and subsequently greatly simplified by using the diff-symmetry to pass to the light-cone gauge, and the kappa-symmetry to remove one half of the fermionic fields [62],

$$X^+(\tau, \sigma) = p^+ \tau, \quad \bar{\gamma}^+ \theta^I = 0, \quad (140)$$

which has led to

$$\begin{aligned} S_{\text{Green-Schwarz light-cone type IIB superstring}} &= \frac{1}{2\pi\alpha'} \int_{-\infty}^{+\infty} d\tau \int_0^1 d\sigma (\mathcal{L}_b + \mathcal{L}_f), \\ \mathcal{L}_b &= \frac{1}{2} \partial_+ X^I \partial_- X^I - \frac{1}{2} m^2 X^I X^I, \quad \mathcal{L}_f = i(\theta^1 \bar{\gamma}^- \partial_+ \theta^1 + \theta^2 \bar{\gamma}^- \partial_- \theta^2 - 2m\theta^1 \bar{\gamma}^- \Pi \theta^2). \end{aligned} \quad (141)$$

Here: $\partial_{\pm} \equiv \partial_{\tau} \pm \partial_{\sigma}$ are the world-sheet light-cone derivatives; for the 32×32 Dirac matrices Γ^{μ} , $\mu = 0, 1, \dots, 9$ ($\text{SO}(1, 9)$ vector indices), we use the chiral representation in terms of the 16×16 (indexed by $\text{SO}(1, 9)$ spinor indices in the chiral representation $\kappa, \kappa' = 1, 2, \dots, 16$) real symmetric matrices,

$$\Gamma^{\mu} = \begin{pmatrix} 0 & \gamma^{\mu} \\ \bar{\gamma}^{\mu} & 0 \end{pmatrix}, \quad \Pi \equiv \gamma^1 \bar{\gamma}^2 \gamma^3 \bar{\gamma}^4, \quad (142)$$

moreover, $X^I(\tau, \sigma)$ are the eight 2-dimensional world-sheet scalars, while $\theta_\kappa^I(\tau, \sigma)$ are the two ($\mathcal{I} = 1, 2$) 10-dimensional real positive-chirality Majorana–Weyl spinors constrained by (140), *i.e.*, from the world-sheet point of view, eight 2-dimensional Majorana spinors. We recognize the above action to govern these eight bosonic and eight fermionic fields so that they are free and massive (of mass $m = p^+ \mu$). Remark also that the limit $\mu \rightarrow 0$ takes us to the flat space.

The quantization, therefore, is straightforward and completely analogous to the flat case, so we will print only the highlights, focusing exclusively on the bosonic sector. First, it is noticed that α' can be removed from the calculations by the rescalings $X^+ \rightarrow X^+$, $X^- \rightarrow 2\pi\alpha' X^-$, $X^I \rightarrow \sqrt{2\pi\alpha'} X^I$, $\theta^I \rightarrow \sqrt{2\pi\alpha'} \theta^I$; then at the end, it is restored by $p^+ \rightarrow 2\pi\alpha' p^+$. The equations of motion ($(\partial_\tau^2 - \partial_\sigma^2 + m^2)X^I = 0$), with periodic b.c., $\sigma \sim \sigma + 1$, are easily solved in terms of an infinite number of harmonic oscillators,

$$X^I(\tau, \sigma) = \cos(m\tau)X_0^I + \frac{1}{m} \sin(m\tau)P_0^I + i \sum_{n \neq 0} \frac{1}{\omega_n} (\varphi_n^1(\tau, \sigma) \alpha_n^{1,I} + \varphi_n^2(\tau, \sigma) \alpha_n^{2,I}), \quad (143)$$

where $\varphi_n^{1,2}(\tau, \sigma) \equiv \exp(-i(\omega_n \tau \mp k_n \sigma))$, with $\omega_n \equiv \text{sign}(n) \sqrt{k_n^2 + m^2}$, $k_n \equiv 2\pi n$. The canonically conjugate momentum follows easily through $P^I(\tau, \sigma) = \partial_\tau X^I(\tau, \sigma)$. The coordinate $X^-(\tau, \sigma)$ is determined from (143) through the Virasoro constraint,

$$P^+ \partial_\sigma X^- + P^I \partial_\sigma X^I + i(\theta^1 \bar{\gamma}^- \partial_\sigma \theta^1 + \theta^2 \bar{\gamma}^- \partial_\sigma \theta^2) = 0, \quad (144)$$

which additionally imposes a restriction on the transverse coordinates by integrating it over σ and using that P^+ is a constant due to (140),

$$\int_0^1 d\sigma (P^I \partial_\sigma X^I + i(\theta^1 \bar{\gamma}^- \partial_\sigma \theta^1 + \theta^2 \bar{\gamma}^- \partial_\sigma \theta^2)) = 0. \quad (145)$$

This system is readily quantized by replacing the classical Poisson brackets with i times quantum commutators,

$$[P_0^I, X_0^J] = -i\delta^{IJ}, \quad [\alpha_m^{\mathcal{I},I}, \alpha_n^{\mathcal{J},J}] = \frac{1}{2} \omega_m \delta_{m+n,0} \delta^{\mathcal{I}\mathcal{J}} \delta^{IJ}, \quad (146)$$

where it is also useful to slightly modify these creation/annihilation operators,

$$a_0^I \equiv \frac{1}{\sqrt{2m}} (P_0^I + imX_0^I), \quad \bar{a}_0^I \equiv \frac{1}{\sqrt{2m}} (P_0^I - imX_0^I), \quad a_{n \geq 1}^{\mathcal{I},I} \equiv \sqrt{\frac{2}{\omega_n}} \alpha_{-n}^{\mathcal{I},I}, \quad \bar{a}_{n \geq 1}^{\mathcal{I},I} \equiv \sqrt{\frac{2}{\omega_n}} \alpha_n^{\mathcal{I},I}. \quad (147)$$

(There are also their fermionic counterparts, $\theta_0, \bar{\theta}_0, \eta_n^{\mathcal{I}}, \bar{\eta}_n^{\mathcal{I}}$.) The light-cone Hamiltonian,

$$H_{\text{light-cone}} = -P^- = \frac{1}{p^+} \int_0^1 d\sigma \left(\frac{1}{2} (P^I P^I + \partial_\sigma X^I \partial_\sigma X^I + m^2 X^I X^I) + i(\theta^1 \bar{\gamma}^- \partial_\tau \theta^1 + \theta^2 \bar{\gamma}^- \partial_\tau \theta^2) \right), \quad (148)$$

upon substituting the oscillators, and restoring the presence of α' , acquires the form

$$\frac{H_{\text{light-cone}}}{\mu} = a_0^I \bar{a}_0^I + 2\bar{\theta}_0 \bar{\gamma}^- \Pi \theta_0 + 4 + \sum_{\mathcal{I}=1,2} \sum_{n \geq 1} \sqrt{1 + \frac{n^2}{(\mu\alpha'p^+)^2}} (a_n^{\mathcal{I},I} \bar{a}_n^{\mathcal{I},I} + \eta_n^{\mathcal{I}} \bar{\gamma}^- \bar{\eta}_n^{\mathcal{I}}). \quad (149)$$

Remark one difference w.r.t. the flat case (which is achieved by sending $\mu \rightarrow 0$), namely, that also the zero-modes are massive and described by harmonic oscillators; this implies that there are no asymptotically free transverse excitations in the theory, but all are bound in a harmonic well; consequently, the notion of the S-matrix becomes problematic. Finally, the Fock vacuum $|0, p^+\rangle$ is defined as being annihilated by all the annihilation operators (also the zero-modes),

$$\bar{a}_0^I |0, p^+\rangle = 0, \quad \bar{\theta}_0^{\kappa} |0, p^+\rangle = 0, \quad \bar{a}_n^{\mathcal{I},I} |0, p^+\rangle = 0, \quad \bar{\eta}_n^{\mathcal{I},\kappa} |0, p^+\rangle = 0, \quad \text{for } n \geq 1, \quad (150)$$

and the physical Fock states are constructed from it by applying the creation operators in a way to comply with the Virasoro constraints stemming from (145),

$$(N^1 - N^2) |\text{physical state}\rangle = 0, \quad \text{where} \quad N^{\mathcal{I}} = \sum_{n \geq 1} k_n (a_n^{\mathcal{I},I} \bar{a}_n^{\mathcal{I},I} + \eta_n^{\mathcal{I}} \bar{\gamma}^- \bar{\eta}_n^{\mathcal{I}}). \quad (151)$$

Their energies,

$$E - J = \frac{E_{\text{light-cone}}}{\mu} = N_0 + \sum_{n \geq 1} (N_n^1 + N_n^2) \sqrt{1 + \frac{n^2}{(\mu\alpha'p^+)^2}}, \quad (152)$$

which is the famous “BMN square-root formula.” For example, the zero-mode (SUGRA) excitations are

$$\begin{aligned} \bar{a}_0^I |0, p^+\rangle, \quad \frac{E_{\text{light-cone}}}{\mu} &= 1, \\ \bar{\theta}_0 |0, p^+\rangle, \quad \frac{E_{\text{light-cone}}}{\mu} &= 1, \\ \bar{a}_0^{I_1} \bar{a}_0^{I_2} \dots \bar{a}_0^{I_N} |0, p^+\rangle, \quad \frac{E_{\text{light-cone}}}{\mu} &= N, \end{aligned} \quad (153)$$

etc., while the lowest truly “stringy” mode reads

$$\bar{a}_n^{1,I} \bar{a}_n^{2,J} |0, p^+\rangle, \quad \frac{E_{\text{light-cone}}}{\mu} = 2 \sqrt{1 + \frac{n^2}{(\mu\alpha'p^+)^2}}. \quad (154)$$

3. The Operators Dual to the PP-Wave Strings

We have thus managed to determine the spectrum of non-interacting strings in the pp-wave background, which is the Penrose limit of $\text{AdS}_5 \times S^5$. Now, a crucial insight is that there must exist an appropriate limit of $\mathcal{N} = 4$ SYM corresponding to the pp-wave strings through AdS/CFT. Before we discuss it, let us note that (152) astonishingly constitutes a prediction for the all-loop (!) scaling dimensions of the operators dual to the pp-wave string states; translating the string-side quantities into their gauge-side counterparts, $p^+ = (E + J)/(2\mu L^2) \approx J/(\mu L^2) = J/(\mu\alpha'\sqrt{\lambda})$, see (137), (106), this prediction reads

$$\Delta - J = N_0 + \sum_{n \geq 1} (N_n^1 + N_n^2) \sqrt{1 + \lambda' n^2}, \quad (155)$$

where the BMN coupling constant λ' (139) has emerged. What remains, however, is to find what is really meant by the “gauge side” here. What is the precise form of the operators whose scaling dimensions are given by (155)? And why does the matching hold at all even though the limits on both sides are taken in the opposite order,

$$\text{first } J \rightarrow \infty, \text{ second } \lambda' = \lambda/J^2 \rightarrow 0 \quad (\text{string side}) \quad \stackrel{(?)}{=} \quad \text{first } \lambda \rightarrow 0, \text{ second } J \rightarrow \infty \quad (\text{gauge side}), \quad (156)$$

and, thus, may not commute [63, 64, 66].

Let us make at this point the following comment: It has been discovered [67, 68] that $\lambda \gg 1$ implies that the 't Hooft expansion (7) on the gauge side will preserve graphs of all genera, and not only planar ones as might seem from having N large; the reason is that the suppression of non-planar diagrams with large N is compensated with their combinatorial abundance as J grows to infinity. Remarkably, this balance is achieved precisely for $J^2 \sim N$, for which also the graphs assemble again into a topology expansion, but with a new genus-counting parameter $g_2 \equiv J^2/N$; any deviation from this double-scaling would trigger dominance of either planar or non-planar diagrams. The corresponding statement on the string side is that there will appear a sector of interacting strings in addition to non-interacting ones; these string interactions stemming from higher-genus world-sheets can then be approached with techniques of string field theory (SFT) [69] adapted to the light-cone pp-wave setting [70–75]. In this subsection, however, we focus only on the simplest planar gauge theory/free strings regime. Another comment is that $\lambda \gg 1$ puts us in a non-perturbative region of the gauge theory, which seems to exclude any tractability of the problem. However, aid comes from recalling that there exists a class of operators which do not receive quantum corrections to their scaling dimensions (14), namely, those belonging to shortened BPS multiplets, such as the half-BPS ones (55), (56). The critical idea is then to construct the operators dual to the pp-wave strings in such a way that they differ only a “little” from being half-BPS, namely, by so-called “impurities” (“magnons”). For such operators, it is then shown that they do receive quantum corrections, yet not through λ , but through the BMN coupling constant λ' , which can stay small even for large λ , only provided that J is sufficiently large. We will explain this below.

	\mathcal{Y}	\mathcal{Y}	$\Phi_{1,2,3,4}$	A_μ	Ψ_K	$\bar{\Psi}_{\bar{K}}$
Δ_{bare}	1	1	1	1	3/2	3/2
J_3	1	-1	0	0	1/2	-1/2
$\Delta_{\text{bare}} - J_3$	0	2	1	1	1	2

TABLE I: The classical scaling dimensions and charges under the R-rotation corresponding to the charge J_3 for the $\mathcal{N} = 4$ SYM supermultiplet. The gluino Ψ splits into 8 components with charge $+1/2$, and 8 with $-1/2$ ($K, \bar{K} = 1, 2, \dots, 8$).

To address the problem of the dual operators, we begin with an already announced observation that the angular momentum $J = J_3$ of our string is the Cartan charge under the symmetry of rotations in the angle ϕ_3 , which means in the plane x^5, x^6 (62), and therefore, AdS/CFT will take it to be the charge associated with R-rotations in the $\mathcal{N} = 4$ SYM's scalars Φ_5, Φ_6 . More generally, if the string's motion is characterized by arbitrary S^5 angular momenta J_1, J_2, J_3 , it is necessary to define the following complex combinations of the six real scalars of the theory,

$$\mathcal{Z} \equiv \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2), \quad \mathcal{W} \equiv \frac{1}{\sqrt{2}}(\Phi_3 + i\Phi_4), \quad \mathcal{Y} \equiv \frac{1}{\sqrt{2}}(\Phi_5 + i\Phi_6), \quad (157)$$

plus their complex conjugations, $\bar{\mathcal{Z}}, \bar{\mathcal{W}}, \bar{\mathcal{Y}}$, as they are charged with charge $+1$ (-1 for the conjugates) under the respective Cartan generators of the R-symmetry group $\text{SO}(6)$. But for simplicity, let us return to the case of just one angular momentum. In order to continue with the analysis, we need to find the classical ($g_{\text{YM}} = 0$) $\text{U}(1)_{J_3}$ -charges of the fundamental fields of $\mathcal{N} = 4$ SYM (15), as well as their classical scaling dimensions; for this, see table I. Knowing these, the guiding principle of constructing the operators dual to the pp-wave strings, *i.e.*, that they should have $\Delta \sim J \gg 1$, is rephrased as the requirement that they are composite operators containing a large number of the \mathcal{Y} fields (they will, thus, be “long”). Moreover, one has to identify $\Delta - J = E_{\text{light-cone}}/\mu$. So, let us find several operators dual to the beginning of the string spectrum (153), (154). The vacuum must have $\Delta - J = 0$, which is realized only by the \mathcal{Y} field, as is clear from table I,

$$|0, p^+\rangle \leftrightarrow \frac{1}{\sqrt{JN^J}} \text{Tr}(\mathcal{Y}^J), \quad (158)$$

where the normalization is picked to reproduce (52). This operator is protected, *i.e.*, $\Delta - J = 0$ remains true at all values of λ . Further, the SUGRA excitations (153), created by the $8 + 8$ bosonic/fermionic zero-modes, have $\Delta - J = 1$, and therefore, their gauge-side duals will arise by inserting into the string of the \mathcal{Y} 's in the vacuum (158) the fields having $\Delta - J = 1$; table I reveals that these are: 4 scalars Φ_i , $i = 1, 2, 3, 4$, 4 gluons A_μ (written as gauge-covariant derivatives $D_\mu \mathcal{Y} = \partial_\mu \mathcal{Y} + [A_\mu, \mathcal{Y}]$ in order to yield a gauge-invariant operator), and 8 gluinos Ψ_K . Consequently,

$$\begin{aligned} \bar{a}_0^i |0, p^+\rangle &\leftrightarrow \frac{1}{\sqrt{N^J}} \text{Tr}(\Phi_i \mathcal{Y}^J), \\ \bar{a}_0^\mu |0, p^+\rangle &\leftrightarrow \frac{1}{\sqrt{N^J}} \text{Tr}(D_\mu \mathcal{Y} \mathcal{Y}^{J-1}), \\ \bar{\theta}_0^K |0, p^+\rangle &\leftrightarrow \frac{1}{\sqrt{N^J}} \text{Tr}(\Psi_K \mathcal{Y}^J), \end{aligned} \quad (159)$$

and analogously for higher zero-modes, for example,

$$\bar{a}_0^i \bar{a}_0^j |0, p^+\rangle \leftrightarrow \frac{1}{\sqrt{JN^J}} \sum_{l=0}^J \text{Tr}(\Phi_i \mathcal{Y}^l \Phi_j \mathcal{Y}^{J-l}), \quad (160)$$

Again, there is no problem with determining their scaling dimensions at any given λ , as they (159) are protected, being a conformal primary, a descendant of (158), and a superdescendant of (158), respectively [77]. An interesting story commences, however, beyond the SUGRA regime, as the dual operators cease to be protected. Consider the simplest “stringy” excitation (154), with $I = i, J = j$. Its dual operator \mathcal{O}_n^{ij} must, therefore, have two scalar impurities, Φ_i and Φ_j , inserted into the vacuum (158),

$$\bar{a}_n^{1,i} \bar{a}_n^{2,j} |0, p^+\rangle \leftrightarrow \mathcal{O}_n^{ij} = \frac{1}{\sqrt{JN^J}} \sum_{l=0}^J \text{Tr}(\Phi_i \mathcal{Y}^l \Phi_j \mathcal{Y}^{J-l}) f(n, l); \quad (161)$$

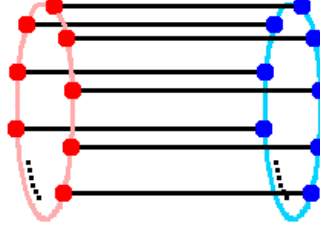


FIG. 9: The planar contribution to the tree-level 2-point correlation function of operators with two scalar impurities (not shown explicitly). The red ring represents the operator at the point x , the blue one at 0.

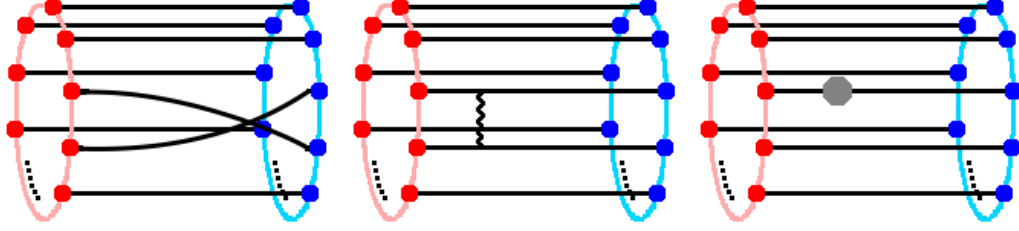


FIG. 10: The planar contributions to the one-loop 2-point correlation function of operators with two scalar impurities: the 4-vertex (LEFT), the gluon exchange (MIDDLE), the self-energy (RIGHT).

we have written it as a linear combination, with some coefficients $f(n, l)$ (obeying $f(0, l) = 1$, so to get back (160)), to ensure that the resulting operator has a definite scaling dimension, *i.e.*, that its 2-point correlation function is of the form (52). The basic task is obviously to calculate this operator's scaling dimension Δ , and compare with the exact prediction (155). The original BMN paper [56] has done it with one-loop (in λ') accuracy, and in the planar limit (we repeat it below (190), proving also that $f(n, l) = \exp(2\pi i n l / J)$), finding perfect agreement with the appropriately expanded string result,

$$\Delta - J = 2 + \lambda' n^2 + \mathcal{O}((\lambda')^2); \quad (162)$$

subsequently, [78] extended the matching to two loops, while [79–81] to three loops, also for a greater (though still small compared to J) number of impurities; even the full square-root formula has been argued [82] to be reproduced on the gauge side under the so-called “dilute-gas approximation.” However, it has then been shown that this approximation can no longer hold due to the presence of the so-called “dressing phase,” which we discuss below, thus explicitly breaking the BMN scaling (this has already been noticed in [83] in the setting of the pp-wave matrix model).

4. The Diagrammatic Approach to the Dilatation Operator

A method of computing scaling dimensions of such “BMN operators” was developed in 2002 [67, 68, 79, 84, 85], and we will briefly review its main points, specifically sticking to the two-scalar-impurity operator $\mathcal{O}_l^{(J)}(x) \equiv \text{Tr}(\Phi_i(x) \mathcal{Y}(x)^l \Phi_j(x) \mathcal{Y}(x)^{J-l})$, compare (161). We aim at calculating the 2-point correlation function at the planar level and up to one quantum loop,

$$\langle \mathcal{O}_{l_1}^{(J)}(x) \bar{\mathcal{O}}_{l_2}^{(J)}(0) \rangle = \left\langle \mathcal{O}_{l_1}^{(J)}(x) \bar{\mathcal{O}}_{l_2}^{(J)}(0) \right\rangle \Big|_{\text{tree-level planar}} + \left\langle \mathcal{O}_{l_1}^{(J)}(x) \bar{\mathcal{O}}_{l_2}^{(J)}(0) \right\rangle \Big|_{\text{one-loop planar}} + \dots \quad (163)$$

This problem is handled diagrammatically, with help of the Feynman rules derived from the $\mathcal{N} = 4$ SYM action (17) in the Feynman gauge; in particular, the scalar and gluon propagators,

$$\langle (\Phi_i(x))_{AB} (\Phi_j(0))_{CD} \rangle_{\text{free-field}} = \frac{g_{\text{YM}}^2}{8\pi^2 |x|^2} \delta_{ij} \delta_{AD} \delta_{BC}, \quad \langle (A_\mu(x))_{AB} (A_\nu(0))_{CD} \rangle_{\text{free-field}} = \frac{g_{\text{YM}}^2}{8\pi^2 |x|^2} \delta_{\mu\nu} \delta_{AD} \delta_{BC}. \quad (164)$$

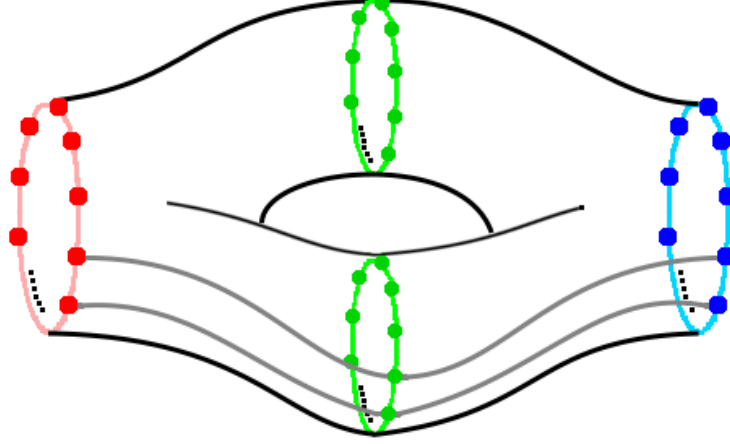


FIG. 11: The toroidal correction to the one-loop 2-point correlation function of operators with two scalar impurities (denoted by gray lines). There is an overlap with double-trace operators of the form (173) (green).

The two composite operators inside the correlator are represented by two rings of dots, corresponding to the positions x and 0 , with lines joining the dots denoting contractions; see figures 9, 10.

A general way of tackling this problem is to encode all the combinatorics in an effective random matrix model, while all the dependence on g_{YM} and x is factored out. To keep track in this zero-dimensional model of the positions x and 0 , we introduce two zero-dimensional fields, $\Phi_i(x) \rightarrow \Phi_i^+$ and $\Phi_i(0) \rightarrow \Phi_i^-$; it is not necessary to repeat this for \mathcal{Y} , as at x there is always $\mathcal{Y}(x)$, while at 0 its conjugate $\mathcal{Y}(0)$. The pertinent matrix model should be complex Gaussian,

$$\langle \mathcal{O} \rangle_{\text{m.m.}} \equiv \int d\mathcal{Y} d\bar{\mathcal{Y}} e^{-\text{Tr}(\mathcal{Y}\bar{\mathcal{Y}})} \mathcal{O}, \quad d\mathcal{Y} d\bar{\mathcal{Y}} \equiv \prod_{A,B=1}^N \frac{1}{\pi} d\text{Re}\mathcal{Y}_{AB} d\text{Im}\mathcal{Y}_{AB}, \quad (165)$$

as its non-zero propagators, $\langle \mathcal{Y}_{AB} \bar{\mathcal{Y}}_{CD} \rangle_{\text{m.m.}} = \delta_{AD} \delta_{BC}$, $\langle (\Phi_i^-)_{AB} (\Phi_j^+)_{CD} \rangle_{\text{m.m.}} = \delta_{ij} \delta_{AD} \delta_{BC}$, correctly reflect the system's combinatorics.

To attack the classical level, we note that we have here just $(J+2)$ propagators, and hence,

$$\left\langle \mathcal{O}_{l_1}^{(J)}(x) \bar{\mathcal{O}}_{l_2}^{(J)}(0) \right\rangle_{\text{tree-level}} = \left(\frac{g_{\text{YM}}^2}{8\pi^2 |x|^2} \right)^{J+2} S_{l_1 l_2}, \quad (166)$$

where the matrix-model correlator is given by

$$S_{l_1 l_2} \equiv \left\langle \mathcal{O}_{l_1}^{(J)} \bar{\mathcal{O}}_{l_2}^{(J)} \right\rangle_{\text{m.m.}} = \left\langle \text{Tr}(\Phi_i^+ \mathcal{Y}^{l_1} \Phi_j^+ \mathcal{Y}^{J-l_1}) \text{Tr}(\Phi_i^- \bar{\mathcal{Y}}^{l_2} \Phi_j^- \bar{\mathcal{Y}}^{J-l_2}) \right\rangle_{\text{m.m.}}, \quad (167)$$

and for example, up to graphs of genus 1, it is derived to be [67, 68],

$$\begin{aligned} S_{l_1 l_2} = & N^{J+2} \delta^{l_1 l_2} + \\ & + N^J \left(\delta^{l_1 l_2} \left(\binom{J-l_1+2}{4} + \binom{l_1+2}{4} \right) + \right. \\ & \left. + \frac{1}{6} l_1 (l_1+1) (3J+1-l_1-3l_2) + (l_2-l_1) (l_1+1) (J-l_2+1) \right) + \mathcal{O}(N^{J-2}), \end{aligned} \quad (168)$$

while higher genera have been addressed in [86]. In particular, at the planar order, the only possible set of contractions is shown in figure 9, from which its diagonal structure in l_1, l_2 is clearly visible; it becomes non-diagonal only after including non-planar diagrams, since then crossing of propagators is allowed. This proportionality to $\delta^{l_1 l_2}$ is an important property, since it means that the classical planar scaling dimension can be readily retrieved (to be $(J+2)$) from (166) according to (52).

At the one-loop order, there are three types of Feynman graphs that contribute to the 2-point correlation function: the scalar 4-vertex, the gluon exchange, and the scalar self-energy; in figure 10, they are shown in the planar limit, in which only interactions between nearest-neighbor lines are taken into account. Without going into details, let us stress the main issues. First, these diagrams are divergent. For example, the vertex includes the following integral, which is divergent at $u = 0$ and $u = x$, and so, should be regularized by, say, introducing a cutoff Λ , *i.e.*, $|x - u|, |u| \geq 1/\Lambda$,

$$\int d^4u \left(\frac{g_{\text{YM}}^2}{8\pi^2|x-u|^2} \right)^2 \left(\frac{g_{\text{YM}}^2}{8\pi^2|u|^2} \right)^2 \sim \frac{g_{\text{YM}}^8}{(8\pi^2)^3|x|^4} \log(\Lambda|x|). \quad (169)$$

This divergence is removed by renormalizing the operators $\mathcal{O}_l^{(J)}(x)$; consequently, despite being in an UV-finite theory (recall (38)), composite operators do undergo renormalization. Now, calculating all the three types of graphs, we obtain

$$\left\langle \mathcal{O}_{l_1}^{(J)}(x) \bar{\mathcal{O}}_{l_2}^{(J)}(0) \right\rangle \Big|_{\text{one-loop}} = -2 \left(\frac{g_{\text{YM}}^2}{8\pi^2|x|^2} \right)^{J+2} T_{l_1 l_2} \log(\Lambda|x|), \quad (170)$$

where the combinatorics is reproduced by the matrix-model factor which is proven to be

$$T_{l_1 l_2} \equiv \left\langle \mathcal{O}_{l_1}^{(J)} H \bar{\mathcal{O}}_{l_2}^{(J)} \right\rangle_{\text{m.m.}}, \quad (171)$$

where the effective matrix-model vertex,

$$H \equiv -\frac{g_{\text{YM}}^2}{8\pi^2} : \text{Tr}([\mathcal{Y}, \Phi_i^+] [\bar{\mathcal{Y}}, \Phi_i^-]) :. \quad (172)$$

In [87], this result is generalized to 2-point functions including any type of $\mathcal{N} = 4$ SYM's fields.

The first observation to make is that (170) is not diagonal in l_1, l_2 , not even at the planar level; this can be seen for example from the 4-vertex, which exchanges the positions of \mathcal{Y} and a real scalar (“hopping”). Thus, the operators $\mathcal{O}_l^{(J)}$ need to be diagonalized in order to find their scaling dimensions. Let us now describe how to approach this diagonalization, as it features an important subtlety, the “operator mixing” [84, 88, 89]. Namely, if we want to compute higher-genus corrections to (171), we need to take into account also multi-trace operators, and not only our original single-trace $\mathcal{O}_l^{(J)}$'s. Figure 11 shows, for example, that double-trace operators will appear while calculating the toroidal corrections to $T_{l_1 l_2}$. In other words, our set of operators should be enlarged, and the diagonalization performed in it. It is shown that the following multi-trace operators should be included,

$$\mathcal{O}_l^{(J_0, J_1, \dots, J_M)} \equiv \text{Tr}(\Phi_i \mathcal{Y}^l \Phi_j \mathcal{Y}^{J_0-l}) \text{Tr}(\mathcal{Y}^{J_1}) \dots \text{Tr}(\mathcal{Y}^{J_M}), \quad \text{where} \quad J_0 + J_1 + \dots + J_M = J. \quad (173)$$

Let α, β denote multi-indices characterizing these operators. Then, their 2-point correlator up to one quantum loop is

$$\langle \mathcal{O}_\alpha(x) \bar{\mathcal{O}}_\beta(0) \rangle = \left(\frac{g_{\text{YM}}^2}{8\pi^2|x|^2} \right)^{J+2} \left(S_{\alpha\beta} - 2T_{\alpha\beta} \log(\Lambda|x|) + \mathcal{O}\left((\lambda')^2\right) \right). \quad (174)$$

We aim at diagonalizing it, *i.e.*, finding a new set of operators,

$$\mathcal{O}_\alpha = V_{\alpha\tilde{\alpha}} \tilde{\mathcal{O}}_{\tilde{\alpha}}, \quad (175)$$

such that they possess definite scaling dimensions,

$$\left\langle \tilde{\mathcal{O}}_{\tilde{\alpha}}(x) \bar{\tilde{\mathcal{O}}}_{\tilde{\beta}}(0) \right\rangle = \frac{C_{\tilde{\alpha}} \delta_{\tilde{\alpha}\tilde{\beta}}}{|x|^{2(J+2+\Delta_{\text{an},\tilde{\alpha}})}} = \frac{C_{\tilde{\alpha}} \delta_{\tilde{\alpha}\tilde{\beta}}}{|x|^{2(J+2)}} \left(1 - 2\Delta_{\text{an},\tilde{\alpha}} \log(\Lambda|x|) + \mathcal{O}\left((\lambda')^2\right) \right). \quad (176)$$

Substituting (175) to (176) and comparing with (174) leads to the relations

$$S = V C V^\dagger, \quad T = V C \Delta_{\text{an}} V^\dagger, \quad (177)$$

where we have defined the diagonal matrices, $C_{\tilde{\alpha}\tilde{\beta}} \equiv C_{\tilde{\alpha}} \delta_{\tilde{\alpha}\tilde{\beta}}$ and $\Delta_{\text{an},\tilde{\alpha}\tilde{\beta}} \equiv \Delta_{\text{an},\tilde{\alpha}} \delta_{\tilde{\alpha}\tilde{\beta}}$. Our goal is the matrix of anomalous dimensions, and for this, let us start from focusing on the matrix $T_{\alpha\beta}$ (171). The vertex (172), while acting on our set of operators \mathcal{O}_α , produces in general some linear combination of these operators,

$$H \circ \mathcal{O}_\alpha = H_{\alpha\beta} \mathcal{O}_\beta. \quad (178)$$

The definition of the action “ \circ ” on the l.h.s. is through contracting the fields $\bar{\mathcal{Y}}$ and Φ_i^- from H with the fields \mathcal{Y} and Φ_i^+ from \mathcal{O}_α , according to the fission/fusion rules of $U(N)$,

$$\text{Tr}(\underbrace{\Phi^+ A \Phi^-}_B) = \text{Tr}(A)\text{Tr}(B), \quad \text{Tr}(\underbrace{\Phi^+ A}_B) \text{Tr}(\underbrace{\Phi^- B}_A) = \text{Tr}(AB). \quad (179)$$

Specifically, if H acts on a single-trace operator, these rules can be shown to yield both single- and double-trace operators,

$$\begin{aligned} H \circ \mathcal{O}_l^{(J)} &= \frac{g_{\text{YM}}^2}{4\pi^2} \left(N \left(2\mathcal{O}_l^{(J)} - \mathcal{O}_{l-1}^{(J)} - \mathcal{O}_{l+1}^{(J)} \right) + \right. \\ &\quad \left. + \sum_{l'=1}^{l-1} \left(\mathcal{O}_{l-l'}^{(J-l', l')} - \mathcal{O}_{l-l'-1}^{(J-l', l')} \right) + \sum_{l'=1}^{J-l-1} \left(\mathcal{O}_l^{(J-l', l')} - \mathcal{O}_{l+1}^{(J-l', l')} \right) \right), \end{aligned} \quad (180)$$

where the boundary contributions are irrelevant if the operators are long. While acting on a double-trace operator, single-, double- and triple-trace ones appear,

$$\begin{aligned} H \circ \mathcal{O}_l^{(J_0, J_1)} &= \frac{g_{\text{YM}}^2}{4\pi^2} \left(N \left(2\mathcal{O}_l^{(J_0, J_1)} - \mathcal{O}_{l-1}^{(J_0, J_1)} - \mathcal{O}_{l+1}^{(J_0, J_1)} \right) + \right. \\ &\quad + \sum_{l'=1}^{l-1} \left(\mathcal{O}_{l-l'}^{(J_0-l', J_1, l')} - \mathcal{O}_{l-l'-1}^{(J_0-l', J_1, l')} \right) + \sum_{l'=1}^{J_0-l-1} \left(\mathcal{O}_l^{(J_0-l', J_1, l')} - \mathcal{O}_{l+1}^{(J_0-l', J_1, l')} \right) + \\ &\quad \left. + J_1 \left(\mathcal{O}_{J_1+l}^{(J)} - \mathcal{O}_{J_1+l-1}^{(J)} + \mathcal{O}_l^{(J)} - \mathcal{O}_{l+1}^{(J)} \right) \right). \end{aligned} \quad (181)$$

We could continue this computation, and a general structure would emerge

$$H = H_0 + H_+ + H_-, \quad (182)$$

where H_0 does not change the number of traces, while H_\pm respectively increases/decreases it by one, thus describing on the string-side geometrical interactions of strings, *i.e.*, their splitting/joining. We will investigate some of these issues in more depth in section VII. Knowing, therefore, how to compute the coefficients $H_{\alpha\beta}$, let us see how they translate into $T_{\alpha\beta}$ according to (171),

$$T_{\alpha\beta} = \langle H \circ \mathcal{O}_\alpha \bar{\mathcal{O}}_\beta \rangle_{\text{m.m.}} = H_{\alpha\gamma} \langle \mathcal{O}_\gamma \bar{\mathcal{O}}_\beta \rangle_{\text{m.m.}} = H_{\alpha\gamma} S_{\gamma\beta}. \quad (183)$$

(Note: S and T are Hermitian by construction, hence, H is non-Hermitian, but related to its Hermitian conjugate by a similarity transformation, $H^\dagger = S^{-1} H S$.) Knowing these allows us in turn to obtain the anomalous dimensions. They are most conveniently encoded in terms of the “dilatation operator,” *i.e.*, an operator with eigenvectors $\tilde{\mathcal{O}}_{\tilde{\alpha}}$ and eigenvalues $\Delta_{\tilde{\alpha}}$,

$$\mathcal{D} \circ \tilde{\mathcal{O}}_{\tilde{\alpha}} = \Delta_{\tilde{\alpha}} \tilde{\mathcal{O}}_{\tilde{\alpha}}, \quad \text{i.e., in the original basis,} \quad \mathcal{D}_{\alpha\beta} = (V \Delta V^{-1})_{\alpha\beta}. \quad (184)$$

In perturbation theory,

$$\mathcal{D} = \sum_{n \geq 0} \mathcal{D}^{(n)}, \quad \text{where} \quad \mathcal{D}^{(n)} \sim g_{\text{YM}}^{2n}, \quad (185)$$

we have thus found for our two-scalar-impurity operators, up to one-loop accuracy,

$$\mathcal{D}_{\alpha\beta}^{(0)} = (J+2)\delta_{\alpha\beta}, \quad \mathcal{D}_{\alpha\beta}^{(1)} = H_{\alpha\beta}, \quad (186)$$

since $\mathcal{D}_{\alpha\beta}^{(1)} = (V \Delta_{\text{an}} V^{-1})_{\alpha\beta} = (V C \Delta_{\text{an}} V^\dagger V^{\dagger,-1} C^{-1} V^{-1})_{\alpha\beta} = (T S^{-1})_{\alpha\beta} = H_{\alpha\beta}$. Observe that $S_{\alpha\beta}$ has completely disappeared from the one-loop dilatation operator; the one-loop anomalous dimensions are simply the eigenvalues of H .

This object H is of major importance. First, crucially, we will soon argue that its planar version is in fact identical with the Hamiltonian of an integrable system (specifically, a spin chain, *i.e.*, a 1-dimensional model describing magnetic interactions between spins). This breakthrough discovery reinforces the quest of diagonalizing H with the

entire powerful machinery of integrability. We will discuss H as an integrable Hamiltonian in subsection I G. Second, it has been proposed [90] that it is the dilatation operator which is dual to the SFT Hamiltonian of the interacting pp-wave string theory, by mimicking (152), (126) at the operator level,

$$\frac{H_{\text{SFT}}}{\mu} \stackrel{(\text{AdS/CFT})}{\leftrightarrow} \mathcal{D} - J. \quad (187)$$

Third, there exists a method of diagonalizing H which does not rely on integrability, but on interpreting it as the Hamiltonian of a certain quantum-mechanical system [85]. Namely, we take the limit of large J in the formulae for the action of H (180), (181), which is done by replacing the discrete variables l, l' , *etc.* with continuous ones, and writing the operators as kets; for example, if we are interested only in the planar level,

$$x \equiv \frac{l}{J}, \quad x \in [0, 1], \quad x \sim x + 1, \quad \text{and} \quad \mathcal{O}_l^{(J)} \leftrightarrow |x\rangle. \quad (188)$$

In this language, the planar version of H acts as the quantum-mechanical Hamiltonian

$$H_{\text{QM}}|_{\text{planar}}|x\rangle = -\frac{\lambda'}{4\pi^2}\partial_x^2|x\rangle, \quad (189)$$

and is straightforwardly diagonalized,

$$H_{\text{QM}}|_{\text{planar}}|n\rangle = \lambda'n^2|n\rangle, \quad |n\rangle = \int_0^1 dx e^{2\pi i n x}|x\rangle, \quad \text{where} \quad n = 1, 2, \dots, \quad (190)$$

precisely agreeing with the implication of the all-loop BMN prediction (162). This derivation has been extended to higher genera [85], discussed on the string side [91], generalized to other types of impurities [92, 93], and taken to two and three quantum loops [78–81].

To summarize, we have sketched a certain regime (of large angular momenta, scaled according to (139)) and a certain class of operators (near-BPS, *i.e.*, half-BPS operators, whose protectedness is violated a “little” by inserting a small number of impurities), which allow for testing of matching of string energies with anomalous dimensions (126) beyond the trivial SUGRA/BPS sector. In the next subsection, we will move even farther from BPS by considering operators with a number of impurities of the same order as the large angular momentum, and show that they again lead to tractable regions on both sides of the AdS/CFT correspondence.

F. Spinning Strings

1. Semiclassical Strings and Long Operators

Another revolutionary step on the way of delving into the structure of AdS/CFT was initiated in 2002 by the work of Gubser, Klebanov and Polyakov (GKP) [94]. They noted that on the string side, the BMN limit describes just the semiclassical expansion around the point-like string solution (134) of the full $\text{AdS}_5 \times S^5$ theory. In other words, by considering classical string theory in $\text{AdS}_5 \times S^5$, and computing the sigma-model one-loop (*i.e.*, $\mathcal{O}(\alpha')$) quantum corrections to it — the same free spectrum of quantum strings in the pp-wave background (152) is found. Higher-order quantum-string orders are suppressed [95, 97, 99, 100, 102, 105, 116, 119], *i.e.*, the semiclassical solution becomes quantum-exact. Indeed, expanding

$$X^M(\tau, \sigma) = X_{\text{classical solution (134)}}^M(\tau, \sigma) + \frac{1}{\lambda^{1/4}} x^M(\tau, \sigma), \quad (191)$$

yields the quadratic fluctuation action precisely equal to the action of type IIB superstring theory in the pp-wave background (141). The inverse-tension expansion of the string energy (128) becomes

$$E = \sqrt{\lambda} \mathcal{E}_0 + \mathcal{E}_1 + \frac{1}{\sqrt{\lambda}} \mathcal{E}_2 + \dots, \quad \text{hence,} \quad E - J = \mathcal{E}_1 + \underbrace{\frac{1}{J} \tilde{\mathcal{E}}_2 + \dots}_{\text{suppressed in the BMN limit}}. \quad (192)$$

In the search for sectors of AdS/CFT beyond BMN admitting direct comparison of string energies and scaling dimensions, the above line of thought has been perfected by Frolov and Tseytlin [95, 99, 100, 102]. Namely, they suggested to consider other classical solutions, and to expand around them; analogously, in the BMN limit $J \sim \sqrt{\lambda} \rightarrow \infty$ (139), the quadratic approximation to the fluctuation action would be enough, allowing for finding the exact spectrum. From this point of view, the Penrose limit (the geometry seen by a point particle revolving around a great circle of S^5) has not been a fundamental choice, but just one of many geometry limits corresponding to other classical string configurations. In this way, through semiclassical quantization, one again obtains predictions for the all-loop scaling dimensions of the dual operators; a drawback, however, is that this means working with patches of the theory according to the chosen configuration to expand around. On the other side of the story, we find long (*i.e.*, having a large R-charge J) operators. Unfortunately, unlike for BMN operators, this arises to a real hindrance if we want to appropriate them with a sort of diagrammatic techniques as described below (163), due to their massive mixing in the degeneracy space w.r.t. their bare scaling dimension. The way out is to use the dilatation operator (184), and its amazing property of integrability, which enables its diagonalization through the so-called “Bethe ansatz”; we discuss it below.

In the following, we will very succinctly present some examples of Frolov–Tseytlin strings. But before that, let us elaborate on the general forms of the series we encounter in this sector. On the string side, we first expand in large J , and then in small λ' (156). We will require of our sector that it submits to the “BMN-scaling assumption,” *i.e.*, that the inverse-tension series can in fact be written as a large- J expansion, according to $J \sim \sqrt{\lambda}$ (in other words, quantizing a string means we retreat from the thermodynamic limit of large J), and that at every large- J order, it admits a regular (analytic) expansion in the BMN coupling λ' ,

$$\begin{aligned} E &= \sqrt{\lambda} \mathcal{E}_0 + \mathcal{E}_1 + \frac{1}{\sqrt{\lambda}} \mathcal{E}_2 + \dots = J \tilde{\mathcal{E}}_0 + \tilde{\mathcal{E}}_1 + \frac{1}{J} \tilde{\mathcal{E}}_2 + \dots = \\ &= J \left(1 + \epsilon_{01} \lambda' + \epsilon_{02} (\lambda')^2 + \dots \right) + \left(\epsilon_{11} \lambda' + \epsilon_{12} (\lambda')^2 + \dots \right) + \frac{1}{J} \left(\epsilon_{21} \lambda' + \epsilon_{22} (\lambda')^2 + \dots \right) + \dots = \\ &= J \left(1 + \sum_{k \geq 1} \left(\epsilon_{0k} + \sum_{n \geq 1} \frac{\epsilon_{nk}}{J^n} \right) (\lambda')^k \right). \end{aligned} \quad (193)$$

On the gauge side, first $\lambda \rightarrow 0$ is taken, and subsequently, $J \rightarrow \infty$. In order to be able to compare the scaling dimensions to the string energies, we must also here assume the BMN scaling,

$$\Delta = J \left(1 + \sum_{k \geq 1} \left(\delta_{0k} + \sum_{n \geq 1} \frac{\delta_{nk}}{J^n} \right) (\lambda')^k \right). \quad (194)$$

In the strict BMN limit, all the finite-size corrections obviously disappear, and we are left with comparing ϵ_{0k} with δ_{0k} , $k = 1, 2, \dots$. We will soon see, however, that there has been discovered a mismatch between these coefficients, pointing to a breaking of the BMN scaling. (Let us remark that these expressions does not hold if the large spins are only in AdS_5 — there is no suppression in $1/S_{1,2}$ in this case; at least one large spin in S^5 is needed.)

2. The Rotating String Ansatz

In order to generalize the point-like solution (134), keeping at the same time the BMN-scaling assumption, the following “rotating string ansatz” is picked,

$$\xi_a(\tau, \sigma) = r_a(\sigma)e^{i(w_a\tau + \alpha_a(\sigma))}, \quad \eta_p(\tau, \sigma) = s_p(\sigma)e^{i(\omega_p\tau + \beta_p(\sigma))}, \quad (195)$$

where $a = 1, 2, 3$, $p = 0, 1, 2$, and ξ_a , η_p are related to the global coordinates through (61), (62). The constraints (58), (59) are translated into

$$\delta^{ab}r_ar_b = 1, \quad \eta^{pq}s_ps_q = -1, \quad (196)$$

with $\eta^{pq} \equiv \text{diag}(-, +, +)$. Moreover, the periodicity requires that

$$r_a(\sigma + 2\pi) = r_a(\sigma), \quad s_p(\sigma + 2\pi) = s_p(\sigma), \quad \alpha_a(\sigma + 2\pi) = \alpha_a(\sigma) + 2\pi m_a, \quad \beta_p(\sigma + 2\pi) = \beta_p(\sigma) + 2\pi k_p, \quad (197)$$

where $m_a \in \mathbb{Z}$, $k_p \in \mathbb{Z}$, $k_0 = 0$ are the “winding numbers.” This ansatz describes a rigid string (r_a , s_p do not depend on τ), rotating with frequencies w_a , ω_p . The global conserved charges (76), (78) are derived to be

$$J_a = \sqrt{\lambda}\mathcal{J}_a = \sqrt{\lambda}w_a \int_0^{2\pi} \frac{d\sigma}{2\pi} r_a^2(\sigma), \quad S_p = \sqrt{\lambda}\mathcal{S}_p = \sqrt{\lambda}\omega_p \int_0^{2\pi} \frac{d\sigma}{2\pi} s_p^2(\sigma), \quad (198)$$

with $p = 1, 2$, and the restraints (196) are reflected in them as

$$\frac{\mathcal{J}_1}{w_1} + \frac{\mathcal{J}_2}{w_2} + \frac{\mathcal{J}_3}{w_3} = 1, \quad \frac{\mathcal{E}}{\kappa} - \frac{\mathcal{S}_1}{\omega_1} - \frac{\mathcal{S}_2}{\omega_2} = 1, \quad (199)$$

where $\kappa \equiv \omega_0$. Let us mention that under this ansatz, the classical string theory in $\text{AdS}_5 \times S^5$ can be mapped to an integrable system, the “Neumann–Rosochatius model.” Actually, the entire classical theory is integrable as shown by Bena, Polchinski and Roiban in 2003 [101], see also [96, 104, 106–108, 110, 112, 113, 122, 123, 126, 132, 135].

3. The $\text{SL}(2)$ Sector and the Goal of this Paper

We do not have time to present solutions based on the ansatz (195); let us just very briefly recapitulate one of them, relevant for the current paper.

Let us begin with recalling that there are three particularly interesting subsectors of the full symmetry group $\text{PSU}(2, 2|4)$, namely, $\text{SU}(2)$, $\text{SL}(2)$, and $\text{SU}(1|1)$, for this reason that the dilatation operator acquires there a relatively simple form; these are the subalgebras of rank one. For all of them, the vacuum is the state composed of J fields \mathcal{Z} (157), *i.e.*, $|0\rangle = |\mathcal{Z}\mathcal{Z}\dots\mathcal{Z}\rangle$; the excitations (“magnons”) arise when some \mathcal{Z} ’s are changed into other fields. In the $\text{SU}(2)$ sector, the excitations are the appearances of another complex combination of two real scalars \mathcal{W} among the \mathcal{Z} ’s. In other words, the single-trace local operators read symbolically $\mathcal{O} = \text{Tr}(\mathcal{Z}^{J_1}\mathcal{W}^{J_2} + \text{permutations})$, where J_1 and J_2 denote the R-charges of \mathcal{O} , and $J_1 + J_2 = J$ (see paragraph IG 1). On the string side, these operators correspond to spinning strings moving in the subspace $S^3 \times \mathbb{R}^1$ of the full space $\text{AdS}_5 \times S^5$; such strings have two independent angular momenta, identified with the R-charges J_1 and J_2 of the operators. A very nice feature of this sector is that under renormalization it is closed perturbatively to all orders in the gauge coupling constant, which follows from the R-charge conservation. The other interesting sector, $\text{SL}(2)$, features, on the gauge side, operators of the form

$$\mathcal{O} = \text{Tr}\left(D_+^{S_1}\mathcal{Z}D_+^{S_2}\mathcal{Z}\dots D_+^{S_J}\mathcal{Z} + \text{permutations}\right), \quad \text{where} \quad S_1 + S_2 + \dots + S_J = S. \quad (200)$$

Notice that in this case, unlike $SU(2)$, there can be multiple (*i.e.*, S_1, S_2, \dots, S_J) excitations sitting at a single site. On the string side, we find strings living in the subspace $AdS_3 \times S^1$, having Lorentz spin S and angular momentum J along S^1 corresponding to the R-charge of the operator. This sector is closed under renormalization, too. Let us not go into details concerning the third special subsector, $SU(1|1)$, mentioning only that it includes fermionic excitations, too. In this article, we focus exclusively on $SL(2)$.

The prime instance of the classical string solution with the $SL(2)$ symmetry is the circular string spinning in AdS_3 and rotating in S^1 [155],

$$\begin{aligned} \rho = \text{const}, \quad t = \tilde{m}\tau, \quad \bar{\psi} = \sqrt{\tilde{m}^2 + m^2}\tau + m\sigma, \quad \varphi_1 = 0, \quad \varphi_2 = 0, \\ \gamma = 0, \quad \phi_3 = \sqrt{\tilde{m}^2 + m^2}\tau - w\sigma, \quad \psi = 0, \quad \phi_1 = 0, \quad \phi_2 = 0. \end{aligned} \quad (201)$$

where the various constants here are related through

$$\sinh^2(\rho) = \frac{\mathcal{S}}{\sqrt{\tilde{m}^2 + m^2}}, \quad (202)$$

$$\mathcal{E}_0 = \frac{\tilde{m}\mathcal{S}}{\sqrt{\tilde{m}^2 + m^2}} + \tilde{m}, \quad (203)$$

$$2\tilde{m}\mathcal{E}_0 - \tilde{m}^2 = 2\sqrt{\tilde{m}^2 + m^2}\mathcal{S} + \mathcal{J}^2 + w^2, \quad (204)$$

$$m\mathcal{S} - w\mathcal{J} = 0. \quad (205)$$

This classical sigma-model solution can now be used for semiclassical quantization, which has been done up to one loop in [155]. It shifts the classical energy $E_0 = \sqrt{\lambda}\mathcal{E}_0$ (203) by a value \mathcal{E}_1 , which is then derived. The two-loop semiclassical quantization around (201), and the pertinent shift \mathcal{E}_2 , are yet to be found. Now, these string theory results can also be accessed in a different way, by making use of the general belief that both planar $\mathcal{N} = 4$ SYM theory and non-interacting string theory on $AdS_5 \times S^5$ are integrable at any order in perturbation theory, and thus, can be quantized with aid of the Bethe ansatz technique. Quantum string Bethe ansatz equations [148, 160], which we discuss below, are discretized versions of their classical counterparts, and are approximate in the sense that they are defined for the system of infinite volume with asymptotic states, which is then taken back to be finite. In this process, one encounters vacuum polarization (Casimir) effects affecting the energy. This impact has been found to be exponentially small with large volume [165, 166, 180, 185]. In [163, 185], the method of the quantum string Bethe ansatz has delivered the shift \mathcal{E}_1 , which agreed with the string theory computation. In this paper, we generalize that pursuit to the next order in large volume, namely, we search for \mathcal{E}_2 ; we give it in paragraph VIB 6, which is our main achievement.

Moreover, at any loop, one can subsequently expand the energy shifts in small BMN coupling λ' . This “fast-moving” limit serves to make a comparison with the perturbative gauge theory, even though on that side the limits are taken in the opposite way: first, small λ , then, large J (156). In our Bethe ansatz approach, we expand the energy shifts \mathcal{E}_1 and \mathcal{E}_2 up to respectively five and three orders in $\sqrt{\lambda'}$, in respectively paragraphs VC 2 and VIA 7. The fractional powers of λ' appear exclusively thanks to the so-called Hernández–López contribution to the string dressing phase (262).

G. Integrability

In this subsection, written unfortunately much too briefly, we introduce the concept of the Bethe ansatz, which employs integrability to solve the otherwise untractable mixing problem for long operators, corresponding for example to spinning strings.

1. The One-Loop Planar Dilatation Operator in the SU(2) Sector

In their seminal paper from 2002, Minahan and Zarembo [138] computed the one-loop planar anomalous dimensions of the single-trace operators belonging to the SO(6) subsector of the full theory, *i.e.*, built out of the six scalars (15),

$$\mathcal{O}_{i_1, i_2, \dots, i_J}(x) \equiv \text{Tr}(\Phi_{i_1}(x)\Phi_{i_2}(x)\dots\Phi_{i_J}(x)), \quad (206)$$

where the length J is large. The derivation resembles very much the one in the 2-impurity case, discussed in paragraph IE 4, and is again based on calculating the three types of Feynman graphs shown in figure 10. It eventually yields

$$\mathcal{D}^{(1)}\Big|_{\text{SO}(6)} = \frac{\lambda}{16\pi^2} \sum_{l=1}^J H_{l,l+1}, \quad H_{l,l+1} \equiv 2I_{l,l+1} + K_{l,l+1} - 2P_{l,l+1}, \quad (207)$$

where the three operators comprising the result act in the space of the indices of the two scalar insertions, at the positions l and $(l+1)$, which is $(\mathbb{R}^6)_l \otimes (\mathbb{R}^6)_{l+1}$, and are respectively the identity, trace, and permutation operators,

$$I_{l,l+1} \equiv \delta_{i_l, j_l} \delta_{i_{l+1}, j_{l+1}}, \quad K_{l,l+1} \equiv \delta_{i_l, i_{l+1}} \delta_{j_l, j_{l+1}}, \quad P_{l,l+1} \equiv \delta_{i_l, j_{l+1}} \delta_{i_{l+1}, j_l}. \quad (208)$$

Now, it is critically noticed that this dilatation operator is exactly equivalent to the Hamiltonian of an integrable system, the SO(6) “spin chain.” This discovery has initiated a whole branch of research, and has given a powerful tool of delving into the intricacies of $\mathcal{N} = 4$ SYM, as well as of the other side of the AdS/CFT duality.

The “Bethe ansatz” [136, 137] is a technique of diagonalizing integrable Hamiltonians. We will sketch how it works on an example of an even narrower closed subsector of the full theory, namely SU(2), and still at one quantum loop. This sector consists of single-trace operators constructed from the two complex scalars, \mathcal{Z} and \mathcal{W} (157); schematically,

$$\mathcal{O} = \text{Tr}(\mathcal{Z}^{J-M}\mathcal{W}^M) + \text{permutations}, \quad (209)$$

where $M \sim J \rightarrow \infty$, *i.e.*, we have a macroscopic number of impurities M . The one-loop planar dilatation operator (207) does not have the trace operator in this case, and so, simplifies to

$$\mathcal{D}^{(1)}\Big|_{\text{SU}(2)} = \frac{\lambda}{8\pi^2} \sum_{l=1}^J (I_{l,l+1} - P_{l,l+1}). \quad (210)$$

Now, this is precisely the Hamiltonian of the integrable ferromagnetic Heisenberg spin chain XXX_{1/2} upon the identification of the \mathcal{Z} fields with spins up, \uparrow , and the \mathcal{W} fields with spins down, \downarrow , as depicted in figure 12,

$$\mathcal{Z} \leftrightarrow \uparrow, \quad \mathcal{W} \leftrightarrow \downarrow. \quad (211)$$

We see moreover that the ferromagnetic vacuum $|0\rangle \equiv |\uparrow\uparrow\dots\uparrow\rangle$ (all spins up) corresponds to the half-BPS operator $\text{Tr}(\mathcal{Z}^J)$ (compare (158)), and insertions of \mathcal{W} represent excitations (“magnons”) over this vacuum.

The Hamiltonian of this spin chain is a $2^J \times 2^J$ matrix H , whose diagonalization for large J is obviously untractable. Here it is where integrability comes into play. Indeed, let there be M impurities, at positions $l_1 < l_2 < \dots < l_M$; such a state, in which we additionally relax the periodicity requirement, we will denote by $|l_1, l_2, \dots, l_M\rangle$. Some linear combination of these M -impurity states will be an eigenstate of our Hamiltonian,

$$H|M\rangle = E_M|M\rangle, \quad \text{where} \quad |M\rangle \equiv \sum_{l_1, l_2, \dots, l_M=1}^J \psi_{l_1, l_2, \dots, l_M} |l_1, l_2, \dots, l_M\rangle. \quad (212)$$

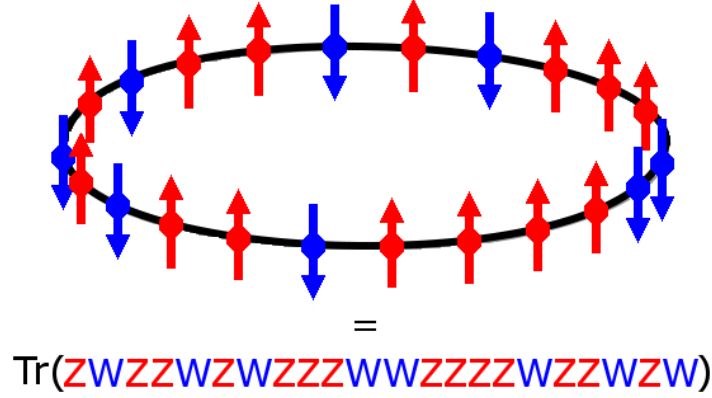


FIG. 12: The mapping of an SU(2) operator (209) to a spin-(1/2) spin chain, according to (211). The dilatation operator (210) is the Hamiltonian of this new system, and it is integrable.

Let us start from solving this eigenequation for $M = 1$; using the plane-wave ansatz, we find

$$H|p\rangle = E_1(p)|p\rangle, \quad \text{where} \quad |p\rangle = \sum_{l=1}^J \psi(l)|l\rangle, \quad \psi(l) = \frac{1}{\sqrt{J}} e^{ipl}, \quad E_1(p) = \frac{\lambda}{2\pi^2} \sin^2\left(\frac{p}{2}\right), \quad (213)$$

as

$$\begin{aligned} H|p\rangle &= \frac{\lambda}{8\pi^2} \sum_{l=1}^J (2\psi(l)|l\rangle - \psi(l)|l-1\rangle - \psi(l)|l+1\rangle) = \frac{\lambda}{8\pi^2} \sum_{l=1}^J (2\psi(l) - \psi(l+1) - \psi(l-1)) |l\rangle = \\ &= \frac{\lambda}{8\pi^2} (2 - e^{-ip} - e^{ip}) \sum_{l=1}^J \psi(l)|l\rangle = \frac{\lambda}{2\pi^2} \sin^2\left(\frac{p}{2}\right) |p\rangle. \end{aligned}$$

Furthermore, restoring the periodicity condition $l \sim l + J$ implies momentum quantization, $p = 2\pi n/J$, with $n \in \mathbb{Z}$.

Proceeding to eigenstates with two magnons, (212) yields two equations for the wave-function,

$$\begin{aligned} 4\psi(l_1, l_2) - \psi(l_1 - 1, l_2) - \psi(l_1 + 1, l_2) - \psi(l_1, l_2 - 1) - \psi(l_1, l_2 + 1) &= E_2 \psi(l_1, l_2), \quad \text{for } l_1 + 1 < l_2, \\ 2\psi(l_1, l_2) - \psi(l_1 - 1, l_2) - \psi(l_1, l_2 + 1) &= E_2 \psi(l_1, l_2), \quad \text{for } l_1 + 1 = l_2, \end{aligned} \quad (214)$$

as well as the consistency condition stemming from them,

$$2\psi(l_1, l_1 + 1) - \psi(l_1, l_1) - \psi(l_1 - 1, l_1) = 0. \quad (215)$$

We approach them again with the ansatz representing two plane waves,

$$\psi(l_1, l_2) = A_0(p_1, p_2) e^{i(p_1 x_1 + p_2 x_2)} + A_0(p_2, p_1) e^{i(p_2 x_1 + p_1 x_2)}, \quad (216)$$

which indeed solves (214) provided that the energy is

$$E_2(p_1, p_2) = E_1(p_1) + E_1(p_2) = \frac{\lambda}{2\pi^2} \left(\sin^2\left(\frac{p_1}{2}\right) + \sin^2\left(\frac{p_2}{2}\right) \right), \quad (217)$$

and the coefficients of the ansatz obey

$$S(p_1, p_2) \equiv \frac{A_0(p_2, p_1)}{A_0(p_1, p_2)} = -\frac{e^{i(p_1 + p_2)} - 2e^{ip_1} + 1}{e^{i(p_1 + p_2)} - 2e^{ip_2} + 1}; \quad (218)$$

this object is called an “S-matrix.” At the end, the periodicity condition is imposed, which can be stated as the following restraint on the two momenta,

$$e^{ip_1 J} = S(p_1, p_2), \quad e^{ip_2 J} = S(p_2, p_1); \quad (219)$$

these are the “Bethe ansatz equations (BAE)”; once solved, they will provide the diagonalization of our Hamiltonian in the 2-magnon sector. However, they should be supplemented by the so-called “zero-momentum condition,”

$$p_1 + p_2 = 0, \quad (220)$$

which stems from the cyclicity condition of the underlying operator.

In general, if we wanted to continue this procedure to higher values of M , it would quickly become too involved. However, the power of integrability is that scatterings of any number of magnons can be reduced to 2-body processes only, *i.e.*, the M -magnon S-matrix is “factorized” into 2-magnon ones. This implies that an arbitrary M -magnon diagonalization problem (212) is solved by the BAE,

$$e^{ip_k J} = \prod_{\substack{j=1 \\ j \neq k}}^M S(p_k, p_j), \quad \text{for all } k = 1, 2, \dots, M \quad (\text{the Bethe ansatz equations}), \quad (221)$$

where S is the 2-body S-matrix (218). In addition to this, we have the zero-momentum condition,

$$\sum_{k=1}^M p_k = 0 \quad (\text{the zero-momentum condition}). \quad (222)$$

Once they are solved for the magnons’ momenta (complex in general), the spectrum of energies is given by

$$E_M(p_1, p_2, \dots, p_M) = \frac{\lambda}{2\pi^2} \sum_{k=1}^M \sin^2\left(\frac{p_k}{2}\right) \quad (\text{the spectrum of the integrable Hamiltonian}), \quad (223)$$

in which way we have finally completed the diagonalization task. Let us also mention that there is another useful language, instead of the momenta, to rewrite (221)–(223), namely, the “rapidities,”

$$u_k \equiv \frac{1}{2} \cot\left(\frac{p_k}{2}\right), \quad (224)$$

which translate the above into

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^J = \prod_{\substack{j=1 \\ j \neq k}}^M \frac{u_k - u_j + i}{u_k - u_j - i}, \quad \text{for all } k = 1, 2, \dots, M \quad (\text{the Bethe ansatz equations}), \quad (225)$$

$$\prod_{k=1}^M \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} = 1 \quad (\text{the zero-momentum condition}), \quad (226)$$

$$E_M(u_1, u_2, \dots, u_M) = \frac{\lambda}{8\pi^2} \sum_{k=1}^M \frac{1}{u_k^2 + \frac{1}{4}} \quad (\text{the spectrum of the integrable Hamiltonian}). \quad (227)$$

2. The BDS Spin Chain

The one-loop planar dilatation operator for the $SU(2)$ sector has subsequently been generalized to higher orders in perturbation theory. Namely, assuming that integrability persists beyond one loop, [79] proposed the following expansion (185),

$$\mathcal{D} = \sum_{n \geq 0} g^{2n} H^{(n)}, \quad \text{with} \quad H^{(n)} = \sum_{l=1}^J H_l^{(n)}, \quad (228)$$

where we have traded the 't Hooft coupling constant λ (5) for a more convenient one, which we will also use in the remaining of this paper,

$$g \equiv \frac{\sqrt{\lambda}}{4\pi}, \quad (229)$$

and where the four leading Hamiltonians read

$$H_l^{(0)} = 1, \quad (230)$$

$$H_l^{(1)} = \frac{1}{2} (1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}), \quad (231)$$

$$H_l^{(2)} = - (1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}) + \frac{1}{4} (1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+2}), \quad (232)$$

$$H_l^{(3)} = \frac{15}{4} (1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}) - \frac{3}{2} (1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+2}) + \frac{1}{4} (1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+3}) - \frac{1}{8} (1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+3}) (1 - \vec{\sigma}_{l+1} \cdot \vec{\sigma}_{l+2}) + \frac{1}{8} (1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+2}) (1 - \vec{\sigma}_{l+1} \cdot \vec{\sigma}_{l+3}), \quad (233)$$

where $\vec{\sigma}_l$ are the three Pauli matrices acting on the spin at the position l . These expressions have been proven in [80, 81, 144]. The fourth order has been recently found [192], too. The three-loop integrability has been shown by discovering [64] that the three-loop $SU(2)$ dilatation operator is equivalent to the Hamiltonian of an integrable system called the “Inozemtsev spin chain” [65]. A drawback of this immersion is that the Inozemtsev chain breaks the BMN scaling at the fourth order.

The Bethe ansatz has also been extended to other sectors of the theory, such as $SU(2|3)$ [80], $SU(1|1)$ [153], $SU(1|2)$, $SU(1,1|2)$ and the full theory $PSU(2,2|4)$ [160].

A natural question is whether one could find an integrable spin chain whose Hamiltonian will reproduce the dilatation operator to all loops in gauge-theory perturbation series, at the same time preserving the BMN scaling, unlike the Inozemtsev chain. Such a system was proposed in 2004 by Beisert, Dippel and Staudacher (BDS) [66], and their conjectured all-loop Bethe ansatz reads

$$e^{ip_k J} = \prod_{\substack{j=1 \\ j \neq k}}^M S_{\text{BDS}}(p_k, p_j), \quad \text{where the 2-body S-matrix,} \quad S_{\text{BDS}}(p_k, p_j) = \frac{u_k - u_j - \frac{i}{g}}{u_k - u_j + \frac{i}{g}}, \quad (234)$$

where the rapidity variables are this time defined through

$$u_k = \frac{1}{2g} \cot\left(\frac{p_k}{2}\right) \sqrt{1 + 16g^2 \sin^2\left(\frac{p_k}{2}\right)}. \quad (235)$$

Today we know that this is not an entirely true proposal, nevertheless, a crucial one — it indeed yields the all-loop Bethe ansatz only without the so-called dressing phase, which we will soon introduce. Moreover, it corresponds to a spin chain with long-range interactions, and hence, it is valid only up to the order $g^{2(L-1)}$, which is the moment when its interactions start to wrap around the entire chain (“wrapping interactions”) [166]. Also, let us mention that the BDS spin chain is equivalent to another system known from solid-state physics, namely, the strongly-coupled half-filled 1-dimensional Hubbard model [169–171].

We have so far described two ways of parameterizing the momenta of the excitations: p_k and u_k . But there exists another method, which we will also employ in the current paper — the “spectral parameters” x_k . They are related to each other with aid of the following mapping,

$$\mathbb{C} \ni x \mapsto u(x; g) \equiv x + \frac{g^2}{x}, \quad (236)$$

which is a 2 : 1 surjection,

$$u(x; g) = u\left(\frac{g^2}{x}; g\right), \quad \text{for all } x \in \mathbb{C}, \quad (237)$$

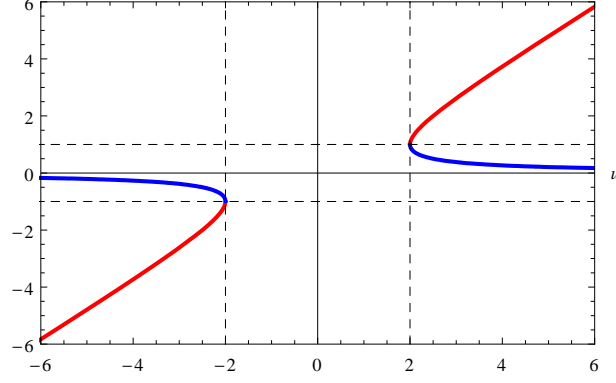


FIG. 13: The two branches, $(-\infty, -2g) \cup (2g, +\infty) \ni u \mapsto x^{\text{branch I}}(u; g)$ (red), $(-\infty, -2g) \cup (2g, +\infty) \ni u \mapsto x^{\text{branch II}}(u; g)$ (blue). The dashed vertical lines are at the positions $2g$ and $-2g$, while the horizontal ones at g and $-g$. All plots for $g = 1$.

and therefore, its inverse function is multi-valued, having two branches,

$$\mathbb{C} \ni u \mapsto x^{\text{branch I}}(u; g) = \frac{u}{2} + \frac{u}{2} \sqrt{1 - \frac{4g^2}{u^2}}, \quad (238)$$

$$\mathbb{C} \ni u \mapsto x^{\text{branch II}}(u; g) = \frac{u}{2} - \frac{u}{2} \sqrt{1 - \frac{4g^2}{u^2}} = \frac{g^2}{x^{\text{branch I}}(u; g)}, \quad (239)$$

where the square roots are the principal ones. The latter part of (239) shows the relation between the two branches, in accordance with (237). We will shortly see that, however, it is only branch I which properly glues the all-loop ($g > 0$) and the one-loop ($g = 0$) case, and hence, we must choose to work with it only. We thus have the relation between the rapidities and the spectral parameters,

$$u_k \equiv u(x_k; g) = x_k + \frac{g^2}{x_k}. \quad (240)$$

Moreover, we define the “shifted spectral parameters,”

$$x_k^\pm \equiv x^{\text{branch I}}\left(u_k \pm \frac{i}{2}; g\right) = x^{\text{branch I}}\left(x_k + \frac{g^2}{x_k} \pm \frac{i}{2}; g\right), \quad (241)$$

through which the magnon momenta are expressed as

$$p_k = -i \log \left(\frac{x_k^+}{x_k^-} \right), \quad (242)$$

where we may assume the logarithm to be the principal one. In this language, the BAEs (234) acquire the form

$$\left(\frac{x_k^+}{x_k^-} \right)^J = \prod_{\substack{j=1 \\ j \neq k}}^M \left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \right)^\eta \frac{1 - \frac{g^2}{x_k^+ x_j^-}}{1 - \frac{g^2}{x_j^+ x_k^-}}, \quad \text{for all } k = 1, 2, \dots, M \quad (\text{BDS BAEs}), \quad (243)$$

where we have extended the analysis from the $\text{SU}(2)$ sector to three important subsectors of the full theory, namely, $\text{SL}(2)$, $\text{SU}(1, 1)$, $\text{SU}(2)$, corresponding respectively to $\eta = -1, 0, +1$; this article deals exclusively with the first of these. Solutions (x_1, x_2, \dots, x_M) to these equations (the “Bethe roots”) yield eigenstates of the spin chain’s Hamiltonian. But not all of these eigenstates have an interpretation on the gauge side of the correspondence — the trace cyclicity condition for $\mathcal{N} = 4$ SYM operators is translated in the spin chain language to the requirement that the spin chain’s Hamiltonian’s eigenstates should also be eigenstates, with eigenvalue 1, of the shift operator $e^{i\hat{P}}$, where \hat{P} is the operator of the total momentum of the magnons,

$$e^{i \sum_{k=1}^M p_k} = 1, \quad i.e., \quad \sum_{k=1}^M p_k = -i \sum_{k=1}^M \log \left(\frac{x_k^+}{x_k^-} \right) = 2\pi w \equiv \varpi, \quad \text{where } w \in \mathbb{Z} \quad (244)$$

(w will be called the “winding number”). Once the Bethe roots (x_1, x_2, \dots, x_M) have been found, not only the energy, but all the infinite family of the local conserved charges of our integrable system can be calculated according to the known formula,

$$Q_t = \sum_{k=1}^M q_t(x_k), \quad \text{for } t = 1, 2, \dots, \quad (245)$$

where the “magnon charges,”

$$q_t(x_k) \equiv \frac{i}{t-1} \left(\frac{1}{(x_k^+)^{t-1}} - \frac{1}{(x_k^-)^{t-1}} \right), \quad \text{for } t = 1, 2, \dots \quad (246)$$

(For $t = 1$, we understand it to be the limit $t \rightarrow 1$, which yields the magnon momentum (242), $q_1(x_k) = p_k$.) The two lowest charges are of particular interest: The first local conserved charge is the total momentum, which according to the momentum condition (244) must be

$$Q_1 = \sum_{k=1}^S p_k = 2\pi w = \varpi, \quad (247)$$

while the second one describes the desired anomalous dimension,

$$\Delta = J + \Delta_{\text{an}}, \quad \Delta_{\text{an}} = 2g^2 Q_2. \quad (248)$$

Finally, let us make the following comment on the reality of the spectral parameters: As announced, we will be working with the $\text{SL}(2)$ sector. There, it can be proven that the Bethe roots are all real (in other sectors, they can acquire complex values). The reality of the x_k ’s implies

$$u_k \in \mathbb{R}, \quad |u_k| \geq 2g, \quad \text{for all } k = 1, 2, \dots, M, \quad (249)$$

and moreover, we will label them in such a way that $u_1 < u_2 < \dots < u_M$. In figure 13, we plot how x_k depends on u_k satisfying (249), when either of the two branches (238), (239) is used; we observe that

$$x_k \in \mathbb{R}, \quad \begin{cases} |x_k| \geq g & \text{for branch I} \\ |x_k| \leq g & \text{for branch II} \end{cases}, \quad \text{for all } k = 1, 2, \dots, M. \quad (250)$$

Therefore, the only way to reasonably realize the transition from $g > 0$ to $g = 0$ is to use branch I. Moreover, since branch I is increasing, they are labeled in such a way that $x_1 < x_2 < \dots < x_M$. Consequently, there will also be

$$x_k^+ = (x_k^-)^*, \quad p_k \in \mathbb{R}, \quad \text{for all } k = 1, 2, \dots, M. \quad (251)$$

3. The Dressing Phase

The BDS Bethe ansatz equations (243) constituted a very good guess; it happens that the true formula differs from them only by a phase factor multiplying the S-matrix, called the “dressing phase.” Investigating the “AdS/CFT S-matrix,” Beisert astonishingly found in 2005 [168] that imposing on it the requirement of unitarity, as well as the “Yang–Baxter equation,” being a reflection of its integrability,

$$S_{12}S_{21} = 1 \quad (\text{unitarity}), \quad S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12} \quad (\text{Yang–Baxter equation}), \quad (252)$$

fixes it up to a phase factor. In other words, only symmetry is needed to almost completely determine the S-matrix. This factor modifies the BAEs in the following way,

$$\left(\frac{x_k^+}{x_k^-} \right)^J = \prod_{\substack{j=1 \\ j \neq k}}^S \left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \right)^\eta \frac{1 - \frac{g^2}{x_k^+ x_j^-}}{1 - \frac{g^2}{x_j^+ x_k^-}} \sigma^2(x_k, x_j; g), \quad \text{for all } k = 1, 2, \dots, S \quad (253)$$

(we have changed M to S to comply with the notation in the $SL(2)$ sector (200)), where $\eta = -1, 0, +1$ for $SL(2)$, $SU(1|1)$, $SU(2)$, respectively, and the dressing phase,

$$\sigma(x_k, x_j; g) = e^{i\theta(x_k, x_j; g)} \quad (254)$$

has to obey a relation stemming from Janik's crucial insight from 2006 [174] that the S-matrix should submit to yet another constraint, the “crossing symmetry,”

$$\theta(x_k^\pm, x_j^\pm; g) + \theta\left(\frac{1}{x_k^\pm}, x_j^\pm; g\right) = -ih(x_k^\pm, x_j^\pm; g), \quad \text{where} \quad h(x_k^\pm, x_j^\pm; g) \equiv \frac{x_j^- x_k^- - x_j^+}{x_j^+ x_k^+ - x_j^-} \frac{1 - \frac{1}{x_k^- x_j^-}}{1 - \frac{1}{x_k^+ x_j^+}}. \quad (255)$$

At strong coupling, which is the regime we will be interested in, the dressing phase has been found to possess the following expansion,

$$\theta(x_k, x_j; g) = \sum_{r \geq 2} \sum_{s \geq r+1} g^{r+s-1} c_{r,s}(g) (q_r(x_k) q_s(x_j) - q_r(x_j) q_s(x_k)), \quad (256)$$

where the magnon charges are defined above (246), while $c_{r,s}(g)$ are some real coefficients depending on the coupling constant. The expression for the charges suggests another way of writing the phase factor,

$$\theta(x_k, x_j; g) = +\chi^a(x_k^+, x_j^+; g) - \chi^a(x_k^+, x_j^-; g) - \chi^a(x_k^-, x_j^+; g) + \chi^a(x_k^-, x_j^-; g), \quad (257)$$

where

$$\chi^a(x_1, x_2; g) \equiv \chi(x_1, x_2; g) - \chi(x_2, x_1; g), \quad (258)$$

$$\chi(x_1, x_2; g) \equiv g \sum_{r \geq 2} \sum_{s \geq r+1} \frac{-c_{r,s}(g)}{(r-1)(s-1)} \left(\frac{g}{x_1}\right)^{r-1} \left(\frac{g}{x_2}\right)^{s-1}. \quad (259)$$

The coefficients $c_{r,s}(g)$ admit a strong-coupling expansion,

$$c_{r,s}(g) = c_{r,s}^{(0)} + \frac{1}{g} c_{r,s}^{(1)} + \sum_{n \geq 1} \frac{1}{g^{2n}} c_{r,s}^{(2n)}, \quad (260)$$

where it can be shown that in order to satisfy the crossing symmetry, only one term with an odd power of $1/g$ is necessary, namely, the first one. The two leading orders have been found to be [148, 177]

$$c_{r,s}^{(0)} = \delta_{r+1,s} \quad (\text{the Arutyunov–Frolov–Staudacher phase}), \quad (261)$$

$$c_{r,s}^{(1)} = \frac{1}{\pi} ((-1)^{r+s} - 1) \frac{(r-1)(s-1)}{(r+s-2)(s-r)} \quad (\text{the Hernández–López phase}), \quad (262)$$

while the higher ones are given by the Beisert–Hernández–López proposal [182],

$$c_{r,s}^{(2n)} = \frac{1}{4} (-1)^n ((-1)^{r+s} - 1) (r-1)(s-1) \frac{B_{2n} \Gamma\left(\frac{s+r-3}{2} + n\right) \Gamma\left(\frac{s-r-1}{2} + n\right)}{\Gamma(2n+1) \Gamma(2n-1) \Gamma\left(\frac{s+r+1}{2} - n\right) \Gamma\left(\frac{s-r+3}{2} - n\right)}, \quad \text{for } n \geq 1, \quad (263)$$

where B_{2n} represent the Bernoulli numbers. For our computation, we will need only the three leading terms of (260), so let us print explicitly also

$$c_{r,s}^{(2)} = \frac{1}{48} (1 - (-1)^{r+s}) (r-1)(s-1). \quad (264)$$

Testing of (262) is the main goal of this article, and our end formulae (1002) and (1025), (1026) provide very sophisticated tools for doing so.

Equivalently, the series (260) can be rewritten as a strong-coupling expansion of the function χ (259),

$$\chi(x_1, x_2; g) = \chi^{(0)}(x_1, x_2; g) + \frac{1}{g} \chi^{(1)}(x_1, x_2; g) + \sum_{n \geq 1} \frac{1}{g^{2n}} \chi^{(2n)}(x_1, x_2; g), \quad (265)$$

and the explicit values of its first three coefficients (261), (262), (264) are translated into

$$\chi^{(0)}(x_1, x_2; g) = -\frac{g^2}{x_2} + \left(-x_1 + \frac{g^2}{x_2}\right) \log\left(1 - \frac{g^2}{x_1 x_2}\right), \quad \text{for } x_1, x_2 \in \mathbb{C}, \quad |x_1 x_2| > g^2, \quad (266)$$

$$\begin{aligned} \chi^{(1)}(x_1, x_2; g) = & \frac{g}{2\pi} \left(\text{Li}_2\left(\frac{\sqrt{x_1} - \frac{g}{\sqrt{x_2}}}{\sqrt{x_1} + \sqrt{x_2}}\right) + \text{Li}_2\left(\frac{\sqrt{x_1} + \frac{g}{\sqrt{x_2}}}{\sqrt{x_1} - \sqrt{x_2}}\right) - \text{Li}_2\left(\frac{\sqrt{x_1} + \frac{g}{\sqrt{x_2}}}{\sqrt{x_1} + \sqrt{x_2}}\right) - \text{Li}_2\left(\frac{\sqrt{x_1} - \frac{g}{\sqrt{x_2}}}{\sqrt{x_1} - \sqrt{x_2}}\right) \right. \\ & \left. + \log\left(\frac{\sqrt{x_1} - \frac{g}{\sqrt{x_2}}}{\sqrt{x_1} + \frac{g}{\sqrt{x_2}}}\right) \left(\log\left(\frac{1}{\sqrt{x_2} + \sqrt{x_1}}\right) - \log\left(\frac{1}{\sqrt{x_2} - \sqrt{x_1}}\right) \right) \right), \\ & \text{for } x_1, x_2 \in \mathbb{R}_+, \quad x_1 x_2 > g^2, \quad x_2 > g, \end{aligned} \quad (267)$$

$$\chi^{(2)}(x_1, x_2; g) = -\frac{g^4 x_2}{24(x_1 x_2 - g^2)(x_2^2 - g^2)}, \quad \text{for } x_1, x_2 \in \mathbb{C}, \quad |x_1 x_2| > g^2, \quad |x_2| > g. \quad (268)$$

For the even coefficients, we indicate the ranges of x_1 and x_2 such that the infinite sums in (259) converge. For the Hernández–López piece (267), the convergence condition is the same as in (268), but the result is given for the simpler narrower sector of x_1 and x_2 real and positive. The infinite sums in the even terms are all easy to find, featuring only geometric series; consequently, all the higher orders are some rational functions of x_1 and x_2 . Only the derivation of the Hernández–López term requires some more care, and we present it in paragraph IIB4. Remark that these functions (266)–(268), despite being coefficients in a large- g series, still depend on g , both directly and indirectly through x_1 and x_2 , and hence, should be further expanded once the dependence of x_1 and x_2 on g has been found.

Let us henceforth restrict ourselves to the $\text{SL}(2)$ case. A standard step in computations such as ours, is to consider logarithm of the S equations (253), which for $\eta = -1$, and after some useful reorganization, yields for every $k = 1, 2, \dots, S$,

$$-iJ \log\left(\frac{x_k^+}{x_k^-}\right) - 2\pi m_k = -2i \sum_{\substack{j=1 \\ j \neq k}}^S \log\left(\frac{1 - \frac{g^2}{x_k^+ x_j^-}}{1 - \frac{g^2}{x_j^+ x_k^-}}\right) + \sum_{\substack{j=1 \\ j \neq k}}^S 2\theta(x_k, x_j; g) - i \sum_{\substack{j=1 \\ j \neq k}}^S \log\left(\frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - \frac{g^2}{x_j^+ x_k^-}}{1 - \frac{g^2}{x_k^+ x_j^-}}\right). \quad (269)$$

The “mode numbers” $m_k \in \mathbb{Z}$ appear since we are free to choose different branches of the logarithm for any of the S equations. We may thus assume all the logarithms above to be principal. Each choice of the mode numbers leads to a different solution to the set of equations, and consequently, to a different string motion. We will now make a crucial assumption that all these mode numbers are equal,

$$m_1 = m_2 = \dots = m_S \equiv m \equiv \frac{\mu}{2\pi}. \quad (270)$$

Moreover, the following identity is noticed (241), helpful for the last term of (269),

$$\frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - \frac{g^2}{x_j^+ x_k^-}}{1 - \frac{g^2}{x_k^+ x_j^-}} = \frac{u_k - u_j - i}{u_k - u_j + i}, \quad (271)$$

which implies a convenient form of our basic equation,

$$\underbrace{-iJ \log\left(\frac{x_k^+}{x_k^-}\right) - \mu}_{\text{l.h.s.}} = \underbrace{\sum_{\substack{j=1 \\ j \neq k}}^S (-2i) \log\left(\frac{1 - \frac{g^2}{x_k^+ x_j^-}}{1 - \frac{g^2}{x_j^+ x_k^-}}\right)}_{\text{r.h.s., term I}} + \underbrace{\sum_{\substack{j=1 \\ j \neq k}}^S 2\theta(x_k, x_j; g)}_{\text{r.h.s., term II (the “phase”)}} + \underbrace{\sum_{\substack{j=1 \\ j \neq k}}^S (-i) \log\left(\frac{u_k - u_j - i}{u_k - u_j + i}\right)}_{\text{r.h.s., term III (the “anomaly”)}} \quad (272)$$

where we have given names to its terms, to be used in subsection IIB.

This equation should be supplemented by the momentum condition (244). Notice that there exists another way to arrive at the momentum condition, namely, to sum both sides of (272) over k . It is easy to see that the r.h.s. is anti-symmetric w.r.t. j and k , and hence, such a sum disappears, leaving us with

$$\sum_{k=1}^S p_k = \mu\alpha, \quad (273)$$

where we denote the so-called “filling fraction,”

$$\alpha \equiv \frac{S}{J}. \quad (274)$$

The two versions of the momentum condition, (244) and (273), must coincide, which puts the following constraint on the various integer parameters of our problem,

$$\mu\alpha = \varpi, \quad i.e., \quad mS = wJ. \quad (275)$$

PROJECT I

II. THE ALL-LOOP STRING BETHE ANSATZ EQUATIONS IN THE $SL(2)$ SECTOR, EXPANDED AT LARGE J UP TO THE ORDER $O(1/J^2)$, AND WRITTEN AS A QUADRATIC EQUATION

A. Introduction

1. The Thermodynamic Limit, the Strong-Coupling Limit, and the Frolov–Tseytlin Limit in the All-Loop Bethe Ansatz Equations in the $SL(2)$ Sector

The equations we want to solve in this paper are the logarithmic string Bethe ansatz equations (272), along with the momentum condition (244), (273). We will approach them in the BMN limit (139), which is the thermodynamic limit, *i.e.*, the limit of a very long spin chain, and the strong-coupling limit — both tuned in such a way that $g \sim J$ as they grow to infinity,

$$J \rightarrow \infty \text{ (thermodynamic)}, \quad g \rightarrow \infty \text{ (strong-coupling)}, \quad \text{such that} \quad \omega \equiv \frac{g}{J} = \text{fixed}. \quad (276)$$

Instead of ω , one also commonly uses the BMN coupling/the rescaled angular momentum $\sqrt{\lambda'} = 1/\mathcal{J} = \sqrt{\lambda}/J = 4\pi\omega$ (139), (78). Remark $\omega \geq 0$.

Moreover, we will focus on the case (the “Frolov–Tseytlin limit”) where the number of magnons S is of order of the length of the spin chain J ,

$$S \rightarrow \infty, \quad \text{such that} \quad \alpha = \frac{S}{J} = \text{fixed} \quad (\text{Frolov–Tseytlin}). \quad (277)$$

Along with the filling fraction α (274), the quantity $\mathcal{S} = S/\sqrt{\lambda} = \alpha/(4\pi\omega)$ (78) is often used, too. Remark $0 \leq \alpha \leq 1$. Notice that the spin chain picture allows α to acquire any value from the interval $[0, 1/2]$, which then can be extended to $[0, 1]$, but only rational values lead to meaningful counterparts on the gauge side of the correspondence, as is clear from (275).

It is well-known that in the limit (276), the Bethe roots (specifically, the spectral parameters) scale linearly with large J ,

$$x_k \rightarrow \infty, \quad \text{such that} \quad y_k \equiv \frac{x_k}{J} = \text{fixed}, \quad \text{for all} \quad k = 1, 2, \dots, S. \quad (278)$$

The y_k ’s will be our new fundamental variables. Recall (250) that in the $SL(2)$ sector, they satisfy

$$y_k \in \mathbb{R}, \quad |y_k| \geq \omega, \quad \text{for all} \quad k = 1, 2, \dots, S. \quad (279)$$

This condition will be important on several occasions. Our end result for the distribution of the y_k ’s will be consistent with it.

In the thermodynamic limit, the y_k ’s condense in separate intervals on the real axis, *i.e.*, instead of a collection of discrete Bethe roots, we have continuous segments, called “Bethe strings” or “cuts.” These cuts describe macroscopic spin waves, dual to semiclassical strings on $AdS_5 \times S^5$. We will shortly make and adhere to the assumption that there is only one cut. In this computation, however, we tackle finite-size (*i.e.*, finite- J) corrections to the logarithmic string Bethe ansatz equations. Hence, it is certain that at some large- J order, the grain structure of the Bethe roots will start to manifest. This happens to be order $O(1/J^2)$, which we consider here, showing how to incorporate this new difficulty into the standard technique of “quadratic equation” of dealing with our type of problems.

The rapidities (240) and the shifted spectral parameters (241), in which the logarithmic string Bethe ansatz equations (272) are written, are expressed through the y_k ’s as

$$u_k = J \left(y_k + \frac{\omega^2}{y_k} \right), \quad (280)$$

$$x_k^\pm = x^{\text{branch I}} \left(J \left(y_k + \frac{\omega^2}{y_k} \right) \pm \frac{i}{2}; J\omega \right). \quad (281)$$

Let us emphasize once again (237) that (280), (281) are invariant under the transformation $y_k \rightarrow \omega^2/y_k$, which means that our equations (272), which are written exclusively in terms of these variables, possess the same symmetry. However, we need to restrict our attention only to the solutions fulfilling (279), which breaks this symmetry.

We will also need, in the course of our calculations, relations between the first and second derivatives w.r.t. u_k and the first and second derivatives w.r.t. y_k ; straightforwardly from (280),

$$\frac{d}{du_k} = \frac{1}{J} \frac{y_k^2}{y_k^2 - \omega^2} \frac{d}{dy_k}, \quad (282)$$

$$\frac{d^2}{du_k^2} = \frac{1}{J^2} \frac{y_k^3}{(y_k^2 - \omega^2)^2} \left(-\frac{2\omega^2}{y_k^2 - \omega^2} \frac{d}{dy_k} + y_k \frac{d^2}{dy_k^2} \right), \quad (283)$$

where the l.h.s. are understood to act on functions of u_k , while the r.h.s. on functions of y_k .

In subsection II B, we will translate the logarithmic string Bethe ansatz equations (272) from the language of the u_k 's and x_k^\pm 's into the one of the y_k 's, and subsequently expand it at large J up to and including the three leading terms.

2. The Density and the Resolvent. Definitions

A solution to our fundamental equations (272) is a set of values of the rescaled spectral parameters (y_1, y_2, \dots, y_S) (278). Equivalently, we may think in terms of the rapidities (u_1, u_2, \dots, u_S) , related to the rescaled spectral parameters through (280). We could also use some other language, such as of the magnon momenta (242), but we will discuss here only the previous two, eventually sticking to the description in terms of rapidities. Either case should be supplemented by hand by the restriction (279).

To begin with, let us assume that J and S are finite.

A standard efficient way to encode either collection of variables, just as is done in random matrix theory, is to define their “densities,” which are the following sums of (real) Dirac delta functions,

$$\mathbb{R} \ni y \mapsto \rho^{\text{resc.spec.par.}}(y) \equiv \frac{1}{J} \sum_{k=1}^S \delta(y - y_k), \quad (284)$$

$$\mathbb{R} \ni v \mapsto \rho^{\text{resc.rapid.}}(v) \equiv \frac{1}{J} \sum_{k=1}^S \delta\left(v - \frac{u_k}{J}\right). \quad (285)$$

Notice that the factors of $1/J$ have been introduced in such a way that both densities and both arguments y, v are of order $O(J^0)$ at large J .

The meaning of these definitions is as follows: Consider some well-behaving functions $y \mapsto f_1(y)$, $v \mapsto f_2(v)$, regular at the points (y_1, y_2, \dots, y_S) , $(u_1/J, u_2/J, \dots, u_S/J)$, respectively. Then (284), (285) can be restated as

$$\frac{1}{J} \sum_{k=1}^S f_1(y_k) = \int_{\mathbb{R}} dy \rho^{\text{resc.spec.par.}}(y) f_1(y), \quad (286)$$

$$\frac{1}{J} \sum_{k=1}^S f_2\left(\frac{u_k}{J}\right) = \int_{\mathbb{R}} dv \rho^{\text{resc.rapid.}}(v) f_2(v). \quad (287)$$

In other words, any summation of functions of our variables can be replaced by an integral of the functions in question with the appropriate density. In particular, setting here $f_1 = 1$, $f_2 = 1$, we obtain the “normalization formulae,”

$$\int_{\mathbb{R}} dy \rho^{\text{resc.spec.par.}}(y) = \alpha, \quad (288)$$

$$\int_{\mathbb{R}} dv \rho^{\text{resc.rapid.}}(v) = \alpha. \quad (289)$$

In this computation, we decide to work exclusively with the density of the rescaled rapidities (285), but we will need it transformed to the space of the rescaled spectral parameters. To this end, recall (280),

$$\frac{u_k}{J} = y_k + \frac{\omega^2}{y_k}, \quad \text{where we impose} \quad |y_k| \geq \omega, \quad (290)$$

and also change the variable v to y by mimicking (290),

$$v(y) \equiv y + \frac{\omega^2}{y}, \quad \text{where we impose} \quad |y| \geq \omega. \quad (291)$$

Expressions (290), (291) yield the following relation between the arguments of the Dirac delta functions in (284), (285),

$$v - \frac{u_k}{J} = (y - y_k) \left(1 - \frac{\omega^2}{yy_k} \right), \quad \text{where we impose} \quad |y_k|, |y| \geq \omega. \quad (292)$$

If we had not imposed the restrictions $|y_k| \geq \omega$, $|y| \geq \omega$, the change of variables formula for Dirac delta function would give

$$\delta \left(v - \frac{u_k}{J} \right) = \frac{y_k^2}{|y_k^2 - \omega^2|} \delta(y - y_k) + \frac{\omega^2}{|y_k^2 - \omega^2|} \delta \left(y - \frac{\omega^2}{y_k} \right) = \frac{y^2}{|y^2 - \omega^2|} \left(\delta(y - y_k) + \delta \left(y - \frac{\omega^2}{y_k} \right) \right), \quad (293)$$

since as a function of y , the r.h.s. of (292) has two roots, $y^{(1)} = y_k$ and $y^{(2)} = \omega^2/y_k$; its derivative reads $(1 - \omega^2/y^2)$; the values of this derivative at the roots $y = y^{(1)}$, $y = y^{(2)}$ are respectively $(1 - \omega^2/y_k^2)$ and $(1 - y_k^2/\omega^2)$. But our requirements remove the root $y^{(2)}$ from the domain of y , leaving us with

$$\delta \left(v - \frac{u_k}{J} \right) = \frac{y_k^2}{y_k^2 - \omega^2} \delta(y - y_k) = \frac{y^2}{y^2 - \omega^2} \delta(y - y_k), \quad \text{where we impose} \quad |y_k|, |y| \geq \omega. \quad (294)$$

We have thus found two alternative ways of how the density of the rescaled rapidities looks like in the language of the rescaled spectral parameters,

$$\begin{aligned} \rho(y) &\equiv \rho^{\text{resc.rapid.}}(v(y)) = \\ &= \frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \delta(y - y_k) = \frac{y^2}{y^2 - \omega^2} \rho^{\text{resc.spec.par.}}(y), \quad \text{where we impose} \quad |y_k|, |y| \geq \omega. \end{aligned} \quad (295)$$

Let us rewrite the summation formula (286) using this new density $\rho(y)$,

$$\frac{1}{J} \sum_{k=1}^S f_1(y_k) = \int_{\mathbb{R}} dy \rho(y) \left(1 - \frac{\omega^2}{y^2} \right) f_1(y), \quad (296)$$

or equivalently,

$$\frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} f_1(y_k) = \int_{\mathbb{R}} dy \rho(y) f_1(y). \quad (297)$$

In particular, setting $f_1 = 1$ in (296) leads to the normalization formula for $\rho(y)$,

$$\int_{\mathbb{R}} dy \rho(y) \left(1 - \frac{\omega^2}{y^2}\right) = \alpha. \quad (298)$$

Alternatively to the densities, there is a second method to encode our variables — by “resolvents” (or “Green functions”), which are complex functions of a complex argument,

$$\mathbb{C} \ni z \mapsto G^{\text{resc.spec.par.}}(z) \equiv \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k}, \quad (299)$$

$$\mathbb{C} \ni w \mapsto G^{\text{resc.rapid.}}(w) \equiv \frac{1}{J} \sum_{k=1}^S \frac{1}{w - \frac{u_k}{J}}. \quad (300)$$

We observe that they are meromorphic with poles at the respective Bethe roots. In the thermodynamic limit, we will see that they turn into holomorphic functions with cuts at the positions of the Bethe strings on the real axis.

Using the Cauchy integral formula, and the fact that f_1, f_2 have been assumed to be regular at the respective Bethe roots, the sums on the l.h.s. of (286), (287) can be written as contour integrals,

$$\frac{1}{J} \sum_{k=1}^S f_1(y_k) = \oint_{\odot \mathcal{C}^y} dz \frac{G^{\text{resc.spec.par.}}(z)}{2\pi i} f_1(z), \quad (301)$$

$$\frac{1}{J} \sum_{k=1}^S f_2\left(\frac{u_k}{J}\right) = \oint_{\odot \mathcal{C}^v} dw \frac{G^{\text{resc.rapid.}}(w)}{2\pi i} f_2(w), \quad (302)$$

where \mathcal{C}^y and \mathcal{C}^v denote contours (A-contours) encircling the respective collections of the Bethe roots; we use a convention that all such loops are counterclockwise. In particular, for $f_1 = 1, f_2 = 1$, we get the normalization formulae,

$$\oint_{\odot \mathcal{C}^y} dz \frac{G^{\text{resc.spec.par.}}(z)}{2\pi i} = \alpha, \quad (303)$$

$$\oint_{\odot \mathcal{C}^v} dw \frac{G^{\text{resc.rapid.}}(w)}{2\pi i} = \alpha. \quad (304)$$

Analogously to (293), let us change variables from the rescaled rapidities to the rescaled spectral parameters in the resolvent (300),

$$G^{\text{resc.rapid.}}(w(z)) = \frac{1}{J} \sum_{k=1}^S \frac{1}{(z - y_k) \left(1 - \frac{\omega^2}{zy_k}\right)} = \frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{1}{z - y_k} + \frac{1}{J} \sum_{k=1}^S \frac{\omega^2}{-y_k^2 + \omega^2} \frac{1}{z - \frac{\omega^2}{y_k}}, \quad (305)$$

where the relation of w to z is the same as v to y (291). And similarly as before (295), the requirement (279) removes the latter part of this expression, leaving us with either of the simplified resolvents,

$$G(z) \equiv \frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{1}{z - y_k}, \quad \text{or alternatively,} \quad G^{\text{alt.}}(z) \equiv \frac{z^2}{z^2 - \omega^2} \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} = \frac{z^2}{z^2 - \omega^2} G^{\text{resc.spec.par.}}(z). \quad (306)$$

Indeed, it is easy to see that

$$\oint_{\odot \mathcal{C}^y} dz \frac{G^{\text{resc.rapid.}}(w(z))}{2\pi i} f_1(z) = \oint_{\odot \mathcal{C}^y} dz \frac{G(z)}{2\pi i} f_1(z) = \oint_{\odot \mathcal{C}^y} dz \frac{G^{\text{alt.}}(z)}{2\pi i} f_1(z), \quad (307)$$

since the singularities $z = \omega^2/y_k$ lie outside of the integration contour.

With these new resolvents, the summation formula (301) can be cast as

$$\frac{1}{J} \sum_{k=1}^S f_1(y_k) = \oint_{\odot \mathcal{C}^y} dz \frac{G(z)}{2\pi i} \left(1 - \frac{\omega^2}{z^2}\right) f_1(z) = \oint_{\odot \mathcal{C}^y} dz \frac{G^{\text{alt.}}(z)}{2\pi i} \left(1 - \frac{\omega^2}{z^2}\right) f_1(z), \quad (308)$$

or equivalently,

$$\frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} f_1(y_k) = \oint_{\odot \mathcal{C}^y} dz \frac{G(z)}{2\pi i} f_1(z) = \oint_{\odot \mathcal{C}^y} dz \frac{G^{\text{alt.}}(z)}{2\pi i} f_1(z). \quad (309)$$

In particular, setting $f_1 = 1$ in (308) gives the normalization formulae,

$$\oint_{\odot \mathcal{C}^y} dz \frac{G(z)}{2\pi i} \left(1 - \frac{\omega^2}{z^2}\right) = \oint_{\odot \mathcal{C}^y} dz \frac{G^{\text{alt.}}(z)}{2\pi i} \left(1 - \frac{\omega^2}{z^2}\right) = \alpha. \quad (310)$$

The first one of these can be conveniently rewritten as

$$\oint_{\odot \mathcal{C}^y} dz \frac{G(z)}{2\pi i} \left(1 - \frac{\omega^2}{z^2}\right) = \lim_{z \rightarrow \infty} (zG(z)) + \omega^2 G'(0) = \alpha, \quad (311)$$

which can be seen either from plugging $f_1(z) = 1$ and $f_1(z) = \omega^2/z^2$ into (309) and using (330),

$$\oint_{\odot \mathcal{C}^y} dz \frac{G(z)}{2\pi i} = \frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} = \lim_{z \rightarrow \infty} (zG(z)), \quad \oint_{\odot \mathcal{C}^y} dz \frac{G(z)}{2\pi i} \frac{\omega^2}{z^2} = \omega^2 \frac{1}{J} \sum_{k=1}^S \frac{1}{y_k^2 - \omega^2} = -\omega^2 G'(0), \quad (312)$$

or directly from the definition of $G(z)$ (306) by just algebraic manipulation,

$$\lim_{z \rightarrow \infty} (zG(z)) + \omega^2 G'(0) = \frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} + \omega^2 \frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{-1}{(0 - y_k)^2} = \frac{1}{J} \sum_{k=1}^S 1 = \alpha. \quad (313)$$

The two languages, of densities and resolvents, are related with aid of the Sokhotsky formula,

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{p \pm i\epsilon} = \mp \pi i \delta(p) + \text{pv} \left(\frac{1}{p} \right), \quad \text{which implies} \quad \delta(p) = -\frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} \left(\frac{1}{p + i\epsilon} - \frac{1}{p - i\epsilon} \right), \quad (314)$$

which allows to derive the densities as the discontinuities of the resolvents,

$$\rho^{\text{resc.spec.par.}}(y) = -\frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} (G^{\text{resc.spec.par.}}(y + i\epsilon) - G^{\text{resc.spec.par.}}(y - i\epsilon)), \quad (315)$$

$$\rho^{\text{resc.rap.}}(v) = -\frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} (G^{\text{resc.rap.}}(v + i\epsilon) - G^{\text{resc.rap.}}(v - i\epsilon)), \quad (316)$$

$$\rho(y) = -\frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} (G(y + i\epsilon) - G(y - i\epsilon)) = -\frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} (G^{\text{alt.}}(y + i\epsilon) - G^{\text{alt.}}(y - i\epsilon)). \quad (317)$$

In the opposite way, we may use the summation formulae for the sums defining the resolvents,

$$G^{\text{resc.spec.par.}}(z) = \int_{\mathbb{R}} dy \rho^{\text{resc.spec.par.}}(y) \frac{1}{z - y} = \oint_{\odot \mathcal{C}^y} dz' \frac{G^{\text{resc.spec.par.}}(z')}{2\pi i} \frac{1}{z - z'}, \quad (318)$$

$$G^{\text{resc.rap.}}(w) = \int_{\mathbb{R}} dv \rho^{\text{resc.rap.}}(v) \frac{1}{w - v} = \oint_{\odot \mathcal{C}^v} dw' \frac{G^{\text{resc.rapid.}}(w')}{2\pi i} \frac{1}{w - w'}, \quad (319)$$

$$G(z) = \int_{\mathbb{R}} dy \rho(y) \frac{1}{z-y} = \oint_{\odot C^y} dz' \frac{G(z')}{2\pi i} \frac{1}{z-z'}. \quad (320)$$

Different publications on the subject use different densities and resolvents from the list above. Here, we will exclusively resort to help of the density $\rho(y)$ (295) and the resolvent $G(z)$ (306), since we believe that our computation is most easily done in their language.

All the above definitions and basic properties have been developed with the supposition that J and S are finite. However, here we are interested in studying the limit of large J and S , along with subleading corrections to it. This poses the question of whether the above formulae survive such a limit, and in what form.

First of all, the definitions. For J and S finite, the density $\rho(y)$ is a sum of a finite number of Dirac delta functions, while for J and S growing to infinity, it can be described by a continuous function receiving large- J corrections,

$$\rho(y) = \rho^{(0)}(y) + \frac{1}{J}\rho^{(1)}(y) + \frac{1}{J^2}\rho^{(2)}(y) + O\left(\frac{1}{J^3}\right). \quad (321)$$

Similarly, for J and S finite, the resolvent $G(z)$ is a rational meromorphic function with poles at the y_k 's, whereas in our limit, it becomes some holomorphic function with cuts at the condensed Bethe strings, which can be computed to some large- J accuracy,

$$G(z) = G^{(0)}(z) + \frac{1}{J}G^{(1)}(z) + \frac{1}{J^2}G^{(2)}(z) + O\left(\frac{1}{J^3}\right). \quad (322)$$

We will take the resolvent as the fundamental quantity, and define the density through the Sokhotsky formula (317).

We will now make the following claim for our computation:

- The summation formulae (296), (297), the normalization formula (298), and the first part of (320), featuring line integration with the density — are valid for J and S finite, and in the limit of large J and S , they are valid only at the leading order.
- The summation formulae (308), (309), the normalization formulae (310), (311), and the second part of (320), featuring contour integration with the resolvent — are valid for J and S finite, and also in the limit of large J and S , to all orders.

The problem which arises at subleading orders is that, as we will see, the subleading terms of the density, *i.e.*, $\rho^{(1)}(y)$, *etc.*, develop singularities at the endpoints of the cut (the leading order $\rho^{(0)}(y)$, on the other hand, vanishes at the endpoints), which the line integrals do not properly take into account, while the contour integrals do. In other words, the line integrals, except for the leading order, miss some boundary contribution. We can thus summarize: We define

$$\rho(y) = -\frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} (G(y + i\epsilon) - G(y - i\epsilon)), \quad (323)$$

which in the opposite way reads

$$G(z) = \int_{\mathbb{R}} dy \rho(y) \frac{1}{z-y} + \left(\begin{array}{c} \text{subleading} \\ \text{boundary} \\ \text{contribution} \end{array} \right) = \oint_{\odot C^y} dz' \frac{G(z')}{2\pi i} \frac{1}{z-z'}. \quad (324)$$

The summation formulae,

$$\frac{1}{J} \sum_{k=1}^S f(y_k) = \int_{\mathbb{R}} dy \rho(y) \left(1 - \frac{\omega^2}{y^2}\right) f(y) + \left(\begin{array}{c} \text{subleading} \\ \text{boundary} \\ \text{contribution} \end{array} \right) = \oint_{\odot C^y} dz \frac{G(z)}{2\pi i} \left(1 - \frac{\omega^2}{z^2}\right) f(z), \quad (325)$$

or equivalently,

$$\frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} f(y_k) = \int_{\mathbb{R}} dy \rho(y) f(y) + \left(\begin{array}{c} \text{subleading} \\ \text{boundary} \\ \text{contribution} \end{array} \right) = \oint_{\odot C^y} dz \frac{G(z)}{2\pi i} f(z), \quad (326)$$

in particular,

$$\int_{\mathbb{R}} dy \rho(y) \left(1 - \frac{\omega^2}{y^2}\right) + \left(\begin{array}{c} \text{subleading} \\ \text{boundary} \\ \text{contribution} \end{array}\right) = \oint_{\mathbb{C}^y} dz \frac{G(z)}{2\pi i} \left(1 - \frac{\omega^2}{z^2}\right) = \alpha = \lim_{z \rightarrow \infty} (zG(z)) + \omega^2 G'(0), \quad (327)$$

which in turn implies

$$\int_{\mathbb{R}} dy \rho(y) + \left(\begin{array}{c} \text{subleading} \\ \text{boundary} \\ \text{contribution} \end{array}\right) = \oint_{\mathbb{C}^y} dz \frac{G(z)}{2\pi i} = \lim_{z \rightarrow \infty} (zG(z)) = \alpha - \omega^2 G'(0). \quad (328)$$

3. The Density and the Resolvent. The Large- J Expansions of the Local Conserved Charges Written Through the Resolvent

In this paragraph, we will practice working with the resolvent, by doing the following useful exercise: Let us expand the local conserved charges Q_t (245), (246) at large J , up to and including the three leading orders, and write the series through the resolvent and its derivatives.

First, let us recall how our resolvent (306) and its arbitrary u -th derivative look like,

$$G'' \dots G^{(u)}(z) = \frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{(-1)^u u!}{(z - y_k)^{u+1}}, \quad \text{for } u = 0, 1, \dots \quad (329)$$

In particular, at $z = 0$,

$$G'' \dots G^{(u)}(0) = \frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{-u!}{y_k^{u+1}}, \quad \text{for } u = 0, 1, \dots \quad (330)$$

Here and in the subsequent calculations, we will use the following technique: If $y \mapsto f(y)$ is any rational function, then the following sum can be written through the resolvent and its derivatives by decomposing $f(y_k)$ into partial fractions, and making use of (329),

$$\frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} f(y_k) \xrightarrow{\text{fractional decomposition of } f(y_k)} \text{resolvent/derivatives.} \quad (331)$$

Remark that while finding the large- J series of any quantity featuring the shifted spectral parameters x_k^\pm (281), such as (334) or any expansion in subsection II B, we encounter the principal square root $\sqrt{(y_k^2 - \omega^2)^2} = |y_k^2 - \omega^2|$, which asks us to use the condition $|y_k| \geq \omega$ (279). Let us print several large- J terms of x_k^\pm as an example,

$$x_k^\pm = J y_k \mp \frac{i y_k^2}{2(\omega^2 - y_k^2)} - \frac{1}{J} \frac{\omega^2 y_k^3}{4(\omega^2 - y_k^2)^3} \pm \frac{1}{J^2} \frac{i \omega^2 y_k^4 (\omega^2 + y_k^2)}{8(\omega^2 - y_k^2)^5} + \mathcal{O}\left(\frac{1}{J^3}\right). \quad (332)$$

If we had used branch II of the mapping x (239), or alternatively, if we had assumed $|y_k| \leq \omega$, we would have obtained an entirely different series (actually, related to (332) through the transformation $y_k \rightarrow \omega^2/y_k$ (237)), eventually leading to wrong results,

$$x_k^\pm = J \frac{\omega^2}{y_k} \pm \frac{i \omega^2}{2(\omega^2 - y_k^2)} + \frac{1}{J} \frac{\omega^2 y_k^3}{4(\omega^2 - y_k^2)^3} \mp \frac{1}{J^2} \frac{i \omega^2 y_k^4 (\omega^2 + y_k^2)}{8(\omega^2 - y_k^2)^5} + \mathcal{O}\left(\frac{1}{J^3}\right), \quad \text{for branch II or } |y_k| \leq \omega. \quad (333)$$

Proceeding to our problem, we expand the magnon charges at large J , for any $t = 1, 2, \dots$ and $k = 1, 2, \dots, S$,

$$q_t(x_k) = \frac{1}{J^t} \frac{1}{y_k^t} \left(\frac{y_k^2}{y_k^2 - \omega^2} - \frac{1}{J^2} \frac{y_k^4 ((t^2 - 5t + 6)\omega^4 + 2(-t^2 + 2t + 3)\omega^2 y_k^2 + t(t+1)y_k^4)}{24(y_k^2 - \omega^2)^5} + \mathcal{O}\left(\frac{1}{J^4}\right) \right). \quad (334)$$

Notice that there are no terms of order $\mathcal{O}(1/J)$ and $\mathcal{O}(1/J^3)$ inside the brackets.

Next, we observe that the r.h.s. of (334) is a rational function of y_k , so, we divide it by $y_k^2/(y_k^2 - \omega^2)$, and decompose into partial fractions (331),

$$\begin{aligned}
Q_t &= \sum_{k=1}^S q_t(x_k) = \frac{1}{J^t} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{1}{y_k^t} - \\
&\quad - \frac{1}{J^{t+2}} \frac{1}{32\omega^t} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \left(\omega^2 \left(\frac{1}{(y_k - \omega)^4} + (-1)^t \frac{1}{(y_k + \omega)^4} \right) + \right. \\
&\quad \left. + \omega \left(\frac{1}{(y_k - \omega)^3} - (-1)^t \frac{1}{(y_k + \omega)^3} \right) + \frac{t(2-t)}{6} \left(\frac{1}{(y_k - \omega)^2} + (-1)^t \frac{1}{(y_k + \omega)^2} \right) + \right. \\
&\quad \left. + \frac{1}{6} \sum_{u=2}^{t-1} \omega^{u-2} (1 + (-1)^{t-u}) (t-u)(t+u-2)(u-1) \frac{1}{y_k^u} \right) + \mathcal{O}\left(\frac{1}{J^{t+4}}\right). \tag{335}
\end{aligned}$$

It is now straightforward to rewrite this expression through the resolvent and its derivatives; expanding these at large J according to (322), we finally obtain the three leading orders of Q_t , for any $t = 1, 2, \dots$,

$$\begin{aligned}
Q_t &= -\frac{1}{J^{t-1}} \frac{G^{(0)''\dots'(t-1)}(0)}{(t-1)!} - \frac{1}{J^t} \frac{G^{(1)''\dots'(t-1)}(0)}{(t-1)!} - \\
&\quad - \frac{1}{J^{t+1}} \left(\frac{G^{(2)''\dots'(t-1)}(0)}{(t-1)!} - \frac{1}{192\omega^t} \left(\omega^2 \left(G^{(0)''''}(\omega) + (-1)^t G^{(0)''''}(-\omega) \right) + \right. \right. \\
&\quad \left. \left. + 3\omega \left(G^{(0)'''}(\omega) - (-1)^t G^{(0)'''}(-\omega) \right) + t(2-t) \left(G^{(0)''}(\omega) + (-1)^t G^{(0)''}(-\omega) \right) + \right. \right. \\
&\quad \left. \left. + \sum_{u=2}^{t-1} \omega^{u-2} (1 + (-1)^{t-u}) (t-u)(t+u-2) \frac{G^{(0)''\dots'(u-1)}(0)}{(u-2)!} \right) \right) + \mathcal{O}\left(\frac{1}{J^{t+2}}\right). \tag{336}
\end{aligned}$$

Let us explicitly write the two lowest charges,

$$\begin{aligned}
Q_1 &= -G^{(0)}(0) - \frac{1}{J} G^{(1)}(0) - \frac{1}{J^2} \left(G^{(2)}(0) - \right. \\
&\quad \left. - \frac{1}{192\omega} \left(\omega^2 \left(G^{(0)''''}(\omega) - G^{(0)''''}(-\omega) \right) + 3\omega \left(G^{(0)'''}(\omega) + G^{(0)'''}(-\omega) \right) + \left(G^{(0)''}(\omega) - G^{(0)''}(-\omega) \right) \right) \right) + \mathcal{O}\left(\frac{1}{J^3}\right), \tag{337}
\end{aligned}$$

and

$$\begin{aligned}
Q_2 &= -\frac{1}{J} G^{(0)'}(0) - \frac{1}{J^2} G^{(1)'}(0) - \\
&\quad - \frac{1}{J^3} \left(G^{(2)'}(0) - \frac{1}{192\omega} \left(\omega \left(G^{(0)''''}(\omega) + G^{(0)''''}(-\omega) \right) + 3 \left(G^{(0)'''}(\omega) - G^{(0)'''}(-\omega) \right) \right) \right) + \mathcal{O}\left(\frac{1}{J^4}\right). \tag{338}
\end{aligned}$$

In particular, for $\omega = 0$, the result (336) simplifies to

$$Q_t|_{\omega=0} = -\frac{1}{J^{t-1}} \frac{G^{(0)''\dots'(t-1)}(0)}{(t-1)!} - \frac{1}{J^t} \frac{G^{(1)''\dots'(t-1)}(0)}{(t-1)!} - \frac{1}{J^{t+1}} \frac{G^{(2)''\dots'(t-1)}(0) - \frac{1}{24} G^{(0)''\dots'(t+1)}(0)}{(t-1)!} + \mathcal{O}\left(\frac{1}{J^{t+2}}\right), \tag{339}$$

which for the two lowest charges reads

$$Q_1|_{\omega=0} = -G^{(0)}(0) - \frac{1}{J}G^{(1)}(0) - \frac{1}{J^2}\left(G^{(2)}(0) - \frac{1}{24}G^{(0)''}(0)\right) + \mathcal{O}\left(\frac{1}{J^3}\right), \quad (340)$$

$$Q_2|_{\omega=0} = -\frac{1}{J}G^{(0)'}(0) - \frac{1}{J^2}G^{(1)'}(0) - \frac{1}{J^3}\left(G^{(2)'}(0) - \frac{1}{24}G^{(0)'''}(0)\right) + \mathcal{O}\left(\frac{1}{J^4}\right). \quad (341)$$

Recall that the first charge Q_1 is the total magnon momentum (247), which according to the momentum condition (244), (273) reads

$$Q_1 = \varpi = \mu\alpha. \quad (342)$$

Having expressed it through the resolvent in (337), (340), we can rewrite the momentum condition as constraints on the large- J terms of the resolvent,

$$G^{(0)}(0) = -\mu\alpha, \quad (343)$$

$$G^{(1)}(0) = 0, \quad (344)$$

$$G^{(2)}(0) = \frac{1}{192\omega}\left(\omega^2\left(G^{(0)'''}(\omega) - G^{(0)'''}(-\omega)\right) + 3\omega\left(G^{(0)''}(\omega) + G^{(0)''}(-\omega)\right) + \left(G^{(0)'}(\omega) - G^{(0)'}(-\omega)\right)\right), \quad (345)$$

etc. These restraints must be taken into account while solving the quadratic equation for the resolvent.

In particular, for $\omega = 0$, these conditions reduce to

$$G^{(0)}(0) = -\mu\alpha, \quad (346)$$

$$G^{(1)}(0) = 0, \quad (347)$$

$$G^{(2)}(0) = \frac{1}{24}G^{(0)''}(0), \quad (348)$$

etc.

The second charge Q_2 describes the string energy according to $E = 2g^2Q_2 = 2\omega^2J^2Q_2$ (248), which is the quantity we eventually want to compute. Therefore, finding the three leading large- J orders of the resolvent allows us (338), (341) to get the three leading terms of the all-loop string energy,

$$E = JE^{(0)} + E^{(1)} + \frac{1}{J}E^{(2)} + \mathcal{O}\left(\frac{1}{J^2}\right), \quad (349)$$

namely,

$$E^{(0)} = -2\omega^2G^{(0)'}(0), \quad (350)$$

$$E^{(1)} = -2\omega^2G^{(1)'}(0), \quad (351)$$

$$E^{(2)} = -2\omega^2\left(G^{(2)'}(0) - \frac{1}{192\omega}\left(\omega\left(G^{(0)'''}(\omega) + G^{(0)'''}(-\omega)\right) + 3\left(G^{(0)''}(\omega) - G^{(0)''}(-\omega)\right)\right)\right). \quad (352)$$

In particular, setting $\omega = 0$ everywhere except for the prefactor ω^2 yields the one-loop string energy,

$$E^{(0)} = -2\omega^2G^{(0)'}(0), \quad (353)$$

$$E^{(1)} = -2\omega^2G^{(1)'}(0), \quad (354)$$

$$E^{(2)} = -2\omega^2\left(G^{(2)'}(0) - \frac{1}{24}G^{(0)'''}(0)\right). \quad (355)$$

B. The Large- J Expansion of the Logarithmic All-Loop String Bethe Ansatz Equations in the $SL(2)$ Sector up to the Order $O(1/J^2)$

In this subsection, we expand our basic equation (272) at large J , keeping in mind the condition (279). The end all-loop result is given in formula (417), while its simplified one-loop version in (418).

1. The l.h.s.

The l.h.s. of (272) is easily expandable,

$$(\text{the l.h.s. of eq. (272)})_k = -iJ \log \left(\frac{x_k^+}{x_k^-} \right) - \mu = -\mu + \frac{y_k}{y_k^2 - \omega^2} - \frac{1}{J^2} \frac{y_k^3 (\omega^4 + 4y_k^2 \omega^2 + y_k^4)}{12 (y_k^2 - \omega^2)^5} + O \left(\frac{1}{J^4} \right). \quad (356)$$

Notice that there are no terms of order $O(1/J)$ and $O(1/J^3)$.

In particular, for $\omega = 0$, (356) reduces to

$$-iJ \log \left(\frac{x_k^+}{x_k^-} \right) - \mu \Big|_{\omega=0} = -\mu + \frac{1}{y_k} - \frac{1}{J^2} \frac{1}{12 y_k^3} + O \left(\frac{1}{J^4} \right). \quad (357)$$

2. The r.h.s., Term I

On the r.h.s. of (272), term I is not problematic either, however somewhat longer,

$$\begin{aligned} (\text{the r.h.s. of eq. (272), term I})_{j,k} &= -2i \log \left(\frac{1 - \frac{q^2}{x_k^+ x_j^-}}{1 - \frac{q^2}{x_j^+ x_k^-}} \right) = \\ &= \frac{1}{J} \frac{2\omega^2 (y_j - y_k) (y_j y_k + \omega^2)}{(y_j^2 - \omega^2) (y_k^2 - \omega^2) (y_j y_k - \omega^2)} - \\ &- \frac{1}{J^3} \frac{\omega^2 (y_j - y_k)}{6 (y_j^2 - \omega^2)^5 (y_k^2 - \omega^2)^5 (y_j y_k - \omega^2)^3} \left((y_j^2 + y_j y_k + y_k^2) \omega^{18} + 4 (y_j^4 + y_j y_k^3 - y_j^2 y_k^2 + y_j^3 y_k + y_k^4) \omega^{16} + \right. \\ &\quad + (y_j^6 - 8y_j y_k^5 - 22y_j^2 y_k^4 - 44y_j^3 y_k^3 - 22y_j^4 y_k^2 - 8y_j^5 y_k + y_k^6) \omega^{14} + \\ &\quad + y_j y_k (-3y_j^6 - 8y_j y_k^5 + 68y_j^2 y_k^4 + 84y_j^3 y_k^3 + 68y_j^4 y_k^2 - 8y_j^5 y_k - 3y_k^6) \omega^{12} + \\ &\quad + y_j^2 y_k^2 (3y_j^6 - 5y_j y_k^5 + 2y_j^2 y_k^4 - 96y_j^3 y_k^3 + 2y_j^4 y_k^2 - 5y_j^5 y_k + 3y_k^6) \omega^{10} + \\ &\quad + y_j^3 y_k^3 (7y_j^6 - 8y_j y_k^5 - 10y_j^2 y_k^4 - 116y_j^3 y_k^3 - 10y_j^4 y_k^2 - 8y_j^5 y_k + 7y_k^6) \omega^8 + \\ &\quad + y_j^5 y_k^5 (-14y_j^4 + 73y_j y_k^3 + 80y_j^2 y_k^2 + 73y_j^3 y_k - 14y_k^4) \omega^6 - \\ &\quad - y_j^6 y_k^6 (12y_j^4 + 5y_j y_k^3 + 44y_j^2 y_k^2 + 5y_j^3 y_k + 12y_k^4) \omega^4 + \\ &\quad \left. + 3y_j^7 y_k^7 (y_j^4 + 2y_j y_k^3 - 7y_j^2 y_k^2 + 2y_j^3 y_k + y_k^4) \omega^2 + 3y_j^9 y_k^9 (y_j^2 + y_k^2) \right) + O \left(\frac{1}{J^4} \right). \quad (358) \end{aligned}$$

Notice that there is no term of order $O(1/J^2)$.

In particular, for $\omega = 0$, (358) completely disappears.

3. The r.h.s., Term II (the “Phase”)

On the r.h.s. of (272), term II equals twice the dressing phase. Substituting to the expression for $\theta(x_k, x_j; J\omega)$ (256) the three leading terms (261), (262), (264) of the large- J expansion (260) of the coefficients $c_{r,s}(J\omega)$, as well as expanding at large J the phase’s magnon–charge part with help of the series (334) of $q_t(x_k)$ — we find term II as a product of two series,

$$\begin{aligned}
 & (\text{the r.h.s. of eq. (272), term II})_{j,k} = 2\theta(x_k, x_j; J\omega) = \frac{2}{J} \sum_{r \geq 2} \sum_{s \geq r+1} \omega^{r+s-1} \cdot \\
 & \cdot \left(\delta_{r+1,s} + \frac{1}{J\omega} \frac{1}{\pi} ((-1)^{r+s} - 1) \frac{(r-1)(s-1)}{(s+r-2)(s-r)} + \frac{1}{J^2 \omega^2} \frac{1}{48} (1 - (-1)^{r+s}) (r-1)(s-1) + \mathcal{O}\left(\frac{1}{J^3}\right) \right) \cdot \\
 & \cdot \frac{y_j^2 y_k^2}{(y_j^2 - \omega^2)(y_k^2 - \omega^2)} \left(\frac{1}{y_k^r y_j^s} - \frac{1}{y_j^r y_k^s} + \right. \\
 & + \frac{1}{J^2} \frac{1}{24 (y_j^2 - \omega^2)^4 (y_k^2 - \omega^2)^4} \left(\frac{1}{y_j^r y_k^s} \left(((r^2 - 5r + 6) y_j^2 + (s^2 - 5s + 6) y_k^2) \omega^{12} - \right. \right. \\
 & - 2((r^2 - 2r - 3) y_j^4 + 2(r^2 - 5r + s^2 - 5s + 12) y_j^2 y_k^2 + (s^2 - 2s - 3) y_k^4) \omega^{10} + \\
 & + (r(r+1) y_j^6 + 2(4r(r-2) + 3(s^2 - 5s + 2)) y_j^4 y_k^2 + 2(4s(s-2) + 3(r^2 - 5r + 2)) y_j^2 y_k^4 + s(s+1) y_k^6) \omega^8 - \\
 & - 4y_j^2 y_k^2 ((r^2 + r + s^2 - 5s + 6) y_j^4 + 3(r^2 - 2r + s^2 - 2s - 6) y_j^2 y_k^2 + (s^2 + s + r^2 - 5r + 6) y_k^4) \omega^6 + \\
 & + y_j^2 y_k^2 ((s^2 - 5s + 6) y_j^6 + 2(3r^2 + 3r + 4s^2 - 8s - 12) y_k^2 y_j^4 + 2(3s^2 + 3s + 4r^2 - 8r - 12) y_k^4 y_j^2 + (r^2 - 5r + 6) y_k^6) \omega^4 - \\
 & \left. \left. - 2y_j^4 y_k^4 ((s^2 - 2s - 3) y_j^4 + 2(r^2 + r + s^2 + s) y_j^2 y_k^2 + (r^2 - 2r - 3) y_k^4) \omega^2 + y_j^6 y_k^6 (s(s+1) y_j^2 + r(r+1) y_k^2) \right) \right) - \\
 & \left. - (r \leftrightarrow s) \right) + \mathcal{O}\left(\frac{1}{J^4}\right) \Bigg) \cdot \tag{359}
 \end{aligned}$$

Notice that in the second series, there are no terms of order $\mathcal{O}(1/J)$ and $\mathcal{O}(1/J^3)$. Remark also that we could have used the language of the function χ , (257), (258), (259), (265), (266), (267), (268).

We will now multiply these two expansions according to the following pattern: Consider the second line of (359), *i.e.*, the series of $c_{r,s}(J\omega)$; it consists of three terms,

$$\underbrace{\delta_{r+1,s}}_{\rightarrow \text{term IIa}} + \underbrace{\frac{1}{J\omega} \frac{1}{\pi} ((-1)^{r+s} - 1) \frac{(r-1)(s-1)}{(r+s-2)(s-r)}}_{\rightarrow \text{term IIb}} + \underbrace{\frac{1}{J^2 \omega^2} \frac{1}{48} (1 - (-1)^{r+s}) (r-1)(s-1)}_{\rightarrow \text{term IIc}}. \tag{360}$$

We will multiply each of these pieces by the remaining part of (359), and then do the summation over r, s for each of these three terms separately; this will give rise to three sub-terms of the current term II, which we will call “IIa,” “IIb,” “IIc.”

Term IIa corresponds to (359) with its second line replaced by just $\delta_{r+1,s}$, and describes the Arutyunov–Frolov–Staudacher contribution. The summation over r, s can then be done explicitly, resulting in

$$2\theta(x_k, x_j; J\omega)|_{\text{term IIa}} = -\frac{1}{J} \frac{2\omega^4 (y_j - y_k)}{(y_j^2 - \omega^2)(y_k^2 - \omega^2)(y_j y_k - \omega^2)} +$$

$$\begin{aligned}
& + \frac{1}{J^3} \frac{\omega^4 (y_j - y_k)}{6 (y_j^2 - \omega^2)^5 (y_k^2 - \omega^2)^5 (y_j y_k - \omega^2)^3} \cdot \\
& \cdot \left((y_j^2 + y_j y_k + y_k^2) \omega^{16} + (4y_j^4 + y_j^3 y_k - 4y_j^2 y_k^2 + y_j y_k^3 + 4y_k^4) \omega^{14} + \right. \\
& + (y_j + y_k)^2 (y_j^4 - 13y_j^3 y_k + 3y_j^2 y_k^2 - 13y_j y_k^3 + y_k^4) \omega^{12} + \\
& + y_j y_k (-3y_j^6 + 4y_j^5 y_k + 62y_j^4 y_k^2 + 84y_j^3 y_k^3 + 62y_j^2 y_k^4 + 4y_j y_k^5 - 3y_k^6) \omega^{10} + \\
& + y_j^2 y_k^2 (3y_j^6 + 4y_j^5 y_k - 46y_j^4 y_k^2 - 132y_j^3 y_k^3 - 46y_j^2 y_k^4 + 4y_j y_k^5 + 3y_k^6) \omega^8 + \\
& + y_j^3 y_k^3 (y_j^6 - 8y_j^5 y_k + 14y_j^4 y_k^2 + 28y_j^3 y_k^3 + 14y_j^2 y_k^4 - 8y_j y_k^5 + y_k^6) \omega^6 + \\
& + y_j^5 y_k^5 (-5y_j^4 + 25y_j^3 y_k + 44y_j^2 y_k^2 + 25y_j y_k^3 - 5y_k^4) \omega^4 - \\
& \left. - 11y_j^7 y_k^7 (y_j^2 + 4y_j y_k + y_k^2) \omega^2 + 3y_j^8 y_k^8 (2y_j^2 + y_j y_k + 2y_k^2) \right) + \mathcal{O}\left(\frac{1}{J^5}\right). \tag{361}
\end{aligned}$$

Notice that there are no terms of order $\mathcal{O}(1/J^2)$ and $\mathcal{O}(1/J^4)$.

To get Term IIc, we take (359) with its second line replaced by just $\frac{1}{J^2 \omega^2} \frac{1}{48} (1 - (-1)^{r+s}) (r-1)(s-1)$. It, too, can be explicitly summed up over r, s , yielding

$$\begin{aligned}
2\theta(x_k, x_j; J\omega)|_{\text{term IIc}} &= -\frac{1}{J^3} \frac{\omega^2 y_j^2 y_k^2 (y_j - y_k)}{6 (y_j^2 - \omega^2)^3 (y_k^2 - \omega^2)^3 (y_j y_k - \omega^2)^3} \cdot \\
& \cdot \left(\omega^8 + 2y_j y_k \omega^6 - 2y_j y_k (y_j^2 + y_j y_k + y_k^2) \omega^4 + 2y_j^3 y_k^3 \omega^2 + y_j^4 y_k^4 \right) + \mathcal{O}\left(\frac{1}{J^5}\right). \tag{362}
\end{aligned}$$

Notice that there is no term of order $\mathcal{O}(1/J^4)$.

Term IIb, corresponding to (359) with its second line replaced by just $\frac{1}{J\omega} \frac{1}{\pi} ((-1)^{r+s} - 1) \frac{(r-1)(s-1)}{(r+s-2)(s-r)}$, and describing the Hernández-López contribution, is more complicated, but the double sum over r, s can be done by employing the same technique as used to derive the Hernández-López function $\chi^{(1)}(x_1, x_2; g)$ (267), which we outline in the next paragraph II B 4,

$$\begin{aligned}
& 2\theta(x_k, x_j; J\omega)|_{\text{term IIb}} = \\
& = \frac{1}{J^2} \frac{1}{\pi} \frac{2y_j^2 y_k^2}{\omega^2 (y_j^2 - \omega^2) (y_k^2 - \omega^2)} \sum_{r \geq 2} \sum_{s \geq r+1} \omega^{r+s} ((-1)^{r+s} - 1) \frac{(r-1)(s-1)}{(r+s-2)(s-r)} \left(\frac{1}{y_k^r y_j^s} - \frac{1}{y_j^r y_k^s} \right) + \mathcal{O}\left(\frac{1}{J^4}\right) = \tag{363} \\
& = -\frac{1}{J^2} \frac{1}{\pi} \frac{y_j^2 y_k^2}{(y_j^2 - \omega^2) (y_k^2 - \omega^2)} \left(\frac{2\omega}{(y_j - y_k) (y_j y_k - \omega^2)} + \right. \\
& \left. + \left(\frac{1}{(y_j - y_k)^2} + \frac{\omega^2}{(y_j y_k - \omega^2)^2} \right) \log \left(\frac{(y_j + \omega) (y_k - \omega)}{(y_j - \omega) (y_k + \omega)} \right) \right) + \mathcal{O}\left(\frac{1}{J^4}\right). \tag{364}
\end{aligned}$$

Notice that there is no term of order $O(1/J^3)$. The double sum (363) is convergent provided the condition (279) is met, which is another reason for its relevance.

We have thus completed in (361), (364), (362) our derivation of the three leading large- J orders of term II. Together with term I (358), we obtain

$$\begin{aligned}
& (\text{the r.h.s. of eq. (272), term I + term II})_{j,k} = \\
& = \frac{1}{J} \frac{2\omega^2 y_j y_k (y_j - y_k)}{(y_j^2 - \omega^2)(y_k^2 - \omega^2)(y_j y_k - \omega^2)} - \\
& - \frac{1}{J^2} \frac{1}{\pi} \frac{y_j^2 y_k^2}{(y_j^2 - \omega^2)(y_k^2 - \omega^2)} \left(\frac{2\omega}{(y_j - y_k)(y_j y_k - \omega^2)} + \left(\frac{1}{(y_j - y_k)^2} + \frac{\omega^2}{(y_j y_k - \omega^2)^2} \right) \log \left(\frac{(y_j + \omega)(y_k - \omega)}{(y_j - \omega)(y_k + \omega)} \right) \right) - \\
& - \frac{1}{J^3} \frac{\omega^2 y_j y_k (y_j - y_k)}{6 (y_j^2 - \omega^2)^5 (y_k^2 - \omega^2)^5 (y_j y_k - \omega^2)^3} \cdot \\
& \cdot \left((3y_j^2 + y_j y_k + 3y_k^2) \omega^{16} + (3y_j^4 - 2y_j^3 y_k - 22y_j^2 y_k^2 - 2y_j y_k^3 + 3y_k^4) \omega^{14} + \right. \\
& + y_j y_k (11y_j^4 - 2y_j^2 y_k^2 + 11y_k^4) \omega^{12} + y_j^2 y_k^2 (-3y_j^4 + 50y_j^3 y_k + 54y_j^2 y_k^2 + 50y_j y_k^3 - 3y_k^4) \omega^{10} + \\
& + 2y_j^2 y_k^2 (2y_j^6 - y_j^5 y_k - 21y_j^4 y_k^2 - 75y_j^3 y_k^3 - 21y_j^2 y_k^4 - y_j y_k^5 + 2y_k^6) \omega^8 + \\
& + y_j^4 y_k^4 (-3y_j^4 + 50y_j^3 y_k + 54y_j^2 y_k^2 + 50y_j y_k^3 - 3y_k^4) \omega^6 - y_j^5 y_k^5 (11y_j^4 - 2y_j^2 y_k^2 + 11y_k^4) \omega^4 + \\
& \left. + y_j^6 y_k^6 (3y_j^4 - 2y_j^3 y_k - 22y_j^2 y_k^2 - 2y_j y_k^3 + 3y_k^4) \omega^2 + y_j^8 y_k^8 (3y_j^2 + y_j y_k + 3y_k^2) \right) + O\left(\frac{1}{J^4}\right). \quad (365)
\end{aligned}$$

In particular, for $\omega = 0$, this whole expression (365) vanishes, *i.e.*, it is only the l.h.s. and term III (the “anomaly”) of (272) which contribute at one loop.

4. The r.h.s., Term II (the “Phase”). Appendix

In this appendix to the previous paragraph, let us describe how to tackle certain double sums appearing in various computations related to the Hernández-López dressing phase.

Let us start from justifying the passage from (363) to (364). Denoting $p \equiv \omega/y_k$, $q \equiv \omega/y_j$, let us prove the following identity,

$$\begin{aligned}
& \sum_{r \geq 2} \sum_{s \geq r+1} (1 - (-1)^{r+s}) \frac{(r-1)(s-1)}{(s+r-2)(s-r)} (p^r q^s - p^s q^r) = \\
& = p^2 q^2 \left(\frac{1}{(p-q)(1-pq)} + \left(\frac{1}{(1-pq)^2} + \frac{1}{(p-q)^2} \right) (\operatorname{arctanh}(q) - \operatorname{arctanh}(p)) \right), \quad (366)
\end{aligned}$$

where p and q must submit to (377). Dividing both sides by $p^2 q^2$, and changing the summation variables from r, s to $t \equiv r-1$, $u \equiv s-r$, which makes the two summation ranges decouple, we equivalently get

$$\sum_{t \geq 1} \sum_{u \geq 1} (1 - (-1)^u) \frac{t(t+u)}{u(2t+u)} p^{t-1} q^{t-1} (q^u - p^u) =$$

$$= \frac{1}{(p-q)(1-pq)} + \left(\frac{1}{(1-pq)^2} + \frac{1}{(p-q)^2} \right) (\operatorname{arctanh}(q) - \operatorname{arctanh}(p)). \quad (367)$$

The sum over u is just over odd values, $u = 2v-1$, due to the prefactor $(1 - (-1)^u)$, and hence, it is further equivalent to

$$\begin{aligned} & \frac{2}{pq} \sum_{t \geq 1} \sum_{v \geq 1} \frac{t(t+2v-1)}{(2v-1)(2t+2v-1)} p^t q^t (q^{2v-1} - p^{2v-1}) = \\ & = \frac{1}{(p-q)(1-pq)} + \left(\frac{1}{(1-pq)^2} + \frac{1}{(p-q)^2} \right) (\operatorname{arctanh}(q) - \operatorname{arctanh}(p)). \end{aligned} \quad (368)$$

This is $(2/(pq))$ times) the anti-symmetrization w.r.t. p and q of the sum

$$\text{sum}_1 \equiv \sum_{t \geq 1} \sum_{v \geq 1} \frac{t(t+2v-1)}{(2v-1)(2t+2v-1)} p^t q^{t+2v-1}, \quad (369)$$

whose closed form we will now attempt to find. Let us be completely general, and allow p, q to acquire any complex values.

The trick is to rewrite

$$p^t q^{t+2v-1} = P^{2t+2v-1} Q^{2v-1}, \quad \text{where} \quad P \equiv \sqrt{p}\sqrt{q}, \quad Q \equiv \frac{\sqrt{q}}{\sqrt{p}}. \quad (370)$$

This equality is true for any complex $p \neq 0, q$. The square roots are understood to be the principal ones, and our convention for their values at the cut $(-\infty, 0)$ is $\sqrt{p} = \lim_{\epsilon \rightarrow 0^+} \sqrt{p + i\epsilon} = +i\sqrt{|p|}$, for p real and negative. This trick is useful because the powers of P and Q match exactly the numbers in the denominator of (369), which suggests that one should differentiate w.r.t. both P and Q , thus canceling the denominator, and reducing the sums to ones easily doable,

$$\frac{d}{dQ} \frac{d}{dP} \text{sum}_1 = \sum_{t \geq 1} \sum_{v \geq 1} t(t+2v-1) P^{2(t+v-1)} Q^{2(v-1)} = \frac{2P^2}{(1-P^2)^3} \frac{1-P^4 Q^2}{(1-P^2 Q^2)^2}. \quad (371)$$

Essentially, after differentiation we obtain two geometric sums, and they converge only for P, Q obeying

$$P, Q \in \mathbb{C}, \quad |PQ| < 1, \quad |P| < 1. \quad (372)$$

The problem is in this way reduced to computing two (indefinite) integrals. First, let us integrate (371) w.r.t. Q ,

$$\frac{d}{dP} \text{sum}_1 = \frac{P^2 Q}{(1-P^2)^2 (1-P^2 Q^2)} + \frac{P(1+P^2)}{(1-P^2)^3} \operatorname{arctanh}(PQ) + f_1(P), \quad (373)$$

where f_1 is an unknown function of P only; using the boundary condition that the l.h.s. of (373) vanishes at $Q = 0$, we determine it to be $f_1(P) = 0$, for all P .

The integral w.r.t. P is slightly more complicated, but can be reduced to ones of the following kind,

$$\int dP \frac{\log(1-P)}{P^2} = -\frac{\log(1-P)}{P} + \log(P-1) - \log(P) + \text{const}, \quad (374)$$

$$\int dP \frac{\log(1-P)}{P^3} = \frac{1}{2P} - \frac{\log(1-P)}{2P^2} + \frac{\log(P-1)}{2} - \frac{\log(P)}{2} + \text{const}, \quad (375)$$

and yields

$$\text{sum}_1 = \frac{1}{4(1-P^2)^2(1-Q^2)^2} \left(PQ(1-P^2)(1-Q^2) - \right.$$

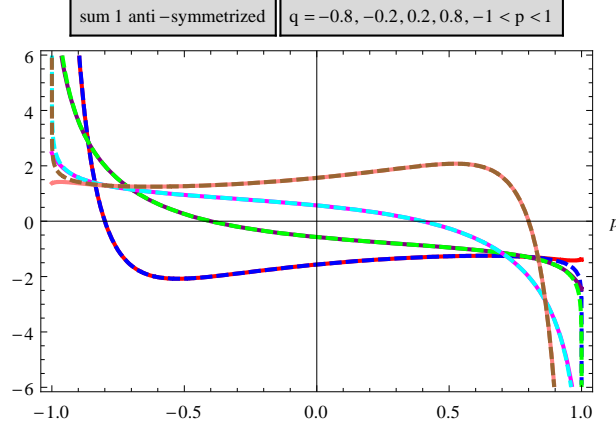


FIG. 14: Plots of both sides (the l.h.s. in solid lines, the r.h.s. in dashed lines) of (379) versus $p \in (-1, 1)$, for four values of $q \in (-1, 1)$, namely $q = -0.8$ (red and dashed blue), -0.2 (purple and dashed green), $+0.2$ (magenta and dashed cyan), $+0.8$ (pink and dashed brown). The two infinite sums are terminated at $t_{\max} = v_{\max} = 20$, which, as we see, is enough to accurately reproduce the exact result. We observe perfect agreement.

$$- (1 - P^2)^2 Q (1 + Q^2) \operatorname{arctanh}(P) + 2 (P^2 + Q^2 + P^2 Q^2 (P^2 + Q^2 - 4)) \operatorname{arctanh}(PQ) \Big) + f_2(Q), \quad (376)$$

where an unknown function f_2 of Q only is then derived from the boundary condition that the l.h.s. of (376) zeroes at $P = 0$ to be $f_2(Q) = 0$, for all Q . The result (376) can be checked by differentiating both sides w.r.t. P . We have also tested it numerically, assuming that the $\operatorname{arctanh}$'s on the r.h.s. denote the principal branch of hyperbolic arcus tangent, where our convention for the value of the principal logarithm at its cut $(-\infty, 0)$ is $\log(p) = \lim_{\epsilon \rightarrow 0^+} \log(p + i\epsilon)$, for p real and negative.

Returning to the variables p, q in (376), and subtracting from it the same equality but with p and q interchanged, we finally prove (368), as desired. It holds for

$$p, q \in \mathbb{C}, \quad |p| < 1, \quad |q| < 1, \quad (377)$$

i.e.,

$$y_j, y_k \in \mathbb{C}, \quad |y_j| > \omega, \quad |y_k| > \omega. \quad (378)$$

Note that in our case y_j and y_k are real, and satisfy the inequalities (378) according to (279). Remark also that restricting to real y_j and y_k allows us to rewrite (366) in a form more often found in literature, which we have also used in (364),

$$\begin{aligned} & \sum_{r \geq 2} \sum_{s \geq r+1} (1 - (-1)^{r+s}) \frac{(r-1)(s-1)}{(s+r-2)(s-r)} (p^r q^s - p^s q^r) = \\ & = p^2 q^2 \left(\frac{1}{(p-q)(1-pq)} + \frac{1}{2} \left(\frac{1}{(1-pq)^2} + \frac{1}{(p-q)^2} \right) \log \left(\frac{(1-p)(1+q)}{(1+p)(1-q)} \right) \right), \quad \text{for } p, q \in (-1, 1). \end{aligned} \quad (379)$$

In figure 14, we numerically verify (379) by plotting both sides as functions of $p \in (-1, 1)$ for several values of $q \in (-1, 1)$.

The same method can be exploited to derive $\chi^{(1)}(x_1, x_2; g)$ (267) from (259) and (262). Denoting this time $p \equiv g/x_1$, $q \equiv g/x_2$, and again assuming them to be arbitrary complex numbers, we aim at finding the double sum

$$\text{sum}_2 \equiv \frac{\chi^{(1)}(x_1, x_2; g)}{g} = \frac{1}{\pi} \sum_{r \geq 2} \sum_{s \geq r+1} (1 - (-1)^{r+s}) \frac{1}{(r+s-2)(s-r)} p^{r-1} q^{s-1}. \quad (380)$$

Proceeding as before from the summation indices r, s to t, v , we obtain equivalently,

$$\text{sum}_2 = \frac{2}{\pi} \sum_{t \geq 1} \sum_{v \geq 1} \frac{1}{(2v-1)(2t+2v-1)} p^t q^{t+2v-1}, \quad (381)$$

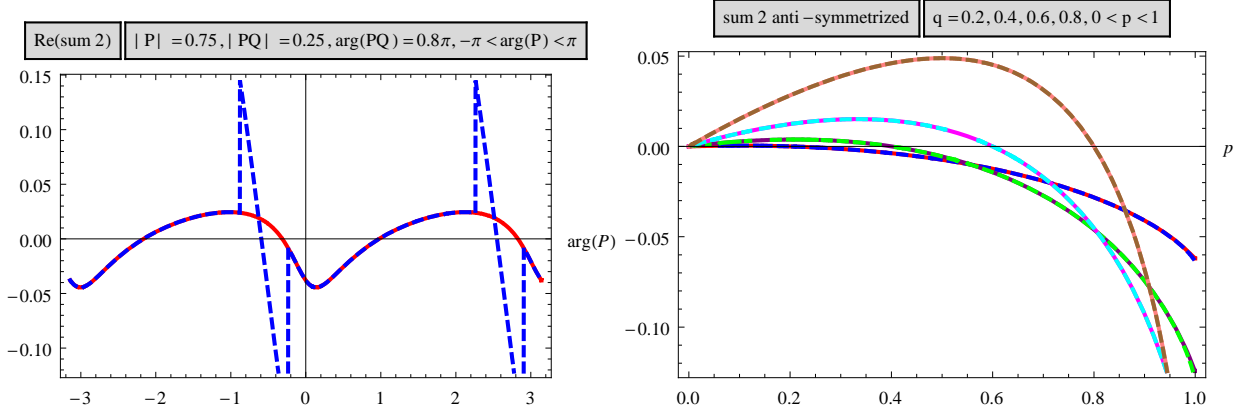


FIG. 15: LEFT: Plots of the real parts of both sides (the l.h.s. in a solid red line, the r.h.s. in a dashed blue line) of (385) versus $\arg(P) \in (-\pi, \pi)$, where we fix $PQ = 0.25e^{0.8\pi i}$ and $|P| = 0.75$, and we assume the principal branches of all the logarithms and dilogarithms. We discover that the principal branches do not always correctly reproduce the sum.

RIGHT: Plots of the anti-symmetrization of both sides (the l.h.s. in solid lines, the r.h.s. in dashed lines) of (387) versus $p \in (0, 1)$, for four values of $q \in (0, 1)$, namely $q = 0.2$ (red and dashed blue), 0.4 (purple and dashed green), 0.6 (magenta and dashed cyan), 0.8 (pink and dashed brown). The two infinite sums are terminated at $t_{\max} = v_{\max} = 20$. We observe perfect agreement.

and the trick (370) yields

$$\frac{d}{dQ} \frac{d}{dP} \text{sum}_2 = \frac{2}{\pi} \sum_{t \geq 1} \sum_{v \geq 1} P^{2(t+v-1)} Q^{2(v-1)} = \frac{2}{\pi} \frac{P^2}{1-P^2} \frac{1}{1-P^2 Q^2}, \quad (382)$$

where the double sum after differentiation is just two geometric series, even simpler than in (371), and convergent under the same assumptions as before (372).

Integrating w.r.t. Q leads to

$$\frac{d}{dP} \text{sum}_2 = \frac{2}{\pi} \frac{P}{1-P^2} \text{arctanh}(PQ) + f_1(P), \quad (383)$$

where again the boundary condition implies $f_1(P) = 0$, for all P .

Even though the integral w.r.t. P looks simpler at first than (373), it is actually more involved. This time, it can be reduced to ones featuring dilogarithm and not usual logarithms (374), (375),

$$\int dP \frac{\log(1-P)}{P} = -\text{Li}_2(P) + \text{const}, \quad (384)$$

and yields

$$\begin{aligned} \text{sum}_2 = & \frac{1}{2\pi} \left(\text{Li}_2 \left(\frac{1-PQ}{1+Q} \right) + \text{Li}_2 \left(\frac{1-PQ}{1-Q} \right) - \text{Li}_2 \left(\frac{1+PQ}{1+Q} \right) - \text{Li}_2 \left(\frac{1+PQ}{1-Q} \right) + \right. \\ & + \log \left(\frac{Q(1-P)}{Q-1} \right) \log(1-PQ) + \log \left(\frac{Q(1+P)}{Q+1} \right) \log(1-PQ) - \\ & \left. - \log \left(\frac{Q(1+P)}{Q-1} \right) \log(1+PQ) - \log \left(\frac{Q(1-P)}{Q+1} \right) \log(1+PQ) \right) + f_2(Q), \end{aligned} \quad (385)$$

where again we find $f_2(Q) = 0$, for all Q . This can be confirmed by differentiating both sides w.r.t. P . The function (385) has a very complicated cut structure. In particular, when testing it numerically, principal branches of logarithm

and dilogarithm (for the latter one, we assume its value at the cut $(1, +\infty)$ to be $\text{Li}_2(p) = \lim_{\epsilon \rightarrow 0^+} \text{Li}_2(p - i\epsilon)$, for p real and greater than 1) do not always give the correct value of the sum, as exemplified in figure 15, left.

To avoid discussing the cut structure of this general expression, let us restrict our attention to P, Q real and obeying (372). Then, and with help of a dilogarithm identity, $\text{Li}_2(p) = \pi^2/6 - \log(p) \log(1-p) - \text{Li}_2(1-p)$, we simplify (385) to be

$$\begin{aligned} \text{sum}_2 = & \frac{1}{2\pi} \left(\text{Li}_2 \left(\frac{Q(1+P)}{Q-1} \right) + \text{Li}_2 \left(\frac{Q(1-P)}{Q+1} \right) - \text{Li}_2 \left(\frac{Q(1+P)}{Q+1} \right) - \text{Li}_2 \left(\frac{Q(1-P)}{Q-1} \right) + \right. \\ & \left. + \log \left(\frac{1-P}{1+P} \right) \left(\log \left(\frac{1}{1+Q} \right) - \log \left(\frac{1}{1-Q} \right) \right) \right), \quad \text{for } P, PQ \in (-1, 1). \end{aligned} \quad (386)$$

Remark that even in this narrower case, Q can be any real number. In terms of p, q , real P, Q are realized provided p, q are real and of the same sign. Let us thus restrict ourselves even more, assuming p, q to be real and positive. Then

$$\begin{aligned} \text{sum}_2 = & \frac{1}{2\pi} \left(\text{Li}_2 \left(\frac{\frac{1}{\sqrt{p}} + \sqrt{q}}{\frac{1}{\sqrt{p}} - \frac{1}{\sqrt{q}}} \right) + \text{Li}_2 \left(\frac{\frac{1}{\sqrt{p}} - \sqrt{q}}{\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}} \right) - \text{Li}_2 \left(\frac{\frac{1}{\sqrt{p}} + \sqrt{q}}{\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}} \right) - \text{Li}_2 \left(\frac{\frac{1}{\sqrt{p}} - \sqrt{q}}{\frac{1}{\sqrt{p}} - \frac{1}{\sqrt{q}}} \right) + \right. \\ & \left. + \log \left(\frac{\frac{1}{\sqrt{p}} - \sqrt{q}}{\frac{1}{\sqrt{p}} + \sqrt{q}} \right) \left(\log \left(\frac{1}{\frac{1}{\sqrt{q}} + \frac{1}{\sqrt{p}}} \right) - \log \left(\frac{1}{\frac{1}{\sqrt{q}} - \frac{1}{\sqrt{p}}} \right) \right) \right), \quad \text{for } q, pq \in (0, 1), \end{aligned} \quad (387)$$

which in the language of x_1, x_2 is printed in (267). In figure 15, right, we pictorially confirm (387) by plotting the anti-symmetrization w.r.t. p and q of both sides as functions of $p \in (0, 1)$ for several values of $q \in (0, 1)$, where we use the principal branches of logarithm and dilogarithm.

5. The r.h.s., Term III (the “Anomaly”). The Non-Anomalous and Anomalous Parts

We now want to expand term III of the r.h.s. of (272) at large J up to and including three leading orders,

$$(\text{the r.h.s. of eq. (272), term III})_{j,k} = -i \log \left(\frac{u_k - u_j - i}{u_k - u_j + i} \right). \quad (388)$$

The problem is that this term, unlike the rest of the equation, features the difference of the rapidities, $(u_k - u_j)$, which may behave in different ways when J increases to infinity, depending on the separation of the indices j and k . We will briefly recall here how the two types of its behavior, the so-called “non-anomalous” and “anomalous” regimes, arise.

For the sake of simplicity, and because only this case is relevant for our computation, let us assume that the Bethe roots condense in just one cut. Also, if there are several cuts, and u_j and u_k belong to different ones, then, as we will see, their difference $(u_k - u_j)$ fits into the non-anomalous regime, and no strange phenomenon happens.

Let us fix the index k as a reference point in such a way that u_k lies far away from the endpoints of the cut,

$$k \sim J, \quad S - k \sim J. \quad (389)$$

We will investigate the difference $(u_k - u_j)$ as a function of the separation

$$n \equiv k - j. \quad (390)$$

A priori, we should not assume this, as the formulae which result from the considerations based on (389) will be used also close to the edges.

A typical (non-anomalous) situation, which occurs for most of the Bethe roots, is when the index j is far away enough from the index k so that the spacing $(u_k - u_j)$ is of order $O(J)$,

$$|n| > N, \quad \text{so that} \quad u_k - u_j \sim J, \quad (391)$$

where $1 \ll N \ll J$ is some integer.

In this case, the large- J series of term III is simply an expansion in large $(u_k - u_j)$, and so, can be easily found,

$$-i \log \left(\frac{u_k - u_j - i}{u_k - u_j + i} \right) \Big|_{\text{non-anomalous}} = -2 \frac{1}{u_k - u_j} + \frac{2}{3} \frac{1}{(u_k - u_j)^3} + O\left(\frac{1}{J^5}\right). \quad (392)$$

When summed over j such that $|n| > N$, it produces

$$-i \sum_{\substack{j: \\ |k-j| > N}} \log \left(\frac{u_k - u_j - i}{u_k - u_j + i} \right) = -2 \sum_{\substack{j: \\ |k-j| > N}} \frac{1}{u_k - u_j} + \frac{2}{3} \sum_{\substack{j: \\ |k-j| > N}} \frac{1}{(u_k - u_j)^3} + O\left(\frac{1}{J^4}\right). \quad (393)$$

Notice that there are no terms of order $O(1/J)$ and $O(1/J^3)$.

The anomalous part of the expansion arises for the indices j and k close enough to each other so that the spacing $(u_k - u_j)$ becomes of the lower order $O(J^0)$,

$$|n| \leq N, \quad \text{so that} \quad u_k - u_j \sim 1. \quad (394)$$

In this case, it can be approximated by its Taylor expansion in $n/J \ll 1$ in the following way: Introduce a smooth function v which describes the large- J distribution of the rescaled rapidities,

$$\frac{u_k}{J} \approx v\left(\frac{k}{J}\right). \quad (395)$$

Now, since j/J is very close to k/J , we can expand,

$$\frac{u_k - u_j}{J} = v\left(\frac{k}{J}\right) - v\left(\frac{j}{J}\right) = v'\left(\frac{k}{J}\right) \frac{n}{J} - \frac{v''\left(\frac{k}{J}\right) n^2}{2} + O\left(\frac{n^3}{J^3}\right). \quad (396)$$

Let us rewrite this series in a language we have already used: Notice (287) that the quantity

$$\frac{1}{v'\left(\frac{k}{J}\right)} \approx \frac{\Delta k}{u_{k+\Delta k} - u_k}, \quad \text{which implies} \quad \int_{\mathbb{R}} dv \frac{1}{v'\left(\frac{k}{J}\right)} f(v) \approx \frac{1}{J} \sum_{k=1}^S f\left(\frac{u_k}{J}\right), \quad (397)$$

describes in the thermodynamic limit the density of the rescaled rapidities $\rho^{\text{resc.rapid.}}(v)$ (285). (This can be true only at large J , as for finite J , (285) is a sum of Dirac delta functions, not a smooth function as (395).) Hence, the two leading coefficients of the above series can be recast as

$$b \equiv v'\left(\frac{k}{J}\right) = \frac{1}{\rho^{\text{resc.rapid.}}\left(\frac{u_k}{J}\right)}, \quad c \equiv v''\left(\frac{k}{J}\right) = -\frac{\rho^{\text{resc.rapid.}'}\left(\frac{u_k}{J}\right)}{\rho^{\text{resc.rapid.}}\left(\frac{u_k}{J}\right)^3}. \quad (398)$$

We just need to express them through our basic density $\rho(y)$ (295), which we do with help of (282),

$$b = \frac{1}{\rho(y_k)}, \quad c = -\frac{y_k^2}{y_k^2 - \omega^2} \frac{\rho'(y_k)}{\rho(y_k)^3}. \quad (399)$$

Using the Taylor series (396), the anomalous part of term III reads

$$\begin{aligned} -i \log \left(\frac{u_k - u_j - i}{u_k - u_j + i} \right) \Big|_{\text{anomalous}} &= -i \log \left(\frac{bn - \frac{1}{2J}cn^2 + O\left(\frac{1}{J^2}\right) - i}{bn - \frac{1}{2J}cn^2 + O\left(\frac{1}{J^2}\right) + i} \right) = \\ &= -i \log \left(\frac{bn - i}{bn + i} \right) - \frac{1}{J} \frac{cn^2}{1 + b^2 n^2} + O\left(\frac{1}{J^2}\right). \end{aligned} \quad (400)$$

Now we aim at summing this expansion term by term over the index j which is such that $0 < |n| = |k - j| \leq N$ (394). The leading term of (400) is odd w.r.t. n , and thus vanishes upon such a summation. The same is true for

the next-to-next-to-leading one, not shown above. The only relevant contributing term is the next-to-leading one, as being even w.r.t. n ,

$$\begin{aligned}
-i \sum_{\substack{j: \\ 0 < |k-j| \leq N}} \log \left(\frac{u_k - u_j - i}{u_k - u_j + i} \right) &= -\frac{2c}{b^2 J} \sum_{n=1}^N \left(1 - \frac{1}{1 + b^2 n^2} \right) + O \left(\frac{1}{J^3} \right) = \\
&= -\frac{2cN}{b^2 J} + \frac{2c}{b^2 J} \sum_{n=1}^{\infty} \frac{1}{1 + b^2 n^2} + O \left(\frac{1}{N} \right) + O \left(\frac{1}{J^3} \right) = \\
&= -\frac{2cN}{b^2 J} + \frac{c}{b^2 J} \left(\frac{\pi}{b} \coth \frac{\pi}{b} - 1 \right) + O \left(\frac{1}{N} \right) + O \left(\frac{1}{J^3} \right). \tag{401}
\end{aligned}$$

Notice that there is no order $O(1/J^2)$ here due to the cancelation described above.

We see that the summation over j in the anomalous regime around k yields a term which is linearly divergent with large N , as well as a convergent sum over n , which can be approximated by an infinite sum, which in turn can be summed up to a hyperbolic cotangent, with an $O(1/N)$ correction, to be disregarded. This linearly divergent term, $-2cN/b^2 J$, can be handled by remarking the equality,

$$-2 \sum_{\substack{j: \\ 0 < |k-j| \leq N}} \frac{1}{u_k - u_j} = -2 \sum_{\substack{n: \\ 0 < |n| \leq N}} \frac{1}{bn - \frac{1}{2J}cn^2 + O\left(\frac{1}{J^2}\right)} = -\frac{2c}{b^2 J} \sum_{n=1}^N 1 + O \left(\frac{1}{J^3} \right) = -\frac{2cN}{b^2 J} + O \left(\frac{1}{J^3} \right). \tag{402}$$

(This is correct up to $O(1/J^3)$, and not just $O(1/J^2)$, due to the same phenomenon as described before eq. (401).) Thanks to it, the anomalous expansion (401) becomes

$$-i \sum_{\substack{j: \\ 0 < |k-j| \leq N}} \log \left(\frac{u_k - u_j - i}{u_k - u_j + i} \right) = -2 \sum_{\substack{j: \\ 0 < |k-j| \leq N}} \frac{1}{u_k - u_j} + \frac{c}{b^2 J} \left(\frac{\pi}{b} \coth \frac{\pi}{b} - 1 \right) + O \left(\frac{1}{J^3} \right), \tag{403}$$

or better, rewriting b and c through the density $\rho(y_k)$ (399),

$$-i \sum_{\substack{j: \\ 0 < |k-j| \leq N}} \log \left(\frac{u_k - u_j - i}{u_k - u_j + i} \right) = -2 \sum_{\substack{j: \\ 0 < |k-j| \leq N}} \frac{1}{u_k - u_j} - \frac{1}{J} \frac{y_k^2}{y_k^2 - \omega^2} \pi \rho'(y_k) \left(\coth(\pi \rho(y_k)) - \frac{1}{\pi \rho(y_k)} \right) + O \left(\frac{1}{J^3} \right). \tag{404}$$

Putting together the non-anomalous (393) and the anomalous (404) parts, term III of the r.h.s. of (272), summed over all $j \neq k$, acquires the form,

$$\begin{aligned}
\sum_{\substack{j=1 \\ j \neq k}}^S (\text{the r.h.s. of eq. (272), term III})_{j,k} &= -i \sum_{\substack{j=1 \\ j \neq k}}^S \log \left(\frac{u_k - u_j - i}{u_k - u_j + i} \right) = \\
&= -2 \underbrace{\sum_{\substack{j: \\ |k-j| > N}} \frac{1}{u_k - u_j} + \frac{2}{3} \sum_{\substack{j: \\ |k-j| > N}} \frac{1}{(u_k - u_j)^3}}_{\text{the non-anomalous part}} - \\
&\quad \underbrace{-2 \sum_{\substack{j: \\ 0 < |k-j| \leq N}} \frac{1}{u_k - u_j} - \frac{1}{J} \frac{y_k^2}{y_k^2 - \omega^2} \pi \rho'(y_k) \left(\coth(\pi \rho(y_k)) - \frac{1}{\pi \rho(y_k)} \right)}_{\text{the anomalous part}} + O \left(\frac{1}{J^3} \right) =
\end{aligned}$$

$$= -2 \sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{u_k - u_j} + \frac{2}{3} \sum_{\substack{j: \\ |k-j| > N}} \frac{1}{(u_k - u_j)^3} - \frac{1}{J} \frac{y_k^2}{y_k^2 - \omega^2} \pi \rho'(y_k) \left(\coth(\pi \rho(y_k)) - \frac{1}{\pi \rho(y_k)} \right) + O\left(\frac{1}{J^3}\right). \quad (405)$$

Remark that this is not yet the final large- J series of the quantity in question: Only the second term in the last line of (405) has a definite large- J order, namely $O(1/J^2)$ due to (391); however, it still needs to undergo a special treatment to be explicitly seen as such, which we do in the next paragraph. The first and the third terms are further expandable at large J , which we postpone until later.

6. The r.h.s., Term III (the “Anomaly”). The Trick

The final line of the expression (405) contains the “cubic” term $\frac{2}{3} \sum_{j: |k-j| > N} \frac{1}{(u_k - u_j)^3}$, which is very inconvenient to work with. To practically deal with it, the following trick is performed:

The expanded logarithmic string Bethe ansatz equations, which consist of the l.h.s. (356), as well as terms I, II (365) and III (405) on the r.h.s., but terminated only at the leading large- J order $O(J^0)$, read

$$\underbrace{-\mu + \frac{y_k}{y_k^2 - \omega^2}}_{\text{the leading order from the l.h.s.}} = \underbrace{\frac{1}{J} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{2\omega^2 y_j y_k (y_j - y_k)}{(y_j^2 - \omega^2) (y_k^2 - \omega^2) (y_j y_k - \omega^2)}}_{\text{the leading order from the r.h.s., term I plus term II}} + \underbrace{\sum_{\substack{j: \\ |k-j| > N}} \frac{-2}{u_k - u_j}}_{\text{the leading order from the r.h.s., term III}} + O\left(\frac{1}{J}\right). \quad (406)$$

This leading-order equation we differentiate twice w.r.t. the variable u_k , which yields a useful expression for our inconvenient “cubic” term,

$$\frac{2}{3} \sum_{\substack{j: \\ |k-j| > N}} \frac{1}{(u_k - u_j)^3} = \frac{1}{6} \frac{d^2}{du_k^2} \left(\mu - \frac{y_k}{y_k^2 - \omega^2} + \frac{1}{J} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{2\omega^2 y_j y_k (y_j - y_k)}{(y_j^2 - \omega^2) (y_k^2 - \omega^2) (y_j y_k - \omega^2)} + O\left(\frac{1}{J}\right) \right). \quad (407)$$

Now the second derivative w.r.t. u_k which appears here should be replaced by the proper combination of the first and second derivatives w.r.t. y_k according to (283). These new derivatives can then be explicitly applied to the function of y_k inside the brackets, giving

$$\begin{aligned} \frac{2}{3} \sum_{\substack{j: \\ |k-j| > N}} \frac{1}{(u_k - u_j)^3} &= -\frac{1}{J^2} \frac{y_k^3 (\omega^4 + 4y_k^2 \omega^2 + y_k^4)}{3(y_k^2 - \omega^2)^5} + \\ &+ \frac{1}{J^3} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{2\omega^2 y_j y_k^3}{3(y_j^2 - \omega^2) (y_k^2 - \omega^2)^5 (y_j y_k - \omega^2)^3} \left((y_j - 3y_k) \omega^8 + y_k^2 (7y_j - 3y_k) \omega^6 - \right. \\ &\left. - 10y_j y_k^3 (y_j - y_k) \omega^4 + y_j^2 y_k^4 (3y_j - 7y_k) \omega^2 + y_j^2 y_k^6 (3y_j - y_k) \right) + O\left(\frac{1}{J^3}\right), \end{aligned} \quad (408)$$

in which way we trade the “cubic” term for something much more tractable.

In particular, for $\omega = 0$, the trick simplifies to

$$\left. \frac{2}{3} \sum_{\substack{j: \\ |k-j| > N}} \frac{1}{(u_k - u_j)^3} \right|_{\omega=0} = \frac{1}{J^2} \frac{-1}{3y_k^3} + O\left(\frac{1}{J^3}\right). \quad (409)$$

In paragraph III C 6, we will explicitly test this trick.

Thanks to the operation (408), we remove the problematic part from term III summed over $j \neq k$ (405), which thus acquires the form

$$\begin{aligned}
& \sum_{\substack{j=1 \\ j \neq k}}^S (\text{the r.h.s. of eq. (272), term III})_{j,k} = -i \sum_{\substack{j=1 \\ j \neq k}}^S \log \left(\frac{u_k - u_j - i}{u_k - u_j + i} \right) = \\
& = -2 \sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{u_k - u_j} - \frac{1}{J^2} \frac{y_k^3 (\omega^4 + 4y_k^2 \omega^2 + y_k^4)}{3(y_k^2 - \omega^2)^5} + \\
& + \frac{1}{J^3} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{2\omega^2 y_j y_k^3}{3(y_j^2 - \omega^2)(y_k^2 - \omega^2)^5(y_j y_k - \omega^2)^3} \left((y_j - 3y_k) \omega^8 + y_k^2 (7y_j - 3y_k) \omega^6 - \right. \\
& \left. - 10y_j y_k^3 (y_j - y_k) \omega^4 + y_j^2 y_k^4 (3y_j - 7y_k) \omega^2 + y_j^2 y_k^6 (3y_j - y_k) \right) - \\
& - \frac{1}{J} \frac{y_k^2}{y_k^2 - \omega^2} \pi \rho'(y_k) \left(\coth(\pi \rho(y_k)) - \frac{1}{\pi \rho(y_k)} \right) + O\left(\frac{1}{J^3}\right). \tag{410}
\end{aligned}$$

We now need just three simple steps to finally polish this finding (410): First, we change the rapidities in its first term into the rescaled spectral parameters (280),

$$\frac{-2}{u_k - u_j} = \frac{1}{J} \frac{1}{y_j - y_k} \frac{2y_j y_k}{y_j y_k - \omega^2}. \tag{411}$$

Second, we substitute the large- J series of the density $\rho(y_k)$ (321) to the term with the hyperbolic cotangent, and expand it accordingly,

$$\begin{aligned}
& -\frac{1}{J} \frac{y_k^2}{y_k^2 - \omega^2} \pi \rho'(y_k) \left(\coth(\pi \rho(y_k)) - \frac{1}{\pi \rho(y_k)} \right) = \\
& = -\frac{1}{J} \frac{y_k^2}{y_k^2 - \omega^2} \pi \rho^{(0)'}(y_k) \left(\coth(\pi \rho^{(0)}(y_k)) - \frac{1}{\pi \rho^{(0)}(y_k)} \right) - \\
& - \frac{1}{J^2} \frac{y_k^2}{y_k^2 - \omega^2} \frac{d}{dy_k} \left(\pi \rho^{(1)}(y_k) \left(\coth(\pi \rho^{(0)}(y_k)) - \frac{1}{\pi \rho^{(0)}(y_k)} \right) \right) + O\left(\frac{1}{J^3}\right). \tag{412}
\end{aligned}$$

Third, we consider the sum over $j \neq k$ in the piece proportional to $1/J^3$, and we change it to a sum over all j . It can be done because the additional $j = k$ term is multiplied by $1/J^3$, and thus, does not affect the formula to the current order. Having applied these modifications, we arrive at

$$\begin{aligned}
& \sum_{\substack{j=1 \\ j \neq k}}^S (\text{the r.h.s. of eq. (272), term III}) = -i \sum_{\substack{j=1 \\ j \neq k}}^S \log \left(\frac{u_k - u_j - i}{u_k - u_j + i} \right) = \\
& \frac{1}{J} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{y_j - y_k} \frac{2y_j y_k}{y_j y_k - \omega^2} - \frac{1}{J} \frac{y_k^2}{y_k^2 - \omega^2} \pi \rho^{(0)'}(y_k) \left(\coth(\pi \rho^{(0)}(y_k)) - \frac{1}{\pi \rho^{(0)}(y_k)} \right) - \frac{1}{J^2} \frac{y_k^3 (\omega^4 + 4y_k^2 \omega^2 + y_k^4)}{3(y_k^2 - \omega^2)^5} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{J^3} \sum_{j=1}^S \frac{2\omega^2 y_j y_k^3}{3 (y_j^2 - \omega^2) (y_k^2 - \omega^2)^5 (y_j y_k - \omega^2)^3} \left((y_j - 3y_k) \omega^8 + y_k^2 (7y_j - 3y_k) \omega^6 - \right. \\
& \quad \left. - 10y_j y_k^3 (y_j - y_k) \omega^4 + y_j^2 y_k^4 (3y_j - 7y_k) \omega^2 + y_j^2 y_k^6 (3y_j - y_k) \right) - \\
& \quad - \frac{1}{J^2} \frac{y_k^2}{y_k^2 - \omega^2} \frac{d}{dy_k} \left(\pi \rho^{(1)}(y_k) \left(\coth \left(\pi \rho^{(0)}(y_k) \right) - \frac{1}{\pi \rho^{(0)}(y_k)} \right) \right) + O\left(\frac{1}{J^3}\right). \tag{413}
\end{aligned}$$

7. Summary

Collecting the l.h.s. (356), as well as term I plus term II (365) and term III (413) on the r.h.s., the logarithmic string Bethe ansatz equations (272) expanded at large J up to and including the order $O(1/J^2)$, after some manipulations, read

$$\begin{aligned}
0 = & \frac{1}{J} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{2y_j y_k (y_j y_k - \omega^2)}{(y_j^2 - \omega^2) (y_k^2 - \omega^2) (y_j - y_k)} + \mu - \frac{y_k}{y_k^2 - \omega^2} - \\
& - \frac{1}{J} \frac{y_k^2}{y_k^2 - \omega^2} \pi \rho^{(0)'}(y_k) \left(\coth \left(\pi \rho^{(0)}(y_k) \right) - \frac{1}{\pi \rho^{(0)}(y_k)} \right) - \\
& - \frac{1}{J^2} \frac{1}{\pi} \sum_{j=1}^S \frac{y_j^2 y_k^2}{(y_j^2 - \omega^2) (y_k^2 - \omega^2)} \left(\frac{2\omega}{(y_j - y_k) (y_j y_k - \omega^2)} + \left(\frac{1}{(y_j - y_k)^2} + \frac{\omega^2}{(y_j y_k - \omega^2)^2} \right) \log \left(\frac{(y_j + \omega) (y_k - \omega)}{(y_j - \omega) (y_k + \omega)} \right) \right) - \\
& - \frac{1}{J^2} \frac{y_k^3 (\omega^4 + 4y_k^2 \omega^2 + y_k^4)}{4 (y_k^2 - \omega^2)^5} + \frac{1}{J^3} \sum_{j=1}^S \frac{\omega^2 y_j y_k}{6 (y_j^2 - \omega^2)^5 (y_k^2 - \omega^2)^5} \left((3y_j^3 - 2y_j^2 y_k - 2y_j y_k^2 + 9y_k^3) \omega^{10} + \right. \\
& \quad + (3y_j^5 + 4y_j^4 y_k - 10y_j^3 y_k^2 - 34y_j^2 y_k^3 + 4y_j y_k^4 + 9y_k^5) \omega^8 - \\
& \quad - 2y_j y_k (y_j^5 + 5y_j^4 y_k - 25y_j^3 y_k^2 - 7y_j^2 y_k^3 + 17y_j y_k^4 + y_k^5) \omega^6 - \\
& \quad - 2y_j^2 y_k^2 (y_j^5 + 17y_j^4 y_k - 7y_j^3 y_k^2 - 25y_j^2 y_k^3 + 5y_j y_k^4 + y_k^5) \omega^4 + \\
& \quad + y_j^3 y_k^3 (9y_j^5 + 4y_j^4 y_k - 34y_j^3 y_k^2 - 10y_j^2 y_k^3 + 4y_j y_k^4 + 3y_k^5) \omega^2 + \\
& \quad \left. + y_j^5 y_k^5 (9y_j^3 - 2y_j^2 y_k - 2y_j y_k^2 + 3y_k^3) \right) - \\
& - \frac{1}{J^2} \frac{y_k^2}{y_k^2 - \omega^2} \frac{d}{dy_k} \left(\pi \rho^{(1)}(y_k) \left(\coth \left(\pi \rho^{(0)}(y_k) \right) - \frac{1}{\pi \rho^{(0)}(y_k)} \right) \right) + O\left(\frac{1}{J^3}\right), \tag{414}
\end{aligned}$$

for any $k = 1, 2, \dots, S$. The terms of this equation are written in the order of non-decreasing powers of $1/J$.

Notice that we have here three sums over j . The last one (*i.e.*, the one multiplied by $1/J^3$) has a summand which is a rational function of y_j , and therefore, the procedure (331) can be applied to rewrite it in the language of the resolvent,

$$\begin{aligned}
& \frac{1}{J^3} \frac{\omega^2 y_k}{6(y_k^2 - \omega^2)^5} \sum_{j=1}^S \frac{y_j^2}{y_j^2 - \omega^2} \left(9y_k^3 (y_k^2 + \omega^2) \frac{1}{y_j} + \frac{y_k (y_k^2 - \omega^2)^2}{2\omega} \left(\frac{(\omega - y_k)^2}{(\omega + y_j)^2} - \frac{(\omega + y_k)^2}{(\omega - y_j)^2} \right) - \right. \\
& \quad \left. - \frac{3(y_k^2 - \omega^2)^4}{8\omega} \left(\frac{1}{(\omega + y_j)^3} + \frac{1}{(\omega - y_j)^3} \right) + \frac{3(y_k^2 - \omega^2)^4}{8} \left(\frac{1}{(\omega + y_j)^4} + \frac{1}{(\omega - y_j)^4} \right) \right) = \\
& = \frac{1}{J^2} \frac{\omega^2 y_k}{6(y_k^2 - \omega^2)^5} \left(-9y_k^3 (y_k^2 + \omega^2) G(0) + \frac{y_k (y_k^2 - \omega^2)^2}{2\omega} \left((\omega + y_k)^2 G'(\omega) - (\omega - y_k)^2 G'(-\omega) \right) - \right. \\
& \quad \left. - \frac{3(y_k^2 - \omega^2)^4}{16\omega} (G''(\omega) - G''(-\omega)) - \frac{(y_k^2 - \omega^2)^4}{16} (G'''(\omega) + G'''(-\omega)) \right). \tag{415}
\end{aligned}$$

The middle sum over j (*i.e.*, the one multiplied by $1/J^2$, representing the Hernández-López contribution) has a non-rational summand, but can be put back to the form (363), which is rational, and allows the procedure (331), which is straightforward here, as the necessary partial fractional decomposition is already performed. (Remark that we have written it as a sum over all j , and not $j \neq k$, since the $j = k$ piece is easily checked to be zero.) Let us show the result, even though we will not need it,

$$\begin{aligned}
& \frac{1}{J^2} \frac{1}{\pi} \sum_{j=1}^S \frac{2y_j^2 y_k^2}{\omega^2 (y_j^2 - \omega^2) (y_k^2 - \omega^2)} \sum_{r \geq 2} \sum_{s \geq r+1} \omega^{r+s} ((-1)^{r+s} - 1) \frac{(r-1)(s-1)}{(r+s-2)(s-r)} \left(\frac{1}{y_k^r y_j^s} - \frac{1}{y_j^r y_k^s} \right) = \\
& = \frac{1}{J} \frac{1}{\pi} \frac{2y_k^2}{\omega^2 (y_k^2 - \omega^2)} \sum_{r \geq 2} \sum_{s \geq r+1} \omega^{r+s} ((-1)^{r+s} - 1) \frac{(r-1)(s-1)}{(r+s-2)(s-r)} \left(\frac{G'' \dots (r-1)(0)}{(r-1)!} \frac{1}{y_k^s} - \frac{G'' \dots (s-1)(0)}{(s-1)!} \frac{1}{y_k^r} \right). \tag{416}
\end{aligned}$$

The first sum (*i.e.*, the one multiplied by $1/J$) is over $j \neq k$, and for $j = k$ is singular, and thus, cannot yet be similarly tackled, but soon will (423).

Substituting (415) to (414), we obtain the most important expression of this subsection,

$$\begin{aligned}
0 & = \underbrace{\frac{1}{J} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{2y_j y_k (y_j y_k - \omega^2)}{(y_j^2 - \omega^2) (y_k^2 - \omega^2) (y_j - y_k)}}_{\text{term I}} + \underbrace{\mu - \frac{y_k}{y_k^2 - \omega^2}}_{\text{term II}} - \\
& \quad - \underbrace{\frac{1}{J} \frac{y_k^2}{y_k^2 - \omega^2} \pi \rho^{(0)'}(y_k) \left(\coth \left(\pi \rho^{(0)}(y_k) \right) - \frac{1}{\pi \rho^{(0)}(y_k)} \right)}_{\text{term III}} - \\
& \quad - \underbrace{\frac{1}{J^2} \frac{1}{\pi} \sum_{j=1}^S \frac{y_j^2 y_k^2}{(y_j^2 - \omega^2) (y_k^2 - \omega^2)} \left(\frac{2\omega}{(y_j - y_k) (y_j y_k - \omega^2)} + \left(\frac{1}{(y_j - y_k)^2} + \frac{\omega^2}{(y_j y_k - \omega^2)^2} \right) \log \left(\frac{(y_j + \omega) (y_k - \omega)}{(y_j - \omega) (y_k + \omega)} \right) \right)}_{\text{term IV}} - \\
& \quad - \underbrace{\frac{1}{J^2} \frac{y_k^3 (\omega^4 + 4y_k^2 \omega^2 + y_k^4)}{4(y_k^2 - \omega^2)^5}}_{\text{term V}} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{J^2} \frac{\omega^2 y_k}{6 (y_k^2 - \omega^2)^5} \underbrace{\left(-9 y_k^3 (y_k^2 + \omega^2) G(0) + \frac{y_k (y_k^2 - \omega^2)^2}{2\omega} \left((\omega + y_k)^2 G'(\omega) - (\omega - y_k)^2 G'(-\omega) \right) - \right.}_{\text{term VI}} \\
& \quad \left. - \frac{3 (y_k^2 - \omega^2)^4}{16\omega} (G'''(\omega) - G'''(-\omega)) - \frac{(y_k^2 - \omega^2)^4}{16} (G'''(\omega) + G'''(-\omega)) \right) -}_{\text{term VI continued}} \\
& \quad \underbrace{- \frac{1}{J^2} \frac{y_k^2}{y_k^2 - \omega^2} \frac{d}{dy_k} \left(\pi \rho^{(1)}(y_k) \left(\coth \left(\pi \rho^{(0)}(y_k) \right) - \frac{1}{\pi \rho^{(0)}(y_k)} \right) \right)}_{\text{term VII}} + O\left(\frac{1}{J^3}\right), \tag{417}
\end{aligned}$$

for any $k = 1, 2, \dots, S$, where we have given names to its terms (not to be confused with the labels in (272)), to be used in subsection II C.

In particular, for $\omega = 0$, terms IV and VI of (417) disappear, and the remaining ones simplify to

$$\begin{aligned}
0 &= \frac{1}{J} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{2}{y_j - y_k} + \mu - \frac{1}{y_k} - \frac{1}{J} \pi \rho^{(0)'}(y_k) \left(\coth \left(\pi \rho^{(0)}(y_k) \right) - \frac{1}{\pi \rho^{(0)}(y_k)} \right) - \\
& \quad - \frac{1}{J^2} \frac{1}{4y_k^3} - \frac{1}{J^2} \frac{d}{dy_k} \left(\pi \rho^{(1)}(y_k) \left(\coth \left(\pi \rho^{(0)}(y_k) \right) - \frac{1}{\pi \rho^{(0)}(y_k)} \right) \right) + O\left(\frac{1}{J^3}\right). \tag{418}
\end{aligned}$$

C. The All-Loop One-Cut Quadratic Equation at the Orders $O(J^0)$, $O(1/J)$, $O(1/J^2)$

1. The Definition of the Quadratic Equation

There are several techniques of solving equations like our fundamental one (417), for example, the “quadratic equation,” the “linear equation” (*i.e.*, the “Riemann–Hilbert method”), the “Baxter equation.” In this computation, we, for the first time along with [194], show how to apply the quadratic equation method to Bethe ansatz equations with next-to-next-to-leading-order finite-size corrections.

The so-called quadratic equation arises by multiplying the k -th equation (417) by $1/(z - y_k)$, where z is a complex parameter, and afterwards adding all of them to each other,

$$\text{the quadratic equation:} \quad 0 = \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} (\text{the r.h.s. of eq. (417)}). \quad (419)$$

In theory, this should work as follows: Since the r.h.s. of (417) can be written as a rational function of y_k , the procedure (331) is able to yield the quadratic equation (419) as an equation for the resolvent $G(z)$. More precisely, since (417) is a large- J series terminated after three leading orders, we need to expand the resolvent, too, according to (322), and thus, the quadratic equation turns to be a perturbative equation for the three leading coefficients $G^{(0)}(z)$, $G^{(1)}(z)$, $G^{(2)}(z)$. We see, according to the powers of $1/J$ multiplying each term in (417), that • an equation for $G^{(0)}(z)$ will come from terms I and II only, • an equation for $G^{(1)}(z)$ will come from terms I, II, III, IV, • an equation for $G^{(2)}(z)$ will come from all the terms. As we have explained in paragraph II A 2, the resolvent captures the distribution of the rescaled spectral parameters (y_1, y_2, \dots, y_S) , and hence, finding the resolvent is equivalent to solving our problem.

In practice, the above program will be slightly modified, as it will not always be profitable to try to rewrite the sum over k in (419), or the sums over j in (417), through the resolvent, instead changing them into integration with an appropriate density (326), supplied when needed by a boundary contribution.

In this subsection, we explicitly derive the quadratic equation for our problem; it is given at all loops in (453)–(462), while for the simplified one-loop case in (463)–(469).

2. Term I

For term I of (417), we symmetrize the summand w.r.t. to the indices j and k , which makes it no longer singular at $j = k$,

$$\begin{aligned} \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} (\text{the r.h.s. of eq. (417), term I}) &= \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \frac{1}{J} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{2y_j y_k (y_j y_k - \omega^2)}{(y_j^2 - \omega^2)(y_k^2 - \omega^2)(y_j - y_k)} = \\ &= -\frac{1}{J^2} \sum_{k=1}^S \sum_{\substack{j=1 \\ j \neq k}}^S \frac{y_j y_k (y_j y_k - \omega^2)}{(z - y_j)(z - y_k)(y_j^2 - \omega^2)(y_k^2 - \omega^2)} = \\ &= -\frac{1}{J^2} \sum_{k=1}^S \sum_{j=1}^S \frac{y_j y_k (y_j y_k - \omega^2)}{(z - y_j)(z - y_k)(y_j^2 - \omega^2)(y_k^2 - \omega^2)} + \frac{1}{J^2} \sum_{k=1}^S \frac{y_k^2}{(z - y_k)^2 (y_k^2 - \omega^2)}. \end{aligned} \quad (420)$$

Now the single sum in (420) gives simply

$$\frac{1}{J^2} \sum_{k=1}^S \frac{y_k^2}{(z - y_k)^2 (y_k^2 - \omega^2)} = -\frac{1}{J^2} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{-1}{(z - y_k)^2} = -\frac{1}{J} G'(z), \quad (421)$$

while the double sum,

$$\begin{aligned}
& -\frac{1}{J^2} \sum_{k=1}^S \sum_{j=1}^S \frac{y_j y_k (y_j y_k - \omega^2)}{(z - y_j)(z - y_k)(y_j^2 - \omega^2)(y_k^2 - \omega^2)} = \\
& = \frac{\omega^2}{z^2} \left(\frac{1}{J} \sum_{j=1}^S \frac{y_j^2}{y_j^2 - \omega^2} \frac{1}{z - y_j} - \frac{1}{J} \sum_{j=1}^S \frac{y_j^2}{y_j^2 - \omega^2} \frac{1}{0 - y_j} \right) \left(\frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{1}{z - y_k} - \frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{1}{0 - y_k} \right) - \\
& \quad - \frac{1}{J} \sum_{j=1}^S \frac{y_j^2}{y_j^2 - \omega^2} \frac{1}{z - y_j} \frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{1}{z - y_k} = \\
& = \frac{\omega^2}{z^2} (G(z) - G(0))^2 - G(z)^2 = \\
& = \left(\frac{\omega^2}{z^2} - 1 \right) G(z)^2 - \frac{2\omega^2}{z^2} G(0)G(z) + \frac{\omega^2}{z^2} G(0)^2. \tag{422}
\end{aligned}$$

Plugging (421) and (422) into (420), we find the contribution of term I to the quadratic equation,

$$\frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} (\text{the r.h.s. of eq. (417), term I}) = \left(\frac{\omega^2}{z^2} - 1 \right) G(z)^2 - \frac{2\omega^2}{z^2} G(0)G(z) + \frac{\omega^2}{z^2} G(0)^2 - \frac{1}{J} G'(z). \tag{423}$$

3. Term II

The input of term II of (417) to the quadratic equation consists of two pieces,

$$\frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} (\text{the r.h.s. of eq. (417), term II}) = \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \left(\mu - \frac{y_k}{y_k^2 - \omega^2} \right). \tag{424}$$

The first subterm yields,

$$\begin{aligned}
\frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \mu &= \mu \frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \left(\left(1 - \frac{\omega^2}{z^2} \right) \frac{1}{z - y_k} + \frac{\omega^2}{z^2} \frac{1}{0 - y_k} + \frac{\omega^2}{z} \frac{-1}{(0 - y_k)^2} \right) = \\
&= \mu \left(\left(1 - \frac{\omega^2}{z^2} \right) G(z) + \frac{\omega^2}{z^2} G(0) + \frac{\omega^2}{z} G'(0) \right), \tag{425}
\end{aligned}$$

and the second one,

$$-\frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \frac{y_k}{y_k^2 - \omega^2} = -\frac{1}{z} \frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \left(\frac{1}{z - y_k} - \frac{1}{0 - y_k} \right) = -\frac{1}{z} (G(z) - G(0)). \tag{426}$$

Hence, (425) and (426) added together give

$$\begin{aligned}
& \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} (\text{the r.h.s. of eq. (417), term II}) = \\
& = \left(\mu \left(1 - \frac{\omega^2}{z^2} \right) - \frac{1}{z} \right) G(z) + \left(\mu \frac{\omega^2}{z^2} + \frac{1}{z} \right) G(0) + \mu \frac{\omega^2}{z} G'(0). \tag{427}
\end{aligned}$$

4. Term I Plus Term II

Adding term I (423) and term II (427), we obtain

$$\begin{aligned} & \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \text{ (the r.h.s. of eq. (417), term I plus term II) =} \\ & = \left(\frac{\omega^2}{z^2} - 1 \right) G(z)^2 + \left(\mu \left(1 - \frac{\omega^2}{z^2} \right) - \frac{1}{z} - \frac{2\omega^2}{z^2} G(0) \right) G(z) + \frac{\omega^2}{z^2} G(0)^2 + \left(\mu \frac{\omega^2}{z^2} + \frac{1}{z} \right) G(0) + \mu \frac{\omega^2}{z} G'(0) - \frac{1}{J} G'(z). \end{aligned} \quad (428)$$

It is a good place to use the large- J series of the resolvent (322) to find the three leading orders of the contribution of term I and term II to the quadratic equation,

$$\begin{aligned} & \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \text{ (the r.h.s. of eq. (417), term I plus term II) =} \\ & = \frac{1}{z^2} \left(- (z^2 - \omega^2) G^{(0)}(z)^2 + \left(\mu z^2 - z - \omega^2 \left(\mu + 2G^{(0)}(0) \right) \right) G^{(0)}(z) + \right. \\ & \quad \left. + \left(G^{(0)}(0) + \mu \omega^2 G^{(0)'}(0) \right) z + \omega^2 G^{(0)}(0) \left(\mu + G^{(0)}(0) \right) \right) + \\ & \quad + \frac{1}{J} \frac{1}{z^2} \left(\left(- 2 (z^2 - \omega^2) G^{(0)}(z) + \mu z^2 - z - \omega^2 \left(\mu + 2G^{(0)}(0) \right) \right) G^{(1)}(z) + \right. \\ & \quad \left. + \left(- 2\omega^2 G^{(0)}(z) + z + \omega^2 \left(\mu + 2G^{(0)}(0) \right) \right) G^{(1)}(0) - z^2 G^{(0)'}(z) + \mu \omega^2 G^{(1)'}(0) z \right) + \\ & \quad + \frac{1}{J^2} \frac{1}{z^2} \left(\left(- 2 (z^2 - \omega^2) G^{(0)}(z) + \mu z^2 - z - \omega^2 \left(\mu + 2G^{(0)}(0) \right) \right) G^{(2)}(z) + \right. \\ & \quad \left. + \left(- 2\omega^2 G^{(0)}(z) + z + \omega^2 \left(\mu + 2G^{(0)}(0) \right) \right) G^{(2)}(0) - \right. \\ & \quad \left. - (z^2 - \omega^2) G^{(1)}(z)^2 - 2\omega^2 G^{(1)}(0) G^{(1)}(z) - z^2 G^{(1)'}(z) + \omega^2 G^{(1)}(0)^2 + \mu \omega^2 G^{(2)'}(0) z \right) + O\left(\frac{1}{J^3}\right). \end{aligned} \quad (429)$$

5. Term III (the “Anomaly”)

The input of term III of (417) to the quadratic equation,

$$\frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \text{ (the r.h.s. of eq. (417), term III) =}$$

$$\begin{aligned}
&= -\frac{1}{J^2} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{1}{z - y_k} \pi \rho^{(0)'}(y_k) \left(\coth \left(\pi \rho^{(0)}(y_k) \right) - \frac{1}{\pi \rho^{(0)}(y_k)} \right) = \\
&= \frac{1}{J} \mathcal{A}^{(1)}(z) + \frac{1}{J^2} \mathcal{A}^{(2,1)}(z) + \mathcal{O} \left(\frac{1}{J^3} \right)
\end{aligned} \tag{430}$$

cannot at this point be expressed through the resolvent, since we do not yet know the large- J leading-order density $\rho^{(0)}(y_k)$. We will see that both at $g = 0$ (533) and $g > 0$ (742), this density possesses a form (namely, its square is a rational function of the argument) which allows for rewriting the summand in the second line of (430) as a rational function of y_k , through appropriately expanding the hyperbolic cotangent (548) — thus enabling us to use the procedure (331) to express term III in the quadratic equation through the resolvent. We will, however, actually follow this program only at $g = 0$ (see paragraphs III B 1 and III C 1), as otherwise it happens to be too involved.

In this latter case, it is reasonable to trade the summation over k in the second line of (430) for integration, according to the summation formula (326),

$$\begin{aligned}
&\frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \text{ (the r.h.s. of eq. (417), term III) } = \\
&= -\frac{1}{J} \int_{\mathbb{R}} dy \rho(y) \frac{1}{z - y} \pi \rho^{(0)'}(y) \left(\coth \left(\pi \rho^{(0)}(y) \right) - \frac{1}{\pi \rho^{(0)}(y)} \right) + \left(\begin{array}{c} \text{subleading} \\ \text{boundary} \\ \text{contribution} \end{array} \right).
\end{aligned} \tag{431}$$

The only J -dependence on the r.h.s. is carried by the density $\rho(y)$, and we replace it by its large- J expansion (321), truncated at the next-to-leading order, obtaining

$$\mathcal{A}^{(1)}(z) = \mathcal{A}^{(1),\text{bulk}}(z) + \mathcal{A}^{(1),\text{boundary}}(z), \tag{432}$$

$$\mathcal{A}^{(2,1)}(z) = \mathcal{A}^{(2,1),\text{bulk}}(z) + \mathcal{A}^{(2,1),\text{boundary}}(z), \tag{433}$$

where the “bulk contributions,”

$$\mathcal{A}^{(1),\text{bulk}}(z) = -\pi \int_{\mathbb{R}} dy \frac{1}{z - y} \rho^{(0)}(y) \rho^{(0)'}(y) \left(\coth \left(\pi \rho^{(0)}(y) \right) - \frac{1}{\pi \rho^{(0)}(y)} \right), \tag{434}$$

$$\mathcal{A}^{(2,1),\text{bulk}}(z) = -\pi \int_{\mathbb{R}} dy \frac{1}{z - y} \rho^{(1)}(y) \rho^{(0)'}(y) \left(\coth \left(\pi \rho^{(0)}(y) \right) - \frac{1}{\pi \rho^{(0)}(y)} \right), \tag{435}$$

while the “boundary contributions” will be considered separately later; we will see that they do not appear for an integration with the leading-order density $\rho^{(0)}(y)$, *i.e.*, $\mathcal{A}^{(1),\text{boundary}}(z) = 0$ (see *e.g.* (562)), while they do if we integrate with any subleading density, *i.e.*, $\mathcal{A}^{(2,1),\text{boundary}}(z) \neq 0$ (see *e.g.* (661), (979)).

6. Term IV (the “Phase”)

For the contribution of term IV of (417) to the quadratic equation, describing the presence of the Hernández-López dressing phase,

$$\frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \text{ (the r.h.s. of eq. (417), term IV) } = \frac{1}{J} \mathcal{P}^{(1)}(z) + \frac{1}{J^2} \mathcal{P}^{(2)}(z) + \mathcal{O} \left(\frac{1}{J^3} \right), \tag{436}$$

we have, analogously as for terms III and VII (the “anomaly”), two ways to proceed.

If we want to express it through the resolvent, we need to use the form before summing over r and s (416),

$$\begin{aligned}
& \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \text{ (the r.h.s. of eq. (417), term IV) } = \\
& = \frac{1}{J} \frac{2}{\pi \omega^2} \sum_{r \geq 2} \sum_{s \geq r+1} \omega^{r+s} ((-1)^{r+s} - 1) \frac{(r-1)(s-1)}{(r+s-2)(s-r)}. \\
& \frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{1}{z - y_k} \left(\frac{G'' \dots' (r-1)(0)}{(r-1)!} \frac{1}{y_k^s} - \frac{G'' \dots' (s-1)(0)}{(s-1)!} \frac{1}{y_k^r} \right) = \\
& = \frac{1}{J} \frac{2}{\pi \omega^2} \sum_{r \geq 2} \sum_{s \geq r+1} \omega^{r+s} ((-1)^{r+s} - 1) \frac{(r-1)(s-1)}{(r+s-2)(s-r)}. \\
& \frac{1}{J} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \left(\frac{G'' \dots' (r-1)(0)}{(r-1)!} \left(\frac{1}{z^s} \frac{1}{z - y_k} + \sum_{u=1}^s \frac{1}{z^{s+1-u}} \frac{1}{y_k^u} \right) - \frac{G'' \dots' (s-1)(0)}{(s-1)!} \left(\frac{1}{z^r} \frac{1}{z - y_k} + \sum_{u=1}^r \frac{1}{z^{r+1-u}} \frac{1}{y_k^u} \right) \right) = \\
& = \frac{1}{J} \frac{2}{\pi \omega^2} \sum_{r \geq 2} \sum_{s \geq r+1} \omega^{r+s} ((-1)^{r+s} - 1) \frac{(r-1)(s-1)}{(r+s-2)(s-r)}. \\
& \cdot \left(\left(\frac{G'' \dots' (r-1)(0)}{(r-1)!} \frac{1}{z^s} - \frac{G'' \dots' (s-1)(0)}{(s-1)!} \frac{1}{z^r} \right) G(z) - \right. \\
& \left. - \left(\frac{G'' \dots' (r-1)(0)}{(r-1)!} \sum_{u=1}^s \frac{1}{z^{s+1-u}} \frac{G'' \dots' (u-1)(0)}{(u-1)!} - \frac{G'' \dots' (s-1)(0)}{(s-1)!} \sum_{u=1}^r \frac{1}{z^{r+1-u}} \frac{G'' \dots' (u-1)(0)}{(u-1)!} \right) \right). \tag{437}
\end{aligned}$$

A disadvantage of this form is that it requires the knowledge of an arbitrary-order derivative of the resolvent at zero, before even summing over r and s .

We will, therefore, resort to the other method, namely, changing summation into integration. But first, we symmetrize the summand w.r.t. j and k ,

$$\begin{aligned}
& \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \text{ (the r.h.s. of eq. (417), term IV) } = \\
& = -\frac{1}{J^3} \frac{1}{\pi} \sum_{j=1}^S \sum_{k=1}^S \frac{1}{z - y_k} \frac{y_j^2 y_k^2}{(y_j^2 - \omega^2)(y_k^2 - \omega^2)}. \\
& \cdot \left(\frac{2\omega}{(y_j - y_k)(y_j y_k - \omega^2)} + \left(\frac{1}{(y_j - y_k)^2} + \frac{\omega^2}{(y_j y_k - \omega^2)^2} \right) \log \left(\frac{(y_j + \omega)(y_k - \omega)}{(y_j - \omega)(y_k + \omega)} \right) \right) = \\
& = \frac{1}{J^3} \frac{1}{2\pi} \sum_{j=1}^S \sum_{k=1}^S \frac{y_j^2}{y_j^2 - \omega^2} \frac{y_k^2}{y_k^2 - \omega^2} \frac{1}{(z - y_j)(z - y_k)} \mathcal{H}(y_j, y_k), \tag{438}
\end{aligned}$$

where we introduce a useful shorthand notation,

$$\mathcal{H}(y_j, y_k) \equiv \frac{2\omega}{y_j y_k - \omega^2} + \left(\frac{1}{y_j - y_k} + \frac{\omega^2 (y_j - y_k)}{(y_j y_k - \omega^2)^2} \right) \log \left(\frac{(y_j + \omega)(y_k - \omega)}{(y_j - \omega)(y_k + \omega)} \right). \quad (439)$$

Now we are ready to use the summation formula (326),

$$\begin{aligned} & \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \text{ (the r.h.s. of eq. (417), term IV) } = \\ & = \frac{1}{J} \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} dy_1 dy_2 \rho(y_1) \rho(y_2) \frac{1}{(z - y_1)(z - y_2)} \mathcal{H}(y_1, y_2) + \left(\begin{array}{c} \text{subleading} \\ \text{boundary} \\ \text{contribution} \end{array} \right). \end{aligned} \quad (440)$$

Expanding the densities $\rho(y_1)$ and $\rho(y_2)$ at large J , up to and including two leading orders, we find

$$\mathcal{P}^{(1)}(z) = \mathcal{P}^{(1),\text{bulk}}(z) + \mathcal{P}^{(1),\text{boundary}}(z), \quad (441)$$

$$\mathcal{P}^{(2)}(z) = \mathcal{P}^{(2),\text{bulk}}(z) + \mathcal{P}^{(2),\text{boundary}}(z), \quad (442)$$

where the “bulk contributions,”

$$\mathcal{P}^{(1),\text{bulk}}(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} dy_1 dy_2 \rho^{(0)}(y_1) \rho^{(0)}(y_2) \frac{1}{(z - y_1)(z - y_2)} \mathcal{H}(y_1, y_2), \quad (443)$$

$$\mathcal{P}^{(2),\text{bulk}}(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} dy_1 dy_2 \left(\rho^{(0)}(y_1) \rho^{(1)}(y_2) + \rho^{(0)}(y_2) \rho^{(1)}(y_1) \right) \frac{1}{(z - y_1)(z - y_2)} \mathcal{H}(y_1, y_2), \quad (444)$$

while the “boundary contributions” will be considered separately later; similarly as for the “anomaly,” there will be $\mathcal{P}^{(1),\text{boundary}}(z) = 0$, but $\mathcal{P}^{(2),\text{boundary}}(z) \neq 0$ (970).

7. Term V

The input of term V of (417) to the quadratic equation,

$$\begin{aligned} & \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \text{ (the r.h.s. of eq. (417), term V) } = \frac{1}{J^3} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{1}{z - y_k} \frac{-y_k (\omega^4 + 4y_k^2 \omega^2 + y_k^4)}{4(y_k^2 - \omega^2)^4} = \\ & = \frac{1}{J^3} \frac{3}{32\omega} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \left(- \frac{8z\omega (z^4 + 4z^2 \omega^2 + \omega^4)}{3(z^2 - \omega^2)^4} \frac{1}{z - y_k} + \right. \\ & + \frac{z^2 + 4z\omega + \omega^2}{6(z - \omega)^4} \frac{1}{\omega - y_k} + \frac{z^2 - 4z\omega + \omega^2}{6(z + \omega)^4} \frac{1}{\omega + y_k} - \frac{z^2 + 4z\omega + \omega^2}{6(z - \omega)^3} \frac{1}{(\omega - y_k)^2} + \frac{z^2 - 4z\omega + \omega^2}{6(z + \omega)^3} \frac{1}{(\omega + y_k)^2} + \\ & \left. + \frac{z\omega}{(z - \omega)^2} \frac{1}{(\omega - y_k)^3} - \frac{z\omega}{(z + \omega)^2} \frac{1}{(\omega + y_k)^3} - \frac{\omega^2}{z - \omega} \frac{1}{(\omega - y_k)^4} + \frac{\omega^2}{z + \omega} \frac{1}{(\omega + y_k)^4} \right) = \\ & = \frac{1}{J^2} \frac{1}{64} \left(- \frac{16z (z^4 + 4z^2 \omega^2 + \omega^4)}{(z^2 - \omega^2)^4} G(z) + \right. \\ & + \frac{z^2 + 4z\omega + \omega^2}{\omega (z - \omega)^4} G(\omega) - \frac{z^2 - 4z\omega + \omega^2}{\omega (z + \omega)^4} G(-\omega) + \frac{z^2 + 4z\omega + \omega^2}{\omega (z - \omega)^3} G'(\omega) - \frac{z^2 - 4z\omega + \omega^2}{\omega (z + \omega)^3} G'(-\omega) + \\ & \left. + \frac{3z}{(z - \omega)^2} G''(\omega) + \frac{3z}{(z + \omega)^2} G''(-\omega) + \frac{\omega}{z - \omega} G'''(\omega) - \frac{\omega}{z + \omega} G'''(-\omega) \right). \end{aligned} \quad (445)$$

8. Term VI

The input of term VI of (417) to the quadratic equation,

$$\begin{aligned}
& \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \text{ (the r.h.s. of eq. (417), term VI) } = \frac{1}{J^3} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{1}{z - y_k} \frac{\omega^2}{6y_k(y_k^2 - \omega^2)^4} \cdot \\
& \cdot \left(-9y_k^3(y_k^2 + \omega^2)G(0) + \frac{y_k(y_k^2 - \omega^2)^2}{2\omega} \left((\omega + y_k)^2 G'(\omega) - (\omega - y_k)^2 G'(-\omega) \right) - \right. \\
& \quad \left. - \frac{3(y_k^2 - \omega^2)^4}{16\omega} (G''(\omega) - G''(-\omega)) - \frac{(y_k^2 - \omega^2)^4}{16} (G'''(\omega) + G'''(-\omega)) \right) = \\
& = \frac{1}{J^3} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \left(\frac{\omega}{96} \left(-\frac{144\omega z^2(z^2 + \omega^2)}{(z^2 - \omega^2)^4} G(0) + 8 \left(\frac{1}{(z - \omega)^2} G'(\omega) - \frac{1}{(z + \omega)^2} G'(-\omega) \right) - \right. \right. \\
& \quad \left. \left. - \frac{3}{z} (G''(\omega) - G''(-\omega)) - \frac{\omega}{z} (G'''(\omega) + G'''(-\omega)) \right) \right) \frac{1}{z - y_k} - \\
& \quad - \frac{\omega}{96z} \left(3(G''(\omega) - G''(-\omega)) + \omega(G'''(\omega) + G'''(-\omega)) \right) \frac{1}{y_k} + \\
& \quad + \frac{\omega(9zG(0) - 4(z - \omega)^2 G'(\omega))}{48(z - \omega)^4} \frac{1}{\omega - y_k} + \frac{\omega(9zG(0) - 4(z + \omega)^2 G'(-\omega))}{48(z + \omega)^4} \frac{1}{\omega + y_k} + \\
& \quad + \frac{\omega(-9zG(0) + 4(z - \omega)^2 G'(\omega))}{48(z - \omega)^3} \frac{1}{(\omega - y_k)^2} + \frac{\omega(9zG(0) - 4(z + \omega)^2 G'(-\omega))}{48(z + \omega)^3} \frac{1}{(\omega + y_k)^2} + \\
& \quad + \frac{3z\omega G(0)}{16(z - \omega)^2} \frac{1}{(\omega - y_k)^3} + \frac{3z\omega G(0)}{16(z + \omega)^2} \frac{1}{(\omega + y_k)^3} - \frac{3\omega^2 G(0)}{16(z - \omega)} \frac{1}{(\omega - y_k)^4} - \frac{3\omega^2 G(0)}{16(z + \omega)} \frac{1}{(\omega + y_k)^4} \Big) = \\
& = \frac{1}{J^2} \frac{\omega}{96} \left(\left(-\frac{144\omega z^2(z^2 + \omega^2)}{(z^2 - \omega^2)^4} G(0) + 8 \left(\frac{1}{(z - \omega)^2} G'(\omega) - \frac{1}{(z + \omega)^2} G'(-\omega) \right) - \right. \right. \\
& \quad \left. \left. - \frac{3}{z} (G''(\omega) - G''(-\omega)) - \frac{\omega}{z} (G'''(\omega) + G'''(-\omega)) \right) \right) G(z) + \\
& \quad + \left(18z \left(\frac{1}{(z - \omega)^4} G(\omega) - \frac{1}{(z + \omega)^4} G(-\omega) \right) + 18z \left(\frac{1}{(z - \omega)^3} G'(\omega) - \frac{1}{(z + \omega)^3} G'(-\omega) \right) + \right. \\
& \quad \left. + \frac{3}{z} \left(\frac{4z^2 - 2z\omega + \omega^2}{(z - \omega)^2} G''(\omega) - \frac{4z^2 + 2z\omega + \omega^2}{(z + \omega)^2} G''(-\omega) \right) + \frac{\omega}{z} \left(\frac{4z - \omega}{z - \omega} G'''(\omega) + \frac{4z + \omega}{z + \omega} G'''(-\omega) \right) \right) G(0) - \\
& \quad - 8 \left(\frac{1}{(z - \omega)^2} G(\omega) G'(\omega) - \frac{1}{(z + \omega)^2} G(-\omega) G'(-\omega) \right) - 8 \left(\frac{1}{z - \omega} G'(\omega)^2 - \frac{1}{z + \omega} G'(-\omega)^2 \right) \Big). \tag{446}
\end{aligned}$$

9. Term V Plus Term VI

Terms V (445) and VI (446) come with the prefactor of $1/J^2$, hence, up to and including the order $O(1/J^2)$, only the leading-order resolvent $G^{(0)}(z)$ will contribute to the quadratic equation through them,

$$\begin{aligned}
& \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} (\text{the r.h.s. of eq. (417), term V plus term VI}) = \\
& = \frac{1}{J^2} \left(\frac{1}{64} \left(- \frac{16z(z^4 + 4z^2\omega^2 + \omega^4)}{(z^2 - \omega^2)^4} G^{(0)}(z) + \right. \right. \\
& + \frac{z^2 + 4z\omega + \omega^2}{\omega(z - \omega)^4} G^{(0)}(\omega) - \frac{z^2 - 4z\omega + \omega^2}{\omega(z + \omega)^4} G^{(0)}(-\omega) + \frac{z^2 + 4z\omega + \omega^2}{\omega(z - \omega)^3} G^{(0)'}(\omega) - \frac{z^2 - 4z\omega + \omega^2}{\omega(z + \omega)^3} G^{(0)'}(-\omega) + \\
& + \frac{3z}{(z - \omega)^2} G^{(0)''}(\omega) + \frac{3z}{(z + \omega)^2} G^{(0)''}(-\omega) + \frac{\omega}{z - \omega} G^{(0)'''}(\omega) - \frac{\omega}{z + \omega} G^{(0)'''}(-\omega) \Big) + \\
& + \frac{\omega}{96} \left(\left(- \frac{144\omega z^2(z^2 + \omega^2)}{(z^2 - \omega^2)^4} G^{(0)}(0) + 8 \left(\frac{1}{(z - \omega)^2} G^{(0)'}(\omega) - \frac{1}{(z + \omega)^2} G^{(0)'}(-\omega) \right) - \right. \right. \\
& \left. \left. - \frac{3}{z} \left(G^{(0)''}(\omega) - G^{(0)''}(-\omega) \right) - \frac{\omega}{z} \left(G^{(0)'''}(\omega) + G^{(0)'''}(-\omega) \right) \right) \right) G^{(0)}(z) + \\
& + \left(18z \left(\frac{1}{(z - \omega)^4} G^{(0)}(\omega) - \frac{1}{(z + \omega)^4} G^{(0)}(-\omega) \right) + 18z \left(\frac{1}{(z - \omega)^3} G^{(0)'}(\omega) - \frac{1}{(z + \omega)^3} G^{(0)'}(-\omega) \right) + \right. \\
& + \frac{3}{z} \left(\frac{4z^2 - 2z\omega + \omega^2}{(z - \omega)^2} G^{(0)''}(\omega) - \frac{4z^2 + 2z\omega + \omega^2}{(z + \omega)^2} G^{(0)''}(-\omega) \right) + \frac{\omega}{z} \left(\frac{4z - \omega}{z - \omega} G^{(0)'''}(\omega) + \frac{4z + \omega}{z + \omega} G^{(0)'''}(-\omega) \right) \Big) G^{(0)}(0) - \\
& - 8 \left(\frac{1}{(z - \omega)^2} G^{(0)}(\omega) G^{(0)'}(\omega) - \frac{1}{(z + \omega)^2} G^{(0)}(-\omega) G^{(0)'}(-\omega) \right) - 8 \left(\frac{1}{z - \omega} G^{(0)'}(\omega)^2 - \frac{1}{z + \omega} G^{(0)'}(-\omega)^2 \right) \Big) + O\left(\frac{1}{J^3}\right). \\
& \tag{447}
\end{aligned}$$

10. Term VII (the “Anomaly”)

The input of term VII of (417) to the quadratic equation,

$$\begin{aligned}
& \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} (\text{the r.h.s. of eq. (417), term VII}) = \\
& = -\frac{1}{J^3} \sum_{k=1}^S \frac{y_k^2}{y_k^2 - \omega^2} \frac{1}{z - y_k} \frac{d}{dy_k} \left(\pi \rho^{(1)}(y_k) \left(\coth \left(\pi \rho^{(0)}(y_k) \right) - \frac{1}{\pi \rho^{(0)}(y_k)} \right) \right) =
\end{aligned}$$

$$= \frac{1}{J^2} \mathcal{A}^{(2,2)}(z) + \mathcal{O}\left(\frac{1}{J^3}\right), \quad (448)$$

similarly to terms III and IV, cannot yet be expressed through the resolvent, to which end we need the knowledge of $\rho^{(0)}(y_k)$ and $\rho^{(1)}(y_k)$.

We will proceed this way, however, only at $g = 0$ (see paragraph III C 2), while at $g > 0$ a more tractable method is to trade the summation over k in the second line of (448) for integration,

$$\begin{aligned} & \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} \text{ (the r.h.s. of eq. (417), term VII) } = \\ & = -\frac{1}{J^2} \int_{\mathbb{R}} dy \rho(y) \frac{1}{z - y} \frac{d}{dy} \left(\pi \rho^{(1)}(y) \left(\coth \left(\pi \rho^{(0)}(y) \right) - \frac{1}{\pi \rho^{(0)}(y)} \right) \right) + \left(\begin{array}{c} \text{subleading} \\ \text{boundary} \\ \text{contribution} \end{array} \right). \end{aligned} \quad (449)$$

Since there is the prefactor of $1/J^2$, it is sufficient to replace $\rho(y)$ by its large- J leading term, which yields

$$\mathcal{A}^{(2,2)}(z) = \mathcal{A}^{(2,2),\text{bulk}}(z) + \mathcal{A}^{(2,2),\text{boundary}}(z), \quad (450)$$

where the “bulk contribution,”

$$\mathcal{A}^{(2,2),\text{bulk}}(z) = -\pi \int_{\mathbb{R}} dy \frac{1}{z - y} \rho^{(0)}(y) \frac{d}{dy} \left(\rho^{(1)}(y) \left(\coth \left(\pi \rho^{(0)}(y) \right) - \frac{1}{\pi \rho^{(0)}(y)} \right) \right), \quad (451)$$

while the “boundary contribution” will be shown to vanish, $\mathcal{A}^{(2,2),\text{boundary}}(z) = 0$, as the integration is with the leading-order density. Finally, remark that it is the sum of (435) and (451) which appears in the quadratic equation, and this sum can be nicely simplified,

$$\begin{aligned} \mathcal{A}^{(2),\text{bulk}}(z) & \equiv \mathcal{A}^{(2,1),\text{bulk}}(z) + \mathcal{A}^{(2,2),\text{bulk}}(z) = \\ & = - \int_{\mathbb{R}} dy \frac{1}{z - y} \frac{d}{dy} \left(\rho^{(1)}(y) \left(\pi \rho^{(0)}(y) \coth \left(\pi \rho^{(0)}(y) \right) - 1 \right) \right). \end{aligned} \quad (452)$$

11. Summary

Collecting the above explicit results for term I plus term II (429) and term V plus term VI (447), as well as the general structure of terms III (430), IV (436) and VII (448), and moreover, simplifying them by substituting for the constants $G^{(0)}(0)$, $G^{(1)}(0)$, $G^{(2)}(0)$ their values according to the momentum conditions (343)–(345) — we finally obtain the quadratic equation for the all-loop SL(2) logarithmic string Bethe ansatz equations with all the mode numbers equal, at the three leading large- J orders to be

- Order $\mathcal{O}(J^0)$: The equation for $G^{(0)}(z)$ is quadratic, which is also the reason for calling this entire method the “quadratic equation,”

$$-(z^2 - \omega^2) G^{(0)}(z)^2 + (z^2 \mu - z + \omega^2 \mu (2\alpha - 1)) G^{(0)}(z) + z \mu \left(-\alpha + \omega^2 \gamma^{(0)} \right) + \omega^2 \mu^2 \alpha (\alpha - 1) = 0, \quad (453)$$

where we have denoted

$$\gamma^{(0)} \equiv G^{(0)'}(0). \quad (454)$$

It must be supplemented by the momentum condition,

$$G^{(0)}(0) = -\mu \alpha. \quad (455)$$

- Order $O(1/J)$: The equation for $G^{(1)}(z)$ is linear, and has already been solved,

$$G^{(1)}(z) = \frac{G^{(0)'}(z) - \frac{\omega^2 \mu \gamma^{(1)}}{z} - (\mathcal{A}^{(1)}(z) + \mathcal{P}^{(1)}(z))}{\frac{1}{z^2} (-2(z^2 - \omega^2) G^{(0)}(z) + z^2 \mu - z + \omega^2 \mu (2\alpha - 1))}, \quad (456)$$

where we have denoted

$$\gamma^{(1)} \equiv G^{(1)'}(0), \quad (457)$$

while $\mathcal{A}^{(1)}(z)$ is given by (430) or (432), (434), and $\mathcal{P}^{(1)}(z)$ by (441), (443). It must be supplemented by the momentum condition,

$$G^{(1)}(0) = 0. \quad (458)$$

- Order $O(1/J^2)$: The equation for $G^{(2)}(z)$ is linear, and has already been solved,

$$G^{(2)}(z) = \frac{\left(1 - \frac{\omega^2}{z^2}\right) G^{(1)}(z)^2 + G^{(1)'}(z) - \frac{\omega^2 \mu \gamma^{(2)}}{z} - (\mathcal{A}^{(2)}(z) + \mathcal{P}^{(2)}(z)) + \mathcal{F}(z)}{\frac{1}{z^2} (-2(z^2 - \omega^2) G^{(0)}(z) + z^2 \mu - z + \omega^2 \mu (2\alpha - 1))}, \quad (459)$$

where we have denoted

$$\gamma^{(2)} \equiv G^{(2)'}(0), \quad (460)$$

and also introduced the following combination of the leading-order resolvents $G^{(0)}(z)$ at various points,

$$\begin{aligned} \mathcal{F}(z) \equiv & \left(\frac{z(z^4 + 4\omega^2 z^2 + \omega^4) - 6\omega^2 \mu \alpha z^2 (z^2 + \omega^2)}{4(z^2 - \omega^2)^4} - \right. \\ & - \frac{\omega}{96} \left(\left(\frac{8}{(z - \omega)^2} - \frac{1}{z^2} \right) G^{(0)'}(\omega) - \left(\frac{8}{(z + \omega)^2} - \frac{1}{z^2} \right) G^{(0)'}(-\omega) \right) + \\ & + \frac{\omega}{32z^2} \left((z + \omega) G^{(0)''}(\omega) - (z - \omega) G^{(0)''}(-\omega) \right) + \frac{\omega^2}{96z^2} \left((z + \omega) G^{(0)'''}(\omega) + (z - \omega) G^{(0)'''}(-\omega) \right) \Big) G^{(0)}(z) - \\ & - \frac{1}{192} \left(\frac{3(z^2 + 4\omega(1 - 3\omega\mu\alpha)z + \omega^2)}{\omega(z - \omega)^4} G^{(0)}(\omega) - \frac{3(z^2 - 4\omega(1 + 3\omega\mu\alpha)z + \omega^2)}{\omega(z + \omega)^4} G^{(0)}(-\omega) + \right. \\ & + \frac{1}{\omega z^2 (z - \omega)^3} \left(4z^4 + z^3 \omega (\omega\mu(1 - 38\alpha) + 9) + 3z^2 \omega^2 (\omega\mu(2\alpha - 1) + 2) - \right. \\ & \left. \left. - z\omega^3 (3\omega\mu(2\alpha - 1) + 1) + \omega^5 \mu(2\alpha - 1) \right) G^{(0)'}(\omega) - \right. \\ & - \frac{1}{\omega z^2 (z + \omega)^3} \left(4z^4 + z^3 \omega (\omega\mu(1 - 38\alpha) - 9) + 3z^2 \omega^2 (-\omega\mu(2\alpha - 1) + 2) + \right. \\ & \left. \left. + z\omega^3 (-3\omega\mu(2\alpha - 1) + 1) - \omega^5 \mu(2\alpha - 1) \right) G^{(0)'}(-\omega) + \right. \\ & \left. + \frac{3}{z^2 (z - \omega)^2} \left(4z^3 (1 - 2\omega\mu\alpha) + z^2 \omega (\omega\mu(2\alpha + 1) - 2) + z\omega^2 (2\omega\mu(\alpha - 1) + 1) - \omega^4 \mu(2\alpha - 1) \right) G^{(0)''}(\omega) + \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{3}{z^2(z+\omega)^2} \left(4z^3(1+2\omega\mu\alpha) + z^2\omega(\omega\mu(2\alpha+1)+2) + z\omega^2(-2\omega\mu(\alpha-1)+1) - \omega^4\mu(2\alpha-1) \right) G^{(0)''}(-\omega) + \\
& + \frac{\omega(4z^2(1-2\omega\mu\alpha) + z\omega(\omega\mu-1) + \omega^3\mu(2\alpha-1))}{z^2(z-\omega)} G^{(0)'''}(\omega) - \\
& - \frac{\omega(4z^2(1+2\omega\mu\alpha) + z\omega(\omega\mu+1) - \omega^3\mu(2\alpha-1))}{z^2(z+\omega)} G^{(0)'''}(-\omega) \Big) + \\
& + \frac{\omega}{12} \left(\frac{1}{(z-\omega)^2} G^{(0)}(\omega) G^{(0)'}(\omega) - \frac{1}{(z+\omega)^2} G^{(0)}(-\omega) G^{(0)'}(-\omega) + \frac{1}{z-\omega} G^{(0)'}(\omega)^2 - \frac{1}{z+\omega} G^{(0)'}(-\omega)^2 \right), \quad (461)
\end{aligned}$$

while $\mathcal{A}^{(2)}(z)$ is given by (430), (448) or (452), and $\mathcal{P}^{(2)}(z)$ by (442), (444). It must be supplemented by the momentum condition,

$$G^{(2)}(0) = \frac{1}{192\omega} \left(\omega^2 \left(G^{(0)'''}(\omega) - G^{(0)'''}(-\omega) \right) + 3\omega \left(G^{(0)''}(\omega) + G^{(0)''}(-\omega) \right) + \left(G^{(0)'}(\omega) - G^{(0)'}(-\omega) \right) \right). \quad (462)$$

In particular, setting here $\omega = 0$ leads to its one-loop counterpart,

- Order $\mathcal{O}(J^0)$:

$$zG^{(0)}(z)^2 + (1-z\mu)G^{(0)}(z) + \mu\alpha = 0, \quad (463)$$

with the momentum condition,

$$G^{(0)}(0) = -\mu\alpha. \quad (464)$$

- Order $\mathcal{O}(1/J)$:

$$G^{(1)}(z) = \frac{G^{(0)'}(z) - \mathcal{A}^{(1)}(z)}{-2G^{(0)}(z) - \frac{1}{z} + \mu}, \quad (465)$$

with the momentum condition,

$$G^{(1)}(0) = 0. \quad (466)$$

- Order $\mathcal{O}(1/J^2)$:

$$G^{(2)}(z) = \frac{G^{(1)}(z)^2 + G^{(1)'}(z) - \mathcal{A}^{(2)}(z) + \mathcal{F}(z)|_{\omega=0}}{-2G^{(0)}(z) - \frac{1}{z} + \mu}, \quad (467)$$

where

$$\mathcal{F}(z)|_{\omega=0} = \frac{3G^{(0)}(z) + 3\mu\alpha - 3zG^{(0)'}(0) - 2z^2G^{(0)''}(0)}{12z^3}, \quad (468)$$

with the momentum condition,

$$G^{(2)}(0) = \frac{1}{24} G^{(0)''}(0). \quad (469)$$

D. The All-Loop One-Cut Linear Equation at the Orders $O(J^0)$, $O(1/J)$, $O(1/J^2)$

1. The One-Loop Master Formulae

Let us start from the one-loop case. By the “master formulae,” we will understand here explicit expressions for the following quantities featuring the difference of the rapidities, $(u_k - u_j)$, where k is a fixed reference point far from the edges of the cut (389),

$$\sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{(u_k - u_j)^t}, \quad \text{where} \quad t = 1, 2, \dots \quad (470)$$

We will handle them by splitting the above sum into the non-anomalous and anomalous regions, just as we have done for the “anomaly” term in the Bethe ansatz equations (see paragraph II B 5),

$$\sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{(u_k - u_j)^t} = \underbrace{\sum_{\substack{j: \\ |k-j| > N}} \frac{1}{(u_k - u_j)^t}}_{\text{non-anomalous}} + \underbrace{\sum_{\substack{j: \\ |k-j| \leq N}} \frac{1}{(u_k - u_j)^t}}_{\text{anomalous}}, \quad (471)$$

and computing each of these pieces separately. We will want to find the three lowest master formulae, up to the three (for $t = 1$) or two (for $t = 2, 3$) leading large- J orders.

To derive the non-anomalous part for any t , we reduce it to the $t = 1$ case in the following way: Recall that at one loop, the relation between the rapidities and rescaled spectral parameters is especially simple (280), $u_k = Jy_k$. Hence,

$$\begin{aligned} \sum_{\substack{j: \\ |k-j| > N}} \frac{1}{(u_k - u_j)^t} &= \frac{1}{J^t} \sum_{\substack{j: \\ |k-j| > N}} \frac{1}{(y_k - y_j)^t} = \frac{1}{J^t} \frac{(-1)^{t-1}}{(t-1)!} \frac{d^{t-1}}{dy_k^{t-1}} \left(\sum_{\substack{j: \\ |k-j| > N}} \frac{1}{y_k - y_j} \right) = \\ &= \frac{1}{J^{t-1}} \frac{(-1)^{t-1}}{(t-1)!} \mathcal{G}^{'' \dots (t-1)}(y_k), \end{aligned} \quad (472)$$

where we have made use of the principal value of the resolvent,

$$\mathcal{G}(y_k) \equiv \frac{1}{2} \lim_{\epsilon \rightarrow 0^+} (G(y_k + i\epsilon) + G(y_k - i\epsilon)) = \frac{1}{J} \sum_{\substack{j: \\ |k-j| > N}} \frac{1}{y_k - y_j}. \quad (473)$$

In the anomalous piece, on the other hand, we exploit the Taylor series of the rapidity u_j around u_k , terminated at the next-to-leading order (396), (399),

$$u_k - u_j = bn - \frac{cn^2}{2J} + O\left(\frac{n^3}{J^2}\right), \quad b = \frac{1}{\rho(y_k)}, \quad c = -\frac{\rho'(y_k)}{\rho(y_k)^3}, \quad (474)$$

which yields

$$\frac{1}{(u_k - u_j)^t} \Big|_{\text{anomalous}} = \frac{1}{b^t n^t} + \frac{1}{J} \frac{tc}{2b^{t+1}n^{t-1}} + O\left(\frac{1}{n^{t-2}J^2}\right). \quad (475)$$

This should be summed over n from minus to plus infinity, hence, all the terms odd in n will disappear upon such a summation, and only the even ones will contribute,

$$\sum_{\substack{j: \\ |k-j| \leq N}} \frac{1}{(u_k - u_j)^t} = \begin{cases} \frac{1}{J} \frac{t\zeta(t-1)c}{b^{t+1}} + O\left(\frac{1}{J^3}\right) & \text{for odd } t, \\ \frac{2\zeta(t)}{b^t} + O\left(\frac{1}{J^2}\right) & \text{for even } t \end{cases} =$$

$$\begin{aligned}
&= \begin{cases} -\frac{1}{J} t \zeta(t-1) \rho(y_k)^{t-2} \rho'(y_k) + O\left(\frac{1}{J^3}\right) & \text{for odd } t, \\ 2\zeta(t) \rho(y_k)^t + O\left(\frac{1}{J^2}\right) & \text{for even } t \end{cases} = \\
&= \begin{cases} -t\zeta(t-1) \left(\frac{1}{J} \rho^{(0)}(y_k)^{t-2} \rho^{(0)'}(y_k) + \right. \\ \quad \left. + \frac{1}{J^2} \rho^{(0)}(y_k)^{t-3} \left(\rho^{(0)}(y_k) \rho^{(1)'}(y_k) + (t-2) \rho^{(0)'}(y_k) \rho^{(1)}(y_k) \right) \right) + O\left(\frac{1}{J^3}\right) & \text{for odd } t, \\ 2\zeta(t) \left(\rho^{(0)}(y_k)^t + \frac{1}{J} t \rho^{(0)}(y_k)^{t-1} \rho^{(1)}(y_k) \right) + O\left(\frac{1}{J^2}\right) & \text{for even } t \end{cases} \quad (476)
\end{aligned}$$

where ζ is the Riemann zeta function, and where in the last line we have used the large- J expansion of the density (321).

Notice an interesting fact: For $t = 1$, we sum over n an expression proportional to n^0 , *i.e.*, divergent, as we have already observed (402). Back there, it served just to destroy another divergent quantity, and there was no need to assign to it any regularized value. In other words, as it seems, it implies that the quadratic equation (463)–(469), so also its solution, do not require any regularization scheme to be applied to this divergent sum. In the above formula (476), however, we have computed it with help of a particular regularization technique, namely, the zeta-function one,

$$\sum_{n=-\infty}^{+\infty} 1 = 2 \sum_{n=1}^{+\infty} 1 = 2\zeta(0) = -1. \quad (477)$$

A priori, it does not seem that there is any principle to single out this scheme. But in paragraph III C 6, we will test the master formula (478), which is based on (477), for the solution to the one-loop one-cut quadratic equation, and it will be fulfilled. This *á posteriori* justifies that the zeta-function regularization method is the right one.

Adding (472) and (476), we arrive at the final result. Let us print it for the three lowest values of t ,

$$\begin{aligned}
\sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{u_k - u_j} &= \mathcal{G}^{(0)}(y_k) + \\
&+ \frac{1}{J} \left(\mathcal{G}^{(1)}(y_k) + \frac{1}{2} \frac{\rho^{(0)'}(y_k)}{\rho^{(0)}(y_k)} \right) + \\
&+ \frac{1}{J^2} \left(\mathcal{G}^{(2)}(y_k) + \frac{1}{2} \frac{d}{dy_k} \left(\frac{\rho^{(1)}(y_k)}{\rho^{(0)}(y_k)} \right) \right) + O\left(\frac{1}{J^3}\right), \quad (478)
\end{aligned}$$

$$\begin{aligned}
\sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{(u_k - u_j)^2} &= \frac{\pi^2}{3} \rho^{(0)}(y_k)^2 + \\
&+ \frac{1}{J} \left(-\mathcal{G}^{(0)'}(y_k) + \frac{2\pi^2}{3} \rho^{(0)}(y_k) \rho^{(1)}(y_k) \right) + O\left(\frac{1}{J^2}\right), \quad (479)
\end{aligned}$$

$$\begin{aligned}
\sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{(u_k - u_j)^3} &= -\frac{1}{J} \frac{\pi^2}{2} \rho^{(0)}(y_k) \rho^{(0)'}(y_k) + \\
&+ \frac{1}{J^2} \left(\frac{1}{2} \mathcal{G}^{(0)''}(y_k) - \frac{\pi^2}{2} \frac{d}{dy_k} \left(\rho^{(0)}(y_k) \rho^{(1)}(y_k) \right) \right) + O\left(\frac{1}{J^3}\right). \quad (480)
\end{aligned}$$

Another important observation is that the successive formulae in this tower are not related to each other through differentiation w.r.t. u_k ,

$$\sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{(u_k - u_j)^t} \neq \frac{(-1)^{t-1}}{(t-1)!} \frac{d^{t-1}}{du_k^{t-1}} \left(\sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{u_k - u_j} \right). \quad (481)$$

This is only true for the non-anomalous parts of the sums (472). This is to be kept in mind while performing tricks like in paragraph II B 6.

2. The One-Loop One-Cut Linear Equation at the Orders $O(J^0)$, $O(1/J)$, $O(1/J^2)$

One application of the first master formula (478) is to obtain the so-called “linear equation” from the expanded one-loop one-cut logarithmic string Bethe ansatz equations (418); this is an equation satisfied by the principal value of the resolvent (473). We easily get it to be

- Order $O(J^0)$:

$$2\mathcal{G}^{(0)}(y_k) = \mu - \frac{1}{y_k}. \quad (482)$$

- Order $O(1/J)$:

$$2\mathcal{G}^{(1)}(y_k) = -\pi\rho^{(0)'}(y_k) \coth\left(\pi\rho^{(0)}(y_k)\right). \quad (483)$$

- Order $O(1/J^2)$:

$$2\mathcal{G}^{(2)}(y_k) = -\frac{1}{4y_k^3} - \frac{d}{dy_k} \left(\pi\rho^{(1)}(y_k) \coth\left(\pi\rho^{(0)}(y_k)\right) \right). \quad (484)$$

We might attempt to approach the problem of finding the three leading large- J orders of the resolvent by solving these three equations; they are nothing but Riemann–Hilbert problems. We will, however, resort to the quadratic equation technique, and afterwards use (482)–(484) as a cross-check.

It is also common to rewrite linear equations through the so-called “quasi-momentum,” instead of the resolvent, which at one loop is defined as

$$p(z) \equiv G(z) + \frac{1}{2z}. \quad (485)$$

Remark that it is only the leading large- J order that is modified here. Hence, the r.h.s. only of (482) will change in this new language,

$$\mathcal{P}^{(0)}(y_k) = \pi m. \quad (486)$$

3. The All-Loop Master Formulae

We would now like to extend the above findings to all loops. Here, the “master formulae” will mean explicit expressions for the following objects,

$$\frac{1}{J^t} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{y_j^2}{y_j^2 - \omega^2} \frac{1}{(y_k - y_j)^t}, \quad \text{where} \quad t = 1, 2, \dots \quad (487)$$

We split them as before into the non-anomalous and anomalous parts (471). Also as before, the former is related through multiple differentiation to the principal value of the resolvent (472),

$$\frac{1}{J^t} \sum_{\substack{j: \\ |k-j| > N}} \frac{y_j^2}{y_j^2 - \omega^2} \frac{1}{(y_k - y_j)^t} = \frac{1}{J^{t-1}} \frac{(-1)^{t-1}}{(t-1)!} \mathcal{G}^{(t-1)'}(y_k), \quad (488)$$

where this time of course

$$\mathcal{G}'(y_k) \equiv \frac{1}{2} \lim_{\epsilon \rightarrow 0^+} (G(y_k + i\epsilon) + G(y_k - i\epsilon)) = \frac{1}{J} \sum_{\substack{j: \\ |k-j| > N}} \frac{y_j^2}{y_j^2 - \omega^2} \frac{1}{y_k - y_j}. \quad (489)$$

In order to tackle the anomalous piece, we need to translate the anomalous expansion of the rapidities (474) into the language of the rescaled spectral parameters (280),

$$y_j = y_k - \frac{by_k^2}{y_k^2 - \omega^2} \frac{n}{J} + \frac{y_k^2}{2(y_k^2 - \omega^2)} \left(c - \frac{2b^2\omega^2 y_k}{(y_k^2 - \omega^2)^2} \right) \frac{n^2}{J^2} + \mathcal{O}\left(\frac{1}{J^3}\right). \quad (490)$$

This gives

$$\begin{aligned} \frac{1}{J^t} \sum_{\substack{j: \\ |k-j| \leq N}} \frac{y_j^2}{y_j^2 - \omega^2} \frac{1}{(y_k - y_j)^t} &= \begin{cases} \frac{1}{J} 2\zeta(t-1) \left(\frac{y_k^2 - \omega^2}{y_k^2} \right)^{t-1} \frac{1}{b^{t-1}} \left(\frac{tc}{2b^2} - \frac{(t-2)\omega^2 y_k}{(y_k^2 - \omega^2)^2} \right) + \mathcal{O}\left(\frac{1}{J^3}\right) & \text{for odd } t, \\ 2\zeta(t) \left(\frac{y_k^2 - \omega^2}{y_k^2} \right)^{t-1} \frac{1}{b^t} + \mathcal{O}\left(\frac{1}{J^2}\right) & \text{for even } t \end{cases} = \\ &= \begin{cases} -\frac{1}{J} \zeta(t-1) \left(\frac{y_k^2 - \omega^2}{y_k^2} \right)^{t-2} \rho(y_k)^{t-2} \left(\frac{2(t-2)\omega^2}{y_k(y_k^2 - \omega^2)} \rho(y_k) + t\rho'(y_k) \right) + \mathcal{O}\left(\frac{1}{J^3}\right) & \text{for odd } t, \\ 2\zeta(t) \left(\frac{y_k^2 - \omega^2}{y_k^2} \right)^{t-1} \rho(y_k)^t + \mathcal{O}\left(\frac{1}{J^2}\right) & \text{for even } t \end{cases} = \\ &= \begin{cases} -\zeta(t-1) \left(\frac{y_k^2 - \omega^2}{y_k^2} \right)^{t-2} \left(\frac{1}{J} \rho^{(0)}(y_k)^{t-2} \left(\frac{2(t-2)\omega^2}{y_k(y_k^2 - \omega^2)} \rho^{(0)}(y_k) + t\rho^{(0)'}(y_k) \right) + \right. \\ \quad \left. + \frac{1}{J^2} \rho^{(0)}(y_k)^{t-3} \left(\frac{2(t-2)(t-1)\omega^2}{y_k(y_k^2 - \omega^2)} \rho^{(0)}(y_k) \rho^{(1)}(y_k) + \right. \right. \\ \quad \left. \left. + t(t-2)\rho^{(0)'}(y_k) \rho^{(1)}(y_k) + t\rho^{(0)}(y_k) \rho^{(1)'}(y_k) \right) \right) + \mathcal{O}\left(\frac{1}{J^3}\right) & \text{for odd } t, \\ 2\zeta(t) \left(\frac{y_k^2 - \omega^2}{y_k^2} \right)^{t-1} \left(\rho^{(0)}(y_k)^t + \frac{1}{J} t\rho^{(0)}(y_k)^{t-1} \rho^{(1)}(y_k) \right) + \mathcal{O}\left(\frac{1}{J^2}\right) & \text{for even } t \end{cases}. \quad (491) \end{aligned}$$

We have again assumed the zeta-function regularization scheme (477) for $t = 1$.

Adding (488) and (491) yields the full result. Let us write it for the three lowest values of t ,

$$\begin{aligned} \frac{1}{J} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{y_j^2}{y_j^2 - \omega^2} \frac{1}{y_k - y_j} &= \mathcal{G}^{(0)}(y_k) + \\ &+ \frac{1}{J} \left(\mathcal{G}^{(1)}(y_k) + \frac{1}{2} \frac{y_k^2}{y_k^2 - \omega^2} \frac{\rho^{(0)'}(y_k)}{\rho^{(0)}(y_k)} - \frac{\omega^2 y_k}{(y_k^2 - \omega^2)^2} \right) + \\ &+ \frac{1}{J^2} \left(\mathcal{G}^{(2)}(y_k) + \frac{1}{2} \frac{y_k^2}{y_k^2 - \omega^2} \frac{d}{dy_k} \left(\frac{\rho^{(1)}(y_k)}{\rho^{(0)}(y_k)} \right) \right) + \mathcal{O}\left(\frac{1}{J^3}\right), \end{aligned} \quad (492)$$

$$\begin{aligned} \frac{1}{J^2} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{y_j^2}{y_j^2 - \omega^2} \frac{1}{(y_k - y_j)^2} &= \frac{\pi^2}{3} \frac{y_k^2 - \omega^2}{y_k^2} \rho^{(0)}(y_k)^2 + \\ &+ \frac{1}{J} \left(-\mathcal{G}^{(0)'}(y_k) + \frac{2\pi^2}{3} \frac{y_k^2 - \omega^2}{y_k^2} \rho^{(0)}(y_k) \rho^{(1)}(y_k) \right) + \mathcal{O}\left(\frac{1}{J^2}\right), \end{aligned} \quad (493)$$

$$\begin{aligned} \frac{1}{J^3} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{y_j^2}{y_j^2 - \omega^2} \frac{1}{(y_k - y_j)^3} &= -\frac{1}{J} \frac{\pi^2}{6} \rho^{(0)}(y_k) \left(\frac{2\omega^2}{y_k^3} \rho^{(0)}(y_k) + \frac{3(y_k^2 - \omega^2)}{y_k^2} \rho^{(0)'}(y_k) \right) + \\ &+ \frac{1}{J^2} \left(\frac{1}{2} \mathcal{G}^{(0)''}(y_k) - \frac{\pi^2}{6} \left(\frac{4\omega^2}{y_k^3} \rho^{(0)}(y_k) \rho^{(1)}(y_k) + \right. \right. \end{aligned} \quad (494)$$

$$\left. + \frac{3(y_k^2 - \omega^2)}{y_k^2} \frac{d}{dy_k} \left(\rho^{(0)}(y_k) \rho^{(1)}(y_k) \right) \right) + \mathcal{O}\left(\frac{1}{J^3}\right). \quad (495)$$

Some manipulation is needed to recast these master formulae in a form analogous to (478)–(480),

$$\begin{aligned} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{u_k - u_j} &= \mathcal{G}^{(0)}(y_k) - G^{(0)}(0) + G^{(0)}\left(\frac{\omega^2}{y_k}\right) + \\ &+ \frac{1}{J} \left(\mathcal{G}^{(1)}(y_k) + \frac{1}{2} \frac{y_k^2}{y_k^2 - \omega^2} \frac{\rho^{(0)'}(y_k)}{\rho^{(0)}(y_k)} - G^{(1)}(0) + G^{(1)}\left(\frac{\omega^2}{y_k}\right) \right) + \\ &+ \frac{1}{J^2} \left(\mathcal{G}^{(2)}(y_k) + \frac{1}{2} \frac{y_k^2}{y_k^2 - \omega^2} \frac{d}{dy_k} \left(\frac{\rho^{(1)}(y_k)}{\rho^{(0)}(y_k)} \right) - G^{(2)}(0) + G^{(2)}\left(\frac{\omega^2}{y_k}\right) \right) + \mathcal{O}\left(\frac{1}{J^3}\right), \end{aligned} \quad (496)$$

$$\begin{aligned} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{(u_k - u_j)^2} &= \frac{\pi^2}{3} \rho^{(0)}(y_k)^2 + \\ &+ \frac{1}{J} \left(\frac{2\pi^2}{3} \rho^{(0)}(y_k) \rho^{(1)}(y_k) - \frac{y_k^2}{y_k^2 - \omega^2} \mathcal{G}^{(0)'}(y_k) + \frac{\omega^2}{y_k^2 - \omega^2} G^{(0)'}\left(\frac{\omega^2}{y_k}\right) \right) + \mathcal{O}\left(\frac{1}{J^2}\right), \end{aligned} \quad (497)$$

$$\begin{aligned} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{(u_k - u_j)^3} &= -\frac{1}{J} \frac{\pi^2}{2} \frac{y_k^2}{y_k^2 - \omega^2} \rho^{(0)}(y_k) \rho^{(0)'}(y_k) + \\ &+ \frac{1}{J^2} \left(-\frac{\pi^2}{2} \frac{y_k^2}{y_k^2 - \omega^2} \frac{d}{dy_k} \left(\rho^{(0)}(y_k) \rho^{(1)}(y_k) \right) - \right. \\ &- \frac{\omega^2 y_k^3}{(y_k^2 - \omega^2)^3} \mathcal{G}^{(0)'}(y_k) + \frac{y_k^4}{2(y_k^2 - \omega^2)^2} \mathcal{G}^{(0)''}(y_k) + \\ &\left. + \frac{\omega^2 y_k^3}{(y_k^2 - \omega^2)^3} G^{(0)'}\left(\frac{\omega^2}{y_k}\right) + \frac{\omega^4}{2(y_k^2 - \omega^2)^2} G^{(0)''}\left(\frac{\omega^2}{y_k}\right) \right) + \mathcal{O}\left(\frac{1}{J^3}\right). \end{aligned} \quad (498)$$

Remark that setting $\omega = 0$ reduces these new expressions to their one-loop counterparts.

4. The All-Loop One-Cut Linear Equation at the Orders $O(J^0)$, $O(1/J)$, $O(1/J^2)$

The first master formula (492) can now be used to rewrite the expanded all-loop one-cut logarithmic string Bethe ansatz equations (417) in the form of a linear equation for every large- J order. Again, this will not be to solve these equations, but to use them as a cross-check of the solution to the quadratic equation.

The pertinent linear equations will be found by relating the first term in (417), *i.e.*, the term featuring the difference $(y_k - y_j)$, and thus, developing a non-anomalous as well as an anomalous part — to the l.h.s. of the master formula (492). A simple manipulation yields for it,

$$\begin{aligned} & \frac{1}{J} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{2y_j y_k (y_j y_k - \omega^2)}{(y_j^2 - \omega^2)(y_k^2 - \omega^2)(y_j - y_k)} = \\ & = -2 \left(\frac{1}{J} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{y_j^2}{y_j^2 - \omega^2} \frac{1}{y_k - y_j} \right) - \frac{2\omega^2}{y_k^2 - \omega^2} G(0) - \frac{1}{J} \frac{2\omega^2 y_k}{(y_k^2 - \omega^2)^2} = \dots, \end{aligned} \quad (499)$$

i.e., explicitly,

$$\begin{aligned} & \dots = -2 \left(\mathcal{G}^{(0)}(y_k) + \frac{\omega^2}{y_k^2 - \omega^2} G^{(0)}(0) \right) - \\ & - \frac{1}{J} \left(2\mathcal{G}^{(1)}(y_k) + \frac{2\omega^2}{y_k^2 - \omega^2} G^{(1)}(0) + \frac{y_k^2}{y_k^2 - \omega^2} \frac{\rho^{(0)'}(y_k)}{\rho^{(0)}(y_k)} \right) - \\ & - \frac{1}{J^2} \left(2\mathcal{G}^{(2)}(y_k) + \frac{2\omega^2}{y_k^2 - \omega^2} G^{(2)}(0) + \frac{y_k^2}{y_k^2 - \omega^2} \frac{d}{dy_k} \left(\frac{\rho^{(1)}(y_k)}{\rho^{(0)}(y_k)} \right) \right) + O\left(\frac{1}{J^3}\right). \end{aligned} \quad (500)$$

The all-loop one-cut linear equations immediately follow,

- Order $O(J^0)$:

$$2\mathcal{G}^{(0)}(y_k) = \mu - \frac{y_k}{y_k^2 - \omega^2} - \frac{2\omega^2}{y_k^2 - \omega^2} G^{(0)}(0). \quad (501)$$

- Order $O(1/J)$:

$$2\mathcal{G}^{(1)}(y_k) = -\frac{2\omega^2}{y_k^2 - \omega^2} G^{(1)}(0) - \frac{y_k^2}{y_k^2 - \omega^2} \pi \rho^{(0)'}(y_k) \coth\left(\pi \rho^{(0)}(y_k)\right) + 2\tilde{\mathcal{P}}^{(1)}(y_k). \quad (502)$$

- Order $O(1/J^2)$:

$$\begin{aligned} 2\mathcal{G}^{(2)}(y_k) &= -\frac{2\omega^2}{y_k^2 - \omega^2} G^{(2)}(0) - \frac{y_k^3 (\omega^4 + 4y_k^2 \omega^2 + y_k^4)}{4(y_k^2 - \omega^2)^5} + \\ &+ \frac{\omega^2 y_k}{6(y_k^2 - \omega^2)^5} \left(-9y_k^3 (y_k^2 + \omega^2) G^{(0)}(0) + \frac{y_k (y_k^2 - \omega^2)^2}{2\omega} \left((\omega + y_k)^2 G^{(0)'}(\omega) - (\omega - y_k)^2 G^{(0)' }(-\omega) \right) - \right. \end{aligned}$$

$$\begin{aligned}
& - \frac{3(y_k^2 - \omega^2)^4}{16\omega} \left(G^{(0)''}(\omega) - G^{(0)''}(-\omega) \right) - \frac{(y_k^2 - \omega^2)^4}{16} \left(G^{(0)'''}(\omega) + G^{(0)'''}(-\omega) \right) \Big) - \\
& - \frac{y_k^2}{y_k^2 - \omega^2} \frac{d}{dy_k} \left(\pi \rho^{(1)}(y_k) \coth \left(\pi \rho^{(0)}(y_k) \right) \right) + 2\tilde{\mathcal{P}}^{(2)}(y_k). \tag{503}
\end{aligned}$$

In the above, we have denoted the Hernández–López phase contribution by

$$\begin{aligned}
& \frac{1}{J} \tilde{\mathcal{P}}^{(1)}(z) + \frac{1}{J^2} \tilde{\mathcal{P}}^{(2)}(z) + \mathcal{O}\left(\frac{1}{J^3}\right) \equiv \\
& \equiv -\frac{1}{J^2} \frac{1}{\pi} \sum_{j=1}^S \frac{y_j^2 z^2}{(y_j^2 - \omega^2)(z^2 - \omega^2)} \left(\frac{2\omega}{(y_j - z)(y_j z - \omega^2)} + \left(\frac{1}{(y_j - z)^2} + \frac{\omega^2}{(y_j z - \omega^2)^2} \right) \log \left(\frac{(y_j + \omega)(z - \omega)}{(y_j - \omega)(z + \omega)} \right) \right), \tag{504}
\end{aligned}$$

while the slashed versions, $\tilde{\mathcal{P}}^{(1)}(y_k)$ and $\tilde{\mathcal{P}}^{(2)}(y_k)$, are defined analogously to (473).

5. Why Is There No Anomalous Contribution From the Hernández–López Phase Term?

One might wonder why the sum over j in the Hernández–López phase term in the expanded logarithmic string Bethe ansatz equations (504) does not possess an anomalous piece, even though the summand depends on the difference $(y_k - y_j)$, where we take $z = y_k$.

The quickest answer is that the pertinent sum vanishes for $j = k$. But let us see it also explicitly, using the zeta–function regularization scheme. We substitute there the anomalous expansion of y_j around y_k (490), obtaining

$$\begin{aligned}
& (\text{the summand on the r.h.s. of eq. (504)})|_{\text{anomalous}} = \\
& = -\frac{n}{J^3} \frac{8b\omega^3 y_k^6}{3\pi(y_k^2 - \omega^2)^6} - \frac{n^2}{J^4} \frac{4\omega^3 y_k^6 \left(-c(y_k^2 - \omega^2)^2 + 6b^2 y_k(y_k^2 + \omega^2) \right)}{3\pi(y_k^2 - \omega^2)^8} + \mathcal{O}\left(\frac{1}{J^5}\right). \tag{505}
\end{aligned}$$

The leading term is odd w.r.t. n , and consequently, vanishes when summed over n from minus to plus infinity. The next-to-leading one is even w.r.t. n , but yields a divergent sum. Similarly, all the further terms in this series depend on n as n^t , where t is a positive integer; the odd powers will vanish upon being summed over n , the even ones will remain, leading to divergent sums. Led by the appropriateness of the zeta–function regularization scheme (477) in the master formulae (478), (492), we decide to assign to these divergent sums their regularized value — which, however, is always zero,

$$\sum_{n=-\infty}^{+\infty} n^{2t} = 2 \sum_{n=1}^{+\infty} n^{2t} = 2\zeta(-2t) = 0, \quad \text{for } t = 1, 2, \dots \tag{506}$$

In this way, the entire anomalous piece of the Hernández–López phase term is proven to be zero as well,

$$\sum_{\substack{j: \\ |k-j| \leq N}} (\text{the summand on the r.h.s. of eq. (504)}) = 0, \tag{507}$$

as expected.

III. SOLVING THE ONE-LOOP ONE-CUT QUADRATIC EQUATION AT THE ORDERS $O(J^0)$, $O(1/J)$, $O(1/J^2)$

In this section, we present an exact solution to the one-loop version (463)–(469) of the one-cut quadratic equation. We do it in order to • show a conceptually easy way of solving this problem, alternative to the Baxter equation method [167], with end results given in simpler forms, • have some practice before attacking the more involved all-loop case, in particular, understand the boundary contributions.

A. The Exact Solution to the One-Loop One-Cut Quadratic Equation at the Order $O(J^0)$

1. The Resolvent $G^{(0)}(z)$

The leading-order equation (463) is quadratic, and is easily solved by

$$G^{(0)}(z) = \frac{-1 + \mu z + \mu \sqrt{P(z)}}{2z}, \quad (508)$$

where $P(z)$ is the quadratic polynomial

$$P(z) = z^2 - \frac{2(1+2\alpha)}{\mu}z + \frac{1}{\mu^2} = (z-a)(z-b), \quad \text{where} \quad a, b \equiv \frac{1}{\mu} (\sqrt{\alpha} \mp \sqrt{1+\alpha})^2. \quad (509)$$

As a solution to a quadratic equation, this function, as it stands, is two-valued. In our computation, however, we adapt the following technique of working with it — we define $\sqrt{P(z)}$ as

$$\sqrt{P(z)} \equiv s\sqrt{z-a}\sqrt{z-b}, \quad (510)$$

where the square roots are principal, and $s = \pm 1$ is a sign, which will be specified below (516). This is similar to how we have handled the logarithmic Bethe ansatz equations (269) by defining the logarithms to be principal, and introducing the mode numbers, which the multi-valuedness is traded for.

The basic features of this solution:

- The roots a, b are real. Remark that μ can have any sign (we assume $\mu \neq 0$, as then the solution obeying the momentum condition would be $G^{(0)}(z) = 0$, which does not satisfy (517), and leads to a trivial density, $\rho^{(0)}(y) = 0$), and a, b are ordered accordingly as

$$b > a > 0, \quad \text{for} \quad \mu > 0, \quad (511)$$

$$b < a < 0, \quad \text{for} \quad \mu < 0. \quad (512)$$

As we see, our solution describes the rescaled spectral parameters condensing in a single interval which is

$$\mathcal{C}^y = [a, b], \quad \text{for} \quad \mu > 0, \quad (513)$$

$$\mathcal{C}^y = [b, a], \quad \text{for} \quad \mu < 0. \quad (514)$$

- For its argument approaching complex infinity, the resolvent has the following expansion,

$$G^{(0)}(z) = \frac{\mu(1+s)}{2} - \frac{1+s(1+2\alpha)}{2} \frac{1}{z} + O\left(\frac{1}{z^2}\right), \quad \text{for} \quad z \rightarrow \infty, \quad (515)$$

hence, in order to have a correct $1/z$ behavior, there must be

$$s = -1, \quad (516)$$

and then

$$G^{(0)}(z) = \frac{\alpha}{z} + O\left(\frac{1}{z^2}\right). \quad (517)$$

Remark that it complies with the relation (313). It also implies that the large- z series of all subleading resolvents, *i.e.*, $G^{(1)}(z)$, $G^{(2)}(z)$, *etc.*, must start from terms of order $O(1/z^2)$.

- Taking the limit $z \rightarrow 0$ of the resolvent with $s = -1$, we reproduce the momentum condition (464). Notice that if there was $s = +1$, the limit would be divergent.
- The first derivative w.r.t. z of the resolvent reads

$$G^{(0)'}(z) = \frac{1}{2z^2} + \frac{1 - \mu(1 + 2\alpha)z}{2\mu z^2 \sqrt{z - a} \sqrt{z - b}}. \quad (518)$$

Its value for z approaching zero,

$$G^{(0)'}(0) = -\mu^2 \alpha (1 + \alpha). \quad (519)$$

- The second derivative w.r.t. z of the resolvent reads

$$G^{(0)''}(z) = -\frac{1}{z^3} + \frac{-1 + 3\mu(1 + 2\alpha)z - 3\mu^2(1 + 2\alpha + 2\alpha^2)z^2 + \mu^3(1 + 2\alpha)z^3}{\mu^3 z^3 (z - a)^{3/2} (z - b)^{3/2}}. \quad (520)$$

Its value for z approaching zero,

$$G^{(0)''}(0) = -2\mu^3 \alpha (1 + \alpha)(1 + 2\alpha) = 2\mu(1 + 2\alpha)G^{(0)'}(0). \quad (521)$$

Notice that it is different from $G^{(0)'}(0)$ by just a simple multiplicative constant. A similar proportionality holds also for $G^{(1)}(z)$ (581), while for $G^{(2)}(z)$ there is just a “small” deviation from it (676).

- We claim that the value at zero of an arbitrary u -th derivative w.r.t. z of the resolvent is given by

$$G^{(0)''\dots t_u}(0) = -\mu^{u+1} u! \sum_{v=0}^u \frac{1}{v+1} \binom{u+v}{u} \binom{u}{v} \alpha^{v+1}, \quad \text{for } u = 0, 1, \dots \quad (522)$$

The above polynomial in α is of order $(u + 1)$, and its coefficients are the “triangle numbers” read by rows, Sloane’s A088617. Let us explicitly print this formula for $u = 3, 4, 5, 6$,

$$G^{(0)''''}(0) = -6\mu^4 \alpha (1 + \alpha) (1 + 5\alpha + 5\alpha^2), \quad (523)$$

$$G^{(0)'''''}(0) = -24\mu^5 \alpha (1 + \alpha)(1 + 2\alpha) (1 + 7\alpha + 7\alpha^2), \quad (524)$$

$$G^{(0)''''''}(0) = -120\mu^6 \alpha (1 + \alpha) (1 + 14\alpha + 56\alpha^2 + 84\alpha^3 + 42\alpha^4), \quad (525)$$

$$G^{(0)'''''''}(0) = -720\mu^7 \alpha (1 + \alpha)(1 + 2\alpha) (1 + 18\alpha + 84\alpha^2 + 132\alpha^3 + 66\alpha^4). \quad (526)$$

Remark a similar structure within pairs, $u = 3$ and $u = 4$, $u = 5$ and $u = 6$, *etc.*, but only for $u = 1$ and $u = 2$ it is a simple proportionality (521). These values yield the leading terms of the local conserved charges (339),

$$Q_t|_{\omega=0}|_{\text{leading}} = -\frac{G^{(0)''\dots t_{(t-1)}}(0)}{(t-1)!}, \quad \text{for } t = 1, 2, \dots, \quad (527)$$

which we plot in figure 16 for several values of t as functions of $\alpha \in [0, 1]$.

- The above results (508), (464), (519), (521) allow us to explicitly write the momentum condition at the level $O(1/J^2)$ (469),

$$G^{(2)}(0) = -\frac{1}{12}\mu^3 \alpha (1 + \alpha)(1 + 2\alpha), \quad (528)$$

as well as the quantity (468),

$$\mathcal{F}(z)|_{\omega=0} = \frac{-3 + 3\mu(1 + 2\alpha)z + 6\mu^2 \alpha (1 + \alpha)z^2 + 8\mu^3 \alpha (1 + \alpha)(1 + 2\alpha)z^3 - 3\mu\sqrt{z - a}\sqrt{z - b}}{24z^4}. \quad (529)$$

Moreover, the denominator in the expressions for $G^{(1)}(z)$ (465) and $G^{(2)}(z)$ (467) is simply

$$-2G^{(0)}(z) - \frac{1}{z} + \mu = \frac{\mu}{z} \sqrt{z - a} \sqrt{z - b}. \quad (530)$$

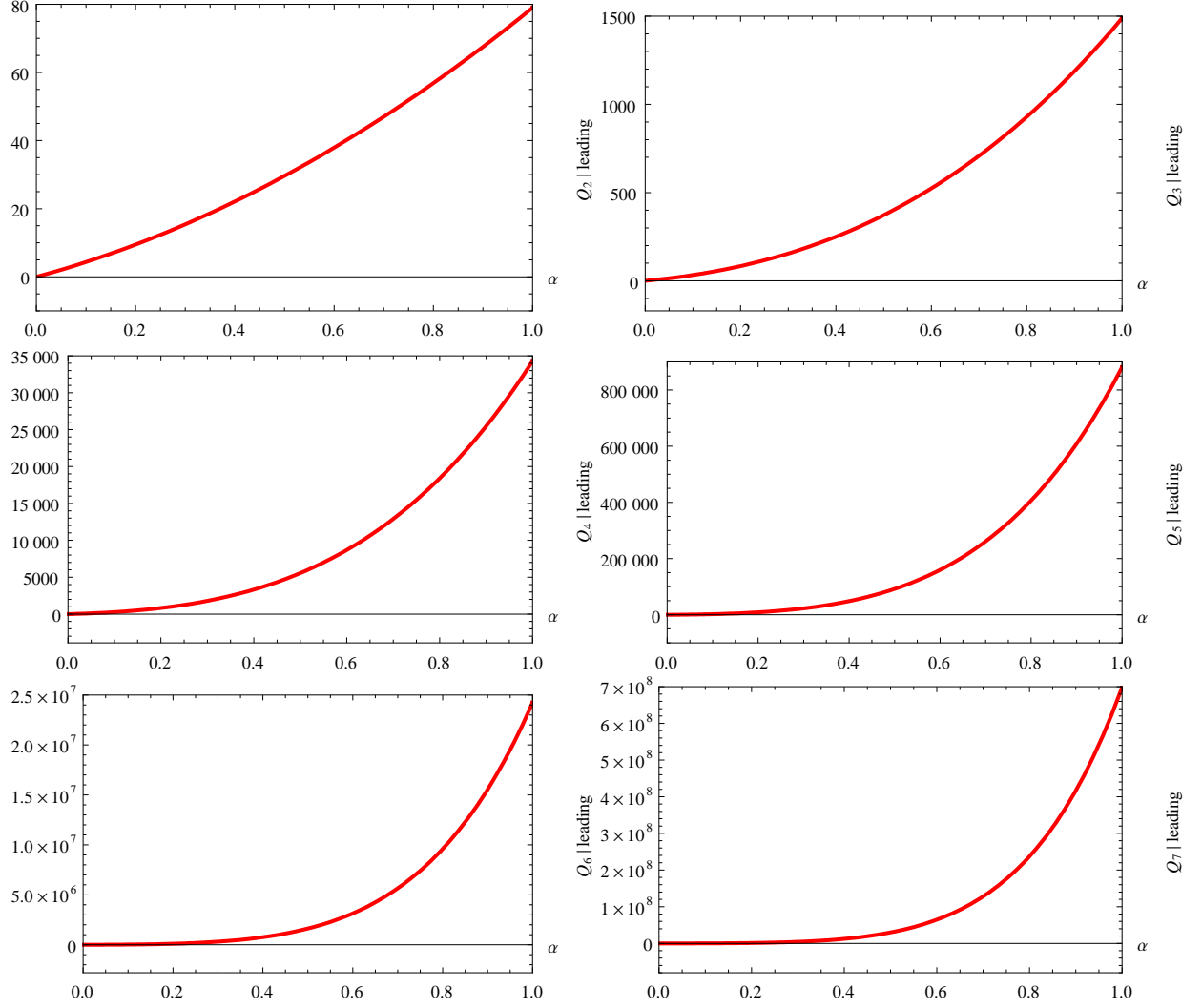


FIG. 16: Graphs of the leading terms of the local conserved charges (527), for $t = 2, 3, 4, 5, 6, 7$, according to (519), (521), (523)–(526), as functions of $\alpha \in [0, 1]$, for $m = 1$.

- A simple investigation of the root structure of $P(z)$, for both signs of μ , reveals that

$$\lim_{\epsilon \rightarrow 0^+} \sqrt{z-a} \sqrt{z-b} \Big|_{z=y \pm i\epsilon} = \pm i \sqrt{(y-a)(b-y)}, \quad \text{for } y \in \mathcal{C}^y. \quad (531)$$

Thanks to it, we can derive the linear equation obeyed by the resolvent,

$$\lim_{\epsilon \rightarrow 0^+} \left(G^{(0)}(y + i\epsilon) + G^{(0)}(y - i\epsilon) \right) = \mu - \frac{1}{y}, \quad \text{for } y \in \mathcal{C}^y, \quad (532)$$

which precisely coincides with (482).

2. The Density $\rho^{(0)}(y)$

The above large- J leading-order resolvent $G^{(0)}(z)$ (508) can now be used to find the large- J leading-order density according to (323) and with help of (531),

$$\rho^{(0)}(y) = \frac{\mu}{2\pi} \frac{\sqrt{(y-a)(b-y)}}{y}, \quad \text{for } y \in \mathcal{C}^y. \quad (533)$$

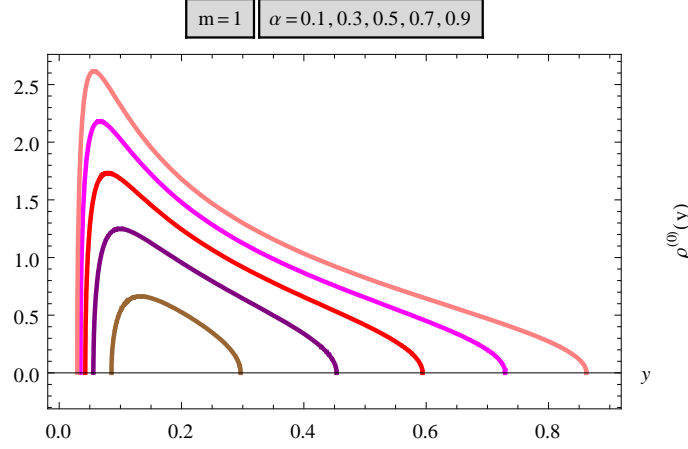


FIG. 17: The density $\mathcal{C}^y \ni y \mapsto \rho^{(0)}(y)$ (533) for $m = 1$ and five values of α : 0.1 (brown), 0.3 (purple), 0.5 (red), 0.7 (magenta), 0.9 (pink).

Notice that it is real and positive-definite, as required. Figure 17 shows some plots of this density.

Let us check the first part of the formula (324). Here, the density vanishes at the endpoints of the cut, and therefore, we do not expect any boundary terms. Using the following general results, easily obtained by the method of residues,

$$\int_{\mathcal{C}^y} dy \sqrt{(y-a)(b-y)} \frac{1}{z-y} = \pi \left(-\frac{a+b}{2} + z - \sqrt{z-a}\sqrt{z-b} \right), \quad \text{for any } z \in \mathbb{C}, \quad (534)$$

$$\int_{\mathcal{C}^y} dy \sqrt{(y-a)(b-y)} \frac{1}{y} = \pi \left(\frac{a+b}{2} - \sqrt{a}\sqrt{b} \right) \quad (535)$$

(note that “ $\int_{\mathcal{C}^y}$ ” means “ \int_a^b ” for $\mu > 0$ and “ \int_b^a ” for $\mu < 0$, as understood from (513), (514)), we indeed find

$$G^{(0),\text{bulk}}(z) \equiv \int_{\mathcal{C}^y} dy \rho^{(0)}(y) \frac{1}{z-y} = G^{(0)}(z). \quad (536)$$

Let us also examine the two normalization formulae (327), (328), which at one loop coincide. The latter one follows directly from (536) and (517), and reads

$$\int_{\mathcal{C}^y} dy \rho^{(0)}(y) = \alpha. \quad (537)$$

3. The Large-Mode-Number, Fixed-Winding-Number Limit

In this paragraph, let us introduce a certain new limit [163, 185], which — even though at one loop not necessary to do calculations, as everything can be found exactly — has been considered as a means to access a non-perturbative regime of the more complicated all-loop case. It will be very helpful to practice its subtleties, advantages and shortcomings in the simpler one-loop setting.

Namely, we will take the mode number μ to be large and the filling fraction α to be small, in such a way that their product, *i.e.*, the winding number ϖ , remains constant,

$$\mu \rightarrow \infty, \quad \alpha \rightarrow 0, \quad \mu\alpha = \varpi = \text{fixed}. \quad (538)$$

We will implement this limit by replacing all the instances of μ by ϖ/α , and taking α to zero.

Let us expand the density $\rho^{(0)}(y)$ in this limit. In trying to do this, we encounter a conceptual problem that the endpoints a, b of the interval \mathcal{C}^y in which the argument y lives — depend on the expansion parameter α . We

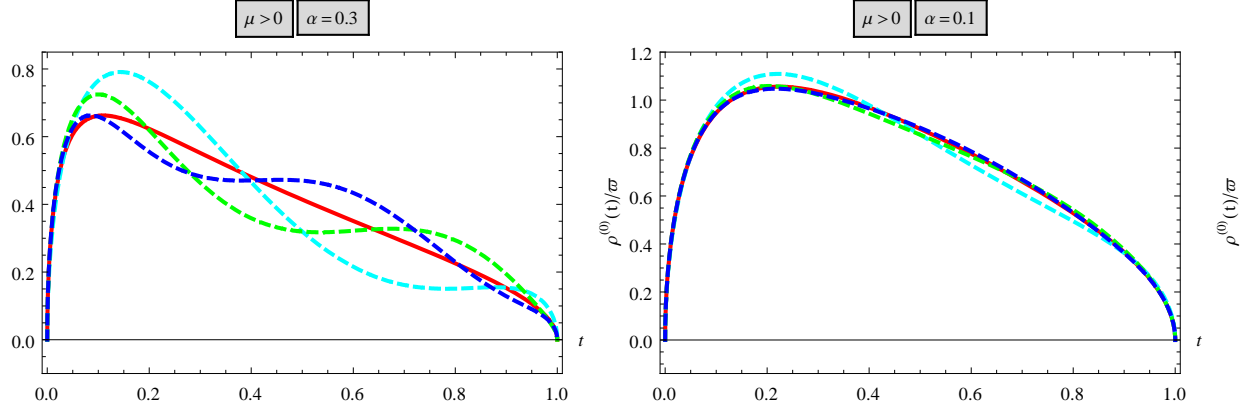


FIG. 18: The exact density divided by ϖ , $[0, 1] \ni t \mapsto \rho^{(0)}(t)/\varpi$ (540) (solid red line), compared to various truncations of its small- α approximation (541): three (dashed cyan), four (dashed green), five (dashed blue) leading orders kept. We assume $\mu > 0$, and set $\alpha = 0.3$ (LEFT), $\alpha = 0.1$ (RIGHT).

circumvent this obstacle through the following idea: The α -dependence of the range of the variable y can be removed by changing y into a new variable t (the “natural parameter” of \mathcal{C}^y) such that

$$\left. \begin{array}{l} y = a + (b - a)t \text{ for } \mu > 0, \\ y = b + (a - b)t \text{ for } \mu < 0 \end{array} \right\}, \quad \text{where} \quad t \in [0, 1], \quad \text{i.e., conversely,} \quad \begin{cases} t = \frac{y-a}{b-a}, \\ t = \frac{y-b}{a-b} \end{cases}. \quad (539)$$

We also assume that t is not correlated with α in any way. This will, for example, be an important clue in the way we will compute residues in paragraph VI A 6.

The density in this new variable reads

$$\rho^{(0)}(t) \equiv \rho^{(0)}(y) \Big|_{y \rightarrow t} = \begin{cases} \frac{2\sqrt{\alpha(1+\alpha)}}{\pi} \frac{1}{a+(b-a)t} \sqrt{t(1-t)} & \text{for } \mu > 0, \\ \frac{2\sqrt{\alpha(1+\alpha)}}{\pi} \frac{1}{b+(a-b)t} \sqrt{t(1-t)} & \text{for } \mu < 0 \end{cases}, \quad (540)$$

and its small- α series is found simply by substituting here the expansions of a , b ,

$$\begin{aligned} \rho^{(0)}(t) = \frac{|\varpi|}{\pi} \left(\frac{2}{\sqrt{\alpha}} \pm 4(1-2t) + \sqrt{\alpha} (5 - 32t + 32t^2) \pm 4\alpha(1-2t) (1 - 16t + 16t^2) + \right. \\ \left. + \frac{1}{4}\alpha^{3/2} (7 - 448t + 2496t^2 - 4096t^3 + 2048t^4) \mp 128\alpha^2 t(1-t)(1-2t) (1 - 8t + 8t^2) + \mathcal{O}(\alpha^{5/2}) \right) \sqrt{t(1-t)}, \end{aligned} \quad (541)$$

where the upper/lower signs correspond to $\mu \gtrless 0$, respectively. The expansion parameter is $\sqrt{\alpha}$, and not α , and it will always be w.r.t. it that we will count orders. Notice that the leading term is of order $\mathcal{O}(1/\sqrt{\alpha})$, i.e., this density explodes in the limit in question, which will be of major importance in paragraph III B 5. In figure 18, we compare this expansion to the exact result.

It is obvious how the leading-order charges (519), (521), (523)–(526) behave in this limit.

Even though it will not attract much of our attention, let us also comment on how to treat functions of a complex argument z in the limit (538). To do this, we change z by mimicking the transformation from y to t (539),

$$\left. \begin{array}{l} z = a + (b - a)Z \text{ for } \mu > 0, \\ z = b + (a - b)Z \text{ for } \mu < 0 \end{array} \right\}, \quad \text{i.e., conversely,} \quad \begin{cases} Z = \frac{z-a}{b-a}, \\ Z = \frac{z-b}{a-b} \end{cases}. \quad (542)$$

Notice that Z can be of any small- α order, unlike t , which is of order $\mathcal{O}(\alpha^0)$.

For example, let us assume that the values of Z are of order $\mathcal{O}(\alpha^0)$, too, and expand the resolvent $G^{(0)}(z)$ (508),

$$G^{(0)}(Z) \equiv G^{(0)}(z) \Big|_{z=a+(b-a)Z} = \frac{1}{\sqrt{\alpha}} \varpi \left(\sqrt{Z} - \sqrt{Z-1} \right)^2 - \varpi \left(\sqrt{Z} - \sqrt{Z-1} \right)^4 + \mathcal{O}(\sqrt{\alpha}). \quad (543)$$

(We show just two leading terms as an example. We adapt $\mu > 0$.) Now, for instance, let us try to reproduce the value of the second charge, *i.e.*, minus the first derivative of the resolvent at $z = 0$ (519). To this end, first, we need to change the derivative w.r.t. z to the one w.r.t. Z ,

$$\left. \begin{aligned} \frac{d}{dz} &= \frac{1}{b-a} \frac{d}{dZ} \quad \text{for } \mu > 0, \\ \frac{d}{dz} &= \frac{1}{a-b} \frac{d}{dZ} \quad \text{for } \mu < 0 \end{aligned} \right\}. \quad (544)$$

Second, observe that Z corresponding to $z = 0$ starts depending on α , and so, we must further expand it,

$$Z|_{z=0} = \mp \frac{1}{4} \frac{1}{\sqrt{\alpha}} + \frac{1}{2} \mp \frac{3}{8} \sqrt{\alpha} \pm \frac{5}{32} \alpha^{3/2} + O(\alpha^{5/2}), \quad (545)$$

where the upper/lower signs correspond to $\mu \gtrless 0$, respectively. This means that each term of the expansion of $G^{(0)'}(z)$, which looks like (543), should be again expanded after the substitution (545). Notice that this may cause the following problem: each term of the original series may contribute to each term (or just some given terms) of the resulting series, thus making it meaningless. We will show an example of such a situation at the end of paragraph IV C 3. But in the case of $G^{(0)'}(0)$, this does not happen. Taking, say, five leading terms of $G^{(0)'}(z)$, and substituting there three leading terms of $Z|_{z=0}$, we get

$$Q_2|_{\omega=0}|_{\text{leading}} = -\frac{1}{b-a} \frac{d}{dZ} G^{(0)}(Z) \Big|_{Z=Z|_{z=0}} = \frac{1}{\alpha} \varpi^2 + \varpi^2 + O(\sqrt{\alpha}), \quad (546)$$

which agrees with the exact expression (519) up to the given order. Remark that the leading-order second charge is, in the large-mode-number, fixed-winding-number limit (538), large as $1/\alpha$.

B. The Exact Solution to the One-Loop One-Cut Quadratic Equation at the Order $O(1/J)$

1. The Anomaly $\mathcal{A}^{(1)}(z)$

The next-to-leading-order resolvent $G^{(1)}(z)$ is given in (465), but what is lacking to fully know it, is a derivation of the anomaly $\mathcal{A}^{(1)}(z)$.

As we have described in paragraph II C 5, there are two ways to do it. The first one is to consider the sum (430), and use the procedure (331) to write it through the known resolvent $G^{(0)}(z)$. The second method is to compute the “bulk part” of the anomaly, $\mathcal{A}^{(1),\text{bulk}}(z)$ (434), which requires doing an integration with the known density $\rho^{(0)}(y)$, and afterwards, examine the “boundary part” separately. At this order, however, we expect no boundary contribution, analogously as in (536), which we prove below (562). In this paragraph, we present both ways.

Let us start from exploiting the first technique. Recall the definition of the pertinent anomaly term (430),

$$\frac{1}{J}\mathcal{A}^{(1)}(z) + \frac{1}{J^2}\mathcal{A}^{(2,1)}(z) + O\left(\frac{1}{J^3}\right) = -\frac{1}{J^2} \sum_{k=1}^S \frac{1}{z-y_k} \pi \rho^{(0)'}(y_k) \left(\coth\left(\pi \rho^{(0)}(y_k)\right) - \frac{1}{\pi \rho^{(0)}(y_k)} \right). \quad (547)$$

First, we need to explicitly recast the summand as a rational function of y_k . To do so, we notice that the bracket with the hyperbolic cotangent can be written as an infinite sum,

$$\coth\left(\pi \rho^{(0)}(y_k)\right) - \frac{1}{\pi \rho^{(0)}(y_k)} = \frac{2}{\pi} \rho^{(0)}(y_k) \sum_{n \geq 1} \frac{1}{\rho^{(0)}(y_k)^2 + n^2}. \quad (548)$$

Using our solution for the density $\rho^{(0)}(y)$ (533), we thus get

$$\frac{1}{J}\mathcal{A}^{(1)}(z) + \frac{1}{J^2}\mathcal{A}^{(2,1)}(z) + O\left(\frac{1}{J^3}\right) = \frac{2}{J^2} \sum_{n \geq 1} \sum_{k=1}^S \frac{1}{z-y_k} \frac{1}{y_k} \frac{1 - \mu(1+2\alpha)y_k}{((\mu^2 - \nu^2)y_k^2 - 2\mu(1+2\alpha)y_k + 1)}, \quad (549)$$

where

$$\nu \equiv 2\pi n. \quad (550)$$

Second, the resulting rational function should be decomposed into partial fractions w.r.t. y_k . We see that in order to do so, we have to solve w.r.t. y and for any integer $n \geq 1$ the quadratic equation,

$$\rho^{(0)}(y)^2 + n^2 = 0, \quad i.e., \quad (\mu^2 - \nu^2)y^2 - 2\mu(1+2\alpha)y + 1 = 0, \quad (551)$$

whose two solutions are

$$a_n, b_n = \frac{\mu(1+2\alpha) \mp \sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}}{\mu^2 - \nu^2}. \quad (552)$$

(Notice $a_0 = a$, $b_0 = b$.) This leads to

$$\begin{aligned} & \frac{1}{J}\mathcal{A}^{(1)}(z) + \frac{1}{J^2}\mathcal{A}^{(2,1)}(z) + O\left(\frac{1}{J^3}\right) = \\ &= \frac{1}{J^2} \sum_{n \geq 1} \sum_{k=1}^S \left(\frac{2a_nb_n - (a_n + b_n)z}{z(z-a_n)(z-b_n)} \frac{1}{z-y_k} + \frac{1}{z-a_n} \frac{1}{a_n - y_k} + \frac{1}{z-b_n} \frac{1}{b_n - y_k} + \frac{2}{z} \frac{1}{y_k} \right) = \\ &= \frac{1}{J} \sum_{n \geq 1} \left(\frac{2a_nb_n - (a_n + b_n)z}{z(z-a_n)(z-b_n)} G(z) + \frac{1}{z-a_n} G(a_n) + \frac{1}{z-b_n} G(b_n) - \frac{2}{z} G(0) \right). \end{aligned} \quad (553)$$

Third, it remains to expand the resolvent at large J (322) to find

$$\mathcal{A}^{(1)}(z) = \sum_{n \geq 1} \left(\frac{2a_n b_n - (a_n + b_n)z}{z(z - a_n)(z - b_n)} G^{(0)}(z) + \frac{1}{z - a_n} G^{(0)}(a_n) + \frac{1}{z - b_n} G^{(0)}(b_n) - \frac{2}{z} G^{(0)}(0) \right), \quad (554)$$

$$\mathcal{A}^{(2,1)}(z) = \sum_{n \geq 1} \left(\frac{2a_n b_n - (a_n + b_n)z}{z(z - a_n)(z - b_n)} G^{(1)}(z) + \frac{1}{z - a_n} G^{(1)}(a_n) + \frac{1}{z - b_n} G^{(1)}(b_n) - \frac{2}{z} G^{(1)}(0) \right). \quad (555)$$

At this point, we must abandon dealing with $\mathcal{A}^{(2,1)}(z)$, as we do not yet know the next-to-leading-order resolvent $G^{(1)}(z)$.

Fourth, we need to calculate the values $G^{(0)}(a_n)$ and $G^{(0)}(b_n)$. Investigating the relative positions of the roots a , b and the roots a_n , b_n , we find concise expressions for the following square roots,

$$\left. \frac{\sqrt{a_n - a} \sqrt{a_n - b}}{\sqrt{b_n - a} \sqrt{b_n - b}} \right\} = \frac{\nu \left(\mp \mu(1 + 2\alpha) + \sqrt{4\mu^2 \alpha(1 + \alpha) + \nu^2} \right)}{\mu(\mu^2 - \nu^2)}, \quad (556)$$

which is true for any μ and any ν . This yields

$$G^{(0)}(a_n, b_n) = -\mu\alpha \pm \frac{\nu - \sqrt{4\mu^2 \alpha(1 + \alpha) + \nu^2}}{2}. \quad (557)$$

Rewriting moreover the first term inside the brackets in (554) with help of (552), and using the momentum condition (464), we arrive at the desired final result,

$$\mathcal{A}^{(1)}(z) = -\frac{1}{z^2} \sum_{n \geq 1} \left(1 + \frac{\nu \sqrt{4\mu^2 \alpha(1 + \alpha) + \nu^2} z^2 + \mu(1 - \mu(1 + 2\alpha)z) \sqrt{z - a} \sqrt{z - b}}{(\mu^2 - \nu^2) z^2 - 2\mu(1 + 2\alpha)z + 1} \right). \quad (558)$$

Remark that the sum in the piece proportional to $\sqrt{z - a} \sqrt{z - b}$ can be explicitly done, as it is of the form (548),

$$\mathcal{A}^{(1)}(z) = - \sum_{n \geq 1} \left(\frac{1}{z^2} + \frac{\nu \sqrt{4\mu^2 \alpha(1 + \alpha) + \nu^2}}{(\mu^2 - \nu^2) z^2 - 2\mu(1 + 2\alpha)z + 1} \right) + \pi \rho^{(0)'}(z) \left(\pi \rho^{(0)}(z) \cot \left(\pi \rho^{(0)}(z) \right) - 1 \right), \quad (559)$$

where for short, $\rho^{(0)}(z) \equiv \mu/(2\pi)(\sqrt{z - a} \sqrt{z - b})/z$. We have not been able to decide whether the remaining sum can be cast in an explicit form.

The same expression can be found by doing the integration in the “bulk part” of the anomaly, $\mathcal{A}^{(1),\text{bulk}}(z)$ (434),

$$\begin{aligned} \mathcal{A}^{(1),\text{bulk}}(z) &= -\pi \int_{\mathcal{C}_y} dy \frac{1}{z - y} \rho^{(0)}(y) \rho^{(0)'}(y) \left(\coth \left(\pi \rho^{(0)}(y) \right) - \frac{1}{\pi \rho^{(0)}(y)} \right) = \\ &= -2 \sum_{n \geq 1} \int_{\mathcal{C}_y} dy \frac{1}{z - y} \frac{\rho^{(0)}(y)^2 \rho^{(0)'}(y)}{\rho^{(0)}(y)^2 + n^2} = \\ &= \frac{\mu}{\pi} \sum_{n \geq 1} \int_{\mathcal{C}_y} dy \sqrt{(y - a)(b - y)} \left(\frac{1 + \mu(1 + 2\alpha)z}{z^2} \frac{1}{y} + \frac{1}{z} \frac{1}{y^2} + \frac{1 - \mu(1 + 2\alpha)a_n}{(\mu^2 - \nu^2) y^{(1)2} (a_n - b_n) (z - a_n)} \frac{1}{y - a_n} - \right. \\ &\quad \left. - \frac{1 - \mu(1 + 2\alpha)b_n}{(\mu^2 - \nu^2) y^{(2)2} (a_n - b_n) (z - b_n)} \frac{1}{y - b_n} - \frac{1 - \mu(1 + 2\alpha)z}{(\mu^2 - \nu^2) z^2 (z - a_n) (z - b_n)} \frac{1}{y - z} \right). \end{aligned} \quad (560)$$

These integrals are of the form (534), (535) and

$$\int_{\mathcal{C}_y} dy \sqrt{(y - a)(b - y)} \frac{1}{y^2} = \pi \left(\frac{a + b}{2\sqrt{a}\sqrt{b}} - 1 \right), \quad (561)$$

which finally yields

$$\mathcal{A}^{(1),\text{bulk}}(z) = \mathcal{A}^{(1)}(z), \quad (562)$$

showing that indeed, at this level there is no boundary contribution to the anomaly.

2. The Resolvent $G^{(1)}(z)$

Substituting the expressions for $G^{(0)'}(z)$ (518), the anomaly $\mathcal{A}^{(1)}(z)$ (558), and the common denominator $\frac{\mu}{z}\sqrt{z-a}\sqrt{z-b}$ (530) into (465) — we find the large- J next-to-leading-order resolvent,

$$G^{(1)}(z) = -\frac{1}{4} \left(\frac{1}{z-a} + \frac{1}{z-b} \right) + \frac{1}{2z} \left(1 + \frac{1}{\mu\sqrt{z-a}\sqrt{z-b}} \right) + \sum_{n \geq 1} \left(-\frac{1}{2} \left(\frac{1}{z-a_n} + \frac{1}{z-b_n} \right) + \frac{1}{z} + \frac{1}{\mu z \sqrt{z-a}\sqrt{z-b}} - \frac{\nu}{2\mu} \frac{1}{\sqrt{z-a}\sqrt{z-b}} \left(\frac{a_n}{z-a_n} - \frac{b_n}{z-b_n} \right) \right), \quad (563)$$

or more explicitly, using the values of a_n and b_n (552),

$$G^{(1)}(z) = -\frac{1}{4} \left(\frac{1}{z-a} + \frac{1}{z-b} \right) + \frac{1}{2z} \left(1 + \frac{1}{\mu\sqrt{z-a}\sqrt{z-b}} \right) + \sum_{n \geq 1} \left(\frac{\frac{1}{z} - \mu(1+2\alpha)}{(\mu^2 - \nu^2)z^2 - 2\mu(1+2\alpha)z + 1} + \frac{1}{\mu z \sqrt{z-a}\sqrt{z-b}} + \frac{\nu\sqrt{4\mu^2\alpha(1+\alpha)} + \nu^2 z}{\mu((\mu^2 - \nu^2)z^2 - 2\mu(1+2\alpha)z + 1)\sqrt{z-a}\sqrt{z-b}} \right). \quad (564)$$

Notice that this expression consists of a piece without the sum over n (which we will dub the “no-sum part”), and the sum over n (whose summand we will call the “one-sum part”). Remark that the no-sum part is precisely equal to half the one-sum part with $n = 0$. Also, the first term of the one-sum part could be explicitly summed up, analogously as in (559).

The basic features of this solution:

- The behavior at infinity,

$$G^{(1)}(z) = \mathcal{O}\left(\frac{1}{z^2}\right), \quad \text{for } z \rightarrow \infty, \quad (565)$$

i.e., it does not spoil the correct $1/z$ term of the large- z series of $G^{(0)}(z)$ (517), as anticipated.

- The limit $z \rightarrow 0$ of the resolvent gives the momentum condition (466).
- The first derivative w.r.t. z of the resolvent,

$$G^{(1)'}(z) = \frac{1}{2\mu^4 z^2 (z-a)^2 (z-b)^2} \left(-1 + 4\mu(1+2\alpha)z - \mu^2(5+8\alpha+8\alpha^2)z^2 + 2\mu^3(1+2\alpha)z^3 - \right. \\ \left. + \mu(-1 + 3\mu(1+2\alpha)z - 2\mu^2 z^2) \sqrt{z-a}\sqrt{z-b} \right) + \\ + \sum_{n \geq 1} \left(\frac{-1 + 3\mu(1+2\alpha)z - 2\mu^2 z^2}{\mu^3 z^2 (z-a)^{3/2} (z-b)^{3/2}} + \right. \\ \left. + \nu\sqrt{4\mu^2\alpha(1+\alpha)} + \nu^2 \right) \\ + \frac{1 - \mu(1+2\alpha)z - (\mu^2(3+8\alpha+8\alpha^2) - \nu^2)z^2 + \mu(1+2\alpha)(5\mu^2 - 3\nu^2)z^3 - 2\mu^2(\mu^2 - \nu^2)z^4}{\mu^3(z-a)^{3/2}(z-b)^{3/2}((\mu^2 - \nu^2)z^2 - 2\mu(1+2\alpha)z + 1)^2} +$$

$$+ \frac{-1 + 4\mu(1 + 2\alpha)z - (\mu^2(5 + 8\alpha + 8\alpha^2) - 3\nu^2)z^2 + 2\mu(1 + 2\alpha)(\mu^2 - \nu^2)z^3}{z^2((\mu^2 - \nu^2)z^2 - 2\mu(1 + 2\alpha)z + 1)^2} \Bigg). \quad (566)$$

In particular, for z tending to zero,

$$G^{(1)'}(0) = \mu^2\alpha(1 + \alpha) + \frac{1}{2} \sum_{n \geq 1} \left(\nu - \sqrt{4\mu^2\alpha(1 + \alpha) + \nu^2} \right)^2. \quad (567)$$

- The resolvent $G^{(2)}(z)$ (467) contains the following quantity,

$$\begin{aligned} G^{(1)}(z)^2 + G^{(1)'}(z) &= \frac{1 + 2\alpha - \mu(2 + 3\alpha + 3\alpha^2)z + \mu^2(1 + 2\alpha)z^2 + \sqrt{z-a}\sqrt{z-b}\mu(1 + 2\alpha - \mu z)}{\mu^3 z(z-a)^2(z-b)^2} + \\ &+ \sum_{n \geq 1} \left(\frac{1}{\mu^3 z^2(z-a)^{3/2}(z-b)^{3/2}((\mu^2 - \nu^2)z^2 - 2\mu(1 + 2\alpha)z + 1)} \right) \cdot \\ &\cdot \left(1 - \mu(1 + 2\alpha)z - \mu^2(3 + 8\alpha + 8\alpha^2)z^2 + \mu(1 + 2\alpha)(5\mu^2 - 2\nu^2)z^3 - 2\mu^2(\mu^2 - \nu^2)z^4 \right) + \\ &+ \frac{2\nu\sqrt{4\mu^2\alpha(1 + \alpha) + \nu^2}}{\mu^3(z-a)^{3/2}(z-b)^{3/2}((\mu^2 - \nu^2)z^2 - 2\mu(1 + 2\alpha)z + 1)^2} \cdot \\ &\cdot \left(1 - 2\mu(1 + 2\alpha)z + \mu(1 + 2\alpha)(2\mu^2 - \nu^2)z^3 - \mu^2(\mu^2 - \nu^2)z^4 \right) + \\ &+ \frac{1}{\mu^2 z^2(z-a)(z-b)((\mu^2 - \nu^2)z^2 - 2\mu(1 + 2\alpha)z + 1)^2} \cdot \\ &\cdot \left(1 - 2\mu(1 + 2\alpha)z - 2\mu^2(1 + 2\alpha + 2\alpha^2)z^2 + 2\mu(1 + 2\alpha)(4\mu^2(1 + \alpha + \alpha^2) - \nu^2)z^3 + \right. \\ &+ \left. (-\mu^4(7 + 20\alpha + 20\alpha^2) + 4\mu^2\nu^2(1 + 3\alpha + 3\alpha^2) + \nu^4)z^4 + 2\mu^3(1 + 2\alpha)(\mu^2 - \nu^2)z^5 \right) + \\ &+ \frac{\nu\sqrt{4\mu^2\alpha(1 + \alpha) + \nu^2}}{\mu^2(z-a)(z-b)((\mu^2 - \nu^2)z^2 - 2\mu(1 + 2\alpha)z + 1)} \Bigg) + \\ &+ \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \frac{1}{((\mu^2 - \nu_1^2)z^2 - 2\mu(1 + 2\alpha)z + 1)((\mu^2 - \nu_2^2)z^2 - 2\mu(1 + 2\alpha)z + 1)} \cdot \\ &\cdot \left(\frac{1}{\mu^2 z^2(z-a)(z-b)} \left(2 - 8\mu(1 + 2\alpha)z + (12\mu^2(1 + 3\alpha + 3\alpha^2) - \nu_1^2 - \nu_2^2)z^2 - \right. \right. \\ &- 2\mu(1 + 2\alpha)(4\mu^2(1 + \alpha + \alpha^2) - \nu_1^2 - \nu_2^2)z^3 + (2\mu^4(1 + 2\alpha + 2\alpha^2) - \mu^2(\nu_1^2 + \nu_2^2) + \nu_1^2\nu_2^2)z^4 + \\ &\left. \left. + z^2\nu_1\sqrt{4\mu^2\alpha(1 + \alpha) + \nu_1^2}(1 - 2\mu(1 + 2\alpha)z + (\mu^2 - \nu_2^2)z^2) + \right. \right. \end{aligned}$$

$$\begin{aligned}
& + z^2 \nu_2 \sqrt{4\mu^2 \alpha(1+\alpha) + \nu_2^2} (1 - 2\mu(1+2\alpha)z + (\mu^2 - \nu_1^2) z^2) + \\
& + z^4 \nu_1 \nu_2 \sqrt{4\mu^2 \alpha(1+\alpha) + \nu_1^2} \sqrt{4\mu^2 \alpha(1+\alpha) + \nu_2^2} + \\
& + \frac{1}{\mu z^2 \sqrt{z-a} \sqrt{z-b}} (1 - \mu(1+2\alpha)z) \cdot \\
& \cdot \left(2 - 4\mu(1+2\alpha)z + (2\mu^2 - \nu_1^2 - \nu_2^2) z^2 + z^2 \left(\nu_1 \sqrt{4\mu^2 \alpha(1+\alpha) + \nu_1^2} + \nu_2 \sqrt{4\mu^2 \alpha(1+\alpha) + \nu_2^2} \right) \right). \quad (568)
\end{aligned}$$

- We can also find an explicit expression for the value at zero of an arbitrary u -th derivative w.r.t. z of this resolvent,

$$\begin{aligned}
G^{(1)''\dots' u}(0) &= \mu^{u+1} u! \left(\frac{1}{2} \sum_{v=0}^{u+1} \frac{(u+1+v)!}{(u+1-v)!} \left(\frac{2^{2v}(u+1)}{u+1+v} \frac{1}{(2v)!} - \frac{1}{v!^2} \right) \alpha^v + \right. \\
&+ \sum_{n \geq 1} \left(\sum_{v=0}^{u+1} \frac{(u+1+v)!}{(u+1-v)!} \left(\frac{2^{2v}(u+1)}{u+1+v} \frac{1}{(2v)!} - \frac{1}{v!^2} \right) \alpha^v + \right. \\
&+ \frac{1}{2} \sum_{v_1=1}^{\lfloor \frac{u+1}{2} \rfloor} \left(\frac{\nu}{\mu} \right)^{2v_1} \sum_{v_2=v_1}^{\lfloor \frac{u+1}{2} \rfloor} 2^{2(v_2-v_1)+1} \binom{u+1}{2v_2} \binom{v_2}{v_1} \alpha^{v_2-v_1} (1+\alpha)^{v_2-v_1} (1+2\alpha)^{u+1-2v_2-} \\
&- \frac{\nu}{2\mu^2} \sqrt{4\mu^2 \alpha(1+\alpha) + \nu^2} \sum_{v_1=0}^u \sum_{v_2=0}^{u-v_1} \sum_{v_3=0}^{\lfloor \frac{v_1-1}{2} \rfloor} \sum_{v_4=0}^{v_3} \cdot \\
&\cdot \left(\frac{\nu}{\mu} \right)^{2v_4} 2^{2(v_3-v_4)+1} \binom{v_1}{2v_3+1} \binom{v_3}{v_4} \binom{u-v_1+v_2}{v_2} \binom{u-v_1}{v_2} \alpha^{v_2+v_3-v_4} (1+\alpha)^{v_3-v_4} (1+2\alpha)^{v_1-2v_3-1} \Bigg). \quad (569)
\end{aligned}$$

Its structure is the following: The no-sum part is a polynomial in α . The one-sum part consists of three pieces: the first one is a polynomial in α ; the second one is a polynomial in ν/μ , whose coefficients are polynomials in α ; the third one is $\sqrt{4\mu^2 \alpha(1+\alpha) + \nu^2}/\nu$ multiplied by a polynomial in $(\nu/\mu)^2$, whose coefficients are polynomials in α . The latter two pieces are not cast explicitly as such polynomials, as we have found it too complicated, but we show how to do it in paragraph III B 3. A sketch of a proof of (569) is given in paragraph III B 3, where we also print this formula for $u = 0, 1, 2, 3, 4, 5, 6$. These quantities yield the next-to-leading-order terms of the local conserved charges (339),

$$Q_t|_{\omega=0}|_{\text{next-to-leading}} = -\frac{G^{(1)''\dots' (t-1)}(0)}{(t-1)!}, \quad \text{for } t = 1, 2, \dots, \quad (570)$$

which we plot in figure 19 for several values of t as functions of $\alpha \in [0, 1]$.

- A short calculation, with help of (533), (548), shows that this resolvent fulfills the linear equation (483),

$$\lim_{\epsilon \rightarrow 0^+} \left(G^{(1)}(y + i\epsilon) + G^{(1)}(y - i\epsilon) \right) = -\pi \rho^{(0)'}(y) \coth \left(\pi \rho^{(0)}(y) \right), \quad \text{for } y \in \mathcal{C}^y. \quad (571)$$

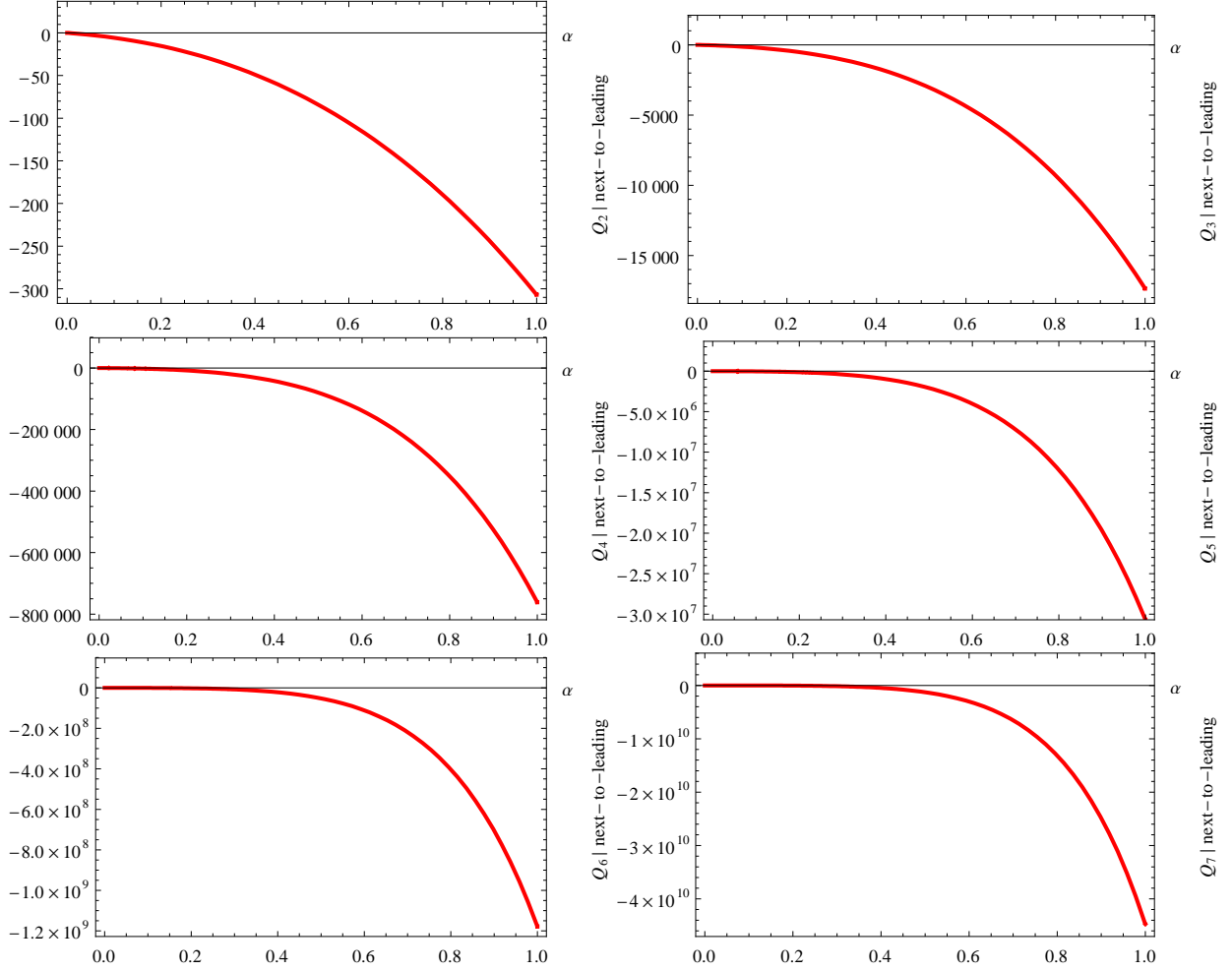


FIG. 19: Graphs of the next-to-leading-order terms of the local conserved charges (570), for $t = 2, 3, 4, 5, 6, 7$, according to (580)–(585), as functions of $\alpha \in [0, 1]$, for $m = 1$. The infinite sum over n has been terminated at $n_{\max} = 10^4$ for $t = 2, 3$, at $n_{\max} = 10^3$ for $t = 4, 5$, and at $n_{\max} = 10^2$ for $t = 6, 7$, which are values large enough to reproduce it correctly, while increasing them causes numerical problems in doing the summation.

3. The Resolvent $G^{(1)}(z)$. Appendix

In this appendix to the previous paragraph, let us comment on the derivation and structure of the result (569). To do so, consider the pertinent resolvent in the form (563).

The u -th derivative at zero of the first term of the no-sum part is easy,

$$-\frac{1}{4} \frac{d^u}{dz^u} \left(\frac{1}{z-a} + \frac{1}{z-b} \right) \Big|_{z=0} = \frac{u!}{4} \left(\frac{1}{a^{u+1}} + \frac{1}{b^{u+1}} \right). \quad (572)$$

We could now apply the Newton identity to $a, b = \frac{1}{\mu}(\sqrt{\alpha} \mp \sqrt{1+\alpha})^2$ (509) in order to express the r.h.s. of (572) through α , but there exists a more elegant way. Denoting $\cosh \vartheta \equiv i\sqrt{\alpha}$, $\sinh \vartheta \equiv i\sqrt{1+\alpha}$ allows us to use the de Moivre identity instead,

$$\frac{u!}{4} \left(\frac{1}{a^{u+1}} + \frac{1}{b^{u+1}} \right) = \frac{u!}{2} \mu^{u+1} (-1)^{u+1} \cosh(2(u+1)\vartheta). \quad (573)$$

It is known that trigonometric and hyperbolic functions with an argument which is an integer multiple of some variable can be effectively handled with aid of the Chebyshev polynomials of the first kind, $T_t(x)$, which have this defining

property that $T_t(\cosh \vartheta) = \cosh(t\vartheta)$, for t a positive integer. Now, the coefficients of $T_t(x)$ for even t are explicitly known (Sloane's A127674), which leads to

$$-\frac{1}{4} \frac{d^u}{dz^u} \left(\frac{1}{z-a} + \frac{1}{z-b} \right) \Big|_{z=0} = \mu^{u+1} (u+1)! \sum_{v=0}^{u+1} \frac{2^{2v-1}}{u+1+v} \binom{u+1+v}{2v} \alpha^v. \quad (574)$$

The u -th derivative at zero of the second term of the no-sum part is given by a sum with “triangle numbers” read by rows, Sloane's A063007,

$$\frac{d^u}{dz^u} \frac{1}{2z} \left(1 + \frac{1}{\mu \sqrt{z-a} \sqrt{z-b}} \right) \Big|_{z=0} = -\frac{1}{2} \mu^{u+1} u! \sum_{v=0}^{u+1} \binom{u+v+1}{v} \binom{u+1}{v} \alpha^v. \quad (575)$$

The above sum is the hypergeometric function ${}_2F_1(-1-u, 2+u; 1; -\alpha)$, but we will not need this fact. Simplifying the sum of (574) and (575) yields the first line of (569).

For the one-sum part, the second and third terms comprise exactly twice the expression (575). The first term looks akin to (572),

$$-\frac{1}{2} \frac{d^u}{dz^u} \left(\frac{1}{z-a_n} + \frac{1}{z-b_n} \right) \Big|_{z=0} = \frac{u!}{2} \left(\left(\frac{1}{a_n} \right)^{u+1} + \left(\frac{1}{b_n} \right)^{u+1} \right). \quad (576)$$

However, we have not found the method of Chebyshev polynomials useful in this case. Although a_n, b_n could be written as full squares, similarly to a, b , and the $\cosh \vartheta, \sinh \vartheta$ parametrization could be introduced, $\cosh \vartheta$ would then be quite involved, rendering the whole technique meaningless. On the other hand, we can now resort to the Newton identity, which quickly leads to the second and third lines of (569).

We see that in this way we obtain a polynomial of order $(u+1)$ in α , plus a polynomial of order $\lfloor (u+1)/2 \rfloor$ in $(\nu/\mu)^2$, whose coefficients are polynomials in α . We have tried to write these coefficients explicitly as such, which requires using the Newton identity to the brackets $(1+\alpha)^{v_2-v_1}$ and $(1+2\alpha)^{u+1-2v_2}$, and properly reorganizing the resulting four sums — but the outcome is somewhat lengthy,

$$\begin{aligned} & \sum_{v_2=v_1}^{\lfloor \frac{u+1}{2} \rfloor} 2^{2(v_2-v_1)+1} \binom{u+1}{2v_2} \binom{v_2}{v_1} \alpha^{v_2-v_1} (1+\alpha)^{v_2-v_1} (1+2\alpha)^{u+1-2v_2} = \\ & = \sum_{v_2=0}^{u+1-2v_1} \alpha^{v_2} \left(\sum_{v_3=v_1}^{\min(\lfloor \frac{u+1}{2} \rfloor, v_1+v_2)} \sum_{v_4=\max(0, v_1+v_2+v_3-u-1)}^{\min(v_3-v_1, v_1+v_2-v_3)} \right. \\ & \quad \left. \cdot 2^{-v_1+v_2+v_3-v_4+1} \binom{u+1}{2v_3} \binom{v_3}{v_1} \binom{v_3-v_1}{v_4} \binom{u+1-2v_3}{v_1+v_2-v_3-v_4} \right). \end{aligned} \quad (577)$$

It is a polynomial of order $(u+1-2v_1)$ in α , with integer coefficients given by the red brackets.

The fourth term of the one-sum part can be calculated similarly to (575),

$$\begin{aligned} & -\frac{\nu}{2\mu} \frac{d^u}{dz^u} \frac{1}{\sqrt{z-a} \sqrt{z-b}} \left(\frac{a_n}{z-a_n} - \frac{b_n}{z-b_n} \right) \Big|_{z=0} = \\ & = -\frac{\nu}{2} u! \sum_{v_1=0}^u \mu^{u-v_1} \left(\left(\frac{1}{a_n} \right)^{v_1} - \left(\frac{1}{b_n} \right)^{v_1} \right) \sum_{v_2=0}^{u-v_1} \binom{u-v_1+v_2}{v_2} \binom{u-v_1}{v_2} \alpha^{v_2}. \end{aligned} \quad (578)$$

Applying the Newton identity quickly yields the fourth and fifth lines of (569). Here also we could reorganize the sums in order to cast this expression as a polynomial in $(\nu/\mu)^2$, with coefficients being polynomials in α , analogously as in (577), but we find it too complicated.

Let us print the formula (569) for the seven lowest values of u in order to get more closely acquainted with it,

- $u = 0$,

$$G^{(1)}(0) = 0, \quad (579)$$

in accordance with the momentum condition (466).

- $u = 1$,

$$G^{(1)'}(0) = \mu^2 \alpha (1 + \alpha) + \frac{1}{2} \sum_{n \geq 1} \left(\nu - \sqrt{4\mu^2 \alpha (1 + \alpha) + \nu^2} \right)^2, \quad (580)$$

agreeing with (567). Notice that here the summand can be nicely reduced to a full square.

- $u = 2$,

$$G^{(1)''}(0) = 6\mu(1 + 2\alpha) \left(\mu^2 \alpha (1 + \alpha) + \frac{1}{2} \sum_{n \geq 1} \left(\nu - \sqrt{4\mu^2 \alpha (1 + \alpha) + \nu^2} \right)^2 \right) = 6\mu(1 + 2\alpha) G^{(1)'}(0). \quad (581)$$

Remark that it is different from $G^{(1)'}(0)$ by just a simple multiplicative constant; recall that there was a similar proportionality between $G^{(0)''}(0)$ and $G^{(0)'}(0)$ (521), and the constant was three times smaller than now. Between the members of other pairs of expressions, namely for $u = 3$ and $u = 4$, $u = 5$ and $u = 6$, *etc.*, we observe analogous structures, but more complex than this proportionality relation, just as we did for the large- J leading-order resolvent (523)–(526).

- $u = 3$,

$$\begin{aligned} G^{(1)'''}(0) = & 6\mu^4 \left(\alpha(1 + \alpha) (6 + 29\alpha + 29\alpha^2) + \right. \\ & + \sum_{n \geq 1} \left(2\alpha(1 + \alpha) (6 + 29\alpha + 29\alpha^2) + 2(3 + 16\alpha + 16\alpha^2) \frac{\nu^2}{\mu^2} + \frac{\nu^4}{\mu^4} - \right. \\ & \left. \left. - \frac{\nu}{\mu^2} \sqrt{4\mu^2 \alpha (1 + \alpha) + \nu^2} \left(6(1 + 5\alpha + 5\alpha^2) + \frac{\nu^2}{\mu^2} \right) \right) \right). \end{aligned} \quad (582)$$

Neither here, nor in any subsequent expression from this list, we have discovered any “full-square simplification” like in (580), (581).

- $u = 4$,

$$\begin{aligned} G^{(1)''''}(0) = & 120\mu^5(1 + 2\alpha) \left(\alpha(1 + \alpha) (2 + 13\alpha + 13\alpha^2) + \right. \\ & + \sum_{n \geq 1} \left(2\alpha(1 + \alpha) (2 + 13\alpha + 13\alpha^2) + 2(1 + 8\alpha + 8\alpha^2) \frac{\nu^2}{\mu^2} + \frac{\nu^4}{\mu^4} - \right. \\ & \left. \left. - \frac{\nu}{\mu^2} \sqrt{4\mu^2 \alpha (1 + \alpha) + \nu^2} \left(2(1 + 7\alpha + 7\alpha^2) + \frac{\nu^2}{\mu^2} \right) \right) \right). \end{aligned} \quad (583)$$

- $u = 5$,

$$G^{(1)'''''}(0) = 120\mu^6 \left(\alpha(1 + \alpha) (15 + 195\alpha + 757\alpha^2 + 1124\alpha^3 + 562\alpha^4) + \right.$$

$$\begin{aligned}
& + \sum_{n \geq 1} \left(2\alpha(1+\alpha) (15 + 195\alpha + 757\alpha^2 + 1124\alpha^3 + 562\alpha^4) \right. \\
& + 3 (5 + 80\alpha + 336\alpha^2 + 512\alpha^3 + 256\alpha^4) \frac{\nu^2}{\mu^2} + 3 (5 + 24\alpha + 24\alpha^2) \frac{\nu^4}{\mu^4} + \frac{\nu^6}{\mu^6} - \\
& \left. - \frac{\nu}{\mu^2} \sqrt{4\mu^2\alpha(1+\alpha) + \nu^2} \left(15 (1 + 14\alpha + 56\alpha^2 + 84\alpha^3 + 42\alpha^4) + 5 (3 + 14\alpha + 14\alpha^2) \frac{\nu^2}{\mu^2} + \frac{\nu^4}{\mu^4} \right) \right) \Bigg) \Bigg). \quad (584)
\end{aligned}$$

• $u = 6$,

$$\begin{aligned}
G^{(1)''''''}(0) &= 5040\mu^7(1+2\alpha) \left(\alpha(1+\alpha) (3 + 49\alpha + 219\alpha^2 + 340\alpha^3 + 170\alpha^4) + \right. \\
& + \sum_{n \geq 1} \left(2\alpha(1+\alpha) (3 + 49\alpha + 219\alpha^2 + 340\alpha^3 + 170\alpha^4) \right. \\
& + (3 + 64\alpha + 320\alpha^2 + 512\alpha^3 + 256\alpha^4) \frac{\nu^2}{\mu^2} + (5 + 32\alpha + 32\alpha^2) \frac{\nu^4}{\mu^4} + \frac{\nu^6}{\mu^6} - \\
& \left. - \frac{\nu}{\mu^2} \sqrt{4\mu^2\alpha(1+\alpha) + \nu^2} \left(3 (1 + 18\alpha + 84\alpha^2 + 132\alpha^3 + 66\alpha^4) + 5 (1 + 6\alpha + 6\alpha^2) \frac{\nu^2}{\mu^2} + \frac{\nu^4}{\mu^4} \right) \right) \Bigg) \Bigg). \quad (585)
\end{aligned}$$

4. The Density $\rho^{(1)}(y)$

The large- J next-to-leading-order density is quickly found from the above resolvent (563),

$$\rho^{(1)}(y) = \frac{1}{\pi\mu} \left(\frac{1}{2} + \sum_{n \geq 1} \left(1 - \frac{\nu y}{2} \left(\frac{a_n}{y - a_n} - \frac{b_n}{y - b_n} \right) \right) \right) \frac{1}{y\sqrt{(y-a)(b-y)}}, \quad \text{for } y \in \mathcal{C}^y. \quad (586)$$

or more explicitly (564),

$$\rho^{(1)}(y) = \frac{1}{\pi\mu} \left(\frac{1}{2} + \sum_{n \geq 1} \left(1 + \frac{\nu\sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}y^2}{(\mu^2 - \nu^2)y^2 - 2\mu(1+2\alpha)y + 1} \right) \right) \frac{1}{y\sqrt{(y-a)(b-y)}}, \quad \text{for } y \in \mathcal{C}^y. \quad (587)$$

Notice that it is real, but can acquire both positive and negative values.

Another important difference from $\rho^{(0)}(y)$ is its behavior close to the endpoints a, b of the cut, as it explodes there with the square-root singularities $1/\sqrt{(y-a)(b-y)}$. More precisely,

$$\rho^{(1)}(y) \sim \frac{1}{\pi} (\sqrt{\alpha} \pm \sqrt{1+\alpha})^2 \left(\frac{1}{2} + \sum_{n \geq 1} \left(1 - \frac{\sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}}{\nu} \right) \right) \frac{1}{\sqrt{(y-a)(b-y)}}, \quad \text{for } y \sim a, b. \quad (588)$$

We may ask about the sign of the above prefactor, *i.e.*, whether the density explodes to plus or minus infinity. We have not been able to determine any closed form of the infinite sum over n in (588), even though we suppose it may

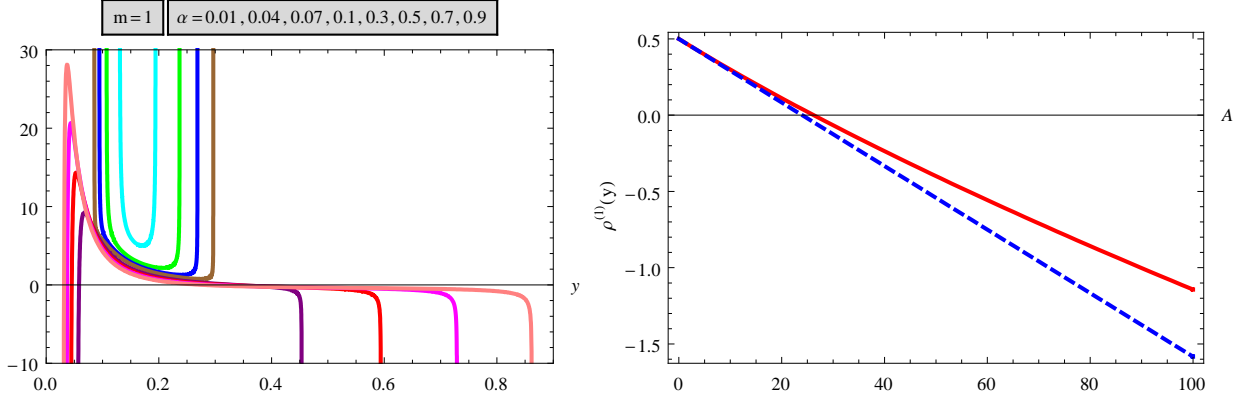


FIG. 20: LEFT: The density $\mathcal{C}^y \ni y \mapsto \rho^{(1)}(y)$ (586), (587) for $m = 1$ and eight values of α : 0.01 (cyan), 0.04 (green), 0.07 (blue), 0.1 (brown), 0.3 (purple), 0.5 (red), 0.7 (magenta), 0.9 (pink). The infinite sum over n has been terminated at $n_{\max} = 20$, which is enough to very accurately reproduce it. We confirm that for small α , it explodes at the endpoints to plus infinity, while for large α , to minus infinity.

RIGHT: The quantity on the l.h.s. of (589) as a function of $A \in [0, 100]$, where the infinite sum over n has been terminated at $n_{\max} = 10^4$ (solid red line). Its lower bound (the r.h.s. of (589)) is shown in dashed blue. It is positive for $A < A_c$ and negative for $A > A_c$, where $A_c \approx 24$, thus determining the endpoint behavior of the density $\rho^{(1)}(y)$.

be related to polylogarithm functions, but it is clear that it acquires only negative values. Adding $1/2$ to it may make it positive for small values of $A \equiv 4\mu^2\alpha(1 + \alpha)$, as there is

$$\frac{1}{2} + \sum_{n \geq 1} \left(1 - \frac{\sqrt{A + \nu^2}}{\nu} \right) \geq \frac{1}{2} - \frac{A}{48}. \quad (589)$$

(We have used $\sqrt{1 + A/\nu^2} \leq 1 + A/(2\nu^2)$ and $\sum_{n \geq 1} 1/\nu^2 = \zeta(2)/(4\pi^2) = 1/24$.) This lower bound is for sure positive when $A \leq 24$. In total, for $A < A_c$, where A_c is slightly larger than 24, the density $\rho^{(1)}(y)$ explodes to plus infinity at the endpoints, while for $A > A_c$ to minus infinity. It is all pictorially described in figure 20. Notice however that if we stick to the values of μ and α allowed by the relation (275), then we always are in the negative regime, as $A \geq 8\varpi^2 \geq 32\pi^2 \approx 315.8$. Remark finally that these singularities are integrable.

Let us again check the first part of the formula (324). Since the density in question explodes at the endpoints of the cut, we expect that the bulk resolvent will differ from the full resolvent. Indeed, using the simple general results,

$$\int_{\mathcal{C}^y} dy \frac{1}{\sqrt{(y-a)(b-y)}} \frac{1}{z-y} = \frac{\pi}{\sqrt{z-a}\sqrt{z-b}}, \quad \text{for any } z \in \mathbb{C}, \quad (590)$$

$$\int_{\mathcal{C}^y} dy \frac{1}{\sqrt{(y-a)(b-y)}} \frac{1}{y} = \frac{\pi}{\sqrt{a}\sqrt{b}}, \quad (591)$$

and performing partial fractional decomposition w.r.t. y of the rational part of the density, we find

$$G^{(1),\text{bulk}}(z) \equiv \int_{\mathcal{C}^y} dy \rho^{(1)}(y) \frac{1}{z-y} = G^{(1)}(z) + \frac{1}{4} \left(\frac{1}{z-a} + \frac{1}{z-b} \right), \quad (592)$$

i.e., the two resolvents differ by the boundary contribution

$$G^{(1),\text{boundary}}(z) \equiv G^{(1)}(z) - G^{(1),\text{bulk}}(z) = -\frac{1}{4} \left(\frac{1}{z-a} + \frac{1}{z-b} \right). \quad (593)$$

Observe that this is precisely the part of $G^{(1)}(z)$ with non-integrable singularities with integer powers, $1/(z-a)$, $1/(z-b)$; the rest of it is either regular or singular as $1/(\sqrt{z-a}\sqrt{z-b})$ at the endpoints of the cut,

$$G^{(1),\text{boundary}}(z) = G^{(1)}(z) \Big|_{\substack{\text{non-integrable} \\ \text{integer-power} \\ \text{singularities}}} \quad (594)$$

We will find the same prescription also in the case of $G^{(2)}(z)$ (686).

Computing bulk quantities (*i.e.*, via integrating with a density) is generally much simpler than computing their full counterparts (*i.e.*, via summing and expressing through the resolvent). At all loops, the latter is virtually impossible. Therefore, it is vital to find an alternative way of deriving boundary contributions. In the particular case of $G^{(1)}(z)$, it happens that such a way is provided by its linear equation (571). Namely, using (548), we rewrite it as

$$\lim_{\epsilon \rightarrow 0^+} \left(G^{(1)}(y + i\epsilon) + G^{(1)}(y - i\epsilon) \right) = - \underbrace{\sum_{n \geq 1} \frac{2\rho^{(0)}(y)\rho^{(0)'}(y)}{\rho^{(0)}(y)^2 + n^2}}_{\text{regular at the endpoints}} - \underbrace{\frac{\rho^{(0)'}(y)}{\rho^{(0)}(y)}}_{=\frac{1}{y} - \frac{1}{2}\left(\frac{1}{y-a} + \frac{1}{y-b}\right)}. \quad (595)$$

The non-integrable integer-power singularities are now visible.

The normalization formulae (327), (328) for this density read therefore,

$$\int_{\mathcal{C}_y} dy \rho^{(1)}(y) = \lim_{z \rightarrow \infty} \left(z G^{(1), \text{bulk}}(z) \right) = \frac{1}{2}. \quad (596)$$

It is not zero, as required, and the boundary piece of the resolvent (593) is necessary to make it zero,

$$\lim_{z \rightarrow \infty} \left(z G^{(1), \text{boundary}}(z) \right) = -\frac{1}{2}. \quad (597)$$

Moreover,

$$\int_{\mathcal{C}_y} dy \frac{\rho^{(1)}(y)}{y} = -G^{(1), \text{bulk}}(0) = -G^{(1)}(0) + G^{(1), \text{boundary}}(0) = \mu \left(\frac{1}{2} + \alpha \right), \quad (598)$$

$$\int_{\mathcal{C}_y} dy \frac{\rho^{(1)}(y)}{y^2} = -G^{(1), \text{bulk}'}(0) = -G^{(1)'}(0) + G^{(1), \text{boundary}'}(0) = -G^{(1)'}(0) + \frac{1}{2}\mu^2 (1 + 8\alpha + 8\alpha^2), \quad (599)$$

where $G^{(1)'}(0)$ is given by (567).

5. The Large-Mode-Number, Fixed-Winding-Number Limit

In this paragraph, we address some subtleties and problems which appear while attempting to take the large-mode-number, fixed-winding-number limit (538) in large- J next-to-leading-order anomaly $\mathcal{A}^{(1)}(z)$ (558).

The most appealing way to proceed, is to notice that we are working with an infinite sum over n , and it naively seems that it is enough to take the limit (538) of each term of this sum separately, and afterwards perform the summation. Then, in principle, any number of small- α orders of $\mathcal{A}^{(1)}(z)$ would be easily derivable. Sadly, this is not possible. To show this, let us change the complex variable z to Z (542), and restrict ourselves to values of Z of order $O(\alpha^0)$. Then, the summand in $\mathcal{A}^{(1)}(z)$ has the following small- α series,

$$\begin{aligned} & \left(\text{the summand of } \mathcal{A}^{(1)}(z) \right)_{n|z=a+(b-a)Z, Z \sim O(\alpha^0)} = \\ & = -\frac{1}{\alpha^2} \varpi^2 \left(1 + \frac{1-2Z}{2\sqrt{Z}\sqrt{Z-1}} \right) - \frac{1}{\alpha^{3/2}} \varpi \left(\frac{\nu}{8Z(Z-1)} + \varpi \left(4(1-2Z) + \frac{1-8Z+8Z^2}{\sqrt{Z}\sqrt{Z-1}} \right) \right) + O\left(\frac{1}{\alpha}\right). \end{aligned} \quad (600)$$

(In this paragraph, for the sake of simplicity, we everywhere assume $\mu > 0$.) The sum over n of any of these terms is clearly divergent. It is, thus, only when we deal with the exact summand that we obtain a convergent sum over n . One might also wonder if some regularization scheme could be applied, like the zeta-function one, which happened to be the “true” scheme in paragraph II D 1, where we have also been dealing with anomaly phenomena; but we have checked that the zeta-function-regularized sums of the terms of (600) have nothing to do with the correct expansion.

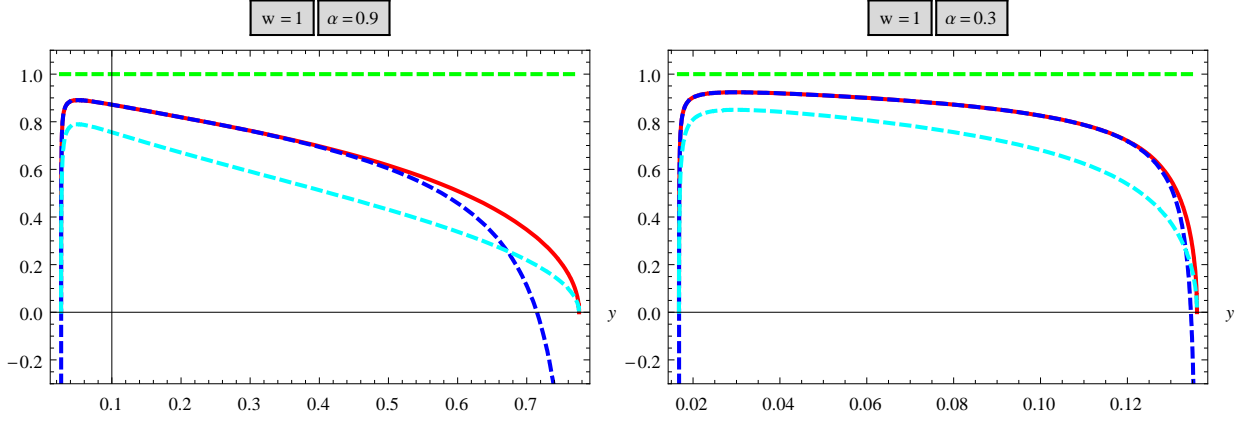


FIG. 21: How to handle the hyperbolic cotangent? Three approaches.

Plots of the function $[a, b] \ni y \mapsto \coth(\pi\rho^{(0)}(y)) - 1/(\pi\rho^{(0)}(y))$ (solid red), approximated either by $(1 - 1/(\pi\rho^{(0)}(y)))$ (602) (dashed blue), or by 1 (629) (dashed green), or by $(2/\pi) \arctan(\pi\rho^{(0)}(y))$ (637) (dashed cyan).

The first description is the most accurate in the bulk, but breaks down at the endpoints of the cut. The second one is much worse, but still provides a correct leading small- α order; at the endpoints, it has a finite value 1, which is, however, different from the true value 0. The third version looks better than the second one, but also correctly delivers only the leading small- α order; it seems to be the best one structurally, as has identical endpoint behavior and cut structure as the exact quantity.

The values of the parameters are: $w = 1$ everywhere, and $\alpha = 0.9$ (LEFT), $\alpha = 0.3$ (RIGHT). Obviously, the smaller α , the better the alignment, and here the α 's are quite large.

This problem is a reflection of the fact that hyperbolic cotangent, when its argument grows to plus infinity, differs from 1 by only an exponentially small quantity,

$$\coth(r) = 1 + 2e^{-2r} + 2e^{-4r} + 2e^{-6r} + \dots, \quad \text{for } r \rightarrow +\infty; \quad (601)$$

this is the asymptotic series of \coth around infinity on the real positive semi-axis. Therefore, it is not possible to work with its “sum version” (548), and separately expand each of its terms.

Since the density $\rho^{(0)}(y)$ is strictly positive for $y \in (a, b)$, *i.e.*, everywhere in its domain except for the endpoints, and since its leading small- α term is of order $O(1/\sqrt{\alpha})$ (541), *i.e.*, large — the way to implement the limit (538) in the derivation of $\mathcal{A}^{(1)}(z)$ is not to expand the hyperbolic cotangent into an infinite sum (548), but instead just replace it by 1, with an exponentially small error,

$$\coth(\pi\rho^{(0)}(y)) = 1 + O(e^{-1/\sqrt{\alpha}}), \quad \text{for } y \in (a, b). \quad (602)$$

There is, however, a subtle modification to this statement, which originates from the zeroing of the density at the endpoints of its domain, which causes the hyperbolic cotangent to explode. This nuance is, thus, yet another “boundary effect,” of course different in nature than those previously introduced. As harmless as it may appear, this is the primordial reason for all our further problems with the approximation (602), to be discussed in detail below. Some of them we have not been able to solve, thus seriously, but not entirely, restricting the applicability of the current approach.

In order to understand a boundary input to (602), let us first look at a simpler function, $[0, +\infty) \ni y \mapsto (1/\alpha)(\coth(\sqrt{y/\alpha}) - 1)$. Pictorially, see figure 22, it seems to tend to the Dirac delta at $y = 0$ as α decreases to zero, and we will now argue for the precise form of this delta, as well as an error with which it comes. This boundary input will, thus, be henceforth dubbed the “Dirac delta contribution.”

The considered function is, for any $\alpha > 0$, infinite at $y = 0$, and it goes to zero with decreasing α at any $y > 0$. Let us first investigate the area under its plot. The indefinite integral is, even for finite $\alpha > 0$, expressible through dilogarithms,

$$\int dy \frac{1}{\alpha} \left(\coth\left(\sqrt{\frac{y}{\alpha}}\right) - 1 \right) = 2\sqrt{\frac{y}{\alpha}} \log\left(1 - e^{-2\sqrt{\frac{y}{\alpha}}}\right) - \text{Li}_2\left(e^{-2\sqrt{\frac{y}{\alpha}}}\right) + \text{const}, \quad \text{for } \alpha > 0. \quad (603)$$

The limit $y \rightarrow 0^+$ of this primitive (we choose $\text{const} = 0$) can also be computed even for finite $\alpha > 0$,

$$\lim_{y \rightarrow 0^+} \left(2\sqrt{\frac{y}{\alpha}} \log\left(1 - e^{-2\sqrt{\frac{y}{\alpha}}}\right) - \text{Li}_2\left(e^{-2\sqrt{\frac{y}{\alpha}}}\right) \right) = -\frac{\pi^2}{6}, \quad \text{for } \alpha > 0. \quad (604)$$

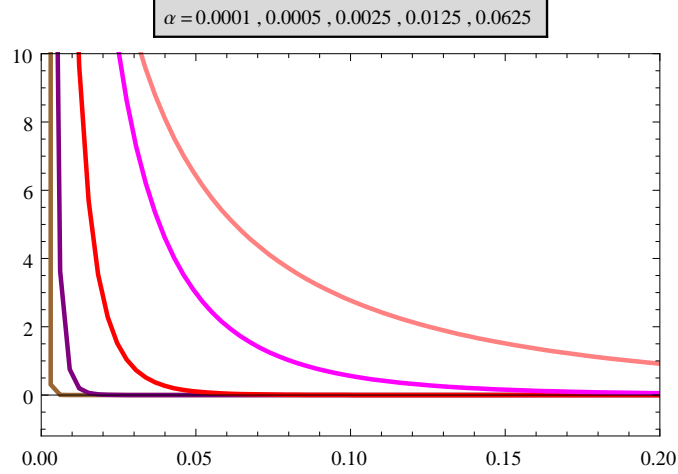


FIG. 22: The function $[0, +\infty) \ni y \mapsto (1/\alpha)(\coth(\sqrt{y/\alpha}) - 1)$ for five values of α : 0.0001 (brown), 0.0005 (purple), 0.0025 (red), 0.0125 (magenta), 0.0625 (pink). We see how it approaches the Dirac delta at $y = 0$ when α goes to zero.

If, on the other hand, we fix some $y > 0$, and expand the primitive in small α , we get only exponentially small terms,

$$2\sqrt{\frac{y}{\alpha}} \log\left(1 - e^{-2\sqrt{\frac{y}{\alpha}}}\right) - \text{Li}_2\left(e^{-2\sqrt{\frac{y}{\alpha}}}\right) =$$

$$= -\left(1 + 2\sqrt{\frac{y}{\alpha}}\right) e^{-2\sqrt{\frac{y}{\alpha}}} - \left(\frac{1}{4} + \sqrt{\frac{y}{\alpha}}\right) e^{-4\sqrt{\frac{y}{\alpha}}} - \left(\frac{1}{9} + \frac{2}{3}\sqrt{\frac{y}{\alpha}}\right) e^{-6\sqrt{\frac{y}{\alpha}}} + \dots = O(\exp), \quad \text{for } \alpha \rightarrow 0^+. \quad (605)$$

(By “ $O(\exp)$,” we denote terms exponentially small when the variable α/y is small.) From this, we may conclude that the small- α behavior of the area under the pertinent plot is

$$\int_{y_-}^{y_+} dy \frac{1}{\alpha} \left(\coth\left(\sqrt{\frac{y}{\alpha}}\right) - 1 \right) = \begin{cases} \frac{\pi^2}{6} + O(\exp) & \text{for } y_- = 0, y_+ > 0 \\ O(\exp) & \text{for } y_-, y_+ > 0 \end{cases}. \quad (606)$$

To ensure the Dirac delta properties, we should also take $y_+ \rightarrow +\infty$, which would cause all the exponential corrections to disappear, thus leaving us with

$$\int_0^\infty dy \frac{1}{\alpha} \left(\coth\left(\sqrt{\frac{y}{\alpha}}\right) - 1 \right) = \frac{\pi^2}{6}. \quad (607)$$

This, up to an exponentially small correction, is the behavior of (a half of) the Dirac delta at $y = 0$ multiplied by $\pi^2/3$ (we are working with the positive semi-axis, and recall $\int_0^\infty dy \delta(y) = 1/2$).

The results (606), (607) are, however, not enough to be sure that we indeed have a Dirac delta. We should find their extension to the case when a suitable test function $f(y; \alpha)$ is present. (Quite unusually for such computations, here, as is clear from (434), we need to take into account test functions dependent on α , and, say, having a Laurent expansion in α around zero.) We have not attempted a general proof, but we propose the following generalization,

$$\int_{y_-}^{y_+} dy \frac{1}{\alpha} \left(\coth\left(\sqrt{\frac{y}{\alpha}}\right) - 1 \right) f(y; \alpha) = \begin{cases} \frac{\pi^2}{6} f(y=0; \alpha) + O(\text{subleading}) + O(\exp) & \text{for } y_- = 0, y_+ > 0 \\ O(\exp) & \text{for } y_-, y_+ > 0 \end{cases}, \quad (608)$$

i.e.,

$$\int_0^\infty dy \frac{1}{\alpha} \left(\coth\left(\sqrt{\frac{y}{\alpha}}\right) - 1 \right) f(y; \alpha) = \frac{\pi^2}{6} f(y=0; \alpha) + O(\text{subleading}). \quad (609)$$

This means that our delta-like function acts, at small α , and with an exponentially small error, as a Dirac delta which convoluted with a test function $f(y; \alpha)$ produces some value, and this value agrees with the number $(\pi^2/6)f(y=0; \alpha)$

at the leading small- α order, while the non-exponential subleading terms may disagree. Remark also that if we integrate from 0 to $+\infty$, there are no longer any exponential corrections.

To clarify the formula (608), let us use it on an example of the test function $f(y; \alpha) = y^p$, where $p \geq 1$ is an integer. The indefinite integral from the l.h.s. of (608) can be found in terms of polylogarithms by the change of variables $x \equiv \exp(2\sqrt{y/\alpha})$,

$$\begin{aligned} \int dy \frac{1}{\alpha} \left(\coth \left(\sqrt{\frac{y}{\alpha}} \right) - 1 \right) y^p &= \frac{\alpha^p}{2^{2p}} \int dx \left(\frac{1}{x-1} - \frac{1}{x} \right) \log(x)^{2p+1} = \\ &= \frac{8(p+1)y^{p+1}}{\alpha} \left(-1 + (2p+2)! \sum_{q=1}^{2p+2} \frac{(-1)^q}{2^q(2p+2-q)!} \text{Li}_q \left(e^{2\sqrt{\frac{y}{\alpha}}} \right) \left(\frac{y}{\alpha} \right)^{-r/2} \right) + \text{const.} \end{aligned} \quad (610)$$

The limit $y \rightarrow 0^+$ of this primitive can be seen to arise only from the $q = 2p+2$ term in the sum, and reads

$$\lim_{y \rightarrow 0^+} (\text{the primitive (610)}) = \frac{(2p+1)!\zeta(2p+2)}{2^{2p}} \alpha^p, \quad \text{for } \alpha > 0, \quad (611)$$

where ζ is the Riemann zeta function. Fixing $y > 0$, and considering small α , produce all sorts of exponentially small terms, and also one non-exponential contribution, which happens to be precisely twice the value at zero,

$$(\text{the primitive (610)}) = 2 \frac{(2p+1)!\zeta(2p+2)}{2^{2p}} \alpha^p + O(\exp), \quad \text{for } \alpha \rightarrow 0^+. \quad (612)$$

Recall that in (605) we have not had any non-exponential terms. Also, remark that, critically, these non-exponential terms do not depend on y ; otherwise, the lower line of (608) would be compromised. Thus, putting things together, we have found the non-exponential part of the small- α series of the following definite integral,

$$\int_{y_-}^{y_+} dy \frac{1}{\alpha} \left(\coth \left(\sqrt{\frac{y}{\alpha}} \right) - 1 \right) y^p = \begin{cases} \frac{(2p+1)!\zeta(2p+2)}{2^{2p}} \alpha^p + O(\exp) & \text{for } y_- = 0, y_+ > 0 \\ O(\exp) & \text{for } y_-, y_+ > 0 \end{cases}, \quad (613)$$

or

$$\int_0^\infty dy \frac{1}{\alpha} \left(\coth \left(\sqrt{\frac{y}{\alpha}} \right) - 1 \right) y^p = \frac{(2p+1)!\zeta(2p+2)}{2^{2p}} \alpha^p. \quad (614)$$

Since $(\pi^2/6)y^p|_{y=0} = 0$, we see that our delta-like function, when applied to the test function $f(y; \alpha) = y^p$, indeed acts as the supposed Dirac delta at $y = 0$, up to the leading small- α order, and exponentially small error.

We have in this way proven (608) for polynomial test functions. We have also numerically tested it with a variety of other types of functions, always obtaining striking agreement. Graphs 23, 26 and 25, to be discussed below, also provide such numerical corroboration. We will, therefore, take the formula to be granted.

Our aim, however, is not to work with the simplified function $[0, +\infty) \ni y \mapsto (1/\alpha)(\coth(\sqrt{y/\alpha}) - 1)$, but rather with $[a, b] \ni y \mapsto (1/\alpha)(\coth(\pi\rho^{(0)}(y)) - 1)$. Given the expression for the density $\rho^{(0)}(y)$ (533), we see that the passage between the two delta-like functions is realized by the change of variables $y \rightarrow \mu^2\alpha(y-a)(b-y)/(4y^2)$ (note that this change of variables depends on α ; this, however, does not affect our reasoning),

$$\begin{aligned} \int_a^b dy \frac{1}{\alpha} \left(\coth \left(\pi\rho^{(0)}(y) \right) - 1 \right) f(y; \alpha) &= \\ &= \int_0^{\mu^2\alpha^2(1+\alpha)} dy \frac{1}{\alpha} \left(\coth \left(\sqrt{\frac{y}{\alpha}} \right) - 1 \right) \cdot \\ &\cdot \left(f \left(\frac{\mu\alpha(1+2\alpha) - 2\sqrt{\alpha(\mu^2\alpha^2(1+\alpha) - y)}}{4y + \mu^2\alpha}; \alpha \right) \right). \end{aligned}$$

The first term is defined as and easily derived to be

$$\mathcal{A}^{(1),\text{bulk,coth}=1}(z) \equiv -\pi \int_a^b dy \frac{1}{z-y} \rho^{(0)}(y) \rho^{(0)'}(y) = \quad (622)$$

$$= -\frac{1}{\pi z^3} \left(\mu \sqrt{\alpha(1+\alpha)} z + (1 - \mu(1+2\alpha)z) \left(\log(\sqrt{\alpha} + \sqrt{1+\alpha}) + \frac{\log(z-a) - \log(z-b)}{4} \right) \right), \quad (623)$$

while the Dirac delta term acquires the form

$$\mathcal{A}^{(1),\text{bulk},\delta}(z) = -\frac{\pi}{12} \left(\frac{1}{z-a} - \frac{1}{z-b} \right) + \text{O(subleading)} + \text{O(exp)}. \quad (624)$$

As mentioned earlier, in order to cast this explicitly as a small- α expansion, one first needs to specify the domain of the complex argument z . Also, (623), (624) may not be reliable for z close to the edges a, b of the cut.

In order to test the approximation (621), let us consider two quantities depending on $\mathcal{A}^{(1),\text{bulk,coth}}(z)$, namely the large- J next-to-leading-order density,

$$\begin{aligned} \rho^{(1)}(t) &\equiv \rho^{(1)}(y) \Big|_{y=a+(b-a)t} = \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{t(1-t)}} \frac{1}{\mu} \left(\frac{a}{a-b} - t \right) \lim_{\epsilon \rightarrow 0^+} \left(\mathcal{A}^{(1),\text{bulk,coth}}(z) \Big|_{z=a+(b-a)Z} \Big|_{Z=t+i\epsilon} + \mathcal{A}^{(1),\text{bulk,coth}}(z) \Big|_{z=a+(b-a)Z} \Big|_{Z=t-i\epsilon} \right), \end{aligned} \quad (625)$$

and (exploiting $(-z/(\mu\sqrt{z-a}\sqrt{z-b}))'|_{z=0} = 1$ and $-z/(\mu\sqrt{z-a}\sqrt{z-b})|_{z=0} = 0$) second charge,

$$Q_2|_{\omega=0}|_{\text{next-to-leading}} = -G^{(1)'}(0) = -\mathcal{A}^{(1),\text{bulk,coth}}(0). \quad (626)$$

Using (621), (623), (624), we immediately find the three leading small- α orders of the density,

$$\rho^{(1)}(t) = \frac{1}{\alpha^2} \rho_{-2}^{(1)}(t) + \frac{1}{\alpha^{3/2}} \rho_{-3/2}^{(1)}(t) + \frac{1}{\alpha} \rho_{-1}^{(1)}(t) + \text{O}\left(\frac{1}{\sqrt{\alpha}}\right) + \text{O(exp)}, \quad (627)$$

where

$$\rho_{-2}^{(1)}(t) = \frac{\varpi^2}{8\pi^2} \frac{1}{\sqrt{t(1-t)}} \left(2 + (-1+2t) \log\left(\frac{1-t}{t}\right) \right), \quad (628)$$

and

$$\rho_{-3/2}^{(1)}(t) = -\frac{\varpi^2}{4\pi^2} \frac{1}{\sqrt{t(1-t)}} \left(4(-1+2t) + (1-8t+8t^2) \log\left(\frac{1-t}{t}\right) \right), \quad (629)$$

and

$$\rho_{-1}^{(1)}(t) = \frac{\varpi^2}{4\pi^2} \frac{1}{\sqrt{t(1-t)}} \left(6(1-8t+8t^2) + (-1+2t)(1-24t+24t^2) \log\left(\frac{1-t}{t}\right) + \underbrace{\frac{\pi^2}{48\varpi^2 t(1-t)}}_{\text{the leading Dirac delta contribution}} \right), \quad (630)$$

and of the second charge,

$$Q_2|_{\omega=0}|_{\text{next-to-leading}} = -\frac{4}{3\pi} \mu^3 (\alpha(1+\alpha))^{3/2} - \underbrace{\frac{\pi}{3} \mu \sqrt{\alpha(1+\alpha)}}_{\text{the leading Dirac delta contribution}} + \text{O(subleading)} + \text{O(exp)} =$$

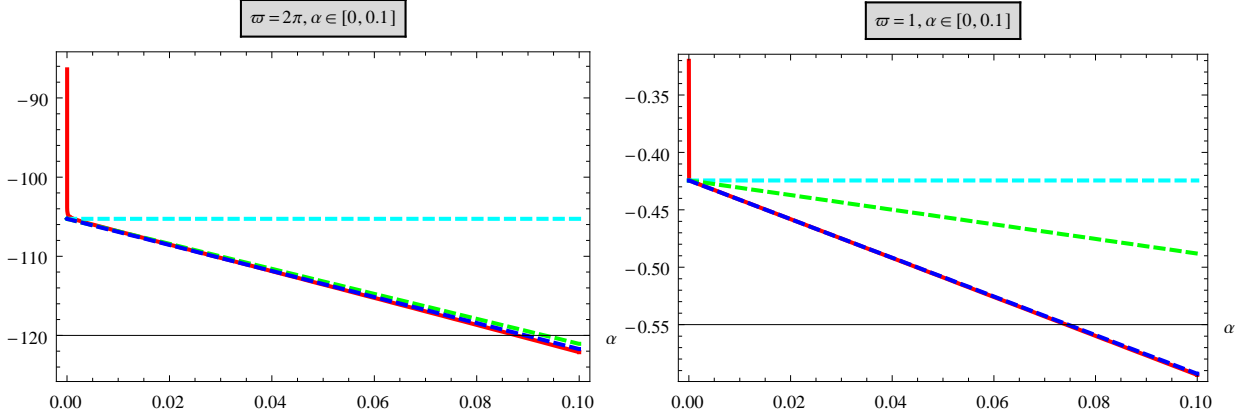


FIG. 23: Numerical support of the approximation “ $\coth \approx 1 + \delta$ ” of the second charge (631).

We plot $[0, 0.1] \ni \alpha \mapsto \alpha^{3/2} Q_2|_{\omega=0}|_{\text{next-to-leading}}$, which should be finite at $\alpha = 0$. (The explosion of the exact quantity at $\alpha = 0$ is a numerical artifact.) The exact value (580) (solid red line) is compared to the series (631) terminated at the leading order (dashed cyan), or the next-to-leading one without the Dirac delta contribution (dashed green) and with it (dashed blue). The agreement is striking, in particular in the RIGHT graph, which strongly confirms our finding.

We have chosen $\varpi = 2\pi$ (LEFT) and $\varpi = 1$ (RIGHT). This latter value is unphysical (a non-integer winding number w), but makes the importance of the Dirac delta input more visible, as is clear from (631). The summation over n in the exact quantity is truncated at $n_{\max} = 10^4$.

$$= -\frac{1}{\alpha^{3/2}} \frac{4\varpi^3}{3\pi} - \frac{1}{\sqrt{\alpha}} \left(\frac{2\varpi^3}{\pi} + \underbrace{\frac{\pi\varpi}{3}}_{\substack{\text{the leading} \\ \text{Dirac delta} \\ \text{contribution}}} \right) + O(\alpha^0) + O(\exp). \quad (631)$$

Remark that this density also explodes at small α , this time as $1/\alpha^2$, compared to $1/\sqrt{\alpha}$ of $\rho^{(0)}(t)$ (541). Similarly, the charge is large as $1/\alpha^{3/2}$, compared to $1/\alpha$ of $Q_2|_{\omega=0}|_{\text{leading}}$ (546), and there is no order $O(1/\alpha)$.

We observe that in both cases, the Dirac delta term starts contributing at the next-to-next-to-leading small- α order; in other words, our method “ $\coth \approx 1 + \delta$ ” (602), (616) allows us to find only three leading orders of objects dependent on $\mathcal{A}^{(1),\text{bulk},\coth}(z)$ in the limit (538). This can be traced back to the presence of the prefactor $1/\alpha$ on the l.h.s. of (606). We will soon be once again convinced, through numerical results, of the great accuracy of the three leading terms obtained within this method; the only unreliable part will be the endpoint behavior of the density.

The picture painted above would be flawless unless a seemingly minute detail which however triggers some serious problems. Namely, making the approximation “ $\coth \approx 1 + \delta$ ” • changes its endpoint behavior, • changes its cut structure. To elaborate on this, recall that in the articles [163, 185, 194], the following approximation has been chosen,

$$\coth\left(\pi\rho^{(0)}(y)\right) - \frac{1}{\pi\rho^{(0)}(y)} \approx 1. \quad (632)$$

It has one structural advantage over ours: The combination on the l.h.s. is finite (and equal to 0) for $y = a$, $y = b$, unlike the bare hyperbolic cotangent. Replacing it by 1, we exchange a quantity finite at the endpoints with another quantity finite at the endpoints, while in our case, an object which diverges at the endpoints is traded for a finite object plus a Dirac-delta singularity, which, however, is a different type of a divergence. This will be responsible for numerous problems, such as the violation of the conditions (596), (598), (599), see (649)–(654), and the appearing of a non-integrable ($\sim 1/(t(1-t))$) endpoint behavior of the density. On the other hand, in our approximation, we replace $\coth(\pi\rho^{(0)}(y))$ by 1 plus a correction which does not have a cut, while in (632), the correction does have it; this makes a difference for example when computing $G^{(1),\text{boundary}}(z)$, see below (647). In figure 21, we observe how much better “ $\coth \approx 1 + \delta$ ” is than this “old” approximation, with an exception of the endpoints, where the former develops an infinite error, partially cured by the Dirac delta contribution.

It is easy to show that our technique and (632) coincide at the leading small- α order, as $1/\pi\rho^{(0)}(y)$ contributes to the density (625) and the second charge (626) only at the next-to-leading one. Since we are worried about the endpoint behavior of the density, let us exploit (632) to repeat the expansions up to the three leading terms of the

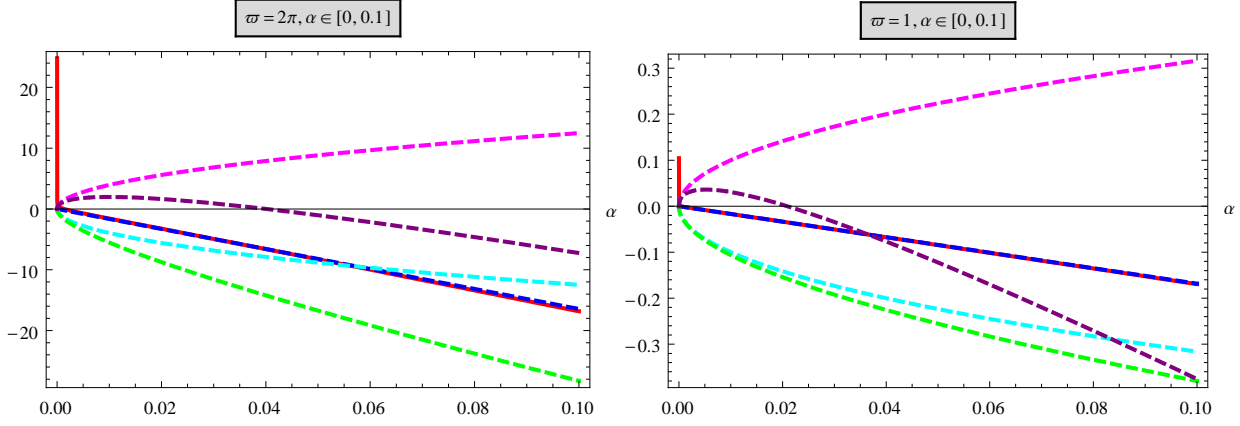


FIG. 24: The supremacy of the approximation “ $\coth \approx 1 + \delta$ ” over the other two, “ $\coth \approx 1 + 1/\pi\rho$ ” and “ $\coth \approx \arctan$ ”. We plot $[0, 0.1] \ni \alpha \mapsto \alpha^{3/2} Q_2|_{\omega=0}|_{\text{next-to-leading}} + 4\varpi^3/(3\pi)$, *i.e.*, without the leading order. The exact value (580) (solid red line) is compared to the “ $\coth \approx 1 + \delta$ ” expansion (631) including all the visible terms, with the Dirac delta contribution (dashed blue), to the “ $\coth \approx 1 + 1/\pi\rho$ ” expansion (636), terminated at two (dashed cyan) or three (dashed green) leading orders, and to the “ $\coth \approx \arctan$ ” expansion (646), similarly terminated at two (dashed magenta) or three (dashed purple) leading orders.

Our Dirac-delta approach proves excellent, while the other two are invalid. It is so visible here since in the former approximation, there is no term $O(1/\alpha)$, while the latter two possess it. Multiplying the charge by $\alpha^{3/2}$, as we do here, yields for them a $\sqrt{\alpha}$ -behavior close to $\alpha = 0$, which is clearly incorrect, as the true behavior is like α .

We have chosen $\varpi = 2\pi$ (LEFT) and $\varpi = 1$ (RIGHT). This latter value is technically unphysical, but provides an even better picture. The summation over n in the exact quantity is truncated at $n_{\max} = 10^4$.

density,

$$\rho_{-2}^{(1), \text{“coth} \approx 1 + 1/\pi\rho\text{”}}(t) = \rho_{-2}^{(1)}(t), \quad (633)$$

and

$$\rho_{-3/2}^{(1), \text{“coth} \approx 1 + 1/\pi\rho\text{”}}(t) = -\frac{\varpi^2}{4\pi^2} \frac{1}{\sqrt{t(1-t)}} \left(4(-1+2t) + (1-8t+8t^2) \log\left(\frac{1-t}{t}\right) - \frac{\pi}{2\varpi} \right), \quad (634)$$

and

$$\rho_{-1}^{(1), \text{“coth} \approx 1 + 1/\pi\rho\text{”}}(t) = \frac{\varpi^2}{4\pi^2} \frac{1}{\sqrt{t(1-t)}} \left(6(1-8t+8t^2) + (-1+2t)(1-24t+24t^2) \log\left(\frac{1-t}{t}\right) + \frac{\pi}{\varpi}(1-2t) \right), \quad (635)$$

and the second charge,

$$Q_2|_{\omega=0}|_{\text{next-to-leading}}|_{\text{“coth} \approx 1 + 1/\pi\rho\text{”}} = -\frac{1}{\alpha^{3/2}} \frac{4\varpi^3}{3\pi} - \frac{1}{\alpha} \varpi^2 - \frac{1}{\sqrt{\alpha}} \frac{2\varpi^3}{\pi} + O(\alpha^0). \quad (636)$$

The leading terms are identical to previously obtained, the subleading ones slightly differ; in particular, in the second charge, the order $O(1/\alpha)$ appears.

Since we have to be sure of the validity of at least the leading-order density $\rho_{-2}^{(1)}(t)$, we have found yet another approximation of the problematic hyperbolic cotangent, which has precisely the same endpoint behavior and the cut structure as the exact quantity, and therefore, will more reliably support our finding for $\rho_{-2}^{(1)}(t)$. It will, however, also yield correctly only the leading orders of both the density and the second charge. The idea behind it is to replace sums by integrals in the following way,

$$\coth\left(\pi\rho^{(0)}(y)\right) - \frac{1}{\pi\rho^{(0)}(y)} = \frac{2}{\pi} \frac{1}{\rho^{(0)}(y)} \sum_{n \geq 1} \frac{1}{1 + \left(\frac{n}{\rho^{(0)}(y)}\right)^2} \approx \frac{2}{\pi} \int_{1/\rho^{(0)}(y)}^{\infty} dx \frac{1}{1+x^2} = \frac{2}{\pi} \arctan\left(\rho^{(0)}(y)\right), \quad (637)$$

which of course mimics the definition of the Riemann integral. As mentioned, we will consider such operations to reliably reproduce only the leading small- α orders. In figure 21, we see that this is not only an accurate description, but also zeroes at the endpoints, just as the exact quantity.

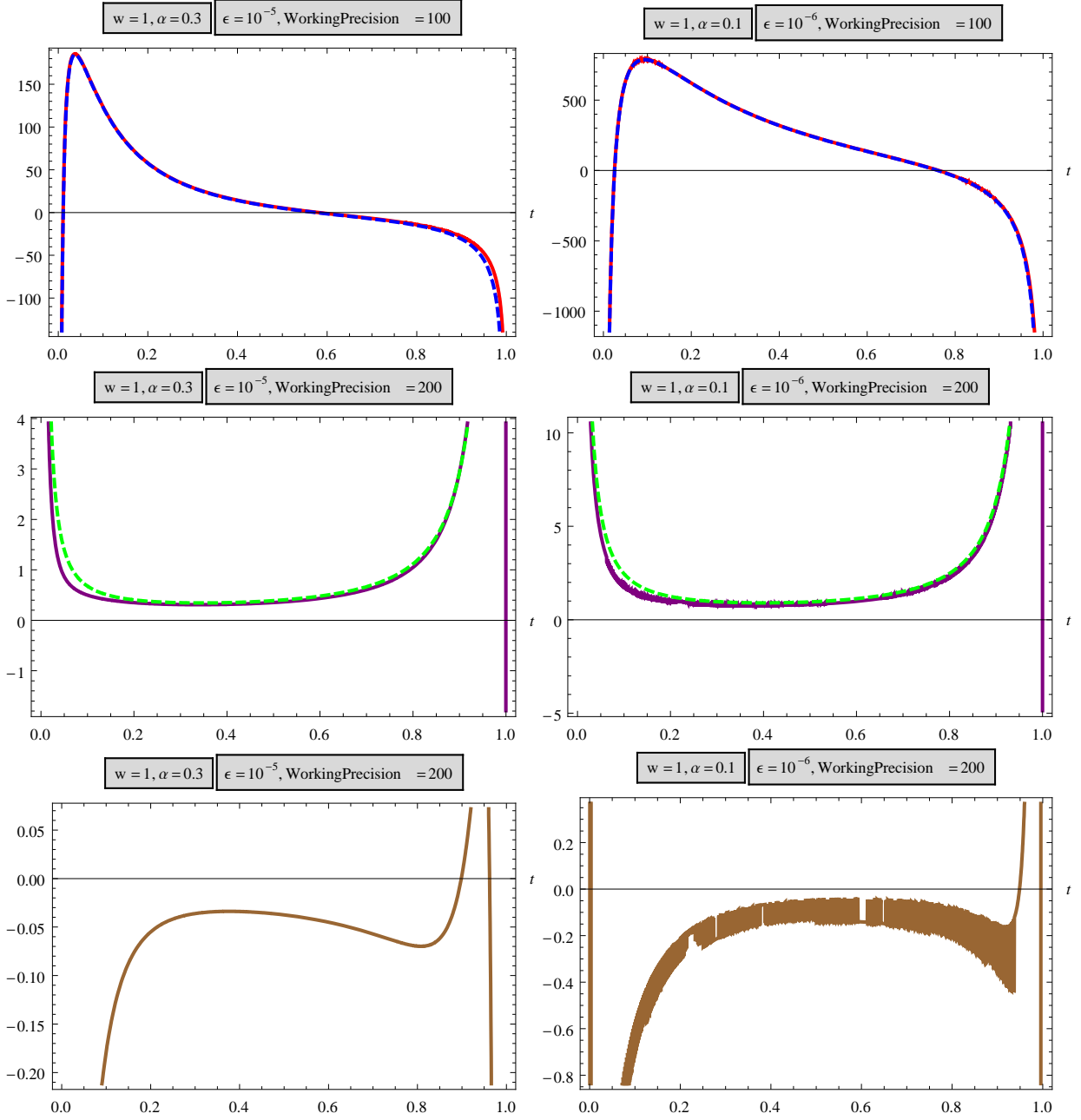


FIG. 25: Tests of the approximation “ $\coth \approx 1 + \delta$ ” (621) for the density.

TOP: The density $[0, 1] \ni t \mapsto \rho^{(1)}(t)$ based on the exact quantity $\mathcal{A}^{(1), \text{bulk}, \coth}(z)$ (solid red) compared to the density based on the approximation $\mathcal{A}^{(1), \text{bulk}, \coth=1}(z)$ (dashed blue). We observe that it is an excellent approximation, *i.e.*, the boundary contribution is very small.

MIDDLE: The difference of the previous two functions (solid purple) compared to the density based on the leading term of the Dirac delta contribution (dashed green). This is an actual numerical confirmation of (621), which, as we see, looks very accurate.

BOTTOM: The difference of the previous two functions (solid brown), showing the existence of subleading terms of the boundary contribution.

The parameters have been fixed to $w = 1$ (*i.e.*, $\varpi = 2\pi$) everywhere, and $\alpha = 0.3$ (LEFT), $\alpha = 0.1$ (RIGHT). Numerical note: We have exploited the integral definition (617), where it was crucial to correctly fine-tune the small parameter ϵ (625) (too small gives a messy plot, too large does not appropriately reproduce the limit $\epsilon \rightarrow 0^+$) and `WorkingPrecision` of doing the integral (too small gives a messy plot, too large requires a very long computation time). We have chosen $\epsilon = 10^{-5}$ (LEFT), $\epsilon = 10^{-6}$ (RIGHT), and `WorkingPrecision` = 100 (TOP), `WorkingPrecision` = 200 (MIDDLE, BOTTOM). This is a very high accuracy (on the computer used, there is `MachinePrecision` ≈ 16), and yet we have obtained some numerical noise in the bottom right graph, under which we can, however, perceive the true line.

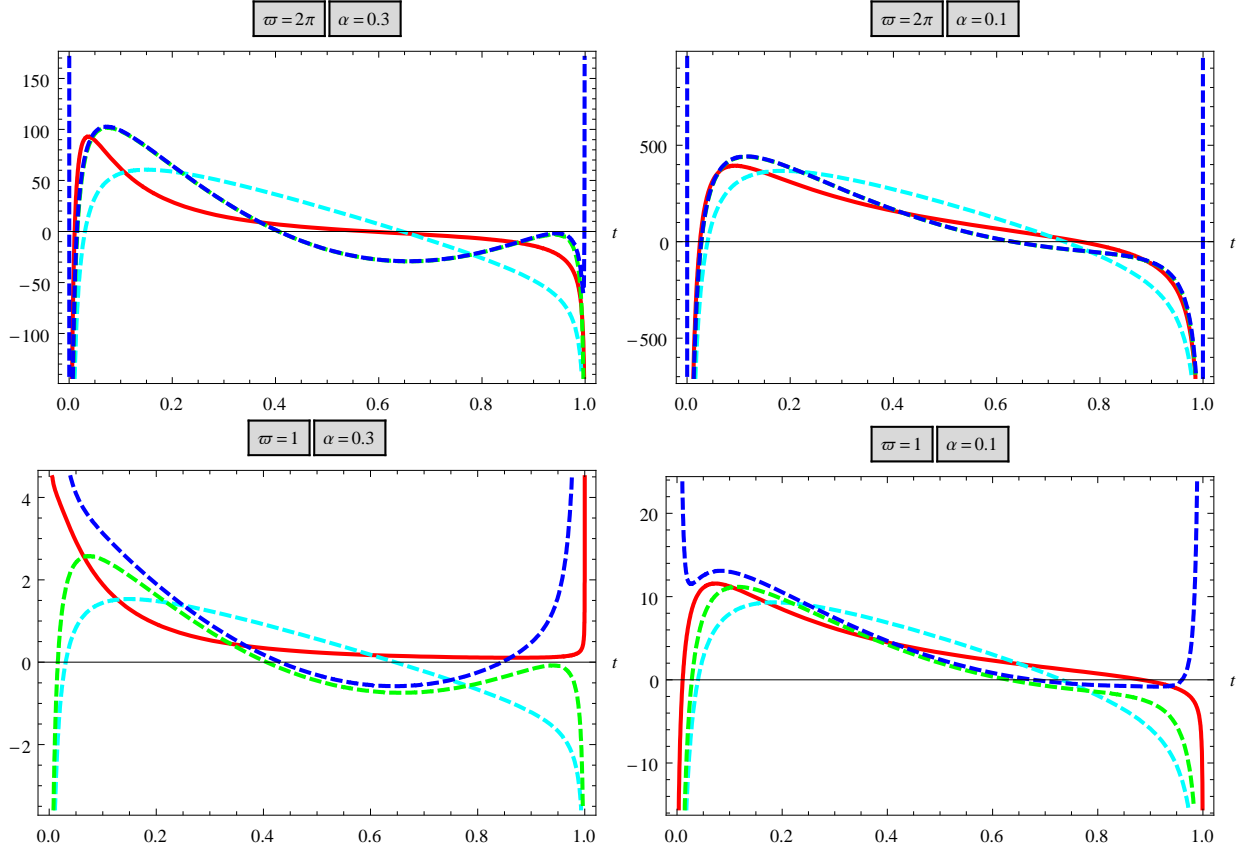


FIG. 26: Numerical support of the approximation “ $\coth \approx 1 + \delta$ ” of the density (627).

Everything is analogous to figure 23.

The values of the parameters are: $\varpi = 2\pi$ (TOP) and $\varpi = 1$ (BOTTOM), and $\alpha = 0.3$ (LEFT), $\alpha = 0.1$ (RIGHT). The summation over n in the exact quantity is truncated at $n_{\max} = 10^3$.

One may wonder whether trading a hyperbolic function for an inverse trigonometric function will do any simplification in computing pertinent integrals. Happily, it does. Let us show how to calculate the anomaly $\mathcal{A}^{(1),\text{bulk}}(z)$ (434) based on the approximation (637). We have

$$\begin{aligned} \mathcal{A}^{(1),\text{bulk},\text{“coth} \approx \arctan\text{”}}(z) &\equiv -2 \int_a^b dy \frac{1}{z-y} \rho^{(0)}(y) \rho^{(0)'}(y) \arctan(\rho^{(0)}(y)) = \\ &= -\frac{1}{2\pi^2} \int_a^b dy \frac{1}{z-y} \frac{1 - \mu(1+2\alpha)y}{y^3} \arctan(\rho^{(0)}(y)). \end{aligned} \quad (638)$$

It is somewhat tricky to handle integrals with arcus tangent like this; they are treated, just to briefly recapitulate the method, by writing an integral in question as a contour one, then deforming the contour so that it encircles the cut of the arctan; its values around its cut are easily found, thus removing the arctan from the integrand, leaving an integral which is much more tractable. This is the same technique as used in computations involving the Hernández-López phase, such as in paragraph V A 1. An important result reads

$$\int_0^{P_{\max}} dP \arctan(P) \frac{P}{\sqrt{P_{\max}^2 - P^2}} \frac{1}{P^2 + W} = \frac{\pi}{2\sqrt{P_{\max}^2 + W}} \operatorname{arctanh} \left(\frac{P_{\max}^2}{\sqrt{W} + \sqrt{P_{\max}^2 + 1} \sqrt{P_{\max}^2 + W}} \right), \quad (639)$$

where W is a complex number, and $P_{\max} \geq 0$. The change of variables $P = \rho^{(0)}(y)$ yields from it

$$\frac{1}{\pi} \int_a^b dy \frac{1}{y-z} \arctan(\rho^{(0)}(y)) = \operatorname{arctanh} \left(\frac{2\mu\alpha(1+\alpha)}{\pi + (1+2\alpha)\sqrt{\mu^2\alpha(1+\alpha) + \pi^2}} \right) +$$

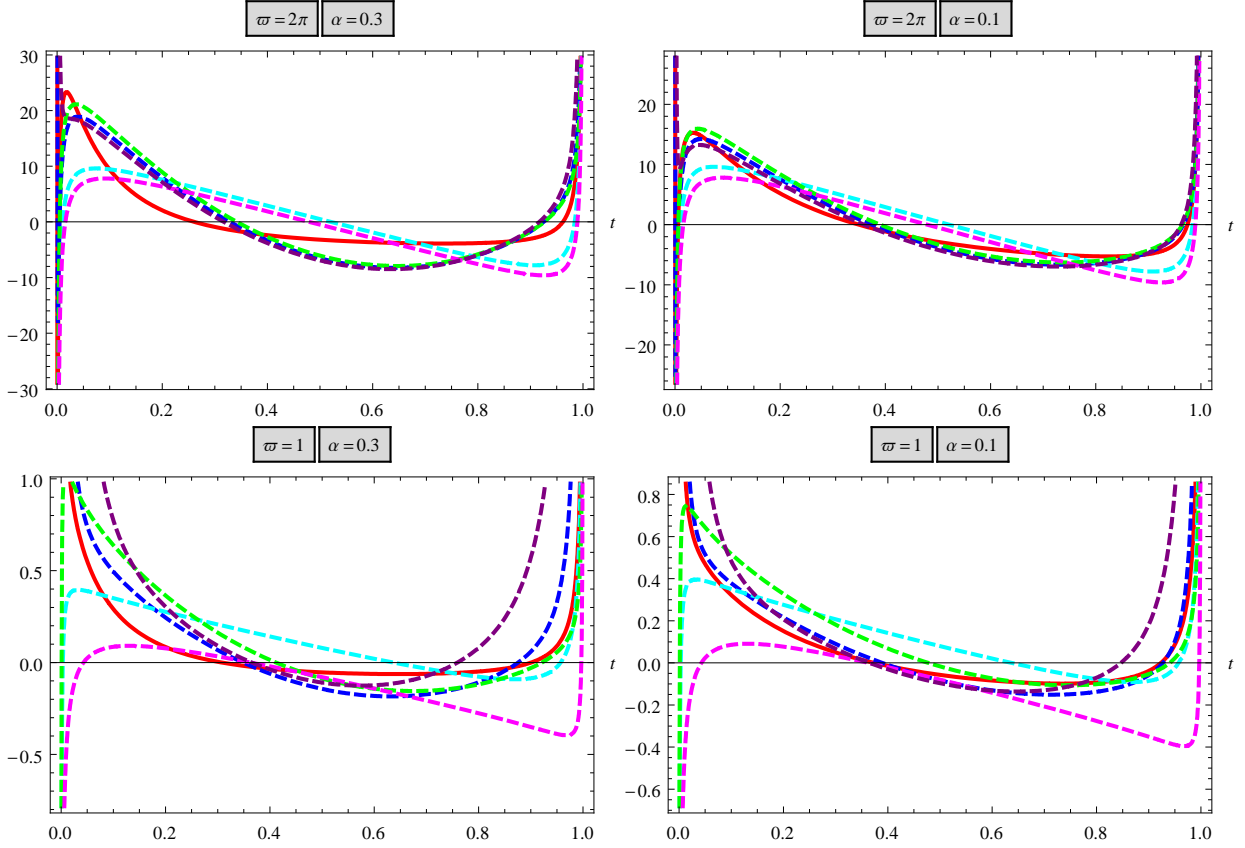


FIG. 27: The supremacy of the approximation “ $\coth \approx 1 + \delta$ ” over the other two, “ $\coth \approx 1 + 1/\pi\rho$ ” and “ $\coth \approx \arctan$ ”. Everything is analogous to figure 24. The values of the parameters are: $\varpi = 2\pi$ (TOP) and $\varpi = 1$ (BOTTOM), and $\alpha = 0.3$ (LEFT), $\alpha = 0.1$ (RIGHT). The summation over n in the exact quantity is truncated at $n_{\max} = 10^3$.

$$+ \frac{\frac{1}{z} - \mu(1 + 2\alpha)}{\sqrt{\left(\frac{1}{z} - \mu(1 + 2\alpha)\right)^2}} \operatorname{arctanh} \left(\frac{2\mu^2\alpha(1 + \alpha)}{\pi\mu\sqrt{\frac{(z-a)(z-b)}{z^2}} + \sqrt{\mu^2\alpha(1 + \alpha) + \pi^2}\sqrt{\left(\frac{1}{z} - \mu(1 + 2\alpha)\right)^2}} \right). \quad (640)$$

(Note an unusual cut structure here; for example, there appears $\sqrt{(z-a)(z-b)}$, and not $\sqrt{z-a}\sqrt{z-b}$, as we are used to, but this is how it should be.) This result allows us to compute (638) to be

$$\begin{aligned} \mathcal{A}^{(1), \text{bulk}, \text{“coth} \approx \arctan\text{”}}(z) &= \frac{1}{2\pi z^2} \left(2 \left(\pi - \sqrt{\mu^2\alpha(1 + \alpha) + \pi^2} \right) + \right. \\ &\quad \left. + \frac{\frac{1}{z} - \mu(1 + 2\alpha)}{\sqrt{\left(\frac{1}{z} - \mu(1 + 2\alpha)\right)^2}} \operatorname{arctanh} \left(\frac{2\mu^2\alpha(1 + \alpha)}{\pi\mu\sqrt{\frac{(z-a)(z-b)}{z^2}} + \sqrt{\mu^2\alpha(1 + \alpha) + \pi^2}\sqrt{\left(\frac{1}{z} - \mu(1 + 2\alpha)\right)^2}} \right) \right). \end{aligned} \quad (641)$$

Now, straightforwardly, the density

$$\begin{aligned} \rho^{(1), \text{“coth} \approx \arctan\text{”}}(y) &= \frac{1}{4\pi^2\mu y\sqrt{(y-a)(b-y)}} \left(-2\pi + 4\sqrt{\mu^2\alpha(1 + \alpha) + \pi^2} + \right. \\ &\quad \left. + \left(\frac{1}{y} - \mu(1 + 2\alpha) \right) \operatorname{arctanh} \left(\frac{4\sqrt{\mu^2\alpha(1 + \alpha) + \pi^2}y(-1 + \mu(1 + 2\alpha)y)}{y^2(4\pi^2 + \mu^2(1 + 8\alpha + 8\alpha^2)) - 2\mu(1 + 2\alpha)y + 1} \right) \right), \end{aligned} \quad (642)$$

which gives

$$\rho_{-2}^{(1), \text{“coth} \approx \arctan\text{”}}(t) = \rho_{-2}^{(1)}(t), \quad (643)$$

and

$$\rho_{-3/2}^{(1), \text{“coth} \approx \arctan\text{”}}(t) = -\frac{\varpi^2}{4\pi^2} \frac{1}{\sqrt{t(1-t)}} \left(4(-1+2t) + (1-8t+8t^2) \log\left(\frac{1-t}{t}\right) + \frac{\pi}{2\varpi} \right), \quad (644)$$

and

$$\begin{aligned} \rho_{-1}^{(1), \text{“coth} \approx \arctan\text{”}}(t) = \frac{\varpi^2}{4\pi^2} \frac{1}{\sqrt{t(1-t)}} & \left(6(1-8t+8t^2) + (-1+2t)(1-24t+24t^2) \log\left(\frac{1-t}{t}\right) - \right. \\ & \left. - \frac{\pi}{\varpi}(1-2t) + \frac{\pi^2}{8\varpi^2} \frac{1}{t(1-t)} \right), \end{aligned} \quad (645)$$

while the second charge,

$$\begin{aligned} Q_2|_{\omega=0}|_{\text{next-to-leading}}|_{\text{“coth} \approx \arctan\text{”}} &= -\mu^2\alpha(1+\alpha) + \frac{2\left(\pi(3\mu^2\alpha(1+\alpha)+2\pi^2) - 2(\mu^2\alpha(1+\alpha)+\pi^2)^{3/2}\right)}{3\pi} = \\ &= -\frac{1}{\alpha^{3/2}} \frac{4\varpi^3}{3\pi} + \frac{1}{\alpha} \varpi^2 - \frac{1}{\sqrt{\alpha}} \left(\frac{2\varpi^3}{\pi} + 2\pi\varpi \right) + \mathcal{O}(\alpha^0). \end{aligned} \quad (646)$$

As anticipated, the leading terms have been confirmed. In particular, due to the good endpoint and cut properties of the approximation “coth \approx arctan,” we will take it as a strong support for the validity of the leading density $\rho_{-2}^{(1)}(t)$. Its subleading terms work very well in the bulk, but have wrong endpoint behavior — in particular, not obeying the three conditions (596), (598), (599) that the density $\rho^{(1)}(y)$ must submit to, as we will show below.

In order to finally advocate our main approach, “coth $\approx 1 + \delta$,” we resort to numerical support. Figures 23 (the second charge) and 26 (the density) corroborate it, in particular showing that the Dirac delta contribution is needed and correct. We repeat this in a different way in figure 25, for the density. Figures 24 (the second charge) and 27 (the density) reveal that the subleading orders of both the “coth $\approx 1 + 1/\pi\rho$ ” and “coth \approx arctan” versions are incorrect.

Let us finally explain why we have been so cautious while dealing with the series of the density (627). As said several times, our approximation “coth $\approx 1 + \delta$,” despite being so excellent and systematically derived, changes the endpoint behavior and cut structure. This made it unclear to us how to calculate the boundary resolvent $G^{(1), \text{boundary}}(z)$. Namely, the r.h.s. of the linear equation (571) no longer contains any piece with integer-power non-integrable singularities, at least in the beginning of the small- α series,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \left(G^{(1)}(y + i\epsilon) + G^{(1)}(y - i\epsilon) \right) &= -\pi\rho^{(0)'}(y) \coth\left(\pi\rho^{(0)}(y)\right) \approx \\ &\approx -\pi\rho^{(0)'}(y) = -\frac{1 - \mu(1+2\alpha)y}{2\mu y^2 \sqrt{(y-a)(b-y)}} + \mathcal{O}\left(\frac{1}{\sqrt{\alpha}}\right) + \mathcal{O}(\exp), \end{aligned} \quad (647)$$

for $y \in [a, b]$. One may wonder whether we are permitted to replace in this way the hyperbolic cotangent which does not appear under an integration sign, as it was the case for $\mathcal{A}^{(1), \text{bulk, coth}}(z)$, and why we have written zero instead of the Dirac deltas (616). The answer is that (647) can be corroborated by a direct computation from the result for $\mathcal{A}^{(1), \text{bulk, coth}=1}(z)$ (623), while the Dirac delta piece $\mathcal{A}^{(1), \text{bulk, } \delta}(z)$ (624) enters the resolvent as an object with a cut, thus invisible in the linear equation. The error $\mathcal{O}(1/\sqrt{\alpha})$ has been derived as follows: The r.h.s. of (647), upon substituting $y = a + (b-a)t$, and expanding at small α , yields a series starting at the order $\mathcal{O}(1/\alpha^2)$, which means that the Dirac delta will start appearing at $\mathcal{O}(1/\alpha)$; its leading term, however, vanishes.

Notice that this is quite problematic. First, knowing the linear equation up to a certain small- α order, it is not clear to us how to derive from it the full dependence of $G^{(1), \text{boundary}}(z)$ on z , as well as its error. Moreover, (647)

does not seem to comply with the exact result (593) for the boundary resolvent $G^{(1),\text{boundary}}(z)$, due to the change of the endpoint behavior mentioned above. We can see it from the small- α series

$$G^{(1),\text{boundary}}(z) \Big|_{z=a+(b-a)Z, Z \sim O(\alpha^0)} = \frac{1}{\alpha^{3/2}} \frac{\varpi(1-2Z)}{16Z(Z-1)} + \frac{1}{\alpha^{1/2}} \frac{\varpi(1-2Z)}{32Z(Z-1)} + O(\sqrt{\alpha}), \quad (648)$$

or more strikingly, by checking the three conditions (596), (598), (599) that the density $\rho^{(1)}(y)$ should be subjected to. Namely, our finding for the density (627) yields

$$\begin{aligned} \int_a^b dy \rho^{(1)}(y) &= \frac{1}{\sqrt{\alpha}} \frac{4}{\varpi} \int_0^1 dt \rho_{-2}^{(1)}(t) + \frac{4}{\varpi} \int_0^1 dt \rho_{-3/2}^{(1)}(t) + \sqrt{\alpha} \frac{2}{\varpi} \int_0^1 dt \left(2\rho_{-1}^{(1)}(t) + \rho_{-2}^{(1)}(t) \right) + O(\alpha) = \\ &= \frac{1}{\sqrt{\alpha}} 0 + 0 + \sqrt{\alpha} 0 + O(\alpha), \end{aligned} \quad (649)$$

and

$$\begin{aligned} \int_a^b dy \frac{\rho^{(1)}(y)}{y} &= \frac{1}{\alpha^{3/2}} 4 \int_0^1 dt \rho_{-2}^{(1)}(t) + \frac{1}{\alpha} 4 \int_0^1 dt \left(2(1-2t)\rho_{-2}^{(1)}(t) + \rho_{-3/2}^{(1)}(t) \right) + \\ &+ \frac{1}{\sqrt{\alpha}} 2 \int_0^1 dt \left(2\rho_{-1}^{(1)}(t) + 4(1-2t)\rho_{-3/2}^{(1)}(t) + (5-32t+32t^2)\rho_{-2}^{(1)}(t) \right) + O(\alpha^0) = \\ &= \frac{1}{\alpha^{3/2}} 0 + \frac{1}{\alpha} 0 + \frac{1}{\sqrt{\alpha}} 0 + O(\alpha^0), \end{aligned} \quad (650)$$

and

$$\begin{aligned} \int_a^b dy \frac{\rho^{(1)}(y)}{y^2} &= \frac{1}{\alpha^{5/2}} 4\varpi \int_0^1 dt \rho_{-2}^{(1)}(t) + \frac{1}{\alpha^2} 4\varpi \int_0^1 dt \left(4(1-2t)\rho_{-2}^{(1)}(t) + \rho_{-3/2}^{(1)}(t) \right) + \\ &+ \frac{1}{\alpha^{3/2}} 2\varpi \int_0^1 dt \left(2\rho_{-1}^{(1)}(t) + 8(1-2t)\rho_{-3/2}^{(1)}(t) + (17-96t+96t^2)\rho_{-2}^{(1)}(t) \right) + O\left(\frac{1}{\alpha}\right) = \\ &= \frac{1}{\alpha^{5/2}} 0 + \frac{1}{\alpha^2} 0 - \frac{1}{\alpha^{3/2}} \frac{4\varpi^3}{3\pi} + O\left(\frac{1}{\alpha}\right). \end{aligned} \quad (651)$$

In the above integrals, the non-integrable piece appearing in $\rho_{-1}^{(1)}(t)$, contributed by the Dirac deltas, is treated with a contour instead of a line integral (check the end of paragraph III C 4 for a similar situation), and giving in this way 0. Eight out of the nine expansion coefficients obtained here are 0, except for the last one in (651), while expanding the r.h.s. of the conditions (596), (598), (599) leads to an entirely different set,

$$\frac{1}{\sqrt{\alpha}} 0 + \frac{1}{2} + \sqrt{\alpha} 0 + O(\alpha), \quad (652)$$

and

$$\frac{1}{\alpha^{3/2}} 0 + \frac{1}{\alpha} \frac{\varpi}{2} + \frac{1}{\sqrt{\alpha}} 0 + O(\alpha^0), \quad (653)$$

and

$$\frac{1}{\alpha^{5/2}} 0 + \frac{1}{\alpha^2} \frac{\varpi^2}{2} - \frac{1}{\alpha^{3/2}} \frac{4\varpi^3}{3\pi} + O\left(\frac{1}{\alpha}\right). \quad (654)$$

Hence, we discover total discrepancy, except for the leading terms (and, strangely, the last terms in (651) and (654)), which we assign to the fact that within our endpoint-behavior-changing approximation, we can no longer use the pertinent expansion of the exact expression (593) for $G^{(1),\text{boundary}}(z)$. We think that some different boundary resolvent should be adapted which would appropriately modify (652)–(654).

We have not been able to overcome this obstacle. Therefore, we will proceed as follows — when computing the density at the level $O(1/J)$, or any object dependent on the anomaly at the level $O(1/J^2)$, we will use the approximation “ $\coth \approx 1 + 1/\pi\rho$ ” (632), and restrict ourselves to only the leading small- α order. The leading density $\rho_{-2}^{(1)}(t)$ and the leading order of the boundary resolvent $G^{(1),\text{boundary}}(z)$ will be taken to be granted. In the all-loop case, this will actually be sufficient to probe into the series far enough to find traces of the Hernández–López phase, whose testing is the main goal of this article.

C. The Exact Solution to the One-Loop One-Cut Quadratic Equation at the Order $O(1/J^2)$

1. The Anomaly $\mathcal{A}^{(2,1)}(z)$

The first step on the way toward the large- J next-to-next-to-leading-order resolvent $G^{(2)}(z)$ (467) is to compute the anomaly term $\mathcal{A}^{(2,1)}(z)$ (430).

In order to find it by the method of summation and expressing through the resolvent, we use the partial result (555), which we obtained in the process of deriving $\mathcal{A}^{(1)}(z)$. What is yet unknown in this expression, are the values of the resolvent $G^{(1)}(z)$ at the roots a_{n_2} , b_{n_2} ; they can be straightforwardly calculated with aid of (556), and read

$$\begin{aligned}
 G^{(1)}(a_{n_2}, b_{n_2}) = & \frac{1}{2\nu_2^2} \left(8\mu^3\alpha(1+\alpha)(1+2\alpha) \mp \mu^2\nu_2(1+8\alpha+8\alpha^2) + 2\mu\nu_2^2(1+2\alpha) \mp \nu_2^3 \pm \right. \\
 & \left. \pm (\mu^2(1+8\alpha+8\alpha^2) \mp 2\mu\nu_2(1+2\alpha) + \nu_2^2) \sqrt{4\mu^2\alpha(1+\alpha) + \nu_2^2} \right) + \\
 & + \sum_{n_1 \geq 1} \frac{1}{\nu_2(\nu_1^2 - \nu_2^2)} \left(-2\mu\nu_2(1+2\alpha)(4\mu^2\alpha(1+\alpha) + \nu_2^2) \mp (\nu_1^2 - \nu_2^2)(\mu^2(1+8\alpha+8\alpha^2) + \nu_2^2) \pm \right. \\
 & \left. \pm \nu_1(\mu^2(1+8\alpha+8\alpha^2) + \nu_2^2) \sqrt{4\mu^2\alpha(1+\alpha) + \nu_1^2} \mp \right. \\
 & \left. \mp (\mu^2\nu_2(1+8\alpha+8\alpha^2) + \nu_2^3 \pm 2\mu(\nu_1^2 - \nu_2^2)(1+2\alpha)) \sqrt{4\mu^2\alpha(1+\alpha) + \nu_2^2} + \right. \\
 & \left. + 2\mu\nu_1(1+2\alpha) \sqrt{4\mu^2\alpha(1+\alpha) + \nu_1^2} \sqrt{4\mu^2\alpha(1+\alpha) + \nu_2^2} \right). \tag{655}
 \end{aligned}$$

Notice that there is an apparent singularity at $\nu_1 = \nu_2$ in the sum over n_1 . Actually, these are 0/0 symbols, which have finite limits as $\nu_1 \rightarrow \nu_2$, which respectively for a_{n_2} , b_{n_2} read

$$\pm \frac{\left(\nu_2 - \sqrt{4\mu^2\alpha(1+\alpha) + \nu_2^2} \right)^2 \left(\mu(1+2\alpha) \pm \sqrt{4\mu^2\alpha(1+\alpha) + \nu_2^2} \right)^2}{2\nu_2^2 \sqrt{4\mu^2\alpha(1+\alpha) + \nu_2^2}}. \tag{656}$$

Now it remains to substitute these values (655), along with the solution for $G^{(1)}(z)$ (564), and the momentum condition (466), into (555), and afterwards symmetrize the double sum — to immediately arrive at the desired result,

$$\begin{aligned}
 \mathcal{A}^{(2,1)}(z) = & \sum_{n \geq 1} \frac{1}{z^2((\mu^2 - \nu^2)z^2 - 2\mu(1+2\alpha)z + 1)} \left(\frac{1}{\mu^2\nu^2(z-a)(z-b)} \cdot \right. \\
 & \cdot \left(\nu^2 - 2\mu\nu^2(1+2\alpha)z + (-4\mu^4\alpha(1+\alpha)(3+16\alpha+16\alpha^2) - 2\mu^2\nu^2(1+8\alpha+8\alpha^2) - \nu^4)z^2 + \right. \\
 & \left. + 8\mu^3(1+2\alpha)^3(4\mu^2\alpha(1+\alpha) + \nu^2)z^3 + \right. \\
 & \left. + \mu^2(-4\mu^4\alpha(1+\alpha)(7+32\alpha+32\alpha^2) + \mu^2\nu^2(-7-20\alpha+44\alpha^2+128\alpha^3+64\alpha^4) + \nu^4(3+16\alpha+16\alpha^2))z^4 + \right.
 \end{aligned}$$

$$\begin{aligned}
& + 2\mu^3(1+2\alpha) \left(4\mu^4\alpha(1+\alpha) + \mu^2\nu^2(1-4\alpha-4\alpha^2) - \nu^4 \right) z^5 \Big) + \\
& + \frac{\sqrt{4\mu^2\alpha(1+\alpha)} + \nu^2}{\nu} z^2 \left(\mu^2(3+16\alpha+16\alpha^2) - 2\mu^3(1+2\alpha)z + \nu^2 + 2\mu\nu^2(1+2\alpha)z \right) + \\
& + \frac{1}{\sqrt{z-a}\sqrt{z-b}} \frac{1-\mu(1+2\alpha)z}{\mu} \Big) + \\
& + \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \frac{1}{z^2((\mu^2 - \nu_1^2)z^2 - 2\mu(1+2\alpha)z + 1)((\mu^2 - \nu_2^2)z^2 - 2\mu(1+2\alpha)z + 1)} \cdot \\
& \cdot \left(2 - 4\mu(1+2\alpha)z - (\mu^2(1+12\alpha+12\alpha^2) + \nu_1^2 + \nu_2^2)z^2 + 8\mu^3(1+2\alpha)^3z^3 + \right. \\
& + (-\mu^4(7+32\alpha+32\alpha^2) + \mu^2(\nu_1^2 + \nu_2^2)(3+16\alpha+16\alpha^2) + \nu_1^2\nu_2^2)z^4 + 2\mu(1+2\alpha)(\mu^2 - \nu_1^2)(\mu^2 - \nu_2^2)z^5 + \\
& + \frac{\sqrt{4\mu^2\alpha(1+\alpha)} + \nu_1^2}{\nu_1} z^2((\mu^2 - \nu_2^2)z^2 - 2\mu(1+2\alpha)z + 1) \cdot \\
& \cdot \left(\mu^2(3+16\alpha+16\alpha^2) - 2\mu^3(1+2\alpha)z + \nu_1^2 + 2\mu\nu_1^2(1+2\alpha)z \right) + \\
& + \frac{\sqrt{4\mu^2\alpha(1+\alpha)} + \nu_2^2}{\nu_2} z^2((\mu^2 - \nu_1^2)z^2 - 2\mu(1+2\alpha)z + 1) \cdot \\
& \cdot \left(\mu^2(3+16\alpha+16\alpha^2) - 2\mu^3(1+2\alpha)z + \nu_2^2 + 2\mu\nu_2^2(1+2\alpha)z \right) + \\
& + \frac{\sqrt{4\mu^2\alpha(1+\alpha)} + \nu_1^2 \sqrt{4\mu^2\alpha(1+\alpha)} + \nu_2^2}{\nu_1\nu_2} z^2 \left(-\mu^2(3+16\alpha+16\alpha^2) + 8\mu^3(1+2\alpha)^3z + \right. \\
& + (-\mu^4(7+32\alpha+32\alpha^2) + \mu^2(\nu_1^2 + \nu_2^2)(3+16\alpha+16\alpha^2) + \nu_1^2\nu_2^2)z^2 + 2\mu(1+2\alpha)(\mu^2 - \nu_1^2)(\mu^2 - \nu_2^2)z^3 \Big) + \\
& + \frac{1}{\sqrt{z-a}\sqrt{z-b}} \frac{1-\mu(1+2\alpha)z}{\mu} \left(2 - 4\mu(1+2\alpha)z + (2\mu^2 - \nu_1^2 - \nu_2^2)z^2 + \right. \\
& + z^2 \left(\nu_1 \sqrt{4\mu^2\alpha(1+\alpha)} + \nu_1^2 + \nu_2 \sqrt{4\mu^2\alpha(1+\alpha)} + \nu_2^2 \right) \Big) \Big) \Big). \tag{657}
\end{aligned}$$

The problem can also be approached differently, by the method of integration with the pertinent density (435),

$$\mathcal{A}^{(2,1),\text{bulk}}(z) = -\pi \int_{\mathcal{C}_y} dy \frac{1}{z-y} \rho^{(1)}(y) \rho^{(0)'}(y) \left(\coth(\pi \rho^{(0)}(y)) - \frac{1}{\pi \rho^{(0)}(y)} \right) =$$

$$\begin{aligned}
&= -2 \int_{\mathcal{C}^y} dy \frac{1}{z-y} \rho^{(1)}(y) \rho^{(0)}(y) \rho^{(0)'}(y) \sum_{n_2 \geq 1} \frac{1}{\rho^{(0)}(y)^2 + n_2^2} = \\
&= \frac{1}{\pi \mu} \int_{\mathcal{C}^y} dy \frac{1}{\sqrt{(y-a)(b-y)}} \frac{1}{z-y} \frac{1-\mu(1+2\alpha)y}{y^2} \left(\sum_{n \geq 1} \frac{1}{(\mu^2 - \nu^2)(y-a_n)(y-b_n)} + \right. \\
&\quad \left. + 2 \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \left(1 - \frac{\nu_1 y}{2} \left(\frac{a_{n_1}}{y-a_{n_1}} - \frac{b_{n_1}}{y-b_{n_1}} \right) \right) \frac{1}{(\mu^2 - \nu_2^2)(y-a_{n_2})(y-b_{n_2})} \right). \tag{658}
\end{aligned}$$

Performing partial fractional decomposition w.r.t. y , symmetrizing the double sum, and then using the integrals (590), (591), and also

$$\int_{\mathcal{C}^y} dy \frac{1}{\sqrt{(y-a)(b-y)}} \frac{1}{y^2} = \frac{\pi \sqrt{a} \sqrt{b}(a+b)}{2a^2 b^2}, \tag{659}$$

lead to the following conclusion: The two-sum parts of $\mathcal{A}^{(2,1)}(z)$ and $\mathcal{A}^{(2,1),\text{bulk}}(z)$ coincide, while the one-sum parts differ by a quantity which depends on n as $1/\nu^2$. This can be explicitly summed up, since $\sum_{n \geq 1} 1/\nu^2 = \zeta(2)/(4\pi^2) = 1/24$, and yields

$$\mathcal{A}^{(2,1),\text{bulk}}(z) = \mathcal{A}^{(2,1)}(z) - \frac{\mu^3 \sqrt{\alpha(1+\alpha)}}{24} \left((\sqrt{\alpha} + \sqrt{1+\alpha})^4 \frac{1}{z-a} - (\sqrt{\alpha} - \sqrt{1+\alpha})^4 \frac{1}{z-b} \right), \tag{660}$$

i.e., the two anomalies differ by the boundary contribution

$$\mathcal{A}^{(2,1),\text{boundary}}(z) = \frac{\mu^3 \sqrt{\alpha(1+\alpha)}}{24} \left((\sqrt{\alpha} + \sqrt{1+\alpha})^4 \frac{1}{z-a} - (\sqrt{\alpha} - \sqrt{1+\alpha})^4 \frac{1}{z-b} \right), \tag{661}$$

as anticipated due to the fact that the density $\rho^{(1)}(y)$ explodes at the endpoints of the cut.

In fact, there is no need to exploit both these techniques. It is enough *e.g.* to use integration, which is normally simpler, and to derive the boundary term (661) from the non-integrable singularities of the resolvent $G^{(1)}(z)$ (593), according to (326). Namely, instead of the second line of (658), we write

$$\begin{aligned}
\mathcal{A}^{(2,1),\text{boundary}}(z) &= -2 \oint_{\mathcal{C}^y} dz' \frac{1}{z-z'} \frac{G^{(1),\text{boundary}}(z')}{2\pi i} \rho^{(0)}(z') \rho^{(0)'}(z') \sum_{n_2 \geq 1} \frac{1}{\rho^{(0)}(z')^2 + n_2^2} = \\
&= (\text{Res}_{z'=a} + \text{Res}_{z'=b}) \left(\frac{1}{2} \left(\frac{1}{z'-a} + \frac{1}{z'-b} \right) \frac{1}{z-z'} \rho^{(0)}(z') \rho^{(0)'}(z') \sum_{n_2 \geq 1} \frac{1}{\rho^{(0)}(z')^2 + n_2^2} \right) = \\
&= \frac{1}{48} \left(\frac{1-\mu(1+2\alpha)a}{a^3} \frac{1}{z-a} + \frac{1-\mu(1+2\alpha)b}{b^3} \frac{1}{z-b} \right) = \\
&= \frac{\mu^3 \sqrt{\alpha(1+\alpha)}}{24} \left((\sqrt{\alpha} + \sqrt{1+\alpha})^4 \frac{1}{z-a} - (\sqrt{\alpha} - \sqrt{1+\alpha})^4 \frac{1}{z-b} \right). \tag{662}
\end{aligned}$$

Observe also that the second line of (662) points to yet another interpretation of the constants multiplying $1/(z-a)$ and $1/(z-b)$ in the boundary term,

$$\mathcal{A}^{(2,1),\text{boundary}}(z) = \frac{\pi^2}{24} \left(\left(\lim_{y \rightarrow a} \frac{\rho^{(0)}(y)^2}{y-a} \right) \frac{1}{z-a} + \left(\lim_{y \rightarrow b} \frac{\rho^{(0)}(y)^2}{y-b} \right) \frac{1}{z-b} \right), \tag{663}$$

i.e., they are $\pi^2/24$ times the leading terms in the series of $\rho^{(0)}(y)^2$ near $y=a$ and $y=b$, respectively.

2. The Anomaly $\mathcal{A}^{(2,2)}(z)$

At this level, there is the second anomaly contribution to be taken into account, $\mathcal{A}^{(2,2)}(z)$ (448),

$$\frac{1}{J^2} \mathcal{A}^{(2,2)}(z) + O\left(\frac{1}{J^3}\right) = -\frac{1}{J^3} \sum_{k=1}^S \frac{1}{z - y_k} \frac{d}{dy_k} \left(\pi \rho^{(1)}(y_k) \left(\coth\left(\pi \rho^{(0)}(y_k)\right) - \frac{1}{\pi \rho^{(0)}(y_k)} \right) \right), \quad (664)$$

or in the “bulk” version (451),

$$\mathcal{A}^{(2,2),\text{bulk}}(z) = -\pi \int_{\mathcal{C}_y} dy \frac{1}{z - y} \rho^{(0)}(y) \frac{d}{dy} \left(\rho^{(1)}(y) \left(\coth\left(\pi \rho^{(0)}(y)\right) - \frac{1}{\pi \rho^{(0)}(y)} \right) \right). \quad (665)$$

Since it is an integration with the large- J leading-order density $\rho^{(0)}(y)$, we do not expect any boundary difference between (664) and (665), and indeed, by explicitly computing both these objects, we have confirmed that

$$\mathcal{A}^{(2,2),\text{bulk}}(z) = \mathcal{A}^{(2,2)}(z). \quad (666)$$

Calculation in this case, by the method of summation and expressing through an appropriate resolvent, deals with

$$\begin{aligned} & \pi \rho^{(1)}(y_k) \left(\coth\left(\pi \rho^{(0)}(y_k)\right) - \frac{1}{\pi \rho^{(0)}(y_k)} \right) = \\ & = -2 \sum_{n \geq 1} \frac{a_n b_n}{a_n - b_n} \left(\frac{1}{y_k - a_n} - \frac{1}{y_k - b_n} \right) - \\ & -4 \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \left(1 - \frac{\nu_1 y_k}{2} \left(\frac{a_{n_1}}{y_k - a_{n_1}} - \frac{b_{n_1}}{y_k - b_{n_1}} \right) \right) \frac{a_{n_2} b_{n_2}}{a_{n_2} - b_{n_2}} \left(\frac{1}{y_k - a_{n_2}} - \frac{1}{y_k - b_{n_2}} \right), \end{aligned} \quad (667)$$

which we need to find the derivative w.r.t. y_k of, multiply by $1/(z - y_k)$, decompose fractionally w.r.t. y_k , symmetrize the double sum, and finally rewrite everything through the resolvent $G^{(0)}(z)$. Since we have already exhaustively explained how such a computation is done, and because here the formulae are quite lengthy, we will not print the result for $\mathcal{A}^{(2,2)}(z)$.

3. The Resolvent $G^{(2)}(z)$

According to (467) — adding to each other $G^{(1)}(z)^2 + G^{(1)'}(z)$ (568), $\mathcal{A}^{(2,1)}(z)$ (657), $\mathcal{A}^{(2,2)}(z)$ (not printed), and $\mathcal{F}(z)|_{\omega=0}$ (529), and dividing by the common denominator $\frac{\mu}{z} \sqrt{z-a} \sqrt{z-b}$ (530) — we finally arrive at the large- J next-to-next-to-leading-order resolvent, which is the most important finding of this section III,

$$\begin{aligned} G^{(2)}(z) = & \frac{1}{24\mu^5 z^3 (z-a)^{5/2} (z-b)^{5/2}} \left(-3 + 15\mu(1+2\alpha)z - 30\mu^2(1+3\alpha+3\alpha^2)z^2 + \right. \\ & \left. + 2\mu(1+2\alpha) \left(12 + \mu^2(15+16\alpha+16\alpha^2) \right) z^3 - \right. \\ & \left. - \mu^2 \left(24(2+3\alpha+3\alpha^2) + \mu^2(15+44\alpha+76\alpha^2+64\alpha^3+32\alpha^4) \right) z^4 + \right. \\ & \left. + \mu^3(1+2\alpha) \left(24 + \mu^2(3+24\alpha+152\alpha^2+256\alpha^3+128\alpha^4) \right) z^5 - \right. \\ & \left. - 2\mu^6\alpha(1+\alpha)(13+64\alpha+64\alpha^2)z^6 + 8\mu^7\alpha(1+\alpha)(1+2\alpha)z^7 \right) + \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{8\mu^4 z^3 (z-a)^2 (z-b)^2} \left(-1 + 4\mu(1+2\alpha)z - 2\mu^2(3+8\alpha+8\alpha^2)z^2 + \right. \\
& \quad \left. + 4\mu(1+2\alpha)(2+\mu^2)z^3 - \mu^2(8+\mu^2)z^4 \right) + \\
& \quad + \sum_{n \geq 1} \frac{1}{\mu^4 (z-a)^2 (z-b)^2 ((\mu^2 - \nu^2)z^2 - 2\mu(1+2\alpha)z + 1)^2} \cdot \\
& \quad \cdot \left(2 \left(2\mu(1+2\alpha) + \left(-2\mu^2(5+16\alpha+16\alpha^2) + \nu^2 \right) z + 2\mu(1+2\alpha) \left(2\mu^2(5+8\alpha+8\alpha^2) - 3\nu^2 \right) z^2 + \right. \right. \\
& \quad \left. + 4\mu^2 \left(-\mu^2(5+16\alpha+16\alpha^2) + \nu^2(3+8\alpha+8\alpha^2) \right) z^3 + \right. \\
& \quad \left. + \mu(1+2\alpha) \left(10\mu^4 - 10\mu^2\nu^2 + \nu^4 \right) z^4 + \mu^2 \left(-2\mu^4 + 3\mu^2\nu^2 - \nu^4 \right) z^5 \right) + \\
& \quad + 2\nu\sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}z \left(1 - 2\mu(1+2\alpha)z + \mu(1+2\alpha)(2\mu^2 - \nu^2)z^3 - \mu^2(\mu^2 - \nu^2)z^4 \right) + \\
& \quad + \frac{\mu}{\nu^2}\sqrt{z-a}\sqrt{z-b} \left(4\mu\nu^2(1+2\alpha) + \left(4\mu^4\alpha(1+\alpha)(3+16\alpha+16\alpha^2) - \mu^2\nu^2(17+52\alpha+52\alpha^2) + 2\nu^4 \right) z + \right. \\
& \quad \left. + \mu(1+2\alpha) \left(-8\mu^4\alpha(1+\alpha)(7+32\alpha+32\alpha^2) + 4\mu^2\nu^2(7+10\alpha+10\alpha^2) - 10\nu^4 \right) z^2 + \right. \\
& \quad \left. + \mu^2 \left(8\mu^4\alpha(1+\alpha)(13+88\alpha+216\alpha^2+256\alpha^3+128\alpha^4) - \right. \right. \\
& \quad \left. \left. - 2\mu^2\nu^2(11+46\alpha+110\alpha^2+128\alpha^3+64\alpha^4) + 4\nu^4(4+11\alpha+11\alpha^2) \right) z^3 + \right. \\
& \quad \left. + \mu(1+2\alpha) \left(-96\mu^6\alpha(1+\alpha)(1+2\alpha)^2 + 8\mu^4\nu^2(1+8\alpha+40\alpha^2+64\alpha^3+32\alpha^4) - 10\mu^2\nu^4 + 2\nu^6 \right) z^4 + \right. \\
& \quad \left. + \mu^2 \left(4\mu^6\alpha(1+\alpha)(11+48\alpha+48\alpha^2) - \mu^4\nu^2(1+56\alpha+312\alpha^2+512\alpha^3+256\alpha^4) + \right. \right. \\
& \quad \left. \left. + 2\mu^2\nu^4(1+6\alpha+38\alpha^2+64\alpha^3+32\alpha^4) - \nu^6 \right) z^5 - \right. \\
& \quad \left. - 8\mu^5\alpha(1+\alpha)(1+2\alpha)(\mu^2 - \nu^2)^2 z^6 \right) + \\
& \quad + \frac{\mu\nu}{\sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}}\sqrt{z-a}\sqrt{z-b}z \left(\mu^2(1+12\alpha+12\alpha^2) + 2\nu^2 - 2\mu(1+2\alpha) \left(2\mu^2(1+6\alpha+6\alpha^2) + \nu^2 \right) z + \right.
\end{aligned}$$

$$\begin{aligned}
& +2\mu^2 \left(\mu^2 (3 + 14\alpha + 14\alpha^2) - 2\nu^2 (1 + 5\alpha + 5\alpha^2) \right) z^2 + \\
& + 2\mu(1 + 2\alpha) \left(-2\mu^4 + 3\mu^2\nu^2 - \nu^4 \right) z^3 + \mu^2 (\mu^2 - \nu^2)^2 z^4 \Big) + \\
& + \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \frac{1}{((\mu^2 - \nu_1^2) z^2 - 2\mu(1 + 2\alpha)z + 1)^2 ((\mu^2 - \nu_2^2) z^2 - 2\mu(1 + 2\alpha)z + 1)^2} \cdot \\
& \cdot \left(\frac{1}{\mu^3(z - a)^{3/2}(z - b)^{3/2}} \left(4\mu(1 + 2\alpha) + \left(-\mu^2 (33 + 116\alpha + 116\alpha^2) + 2(\nu_1^2 + \nu_2^2) \right) z + \right. \right. \\
& \quad \left. \left. + 6\mu(1 + 2\alpha) \left(4\mu^2 (5 + 13\alpha + 13\alpha^2) - 3(\nu_1^2 + \nu_2^2) \right) z^2 + \right. \right. \\
& \quad \left. \left. + 4 \left(-\mu^4 (63 + 331\alpha + 699\alpha^2 + 736\alpha^3 + 368\alpha^4) + \mu^2 (\nu_1^2 + \nu_2^2) (17 + 59\alpha + 59\alpha^2) - \nu_1^2 \nu_2^2 \right) z^3 + \right. \right. \\
& \quad \left. \left. + 2\mu(1 + 2\alpha) \left(8\mu^4 (21 + 73\alpha + 113\alpha^2 + 80\alpha^3 + 40\alpha^4) - \mu^2 (\nu_1^2 + \nu_2^2) (71 + 168\alpha + 168\alpha^2) + \nu_1^4 + 13\nu_1^2 \nu_2^2 + \nu_2^4 \right) z^4 + \right. \right. \\
& \quad \left. \left. + \mu^2 \left(-6\mu^4 (49 + 258\alpha + 546\alpha^2 + 576\alpha^3 + 288\alpha^4) - (\nu_1^4 + \nu_2^4) (9 + 32\alpha + 32\alpha^2) - \right. \right. \right. \\
& \quad \left. \left. \left. - 4\nu_1^2 \nu_2^2 (17 + 57\alpha + 57\alpha^2) + 4\mu^2 (\nu_1^2 + \nu_2^2) (45 + 210\alpha + 386\alpha^2 + 352\alpha^3 + 176\alpha^4) \right) z^5 + \right. \right. \\
& \quad \left. \left. + 2\mu(1 + 2\alpha) \left(4\mu^6 (21 + 55\alpha + 55\alpha^2) - \mu^4 (\nu_1^2 + \nu_2^2) (71 + 168\alpha + 168\alpha^2) + \right. \right. \right. \\
& \quad \left. \left. \left. 2\mu^2 (4(\nu_1^4 + \nu_2^4) (1 + 2\alpha + 2\alpha^2) + \nu_1^2 \nu_2^2 (23 + 42\alpha + 42\alpha^2)) - 2\nu_1^2 \nu_2^2 (\nu_1^2 + \nu_2^2) \right) z^6 + \right. \right. \\
& \quad \left. \left. + 2\mu^2 \left(-2\mu^6 (15 + 53\alpha + 53\alpha^2) + 2\mu^4 (\nu_1^2 + \nu_2^2) (17 + 59\alpha + 59\alpha^2) - \right. \right. \right. \\
& \quad \left. \left. \left. - \mu^2 ((\nu_1^4 + \nu_2^4) (7 + 24\alpha + 24\alpha^2) + 2\nu_1^2 \nu_2^2 (17 + 57\alpha + 57\alpha^2)) + \nu_1^2 \nu_2^2 (\nu_1^2 + \nu_2^2) (5 + 16\alpha + 16\alpha^2) \right) z^7 + \right. \right. \\
& \quad \left. \left. + 2\mu(1 + 2\alpha) (\mu^2 - \nu_1^2) (\mu^2 - \nu_2^2) \left(6\mu^4 - 3\mu^2 (\nu_1^2 + \nu_2^2) + \nu_1^2 \nu_2^2 \right) z^8 - \right. \right. \\
& \quad \left. \left. - \mu^2 (\mu^2 - \nu_1^2)^2 (\mu^2 - \nu_2^2)^2 z^9 + \right. \right. \\
& \quad \left. \left. + \frac{\nu_1}{\sqrt{4\mu^2\alpha(1 + \alpha) + \nu_1^2}} z ((\mu^2 - \nu_2^2) z^2 - 2\mu(1 + 2\alpha)z + 1)^2 \cdot \right. \right. \\
& \quad \left. \cdot \left(\mu^2 (1 + 12\alpha + 12\alpha^2) + 2\nu_1^2 - 2\mu(1 + 2\alpha) \left(2\mu^2 (1 + 6\alpha + 6\alpha^2) + \nu_1^2 \right) z + \right. \right.
\end{aligned}$$

$$\begin{aligned}
& +2\mu^2 \left(\mu^2 (3 + 14\alpha + 14\alpha^2) - 2\nu_1^2 (1 + 5\alpha + 5\alpha^2) \right) z^2 + \\
& + 2\mu(1 + 2\alpha) \left(-2\mu^4 + 3\mu^2\nu_1^2 - \nu_1^4 \right) z^3 + \mu^2 (\mu^2 - \nu_1^2)^2 z^4 \Big) + \\
& + \frac{\nu_2}{\sqrt{4\mu^2\alpha(1+\alpha)} + \nu_2^2} z \left((\mu^2 - \nu_1^2) z^2 - 2\mu(1 + 2\alpha)z + 1 \right)^2 \cdot \\
& \cdot \left(\mu^2 (1 + 12\alpha + 12\alpha^2) + 2\nu_2^2 - 2\mu(1 + 2\alpha) \left(2\mu^2 (1 + 6\alpha + 6\alpha^2) + \nu_2^2 \right) z + \right. \\
& + 2\mu^2 \left(\mu^2 (3 + 14\alpha + 14\alpha^2) - 2\nu_2^2 (1 + 5\alpha + 5\alpha^2) \right) z^2 + \\
& + 2\mu(1 + 2\alpha) \left(-2\mu^4 + 3\mu^2\nu_2^2 - \nu_2^4 \right) z^3 + \mu^2 (\mu^2 - \nu_2^2)^2 z^4 \Big) - \\
& - \frac{\nu_1\nu_2}{\sqrt{4\mu^2\alpha(1+\alpha)} + \nu_1^2\sqrt{4\mu^2\alpha(1+\alpha)} + \nu_2^2} z \left(\mu^2(1 + 2\alpha)^2 - 8\mu^3(1 + 2\alpha)^3 z + \right. \\
& + \left(4\mu^4 (7 + 51\alpha + 127\alpha^2 + 152\alpha^3 + 76\alpha^4) - 2\mu^2 (\nu_1^2 + \nu_2^2) (1 + 12\alpha + 12\alpha^2) - 4\nu_1^2\nu_2^2 \right) z^2 - \\
& - 2\mu(1 + 2\alpha) \left(4\mu^4 (7 + 41\alpha + 65\alpha^2 + 48\alpha^3 + 24\alpha^4) - 2\mu^2 (\nu_1^2 + \nu_2^2) (3 + 26\alpha + 26\alpha^2) - 7\nu_1^2\nu_2^2 \right) z^3 + \\
& + \left(2\mu^6 (35 + 270\alpha + 702\alpha^2 + 736\alpha^3 + 48\alpha^4 - 384\alpha^5 - 128\alpha^6) - 2\mu^4 (\nu_1^2 + \nu_2^2) (15 + 136\alpha + 368\alpha^2 + 464\alpha^3 + 232\alpha^4) + \right. \\
& + \mu^2 ((\nu_1^4 + \nu_2^4) (1 + 12\alpha + 12\alpha^2) - 4\nu_1^2\nu_2^2 (3 + \alpha + \alpha^2)) + 2\nu_1^2\nu_2^2 (\nu_1^2 + \nu_2^2) \Big) z^4 - \\
& - 2\mu(1 + 2\alpha) \left(4\mu^6 (7 + 34\alpha + 42\alpha^2 + 16\alpha^3 + 8\alpha^4) - 4\mu^4 (\nu_1^2 + \nu_2^2) (5 + 26\alpha + 30\alpha^2 + 8\alpha^3 + 4\alpha^4) + \right. \\
& + 2\mu^2 ((\nu_1^4 + \nu_2^4) (1 + 6\alpha + 6\alpha^2) + 3\nu_1^2\nu_2^2 (1 + 8\alpha + 8\alpha^2)) + \nu_1^2\nu_2^2 (\nu_1^2 + \nu_2^2) \Big) z^5 + \\
& + 2\mu^2 \left(2\mu^6 (7 + 37\alpha + 65\alpha^2 + 56\alpha^3 + 28\alpha^4) - \mu^4 (\nu_1^2 + \nu_2^2) (15 + 76\alpha + 116\alpha^2 + 80\alpha^3 + 40\alpha^4) + \right. \\
& + \mu^2 ((\nu_1^4 + \nu_2^4) (3 + 14\alpha + 14\alpha^2) + 2\nu_1^2\nu_2^2 (7 + 35\alpha + 47\alpha^2 + 24\alpha^3 + 12\alpha^4)) - \\
& - 2\nu_1^2\nu_2^2 (\nu_1^2 + \nu_2^2) (1 + 5\alpha + 5\alpha^2) \Big) z^6 - \\
& - 2\mu(1 + 2\alpha) (\mu^2 - \nu_1^2) (\mu^2 - \nu_2^2) \left(4\mu^4 (1 + \alpha + \alpha^2) - 2\mu^2 (\nu_1^2 + \nu_2^2) + \nu_1^2\nu_2^2 \right) z^7 +
\end{aligned}$$

$$\begin{aligned}
& + \mu^2 (\mu^2 - \nu_1^2)^2 (\mu^2 - \nu_2^2)^2 z^8 \Big) + \\
& + 2 \Big(2\mu(1+2\alpha) + \Big(-2\mu^2(5+16\alpha+16\alpha^2) + \nu_1^2 + \nu_2^2 \Big) z + \\
& + 2\mu(1+2\alpha) \Big(2\mu^2(5+8\alpha+8\alpha^2) - 3(\nu_1^2 + \nu_2^2) \Big) z^2 + \\
& + \Big(-4\mu^4(5+16\alpha+16\alpha^2) + 4\mu^2(\nu_1^2 + \nu_2^2)(3+8\alpha+8\alpha^2) - 4\nu_1^2\nu_2^2 \Big) z^3 + \\
& + \mu(1+2\alpha) \Big(10\mu^4 - 10\mu^2(\nu_1^2 + \nu_2^2) + \nu_1^4 + 8\nu_1^2\nu_2^2 + \nu_2^4 \Big) z^4 + \\
& + \Big(-2\mu^6 + 3\mu^4(\nu_1^2 + \nu_2^2) - \mu^2(\nu_1^4 + 4\nu_1^2\nu_2^2 + \nu_2^4) + \nu_1^2\nu_2^2(\nu_1^2 + \nu_2^2) \Big) z^5 + \\
& + \Big(\nu_1\sqrt{4\mu^2\alpha(1+\alpha) + \nu_1^2} + \nu_2\sqrt{4\mu^2\alpha(1+\alpha) + \nu_2^2} \Big) z \cdot \\
& \cdot \Big(1 - 2\mu(1+2\alpha)z + \mu(1+2\alpha) \Big(2\mu^2 - \nu_1^2 - \nu_2^2 \Big) z^3 + \Big(-\mu^4 + \mu^2(\nu_1^2 + \nu_2^2) - \nu_1^2\nu_2^2 \Big) z^4 \Big) \Big). \tag{668}
\end{aligned}$$

The basic features of this solution:

- The behavior at infinity of the various parts of the resolvent is

$$G^{(2)}(z) \Big|_{\text{the no-sum part}} = \frac{1}{3}\mu^2\alpha(1+\alpha)(1+2\alpha)\frac{1}{z} + O\left(\frac{1}{z^2}\right), \quad \text{for } z \rightarrow \infty, \tag{669}$$

$$G^{(2)}(z) \Big|_{\text{the one-sum part}} = -\frac{8}{\nu^2}\mu^2\alpha(1+\alpha)(1+2\alpha)\frac{1}{z} + O\left(\frac{1}{z^2}\right), \quad \text{for } z \rightarrow \infty, \tag{670}$$

$$G^{(2)}(z) \Big|_{\text{the two-sum part}} = O\left(\frac{1}{z^2}\right), \quad \text{for } z \rightarrow \infty, \tag{671}$$

which, upon using $\sum_{n \geq 1} 1/\nu^2 = 1/24$, gives in total

$$G^{(2)}(z) = O\left(\frac{1}{z^2}\right), \quad \text{for } z \rightarrow \infty, \tag{672}$$

identically as for $G^{(1)}(z)$ (565), which is necessary in order not to change the correct $1/z$ term of the large- z series of $G^{(0)}(z)$ (517). This is actually how we have been led to discover the existence of boundary contributions such as $\mathcal{A}^{(2,1),\text{boundary}}(z)$ (661); we have initially computed the resolvent $G^{(2)}(z)$ using only the bulk anomalies, and we observed its wrong large- z behavior, deprived of the term

$$\frac{-\mathcal{A}^{(2,1),\text{boundary}}(z)}{\frac{\mu}{z}\sqrt{z-a}\sqrt{z-b}} = -\frac{1}{3}\mu^2\alpha(1+\alpha)(1+2\alpha)\frac{1}{z} + O\left(\frac{1}{z^2}\right), \quad \text{for } z \rightarrow \infty. \tag{673}$$

- Taking the limit $z \rightarrow 0$ of the resolvent, we find that the one-sum and two-sum parts yield zero, while the no-sum part reproduces precisely the momentum condition (528).

- We have not attempted to look for any more general expressions for derivatives of this resolvent, except for the values at zero of the first and second ones, which after some simplifications read

$$G^{(2)'}(0) = -\mu^2\alpha(1+\alpha) \left(1 + \frac{1}{12}\mu^2(4+21\alpha+21\alpha^2) \right) + \\ + \mu^2(1+2\alpha)^2 \sum_{n \geq 1} \left(1 - \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2}} \right) + \mu^2(1+2\alpha)^2 \left(\sum_{n \geq 1} \left(1 - \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2}} \right) \right)^2, \quad (674)$$

and

$$G^{(2)''}(0) = -\frac{1}{2}\mu^3\alpha(1+\alpha)(1+2\alpha)(28+\mu^2(3+22\alpha+22\alpha^2)) + \\ + 6\mu^3(1+2\alpha)^3 \sum_{n \geq 1} \left(1 - \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2}} \right) + 6\mu^3(1+2\alpha)^3 \left(\sum_{n \geq 1} \left(1 - \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2}} \right) \right)^2. \quad (675)$$

Notice that in both cases, the two-sum parts are reduced to a single sum, the same as in the one-sum parts, squared. Recall that there have been proportionality relations between $G^{(0)''}(0)$ and $G^{(0)'}(0)$ (521), with the constant $2\mu(1+2\alpha)$, as well as between $G^{(1)''}(0)$ and $G^{(1)'}(0)$ (581), with the constant $6\mu(1+2\alpha)$; here, the one-sum and two-sum parts indeed display such a proportionality, with the constant $6\mu(1+2\alpha)$, but the no-sum part unfortunately does not,

$$G^{(2)''}(0) = 6\mu(1+2\alpha) \left(G^{(2)'}(0) - \frac{1}{12}\mu^2\alpha(1+\alpha)(16+\mu^2(-1+\alpha+\alpha^2)) \right). \quad (676)$$

Furthermore, these values allow us to find the next-to-next-to-leading-order terms of the local conserved charges (339),

$$Q_t|_{\omega=0}|_{\text{next-to-next-to-leading}} = -\frac{G^{(2)''\dots'(t-1)}(0) - \frac{1}{24}G^{(0)''\dots'(t+1)}(0)}{(t-1)!}, \quad \text{for } t = 1, 2, \dots, \quad (677)$$

which for $t = 2, 3$, upon using (523), (524), are

$$Q_2|_{\omega=0}|_{\text{next-to-next-to-leading}} = \mu^2\alpha(1+\alpha) \left(1 + \frac{1}{12}\mu^2(1+6\alpha+6\alpha^2) \right) - \\ - \mu^2(1+2\alpha)^2 \sum_{n \geq 1} \left(1 - \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2}} \right) - \mu^2(1+2\alpha)^2 \left(\sum_{n \geq 1} \left(1 - \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2}} \right) \right)^2, \quad (678)$$

and

$$Q_3|_{\omega=0}|_{\text{next-to-next-to-leading}} = \mu^3\alpha(1+\alpha)(1+2\alpha) \left(7 + \frac{1}{4}\mu^2(1+8\alpha+8\alpha^2) \right) - \\ - 3\mu^3(1+2\alpha)^3 \sum_{n \geq 1} \left(1 - \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2}} \right) - 3\mu^3(1+2\alpha)^3 \left(\sum_{n \geq 1} \left(1 - \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2}} \right) \right)^2. \quad (679)$$

We plot them as functions of $\alpha \in [0, 1]$ in figure 28. Moreover, Gromov and Kazakov [167] give several numerical values of the second charge, and we have reproduced them with our formula (678), as shown in table II.

- The linear equation satisfied by this resolvent is checked to be (484),

$$\lim_{\epsilon \rightarrow 0^+} \left(G^{(2)}(y + i\epsilon) + G^{(2)}(y - i\epsilon) \right) = -\frac{d}{dy} \left(\pi\rho^{(1)}(y) \coth \left(\pi\rho^{(0)}(y) \right) \right) - \frac{1}{4y^3}, \quad \text{for } y \in \mathcal{C}^y. \quad (680)$$

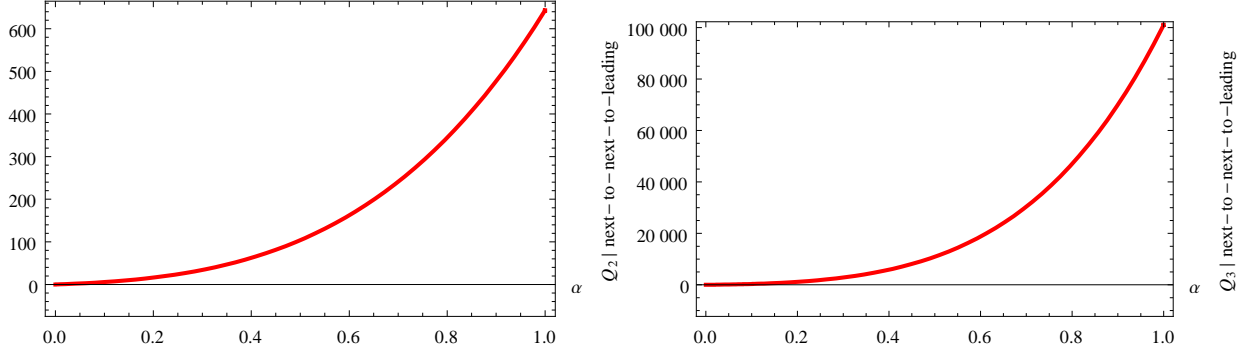


FIG. 28: Graphs of the next-to-next-to-leading-order terms of the local conserved charges (677), for $t = 2, 3$, according to (678), (679), as functions of $\alpha \in [0, 1]$, for $m = 1$. The infinite sums over n have been terminated at $n_{\max} = 10^4$.

(m, α)	(2, 0.5)	(1, 2)	(3, 1/3)	(2, 1)	(5, 0.2)
Gromov and Kazakov	1160	5464	1592	8982	1504
our formula (678) with $n_{\max} = 10^4$	1165.04	5476.17	1598.18	9009.81	1528.67
our formula (678) with $n_{\max} = 10^5$	1162.66	5465.69	1592.47	8986.60	1508.13
our formula (678) with $n_{\max} = 10^6$	1162.42	5464.64	1591.90	8984.28	1506.08
our formula (678) with $n_{\max} = 10^7$	1162.40	5464.53	1591.84	8984.04	1505.87

TABLE II: Comparison of the several values of the second charge $Q_2|_{\omega=0}|_{\text{next-to-next-to-leading}}$ as given by Gromov and Kazakov [167], and as found from (678). We also show how the infinite sums over n in (678) converge by considering their four different truncations, to $n_{\max} = 10^4$, $n_{\max} = 10^5$, $n_{\max} = 10^6$, $n_{\max} = 10^7$. We observe a very good agreement.

4. The Density $\rho^{(2)}(y)$

It is straightforward to read the large- J next-to-next-to-leading-order density from the above resolvent (668) — it is comprised of the part of the resolvent containing the square root $\sqrt{z-a}\sqrt{z-b}$, where each $1/((z-a)^{5/2}(z-b)^{5/2})$ is replaced by $(1/\pi)1/((y-a)(b-y))^{5/2}$, and each $1/((z-a)^{3/2}(z-b)^{3/2})$ by $(-1/\pi)1/((y-a)(b-y))^{3/2}$, and thus, we will not explicitly print it. Some of its plots are presented in figure 29.

Again, this density explodes at the endpoints of the cut. This time it always is a divergence to plus infinity, and more severe than for $\rho^{(1)}(y)$, namely as $1/((y-a)(b-y))^{5/2}$,

$$\rho^{(2)}(y) \sim \frac{1}{\pi} \frac{5\alpha(1+\alpha)(\sqrt{\alpha} \mp \sqrt{1+\alpha})^2}{\mu^4} \frac{1}{((y-a)(b-y))^{5/2}} - \frac{1}{\pi} \frac{(\sqrt{\alpha} \pm \sqrt{1+\alpha})^2}{\mu^2} \left(\sum_{n \geq 1} \left(1 - \frac{\sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}}{\nu} \right) + \left(\sum_{n \geq 1} \left(1 - \frac{\sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}}{\nu} \right) \right)^2 \right) \cdot \frac{1}{((y-a)(b-y))^{3/2}}, \quad \text{for } y \sim a, b. \quad (681)$$

If we want to compute $G^{(2),\text{bulk}}(z)$, we face the problem that $\rho^{(2)}(y)$ contains a part which is a non-integrable function over \mathcal{C}^y , *i.e.*, which diverges at the edges of the cut like $1/((y-a)(b-y))^p$, where $p \geq 1$ (here, p is an integer or a half-integer),

$$\rho^{(2)}(y) = \rho^{(2)}(y) \Big|_{\text{integrable}} + \rho^{(2)}(y) \Big|_{\text{non-integrable singularities}}. \quad (682)$$

The integrable part can be handled as before, and it happens that the right way to deal with the non-integrable one is to change line integration with contour integration; define, thus,

$$G^{(2),\text{bulk}}(z) \equiv \int_{\mathcal{C}^y} dy \rho^{(2)}(y) \Big|_{\text{integrable}} \frac{1}{z-y} + \oint_{\mathcal{C}^y} dy \tilde{\rho}^{(2)}(y) \Big|_{\text{non-integrable singularities}} \frac{1}{z-y}, \quad (683)$$

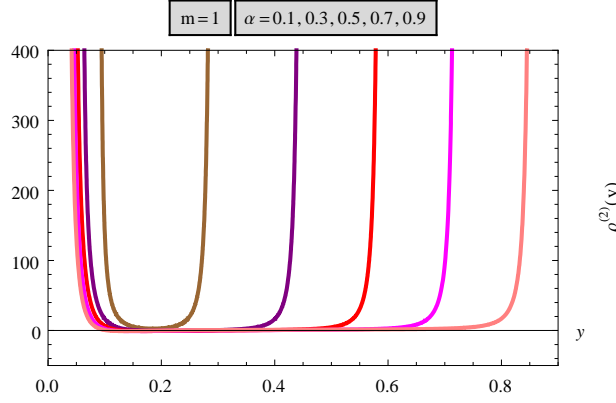


FIG. 29: The density $\mathcal{C}^y \ni y \mapsto \rho^{(2)}(y)$ for $m = 1$ and five values of α : 0.1 (brown), 0.3 (purple), 0.5 (red), 0.7 (magenta), 0.9 (pink). The infinite sums over n have been terminated at $n_{\max} = 20$, which is enough to very accurately reproduce it.

where the tilde in the second term indicates that we have changed all the square roots $\sqrt{(y-a)(b-y)}$ to $\sqrt{y-a}\sqrt{y-b}$, and y has become complex. Now, by a direct calculation, we can show that

$$\int_{\mathcal{C}^y} dy \rho^{(2)}(y) \Big|_{\text{integrable}} \frac{1}{z-y} = G^{(2)}(z) \Big|_{\text{integrable}}, \quad (684)$$

$$\oint_{\mathcal{C}^y} dy \tilde{\rho}^{(2)}(y) \Big|_{\substack{\text{non-integrable} \\ \text{singularities}}} \frac{1}{z-y} = G^{(2)}(z) \Big|_{\substack{\text{non-integrable} \\ \text{half-integer-power} \\ \text{singularities}}}. \quad (685)$$

This implies that the bulk resolvent differs from the full one by a boundary term which is precisely the part of the full resolvent with non-integrable integer-power singularities,

$$G^{(2),\text{boundary}}(z) = G^{(2)}(z) \Big|_{\substack{\text{non-integrable} \\ \text{integer-power} \\ \text{singularities}}} \quad (686)$$

just as it was true for $G^{(1)}(z)$ (594). Explicitly, these terms come from the no-sum and one-sum pieces of the resolvent, and yield

$$G^{(2),\text{boundary}}(z) = \frac{1}{(z-a)^2(z-b)^2} \frac{2(1+2\alpha-\mu z)}{\mu^3} \left(\frac{1}{2} + \sum_{n \geq 1} \left(1 - \frac{\sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}}{\nu} \right) \right). \quad (687)$$

Let us illustrate this calculation by presenting it for the no-sum piece. The no-sum part of the density can be split as

$$\rho^{(2)}(y) \Big|_{\text{no-sum}} \Big|_{\text{integrable}} = \frac{1}{\pi} \frac{1}{\sqrt{(y-a)(b-y)}} \frac{1}{24\mu y^3} \cdot (-3 + 3\mu(1+2\alpha)y + 6\mu^2\alpha(1+\alpha)y^2 + 8\mu^3\alpha(1+\alpha)(1+2\alpha)y^3), \quad (688)$$

$$\tilde{\rho}^{(2)}(y) \Big|_{\text{no-sum}} \Big|_{\substack{\text{non-integrable} \\ \text{singularities}}} = \frac{1}{\pi} \frac{1}{\sqrt{y-a}\sqrt{y-b}} \left(\frac{5\mu\alpha(1+\alpha)y}{\mu^4(y-a)^2(y-b)^2} + \frac{1+2\alpha}{\mu^2(y-a)(y-b)} \right). \quad (689)$$

For the no-sum part of the resolvent,

$$G^{(2)}(z) \Big|_{\text{no-sum}} \Big|_{\text{integrable}} = \frac{1}{\sqrt{z-a}\sqrt{z-b}} \frac{1}{24\mu z^3} \cdot (-3 + 3\mu(1+2\alpha)z + 6\mu^2\alpha(1+\alpha)z^2 + 8\mu^3\alpha(1+\alpha)(1+2\alpha)z^3) - \frac{1}{8z^3}, \quad (690)$$

$$G^{(2)}(z) \Big|_{\text{no-sum}} \Big|_{\substack{\text{non-integrable} \\ \text{half-integer-power} \\ \text{singularities}}} = \frac{1}{\sqrt{z-a}\sqrt{z-b}} \left(\frac{5\mu\alpha(1+\alpha)z}{\mu^4(z-a)^2(z-b)^2} + \frac{1+2\alpha}{\mu^2(z-a)(z-b)} \right), \quad (691)$$

$$G^{(2)}(z) \Big|_{\text{no-sum}} \Big|_{\substack{\text{non-integrable} \\ \text{integer-power} \\ \text{singularities}}} = \frac{1+2\alpha-\mu z}{\mu^3(z-a)^2(z-b)^2}. \quad (692)$$

Now, the integrable part of the no-sum density leads to the integrable part of the no-sum resolvent (684), as can be checked with help of the integrals (590), (591), (659), and also

$$\int_{\mathcal{C}_y} dy \frac{1}{\sqrt{(y-a)(b-y)}} \frac{1}{y^3} = \frac{\pi\sqrt{a}\sqrt{b}(3a^2+2ab+3b^2)}{8a^3b^3}. \quad (693)$$

For the non-integrable part of the no-sum density, we deform the integration contour so that it encircles the point at infinity. The residue at infinity is, however, zero, and only the residue at $y = z$ contributes, being responsible for (685). An analogous reasoning can be performed for the one-sum and two-sum pieces as well.

As we already know (see paragraph III B 4), the non-integrable integer-power singularities of the resolvent can be read from the pertinent linear equation. And indeed, we check that here (680),

$$\lim_{\epsilon \rightarrow 0^+} \left(G^{(2)}(y+i\epsilon) + G^{(2)}(y-i\epsilon) \right) = - \underbrace{\frac{d}{dy} \left(\sum_{n \geq 1} \frac{2\rho^{(0)}(y)\rho^{(1)}(y)}{\rho^{(0)}(y)^2 + n^2} \right)}_{\substack{\text{regular} \\ \text{at the endpoints}}} - \frac{1}{4y^3} \underbrace{\frac{d}{dy} \left(\frac{\rho^{(1)}(y)}{\rho^{(0)}(y)} \right)}_{\substack{\text{includes a part} \\ \text{singular} \\ \text{at the endpoints}}}, \quad (694)$$

leads exactly to (687).

At the end, notice that

$$\lim_{z \rightarrow \infty} \left(z G^{(2), \text{boundary}}(z) \right) = 0, \quad (695)$$

which means that the generalized normalization formula for the density in question gives zero,

$$\int_{\mathcal{C}_y} dy \rho^{(2)}(y) \Big|_{\text{integrable}} + \oint_{\mathcal{C}_y} dy \tilde{\rho}^{(2)}(y) \Big|_{\substack{\text{non-integrable} \\ \text{singularities}}} = 0, \quad (696)$$

even though the area under its plot is, of course, infinite.

5. The Large-Mode-Number, Fixed-Winding-Number Limit

In paragraph III B 5, we have extensively discussed how to correctly apply the large-mode-number, fixed-winding-number limit to the anomaly $\mathcal{A}^{(1), \text{bulk}}(z)$. To work with the bulk anomaly $\mathcal{A}^{(2), \text{bulk}}(z)$ (452) and the boundary anomaly $\mathcal{A}^{(2), \text{boundary}}(z)$ (433), one needs to know the density $\rho^{(1)}(y)$ and the boundary resolvent $G^{(1), \text{boundary}}(z)$. As explained, we can rely only on the leading small- α terms of these objects, and for this, it is enough to consider the approximation “ $\coth \approx 1 + 1/\pi\rho$ ” (632). In this paragraph, we will see further evidence of problems caused by subleading orders of the density in our most accurate approximation “ $\coth \approx 1 + \delta$.”

Our goal for now will be to find the leading small- α term of the second charge $Q_2|_{\omega=0}|_{\text{next-to-next-to-leading}}$ (678), which is related to minus the derivative at $z = 0$ of the resolvent $G^{(2)}(z)$ (467). Looking at this last formula, it is clear

that there are four contributions to the charge, *i.e.*, from $\mathcal{F}(z)|_{\omega=0}$, from $(G^{(1)}(z)^2 + G^{(1)'}(z))$, from $\mathcal{A}^{(2),\text{boundary}}(z)$, and from $\mathcal{A}^{(2),\text{bulk}}(z)$.

Since the resolvent in question has the form $G^{(2)}(z) = zf(z)/(\mu\sqrt{z-a}\sqrt{z-b})$, it naively seems that $G^{(2)'}(0) = -f(0)$. This, however, is only true when $f(0)$ and $f'(0)$ are finite, which holds for three out four contributions mentioned above, with an exception of $\mathcal{F}(z)|_{\omega=0}$. This one should then be handled separately, and yields

$$\left. \frac{d}{dz} \left(\frac{\mathcal{F}(z)|_{\omega=0}}{\frac{\mu}{z}\sqrt{z-a}\sqrt{z-b}} \right) \right|_{z=0} = \frac{1}{12}\mu^4\alpha(1+\alpha)(2+11\alpha+11\alpha^2) = \frac{1}{\alpha^3}\frac{\varpi^4}{6} + \mathcal{O}\left(\frac{1}{\alpha^2}\right). \quad (697)$$

The contribution from the resolvent $G^{(1)}(z)$ is found simply from the momentum condition (469) and the series (631),

$$-G^{(1)}(0)^2 - G^{(1)'}(0) = -G^{(1)'}(0) = -\frac{1}{\alpha^{3/2}}\frac{4\varpi^3}{3\pi} + \mathcal{O}\left(\frac{1}{\alpha}\right). \quad (698)$$

(One can check that the momentum condition can still be used, even if we work with the approximation “ $\coth \approx 1 + \delta$.”) This is much smaller than (697), and will, therefore, be negligible for our purposes.

The boundary anomaly $\mathcal{A}^{(2),\text{boundary}}(z)$ is known exactly (661), but, as discussed, only its leading small- α term is certain. We get

$$\mathcal{A}^{(2),\text{boundary}}(0) = \mathcal{A}^{(2,1),\text{boundary}}(0) = -\frac{1}{6}\mu^4\alpha(1+\alpha)(1+4\alpha)(3+4\alpha) = -\frac{1}{\alpha^3}\frac{\varpi^4}{2} + \mathcal{O}\left(\frac{1}{\alpha^2}\right). \quad (699)$$

Let us now proceed to the most problematic part, *i.e.*, the contribution from the bulk anomaly $\mathcal{A}^{(2),\text{bulk}}(z)$. The first step is to integrate by parts (we assume $\mu > 0$),

$$\begin{aligned} \mathcal{A}^{(2),\text{bulk}}(z) &= -\pi \int_a^b dy \frac{1}{z-y} \frac{d}{dy} \left(\rho^{(0)}(y)\rho^{(1)}(y) \left(\coth(\pi\rho^{(0)}(y)) - \frac{1}{\pi\rho^{(0)}(y)} \right) \right) = \\ &= \pi \int_a^b dy \frac{1}{(z-y)^2} \rho^{(0)}(y)\rho^{(1)}(y) \left(\coth(\pi\rho^{(0)}(y)) - \frac{1}{\pi\rho^{(0)}(y)} \right). \end{aligned} \quad (700)$$

In doing this, we have noticed that the boundary piece from such an integration is zero when we work with the exact expression $\coth(\pi\rho^{(0)}(y)) - 1/(\pi\rho^{(0)}(y))$. If, however, we try to use either “ $\coth \approx 1 + \delta$ ” or “ $\coth \approx 1 + 1/\pi\rho$,” the boundary piece is no longer zero, and in the former case, it even diverges. We will disregard this, sticking to the leading order of the result only, similarly as we have done for $G^{(1),\text{boundary}}(z)$ at the end of paragraph III B 5. We know that to arrive at the leading term, it is enough to approximate “ $\coth \approx 1 + 1/\pi\rho$ ” (632), hence, we consider

$$\mathcal{A}^{(2),\text{bulk}}(0) = \pi \int_a^b dy \frac{1}{y^2} \rho^{(0)}(y)\rho^{(1)}(y) + \mathcal{O}(\text{subleading}). \quad (701)$$

Changing the integration variable y to t (539), as customary, and exploiting the leading orders of the two densities (541) and (628), we easily find

$$\begin{aligned} \mathcal{A}^{(2),\text{bulk}}(0) &= \pi \int_0^1 dt \frac{b-a}{(a+(b-a)t)^2} \rho^{(0)}(t)\rho^{(1)}(t) + \mathcal{O}(\text{subleading}) = \\ &= \frac{1}{\alpha^3} \frac{\varpi^4}{\pi^2} \int_0^1 dt \left(2 + (-1+2t) \log\left(\frac{1-t}{t}\right) \right) + \mathcal{O}\left(\frac{1}{\alpha^{5/2}}\right) = \frac{1}{\alpha^3} \frac{\varpi^4}{\pi^2} + \mathcal{O}\left(\frac{1}{\alpha^{5/2}}\right). \end{aligned} \quad (702)$$

Let us emphasize that doing the approximation “ $\coth \approx 1 + 1/\pi\rho$ ” is possible only after the integration by parts, since otherwise we would have gotten a term of order $\mathcal{O}(1/\alpha^{7/2})$ being logarithmically divergent. These complications add to our list of problems caused by the subleading terms of the density $\rho^{(1)}(t)$ in any of our three approximations from paragraph III B 5. As excellent as they are in the bulk, to be able to use them, one would need to discover some

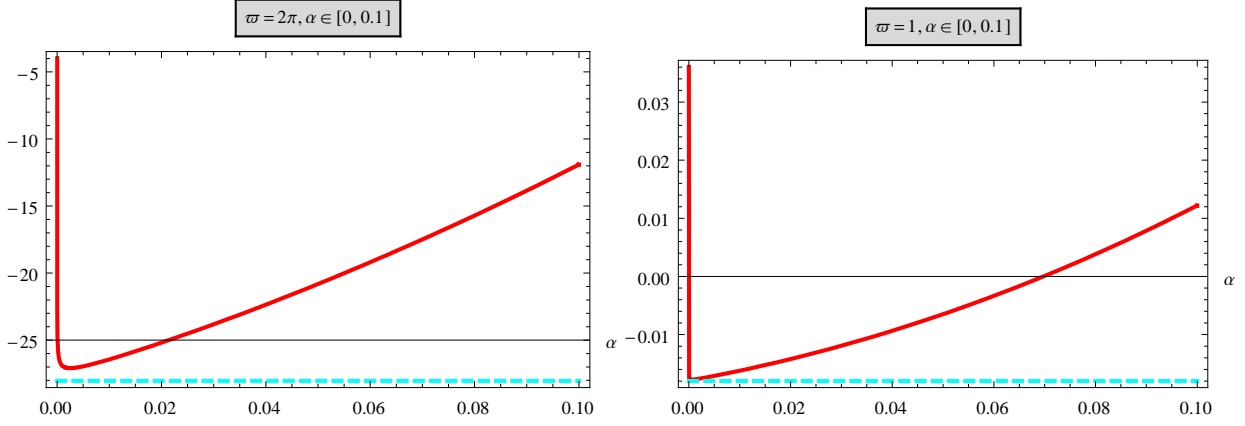


FIG. 30: Numerical check of the leading small- α term of the second charge (705).

We plot $[0, 0.1] \ni \alpha \mapsto \alpha^3 Q_2|_{\omega=0}|_{\text{next-to-next-to-leading}}$, which should be finite at $\alpha = 0$. (The explosion of the exact quantity at $\alpha = 0$ is a numerical artifact.) The exact value (678) (solid red line) is compared to the leading order of the series (705) (dashed cyan).

We have chosen $\varpi = 2\pi$ (LEFT) and $\varpi = 1$ (RIGHT). This latter value is unphysical. The summation over n in the exact quantity is truncated at $n_{\max} = 10^4$.

further approximation; the usability of the leading density has, on the other hand, been extensively tested, and our end result here (705) will confirm it once again.

Collecting the leading terms of (697), (699), (702), we finally arrive at the large-mode-number, fixed-winding-number series of

$$G^{(2)'}(0) = \frac{1}{\alpha^3} \varpi^4 \left(\frac{1}{\pi^2} - \frac{1}{3} \right) + O\left(\frac{1}{\alpha^{5/2}}\right). \quad (703)$$

In order to get from it the second charge, we must add to it

$$-\frac{1}{24} G^{(0)'''}(0) = \frac{1}{4} \mu^4 \alpha (1 + \alpha) (1 + 5\alpha + 5\alpha^2) = \frac{1}{\alpha^3} \frac{\varpi^4}{4} + O\left(\frac{1}{\alpha^2}\right), \quad (704)$$

and negate the everything, as (341) states, which yields

$$Q_2|_{\omega=0}|_{\text{next-to-next-to-leading}} = \frac{1}{\alpha^3} \varpi^4 \left(\frac{1}{12} - \frac{1}{\pi^2} \right) + O\left(\frac{1}{\alpha^{5/2}}\right). \quad (705)$$

It diverges strongly with small α , as $1/\alpha^3$, as compared to $1/\alpha^{3/2}$ of the large- J next-to-leading second charge (631), and $1/\alpha$ of the large- J leading one (546). Figure 30 pictorially proves this value.

Let us corroborate this result by applying the approximation method (637) to the exact expression for $Q_2|_{\omega=0}|_{\text{next-to-next-to-leading}}$ (678). Namely, the leading order can be found by changing the summation to an integration in

$$\begin{aligned} \sum_{n \geq 1} \left(1 - \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}} \right) &= \frac{\mu\sqrt{\alpha(1+\alpha)}}{\pi} \frac{\pi}{\mu\sqrt{\alpha(1+\alpha)}} \sum_{n \geq 1} \left(1 - \frac{\frac{\pi n}{\mu\sqrt{\alpha(1+\alpha)}}}{\sqrt{1 + \left(\frac{\pi n}{\mu\sqrt{\alpha(1+\alpha)}} \right)^2}} \right) \approx \\ &\approx \frac{\mu\sqrt{\alpha(1+\alpha)}}{\pi} \int_0^\infty dx \left(1 - \frac{x}{\sqrt{1+x^2}} \right) + O(\text{subleading}) = \frac{\mu\sqrt{\alpha(1+\alpha)}}{\pi} + O(\text{subleading}) = \frac{1}{\sqrt{\alpha}} \frac{\varpi}{\pi} + O(\alpha^0). \end{aligned} \quad (706)$$

This leads to precisely the same leading order as in (705). The computation is, thus, finished, and our technique — established.

6. Check of the Master Formulae

In this paragraph, we explicitly check that the one-loop $t = 1, 3$ master formulae (478), (480) are indeed fulfilled by the resolvents we have found. Our motivation is that • a test of the $t = 1$ one will prove the applicability of the zeta-function regularization scheme (477), under which it has been obtained, and • a test of the $t = 3$ formula will be a confirmation of the validity of the “cubic term trick” (409), which constitutes a crucial foundation of our entire computation based on the quadratic equation method.

A technique to be exploited here will be to form “quadratic equations” from our master formulae, *i.e.*, to multiply them by $1/(J(z - y_k))$, where z is a complex parameter, and sum over all $k = 1, 2, \dots, S$; such relations will then be proven. An important ingredient will be the procedure (331) of rewriting such sums in terms of the resolvent and its derivatives. It will actually work because the problematic terms $1/(y_k - y_j)$ and $1/(y_k - y_j)^3$ disappear when the double sum over j, k is symmetrized. This is also the reason why the $t = 2$ master formula cannot be checked in a similar fashion; the term $1/(y_k - y_j)^2$ remains in the “quadratic-equation version” of the formula even after the symmetrization.

The l.h.s. of the quadratic-equation version of the one-loop $t = 1$ master formula reads

$$\begin{aligned} \frac{1}{J^2} \sum_{k=1}^S \sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{z - y_k} \frac{1}{y_k - y_j} &= \frac{1}{2} \frac{1}{J^2} \sum_{k=1}^S \left(\sum_{j=1}^S \frac{1}{(z - y_j)(z - y_k)} - \frac{1}{(z - y_k)^2} \right) = \\ &= \frac{1}{2} \left(G(z)^2 + \frac{1}{J} G'(z) \right) = \\ &= \frac{1}{2} G^{(0)}(z)^2 + \frac{1}{J} \frac{1}{2} \left(G^{(0)'}(z) + 2G^{(0)}(z)G^{(1)}(z) \right) + \frac{1}{J^2} \frac{1}{2} \left(G^{(1)'}(z) + G^{(1)}(z)^2 + 2G^{(0)}(z)G^{(2)}(z) \right) + O\left(\frac{1}{J^3}\right). \end{aligned} \quad (707)$$

With the r.h.s. we proceed by using the linear equations (482)–(484) (or (532), (571), (680)),

$$\begin{aligned} \frac{1}{J} \sum_{k=1}^S \frac{1}{z - y_k} &\left(\mathcal{G}^{(0)}(y_k) + \frac{1}{J} \left(\mathcal{G}^{(1)}(y_k) + \frac{1}{2} \frac{\rho^{(0)'}(y_k)}{\rho^{(0)}(y_k)} \right) + \right. \\ &\left. + \frac{1}{J^2} \left(\mathcal{G}^{(2)}(y_k) + \frac{1}{2} \frac{d}{dy_k} \left(\frac{\rho^{(1)}(y_k)}{\rho^{(0)}(y_k)} \right) \right) \right) + O\left(\frac{1}{J^3}\right) = \\ &= \frac{1}{2} \left(\left(\mu - \frac{1}{z} \right) G^{(0)}(z) + \frac{1}{z} G^{(0)}(0) \right) + \\ &+ \frac{1}{J} \frac{1}{2} \left(\left(\mu - \frac{1}{z} \right) G^{(1)}(z) + \frac{1}{z} G^{(1)}(0) + \mathcal{A}^{(1)}(z) \right) + \\ &+ \frac{1}{J^2} \frac{1}{2} \left(\left(\mu - \frac{1}{z} \right) G^{(2)}(z) + \frac{1}{z} G^{(2)}(0) + \mathcal{A}^{(2,1)}(z) + \mathcal{A}^{(2,2)}(z) + \right. \\ &\left. + \frac{1}{4z} \left(\frac{1}{2} G^{(0)''}(0) + \frac{1}{z} G^{(0)'}(0) + \frac{1}{z^2} G^{(0)}(0) - \frac{1}{z^2} G^{(0)}(z) \right) \right) + O\left(\frac{1}{J^3}\right). \end{aligned} \quad (708)$$

Comparing both sides order by order, we find that their being equal to each other is precisely equivalent to the one-loop one-cut quadratic equation (463)–(469). This completes the proof.

Let us move to establish the validity of the one-loop $t = 3$ master formula in its quadratic-equation version. The l.h.s. reads

$$\begin{aligned}
& \frac{1}{J^4} \sum_{k=1}^S \sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{z - y_k} \frac{1}{(y_k - y_j)^3} = \frac{1}{2} \frac{1}{J^4} \sum_{k=1}^S \sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{(z - y_j)(z - y_k)(y_k - y_j)^2} = \\
& = \frac{1}{2} \frac{1}{J^3} \sum_{k=1}^S \frac{1}{(z - y_k)^2} \left(\frac{1}{z - y_k} \left(\frac{1}{J} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{z - y_j} - \frac{1}{J} \frac{1}{z - y_k} \right) - \right. \\
& \quad \left. - \frac{1}{z - y_k} \left(\frac{1}{J} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{y_k - y_j} \right) + \left(\frac{1}{J} \sum_{\substack{j=1 \\ j \neq k}}^S \frac{1}{(y_k - y_j)^2} \right) \right) = \dots
\end{aligned} \tag{709}$$

Substituting here the one-loop $t = 1$ and $t = 2$ master formulae (478), (479), and keeping only the two leading large- J orders, we further get

$$\begin{aligned}
& \dots = \frac{1}{2} \frac{1}{J} \sum_{k=1}^S \frac{1}{(z - y_k)^2} \left(\frac{1}{J} \frac{\pi^2}{3} \rho^{(0)}(y_k)^2 + \right. \\
& \quad \left. + \frac{1}{J^2} \left(\frac{2\pi^2}{3} \rho^{(0)}(y_k) \rho^{(1)}(y_k) + \frac{1}{z - y_k} G^{(0)}(z) - \frac{1}{z - y_k} \mathcal{G}^{(0)}(y_k) - \mathcal{G}^{(0)'}(y_k) \right) \right) + \mathcal{O}\left(\frac{1}{J^3}\right).
\end{aligned} \tag{710}$$

To deal with the leading term of (710), we substitute the expression for the density $\rho^{(0)}(y)$ (533), and exploit the procedure (331) to write the sum through the resolvent,

$$\begin{aligned}
& \frac{\pi^2}{6} \frac{1}{J^2} \sum_{k=1}^S \frac{1}{(z - y_k)^2} \rho^{(0)}(y_k)^2 = \\
& = -\frac{1}{J^2} \frac{1}{24z^2} \sum_{k=1}^S \left(\frac{2(1 - \mu(1 + 2\alpha)z)}{z} \frac{1}{y_k} + \frac{1}{y_k^2} + \frac{2(1 - \mu(1 + 2\alpha)z)}{z} \frac{1}{z - y_k} + \mu^2(z - a)(z - b) \frac{1}{(z - y_k)^2} \right) = \\
& = \frac{1}{J} \frac{1}{24z^2} \left(\frac{2(1 - \mu(1 + 2\alpha)z)}{z} G^{(0)}(0) + G^{(0)'}(0) - \frac{2(1 - \mu(1 + 2\alpha)z)}{z} G^{(0)}(z) + \mu^2(z - a)(z - b) G^{(0)'}(z) \right) + \\
& + \frac{1}{J^2} \frac{1}{24z^2} \left(\frac{2(1 - \mu(1 + 2\alpha)z)}{z} G^{(1)}(0) + G^{(1)'}(0) - \frac{2(1 - \mu(1 + 2\alpha)z)}{z} G^{(1)}(z) + \mu^2(z - a)(z - b) G^{(1)'}(z) \right) + \mathcal{O}\left(\frac{1}{J^3}\right).
\end{aligned} \tag{711}$$

The next-to-leading term of (710) consists of a piece with resolvents, and a piece with densities. The former one is tackled by using the linear equation (532) and the procedure (331),

$$\begin{aligned}
& \frac{1}{2} \frac{1}{J^3} \sum_{k=1}^S \frac{1}{(z - y_k)^2} \left(\frac{1}{z - y_k} G^{(0)}(z) - \frac{1}{z - y_k} \mathcal{G}^{(0)}(y_k) - \mathcal{G}^{(0)'}(y_k) \right) = \\
& = \frac{1}{J^2} \left(\frac{1}{4} G^{(0)}(z) G^{(0)''}(z) + \frac{1}{4z} \left(\frac{1}{z^2} G^{(0)}(0) + \frac{1}{z} G^{(0)'}(0) - \frac{1}{z^2} G^{(0)}(z) + \frac{1 - \mu z}{2} G^{(0)''}(z) \right) \right) + \mathcal{O}\left(\frac{1}{J^3}\right).
\end{aligned} \tag{712}$$

The latter one is rewritten as an integral with the density $\rho^{(0)}(y)$, since we are interested only in the leading order, which does not produce any boundary contribution. But then, it is treated as an integral with the density $\rho^{(1)}(y)$, and thus, expressed through the bulk resolvent $G^{(1),\text{bulk}}(z)$,

$$\begin{aligned}
\frac{\pi^2}{3} \frac{1}{J^3} \sum_{k=1}^S \frac{1}{(z-y_k)^2} \rho^{(0)}(y_k) \rho^{(1)}(y_k) &= \frac{\pi^2}{3} \frac{1}{J^2} \int_{\mathcal{C}_y} dy \rho^{(0)}(y)^2 \rho^{(1)}(y) \frac{1}{(z-y)^2} + \mathcal{O}\left(\frac{1}{J^3}\right) = \\
&= -\frac{1}{J^2} \frac{1}{12z^2} \int_{\mathcal{C}_y} dy \rho^{(1)}(y) \left(\frac{2(1-\mu(1+2\alpha)z)}{z} \frac{1}{y} + \frac{1}{y^2} + \right. \\
&\quad \left. + \frac{2(1-\mu(1+2\alpha)z)}{z} \frac{1}{z-y} + \mu^2(z-a)(z-b) \frac{1}{(z-y)^2} \right) + \mathcal{O}\left(\frac{1}{J^3}\right) = \\
&= \frac{1}{J^2} \frac{1}{12z^2} \left(\frac{2(1-\mu(1+2\alpha)z)}{z} G^{(1),\text{bulk}}(0) + G^{(1),\text{bulk}'}(0) - \right. \\
&\quad \left. - \frac{2(1-\mu(1+2\alpha)z)}{z} G^{(1),\text{bulk}}(z) + \mu^2(z-a)(z-b) G^{(1),\text{bulk}'}(z) \right) + \mathcal{O}\left(\frac{1}{J^3}\right) = \\
&= \frac{1}{J^2} \frac{1}{12z^2} \left(\frac{2(1-\mu(1+2\alpha)z)}{z} G^{(1)}(0) + G^{(1)'}(0) - \right. \\
&\quad \left. - \frac{2(1-\mu(1+2\alpha)z)}{z} G^{(1)}(z) + \mu^2(z-a)(z-b) G^{(1)'}(z) \right) + \mathcal{O}\left(\frac{1}{J^3}\right), \tag{713}
\end{aligned}$$

where in the last passage we have used a surprising cancelation of the contribution from $G^{(1),\text{boundary}}(z)$. The l.h.s. of our $t=3$ master formula in its quadratic-equation form, up to two leading large- J orders, is, thus, given by the sum of (711), (712), (713).

On the r.h.s., we proceed in a very similar way. To the leading order in (480), and the first term of the next-to-leading one, we apply the method (331), while the second term there is handled by changing summation to integration, and in this way writing it through $G^{(1),\text{bulk}}(z)$. This yields

$$\begin{aligned}
\frac{1}{J} \sum_{k=1}^S \frac{1}{z-y_k} &\left(-\frac{1}{J} \frac{\pi^2}{2} \rho^{(0)}(y_k) \rho^{(0)'}(y_k) + \frac{1}{J^2} \left(-\frac{1}{2y_k^3} - \frac{\pi^2}{2} \frac{d}{dy_k} \left(\rho^{(0)}(y_k) \rho^{(1)}(y_k) \right) \right) \right) + \mathcal{O}\left(\frac{1}{J^3}\right) = \\
&= \frac{1}{J} \frac{1}{8} \left(\frac{1-\mu(1+2\alpha)z}{z^3} G^{(0)}(0) + \frac{1-\mu(1+2\alpha)z}{z^2} G^{(0)'}(0) + \frac{1}{2z} G^{(0)''}(0) - \frac{1-\mu(1+2\alpha)z}{z^3} G^{(0)}(z) \right) + \\
&+ \frac{1}{J^2} \frac{1}{2} \left(\frac{1-\mu(1+2\alpha)z}{2z^3} G^{(1)}(0) + \frac{1}{4z^2} G^{(1)'}(0) - \frac{1-\mu(1+2\alpha)z}{2z^3} G^{(1)}(z) + \frac{\mu^2(z-a)(z-b)}{4z^2} G^{(1)'}(z) + \right. \\
&\quad \left. + \frac{1}{z^3} G^{(0)}(0) + \frac{1}{z^2} G^{(0)'}(0) + \frac{1}{2z} G^{(0)''}(0) - \frac{1}{z^3} G^{(0)}(z) + \right. \\
&\quad \left. + \frac{\mu^3 \sqrt{\alpha(1+\alpha)}}{8} \left((\sqrt{\alpha} + \sqrt{1+\alpha})^4 \frac{1}{z-a} - (\sqrt{\alpha} - \sqrt{1+\alpha})^4 \frac{1}{z-b} \right) \right) + \mathcal{O}\left(\frac{1}{J^3}\right). \tag{714}
\end{aligned}$$

Remark that the last line is a contribution from $G^{(1),\text{boundary}}(z)$. Comparing this r.h.s. with the previously computed l.h.s., we confirm that they are equal, thus finishing the proof of the one-loop $t=3$ master formula for our one-cut case.

IV. SOLVING THE ALL-LOOP ONE-CUT QUADRATIC EQUATION AT THE ORDER $O(J^0)$

In sections IV, V, VI, we solve the all-loop one-cut quadratic equation (453)–(462) at the three leading large- J levels, one in each section. All the formulae are much lengthier than for one loop, but the main complication happens to be the hyperbolic cotangent $\coth(\pi\rho^{(0)}(y))$, appearing in anomaly computations. For these reasons, we will need to be reinforced by some simplifying assumptions; we will consider two different limits: • the small- ω (weak-coupling) approximation, • the large-mode-number, fixed-winding-number approximation (which probes the strong-coupling regime).

A. The Exact Solution to the All-Loop One-Cut Quadratic Equation at the Order $O(J^0)$

1. The Exact Resolvent $G^{(0)}(z)$

The leading-order equation (453) is quadratic, and gives

$$G^{(0)}(z) = \frac{\mu}{2(z^2 - \omega^2)} \left(z^2 - \frac{1}{\mu}z + \omega^2(2\alpha - 1) + s_1\sqrt{P(z)} \right), \quad (715)$$

where $P(z)$ is the quartic polynomial

$$P(z) \equiv z^4 - 2\frac{1}{\mu} \left(1 + 2\alpha - 2\omega^2\gamma^{(0)} \right) z^3 + \left(\frac{1}{\mu^2} + 2\omega^2(2\alpha^2 - 1) \right) z^2 + 2\omega^2\frac{1}{\mu} \left(1 - 2\omega^2\gamma^{(0)} \right) z + \omega^4. \quad (716)$$

This increase in the order of $P(z)$, from two at one loop (509) to four at all loop, is the primordial source of complication in the computation to follow. In order to conveniently handle the two-valuedness of this solution, we define $\sqrt{P(z)}$ as

$$\sqrt{P(z)} \equiv \sqrt{z-a}\sqrt{z-b}\sqrt{z-c_1}\sqrt{z-c_2}, \quad (717)$$

where a, b, c_1, c_2 denote the four roots of $P(z)$ (see below (740) for the justification of this notation), the square roots on the r.h.s. are principal, and $s_1 = \pm 1$ is a sign, to be specified below (746).

The basic features of this solution:

- At complex infinity, the resolvent behaves as

$$G^{(0)}(z) = \frac{\mu(1+s_1s_2)}{2} + \left(-s_1s_2 \left(\alpha - \omega^2\gamma^{(0)} \right) - \frac{1+s_1s_2}{2} \right) \frac{1}{z} + O\left(\frac{1}{z^2}\right), \quad (718)$$

where $s_2 = \pm 1$ is some sign coming from

$$\lim_{z \rightarrow \infty} \frac{1}{z^4} P(z) = 1, \quad \text{which implies} \quad \lim_{z \rightarrow \infty} \frac{1}{z^2} \sqrt{P(z)} = s_2. \quad (719)$$

Hence, in order to have a correct $1/z$ behavior, the roots of $P(z)$ and the sign s_1 must interplay in such a way that

$$s_1s_2 = -1, \quad (720)$$

since then the $O(z^0)$ term disappears, and we are left with

$$G^{(0)}(z) = \left(\alpha - \omega^2\gamma^{(0)} \right) \frac{1}{z} + O\left(\frac{1}{z^2}\right). \quad (721)$$

This is precisely what the relation (313) tells us. Remark that this, unlike at one loop, does not imply that the large- z series of the subleading resolvents, *i.e.*, $G^{(1)}(z)$, $G^{(2)}(z)$, *etc.*, must begin at the order $O(1/z^2)$; they have also to appropriately modify the $1/z$ term.

- Computing the resolvent's value at $z = 0$, we find

$$G^{(0)}(0) = \mu \left(-\alpha + \frac{1 - s_1 s_3}{2} \right), \quad (722)$$

where $s_3 = \pm 1$ is some sign coming from

$$P(0) = \omega^4, \quad \text{which implies} \quad \sqrt{P(0)} = s_3 \omega^2. \quad (723)$$

Hence, in order to have the value at zero consistent with the momentum condition (455), the roots of $P(z)$ and the sign s_1 must be such that

$$s_1 s_3 = +1, \quad (724)$$

since then

$$G^{(0)}(0) = -\mu\alpha, \quad (725)$$

as required.

- The first derivative w.r.t. z of the resolvent is

$$\begin{aligned} G^{(0)'}(z) = & \frac{1}{2(z^2 - \omega^2)^2} \left(z^2 - 4\omega^2 \mu \alpha z + \omega^2 + \right. \\ & + s_1 \frac{1}{\mu \sqrt{P(z)}} \left(\mu (1 + 2\alpha - 2\omega^2 \gamma^{(0)}) z^4 - (1 + 4\omega^2 \mu^2 \alpha^2) z^3 + \right. \\ & \left. \left. + 6\omega^2 \mu \alpha z^2 - \omega^2 (1 + 4\omega^2 \mu^2 \alpha^2) z - \omega^4 \mu (1 - 2\omega^2 \gamma^{(0)}) \right) \right). \end{aligned} \quad (726)$$

Its value at $z = 0$,

$$G^{(0)'}(0) = s_1 s_3 \gamma^{(0)} + \frac{1 - s_1 s_3}{2\omega^2}. \quad (727)$$

We see that only when the same condition as before (724) is satisfied, does this quantity comply with the definition of $\gamma^{(0)}$ (454),

$$G^{(0)'}(0) = \gamma^{(0)}, \quad (728)$$

as required by the resolvent's self-consistency. This also means that the constant $\gamma^{(0)}$ is, without any additional constraints, arbitrary. Such constraints will, however, be imposed (740), thus fixing $\gamma^{(0)}$ (758).

- Let us explicitly print the values at $z = 0$ of a few further derivatives (we everywhere use (724)),

$$G^{(0)''}(0) = -\frac{2}{\omega^2 \mu} \left(\mu^2 \alpha (1 + \alpha) + \gamma^{(0)} - \omega^2 \gamma^{(0)2} \right), \quad (729)$$

$$G^{(0)'''}(0) = \frac{6}{\omega^4 \mu^2} \left(\mu^2 \alpha (1 + \alpha) + (1 + \omega^2 \mu^2 (1 - 2\alpha^2)) \gamma^{(0)} - 3\omega^2 \gamma^{(0)2} + 2\omega^4 \gamma^{(0)3} \right), \quad (730)$$

$$\begin{aligned} G^{(0)''''}(0) = & \frac{24}{\omega^6 \mu^3} \left(\mu^2 \alpha (1 + \alpha) \left(-1 + \omega^2 \mu^2 (-1 - \alpha + \alpha^2) \right) + \left(-1 + 2\omega^2 \mu^2 (-1 + \alpha + 3\alpha^2) \right) \gamma^{(0)} + \right. \\ & \left. + 2\omega^2 \left(3 + \omega^2 \mu^2 (1 - 3\alpha^2) \right) \gamma^{(0)2} - 10\omega^4 \gamma^{(0)3} + 5\omega^6 \gamma^{(0)4} \right). \end{aligned} \quad (731)$$

Once the constant $\gamma^{(0)}$ is determined, these values will allow us to find the large- J leading-order terms of a few lowest local conserved charges (336),

$$Q_t|_{\text{leading}} = -\frac{G^{(0)''\dots'(t-1)}(0)}{(t-1)!}, \quad \text{for} \quad t = 1, 2, \dots \quad (732)$$

- In the course of the computation, we will need the values of this resolvent and its three lowest derivatives at the points $z = \pm\omega$. Naively, it seems that they diverge due to the vanishing of the denominator, but the numerator can be checked to be zero as well, and so, they are in fact 0/0 symbols. Applying the l'Hôpital rule leads to

$$G^{(0)}(\omega) = \frac{\mu}{-1 + 2\omega\mu\alpha} \left(\alpha + \omega\mu\alpha(1 - \alpha) - \gamma^{(0)}\omega^2 \right), \quad (733)$$

$$G^{(0)'}(\omega) = \frac{\mu^2}{(-1 + 2\omega\mu\alpha)^3} \left(\alpha(1 + \alpha) + 2\omega\mu\alpha(1 - \alpha^2) - 2\omega^2\mu^2\alpha^2(1 - \alpha^2) - 2\gamma^{(0)}\omega^2(1 + \alpha) + 2\gamma^{(0)2}\omega^4 \right), \quad (734)$$

$$G^{(0)''}(\omega) = \frac{2\mu^2}{(-1 + 2\omega\mu\alpha)^5} \left(\mu\alpha(1 + \alpha)(1 + 2\alpha) + \omega\mu^2\alpha(1 + \alpha)(4 + 5\alpha - 5\alpha^2) + 4\omega^2\mu^3\alpha^2(1 + \alpha)(-1 - 2\alpha + 2\alpha^2) + 4\omega^3\mu^4\alpha^4(1 - \alpha^2) + \gamma^{(0)}\omega \left(1 - 2\omega\mu(2 + 8\alpha + 3\alpha^2) + 4\omega^2\mu^2\alpha(-2 + \alpha) + 8\omega^3\mu^3\alpha^2 \right) + \gamma^{(0)2}\omega^3 \left(-1 + 4\omega\mu(3 + 4\alpha) - 4\omega^2\mu^2\alpha^2 \right) - 8\gamma^{(0)3}\omega^6\mu \right), \quad (735)$$

$$G^{(0)'''}(\omega) = \frac{6\mu^2}{(-1 + 2\omega\mu\alpha)^7} \left(5\mu^2\alpha^2(1 + \alpha)^2 + 2\omega\mu^3\alpha(1 + \alpha)^2(4 + 14\alpha - 7\alpha^2) + 28\omega^2\mu^4\alpha^3(-2 + \alpha)(1 + \alpha)^2 - 8\omega^3\mu^5\alpha^3(1 + \alpha)^2(2 - 6\alpha + 3\alpha^2) + 8\omega^4\mu^6\alpha^4(1 - \alpha^2)^2 + \gamma^{(0)} \left(-1 + 2\omega\mu(2 + 7\alpha) - 4\omega^2\mu^2(2 + 14\alpha + 27\alpha^2 + 5\alpha^3) + 8\omega^3\mu^3\alpha(-6 - 9\alpha + 7\alpha^2) + 16\omega^4\mu^4\alpha^2(3 + 7\alpha - \alpha^2) - 32\omega^5\mu^5\alpha^4 \right) + \gamma^{(0)2}\omega^2 \left(1 - 2\omega\mu(6 + 7\alpha) + 12\omega^2\mu^2(4 + 14\alpha + 9\alpha^2) + 8\omega^3\mu^3\alpha(6 - 6\alpha - 7\alpha^2) + 16\omega^4\mu^4\alpha^2(-3 + \alpha^2) \right) + 8\gamma^{(0)3}\omega^5\mu \left(1 - 2\omega\mu(5 + 7\alpha) + 4\omega^2\mu^2\alpha^2 \right) + 40\gamma^{(0)4}\omega^8\mu^2 \right), \quad (736)$$

while the values at $z = -\omega$ are obtained from these ones through exchanging ω with $-\omega$. We have already assumed $s_4 = -1$, $s_5 = +1$ (745), which are yet to be defined and determined.

- The resolvent is easily checked to fulfill the pertinent linear equation (501),

$$\lim_{\epsilon \rightarrow 0^+} \left(G^{(0)}(y + i\epsilon) + G^{(0)}(y - i\epsilon) \right) = \mu + \frac{2\omega^2\mu\alpha - y}{y^2 - \omega^2}, \quad \text{for } y \in \mathcal{C}^y, \quad (737)$$

where we have used the condition that there is only one cut (740).

- In particular, for $\omega = 0$, this solution reduces to our one-loop result (508). Recall that we have found for it $s = -1$ (516), which is the true value here as well.
- It also precisely agrees with the formulae (3.13), (3.14) of [163] under the following substitutions of this article's quantities with our quantities,

$$\mathcal{J} \rightarrow \frac{1}{4\pi\omega}, \quad \mathcal{S} \rightarrow \frac{\alpha}{4\pi\omega}, \quad \mathcal{E} \rightarrow \frac{1 + \alpha - 2\omega^2\gamma^{(0)}}{4\pi\omega}, \quad k \rightarrow \frac{\mu}{2\pi}, \quad m \rightarrow -\frac{\mu\alpha}{2\pi}, \quad z \rightarrow \frac{z}{\omega}. \quad (738)$$

2. The Exact Density $\rho^{(0)}(y)$ and the Definition of the Double-Root Consistency Condition

This exact large- J leading-order resolvent (715) produces the exact large- J leading-order density according to (323),

$$\rho^{(0)}(y) = (-s_1) \frac{\mu}{2\pi} \frac{1}{y^2 - \omega^2} \lim_{\epsilon \rightarrow 0^+} \frac{\sqrt{P(z)} \Big|_{z=y+i\epsilon} - \sqrt{P(z)} \Big|_{z=y-i\epsilon}}{2i}, \quad (739)$$

for a certain domain of y . This density must be real and positive-definite. It will be real when the roots of $P(z)$ are all real.

Since the polynomial $P(z)$ is quartic, the resulting density $\rho^{(0)}(y)$ is in general supported on a domain consisting of two intervals. However, since the beginning of our computation, we have assumed that there is actually only one interval (the “one-cut case”). This requirement puts an additional constraint on $P(z)$ that it should have two single roots a, b , and one double root c (the “double-root consistency condition”),

$$P(z) = (z - a)(z - b)(z - c)^2. \quad (740)$$

In order to fulfill this, the otherwise arbitrary constant $\gamma^{(0)}$ needs to be fixed in the way described below in paragraph IV A 3. We will adhere to this assumption everywhere in the subsequent calculations.

The density’s domain is now the interval

$$\mathcal{C}^y = [\min(a, b), \max(a, b)], \quad (741)$$

and the density itself is obtained from the discontinuity of the square root $\sqrt{P(z)} = (z - c)\sqrt{z - a}\sqrt{z - b}$,

$$\rho^{(0)}(y) = (-s_1) \frac{\mu}{2\pi} \frac{y - c}{y^2 - \omega^2} \sqrt{(y - a)(b - y)}, \quad \text{for } y \in \mathcal{C}^y. \quad (742)$$

We can corroborate our derivation of the density $\rho^{(0)}(y)$ by checking whether it indeed yields back the resolvent $G^{(0)}(y)$ upon applying (324). No boundary term is expected. The integral (534) allows us to find

$$\begin{aligned} G^{(0),\text{bulk}}(z) &\equiv \int_{\mathcal{C}^y} dy \rho^{(0)}(y) \frac{1}{z - y} = \\ &= (-s_1) \frac{\mu}{2(z^2 - \omega^2)} \left(z^2 + \frac{\sqrt{P(\omega)} - \sqrt{P(-\omega)}}{2\omega} z + \frac{\sqrt{P(\omega)} + \sqrt{P(-\omega)}}{2} - \omega^2 - \sqrt{P(z)} \right). \end{aligned} \quad (743)$$

Now (716) gives

$$P(\omega) = \omega^2 \left(\frac{1}{\mu} - 2\omega\alpha \right)^2, \quad P(-\omega) = \omega^2 \left(\frac{1}{\mu} + 2\omega\alpha \right)^2, \quad (744)$$

hence, there will for sure be

$$\sqrt{P(\omega)} = s_4 \omega \left(\frac{1}{\mu} - 2\omega\alpha \right), \quad \sqrt{P(-\omega)} = s_5 \omega \left(\frac{1}{\mu} + 2\omega\alpha \right), \quad (745)$$

where $s_4, s_5 = \pm 1$ are signs originating from the root structure of $P(z)$. Plugging (745) into (743), we get that

$$G^{(0),\text{bulk}}(z) = G^{(0)}(z), \quad \text{if and only if} \quad s_1 = -1, s_4 = -1, s_5 = +1, \quad (746)$$

which also in turn implies (720), (724) that

$$s_2 = +1, s_3 = -1. \quad (747)$$

In this way, all the unknown signs (which follow from the order in which the roots a, b, c of $P(z)$ lie on the real axis w.r.t. each other and the points $0, \omega, -\omega$) have been determined by the requirements that our solution for $G^{(0)}(z)$ • has a correct large- z behavior, • is consistent with the momentum condition, • is consistent with the definition of $\gamma^{(0)}$, • leads to a density which yields it back upon (324). Moreover, the roots a, b, c should be real, to make the density real, and the density should be positive-definite,

$$\mu \frac{y - c}{y^2 - \omega^2} \geq 0, \quad \text{for all } y \in \mathcal{C}^y. \quad (748)$$

We will investigate and impose these conditions below.

3. The Exact Solution to the Double-Root Consistency Condition

In order to satisfy the double-root consistency condition (740), the constant $\gamma^{(0)}$ should acquire a certain value, which we will now find.

It is convenient to reformulate the condition so as to force both $P(z)$ and its derivative to vanish at $z = c$,

$$P(c) = 0, \quad P'(c) = 0, \quad (749)$$

i.e., explicitly,

$$c^4 - 2c^3 \frac{1}{\mu} \left(1 + 2\alpha - 2\omega^2 \gamma^{(0)}\right) + c^2 \left(\frac{1}{\mu^2} + 2\omega^2 (2\alpha^2 - 1)\right) + 2c\omega^2 \frac{1}{\mu} \left(1 - 2\omega^2 \gamma^{(0)}\right) + \omega^4 = 0, \quad (750)$$

$$4c^3 - 6c^2 \frac{1}{\mu} \left(1 + 2\alpha - 2\omega^2 \gamma^{(0)}\right) + 2c \left(\frac{1}{\mu^2} + 2\omega^2 (2\alpha^2 - 1)\right) + 2\omega^2 \frac{1}{\mu} \left(1 - 2\omega^2 \gamma^{(0)}\right) = 0, \quad (751)$$

which are two equations with two unknowns, $\gamma^{(0)}$ and c . Instead of this set of a quartic and a cubic equation in c , we can have two cubic equations by multiplying the latter one by $c/4$ and subtracting it from the former one,

$$-c^3 \frac{1}{\mu} \left(1 + 2\alpha - 2\omega^2 \gamma^{(0)}\right) + c^2 \left(\frac{1}{\mu^2} + 2\omega^2 (2\alpha^2 - 1)\right) + 3c\omega^2 \frac{1}{\mu} \left(1 - 2\omega^2 \gamma^{(0)}\right) + 2\omega^4 = 0. \quad (752)$$

Now the set of (751) and (752) fully captures the double-root consistency condition (740).

We proceed in the following way: We find $\gamma^{(0)}$ from (752), which is a linear equation for $\gamma^{(0)}$,

$$\gamma^{(0)} = \frac{c^3 \mu (2\alpha + 1) - c^2 (1 + 2\omega^2 \mu^2 (2\alpha^2 - 1)) - 3c\omega^2 \mu - 2\omega^4 \mu^2}{2c\omega^2 \mu (c^2 - 3\omega^2)}, \quad (753)$$

which we then substitute to (751), obtaining a sixth-order equation for the unknown c/ω ,

$$\left(\frac{c}{\omega}\right)^6 - \left(\frac{c}{\omega}\right)^4 \left(\frac{1}{\omega^2 \mu^2} + 4\alpha^2 + 1\right) + \left(\frac{c}{\omega}\right)^3 \frac{8\alpha}{\omega \mu} - \left(\frac{c}{\omega}\right)^2 \left(\frac{1}{\omega^2 \mu^2} + 4\alpha^2 + 1\right) + 1 = 0. \quad (754)$$

This equation has a high order, but also possesses a crucial property that it is self-reciprocal, *i.e.*, the coefficients by the powers $(6 - t)$ and t , for any $t = 0, 1, 2, 3$, are identical. (We will see how reciprocity reappears in our further calculations.) This feature allows us to simplify it greatly according to a standard procedure. Namely, some regrouping and introducing a new variable

$$d \equiv \frac{c}{\omega} + \frac{\omega}{c}, \quad (755)$$

make it equivalent to a depressed (*i.e.*, having no quadratic term) cubic equation for d ,

$$d^3 - d \left(\frac{1}{\omega^2 \mu^2} + 4(\alpha^2 + 1)\right) + \frac{8\alpha}{\omega \mu} = 0. \quad (756)$$

This equation yields some three solutions for d , call them $d^{(1)}$, $d^{(2)}$, $d^{(3)}$, and each of them gives two choices for c by inverting (755),

$$c^{(i,s_6)} = \frac{\omega}{2} \left(d^{(i)} + s_6 \sqrt{d^{(i)2} - 4}\right), \quad \text{for } i = 1, 2, 3, \quad s_6 = \pm 1. \quad (757)$$

These are the six solutions of (754). (Let us assume that the square root $\sqrt{d^{(i)2} - 4}$ denotes actually, for $(d^{(i)2} - 4)$ real and positive, the positive square root $|\sqrt{d^{(i)2} - 4}|$, a possible sign being taken into account by s_6 . For complex $(d^{(i)2} - 4)$, we could define it as the principal square root, but this will be irrelevant for our discussion.)

Now that we have found c , we plug it back into (753), and simplify with help of (756), which gives

$$\gamma^{(0)(i,s_6)} = \frac{\mu}{2\omega} \left(\frac{\alpha + 1}{\omega \mu} - s_6 \sqrt{d^{(i)2} - 4} \left(\frac{1}{3} + \frac{-\frac{3\alpha}{\omega \mu} d^{(i)} + 2 \left(\frac{1}{\omega^2 \mu^2} + 4(\alpha^2 - \frac{1}{3})\right)}{3d^{(i)2} - 16} \right) \right), \quad (758)$$

which is the value necessary for $P(z)$ to have a double root.

The value of the double root c is shown in (757), and the two remaining single roots can also be found. We take $P(z)$ (716), fix in it $\gamma^{(0)}$ with one of the values $\gamma^{(0)(i,s_6)}$ (758), and divide it by the polynomial $(z - c^{(i,s_6)})^2$,

$$\begin{aligned}
& \left(z - a^{(i,s_6)} \right) \left(z - b^{(i,s_6)} \right) = z^2 + z \left(-\frac{2\alpha}{\mu} + s_6 \frac{\omega}{3} \sqrt{d^{(i)2} - 4} + \right. \\
& + \frac{1}{3d^{(i)2} - 16} \left(-\frac{24\alpha}{\mu} + \left(\frac{3}{\omega\mu^2} + 4\omega(3\alpha^2 - 1) \right) d^{(i)} - 4s_6 \left(\frac{1}{\omega\mu^2} + 4\omega \left(\alpha^2 - \frac{1}{3} \right) \right) \sqrt{d^{(i)2} - 4} + s_6 \frac{6\alpha}{\mu} d^{(i)} \sqrt{d^{(i)2} - 4} \right) \Big) + \\
& + \left(\frac{1}{2\mu^2} + \omega^2 \left(2\alpha^2 - \frac{5}{3} \right) + \right. \\
& + \left. \frac{1}{3d^{(i)2} - 16} \left(8 \left(\frac{1}{\mu^2} + 4\omega^2 \left(\alpha^2 - \frac{1}{3} \right) \right) - \frac{12\omega\alpha}{\mu} d^{(i)} + s_6 \frac{12\omega\alpha}{\mu} \sqrt{d^{(i)2} - 4} - s_6 \left(\frac{3}{2\mu^2} + 2\omega^2(3\alpha^2 - 1) \right) d^{(i)} \sqrt{d^{(i)2} - 4} \right) \right). \tag{759}
\end{aligned}$$

We have thus explicitly solved the double-root consistency condition (740): In order to ensure it, the constant $\gamma^{(0)}$ must be given by the formula (758), with some $i = 1, 2, 3$ and some $s_6 = \pm 1$, where $d^{(i)}$ is the i -th solution of the depressed cubic equation (756). Then, the double root c is expressed by (757), while the remaining roots a, b by (759).

This is not the end of the story, since having found the roots a, b, c , we need to impose on them the conditions from the end of paragraph IV A 2, *i.e.*, that they be real and obey (746)–(748). It is these requirements that will decide which root number i and which sign s_6 have to be chosen. They can probably be imposed in full generality, but it will be simpler and enough for us to do so within one of the approximations which we are about to make.

B. The Weak-Coupling Limit

1. Introduction

The double-root consistency condition (740), solved by (758) and (756), is quite a complicated implicit constraint on the quartic polynomial $P(z)$ (716). Therefore, in order to practically proceed further, we need to restrict ourselves to some sector of the parameters ω , μ , α where it simplifies significantly enough. In this subsection, we will consider the limit of

$$\omega \rightarrow 0, \quad \text{i.e., in other words,} \quad \mathcal{J} = \frac{1}{4\pi\omega} \rightarrow \infty. \quad (760)$$

Actually, we will discover that a more convenient expansion parameter is

$$\eta \equiv \omega\mu, \quad (761)$$

and we will need to have $\eta \ll 1$. The leading order of this approximation has of course been found in section III, but here we will want to derive its several subleading terms.

2. The Perturbative Values of the Roots a , b , c and the Constant $\gamma^{(0)}$

We start from solving perturbatively the depressed cubic equation (756). We observe that the leading term of any solution should be of order $O(1/\omega)$. Forming an ansatz $d = d_{-1}/\omega$, and retaining only the leading small- ω part of the equation, we get the following three solutions,

$$\frac{1}{\omega^3} \left(d_{-1}^3 - \frac{d_{-1}}{\mu^2} \right) + O\left(\frac{1}{\omega}\right) = 0, \quad \text{solved by} \quad d_{-1}^{(i)} = \begin{cases} 0 & \text{for } i = 1 \\ \frac{1}{\mu} & \text{for } i = 2 \\ -\frac{1}{\mu} & \text{for } i = 3 \end{cases}. \quad (762)$$

Now we aim at finding the small- ω perturbative solutions around each of these values,

$$d^{(i)} = \frac{1}{\omega} d_{-1}^{(i)} + \omega d_1^{(i)} + \omega^3 d_3^{(i)} + \omega^5 d_5^{(i)} + O(\omega^7). \quad (763)$$

We have already indicated that there will be no even powers in the series. We also decide to restrict our interest to the four leading terms. Plugging (763) into (756), and solving order by order, we find

$$d^{(1)} = 8\eta\alpha - 32\eta^3\alpha(1 + \alpha^2) + 128\eta^5\alpha(1 + 6\alpha^2 + \alpha^4) + O(\eta^7), \quad (764)$$

$$d^{(2)} = \frac{1}{\eta} + 2\eta(1 - \alpha)^2 - 2\eta^3(1 - \alpha)^2(1 - 6\alpha + \alpha^2) + 4\eta^5(1 - \alpha)^2(1 - 14\alpha + 34\alpha^2 - 14\alpha^3 + \alpha^4) + O(\eta^7), \quad (765)$$

$$d^{(3)} = -\frac{1}{\eta} - 2\eta(1 + \alpha)^2 + 2\eta^3(1 + \alpha)^2(1 + 6\alpha + \alpha^2) - 4\eta^5(1 + \alpha)^2(1 + 14\alpha + 34\alpha^2 + 14\alpha^3 + \alpha^4) + O(\eta^7). \quad (766)$$

We notice that the expansion parameter is actually small $\eta = \omega\mu$, not ω alone, as we will also see in all the subsequent series. We do not know whether it is possible to determine a formula for the coefficients of the polynomials in α which appear at any given order, but we remark that they are self-reciprocal. Notice that $d^{(3)}$ can at each order be obtained from the corresponding order of $d^{(2)}$ by changing the signs of μ and α ; $d^{(1)}$, however, has a different structure.

The small- η expansions of the roots a , b , c , and the constant $\gamma^{(0)}$ follow from substituting (764)–(766) into the exact formulae (759), (757), (758). Let us start from c . The solution $d^{(1)}$ yields

$$c^{(1,s_6)} = is_6\omega + O(\omega^2), \quad (767)$$

which is not real, and thus, should be removed. The $i = 2$ case differs only by changing the signs of μ and α from $i = 3$, which we print below,

$$c^{(3,s_6)} = \begin{cases} \frac{1}{\mu} \left(-\eta^2 + \eta^4 (1 + 4\alpha + 2\alpha^2) - 2\eta^6 (1 + 10\alpha + 23\alpha^2 + 16\alpha^3 + 3\alpha^4) + \right. \\ \quad \left. + \eta^8 (5 + 88\alpha + 412\alpha^2 + 752\alpha^3 + 590\alpha^4 + 192\alpha^5 + 20\alpha^6) + O(\eta^{10}) \right) & \text{for } s_6 = \text{sign}(\mu), \\ \frac{1}{\mu} \left(-1 - \eta^2 (1 + 4\alpha + 2\alpha^2) + \eta^4 (1 + 12\alpha + 26\alpha^2 + 16\alpha^3 + 2\alpha^4) - \right. \\ \quad \left. - 2\eta^6 (1 + 22\alpha + 103\alpha^2 + 176\alpha^3 + 123\alpha^4 + 32\alpha^5 + 2\alpha^6) + O(\eta^8) \right) & \text{for } s_6 = -\text{sign}(\mu) \end{cases}. \quad (768)$$

Since at one loop, the double root is zero, there must be $s_6 = \text{sign}(\mu)$ if $i = 3$, or $s_6 = -\text{sign}(\mu)$ if $i = 2$.

Before calculating the series of the single roots a , b , it is useful to consider the discriminant of the quadratic polynomial (759), in order to see whether they are real or not. We find that for $i = 2$, the leading small- η order of the discriminant is always negative, while for $i = 3$, always positive. Therefore, the $i = 2$ case is inappropriate. In this way, we have found the correct root number and the correct sign,

$$i = 3, \quad s_6 = \text{sign}(\mu). \quad (769)$$

The small- η expansions of a , b read then

$$\begin{aligned} a, b = \frac{1}{\mu} & \left((\sqrt{\alpha} \mp \sqrt{1+\alpha})^2 + \eta^2 (1 + 2\alpha + 2\alpha^2 \mp 2\alpha\sqrt{\alpha(1+\alpha)}) - \right. \\ & - \eta^4 \left((1 + 6\alpha + 10\alpha^2 + 8\alpha^3 + 2\alpha^4) \mp \sqrt{\alpha(1+\alpha)} (-1 + 2\alpha + 7\alpha^2 + 2\alpha^3) \right) + \\ & + \eta^6 \left(2(1 + 12\alpha + 39\alpha^2 + 56\alpha^3 + 43\alpha^4 + 16\alpha^5 + 2\alpha^6) \mp \right. \\ & \left. \mp \sqrt{\alpha(1+\alpha)} (-4 - 11\alpha + 22\alpha^2 + 59\alpha^3 + 30\alpha^4 + 4\alpha^5) \right) + O(\eta^8) \end{aligned} \quad (770)$$

(it is arbitrary which root to call “ a ” and which one “ b ”; here, we adopt the names to match the one-loop terminology (509)), while of the constant $\gamma^{(0)}$,

$$\begin{aligned} \gamma^{(0)} = \mu^2 & \left(-\alpha(1+\alpha) + \eta^2 \alpha(1+\alpha) (1 + 3\alpha + \alpha^2) - \right. \\ & \left. - 2\eta^4 \alpha(1+\alpha) (1 + 7\alpha + 13\alpha^2 + 7\alpha^3 + \alpha^4) + O(\eta^6) \right). \end{aligned} \quad (771)$$

We observe that all the leading terms precisely coincide with the one-loop results. The upper line of (768) and (771) agree with the formulae (3.34) and (3.35) of [163] under the translation (738) and $c \rightarrow c/\eta$. Remark that terminating the series of d at the fourth leading order leads to four leading terms of a , b , c , and three for $\gamma^{(0)}$. More orders could easily be derived. An important finding is that in all the above expansions, there appear only even powers of η . Figure 31 compares the exact values of the quantities in question to their small- η approximations.

At the end of this paragraph, a small comment: Notice that we have now two different ways of approximately writing the polynomial $P(z)$, through the roots (740) or through the constant $\gamma^{(0)}$ (716). It can, however, be checked that these two versions coincide up to the order of the truncations,

$$(z-a)(z-b)(z-c)^2 \Bigg|_{\substack{a, b, c \text{ given by} \\ \text{the expansions (770), (768)}}} - P(z) \Bigg|_{\substack{\gamma^{(0)} \text{ given by} \\ \text{the expansion (771)}}} = O(\eta^8). \quad (772)$$

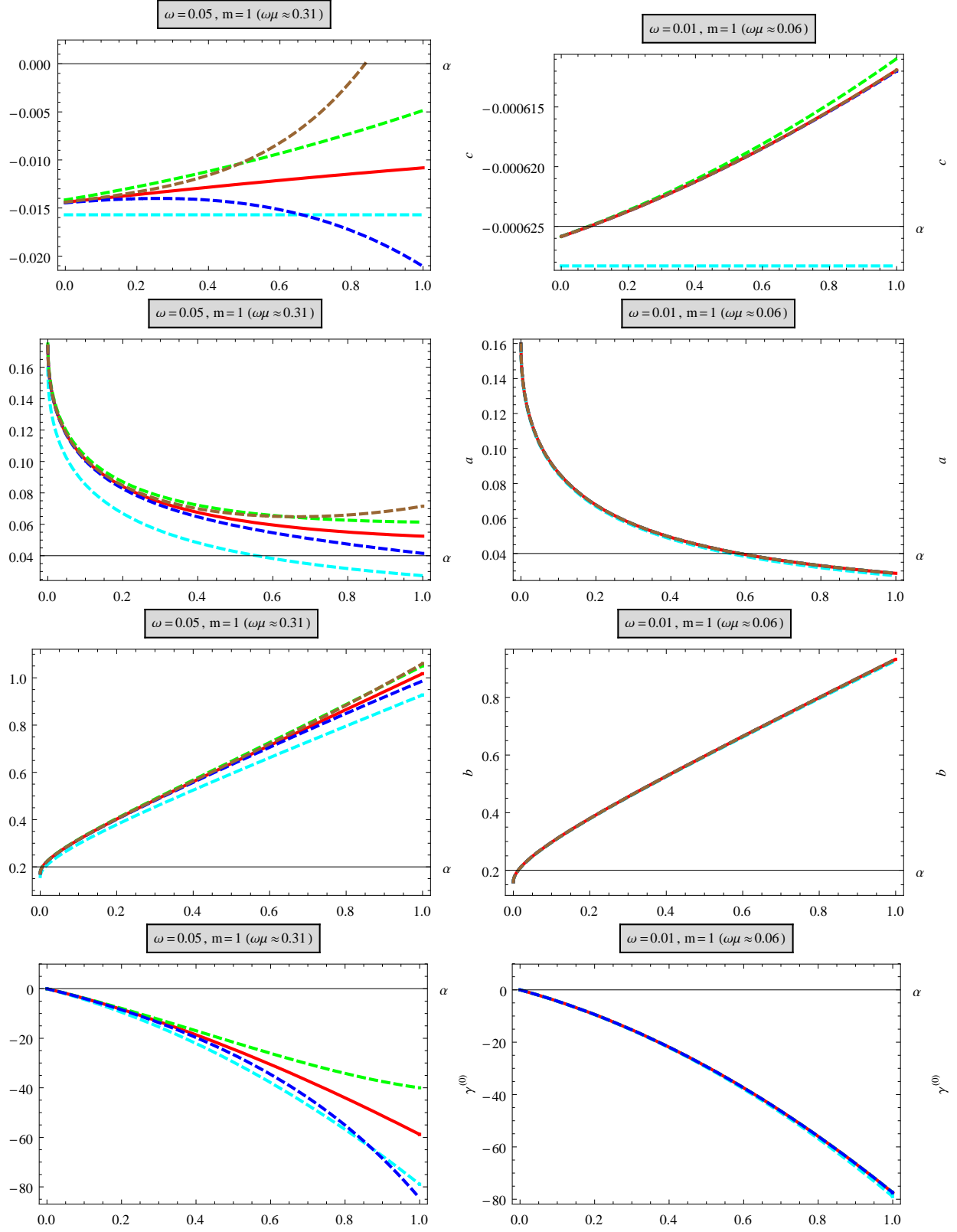


FIG. 31: Plots w.r.t. $\alpha \in [0, 1]$ of the exact values (solid red), and the small- η series truncated up to one (dashed cyan), two (dashed green), three (dashed blue), or four (dashed brown) leading orders, of the quantities c (FIRST ROW), a (SECOND ROW), b (THIRD ROW), $\gamma^{(0)}$ (FOURTH ROW). The parameters are set to $m = 1$ everywhere, and $\omega = 0.05$ (*i.e.*, $\eta = 0.1\pi \approx 0.31$) (LEFT), $\omega = 0.01$ (*i.e.*, $\eta = 0.02\pi \approx 0.06$) (RIGHT).

3. The Perturbative Density $\rho^{(0)}(y)$

In order to find the small- η expansion of the large- J leading-order density $\rho^{(0)}(y)$, we change the variable y to t according to (539),

$$\rho^{(0)}(t) \equiv \rho^{(0)}(y) \Big|_{y \rightarrow t} = \begin{cases} \frac{\mu}{2\pi} \frac{(a-c+(b-a)t)(b-a)}{(a+(b-a)t)^2 - \omega^2} \sqrt{t(1-t)} & \text{for } \mu > 0, \\ \frac{\mu}{2\pi} \frac{(b-c+(a-b)t)(a-b)}{(b+(a-b)t)^2 - \omega^2} \sqrt{t(1-t)} & \text{for } \mu < 0 \end{cases}. \quad (773)$$

Now, it remains to use the small- η series of the roots (768), (770), to obtain (we print the four leading terms),

$$\rho^{(0)}(t) = \mu \left(\rho_0^{(0)}(t) + \rho_2^{(0)}(t)\eta^2 + \rho_4^{(0)}(t)\eta^4 + \rho_6^{(0)}(t)\eta^6 + \mathcal{O}(\eta^8) \right), \quad (774)$$

where (we take $\mu > 0$),

$$\rho_0^{(0)}(t) = \frac{1}{\pi} \sqrt{t(1-t)} \frac{2\sqrt{\alpha(1+\alpha)}}{1 + 2\alpha + 2(-1 + 2t)\sqrt{\alpha(1+\alpha)}}, \quad (775)$$

$$\begin{aligned} \rho_2^{(0)}(t) = & -\frac{1}{\pi} \sqrt{t(1-t)} \frac{2}{\left(1 + 2\alpha + 2(-1 + 2t)\sqrt{\alpha(1+\alpha)}\right)^3} \cdot \\ & \cdot \left(2\alpha^2(1+\alpha)(-1 + 2t) + \sqrt{\alpha(1+\alpha)}(-1 + \alpha + 2\alpha^2) \right), \end{aligned} \quad (776)$$

$$\begin{aligned} \rho_4^{(0)}(t) = & \frac{1}{\pi} \sqrt{t(1-t)} \frac{1 + \alpha}{\left(1 + 2\alpha + 2(-1 + 2t)\sqrt{\alpha(1+\alpha)}\right)^5} \cdot \\ & \cdot \left(2\alpha \left(5 - 6\alpha - 89\alpha^2 - 184\alpha^3 - 112\alpha^4 + \right. \right. \\ & + 2t \left(-5 + 14\alpha + 129\alpha^2 + 272\alpha^3 + 168\alpha^4 \right) - \\ & - 48t^2\alpha(1+\alpha)(1 + 4\alpha + 7\alpha^2) + 32t^3\alpha(1+\alpha)(1 + 4\alpha + 7\alpha^2) \Big) + \\ & + \sqrt{\alpha(1+\alpha)} \left(-3 - 9\alpha + 78\alpha^2 + 256\alpha^3 + 224\alpha^4 - \right. \\ & \left. \left. - 16t\alpha(1 + 14\alpha + 47\alpha^2 + 42\alpha^3) + 16t^2\alpha(1 + 14\alpha + 47\alpha^2 + 42\alpha^3) \right) \right), \end{aligned} \quad (777)$$

$$\begin{aligned}
\rho_6^{(0)}(t) = & \frac{1}{\pi} \sqrt{t(1-t)} \frac{1+\alpha}{\left(1+2\alpha+2(-1+2t)\sqrt{\alpha(1+\alpha)}\right)^7} \cdot \\
& \cdot \left(2\alpha \left(-21 - 49\alpha + 1611\alpha^2 + 12641\alpha^3 + 40566\alpha^4 + \right. \right. \\
& + 67440\alpha^5 + 59616\alpha^6 + 25344\alpha^7 + 3584\alpha^8 - \\
& - 2t \left(-21 + 119\alpha + 4275\alpha^2 + 30265\alpha^3 + 97678\alpha^4 + \right. \\
& + 165328\alpha^5 + 148160\alpha^6 + 63360\alpha^7 + 8960\alpha^8 \left. \right) + \\
& + 16t^2\alpha(1+\alpha) \left(63 + 1000\alpha + 6169\alpha^2 + \right. \\
& + 17296\alpha^3 + 23220\alpha^4 + 13600\alpha^5 + 2240\alpha^6 \left. \right) - \\
& - 32t^3\alpha(1+\alpha) \left(21 + 440\alpha + 2883\alpha^2 + \right. \\
& + 8352\alpha^3 + 11500\alpha^4 + 6800\alpha^5 + 1120\alpha^6 \left. \right) + \\
& + 1280t^4\alpha^2(1+\alpha)^2(1+2\alpha) \left(4 + 19\alpha + 32\alpha^2 + 7\alpha^3 \right) - \\
& - 512t^5\alpha^2(1+\alpha)^2(1+2\alpha) \left(4 + 19\alpha + 32\alpha^2 + 7\alpha^3 \right) \left. \right) + \\
& + \sqrt{\alpha(1+\alpha)} \left(7 + 110\alpha - 601\alpha^2 - 10132\alpha^3 - 44524\alpha^4 - \right. \\
& - 92032\alpha^5 - 96576\alpha^6 - 47104\alpha^7 - 7168\alpha^8 + \\
& + 16t\alpha \left(-9 + 262\alpha + 3061\alpha^2 + 13306\alpha^3 + 28052\alpha^4 + \right. \\
& + 29960\alpha^5 + 14720\alpha^6 + 2240\alpha^7 \left. \right) - \\
& - 16t^2\alpha \left(-9 + 550\alpha + 5797\alpha^2 + \right. \\
& + 25354\alpha^3 + 54692\alpha^4 + 59480\alpha^5 + 29440\alpha^6 + 4480\alpha^7 \left. \right) + \\
& + 512t^3\alpha^2(1+\alpha) \left(18 + 153\alpha + 600\alpha^2 + 1065\alpha^3 + 780\alpha^4 + 140\alpha^5 \right) - \\
& - 256t^4\alpha^2(1+\alpha) \left(18 + 153\alpha + 600\alpha^2 + 1065\alpha^3 + 780\alpha^4 + 140\alpha^5 \right) \left. \right) \left. \right). \tag{778}
\end{aligned}$$

The leading term is of course equal to the one-loop density (533). In figure 32, we test this small- η approximation of the density.

4. The Perturbative Quantities $G^{(2)}(0)$ and $\mathcal{F}(z)$

The exact values (733)–(736), combined with the small- η expansions of c (768), a , b (770), and $\gamma^{(0)}$ (771), allow us to find the small- η series of the following important objects:

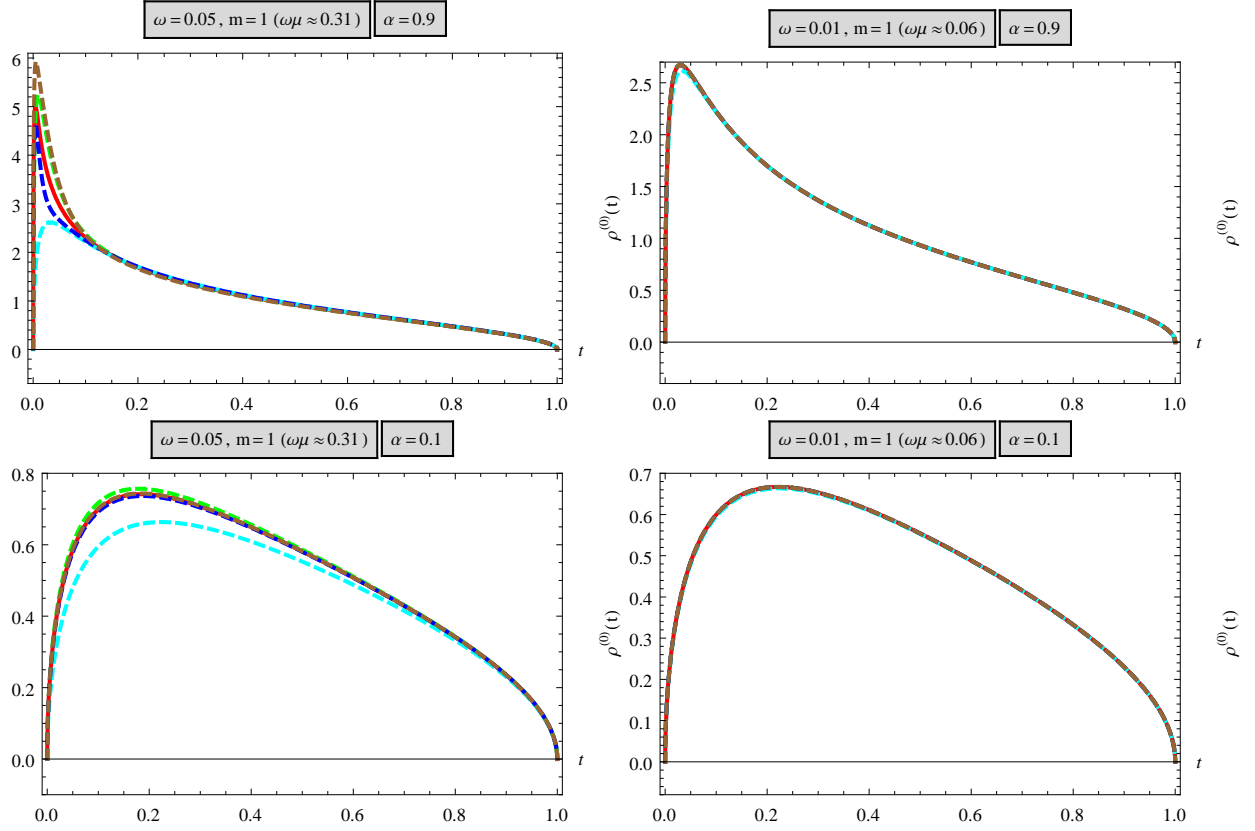


FIG. 32: The exact density, $[0, 1] \ni t \mapsto \rho^{(0)}(t)$ (742) (solid red line), compared to various truncations of its small- η approximation (774): one (dashed cyan), two (dashed green), three (dashed blue), four (dashed brown) leading orders kept. The values of the parameters are: $m = 1$ everywhere, $\alpha = 0.9$ (TOP), $\alpha = 0.1$ (BOTTOM), and $\omega = 0.05$ (*i.e.*, $\eta = 0.1\pi \approx 0.31$) (LEFT), $\omega = 0.01$ (*i.e.*, $\eta = 0.02\pi \approx 0.06$) (RIGHT).

First, the r.h.s. of the momentum condition at the level $O(1/J^2)$ (462),

$$\begin{aligned}
 G^{(2)}(0) = & \mu^3 \left(-\frac{1}{12}\alpha(1+\alpha)(1+2\alpha) - \frac{1}{6}\eta^2\alpha(1+\alpha) \left(3 + 30\alpha + 75\alpha^2 + 53\alpha^3 \right) - \right. \\
 & - \frac{1}{6}\eta^4\alpha^2(1+\alpha) \left(57 + 591\alpha + 2011\alpha^2 + 2757\alpha^3 + 1312\alpha^4 \right) - \\
 & - \frac{1}{3}\eta^6\alpha^2(1+\alpha) \left(-27 - 69\alpha + 1723\alpha^2 + \right. \\
 & \left. \left. + 11298\alpha^3 + 26967\alpha^4 + 28235\alpha^5 + 10889\alpha^6 \right) + O(\eta^8) \right). \tag{779}
 \end{aligned}$$

The leading term is checked to be the same as for one loop (528). Only even powers of η are present.

Second, recall the quantity $\mathcal{F}(z)$ (461). Since it depends on z , before expanding, we must restrict it to some domain of this complex argument having a definite dependence on ω . Let us start from considering this term's contribution to the second local conserved charge, *i.e.*, to $\gamma^{(2)} = G^{(2)'}(0)$. It seems (459) that in order to reach this goal, we need to calculate

$$\lim_{z \rightarrow 0} \frac{d}{dz} \left(\frac{\mathcal{F}(z)}{\frac{\mu}{z^2} \sqrt{P(z)}} \right) = 0, \tag{780}$$

where we have used that the denominator $\frac{1}{z^2}(-2(z^2 - \omega^2)G^{(0)}(z) + z^2\mu - z + \omega^2\mu(2\alpha - 1)) = \frac{\mu}{z^2}\sqrt{P(z)}$. Surprisingly, for any ω , this limit is zero! Notice at this point that for one loop, *i.e.*, taking first the limit $\omega \rightarrow 0$, the above limit

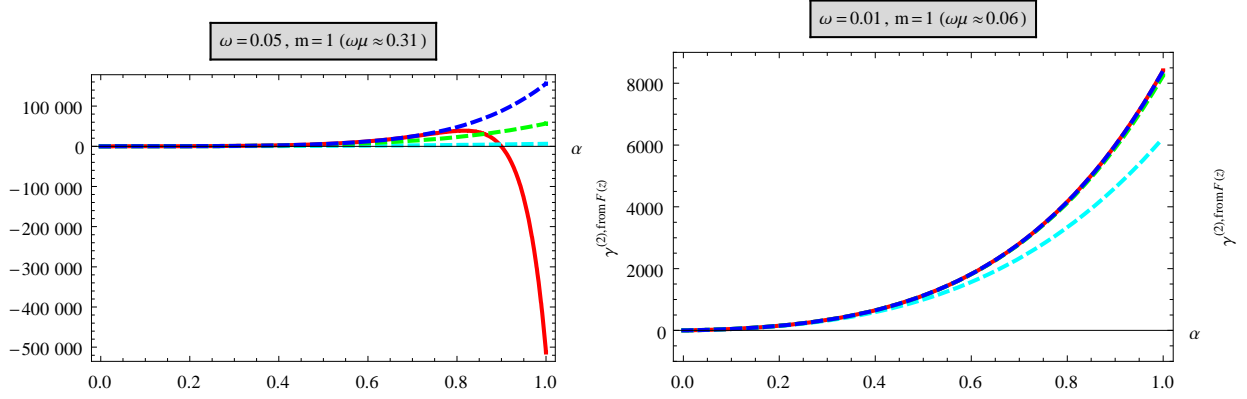


FIG. 33: The contribution of the term $\mathcal{F}(z)$ to $\gamma^{(2)}$ (782), as a function of $\alpha \in [0, 1]$. The exact value (solid red) is compared to its small- η expansion, terminated at one (dashed cyan), two (dashed green), three (dashed blue) leading orders. The values of the parameters are: $m = 1$ everywhere, $\omega = 0.05$ (*i.e.*, $\eta = 0.1\pi \approx 0.31$) (LEFT), $\omega = 0.01$ (*i.e.*, $\eta = 0.02\pi \approx 0.06$) (RIGHT).

$z \rightarrow 0$ is finite and non-zero (697),

$$\lim_{z \rightarrow 0} \lim_{\omega \rightarrow 0} \frac{d}{dz} \left(\frac{\mathcal{F}(z)}{\frac{\mu}{z^2} \sqrt{P(z)}} \right) = \frac{1}{12} \mu^4 \alpha (1 + \alpha) (2 + 11\alpha + 11\alpha^2). \quad (781)$$

This means that, as far as this object is concerned, the two limits do not commute. However, this will not be the way to derive $\gamma^{(2)}$. We will soon understand that the proper contribution of the term $\mathcal{F}(z)$ to the large- J next-to-next-to-leading-order energy is proportional to its value at the double root, $z = c$,

$$\begin{aligned} \gamma^{(2), \text{from } \mathcal{F}(z)} \equiv \frac{c}{\omega^2 \mu} \mathcal{F}(c) = \mu^4 & \left(\frac{1}{12} \alpha (1 + \alpha) (2 + 11\alpha + 11\alpha^2) + \right. \\ & + \frac{1}{6} \eta^2 \alpha (1 + \alpha) \left(2 + 57\alpha + 270\alpha^2 + 432\alpha^3 + 222\alpha^4 \right) + \\ & + \frac{1}{6} \eta^4 \alpha (1 + \alpha) \left(-10 - 166\alpha - 418\alpha^2 + \right. \\ & \left. \left. + 1343\alpha^3 + 6574\alpha^4 + 8623\alpha^5 + 3673\alpha^6 \right) + \mathcal{O}(\eta^6) \right). \end{aligned} \quad (782)$$

Remark that the leading term coincides with the one-loop result (781). We again observe only even powers of η . Figure 33 presents tests of this expansion.

Another effect of $\mathcal{F}(z)$ is its contribution to the density $\rho^{(2)}(t)$, which according to (459) reads

$$\rho^{(2), \text{from } \mathcal{F}(z)}(t) \equiv -\frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} \left(\left(\frac{\mathcal{F}(z)}{\frac{\mu}{z^2} \sqrt{P(z)}} \right) \Big|_{z \rightarrow Z} \Big|_{Z=t+i\epsilon} \right) - \left(\frac{\mathcal{F}(z)}{\frac{\mu}{z^2} \sqrt{P(z)}} \Big|_{z \rightarrow Z} \Big|_{Z=t-i\epsilon} \right). \quad (783)$$

We will not print it explicitly, as it is somewhat lengthy.

C. The Large-Mode-Number, Fixed-Winding-Number Limit

1. Introduction

In this subsection, let us investigate how the all-loop computation can be tackled with assistance of yet another approximation — the large-mode-number, fixed-winding-number limit, which we have introduced and discussed for one loop in paragraphs III A 3, III B 5, III C 5,

$$\mu \rightarrow \infty, \quad \alpha \rightarrow 0, \quad \mu\alpha = \varpi = \text{fixed}. \quad (784)$$

Instead of ϖ , it happens that a better fixed parameter will be

$$\kappa \equiv \sqrt{2\omega|\varpi|}. \quad (785)$$

At one loop, we have removed all the instances of μ in favor of α . Here, let us do otherwise, namely, replace all the α 's with $\kappa^2/(2\omega|\mu|)$, and take μ to be large. Using α as an expansion parameter would cause for example the terms of the series of a, b, c diverge for κ close to zero, which does not happen if μ is used. Also, this is how this limit has been formulated in the literature [163, 185]. Moreover, we will find that μ appears only as the combination $\eta = \omega\mu$ (761), which now should be $\eta \gg 1$.

The application of this limit to our exact formulae will be precisely parallel to how we have proceeded for weak coupling in subsection IV B, so, only the highlights will be printed.

2. The Perturbative Values of the Roots a, b, c and the Constant $\gamma^{(0)}$

The depressed cubic equation (756) is solved perturbatively at large η to yield

$$d^{(1)} = \frac{1}{\eta^2}\kappa^2 - \frac{1}{2^2}\frac{1}{\eta^4}\kappa^2(1 + \kappa^4) + \frac{1}{2^4}\frac{1}{\eta^6}\kappa^2(1 + 6\kappa^4 + \kappa^8) + \mathcal{O}\left(\frac{1}{\eta^8}\right), \quad (786)$$

$$d^{(2)} = -d^{(3)}\Big|_{\kappa^2 \rightarrow -\kappa^2}, \quad (787)$$

$$\begin{aligned} d^{(3)} = & -2 - \frac{1}{2^2}\frac{1}{\eta^2}(1 + \kappa^2)^2 + \frac{1}{2^6}\frac{1}{\eta^4}(1 + \kappa^2)^2(1 + 6\kappa^2 + \kappa^4) - \\ & - \frac{1}{2^9}\frac{1}{\eta^6}(1 + \kappa^2)^2(1 + 14\kappa^2 + 34\kappa^4 + 14\kappa^6 + \kappa^8) + \mathcal{O}\left(\frac{1}{\eta^8}\right), \end{aligned} \quad (788)$$

where we terminate the series to include only four leading orders. Remark that the polynomials in κ are self-reciprocal.

The expansion of the double root c founded on the solution $d^{(1)}$ again happens to be non-real, and hence, must be disregarded. Based on the positivity of the discriminant of the polynomial $(z - a)(z - b)$, we find that there are four possibilities left to investigate: • case A: $\mu > 0$, $s_6 = +1$, $d = d^{(3)}$, • case B: $\mu > 0$, $s_6 = -1$, $d = d^{(3)}$, • case C: $\mu < 0$, $s_6 = +1$, $d = d^{(2)}$, • case D: $\mu < 0$, $s_6 = -1$, $d = d^{(2)}$. Were there no other restraints on the roots, all of these sectors would be correct. However, we know that the density must adhere to the conditions (746)–(748), *i.e.*, that it is positive-definite, and that the signs $s_2 = +1$, $s_3 = -1$, $s_4 = -1$, $s_5 = +1$. Let us enforce the first of them. The order of the roots w.r.t. to each other and the points 0, ω , $-\omega$ is respectively,

$$-\omega < c < 0 < \omega < a < b, \quad \text{for case A,} \quad (789)$$

$$c < -\omega < 0 < b < a < \omega, \quad \text{for case B,} \quad (790)$$

$$-\omega < a < b < 0 < \omega < c, \quad \text{for case C,} \quad (791)$$

$$b < a < -\omega < 0 < c < \omega, \quad \text{for case D,} \quad (792)$$

which implies that only in cases A and D the density is positive-definite,

$$\mu \frac{y - c}{y^2 - \omega^2} \begin{cases} \geq 0 \\ \leq 0 \end{cases}, \quad \text{for all } y \in \mathcal{C}^y, \quad \text{for } \begin{cases} \text{cases A and D,} \\ \text{cases B and C} \end{cases}. \quad (793)$$

This fixes the root number i and the sign s_6 for any given sign of μ .

The appropriate large- η series of the roots are then derived to be

$$\begin{aligned}
 a = \text{sign}(\mu)\omega & \left(1 + \frac{1}{2}\frac{1}{\eta}(1-\kappa)^2 + \frac{1}{2^3}\frac{1}{\eta^2}(1-\kappa)^4 - \right. \\
 & - \frac{1}{2^4}\frac{1}{\eta^3}(1-\kappa)^2\kappa(1-\kappa+\kappa^2) - \frac{1}{2^7}\frac{1}{\eta^4}(1-\kappa)^4(1+\kappa^2)^2 + \\
 & + \frac{1}{2^9}\frac{1}{\eta^5}(1-\kappa)^2\kappa(3-4\kappa+13\kappa^2-10\kappa^3+13\kappa^4-4\kappa^5+3\kappa^6) + \\
 & \left. + \frac{1}{2^{10}}\frac{1}{\eta^6}(1-\kappa)^4(1+\kappa^2)^2(1-\kappa+4\kappa^2-\kappa^3+\kappa^4) + \mathcal{O}\left(\frac{1}{\eta^7}\right) \right), \tag{794}
 \end{aligned}$$

$$b = a|_{\kappa \rightarrow -\kappa}, \tag{795}$$

$$\begin{aligned}
 c = \text{sign}(\mu)\omega & \left(-1 + \frac{1}{2}\frac{1}{\eta}(1+\kappa^2) - \frac{1}{2^3}\frac{1}{\eta^2}(1+\kappa^2)^2 - \frac{1}{2^4}\frac{1}{\eta^3}\kappa^2(1+\kappa^2) + \right. \\
 & + \frac{1}{2^7}\frac{1}{\eta^4}(1+\kappa^2)^2(1+6\kappa^2+\kappa^4) + \frac{3}{2^8}\frac{1}{\eta^5}\kappa^2(1+\kappa^2)(1+3\kappa^2+\kappa^4) - \\
 & \left. - \frac{1}{2^{10}}\frac{1}{\eta^6}(1+\kappa^2)^2(1+14\kappa^2+34\kappa^4+14\kappa^6+\kappa^8) + \mathcal{O}\left(\frac{1}{\eta^7}\right) \right). \tag{796}
 \end{aligned}$$

These remaining requirements on the density must now be checked as well. We find that the expansions (794)–(796) give

$$\lim_{z \rightarrow \infty} \frac{1}{z^2} \sqrt{P(z)} = +1 \quad (\text{exactly}), \tag{797}$$

$$\sqrt{P(0)} = -\omega^2 + \mathcal{O}\left(\frac{1}{\eta^7}\right), \tag{798}$$

$$\sqrt{P(\omega)} = \begin{cases} 2\omega^2 \alpha \frac{-|1-\kappa^2|}{\kappa^2} + \mathcal{O}\left(\frac{1}{\eta^7}\right) & \text{for case A,} \\ 2\omega^2 \alpha \frac{1+\kappa^2}{\kappa^2} + \mathcal{O}\left(\frac{1}{\eta^7}\right) & \text{for case D} \end{cases}, \tag{799}$$

$$\sqrt{P(-\omega)} = \begin{cases} 2\omega^2 \alpha \frac{1+\kappa^2}{\kappa^2} + \mathcal{O}\left(\frac{1}{\eta^7}\right) & \text{for case A,} \\ 2\omega^2 \alpha \frac{-|1-\kappa^2|}{\kappa^2} + \mathcal{O}\left(\frac{1}{\eta^7}\right) & \text{for case D} \end{cases}. \tag{800}$$

Therefore, the derived large- η series of the roots obey the first two conditions in question, but the other two are fulfilled only if an additional constraint is introduced,

$$\kappa \leq 1, \quad \text{i.e., in other words,} \quad |w| \leq \mathcal{J}, \tag{801}$$

which is physically justified. This also reappears in various situations as a necessary condition of self-consistency of our approximate calculations.

We may finally proceed to the large- η energy, which in both cases reads

$$\gamma^{(0)} = \frac{1}{\omega^2} \left(-\frac{\kappa^2}{2} + \frac{1}{2^2}\frac{1}{\eta}\kappa^2 - \frac{1}{2^4}\frac{1}{\eta^2}\kappa^2(1+\kappa^2) + \frac{1}{2^8}\frac{1}{\eta^4}\kappa^2(1+\kappa^2)(1+3\kappa^2+\kappa^4) + \mathcal{O}\left(\frac{1}{\eta^6}\right) \right). \tag{802}$$

All the polynomials in κ above are self-reciprocal.

In figure 34, we provide a pictorial proof that even a few leading terms of the above expansions, and even for a small parameter η , constitute very accurate approximations of the exact values.

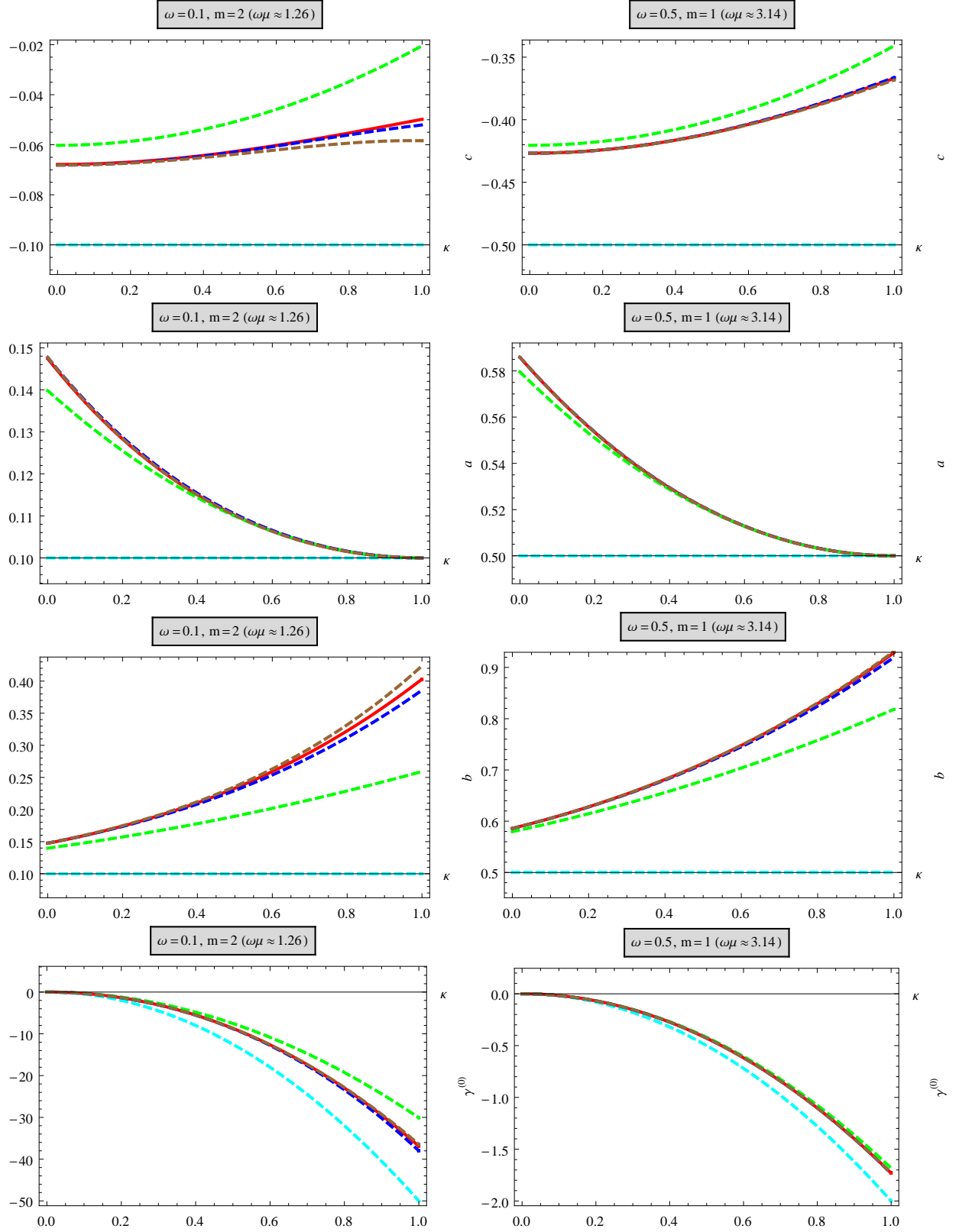


FIG. 34: Plots w.r.t. $\kappa \in [0, 1]$ of the exact values (solid red), and the large- η series truncated up to one (dashed cyan), two (dashed green), three (dashed blue), or four (dashed brown) leading orders, of the quantities c (FIRST ROW), a (SECOND ROW), b (THIRD ROW), $\gamma^{(0)}$ (FOURTH ROW). The parameters are set to $\omega = 0.1, m = 2$ (i.e., $\eta = 0.4\pi \approx 1.26$) (LEFT), $\omega = 0.5, m = 1$ (i.e., $\eta = \pi \approx 3.14$) (RIGHT).

3. The Perturbative Density $\rho^{(0)}(y)$

The large- η series of the roots a, b, c (794)–(796), plus the procedure of changing the variable y to t (539), allow us now to find the pertinent expansion of the large- J leading-order density (742). Let us write the four leading terms, even though the above truncations could lead to six ones,

$$\rho^{(0)}(t) = \frac{1}{\omega} \left(\eta \rho_{-1}^{(0)}(t) + \rho_0^{(0)}(t) + \frac{1}{\eta} \rho_1^{(0)}(t) + \frac{1}{\eta^2} \rho_2^{(0)}(t) + \mathcal{O}\left(\frac{1}{\eta^3}\right) \right), \quad (803)$$

where (let us restrict ourselves to $\mu > 0$),

$$\rho_{-1}^{(0)}(t) = \frac{1}{\pi} \sqrt{t(1-t)} \frac{1}{(1-\kappa)^2 + 4\kappa t} 2\kappa, \quad (804)$$

$$\rho_0^{(0)}(t) = \frac{1}{\pi} \sqrt{t(1-t)} \frac{1}{((1-\kappa)^2 + 4\kappa t)^2} \kappa^2 \left((1-\kappa)^2 - 2(1+\kappa^2)t \right), \quad (805)$$

$$\begin{aligned} \rho_1^{(0)}(t) = & \frac{1}{\pi} \sqrt{t(1-t)} \frac{1}{((1-\kappa)^2 + 4\kappa t)^3} \frac{\kappa}{8} \cdot \\ & \cdot \left((1-\kappa)^4 \left(1 - 2\kappa + 6\kappa^2 - 2\kappa^3 + \kappa^4 \right) + \right. \\ & + 4\kappa(1-\kappa)^2(1+\kappa^2) \left(3 - 10\kappa + 3\kappa^2 \right) t + \\ & + 32\kappa^2(1+\kappa^2) \left(2 - 3\kappa + 2\kappa^2 \right) t^2 + \\ & \left. + 64\kappa^3(1+\kappa^2)t^3 \right), \end{aligned} \quad (806)$$

$$\begin{aligned} \rho_2^{(0)}(t) = & \frac{1}{\pi} \sqrt{t(1-t)} \frac{1}{((1-\kappa)^2 + 4\kappa t)^4} \frac{-\kappa^3}{8} \cdot \\ & \cdot \left((1-\kappa)^6 \left(1 - \kappa + \kappa^2 \right) - \right. \\ & - 2(1-\kappa)^4 \left(4 - 15\kappa + 4\kappa^2 - 15\kappa^3 + 4\kappa^4 \right) t + \\ & + 8(1-\kappa)^2 \left(1 - 16\kappa + 21\kappa^2 - 30\kappa^3 + 21\kappa^4 - 16\kappa^5 + \kappa^6 \right) t^2 + \\ & + 32\kappa(1+\kappa^2) \left(3 - 16\kappa + 25\kappa^2 - 16\kappa^3 + 3\kappa^4 \right) t^3 + \\ & + 128\kappa^2(1+\kappa^2) \left(2 - 5\kappa + 2\kappa^2 \right) t^4 + \\ & \left. + 256\kappa^3(1+\kappa^2)t^5 \right). \end{aligned} \quad (807)$$

An important observation is that the leading order here is $\mathcal{O}(\eta)$, *i.e.*, the density explodes with μ growing to infinity. Remark that self-reciprocal polynomials of κ reappear also here, at any order.

In figure 35, we test the above approximation. Even though the expansion parameter η is very small in all the figures, we observe accurate agreement, even when only the leading order $\rho_{-1}^{(0)}(t)$ (cyan color) is compared to the exact function (red color).

Finally, let us mention that if we use the large- η terms of the density to compute the corresponding bulk resolvent in the variable Z (542),

$$G_p^{(0),\text{bulk}}(Z) \equiv \int_0^1 dt \rho_p^{(0)}(t) \frac{1}{Z-t}, \quad \text{for } p = -1, 0, 1, 2, \dots, \quad (808)$$

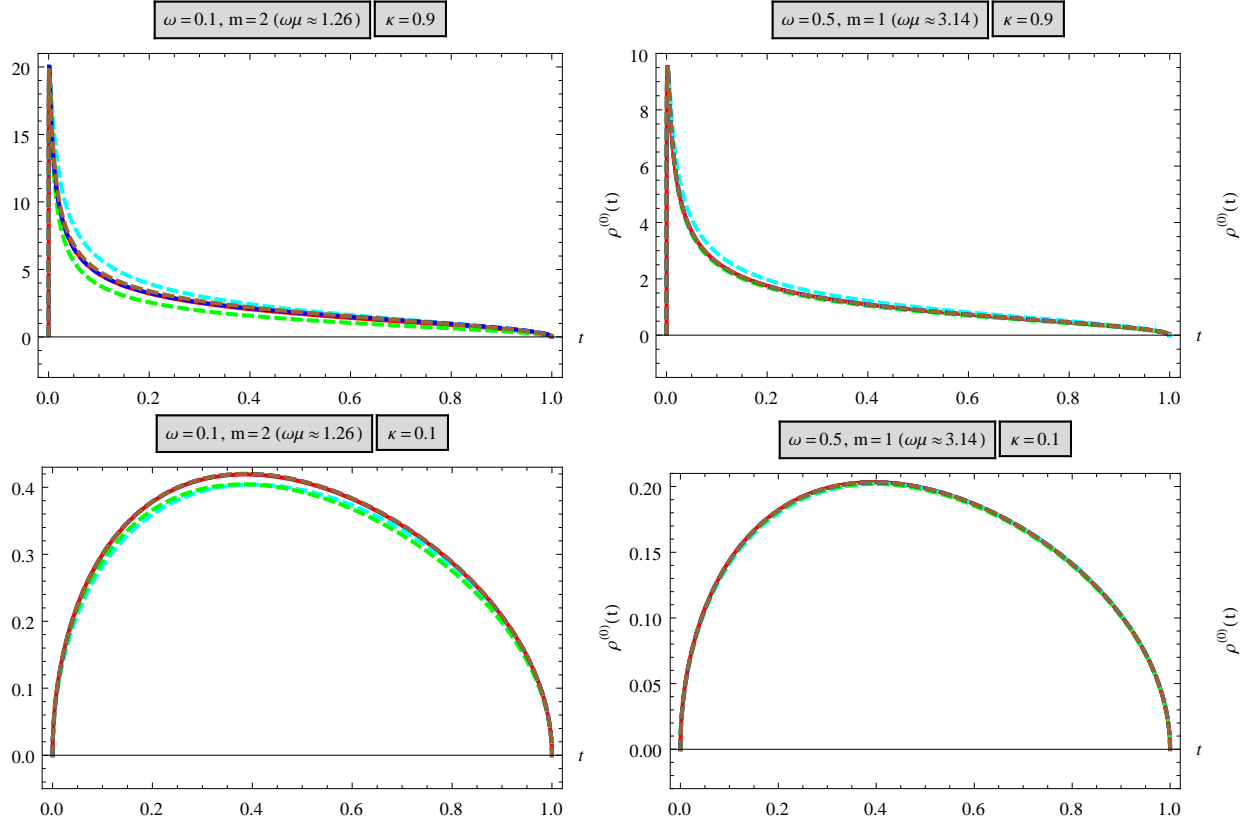


FIG. 35: The exact density, $[0, 1] \ni t \mapsto \rho^{(0)}(t)$ (773) (solid red line), compared to various truncations of its large- η approximation (803): one (dashed cyan), two (dashed green), three (dashed blue), four (dashed brown) leading orders kept. The values of the parameters are: $\kappa = 0.9$ (TOP), $\kappa = 0.1$ (BOTTOM), and $\omega = 0.1, m = 2$ (*i.e.*, $\eta = 0.4\pi \approx 1.26$) (LEFT), $\omega = 0.5, m = 1$ (*i.e.*, $\eta = \pi \approx 3.14$) (RIGHT).

and then we form from them $G^{(0),\text{bulk}}(Z)$,

$$G^{(0),\text{bulk}}(Z) = \frac{1}{\omega} \left(\eta G_{-1}^{(0),\text{bulk}}(Z) + G_0^{(0),\text{bulk}}(Z) + \frac{1}{\eta} G_1^{(0),\text{bulk}}(Z) + \frac{1}{\eta^2} G_2^{(0),\text{bulk}}(Z) + \mathcal{O}\left(\frac{1}{\eta^3}\right) \right), \quad (809)$$

it will be correct, *i.e.*, equal to $G^{(0)}(z)|_{z=a+(b-a)Z}$, only when the condition $\kappa \leq 1$ (801) is satisfied. Let us print the explicit expressions for the four leading coefficients, for $\mu > 0$, without assuming (801), in order to show that the integration (808) produces factors of $|1 - \kappa^2|$,

$$G_{-1}^{(0),\text{bulk}}(Z) = \frac{1}{2} - \frac{|1 - \kappa^2|}{2((1 - \kappa)^2 + 4\kappa Z)} - \frac{1}{(1 - \kappa)^2 + 4\kappa Z} \sqrt{Z} \sqrt{Z - 1} 2\kappa, \quad (810)$$

$$G_0^{(0),\text{bulk}}(Z) = -\frac{1}{8} (1 + \kappa^2) + \frac{|1 - \kappa^2|}{8((1 - \kappa)^2 + 4\kappa Z)^2} \left((1 - \kappa)^4 + 8\kappa (1 + \kappa^2) Z \right) - \frac{1}{((1 - \kappa)^2 + 4\kappa Z)^2} \sqrt{Z} \sqrt{Z - 1} \kappa^2 \left((1 - \kappa)^2 - 2(1 + \kappa^2) Z \right), \quad (811)$$

$$\begin{aligned}
G_1^{(0),\text{bulk}}(Z) = & \frac{1}{32} (1 + \kappa^2) ((1 - \kappa)^2 + 4\kappa Z) - \\
& - \frac{|1 - \kappa^2|}{32 ((1 - \kappa)^2 + 4\kappa Z)^3} \cdot \\
& \cdot \left((1 - \kappa)^4 (1 - 2\kappa + 4\kappa^2 - 2\kappa^3 + \kappa^4) + \right. \\
& + 4\kappa(1 - \kappa)^2 (3 - 6\kappa + 2\kappa^2 - 6\kappa^3 + 3\kappa^4) Z + \\
& + 16\kappa^2 (3 + 4\kappa^2 + 3\kappa^4) Z^2 \Big) - \\
& - \frac{1}{8 ((1 - \kappa)^2 + 4\kappa Z)^3} \sqrt{Z} \sqrt{Z - 1} \kappa \cdot \\
& \cdot \left((1 - \kappa)^4 (1 - 2\kappa + 6\kappa^2 - 2\kappa^3 + \kappa^4) + \right. \\
& + 4\kappa(1 - \kappa)^2 (1 + \kappa^2) (3 - 10\kappa + 3\kappa^2) Z + \\
& + 32\kappa^2 (1 + \kappa^2) (2 - 3\kappa + 2\kappa^2) Z^2 + \\
& + 64\kappa^3 (1 + \kappa^2) Z^3 \Big), \tag{812}
\end{aligned}$$

$$\begin{aligned}
G_2^{(0),\text{bulk}}(Z) = & - \frac{1}{8} \kappa^2 (1 + \kappa^2) Z(Z - 1) - \\
& - \frac{|1 - \kappa^2|^3}{8 ((1 - \kappa)^2 + 4\kappa Z)^4} \kappa^2 Z \cdot \\
& \cdot \left((1 - \kappa)^4 - \right. \\
& - (1 - 12\kappa + 6\kappa^2 - 12\kappa^3 + \kappa^4) Z - \\
& - 8\kappa (1 + \kappa^2) Z^2 \Big) + \\
& + \frac{1}{8 ((1 - \kappa)^2 + 4\kappa Z)^4} \sqrt{Z} \sqrt{Z - 1} \kappa^3 \cdot \\
& \cdot \left((1 - \kappa)^6 (1 - \kappa + \kappa^2) - \right. \\
& - 2(1 - \kappa)^4 (4 - 15\kappa + 4\kappa^2 - 15\kappa^3 + 4\kappa^4) Z + \\
& + 8(1 - \kappa)^2 (1 - 16\kappa + 21\kappa^2 - 30\kappa^3 + 21\kappa^4 - 16\kappa^5 + \kappa^6) Z^2 + \\
& + 32\kappa (1 + \kappa^2) (3 - 16\kappa + 25\kappa^2 - 16\kappa^3 + 3\kappa^4) Z^3 + \\
& + 128\kappa^2 (1 + \kappa^2) (2 - 5\kappa + 2\kappa^2) Z^4 + \\
& + 256\kappa^3 (1 + \kappa^2) Z^5 \Big). \tag{813}
\end{aligned}$$

The agreement, for $\kappa \leq 1$, of these coefficients with the expansion of the exact result, serves as a confirmation of the validity of the general technique we employ, namely, the method of changing variables from y to t .

Let us briefly reiterate on the discussion from the end of paragraph III A 3. Namely, recall that Z , unlike t , may be of any order at large η , depending on how we define a domain of the underlying complex argument z . Therefore, if we are dealing with a large- η series whose terms are functions of Z , each term needs to be further expanded. This may cause the “problem of terms of equal order,” *i.e.*, that an infinite number of terms of the initial series will contribute to a given term of the resulting expansion. Let us show an example of this situation. Consider the denominator

appearing in the resolvents $G^{(1)}(z)$ (456), $G^{(2)}(z)$ (459), and its value at $z = \omega$, which can easily be exactly computed,

$$\frac{1}{z^2} \left(-2(z^2 - \omega^2) G^{(0)}(z) + z^2 \mu - z + \omega^2 \mu (2\alpha - 1) \right) \Big|_{z=\omega} = \frac{\mu}{\omega^2} \sqrt{P(\omega)} = \frac{1 + \kappa^2}{\omega}. \quad (814)$$

Now, we try to derive it approximately: We change z to Z , and instead of $G^{(0)}(Z)$, we use the expansion (809). We get the following structure,

$$(\text{the l.h.s. of eq. (814)}) = \frac{1}{\omega} \left((\text{term I})_Z + \frac{1}{\eta} (\text{term II})_Z + \frac{1}{\eta^2} (\text{term III})_Z + \left(\frac{1}{\eta^3} \right) \right). \quad (815)$$

But Z corresponding to $z = \omega$ depends on η ,

$$Z|_{z=\omega} = \frac{\omega - a}{b - a} = -\frac{(1 - \kappa)^2}{4\kappa} + \frac{1}{\eta} \frac{(1 - \kappa^2)^2}{16\kappa} - \frac{1}{\eta^2} \frac{(1 - \kappa^2)(1 - \kappa^4)}{64\kappa} - \frac{1}{\eta^3} \frac{\kappa(1 - \kappa^2)^2}{128} + \mathcal{O}\left(\frac{1}{\eta^4}\right), \quad (816)$$

and consequently, each term must be further expanded,

$$\begin{aligned} (\text{term I})_Z &= -4\eta + (1 + \kappa^2) - \frac{1}{2} \frac{1}{\eta} (1 + \kappa^2)^2 + \\ &\quad + \frac{1}{16} \frac{1}{\eta^2} (1 + \kappa^2)(1 + \kappa^4) + \mathcal{O}\left(\frac{1}{\eta^3}\right), \end{aligned} \quad (817)$$

$$\begin{aligned} \frac{1}{\eta} (\text{term II})_Z &= -12\eta + 8(1 + \kappa^2) - \\ &\quad - \frac{1}{4} \frac{1}{\eta} (17 + 40\kappa^2 + 17\kappa^4) + \\ &\quad + \frac{1}{8} \frac{1}{\eta^2} (1 + \kappa^2)(11 + 16\kappa^2 + 11\kappa^4) + \mathcal{O}\left(\frac{1}{\eta^3}\right), \end{aligned} \quad (818)$$

$$\begin{aligned} \frac{1}{\eta^2} (\text{term III})_Z &= -32\eta + 32(1 + \kappa^2) - \\ &\quad - \frac{1}{4} \frac{1}{\eta} (91 + 204\kappa^2 + 91\kappa^4) + \\ &\quad + \frac{1}{16} \frac{1}{\eta^2} (1 + \kappa^2)(171 + 332\kappa^2 + 171\kappa^4) + \mathcal{O}\left(\frac{1}{\eta^3}\right), \end{aligned} \quad (819)$$

etc. Unfortunately, each of these series commences at the same order, and hence, the entire expansion (815) would be necessary to recover even the leading term of the quantity in question.

4. The Perturbative Quantities $G^{(2)}(0)$ and $\mathcal{F}(z)$

Similarly as in paragraph IV B 4, we are now able to calculate the large- η expansions of two important objects, namely, the r.h.s. of the momentum condition at the level $\mathcal{O}(1/J^2)$ (462),

$$\begin{aligned} G^{(2)}(0) &= \frac{1}{\omega^3} \left(-\eta^4 \frac{\kappa^2(1 + 3\kappa^2 + \kappa^4)}{4(1 - \kappa^2)^7} - \eta^2 \frac{\kappa^2(2 + 7\kappa^2 + 12\kappa^4 + 7\kappa^6 + 2\kappa^8)}{48(1 - \kappa^2)^7} + \right. \\ &\quad \left. + \frac{\kappa^2(6 + 22\kappa^2 + 69\kappa^4 + 8\kappa^6 + 20\kappa^8 - 6\kappa^{10} + \kappa^{12})}{1536(1 - \kappa^2)^7} + \mathcal{O}\left(\frac{1}{\eta^2}\right) \right). \end{aligned} \quad (820)$$

and the contribution of the quantity $\mathcal{F}(z)$ (461) to the energy at the level $\mathcal{O}(1/J^2)$, *i.e.*, to $\gamma^{(2)}$,

$$\begin{aligned} \gamma^{(2), \text{from } \mathcal{F}(z)} &= \frac{c}{\omega^2 \mu} \mathcal{F}(c) = \frac{1}{\omega^4} \left(-\eta^4 \frac{\kappa^2(1 + 3\kappa^2 + \kappa^4)}{4(1 - \kappa^2)^7} + \eta^3 \frac{\kappa^2(3 + \kappa^2)(1 + 3\kappa^2)}{6(1 - \kappa^2)^6} - \right. \\ &\quad \left. - \eta^2 \frac{\kappa^2(25 + 40\kappa^2 + 12\kappa^4 + 4\kappa^6 - 21\kappa^8)}{96(1 - \kappa^2)^7} + \mathcal{O}(\eta) \right). \end{aligned} \quad (821)$$

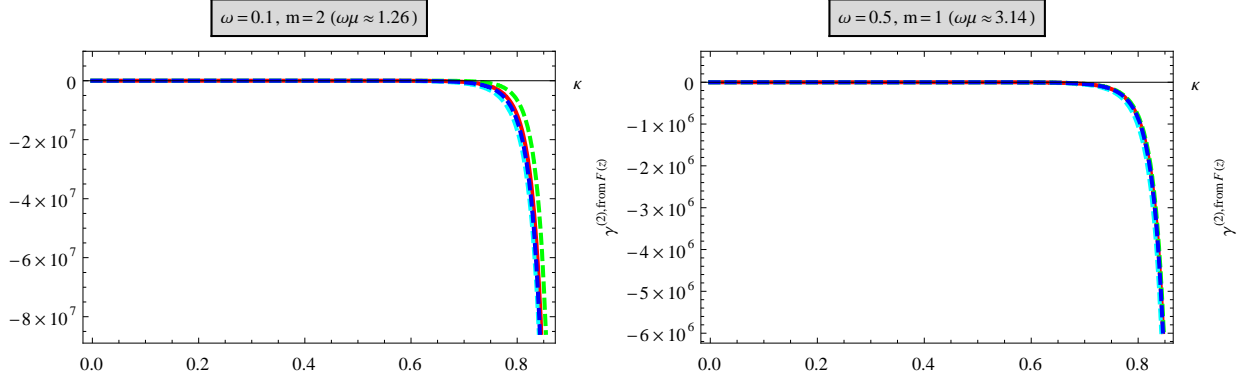


FIG. 36: The contribution of the term $\mathcal{F}(z)$ to $\gamma^{(2)}$ (821), as a function of $\kappa \in [0, 1]$. The exact value (solid red) is compared to its large- η expansion, terminated at one (dashed cyan), two (dashed green), three (dashed blue) leading orders. The values of the parameters are: $\omega = 0.1$, $m = 2$ (*i.e.*, $\eta = 0.4\pi \approx 1.26$) (LEFT), $\omega = 0.5$, $m = 1$ (*i.e.*, $\eta = \pi \approx 3.14$) (RIGHT).

We have assumed $\mu > 0$. Remark that they both diverge as η^4 at large η . This is particularly important for the latter one, as we will see that it constitutes the leading large- η contribution to the energy. Figure 36 shows how accurate these approximations are, even when the expansion parameter η is not very large.

The quantity $\mathcal{F}(z)$ is composed of the values of the resolvent and its three lowest derivatives at the points $\pm\omega$ (733)–(736). Let us also print their large- η series, as follow from the adapted truncation of $\gamma^{(0)}$ (802),

$$G^{(0)}(\omega) = -\frac{1}{\omega} \left(\eta \frac{\kappa^2}{1-\kappa^2} + \frac{\kappa^2}{4} + \frac{1}{\eta} \frac{\kappa^2(1+\kappa^2)}{16(1-\kappa^2)} + \mathcal{O}\left(\frac{1}{\eta^3}\right) \right), \quad (822)$$

$$G^{(0)}(-\omega) = -\frac{1}{\omega} \left(\frac{\kappa^2}{4} + \frac{1}{\eta} \frac{\kappa^2}{16} + \mathcal{O}\left(\frac{1}{\eta^3}\right) \right), \quad (823)$$

$$G^{(0)'}(\omega) = -\frac{1}{\omega^2} \left(\eta^2 \frac{2\kappa^2}{(1-\kappa^2)^3} + \frac{\kappa^2(1+3\kappa^2-\kappa^4+\kappa^6)}{8(1-\kappa^2)^3} + \mathcal{O}\left(\frac{1}{\eta^2}\right) \right), \quad (824)$$

$$G^{(0)'}(-\omega) = -\frac{1}{\omega^2} \left(\frac{\kappa^2}{8} + \mathcal{O}\left(\frac{1}{\eta^2}\right) \right), \quad (825)$$

$$G^{(0)''}(\omega) = -\frac{1}{\omega^3} \left(\eta^3 \frac{8\kappa^2(1+\kappa^2)}{(1-\kappa^2)^5} - \eta^2 \frac{2\kappa^2}{(1-\kappa^2)^3} + \eta \frac{\kappa^2(1+\kappa^2)(1+\kappa^2+\kappa^4)}{(1-\kappa^2)^5} - \frac{\kappa^2(1+3\kappa^2-\kappa^4+\kappa^6)}{8(1-\kappa^2)^3} + \mathcal{O}\left(\frac{1}{\eta}\right) \right), \quad (826)$$

$$G^{(0)''}(-\omega) = -\frac{1}{\omega^3} \left(\frac{\kappa^2}{8} + \mathcal{O}\left(\frac{1}{\eta}\right) \right), \quad (827)$$

$$\begin{aligned}
G^{(0)'''}(\omega) = & -\frac{1}{\omega^4} \left(\eta^4 \frac{48\kappa^2 (1 + 3\kappa^2 + \kappa^4)}{(1 - \kappa^2)^7} - \eta^3 \frac{24\kappa^2 (1 + \kappa^2)}{(1 - \kappa^2)^5} + \right. \\
& + \eta^2 \frac{12\kappa^2 (1 + \kappa^2 + 6\kappa^4 + \kappa^6 + \kappa^8)}{(1 - \kappa^2)^7} - \\
& \left. - \eta \frac{3\kappa^2 (1 + \kappa^2) (1 + \kappa^2 + \kappa^4)}{(1 - \kappa^2)^5} + O(\eta^0) \right), \tag{828}
\end{aligned}$$

$$G^{(0)''' }(-\omega) = O(\eta^0). \tag{829}$$

Remark that for $z = -\omega$, there occurs a special cancelation which causes the highest order to be $O(\eta^0)$ for the resolvent and all the three derivatives, even though it naively looks like the highest order might be $O(\eta^4)$.

V. SOLVING THE ALL-LOOP ONE-CUT QUADRATIC EQUATION AT THE ORDER $O(1/J)$

The all-loop quadratic equation at the level $O(1/J)$ is actually linear and trivially solved (456). What remains to be done, however, is to calculate • the “phase” term $\mathcal{P}^{(1)}(z)$ (441), (443), and • the “anomaly” term, $\mathcal{A}^{(1)}(z)$ (430), (432), (434). For simplicity, we will henceforth assume $\mu > 0$.

A. The Phase $\mathcal{P}^{(1)}(z)$

1. The Exact Phase $\mathcal{P}^{(1)}(z)$

As we have discussed in paragraph II C 6, the practical way to handle the large- J leading-order Hernández-López phase input to the all-loop quadratic equation, $\mathcal{P}^{(1)}(z)$, is to consider its integral form (443), which, as we have already understood, does not give rise to any boundary terms, $\mathcal{P}^{(1),\text{bulk}}(z) = \mathcal{P}^{(1)}(z)$, and thus, may be safely exploited.

It is an involved double integral, but it happens that it is exactly doable! Let us first introduce a new notation; by a hat we will denote variables rescaled by ω ,

$$\hat{a} \equiv \frac{a}{\omega}, \quad \hat{b} \equiv \frac{b}{\omega}, \quad \hat{c} \equiv \frac{c}{\omega}, \quad \hat{z} \equiv \frac{z}{\omega}, \quad \hat{y} \equiv \frac{y}{\omega}, \quad (830)$$

and by a check, the following combination of our variables,

$$\check{a} \equiv \frac{a + \omega}{a - \omega}, \quad \check{b} \equiv \frac{b + \omega}{b - \omega}, \quad \check{c} \equiv \frac{c + \omega}{c - \omega}, \quad \check{z} \equiv \frac{z + \omega}{z - \omega}, \quad \check{y} \equiv \frac{y + \omega}{y - \omega}. \quad (831)$$

Note: The ordering of the roots of $P(z)$ true for $\mu > 0$, i.e., $-\omega < c < 0 < \omega < a < b$ (789), translates in these new variables to $1 < \check{b} < \check{a}$ and $-1 < \check{c} < 0$. Now, using the exact large- J leading-order density (742), the relevant integral reads

$$\begin{aligned} \mathcal{P}^{(1)}(z) = & \frac{\mu^2}{8\pi^3\omega} \int_{\hat{a}}^{\hat{b}} \int_{\hat{a}}^{\hat{b}} d\hat{y}_1 d\hat{y}_2 \frac{\hat{y}_1 - \hat{c}}{\hat{y}_1^2 - 1} \sqrt{(\hat{y}_1 - \hat{a})(\hat{b} - \hat{y}_1)} \frac{\hat{y}_2 - \hat{c}}{\hat{y}_2^2 - 1} \sqrt{(\hat{y}_2 - \hat{a})(\hat{b} - \hat{y}_2)} \frac{1}{(\hat{z} - \hat{y}_1)(\hat{z} - \hat{y}_2)} \\ & \cdot \left(\frac{-2}{1 - \hat{y}_1 \hat{y}_2} + \left(\frac{1}{\hat{y}_1 - \hat{y}_2} + \frac{\hat{y}_1 - \hat{y}_2}{(1 - \hat{y}_1 \hat{y}_2)^2} \right) \log \left(\frac{(\hat{y}_1 + 1)(\hat{y}_2 - 1)}{(\hat{y}_1 - 1)(\hat{y}_2 + 1)} \right) \right). \end{aligned} \quad (832)$$

A somewhat long computation yields the following exact result,

$$\begin{aligned} \mathcal{P}^{(1)}(z) = & \frac{\mu^2}{16\pi\omega} \frac{(1 - \check{z})^2}{\check{z}^3(1 - \check{a})(1 - \check{b})(1 - \check{c})^2} \\ & \cdot \left(2 \left(\sqrt{\check{a}} - \sqrt{\check{b}} \right)^2 \check{z} (\check{c}^2 - \check{c}\check{z} + \check{z}^2) + \right. \\ & + \left(-6\check{a}\check{b}\check{c}^2 + 3(2\check{a}\check{b}\check{c} + (\check{a} + \check{b})\check{c}^2) \check{z} + 16\sqrt{\check{a}\check{b}\check{c}}\check{z}^2 + 3(\check{a} + \check{b} + 2\check{c})\check{z}^3 - 6\check{z}^4 + \right. \\ & \left. + 8\sqrt{\check{z} - \check{a}}\sqrt{\check{z} - \check{b}}(\check{c} - \check{z}) \left(\sqrt{\check{a}\check{b}\check{c}} + \check{z}^2 \right) \right) \log(2) + \\ & + 4 \left(\check{a}\check{b}\check{c}^2 - \check{a}\check{b}\check{c}\check{z} - \left(\check{a} + \check{b} + 4\sqrt{\check{a}\check{b}} + \check{c} \right) \check{c}\check{z}^2 - (\check{a} + \check{b} + \check{c})\check{z}^3 + 2\check{z}^4 - \right. \end{aligned}$$

$$\begin{aligned}
& -\sqrt{\check{z}-\check{a}}\sqrt{\check{z}-\check{b}}(\check{c}-\check{z})\left(4\check{z}^2-\check{c}\sqrt{\check{z}+\check{a}}\sqrt{\check{z}+\check{b}}\right)\log\left(\sqrt{\check{a}}+\sqrt{\check{b}}\right)+ \\
& +\left(-2\check{a}\check{b}\check{c}^2+(2\check{a}\check{b}+(\check{a}+\check{b})\check{c})\check{c}\check{z}+(\check{a}+\check{b}+2\check{c})\check{z}^3-2\check{z}^4\right)\log(\check{a}+\check{b})+ \\
& +\check{c}\left(-2\check{a}\check{b}\check{c}+(2\check{a}\check{b}+(\check{a}+\check{b})\check{c})\check{z}+4\sqrt{\check{a}\check{b}}\check{z}^2+4\sqrt{\check{a}\check{b}}\sqrt{\check{z}-\check{a}}\sqrt{\check{z}-\check{b}}(\check{c}-\check{z})\right)\log(\check{a}\check{b})+ \\
& +4(\check{c}-\check{z})\left(\check{a}\check{b}\check{c}-(\check{a}+\check{b})\check{c}\check{z}+\check{c}\check{z}^2-2\left(\sqrt{\check{a}\check{b}\check{c}}-\check{z}^2\right)\sqrt{\check{z}-\check{a}}\sqrt{\check{z}-\check{b}}\right)\log\left(\sqrt{\check{a}\check{b}}+\check{z}-\sqrt{\check{z}-\check{a}}\sqrt{\check{z}-\check{b}}\right)+ \\
& +4\check{c}(\check{c}-\check{z})\sqrt{\check{a}^2-\check{z}^2}\sqrt{\check{b}^2-\check{z}^2}\log\left(\sqrt{\check{a}\check{b}}-\check{z}+\sqrt{\check{z}+\check{a}}\sqrt{\check{z}+\check{b}}\right)- \\
& -\left(2\check{a}^2\check{b}^2\check{c}-\check{a}\check{b}(\check{a}+\check{b})\check{c}\check{z}-\check{a}\check{b}(\check{a}+\check{b})\check{z}^2+(2(\check{a}^2+\check{b}^2)+(\check{a}+\check{b})\check{c})\check{z}^3+(\check{a}+\check{b}-2\check{c})\check{z}^4-4\check{z}^5\right). \\
& \cdot\frac{\check{c}-\check{z}}{\sqrt{\check{a}^2-\check{z}^2}\sqrt{\check{b}^2-\check{z}^2}}\log\left(\frac{\check{a}\check{b}+\check{z}^2+\sqrt{\check{a}^2-\check{z}^2}\sqrt{\check{b}^2-\check{z}^2}}{\check{a}+\check{b}}\right)- \\
& -\check{z}(\check{a}+\check{b}-2\check{z})(\check{c}-\check{z})^2\log\left(\frac{\left(\sqrt{\check{z}-\check{a}}+\sqrt{\check{z}-\check{b}}\right)^2}{-2\check{a}\check{b}+(\check{a}+\check{b})\check{z}+2\sqrt{\check{a}\check{b}}\sqrt{\check{z}-\check{a}}\sqrt{\check{z}-\check{b}}}\right)- \\
& -\frac{\sqrt{\check{z}-\check{a}}\sqrt{\check{z}-\check{b}}\check{z}(\check{c}^2-\check{z}^2)(\check{a}+\check{b}+2\check{z})}{\sqrt{\check{z}+\check{a}}\sqrt{\check{z}+\check{b}}}\log\left(\frac{\left(\sqrt{\check{z}+\check{a}}+\sqrt{\check{z}+\check{b}}\right)^2}{2\check{a}\check{b}+(\check{a}+\check{b})\check{z}+2\sqrt{\check{a}\check{b}}\sqrt{\check{z}+\check{a}}\sqrt{\check{z}+\check{b}}}\right)- \\
& -\frac{\check{z}^2(\check{c}^2-\check{z}^2)(\check{a}^2+\check{b}^2-2\check{z}^2)}{2\sqrt{\check{a}^2-\check{z}^2}\sqrt{\check{b}^2-\check{z}^2}}\log\left(\frac{\left(\sqrt{\check{a}^2-\check{z}^2}+\sqrt{\check{b}^2-\check{z}^2}\right)^2}{2\check{a}^2\check{b}^2-(\check{a}^2+\check{b}^2)\check{z}^2+2\check{a}\check{b}\sqrt{\check{a}^2-\check{z}^2}\sqrt{\check{b}^2-\check{z}^2}}\right)\Bigg). \tag{833}
\end{aligned}$$

It would take too much space to describe in detail the derivation. Let us just mention that we have exploited a certain technique to deal with integrals containing logarithms. Namely, we rewrite them as contour integrals over the interval $[a, b]$, and subsequently, we deform the contour in such a way that it encircles the cut of the logarithm, and possibly some other poles. The difference of the values of the logarithm above and below its cut is $2\pi i$, in which way the logarithm is removed, and the integral reduced to a rational one. To ensure the reader about the agreement between (833) and (832), in figure 37, we plot the quantities (838), (839) based on either of these formulae, observing perfect alignment.

Comments:

- The domain of the variable z is

$$z \in \mathbb{C}, \quad z \notin [a, b], \quad i.e., \quad \check{z} \in \mathbb{C}, \quad \check{z} \notin [\check{b}, \check{a}], \tag{834}$$

since otherwise z would lie on the integration interval $[a, b]$, and the integral (832) would develop a singularity.

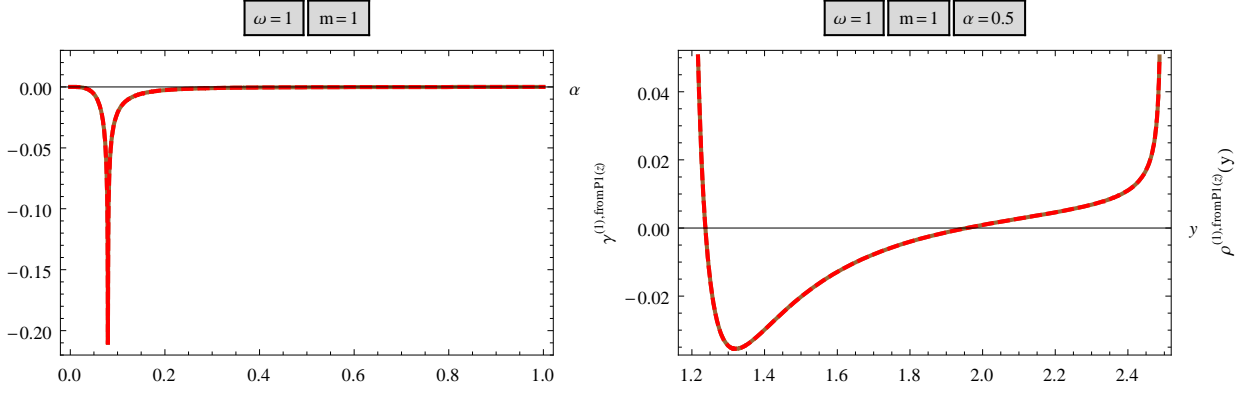


FIG. 37: LEFT: The contribution of the term $\mathcal{P}^{(1)}(z)$ to the energy $\gamma^{(1)}$, as a function of $\alpha \in [0, 1]$. Its definition, based on the integral (832), is plotted in solid brown, while its explicit formula, based on the result (833), in dashed red. The values of the parameters are: $\omega = 1$, $m = 1$. The numerical integration has been performed with `WorkingPrecision = MachinePrecision`.

RIGHT: The contribution of the term $\mathcal{P}^{(1)}(z)$ to the density $\rho^{(1)}(y)$, as a function of $y \in [a, b]$. The colors of the lines as in the left figure.

The values of the parameters are: $\omega = 1$, $m = 1$, $\alpha = 0.5$, which amounts for $a \approx 1.19$, $b \approx 2.49$, $c \approx -0.58$. The numerical integration has been performed with `WorkingPrecision = MachinePrecision`, and $\epsilon = 10^{-4}$.

- The integration process that has led to the expression (833) is valid only for $z \neq \omega$ and $z \neq -\omega$ (i.e., $\tilde{z} \neq \infty$ and $\tilde{z} \neq 0$). It can quickly be seen from (832) — for these two particular arguments, the multiplicity of the roots w.r.t. \hat{y}_1, \hat{y}_2 in the denominator changes, and hence, the necessary fractional decomposition, and consequently the integration process, should be separately repeated. We then obtain

$$\begin{aligned} \mathcal{P}^{(1)}(\omega) &= \frac{\mu^2}{96\pi\omega} \frac{1}{(1-\tilde{a})(1-\tilde{b})(1-\tilde{c})^2} \cdot \\ &\cdot \left(2 \left((\tilde{a} + \tilde{b})^3 - 12\tilde{a}\tilde{b}\tilde{c} + 6(\tilde{a} + \tilde{b})\tilde{c}^2 - \sqrt{\tilde{a}\tilde{b}}(3\tilde{a}^2 + 2\tilde{a}\tilde{b} + 3\tilde{b}^2 - 6(\tilde{a} + \tilde{b})\tilde{c} + 12\tilde{c}^2) \right) + \right. \\ &+ 3 \left(-(\tilde{a} - \tilde{b})^2(\tilde{a} + \tilde{b}) + 2(\tilde{a}^2 + 10\tilde{a}\tilde{b} + \tilde{b}^2)\tilde{c} + 12(\tilde{a} + \tilde{b})\tilde{c}^2 + 16\sqrt{\tilde{a}\tilde{b}}\tilde{c}(\tilde{a} + \tilde{b} + 2\tilde{c}) \right) \log(2) - \\ &- 6(\tilde{a} + \tilde{b} + 2\tilde{c}) \left(-(\tilde{a} - \tilde{b})^2 + 4(\tilde{a} + \tilde{b})\tilde{c} + 8\sqrt{\tilde{a}\tilde{b}}\tilde{c} \right) \log(\sqrt{\tilde{a}} + \sqrt{\tilde{b}}) - \\ &- 3(\tilde{a} + \tilde{b}) \left((\tilde{a} - \tilde{b})^2 - 2(\tilde{a} + \tilde{b})\tilde{c} - 4\tilde{c}^2 \right) \log(\tilde{a} + \tilde{b}) + \\ &\left. + 6\tilde{c} \left(2\tilde{a}\tilde{b} + (\tilde{a} + \tilde{b})\tilde{c} + 2\sqrt{\tilde{a}\tilde{b}}(\tilde{a} + \tilde{b} + 2\tilde{c}) \right) \log(\tilde{a}\tilde{b}) \right), \end{aligned} \quad (835)$$

and

$$\mathcal{P}^{(1)}(-\omega) = \mathcal{P}^{(1)}(\omega) \Big|_{\omega \rightarrow -\omega} = \mathcal{P}^{(1)}(\omega) \Big|_{\tilde{a} \rightarrow 1/\tilde{a}, \tilde{b} \rightarrow 1/\tilde{b}, \tilde{c} \rightarrow 1/\tilde{c}}. \quad (836)$$

This complication implies in particular that we need to check the continuity of the function $z \mapsto \mathcal{P}^{(1)}(z)$ at these two points. To this end, we have to take the limits $z \rightarrow \pm\omega$ of the general result (833). It naively seems to explode there, but a careful investigation reveals it to be a $0/0$ symbol, which upon triple application of the l'Hôpital rule yields the finite values identical to (835), (836),

$$\lim_{z \rightarrow \pm\omega} \mathcal{P}^{(1)}(z) = \mathcal{P}^{(1)}(\pm\omega), \quad (837)$$

and therefore, the function in question is continuous.

- There are two important quantities based on $\mathcal{P}^{(1)}(z)$. The first one is its contribution to the large- J next-to-leading-order energy, *i.e.*, to $\gamma^{(1)}$, which, as we will shortly see in paragraph V C 1, is proportional to the value of the function at the double root, $z = c$,

$$\gamma^{(1), \text{from } \mathcal{P}^{(1)}(z)} \equiv -\frac{c}{\omega^2 \mu} \mathcal{P}^{(1)}(c) = \frac{\mu}{8\pi\omega^2} \frac{1 + \check{c}}{(1 - \check{a})(1 - \check{b})(1 - \check{c})}.$$

$$\cdot \left(\left(\sqrt{\check{a}} - \sqrt{\check{b}} \right)^2 + \left(3(\check{a} + \check{b}) + 8\sqrt{\check{a}\check{b}} \right) \log(2) - 4 \left(\sqrt{\check{a}} + \sqrt{\check{b}} \right)^2 \log \left(\sqrt{\check{a}} + \sqrt{\check{b}} \right) + \right.$$

$$\left. + (\check{a} + \check{b}) \log (\check{a} + \check{b}) + \left(\frac{\check{a} + \check{b}}{2} + 2\sqrt{\check{a}\check{b}} \right) \log (\check{a}\check{b}) \right). \quad (838)$$

- The second crucial object is the phase's input to the large- J next-to-leading-order density, which according to (456) reads

$$\rho^{(1), \text{from } \mathcal{P}^{(1)}(z)}(y) \equiv$$

$$\equiv -\frac{1}{8\pi\mu} \frac{\sqrt{(1 - \check{a})(1 - \check{b})(1 - \check{c})(1 + \check{y})^2}}{\sqrt{(\check{a} - \check{y})(\check{y} - \check{b})(\check{y} - \check{c})}} \lim_{\epsilon \rightarrow 0^+} \left(\mathcal{P}^{(1)}(z = y + i\epsilon) + \mathcal{P}^{(1)}(z = y - i\epsilon) \right) =$$

$$= \frac{\mu}{64\pi^2\omega} \frac{(1 - \check{y}^2)^2}{\sqrt{(1 - \check{a})(1 - \check{b})(1 - \check{c})\check{y}^3}}.$$

$$\cdot \left(\frac{1}{\sqrt{(\check{a} - \check{y})(\check{y} - \check{b})(\check{c} - \check{y})}} \left(2 \left(\sqrt{\check{a}} - \sqrt{\check{b}} \right)^2 \check{y} (\check{c}^2 - \check{c}\check{y} + \check{y}^2) + \right.$$

$$+ \left(-6\check{a}\check{b}\check{c}^2 + 3(2\check{a}\check{b}\check{c} + (\check{a} + \check{b})\check{c}^2) \check{y} + 16\sqrt{\check{a}\check{b}}\check{c}\check{y}^2 + 3(\check{a} + \check{b} + 2\check{c})\check{y}^3 - 6\check{y}^4 \right) \log(2) +$$

$$+ 4 \left(2\check{a}\check{b}\check{c}^2 - (2\check{a}\check{b} + (\check{a} + \check{b})\check{c}) \check{c}\check{y} - 4\sqrt{\check{a}\check{b}}\check{c}\check{y}^2 - (\check{a} + \check{b} + 2\check{c})\check{y}^3 + 2\check{y}^4 \right) \log \left(\sqrt{\check{a}} + \sqrt{\check{b}} \right) +$$

$$+ \left(-2\check{a}\check{b}\check{c}^2 + (2\check{a}\check{b} + (\check{a} + \check{b})\check{c}) \check{c}\check{y} + (\check{a} + \check{b} + 2\check{c})\check{y}^3 - 2\check{y}^4 \right) \log (\check{a} + \check{b}) +$$

$$+ \check{c} \left(-2\check{a}\check{b}\check{c} + (2\check{a}\check{b} + (\check{a} + \check{b})\check{c}) \check{y} + 4\sqrt{\check{a}\check{b}}\check{y}^2 \right) \log (\check{a}\check{b}) +$$

$$+ (\check{c} - \check{y}) \left(2\check{a}\check{b}\check{c} - (\check{a} + \check{b})\check{c}\check{y} - (\check{a} + \check{b})\check{y}^2 + 2\check{y}^3 \right) \log (\check{y}) \Big) -$$

$$- 8 \left(\sqrt{\check{a}\check{b}\check{c}} - \check{y}^2 \right) \arctan \left(\frac{\sqrt{(\check{a} - \check{y})(\check{y} - \check{b})}}{\sqrt{\check{a}\check{b}} + \check{y}} \right) +$$

$$+ \left(2\check{a}\check{b}\check{c} + (\check{a} + \check{b})\check{c}\check{y} - (\check{a} + \check{b})\check{y}^2 - 2\check{y}^3 \right) \frac{1}{\sqrt{(\check{a} + \check{y})(\check{b} + \check{y})}} \arctan \left(\frac{\sqrt{(\check{a}^2 - \check{y}^2)(\check{y}^2 - \check{b}^2)}}{\check{a}\check{b} + \check{y}^2} \right) \Big). \quad (839)$$

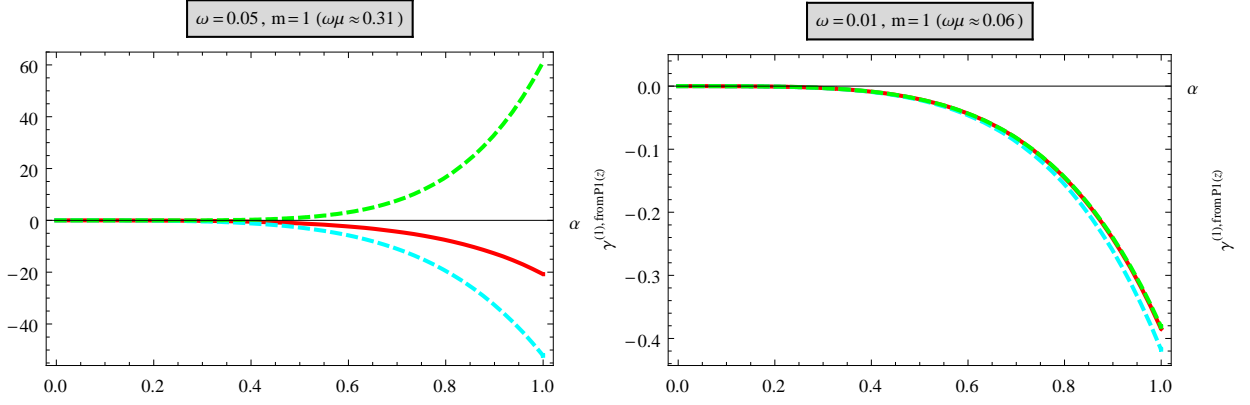


FIG. 38: The contribution of the term $\mathcal{P}^{(1)}(z)$ to the energy $\gamma^{(1)}$, as a function of $\alpha \in [0, 1]$. The exact value (838) (solid red) is compared to its small- η expansion (840), terminated at one (dashed cyan), or two (dashed green) leading orders. The values of the parameters are: $m = 1$ everywhere, $\omega = 0.05$ (*i.e.*, $\eta = 0.1\pi \approx 0.31$) (LEFT), $\omega = 0.01$ (*i.e.*, $\eta = 0.02\pi \approx 0.06$) (RIGHT).

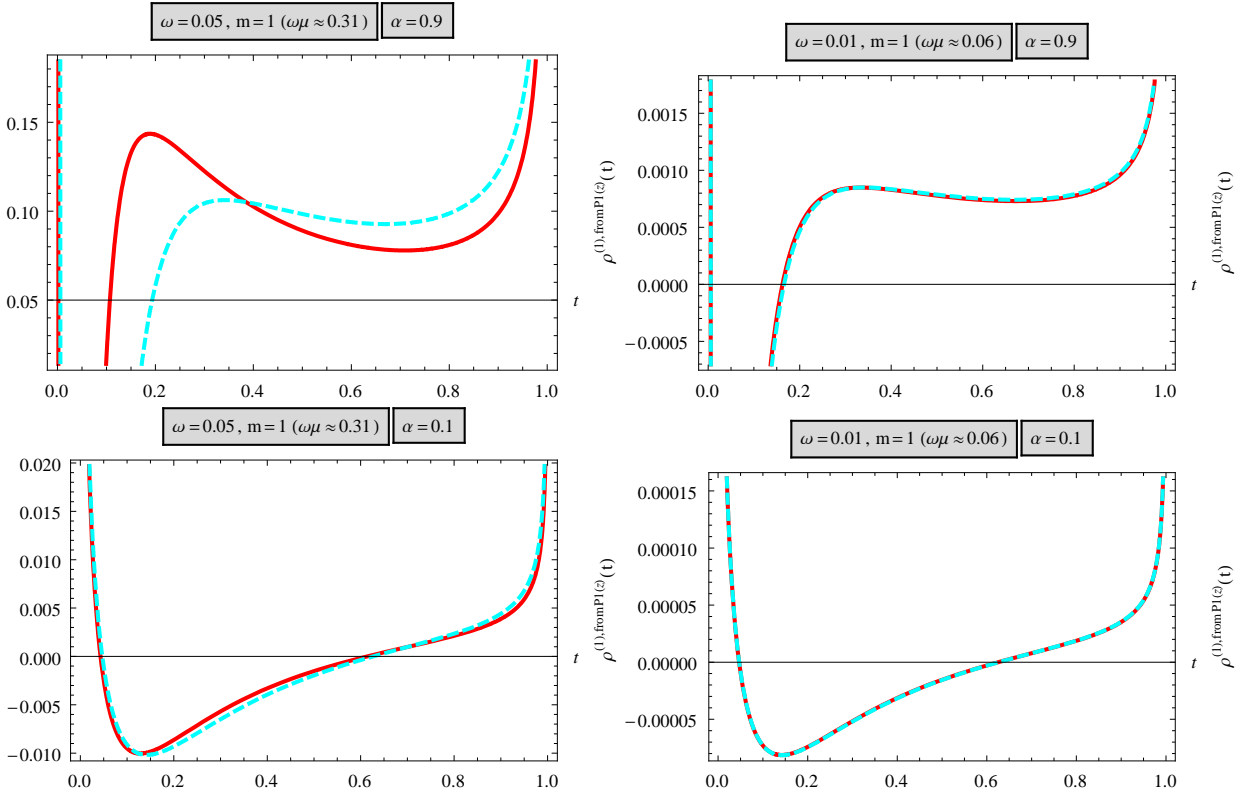


FIG. 39: The contribution of the term $\mathcal{P}^{(1)}(z)$ to the density $\rho^{(1)}(t)$, as a function of $t \in [0, 1]$. The exact value (839) (solid red) is compared to the leading-order of its small- η expansion (841) (dashed cyan). The values of the parameters are: $m = 1$ everywhere, $\alpha = 0.9$ (TOP), $\alpha = 0.1$ (BOTTOM), and $\omega = 0.05$ (*i.e.*, $\eta = 0.1\pi \approx 0.31$) (LEFT), $\omega = 0.01$ (*i.e.*, $\eta = 0.02\pi \approx 0.06$) (RIGHT).

2. The Weak-Coupling Limit

Having the exact formulae for our two important objects (838), (839), it is straightforward, though time-consuming, to obtain their series in various parameters. In this paragraph, let us expand them at weak coupling, *i.e.*, at small ω .

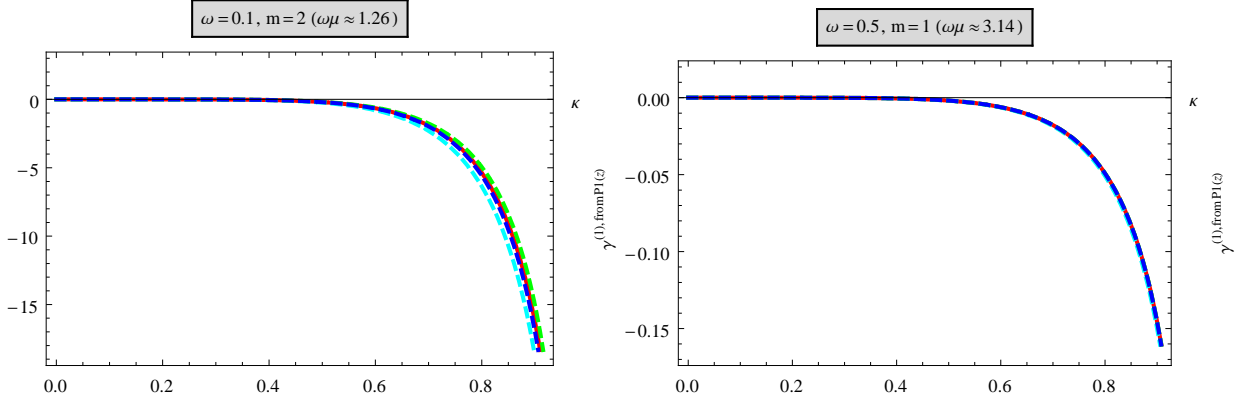


FIG. 40: The contribution of the term $\mathcal{P}^{(1)}(z)$ to the energy $\gamma^{(1)}$, as a function of $\kappa \in [0, 1]$. The exact value (838) (solid red) is compared to its large- η expansion (843), terminated at one (dashed cyan), two (dashed green), or three (dashed blue) leading orders. The values of the parameters are: $\omega = 0.1$, $m = 2$ (i.e., $\eta = 0.4\pi \approx 1.26$) (LEFT), $\omega = 0.5$, $m = 1$ (i.e., $\eta = \pi \approx 3.14$) (RIGHT).

The contribution to the energy at the level $O(1/J)$,

$$\gamma^{(1), \text{from } \mathcal{P}^{(1)}(z)} = \mu^3 \left(-\eta^3 \frac{8}{3\pi} \alpha^3 (1+\alpha)^3 + \eta^5 \frac{16}{3\pi} \alpha^3 (1+\alpha)^3 (3+8\alpha) + O(\eta^6) \right). \quad (840)$$

Notice that the Hernández-López phase commences only at the order $O(\eta^3)$. Moreover, unlike the weak-coupling series of the energy at the level $O(J^0)$ (771), and unlike the contributions of the anomaly $\mathcal{A}^{(1)}(z)$ (856) and of $G^{(0)'}(z)$ (892) to the energy at the level $O(1/J)$, where η appears only in even powers — here we find odd powers of η .

The contribution to the density at the level $O(1/J)$, necessarily in the variable t , happens to be quite simple,

$$\rho^{(1), \text{from } \mathcal{P}^{(1)}(z)}(t) = \eta^3 \rho_3^{(1), \text{from } \mathcal{P}^{(1)}(z)}(t) + O(\eta^4), \quad (841)$$

where

$$\rho_3^{(1), \text{from } \mathcal{P}^{(1)}(z)}(t) = \frac{1}{\sqrt{t(1-t)}} \frac{\mu^2}{96\pi^2} \frac{1}{\left(t - \frac{-(\sqrt{\alpha}-\sqrt{1+\alpha})^2}{4\sqrt{\alpha(1+\alpha)}} \right)^3} \cdot \left((\sqrt{\alpha}-\sqrt{1+\alpha})^4 - 8 \left(1+2\alpha+2\alpha^2 - (1+2\alpha)\sqrt{\alpha(1+\alpha)} \right) t + 8 \left(1+2\alpha+2\alpha^2 \right) t^2 \right). \quad (842)$$

Again, the leading weak-coupling term is $O(\eta^3)$.

Figures 38 and 39 reveal how accurate these leading-order approximations are.

3. The Large-Mode-Number, Fixed-Winding-Number Limit

Analogously, the two exact quantities (838), (839) can be expanded in the limit (784), i.e., at large μ , small α , fixed $\kappa = \sqrt{2\omega|\varpi|}$ (which we take to obey $\kappa \leq 1$ (801)). In both cases, we have found the three leading terms.

The contribution to the energy at the level $O(1/J)$,

$$\gamma^{(1), \text{from } \mathcal{P}^{(1)}(z)} = \frac{1}{\omega^3} \left(\frac{1}{16\pi} \left(2\kappa^2 + (3-\kappa^2) \log(1-\kappa^2) + (1+\kappa^2) \log(1+\kappa^2) \right) + \right.$$

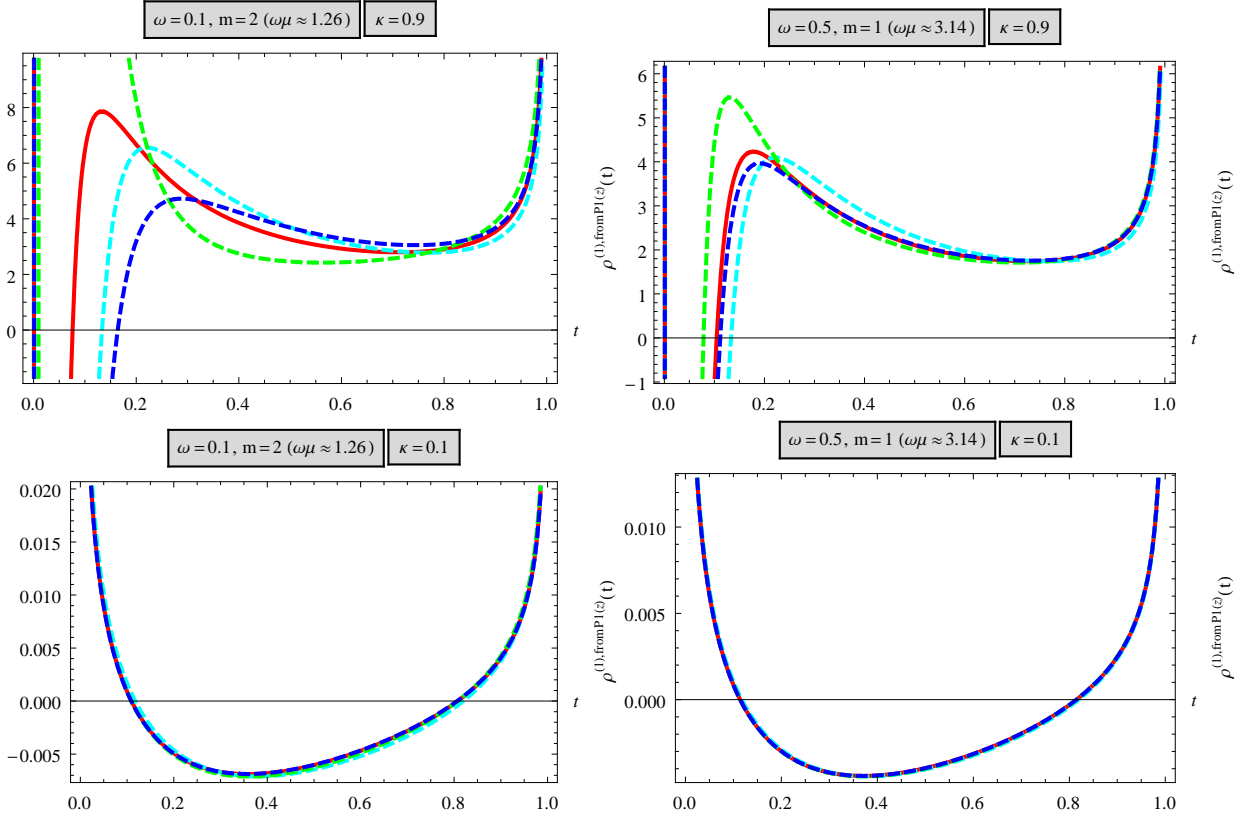


FIG. 41: The contribution of the term $\mathcal{P}^{(1)}(z)$ to the density $\rho^{(1)}(t)$, as a function of $t \in [0, 1]$. The exact value (839) (solid red) is compared to its large- η expansion (844), terminated at one (dashed cyan), two (dashed green), or three (dashed blue) leading orders.

The values of the parameters are: $\kappa = 0.9$ (TOP), $\kappa = 0.1$ (BOTTOM), and $\omega = 0.1$, $m = 2$ (i.e., $\eta = 0.4\pi \approx 1.26$) (LEFT), $\omega = 0.5$, $m = 1$ (i.e., $\eta = \pi \approx 3.14$) (RIGHT).

$$\begin{aligned}
 & + \frac{1}{\eta^2} \frac{\kappa^2}{128\pi} \left(2\kappa^2 + (1 - 3\kappa^2) \log(1 - \kappa^2) - (1 + \kappa^2) \log(1 + \kappa^2) \right) + \\
 & + \frac{1}{\eta^4} \frac{\kappa^2}{2048\pi} \left(-2\kappa^2 \left(3 + 9\kappa^2 + 5\kappa^4 \right) + \left(-3 + 4\kappa^2 + 20\kappa^4 + 9\kappa^6 \right) \log(1 - \kappa^2) + \right. \\
 & \left. + 3(1 + \kappa^2) \left(1 + 3\kappa^2 + \kappa^4 \right) \log(1 + \kappa^2) \right) + \mathcal{O}\left(\frac{1}{\eta^6}\right). \tag{843}
 \end{aligned}$$

We discover only even powers of η , and that the series begins at $\mathcal{O}(\eta^0)$, i.e., it is finite in the strict limit in question. In figure 40, this series is put to a test.

The contribution to the density at the level $\mathcal{O}(1/J)$,

$$\rho^{(1), \text{from } \mathcal{P}^{(1)}(z)}(t) = \frac{1}{\omega^2} \left(\eta^3 \rho_{-3}^{(1), \text{from } \mathcal{P}^{(1)}(z)}(t) + \eta^2 \rho_{-2}^{(1), \text{from } \mathcal{P}^{(1)}(z)}(t) + \eta \rho_{-1}^{(1), \text{from } \mathcal{P}^{(1)}(z)}(t) + \mathcal{O}(\eta^0) \right), \tag{844}$$

where the leading order reads

$$\begin{aligned}
 \rho_{-3}^{(1), \text{from } \mathcal{P}^{(1)}(z)}(t) &= \frac{1}{4\pi^2 ((1 - \kappa)^2 + 4\kappa t)^3} \cdot \left(-\frac{2\kappa ((1 - \kappa)^2 + 4\kappa t)}{\sqrt{t(1 - t)}} + 8(1 - \kappa^2) \arctan\left(\frac{2\kappa \sqrt{t(1 - t)}}{1 - \kappa + 2\kappa t}\right) - \right.
 \end{aligned}$$

$$\begin{aligned}
& - \frac{2 \left((1-\kappa)^2 (1+\kappa+\kappa^2) + 2\kappa (1+\kappa^2) t \right)}{\sqrt{(1+\kappa^2+2\kappa t) ((1-\kappa)^2+2\kappa t)}} \arctan \left(\frac{4\kappa \sqrt{t(1-t)} (1+\kappa^2+2\kappa t) ((1-\kappa)^2+2\kappa t)}{(1-\kappa)^2 (1+\kappa^2) + 4\kappa (1-\kappa)^2 t + 8\kappa^2 t^2} \right) - \\
& - \frac{(1-\kappa)^2 - 2 (1+\kappa^2) t}{\sqrt{t(1-t)}} \log \left(\frac{(1-\kappa)^2 + 4\kappa t}{(1-\kappa^2)^2 (1+\kappa^2)} \right), \tag{845}
\end{aligned}$$

and the next-to-leading order,

$$\begin{aligned}
& \rho_{-2}^{(1), \text{from } \mathcal{P}^{(1)}(z)}(t) = \frac{\kappa}{8\pi^2 ((1-\kappa)^2 + 4\kappa t)^4} \cdot \\
& \cdot \left(\frac{2\kappa ((1-\kappa)^2 + 4\kappa t)}{(1+\kappa^2+2\kappa t) ((1-\kappa)^2+2\kappa t) \sqrt{t(1-t)}} \left((1-\kappa)^4 (1+\kappa^2) + 2(1-\kappa)^2 \left(-1 + 16\kappa - 4\kappa^2 + 18\kappa^3 - \kappa^4 \right) t + \right. \right. \\
& + 4\kappa \left(-9 + 45\kappa - 53\kappa^2 + 51\kappa^3 - 10\kappa^4 \right) t^2 + 8\kappa^2 (1-\kappa) (-15 + 17\kappa) t^3 - 128\kappa^3 t^4 \Big) - \\
& - 192\kappa (1-\kappa^2) t(1-t) \arctan \left(\frac{2\kappa \sqrt{t(1-t)}}{1-\kappa+2\kappa t} \right) + \\
& + \frac{8\kappa t(1-t)}{((1+\kappa^2+2\kappa t) ((1-\kappa)^2+2\kappa t))^{3/2}} \left((1-\kappa)^4 \left(6 + 6\kappa + 11\kappa^2 + 6\kappa^3 + 6\kappa^4 \right) + \right. \\
& + 4\kappa (1-\kappa)^2 \left(9 + 10\kappa^2 + 9\kappa^4 \right) t + 8\kappa^2 \left(9 - 9\kappa + 10\kappa^2 - 9\kappa^3 + 9\kappa^4 \right) t^2 + \\
& + 48\kappa^3 (1+\kappa^2) t^3 \Big) \arctan \left(\frac{4\kappa \sqrt{t(1-t)} (1+\kappa^2+2\kappa t) ((1-\kappa)^2+2\kappa t)}{(1-\kappa)^2 (1+\kappa^2) + 4\kappa (1-\kappa)^2 t + 8\kappa^2 t^2} \right) + \\
& + \frac{1}{\sqrt{t(1-t)}} \left((1-\kappa)^4 + 32\kappa (1-\kappa)^2 t - 8\kappa (9 - 8\kappa + 9\kappa^2) t^2 + 48\kappa (1+\kappa^2) t^3 \right) \log \left(\frac{(1-\kappa)^2 + 4\kappa t}{(1-\kappa^2)^2 (1+\kappa^2)} \right) \Big), \tag{846}
\end{aligned}$$

and the next-to-next-to-leading order,

$$\begin{aligned}
& \rho_{-1}^{(1), \text{from } \mathcal{P}^{(1)}(z)}(t) = \frac{1}{4\pi^2 ((1-\kappa)^2 + 4\kappa t)^5} \cdot \\
& \cdot \left(- \frac{\kappa}{8 (1+\kappa^2+2\kappa t)^2 ((1-\kappa)^2+2\kappa t)^2 \sqrt{t(1-t)}} \cdot \right. \\
& \cdot \left((1-\kappa)^{10} (1+\kappa^2)^2 \left(2\kappa^4 + 3\kappa^2 + 2\kappa + 1 \right) + \right. \\
& + 4\kappa (1-\kappa)^8 (1+\kappa^2)^2 \left(10\kappa^4 + 4\kappa^3 - 6\kappa^2 + 15\kappa + 4 \right) t + \\
& + 4\kappa^2 (1-\kappa)^6 \left(76\kappa^8 + 132\kappa^7 + 133\kappa^6 + 658\kappa^5 - 25\kappa^4 + 776\kappa^3 - 65\kappa^2 + 234\kappa + 17 \right) t^2 +
\end{aligned}$$

$$\begin{aligned}
& +32\kappa^3(1-\kappa)^4 \left(27\kappa^8 + 122\kappa^7 + 146\kappa^6 + 482\kappa^5 - 8\kappa^4 + 638\kappa^3 - 98\kappa^2 + 206\kappa - 11 \right) t^3 + \\
& +32\kappa^4(1-\kappa)^2 \left(-7\kappa^8 + 216\kappa^7 + 1298\kappa^6 - 550\kappa^5 + 1308\kappa^4 + 700\kappa^3 - 170\kappa^2 + 690\kappa - 125 \right) t^4 + \\
& +128\kappa^5 \left(-23\kappa^8 - 336\kappa^7 + 2295\kappa^6 - 3996\kappa^5 + 4273\kappa^4 - 2544\kappa^3 + 877\kappa^2 + 156\kappa - 94 \right) t^5 + \\
& +256\kappa^6 \left(56\kappa^6 - 1001\kappa^5 + 2682\kappa^4 - 2954\kappa^3 + 1992\kappa^2 - 525\kappa - 26 \right) t^6 + \\
& +512\kappa^7 \left(143\kappa^4 - 864\kappa^3 + 1094\kappa^2 - 672\kappa + 75 \right) t^7 + \\
& +12288\kappa^8 \left(9\kappa^2 - 21\kappa + 7 \right) t^8 + 57344\kappa^9 t^9 \Big) + \\
& + \left((1-\kappa)^5 \left(3\kappa^4 - \kappa^3 + 3\kappa^2 + 1 \right) + \right. \\
& + 2\kappa(1-\kappa)^3 \left(-3\kappa^5 + 27\kappa^4 - 44\kappa^3 - 12\kappa^2 + 3\kappa + 5 \right) t + \\
& + 24\kappa^2(1-\kappa) \left(-3\kappa^5 + 21\kappa^4 + 4\kappa^3 + 20\kappa^2 + 9\kappa + 1 \right) t^2 - \\
& - 128\kappa^3(1-\kappa) \left(3\kappa^3 + 3\kappa^2 + 5\kappa + 1 \right) t^3 + 256\kappa^4 t^4 \Big) \arctan \left(\frac{2\kappa\sqrt{t(1-t)}}{1-\kappa+2\kappa t} \right) - \\
& - \frac{1}{8(1+\kappa^2+2\kappa t)^{5/2}((1-\kappa)^2+2\kappa t)^{5/2}} \cdot \\
& \cdot \left((1-\kappa)^{10} (1+\kappa^2)^3 (2+\kappa^2) (1+2\kappa^2) + \right. \\
& + 4\kappa(1-\kappa)^8 (1+\kappa^2) \left(10\kappa^8 - 2\kappa^7 + 17\kappa^6 - 45\kappa^5 + 20\kappa^4 - 45\kappa^3 + 17\kappa^2 - 2\kappa + 10 \right) t + \\
& + 4\kappa^2(1-\kappa)^6 (1+\kappa^2) \left(82\kappa^8 + 48\kappa^7 + 263\kappa^6 + 32\kappa^5 + 686\kappa^4 + 32\kappa^3 + 263\kappa^2 + 48\kappa + 82 \right) t^2 + \\
& + 16\kappa^3(1-\kappa)^4 \left(74\kappa^{10} + 92\kappa^9 + 449\kappa^8 + 260\kappa^7 + 1165\kappa^6 + 272\kappa^5 + 1165\kappa^4 + 260\kappa^3 + 449\kappa^2 + 92\kappa + 74 \right) t^3 + \\
& + 32\kappa^4(1-\kappa)^2 \left(51\kappa^{10} + 72\kappa^9 + 763\kappa^8 - 256\kappa^7 + 1362\kappa^6 - 336\kappa^5 + 1362\kappa^4 - 256\kappa^3 + 763\kappa^2 + 72\kappa + 51 \right) t^4 + \\
& + 128\kappa^5 \left(3\kappa^{10} - 105\kappa^9 + 773\kappa^8 - 1168\kappa^7 + 1320\kappa^6 - 1198\kappa^5 + 1320\kappa^4 - 1168\kappa^3 + 773\kappa^2 - 105\kappa + 3 \right) t^5 +
\end{aligned}$$

$$\begin{aligned}
& +128\kappa^6 \left(35\kappa^8 - 602\kappa^7 + 1584\kappa^6 - 1526\kappa^5 + 1594\kappa^4 - 1526\kappa^3 + 1584\kappa^2 - 602\kappa + 35 \right) t^6 + \\
& +512\kappa^7 \left(43\kappa^6 - 256\kappa^5 + 301\kappa^4 - 256\kappa^3 + 301\kappa^2 - 256\kappa + 43 \right) t^7 + \\
& +8192\kappa^8 \left(4\kappa^4 - 9\kappa^3 + 4\kappa^2 - 9\kappa + 4 \right) t^8 + \\
& +16384\kappa^9 \left(1 + \kappa^2 \right) t^9 \arctan \left(\frac{4\kappa\sqrt{t(1-t)}(1+\kappa^2+2\kappa t)\left((1-\kappa)^2+2\kappa t\right)}{(1-\kappa)^2(1+\kappa^2)+4\kappa(1-\kappa)^2t+8\kappa^2t^2} \right) + \\
& + \frac{1}{16\kappa\sqrt{t(1-t)}} \cdot \left(\left((1-\kappa)^6 (1+\kappa^2) \left(\kappa^4 - 2\kappa^3 + 6\kappa^2 - 2\kappa + 1 \right) + \right. \right. \\
& + 8\kappa(1-\kappa)^4 \left(2\kappa^6 - 7\kappa^5 + 18\kappa^4 - 22\kappa^3 + 18\kappa^2 - 7\kappa + 2 \right) t + \\
& + 8\kappa^2(1-\kappa)^2 \left(15\kappa^6 - 72\kappa^5 + 209\kappa^4 - 32\kappa^3 + 209\kappa^2 - 72\kappa + 15 \right) t^2 + \\
& + 32\kappa^3 \left(17\kappa^6 - 112\kappa^5 + 87\kappa^4 - 96\kappa^3 + 87\kappa^2 - 112\kappa + 17 \right) t^3 + \\
& + 256\kappa^4 \left(7\kappa^4 + 6\kappa^2 + 7 \right) t^4 \Big) \log(1+\kappa^2) - \\
& - \left((1-\kappa)^6 (1+\kappa^2) \left(\kappa^4 - 4\kappa^3 - 6\kappa^2 + 4\kappa - 3 \right) + \right. \\
& + 4\kappa(1-\kappa)^4 \left(5\kappa^6 - 8\kappa^5 - 29\kappa^4 + 48\kappa^3 - 45\kappa^2 + 24\kappa - 11 \right) t + \\
& + 16\kappa^2(1-\kappa)^2 \left(7\kappa^6 + 8\kappa^5 - 127\kappa^4 - 48\kappa^3 - 151\kappa^2 + 56\kappa - 17 \right) t^2 + \\
& + 64\kappa^3 \left(\kappa^6 + 56\kappa^5 - 9\kappa^4 + 16\kappa^3 - 25\kappa^2 + 88\kappa - 15 \right) t^3 + \\
& + 256\kappa^4(1+\kappa) \left(-7\kappa^3 - 3\kappa^2 + \kappa - 11 \right) t^4 + \\
& + 1024\kappa^5(1+\kappa^2) t^5 \Big) \log(1-\kappa^2) + \\
& + 2\kappa \left(- (1-\kappa)^6 (1+\kappa^2)^2 + \right.
\end{aligned}$$

$$\begin{aligned}
& +2(1-\kappa)^4 \left(\kappa^6 - 6\kappa^5 - \kappa^4 + 4\kappa^3 - \kappa^2 - 6\kappa + 1 \right) t + \\
& +4\kappa(1-\kappa)^2 \left(5\kappa^6 - 8\kappa^5 - 69\kappa^4 - 128\kappa^3 - 69\kappa^2 - 8\kappa + 5 \right) t^2 + \\
& +16\kappa^2 \left(3\kappa^6 + 32\kappa^5 + 53\kappa^4 - 64\kappa^3 + 53\kappa^2 + 32\kappa + 3 \right) t^3 - \\
& -256\kappa^3 \left(\kappa^4 + 5\kappa^3 - 2\kappa^2 + 5\kappa + 1 \right) t^4 + \\
& + 512\kappa^4 \left(1 + \kappa^2 \right) t^5 \log \left((1-\kappa)^2 + 4\kappa t \right) \Big). \tag{847}
\end{aligned}$$

Notice that this contribution commences at the order $O(\eta^3)$, *i.e.*, it explodes in the considered limit.

What is of major importance is that, as we have explained in paragraph III B 5, even though these terms indeed provide an excellent approximation to the density, here both in the bulk and at the edges of the domain, as shown in figure 41 — it is only the leading order (845) which will be usable for further computations at the level $O(1/J^2)$, since the subleading orders of the anomaly's piece in the density (911) cause numerous problems associated with their endpoint behavior, which may not be rendered well.

B. The Anomaly $\mathcal{A}^{(1)}(z)$

1. The Weak-Coupling Limit

The large- J leading-order input of the anomaly to the all-loop quadratic equation, $\mathcal{A}^{(1)}(z)$, is too complicated to be tackled exactly, due to the presence of the hyperbolic cotangent $\coth(\pi\rho^{(0)}(y))$, and therefore, we must be reinforced by one of the approximations we have introduced. In this paragraph, we search for several leading terms of the small- ω expansion of this anomaly.

We will consider the integral form (434), and expand the hyperbolic cotangent as a sum over n (548),

$$\begin{aligned}\mathcal{A}^{(1)}(z) &= \mathcal{A}^{(1),\text{bulk}}(z) = -\pi \int_a^b dy \frac{1}{z-y} \rho^{(0)}(y) \rho^{(0)'}(y) \left(\coth(\pi\rho^{(0)}(y)) - \frac{1}{\pi\rho^{(0)}(y)} \right) = \\ &= -2 \sum_{n \geq 1} \int_a^b dy \frac{1}{z-y} \rho^{(0)}(y)^2 \rho^{(0)'}(y) \frac{1}{\rho^{(0)}(y)^2 + n^2},\end{aligned}\quad (848)$$

where the exact density $\rho^{(0)}(y)$ is given by (742).

The integrand here is $\sqrt{(y-a)(b-y)}$ times a rational function of y , and hence, the computation can be performed by writing the integral as a contour one, and using the method of residues. To this end, it is necessary to factorize the quartic polynomial originating from the last term in (848),

$$\frac{1}{\rho^{(0)}(y)^2 + n^2} = -4\pi^2 (y^2 - \omega^2)^2 \frac{1}{Q_n(y)}, \quad (849)$$

where

$$\begin{aligned}Q_n(y) &\equiv (\mu^2 abc^2 - \omega^4 \nu^2) - \mu^2 (2ab + (a+b)c) cy + \\ &+ \mu^2 \left(ab + 2(a+b)c + c^2 + \frac{2\omega^2 \nu^2}{\mu^2} \right) y^2 - \mu^2 (a+b+2c)y^3 + (\mu^2 - \nu^2) y^4 \equiv \\ &\equiv (\mu^2 - \nu^2) (y - a_n) (y - b_n) (y - c_n) (y - d_n),\end{aligned}\quad (850)$$

where recall $\nu = 2\pi n$ (550). Of course, this task is too involved to be undertaken exactly, especially that a, b, c are themselves roots of a quartic polynomial, but it is straightforward to solve it perturbatively at small $\eta = \omega\mu$ (761), using the small- η series of a, b, c (770), (768),

$$\begin{aligned}a_n &= \frac{\mu(1+2\alpha) - \sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}}{\mu^2 - \nu^2} + \\ &+ \eta^2 \frac{1}{\mu^2(\mu^2 - \nu^2)} \left(\mu \left(\mu^2(1+2\alpha+2\alpha^2) - \nu^2 \right) - \right. \\ &\quad \left. - \frac{1}{\sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}} \left(4\mu^4\alpha^2(1+\alpha) + \mu^2\nu^2(-1+2\alpha+2\alpha^2) + \nu^4 \right) \right) + \\ &+ \eta^4 \frac{1}{\mu^4(\mu^2 - \nu^2)} \left(\mu \left(-\mu^4(1+6\alpha+10\alpha^2+8\alpha^3+2\alpha^4) + 2\mu^2\nu^2(-1+\alpha+\alpha^2) + \nu^4(3+2\alpha) \right) \right. \\ &\quad \left. + \frac{1}{(4\mu^2\alpha(1+\alpha) + \nu^2)^{3/2}} \left(8\mu^8\alpha^2(1+\alpha)^2(-1+2\alpha+7\alpha^2+2\alpha^3) + \right. \right. \\ &\quad \left. \left. + 4\mu^6\nu^2\alpha(1+\alpha)(-4-4\alpha+9\alpha^2+7\alpha^3) + \mu^4\nu^4(-3+6\alpha+52\alpha^2+60\alpha^3+18\alpha^4) + \right. \right. \\ &\quad \left. \left. + 2\mu^2\nu^6(1+6\alpha+4\alpha^2) + \nu^8 \right) \right) + \mathcal{O}(\eta^6),\end{aligned}\quad (851)$$

$$b_n = a_n \Big|_{\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2} \rightarrow -\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2}}, \quad (852)$$

$$\begin{aligned} c_n = & -\eta^2 \frac{\mu+\nu}{\mu^2} + \eta^4 \frac{\mu+\nu}{\mu^4} \left(\mu^2 (1+4\alpha+2\alpha^2) + 2\mu\nu(1+\alpha) + \nu^2 \right) - \\ & -\eta^6 \frac{2(\mu+\nu)}{\mu^6} \left(\mu^4 (1+10\alpha+23\alpha^2+16\alpha^3+3\alpha^4) + \mu^3\nu(1+\alpha) (4+17\alpha+8\alpha^2) + \right. \\ & \left. + 2\mu^2\nu^2(1+2\alpha)(3+2\alpha) + 4\mu\nu^3(1+\alpha) + \nu^4 \right) + \mathcal{O}(\eta^8), \end{aligned} \quad (853)$$

$$d_n = c_n \Big|_{\nu \rightarrow -\nu}, \quad (854)$$

where we have included everywhere the three leading terms. In the leading orders of a_n, b_n we recognize their one-loop counterparts (552), while c_n, d_n are zero at one loop. Remark that there are only even powers of η in all the series.

The integrand in (848) has seven poles outside of the contour encircling the interval $[a, b]$ which yield non-zero residues, namely, at $z, \omega, -\omega, a_n, b_n, c_n, d_n$. The sum of the residues is much too long to write it here, and we will explicitly give only the small- η expansions of its contribution to the two important objects, the large- J next-to-leading-order energy $\gamma^{(1)}$ and density $\rho^{(1)}(y)$. An important observation is that the sums over n of the separate terms of the small- η series are convergent; in particular, this will not be true for the large-mode-number, fixed-winding-number limit, as we have already discovered in paragraph III B 5, and will see again in paragraph V B 2, thus forcing us to use another approach.

Indeed, let us define

$$\gamma^{(1), \text{from } \mathcal{A}^{(1)}(z)} \equiv -\frac{c}{\omega^2 \mu} \mathcal{A}^{(1)}(c), \quad (855)$$

which will shortly be justified (see paragraph V C 1). Then, we find

$$\begin{aligned} \gamma^{(1), \text{from } \mathcal{A}^{(1)}(z)} = & \frac{1}{2} \sum_{n \geq 1} \left(\nu - \sqrt{4\mu^2\alpha(1+\alpha)+\nu^2} \right)^2 + \\ & + \frac{\eta^2}{\mu^2} \sum_{n \geq 1} \left(- \left(2\mu^4\alpha(1+\alpha) (4+11\alpha+3\alpha^2) + 2\mu^2\nu^2 (3+10\alpha+5\alpha^2) + \nu^4 \right) + \right. \\ & \left. + \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2}} \left(4\mu^4\alpha(1+\alpha) (5+15\alpha+6\alpha^2) + 2\mu^2\nu^2 (3+11\alpha+6\alpha^2) + \nu^4 \right) \right) + \\ & + 2 \frac{\eta^4}{\mu^4} \sum_{n \geq 1} \left(2\mu^6\alpha(1+\alpha) (8+52\alpha+89\alpha^2+42\alpha^3+5\alpha^4) + \right. \\ & + \mu^4\nu^2 (15+128\alpha+279\alpha^2+202\alpha^3+44\alpha^4) + \mu^2\nu^4 (15+38\alpha+19\alpha^2) + \nu^6 - \\ & - \frac{\nu}{(4\mu^2\alpha(1+\alpha)+\nu^2)^{3/2}} \left(4\mu^8\alpha^2(1+\alpha)^2 (45+324\alpha+621\alpha^2+370\alpha^3+60\alpha^4) + \right. \\ & \left. + 2\mu^6\nu^2\alpha(1+\alpha) (53+481\alpha+1083\alpha^2+815\alpha^3+192\alpha^4) + \right. \\ & \left. \left. + \mu^4\nu^4 (15+218\alpha+603\alpha^2+556\alpha^3+164\alpha^4) + \mu^2\nu^6 (15+44\alpha+25\alpha^2) + \nu^8 \right) \right) + \mathcal{O}(\eta^6). \end{aligned} \quad (856)$$

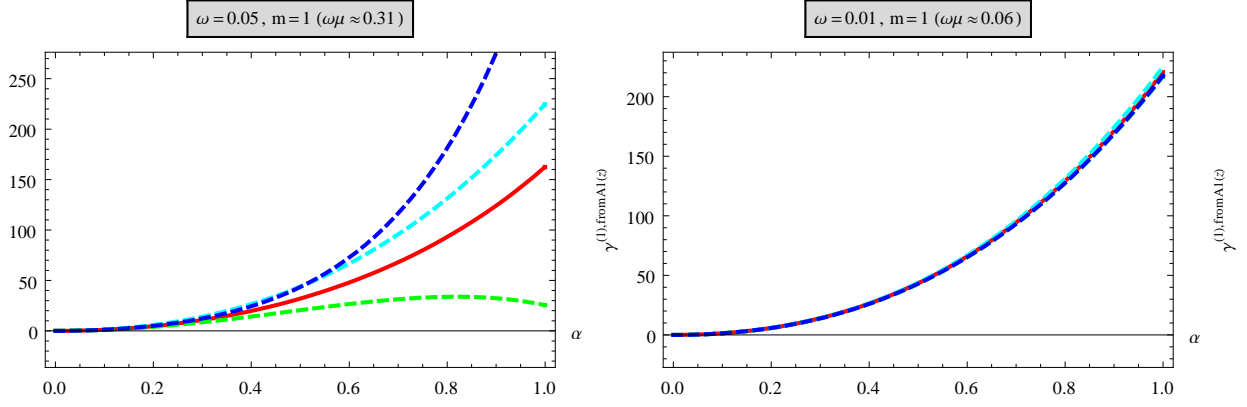


FIG. 42: The contribution of the term $\mathcal{A}^{(1)}(z)$ to the energy $\gamma^{(1)}$, as a function of $\alpha \in [0, 1]$. The exact value (855) (solid red) is compared to its small- η expansion (856), terminated at one (dashed cyan), two (dashed green), or three (dashed blue) leading orders.

The values of the parameters are: $m = 1$ everywhere, $\omega = 0.05$ (*i.e.*, $\eta = 0.1\pi \approx 0.31$) (LEFT), $\omega = 0.01$ (*i.e.*, $\eta = 0.02\pi \approx 0.06$) (RIGHT). The numerical integration in the exact value is performed with `WorkingPrecision` = `MachinePrecision` ≈ 16 , while the numerical summations over n in the terms of the series are truncated at $n_{\max} = 100$.

Notice that there are only even powers of η here. The accuracy of this series is pictorially tested in figure 42.

The contribution to the density is

$$\rho^{(1), \text{from } \mathcal{A}^{(1)}(z)}(y) \equiv -\frac{1}{2\pi\mu} \frac{y^2}{(y-c)\sqrt{(y-a)(b-y)}} \lim_{\epsilon \rightarrow 0^+} \left(\mathcal{A}^{(1)}(y+i\epsilon) + \mathcal{A}^{(1)}(y-i\epsilon) \right). \quad (857)$$

Before expanding, we change y to t (539), as usual. After a tedious computation, we arrive at the following series,

$$\rho^{(1), \text{from } \mathcal{A}^{(1)}(z)}(t) = \rho_0^{(1), \text{from } \mathcal{A}^{(1)}(z)}(t) + \eta^2 \rho_2^{(1), \text{from } \mathcal{A}^{(1)}(z)}(t) + \mathcal{O}(\eta^4). \quad (858)$$

Again, there appear no odd powers of η . The leading order reads

$$\rho_0^{(1), \text{from } \mathcal{A}^{(1)}(z)}(t) = \frac{\mu}{4\pi\sqrt{t(1-t)}\sqrt{\alpha(1+\alpha)}} \frac{1}{(\sqrt{\alpha} - \sqrt{1+\alpha})^2 + 4t\sqrt{\alpha(1+\alpha)}} \cdot \sum_{n \geq 1} \left(1 - \frac{\nu\sqrt{4\mu^2\alpha(1+\alpha) + \nu^2} \left(1 + 8\alpha + 8\alpha^2 - 16\alpha(1+\alpha)t(1-t) + 4(-1+2t)(1+2\alpha)\sqrt{\alpha(1+\alpha)} \right)}{\nu^2(1+8\alpha+8\alpha^2) + 16(\mu^2 - \nu^2)\alpha(1+\alpha)t(1-t) + 4\nu^2(-1+2t)(1+2\alpha)\sqrt{\alpha(1+\alpha)}} \right). \quad (859)$$

It can be checked to comply with the one-loop anomaly result (558). The subsequent non-zero term is more lengthy,

$$\rho_2^{(1), \text{from } \mathcal{A}^{(1)}(z)}(t) = \frac{1}{4\pi\mu\sqrt{t(1-t)}\sqrt{\alpha(1+\alpha)}} \frac{1}{\left((\sqrt{\alpha} - \sqrt{1+\alpha})^2 + 4t\sqrt{\alpha(1+\alpha)} \right)^3} \cdot \sum_{n \geq 1} \frac{1}{\left(\nu^2(1+8\alpha+8\alpha^2) + 16(\mu^2 - \nu^2)\alpha(1+\alpha)t(1-t) + 4\nu^2(-1+2t)(1+2\alpha)\sqrt{\alpha(1+\alpha)} \right)^2} \cdot \left(R_1 + 2^2 t \alpha^{1/2} (1+\alpha)^{1/2} R_2 + 2^4 t^2 \alpha (1+\alpha) R_3 + 2^6 t^3 \alpha^{3/2} (1+\alpha)^{3/2} R_4 + \right. \\ \left. + 2^8 t^4 \alpha^2 (1+\alpha)^2 R_5 + 2^{10} t^5 \alpha^{5/2} (1+\alpha)^{5/2} R_6 + 2^{12} t^6 \alpha^3 (1+\alpha)^3 R_7 \right). \quad (860)$$

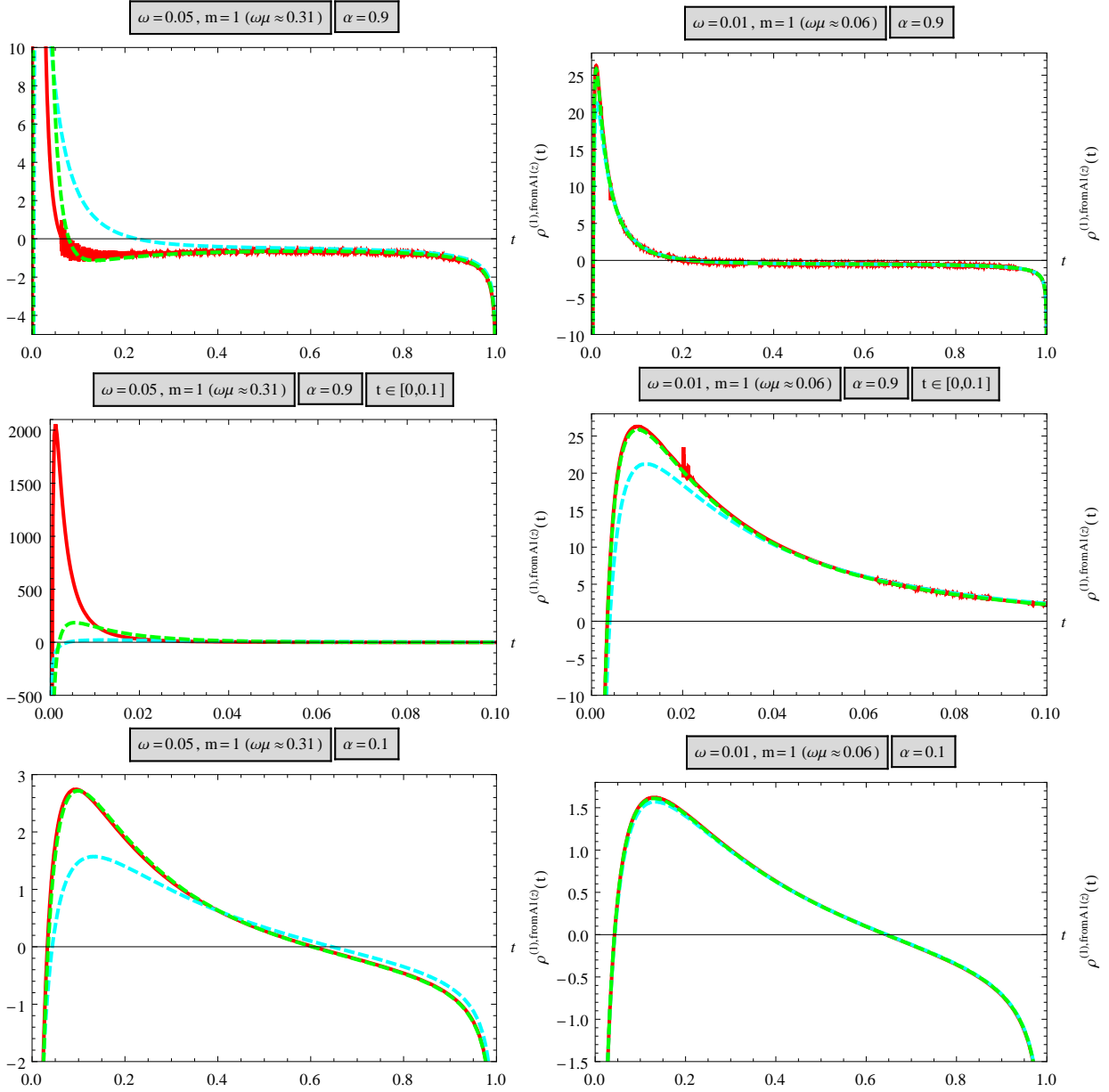


FIG. 43: The contribution of the term $\mathcal{A}^{(1)}(z)$ to the density $\rho^{(1)}(t)$, as a function of $t \in [0, 1]$. The exact value (857) (solid red) is compared to its small- η expansion (858), terminated at one (dashed cyan), or two (dashed green) leading orders. The values of the parameters are: $m = 1$ everywhere, $\alpha = 0.9$ (TOP, MIDDLE), $\alpha = 0.1$ (BOTTOM), and $\omega = 0.05$ (*i.e.*, $\eta = 0.1\pi \approx 0.31$) (LEFT), $\omega = 0.01$ (*i.e.*, $\eta = 0.02\pi \approx 0.06$) (RIGHT). In the MIDDLE row, we zoom into the peak for $t \in [0, 0.1]$, not visible in the TOP row. In the numerical integration in the exact value, we have tuned the parameters to be `WorkingPrecision` = 200 and $\epsilon = 10^{-5}$, while the numerical summations in the terms of the series are truncated at $n_{\max} = 100$.

Inside the red brackets, we have a sixth-order polynomial in t ; each of its coefficients contains the following structure,

$$R_p \equiv \frac{1}{\sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}} \left(R_{p,1} + \sqrt{\alpha(1+\alpha)} R_{p,2} \right) + R_{p,3} + \sqrt{\alpha(1+\alpha)} R_{p,4}, \quad \text{for } p = 1, 2, \dots, 7, \quad (861)$$

where the 28 symbols $R_{p,q}$, for $p = 1, 2, \dots, 7$, $q = 1, 2, 3, 4$, denote polynomial functions of the parameters μ , α , and

the summation variable ν , which we list below,

$$R_{1,1} = \nu^3 \left(8\alpha(1+\alpha) (1024\alpha^7 + 3328\alpha^6 + 4096\alpha^5 + 2224\alpha^4 + 364\alpha^3 - 99\alpha^2 - 31\alpha - 1) \mu^4 - \right. \\ \left. - (4096\alpha^8 + 12288\alpha^7 + 11264\alpha^6 - 256\alpha^5 - 6096\alpha^4 - 3328\alpha^3 - 568\alpha^2 - 9\alpha + 1) \nu^2 \mu^2 - \right. \\ \left. - (8\alpha^2 + 8\alpha + 1) (256\alpha^4 + 512\alpha^3 + 320\alpha^2 + 64\alpha + 1) \nu^4 \right), \quad (862)$$

$$R_{1,2} = 2\nu^3 \left(4\alpha(1+\alpha) (-1024\alpha^6 - 2816\alpha^5 - 2816\alpha^4 - 1104\alpha^3 - 28\alpha^2 + 69\alpha + 8) \mu^4 + \right. \\ \left. + (2048\alpha^7 + 5120\alpha^6 + 3328\alpha^5 - 1280\alpha^4 - 2232\alpha^3 - 772\alpha^2 - 65\alpha + 2) \nu^2 \mu^2 + \right. \\ \left. + 2(2\alpha + 1)(4\alpha + 1)(4\alpha + 3) (16\alpha^2 + 16\alpha + 1) \nu^4 \right), \quad (863)$$

$$R_{1,3} = \nu^4 \left((-4096\alpha^7 - 14848\alpha^6 - 21248\alpha^5 - 14944\alpha^4 - 5152\alpha^3 - 714\alpha^2 - 11\alpha + 1) \mu^2 + \right. \\ \left. + (8\alpha^2 + 8\alpha + 1) (256\alpha^4 + 512\alpha^3 + 320\alpha^2 + 64\alpha + 1) \nu^2 \right), \quad (864)$$

$$R_{1,4} = 2\nu^4 \left((2048\alpha^6 + 6400\alpha^5 + 7680\alpha^4 + 4304\alpha^3 + 1064\alpha^2 + 77\alpha - 2) \mu^2 - \right. \\ \left. - 2(2\alpha + 1)(4\alpha + 1)(4\alpha + 3) (16\alpha^2 + 16\alpha + 1) \nu^2 \right), \quad (865)$$

$$R_{2,1} = \nu \left(-32\alpha^3(1+\alpha)^2 (64\alpha^3 + 80\alpha^2 + 24\alpha + 1) \mu^6 + \right. \\ \left. + 4\alpha(1+\alpha) (3072\alpha^6 + 8576\alpha^5 + 8384\alpha^4 + 3008\alpha^3 - 68\alpha^2 - 207\alpha - 16) \nu^2 \mu^4 - \right. \\ \left. - (6144\alpha^7 + 15360\alpha^6 + 9216\alpha^5 - 5440\alpha^4 - 7752\alpha^3 - 2516\alpha^2 - 193\alpha + 2) \nu^4 \mu^2 - \right. \\ \left. - 6(2\alpha + 1) (256\alpha^4 + 512\alpha^3 + 304\alpha^2 + 48\alpha + 1) \nu^6 \right), \quad (866)$$

$$R_{2,2} = 4\nu \left(8\alpha^3(1+\alpha) (64\alpha^3 + 112\alpha^2 + 56\alpha + 7) \mu^6 - \right. \\ \left. - (3072\alpha^7 + 10112\alpha^6 + 12288\alpha^5 + 6320\alpha^4 + 804\alpha^3 - 344\alpha^2 - 79\alpha - 1) \nu^2 \mu^4 + \right. \\ \left. + 2 (768\alpha^6 + 1536\alpha^5 + 480\alpha^4 - 776\alpha^3 - 587\alpha^2 - 109\alpha - 2) \nu^4 \mu^2 + \right. \\ \left. + 3 (16\alpha^2 + 12\alpha + 1) (16\alpha^2 + 20\alpha + 5) \nu^6 \right), \quad (867)$$

$$R_{2,3} = \nu^2 \left(16\alpha^2(1+\alpha) (128\alpha^3 + 272\alpha^2 + 192\alpha + 43) \mu^4 - \right. \\ \left. - (6144\alpha^6 + 19584\alpha^5 + 23680\alpha^4 + 13152\alpha^3 + 3128\alpha^2 + 205\alpha - 2) \nu^2 \mu^2 + \right. \\ \left. + 6(2\alpha + 1) (256\alpha^4 + 512\alpha^3 + 304\alpha^2 + 48\alpha + 1) \nu^4 \right), \quad (868)$$

$$R_{2,4} = 4\nu^2 \left(-2 (256\alpha^5 + 672\alpha^4 + 624\alpha^3 + 226\alpha^2 + 19\alpha - 1) \mu^4 + \right. \\ \left. + 4 (384\alpha^5 + 1032\alpha^4 + 1012\alpha^3 + 421\alpha^2 + 62\alpha + 1) \nu^2 \mu^2 - \right. \\ \left. - 3 (16\alpha^2 + 12\alpha + 1) (16\alpha^2 + 20\alpha + 5) \nu^4 \right), \quad (869)$$

$$\begin{aligned}
R_{3,1} = & \nu \left(-8\alpha^3(1+\alpha)(320\alpha^3+576\alpha^2+288\alpha+31)\mu^6 + \right. \\
& + (7680\alpha^7+25600\alpha^6+30720\alpha^5+14800\alpha^4+1140\alpha^3-1032\alpha^2-175\alpha-1)\nu^2\mu^4 - \\
& - 2(1920\alpha^6+3840\alpha^5+960\alpha^4-2320\alpha^3-1611\alpha^2-261\alpha-2)\nu^4\mu^2 - \\
& \left. - 15(128\alpha^4+256\alpha^3+160\alpha^2+32\alpha+1)\nu^6 \right), \tag{870}
\end{aligned}$$

$$\begin{aligned}
R_{3,2} = & 2\nu \left(4\alpha^2(1+\alpha)(320\alpha^3+416\alpha^2+120\alpha+3)\mu^6 - \right. \\
& - (3840\alpha^6+10880\alpha^5+10400\alpha^4+3320\alpha^3-320\alpha^2-271\alpha-12)\nu^2\mu^4 + \\
& + 2(960\alpha^5+1440\alpha^4-120\alpha^3-980\alpha^2-383\alpha-24)\nu^4\mu^2 + \\
& \left. + 60(2\alpha+1)(8\alpha^2+8\alpha+1)\nu^6 \right), \tag{871}
\end{aligned}$$

$$\begin{aligned}
R_{3,3} = & -16\alpha(1+\alpha)(16\alpha^3+26\alpha^2+11\alpha-1)\mu^6 + \\
& + 2(1280\alpha^5+3264\alpha^4+2880\alpha^3+970\alpha^2+75\alpha-1)\nu^2\mu^4 - \\
& - 4(960\alpha^5+2640\alpha^4+2600\alpha^3+1053\alpha^2+138\alpha+1)\nu^4\mu^2 + \\
& + 15(128\alpha^4+256\alpha^3+160\alpha^2+32\alpha+1)\nu^6, \tag{872}
\end{aligned}$$

$$\begin{aligned}
R_{3,4} = & 4 \left(8\alpha(1+\alpha)(8\alpha^2+9\alpha+2)\mu^6 - \alpha(640\alpha^3+1312\alpha^2+864\alpha+177)\nu^2\mu^4 + \right. \\
& \left. + (960\alpha^4+2160\alpha^3+1640\alpha^2+443\alpha+24)\nu^4\mu^2 - 30(2\alpha+1)(8\alpha^2+8\alpha+1)\nu^6 \right), \tag{873}
\end{aligned}$$

$$\begin{aligned}
R_{4,1} = & \nu \left(-4\alpha^2(1+\alpha)(320\alpha^3+432\alpha^2+120\alpha+1)\mu^6 + \right. \\
& + (2560\alpha^6+7360\alpha^5+6880\alpha^4+1880\alpha^3-400\alpha^2-197\alpha-4)\nu^2\mu^4 - \\
& - 2(640\alpha^5+960\alpha^4-160\alpha^3-740\alpha^2-261\alpha-8)\nu^4\mu^2 - \\
& \left. - 20(2\alpha+1)(16\alpha^2+16\alpha+1)\nu^6 \right), \tag{874}
\end{aligned}$$

$$\begin{aligned}
R_{4,2} = & 8\nu \left(4\alpha^2(1+\alpha)(4\alpha+3)(10\alpha+1)\mu^6 - (320\alpha^5+760\alpha^4+520\alpha^3+50\alpha^2-45\alpha-6)\nu^2\mu^4 + \right. \\
& \left. + (160\alpha^4+160\alpha^3-100\alpha^2-125\alpha-19)\nu^4\mu^2 + 5(4\alpha+1)(4\alpha+3)\nu^6 \right), \tag{875}
\end{aligned}$$

$$\begin{aligned}
R_{4,3} = & 2 \left(-16\alpha(1+\alpha)(8\alpha^2+9\alpha+2)\mu^6 + \alpha(640\alpha^3+1264\alpha^2+768\alpha+139)\nu^2\mu^4 - \right. \\
& \left. - (640\alpha^4+1480\alpha^3+1120\alpha^2+281\alpha+8)\nu^4\mu^2 + 10(2\alpha+1)(16\alpha^2+16\alpha+1)\nu^6 \right), \tag{876}
\end{aligned}$$

$$\begin{aligned}
R_{4,4} = & 8 \left((2\alpha+1)(16\alpha^2+13\alpha-1)\mu^6 - (160\alpha^3+236\alpha^2+94\alpha+7)\nu^2\mu^4 + \right. \\
& \left. + (160\alpha^3+290\alpha^2+155\alpha+19)\nu^4\mu^2 - 5(4\alpha+1)(4\alpha+3)\nu^6 \right), \tag{877}
\end{aligned}$$

$$\begin{aligned}
R_{5,1} = & \nu \left(-4\alpha^2(1+\alpha)(80\alpha^2+74\alpha+3)\mu^6 + \right. \\
& + (480\alpha^5+1160\alpha^4+760\alpha^3+10\alpha^2-85\alpha-6)\nu^2\mu^4 - \\
& \left. - (240\alpha^4+240\alpha^3-180\alpha^2-205\alpha-19)\nu^4\mu^2 - 15(8\alpha^2+8\alpha+1)\nu^6 \right), \tag{878}
\end{aligned}$$

$$\begin{aligned}
R_{5,2} = & 10\nu \left(4\alpha^2(1+\alpha)(8\alpha+3)\mu^6 - (2\alpha+1)(24\alpha^3+34\alpha^2+\alpha-4)\nu^2\mu^4 + \right. \\
& \left. + (24\alpha^3+12\alpha^2-21\alpha-10)\nu^4\mu^2 + 6(2\alpha+1)\nu^6 \right), \tag{879}
\end{aligned}$$

$$\begin{aligned}
R_{5,3} = & (-112\alpha^3-122\alpha^2-11\alpha+1)\mu^6 + \\
& + (320\alpha^3+436\alpha^2+134\alpha+7)\nu^2\mu^4 - \\
& - (240\alpha^3+450\alpha^2+235\alpha+19)\nu^4\mu^2 + 15(8\alpha^2+8\alpha+1)\nu^6, \tag{880}
\end{aligned}$$

$$R_{5,4} = 10(\mu^2 - \nu^2) \left((8\alpha^2+9\alpha+2)\mu^4 - (24\alpha^2+21\alpha+4)\nu^2\mu^2 + 6(2\alpha+1)\nu^4 \right), \tag{881}$$

$$\begin{aligned}
R_{6,1} = & \nu \left(-4\alpha^2(1+\alpha)(8\alpha+3)\mu^6 + (2\alpha+1)(24\alpha^3+34\alpha^2+\alpha-4)\nu^2\mu^4 - \right. \\
& \left. - (24\alpha^3+12\alpha^2-21\alpha-10)\nu^4\mu^2 - 6(2\alpha+1)\nu^6 \right), \tag{882}
\end{aligned}$$

$$R_{6,2} = 12(\mu^2 - \nu^2) \left(4\alpha^2(1+\alpha)\mu^4 - (2\alpha^2-1)\nu^2\mu^2 - \nu^4 \right), \tag{883}$$

$$R_{6,3} = (\mu^2 - \nu^2) \left((-8\alpha^2-9\alpha-2)\mu^4 + (24\alpha^2+21\alpha+4)\nu^2\mu^2 - 6(2\alpha+1)\nu^4 \right), \tag{884}$$

$$R_{6,4} = 12(\mu^2 - \nu^2)^2(2\alpha\mu^2 - \nu^2), \tag{885}$$

$$R_{7,1} = \nu(\mu^2 - \nu^2) \left(-4\alpha^2(1+\alpha)\mu^4 + (2\alpha^2-1)\nu^2\mu^2 + \nu^4 \right), \tag{886}$$

$$R_{7,2} = 0, \tag{887}$$

$$R_{7,3} = (\mu^2 - \nu^2)^2(-2\mu^2\alpha + \nu^2), \tag{888}$$

$$R_{7,4} = 0. \tag{889}$$

Figure 43 presents a numerical check of the above formulae.

Except for the phase and the anomaly terms, in the numerator of the resolvent $G^{(1)}(z)$ (456) we find also the derivative $G^{(0)'}(z)$. Its contributions to the pertinent energy and density read

$$\gamma^{(1), \text{from } G^{(0)'}(z)} \equiv \frac{c}{\omega^2 \mu} G^{(0)'}(c), \tag{890}$$

$$\rho^{(1), \text{from } G^{(0)'}(z)}(y) \equiv \frac{1}{2\pi\mu} \frac{y^2}{(y-c)\sqrt{(y-a)(b-y)}} \lim_{\epsilon \rightarrow 0^+} \left(G^{(0)'}(y+i\epsilon) + G^{(0)'}(y-i\epsilon) \right). \quad (891)$$

The weak-coupling expansions are easily found,

$$\begin{aligned} \gamma^{(1), \text{from } G^{(0)'}(z)} = & \mu^2 \left(\alpha(1+\alpha) - \eta^2 \alpha(1+\alpha) (4 + 11\alpha + 3\alpha^2) + \right. \\ & \left. + 2\eta^4 \alpha(1+\alpha) (8 + 52\alpha + 89\alpha^2 + 42\alpha^3 + 5\alpha^4) + \mathcal{O}(\eta^6) \right), \end{aligned} \quad (892)$$

and

$$\rho^{(1), \text{from } G^{(0)'}(z)}(t) = \rho_0^{(1), \text{from } G^{(0)'}(z)}(t) + \eta^2 \rho_2^{(1), \text{from } G^{(0)'}(z)}(t) + \mathcal{O}(\eta^4), \quad (893)$$

where the leading term,

$$\rho_0^{(1), \text{from } G^{(0)'}(z)}(t) = \frac{1}{8\pi\sqrt{t(1-t)}\sqrt{\alpha(1+\alpha)}} \frac{1}{(\sqrt{\alpha} - \sqrt{1+\alpha})^2 + 4t\sqrt{\alpha(1+\alpha)}}, \quad (894)$$

and the next-to-leading term,

$$\begin{aligned} \rho_2^{(1), \text{from } G^{(0)'}(z)}(t) = & -\frac{1}{8\pi\sqrt{t(1-t)}\sqrt{\alpha(1+\alpha)}} \frac{1}{\left((\sqrt{\alpha} - \sqrt{1+\alpha})^2 + 4t\sqrt{\alpha(1+\alpha)} \right)^3} \\ & \cdot \left(-1 + 11\alpha + 26\alpha^2 + 16\alpha^3 - 32t(1-t)\alpha^2(1+\alpha) + 2(-1+2t)(2+9\alpha+8\alpha^2)\sqrt{\alpha(1+\alpha)} \right). \end{aligned} \quad (895)$$

2. The Large-Mode-Number, Fixed-Winding-Number Limit

In this paragraph, we aim at expanding the two quantities (855), (857), based on the anomaly $\mathcal{A}^{(1)}(z)$ (848), in the series of large $\eta = \omega\mu$, small α , fixed $\kappa = \sqrt{2\omega|\varpi|} \leq 1$ (784).

The roots of the quartic polynomial $Q_n(y)$ (850) can again be found perturbatively, and let us print a number of their leading orders,

$$\begin{aligned} a_n = & \omega \left(1 + \frac{1}{2}(1-\kappa)^2 \frac{1}{\eta} + \frac{1}{8}(1-\kappa)^4 \frac{1}{\eta^2} - \right. \\ & \left. - \frac{1}{16\kappa}(1-\kappa)^2 (\kappa^2 (1-\kappa+\kappa^2) + 2(1-\kappa)^2 \nu^2 \omega^2) \frac{1}{\eta^3} + \mathcal{O}\left(\frac{1}{\eta^4}\right) \right), \end{aligned} \quad (896)$$

$$b_n = a_n \big|_{\kappa \rightarrow -\kappa}, \quad (897)$$

$$c_n = \omega \left(-1 + \frac{1}{2}(1+\kappa^2) \frac{1}{\eta} - \frac{1}{8}(1+\kappa^2)(1+\kappa^2+4\nu\omega) \frac{1}{\eta^2} + \mathcal{O}\left(\frac{1}{\eta^3}\right) \right), \quad (898)$$

$$d_n = c_n \big|_{\nu \rightarrow -\nu}. \quad (899)$$

In the present case, however, these will lead to divergent sums over n , similarly to what we have already discovered at one loop (600). Thus, one needs to resort to the approximation “ $\coth \approx 1 + \delta$ ” (602), (616), as described in detail in paragraph III B 5.

Since the density $\rho^{(0)}(y)$ explodes at large μ (803), we are allowed to approximate, with an exponentially small error, the hyperbolic cotangent by 1, to which, however, we must add the Dirac delta contribution. This prescription is best implemented in the language (619) of

$$\mathcal{A}^{(1),\coth}(z) = \mathcal{A}^{(1)}(z) - G^{(0)'}(z), \quad (900)$$

and gives (621)

$$\mathcal{A}^{(1),\coth}(z) \approx \mathcal{A}^{(1),\coth=1}(z) + \mathcal{A}^{(1),\delta}(z), \quad (901)$$

where (622)

$$\mathcal{A}^{(1),\coth=1}(z) \equiv -\pi \int_a^b dy \frac{1}{z-y} \rho^{(0)}(y) \rho^{(0)'}(y). \quad (902)$$

Remark that this is really a large- μ approximation, not only large- η ; we will see it pictorially in figures 44 and 45, which show how accurate (901) is, even for very small μ .

Having removed the hyperbolic cotangent, the remaining integral (902) is of a rational function of y , and so, can be exactly performed. One may wonder why we do this integral exactly, even though we have assumed the large- μ approximation, *i.e.*, despite only its several leading orders will be meaningful. It is simply a question of economy — to first do the integral exactly and subsequently expand the result, rather than conversely, which would require doing a number of similar integrals for each large- μ term we want to calculate. Now, the easiest way to approach (902) is to integrate by parts (the density $\rho^{(0)}(y)$ vanishes at the endpoints), and perform partial fractional decomposition w.r.t. \hat{y} , which reduces the integral to elementary ones, yielding in the rescaled variables (830),

$$\begin{aligned} \mathcal{A}^{(1),\coth=1}(z) &= \frac{\pi}{2} \int_a^b dy \frac{1}{(z-y)^2} \rho^{(0)}(y)^2 = \frac{\mu^2}{8\pi\omega} \int_{\hat{a}}^{\hat{b}} d\hat{y} \frac{(\hat{y}-\hat{c})^2(\hat{y}-\hat{a})(\hat{y}-\hat{b})}{(\hat{z}-\hat{y})^2(\hat{y}^2-1)^2} = \\ &= -\frac{\mu^2}{8\pi\omega} \left(\frac{(\hat{b}-\hat{a})(1+3\hat{c}^2-8\hat{c}\hat{z}+(\hat{c}^2+3)\hat{z}^2)}{2(\hat{z}^2-1)^2} + \right. \\ &+ \frac{\hat{c}-\hat{z}}{(\hat{z}^2-1)^3} \left(-\left(2\hat{a}\hat{b}+(\hat{a}+\hat{b})\hat{c}\right) + \left(3(\hat{a}+\hat{b})+2(1+2\hat{a}\hat{b})\hat{c}\right)\hat{z} - \left(2(\hat{a}\hat{b}+2)+3(\hat{a}+\hat{b})\hat{c}\right)\hat{z}^2 + \left(\hat{a}+\hat{b}+2\hat{c}\right)\hat{z}^3 \right) \\ &\quad \cdot \left(\log(\hat{z}-\hat{a}) - \log(\hat{z}-\hat{b}) \right) + \\ &+ \frac{\hat{c}+1}{4(\hat{z}+1)^3} \left(1-\hat{a}\hat{b} - \left(1+2(\hat{a}+\hat{b})+3\hat{a}\hat{b}\right)\hat{c} + \left(3+2(\hat{a}+\hat{b})+\hat{a}\hat{b}+(1-\hat{a}\hat{b})\hat{c}\right)\hat{z} \right) \log\left(\frac{\hat{a}+1}{\hat{b}+1}\right) + \\ &+ \frac{\hat{c}-1}{4(\hat{z}-1)^3} \left(-1+\hat{a}\hat{b} - \left(1-2(\hat{a}+\hat{b})+3\hat{a}\hat{b}\right)\hat{c} + \left(3-2(\hat{a}+\hat{b})+\hat{a}\hat{b}-(1-\hat{a}\hat{b})\hat{c}\right)\hat{z} \right) \log\left(\frac{\hat{a}-1}{\hat{b}-1}\right) \Big). \end{aligned} \quad (903)$$

Comments:

- The domain of the variable z is

$$z \in \mathbb{C}, \quad z \notin [a, b]. \quad (904)$$

- To be precise, the fractional decomposition that has led to the result (903) is valid for $z \neq \omega$ and $z \neq -\omega$, for which arguments we must integrate separately,

$$\mathcal{A}^{(1),\coth=1}(\omega) = -\frac{\mu^2}{384\pi\omega} \left(12(\hat{b}-\hat{a})(\hat{c}^2-2\hat{c}+3) + \right.$$

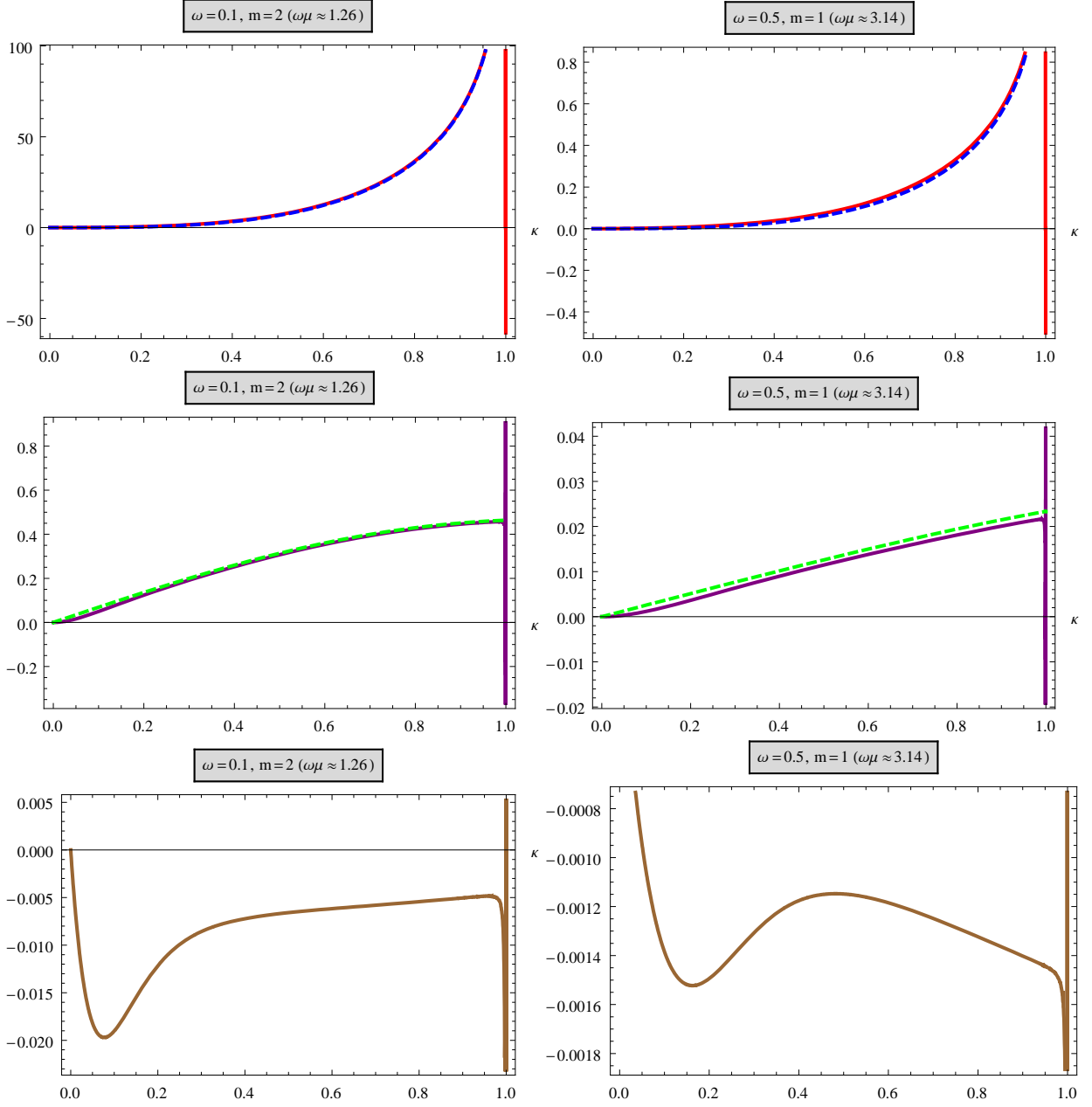


FIG. 44: Tests of the approximation (901).

TOP: The contribution to the energy $[0, 1] \ni \kappa \mapsto \gamma^{(1)}$ from the exact quantity $\mathcal{A}^{(1),\text{coth}}(z)$ (solid red) compared to the contribution from the approximation $\mathcal{A}^{(1),\text{coth}=1}(z)$ (dashed blue). We observe that they are very close to each other, *i.e.*, the Dirac delta term $\mathcal{A}^{(1),\delta}(z)$ is very small.

MIDDLE: The difference of the previous two functions (solid purple) compared to the energy based on the leading term of the Dirac delta term $\mathcal{A}^{(1),\delta}(z)$ (dashed green). This is an actual numerical confirmation of (901).

BOTTOM: The difference of the previous two functions (solid brown), showing the existence of subleading terms of the Dirac delta term.

The values of the parameters are: $\omega = 0.1, m = 2$ (*i.e.*, $\eta = 0.4\pi \approx 1.26$) (LEFT), $\omega = 0.5, m = 1$ (*i.e.*, $\eta = \pi \approx 3.14$) (RIGHT). The agreement is better in the left column, because μ is larger there, even though η is smaller. The integral comprising $\mathcal{A}^{(1),\text{coth}}(z)$ has been done everywhere with `WorkingPrecision = MachinePrecision` ≈ 16 .

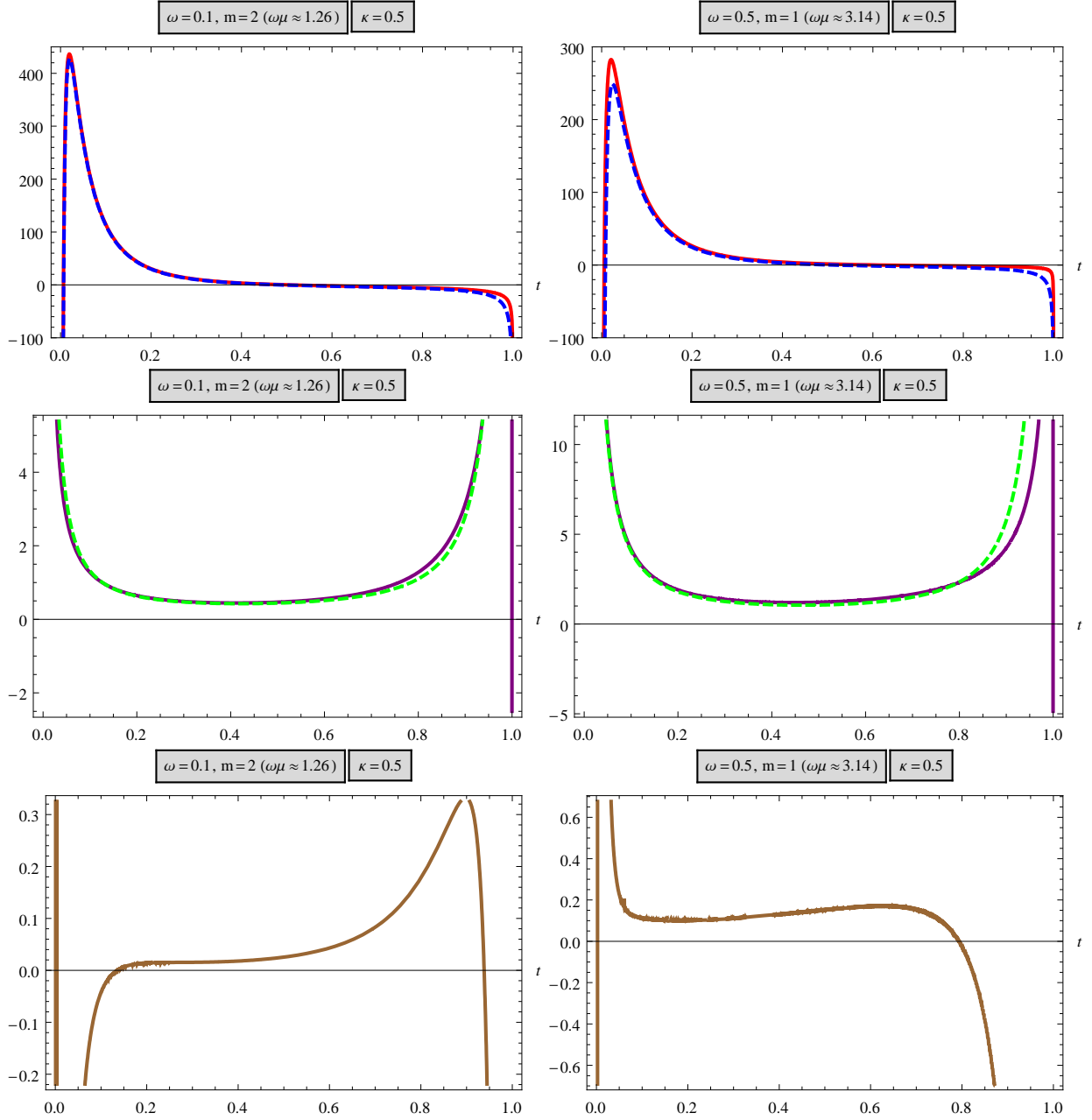


FIG. 45: Analogously as figure 44, but here we compare the various contributions to the density $[0, 1] \ni t \mapsto \rho^{(1)}(t)$. The values of the parameters are: $\kappa = 0.5$ everywhere, $\omega = 0.1$, $m = 2$ (i.e., $\eta = 0.4\pi \approx 1.26$) (LEFT), $\omega = 0.5$, $m = 1$ (i.e., $\eta = \pi \approx 3.14$) (RIGHT). Again, the agreement is slightly better in the left column, even though in both it is very good despite very small μ . The integral comprising $\mathcal{A}^{(1),\text{coth}}(z)$ has been done everywhere with `WorkingPrecision` = 200 and $\epsilon = 10^{-5}$.

$$\begin{aligned}
& + \frac{2(\hat{b} - \hat{a})(\hat{c} - 1)}{(\hat{a} - 1)^2(\hat{b} - 1)^2} \left(12 - 27(\hat{a} + \hat{b}) + 16(\hat{a}^2 + \hat{b}^2) + \hat{a}\hat{b}(58 - 33(\hat{a} + \hat{b}) + 18\hat{a}\hat{b}) - \right. \\
& \quad \left. - \left(6 - 9(\hat{a} + \hat{b}) + 4(\hat{a}^2 + \hat{b}^2) + \hat{a}\hat{b}(10 - 3(\hat{a} + \hat{b})) \right) \hat{c} \right) + \\
& \quad + 3(\hat{c} + 1) \left(-2 - (\hat{a} + \hat{b}) + (\hat{a} + \hat{b} + 2\hat{a}\hat{b})\hat{c} \right) \log \left(\frac{(\hat{a} - 1)(\hat{b} + 1)}{(\hat{a} + 1)(\hat{b} - 1)} \right), \tag{905}
\end{aligned}$$

and

$$\mathcal{A}^{(1),\text{coth}=1}(-\omega) = \mathcal{A}^{(1),\text{coth}=1}(\omega) \Big|_{\omega \rightarrow -\omega}. \quad (906)$$

This puts a question mark over the continuity of the function $z \mapsto \mathcal{A}^{(1),\text{coth}=1}(z)$ at these points. To clarify the matter, we need to compute the limits $z \rightarrow \pm\omega$ of the general result (903). It naively seems to explode, due to the denominator $(\hat{z}^2 - 1)^3$, but the numerator can be checked to vanish as well, and so, we actually have 0/0 symbols. (Note that $z = \pm\omega$ lie on the cuts of both $\log(z-a)$ and $\log(z-b)$, and thus, we need to approach them from either the upper or lower half-plane; in both cases the numerator zeroes.) Applying thrice the l'Hôpital rule leads to finite values identical to (905), (906),

$$\lim_{z \rightarrow \pm\omega} \mathcal{A}^{(1),\text{coth}=1}(z) = \mathcal{A}^{(1),\text{coth}=1}(\pm\omega), \quad (907)$$

thus proving the function $z \mapsto \mathcal{A}^{(1),\text{coth}=1}(z)$ to be continuous.

- We will now consider our two important quantities related to $\mathcal{A}^{(1),\text{coth}=1}(z)$. Firstly, its contribution to the large- J next-to-leading-order energy,

$$\begin{aligned} \gamma^{(1),\text{from } \mathcal{A}^{(1),\text{coth}=1}(z)} &\equiv -\frac{c}{\omega^2 \mu} \mathcal{A}^{(1),\text{coth}=1}(c) = \\ &= \frac{\mu}{16\pi\omega^2 \hat{c}} \left(\hat{b} - \hat{a} + \frac{1 - \hat{a}\hat{b}}{2} \log \left(\frac{(\hat{a}+1)(\hat{b}-1)}{(\hat{a}-1)(\hat{b}+1)} \right) \right). \end{aligned} \quad (908)$$

- Secondly, its contribution to the large- J next-to-leading-order density,

$$\begin{aligned} \rho^{(1),\text{from } \mathcal{A}^{(1),\text{coth}=1}(z)}(y) &\equiv \\ &\equiv -\frac{1}{2\pi\mu} \frac{y^2}{(y-c)\sqrt{(y-a)(b-y)}} \lim_{\epsilon \rightarrow 0^+} \left(\mathcal{A}^{(1),\text{coth}=1}(z = y + i\epsilon) + \mathcal{A}^{(1),\text{coth}=1}(z = y - i\epsilon) \right) = \\ &= \frac{\mu}{8\pi^2\omega} \frac{\hat{y}^2}{(\hat{y} - \hat{c})\sqrt{(\hat{y} - \hat{a})(\hat{b} - \hat{y})}} \left(\frac{(\hat{b} - \hat{a})(\hat{y}^2(\hat{c}^2 + 3) - 8\hat{c}\hat{y} + 3\hat{c}^2 + 1)}{2(\hat{y}^2 - 1)^2} + \right. \\ &+ \frac{\hat{c} - \hat{y}}{(\hat{y}^2 - 1)^3} \left(- (2\hat{a}\hat{b} + (\hat{a} + \hat{b})\hat{c}) + (3(\hat{a} + \hat{b}) + 2(1 + 2\hat{a}\hat{b})\hat{c})\hat{y} - (2(\hat{a}\hat{b} + 2) + 3(\hat{a} + \hat{b})\hat{c})\hat{y}^2 + (\hat{a} + \hat{b} + 2\hat{c})\hat{y}^3 \right) \\ &\quad \cdot \log \left(\frac{\hat{y} - \hat{a}}{\hat{b} - \hat{y}} \right) + \\ &+ \frac{\hat{c} + 1}{4(\hat{y} + 1)^3} \left(1 - \hat{a}\hat{b} - (1 + 2(\hat{a} + \hat{b}) + 3\hat{a}\hat{b})\hat{c} + (3 + 2(\hat{a} + \hat{b}) + \hat{a}\hat{b} + (1 - \hat{a}\hat{b})\hat{c})\hat{y} \right) \log \left(\frac{\hat{a} + 1}{\hat{b} + 1} \right) + \\ &+ \frac{\hat{c} - 1}{4(\hat{y} - 1)^3} \left(-1 + \hat{a}\hat{b} - (1 - 2(\hat{a} + \hat{b}) + 3\hat{a}\hat{b})\hat{c} + (3 - 2(\hat{a} + \hat{b}) + \hat{a}\hat{b} - (1 - \hat{a}\hat{b})\hat{c})\hat{y} \right) \log \left(\frac{\hat{a} - 1}{\hat{b} - 1} \right) \Big). \end{aligned} \quad (909)$$

Now, it is straightforward to derive the large- η series of (908) and (909) up to any desired order: The level- $\mathcal{O}(1/J)$ energy,

$$\gamma^{(1),\text{from } \mathcal{A}^{(1),\text{coth}=1}(z)} = \frac{1}{\omega^3} \left(\frac{1}{16\pi} \left(-2\kappa + (1 + \kappa^2) \log \left(\frac{1 + \kappa}{1 - \kappa} \right) \right) - \right.$$

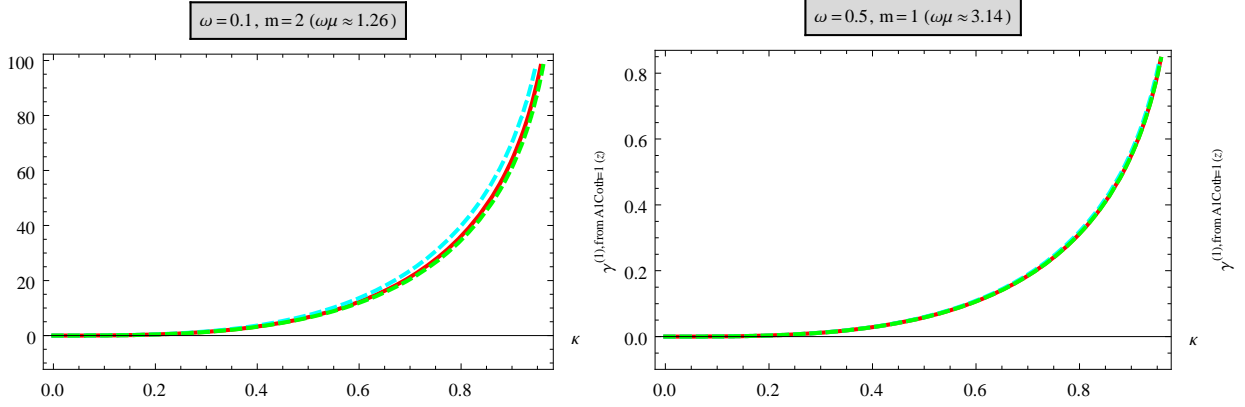


FIG. 46: The contribution of the term $\mathcal{A}^{(1),\text{coth}=1}(z)$ to the energy $\gamma^{(1)}$, as a function of $\kappa \in [0, 1]$. The exact value (908) (solid red) is compared to its large- η expansion (910), terminated at one (dashed cyan), or two (dashed green) leading orders. The values of the parameters are: $\omega = 0.1$, $m = 2$ (*i.e.*, $\eta = 0.4\pi \approx 1.26$) (LEFT), $\omega = 0.5$, $m = 1$ (*i.e.*, $\eta = \pi \approx 3.14$) (RIGHT).

$$- \frac{1}{\eta^2} \frac{1}{128\pi} \kappa^2 \left(2\kappa + (1 + \kappa^2) \log \left(\frac{1 + \kappa}{1 - \kappa} \right) \right) + O \left(\frac{1}{\eta^4} \right). \quad (910)$$

There are only even powers of η , and the expansion begins at the order $O(\eta^0)$, *i.e.*, it is finite in the strict limit $\eta \rightarrow \infty$. It is tested in figure 46. The level- $O(1/J)$ density,

$$\begin{aligned} \rho_{-2}^{(1),\text{from } \mathcal{A}^{(1),\text{coth}=1}(z)}(t) &= \frac{1}{\omega^2} \left(\eta^3 \rho_{-3}^{(1),\text{from } \mathcal{A}^{(1),\text{coth}=1}(z)}(t) + \eta^2 \rho_{-2}^{(1),\text{from } \mathcal{A}^{(1),\text{coth}=1}(z)}(t) + \right. \\ &\quad \left. + \eta \rho_{-1}^{(1),\text{from } \mathcal{A}^{(1),\text{coth}=1}(z)}(t) + O(\eta^0) \right), \end{aligned} \quad (911)$$

where the leading order,

$$\begin{aligned} \rho_{-3}^{(1),\text{from } \mathcal{A}^{(1),\text{coth}=1}(z)}(t) &= \frac{1}{4\pi^2 \sqrt{t(1-t)} ((1-\kappa)^2 + 4\kappa t)^3} \\ &\cdot \left(2((1-\kappa)^2 + 4\kappa t) + \left((1-\kappa)^2 - 2(1+\kappa^2)t \right) \log \left(\left(\frac{1+\kappa}{1-\kappa} \right)^2 \frac{t}{1-t} \right) \right), \end{aligned} \quad (912)$$

the next-to-leading order,

$$\begin{aligned} \rho_{-2}^{(1),\text{from } \mathcal{A}^{(1),\text{coth}=1}(z)}(t) &= - \frac{\kappa}{8\pi^2 \sqrt{t(1-t)} ((1-\kappa)^2 + 4\kappa t)^4} \left(48\kappa t(1-t) ((1-\kappa)^2 + 4\kappa t) + \right. \\ &\quad \left. + \left((1-\kappa)^4 + 32\kappa(1-\kappa)^2 t - 8\kappa(9-8\kappa+9\kappa^2)t^2 + 48\kappa(1+\kappa^2)t^3 \right) \log \left(\left(\frac{1+\kappa}{1-\kappa} \right)^2 \frac{t}{1-t} \right) \right), \end{aligned} \quad (913)$$

the next-to-next-to-leading order,

$$\begin{aligned} \rho_{-1}^{(1),\text{from } \mathcal{A}^{(1),\text{coth}=1}(z)}(t) &= \frac{1}{128\pi^2 \sqrt{t(1-t)} \kappa ((1-\kappa)^2 + 4\kappa t)^5} \\ &\cdot \left(4\kappa((1-\kappa)^2 + 4\kappa t) \left((1-\kappa)^4 (1+2\kappa-2\kappa^2+2\kappa^3+\kappa^4) + 4\kappa(1-\kappa)^2 (1+14\kappa-46\kappa^2+14\kappa^3+\kappa^4) t - \right. \right. \end{aligned}$$

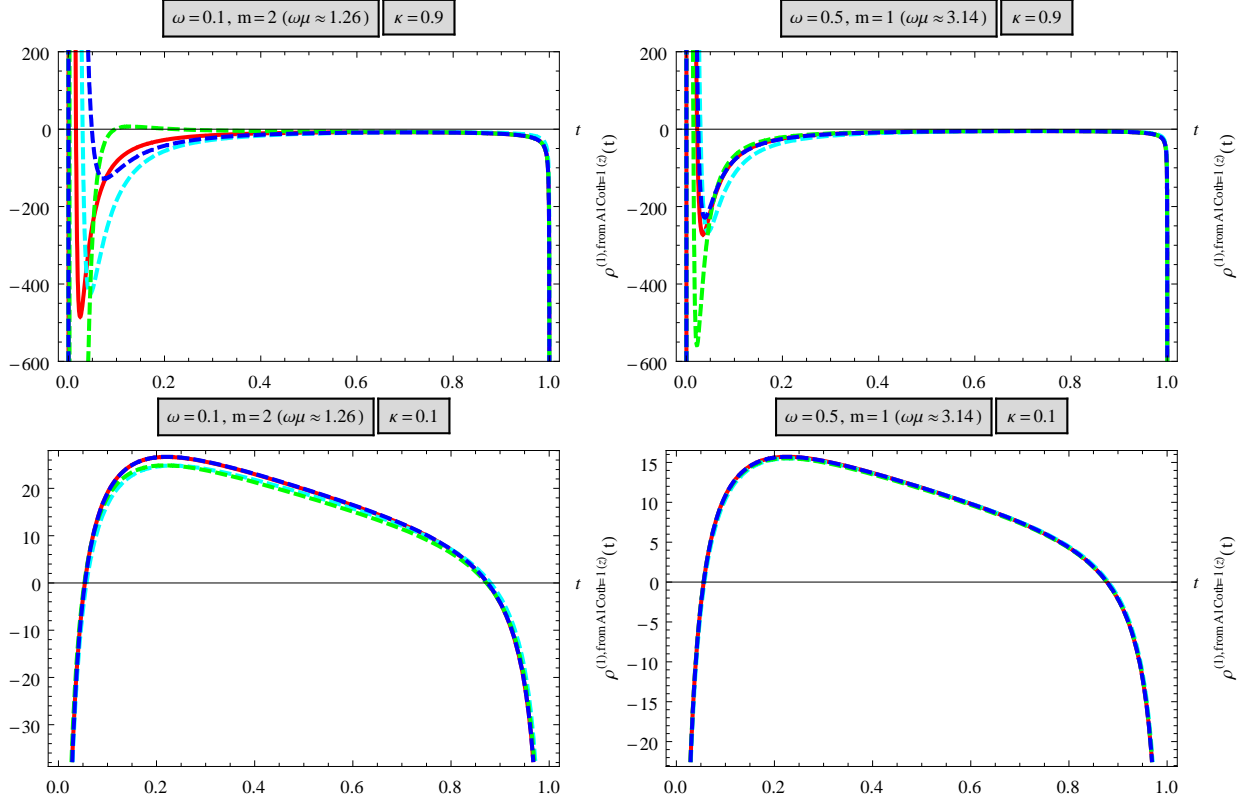


FIG. 47: The contribution of the term $\mathcal{A}^{(1),\text{coth}=1}(z)$ to the density $\rho^{(1)}(t)$, as a function of $t \in [0, 1]$. The exact value (909) (solid red) is compared to its large- η expansion (911), terminated at one (dashed cyan), two (dashed green), or three (dashed blue) leading orders. The values of the parameters are: $\kappa = 0.9$ (TOP), $\kappa = 0.1$ (BOTTOM), and $\omega = 0.1, m = 2$ (i.e., $\eta = 0.4\pi \approx 1.26$) (LEFT), $\omega = 0.5, m = 1$ (i.e., $\eta = \pi \approx 3.14$) (RIGHT).

$$\begin{aligned}
& -16\kappa^2 (3 - 48\kappa + 14\kappa^2 - 48\kappa^3 + 3\kappa^4) t^2 - 512\kappa^3 (1 + \kappa + \kappa^2) t^3 + 256\kappa^4 t^4 \Big) + \\
& + \Big((1 - \kappa)^6 (1 + \kappa^2) (1 + 6\kappa^2 + \kappa^4) + 4\kappa(1 - \kappa)^4 (3 - 8\kappa + 37\kappa^2 - 48\kappa^3 + 37\kappa^4 - 8\kappa^5 + 3\kappa^6) t + \\
& + 16\kappa^2(1 - \kappa)^2 (5 - 32\kappa + 139\kappa^2 + 48\kappa^3 + 139\kappa^4 - 32\kappa^5 + 5\kappa^6) t^2 + \\
& + 64\kappa^3 (7 - 72\kappa + 17\kappa^2 - 16\kappa^3 + 17\kappa^4 - 72\kappa^5 + 7\kappa^6) t^3 + \\
& + 256\kappa^4 (1 + \kappa)^2 (9 - 8\kappa + 9\kappa^2) t^4 - 1024\kappa^5 (1 + \kappa^2) t^5 \Big) \log \left(\left(\frac{1 + \kappa}{1 - \kappa} \right)^2 \right) - \\
& - 4\kappa \Big(- (1 - \kappa)^6 (1 + \kappa^2)^2 + 2(1 - \kappa)^4 (1 - 6\kappa - \kappa^2 + 4\kappa^3 - \kappa^4 - 6\kappa^5 + \kappa^6) t + \\
& + 4\kappa(1 - \kappa)^2 (5 - 8\kappa - 69\kappa^2 - 128\kappa^3 - 69\kappa^4 - 8\kappa^5 + 5\kappa^6) t^2 + \\
& + 16\kappa^2 (3 + 32\kappa + 53\kappa^2 - 64\kappa^3 + 53\kappa^4 + 32\kappa^5 + 3\kappa^6) t^3 -
\end{aligned}$$

$$- 256\kappa^3 (1 + 5\kappa - 2\kappa^2 + 5\kappa^3 + \kappa^4) t^4 + 512\kappa^4 (1 + \kappa^2) t^5 \log \left(\frac{t}{1-t} \right). \quad (914)$$

Similarly as for the contribution from the Hernández-López phase (844), this density explodes at large η as η^3 . The three above terms are tested in figure 47. All of them yield an accurate description of the exact object; however, it is only the leading one (912) that can be subsequently used at the level $O(1/J^2)$ in our pursuit of the energy $E^{(2)}$, as the subleading orders, however accurate, cause numerous structural problems, discovered in paragraphs III B 5 and III C 5.

Let us finally move to the Dirac delta contribution in (901). Surprisingly, we obtain the same expression as in the one loop case (624),

$$\mathcal{A}^{(1),\delta}(z) = -\frac{\pi}{12} \left(\frac{1}{z-a} - \frac{1}{z-b} \right) + O(\text{subleading}) + O(\text{exp}). \quad (915)$$

Our two fundamental objects should be defined as

$$\gamma^{(1),\text{from } \mathcal{A}^{(1),\delta}(z)} \equiv -\frac{c}{\omega^2 \mu} \mathcal{A}^{(1),\delta}(c), \quad (916)$$

$$\rho^{(1),\text{from } \mathcal{A}^{(1),\delta}(z)}(y) \equiv -\frac{1}{2\pi\mu} \frac{y^2}{(y-c)\sqrt{(y-a)(b-y)}} \lim_{\epsilon \rightarrow 0^+} \left(\mathcal{A}^{(1),\delta}(y+i\epsilon) + \mathcal{A}^{(1),\delta}(y-i\epsilon) \right). \quad (917)$$

And it is straightforward to derive their large- η series, from which obviously only the leading terms are correct,

$$\gamma^{(1),\text{from } \mathcal{A}^{(1),\delta}(z)} = \frac{1}{\omega} \left(\frac{1}{\eta^2} \frac{\pi\kappa}{24} + O\left(\frac{1}{\eta^3}\right) \right) + O(\text{exp}), \quad (918)$$

$$\rho^{(1),\text{from } \mathcal{A}^{(1),\delta}(z)}(t) = \eta \frac{1}{96\kappa^2 (t(1-t))^{3/2}} + O(\eta^0) + O(\text{exp}). \quad (919)$$

As at one loop, the Dirac delta input commences at the next-to-next-to-leading order.

C. The Energy $E^{(1)}$

1. The Holomorphicity Consistency Condition for the Resolvent $G^{(1)}(z)$

In subsections V A and V B, we have completed the derivation of the two critical ingredients of the resolvent $G^{(1)}(z)$ (456), the phase $\mathcal{P}^{(1)}(z)$ and the anomaly $\mathcal{A}^{(1)}(z)$, in either the weak-coupling limit, or the large-mode-number, fixed-winding-number limit.

Let us recall the solution for this resolvent,

$$G^{(1)}(z) = \frac{-\mathcal{P}^{(1)}(z) - \mathcal{A}^{(1)}(z) + G^{(0)'}(z) - \frac{\omega^2 \mu \gamma^{(1)}}{z}}{\frac{\mu}{z^2}(z-c)\sqrt{z-a}\sqrt{z-b}}. \quad (920)$$

It features an arbitrary constant $\gamma^{(1)}$. It may be checked by differentiating the above relation w.r.t. z and setting $z = 0$ that indeed $G^{(1)'}(0) = \gamma^{(1)}$, as required (457). A similar situation occurred for the resolvent $G^{(0)}(z)$ (715), (728). There, it was the double-root (*i.e.*, one-cut) consistency condition (740) that fixed the constant $\gamma^{(0)}$ (758). Here, the resulting density $\rho^{(1)}(y)$ will of course be supported on one interval, so, that condition is automatically fulfilled. There exists, however, another constraint that the resolvent $G^{(1)}(z)$ must submit to, namely

$$G^{(1)}(z) \text{ — holomorphic everywhere except the cut } [a, b]. \quad (921)$$

We observe, however, that the solution (920) seems to violate it due to the apparent pole at $z = c$. In order to avoid this situation, we have to assume that the numerator of the fraction on the r.h.s. of (920) includes a factor of $(z - c)$, which implies in turn that it vanishes at $z = c$. This condition determines then the constant $\gamma^{(1)}$,

$$\gamma^{(1)} = \frac{c}{\omega^2 \mu} \left(-\mathcal{P}^{(1)}(c) - \mathcal{A}^{(1)}(c) + G^{(0)'}(c) \right), \quad (922)$$

i.e., using a previously introduced notation,

$$\gamma^{(1)} = \gamma^{(1), \text{from } \mathcal{P}^{(1)}(z)} + \gamma^{(1), \text{from } \mathcal{A}^{(1)}(z)} + \gamma^{(1), \text{from } G^{(0)'}(z)}, \quad (923)$$

or

$$\gamma^{(1)} = \gamma^{(1), \text{from } \mathcal{P}^{(1)}(z)} + \gamma^{(1), \text{from } \mathcal{A}^{(1), \text{coth}}(z)}. \quad (924)$$

In this way, the solution for $G^{(1)}(z)$ is fully found, and moreover, the second local conserved charge (338), *i.e.*, the energy (351), at the level $O(1/J)$ is determined,

$$Q_2|_{\text{next-to-leading}} = -\gamma^{(1)}, \quad \textit{i.e.}, \quad E^{(1)} = -2\omega^2 \gamma^{(1)}. \quad (925)$$

2. The Weak-Coupling Limit of the Energy $E^{(1)}$

Collecting the weak-coupling expansions of $\gamma^{(1), \text{from } \mathcal{P}^{(1)}(z)}$ (840), $\gamma^{(1), \text{from } \mathcal{A}^{(1)}(z)}$ (856), $\gamma^{(1), \text{from } G^{(0)'}(z)}$ (892), we find five leading terms of the constant $\gamma^{(1)}$,

$$\gamma^{(1)} = \gamma_0^{(1)} + \eta \gamma_1^{(1)} + \eta^2 \gamma_2^{(1)} + \eta^3 \gamma_3^{(1)} + \eta^4 \gamma_4^{(1)} + \eta^5 \gamma_5^{(1)} + O(\eta^6), \quad (926)$$

where

$$\gamma_0^{(1)} = \mu^2 \alpha (1 + \alpha) + \frac{1}{2} \sum_{n \geq 1} \left(\nu - \sqrt{4\mu^2 \alpha (1 + \alpha) + \nu^2} \right)^2, \quad (927)$$

$$\gamma_1^{(1)} = 0, \quad (928)$$

$$\begin{aligned}
\gamma_2^{(1)} = & -\mu^2\alpha(1+\alpha)(4+11\alpha+3\alpha^2) + \\
& + \frac{1}{\mu^2} \sum_{n \geq 1} \left(- \left(2\mu^4\alpha(1+\alpha)(4+11\alpha+3\alpha^2) + 2\mu^2\nu^2(3+10\alpha+5\alpha^2) + \nu^4 \right) + \right. \\
& \left. + \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2}} \left(4\mu^4\alpha(1+\alpha)(5+15\alpha+6\alpha^2) + 2\mu^2\nu^2(3+11\alpha+6\alpha^2) + \nu^4 \right) \right), \quad (929)
\end{aligned}$$

$$\gamma_3^{(1)} = -\frac{8}{3\pi}\mu^3\alpha^3(1+\alpha)^3, \quad (930)$$

$$\begin{aligned}
\gamma_4^{(1)} = & 2\mu^2\alpha(1+\alpha)(8+52\alpha+89\alpha^2+42\alpha^3+5\alpha^4) + \\
& + \frac{2}{\mu^4} \sum_{n \geq 1} \left(2\mu^6\alpha(1+\alpha)(8+52\alpha+89\alpha^2+42\alpha^3+5\alpha^4) + \right. \\
& + \mu^4\nu^2(15+128\alpha+279\alpha^2+202\alpha^3+44\alpha^4) + \mu^2\nu^4(15+38\alpha+19\alpha^2) + \nu^6 - \\
& - \frac{\nu}{(4\mu^2\alpha(1+\alpha)+\nu^2)^{3/2}} \left(4\mu^8\alpha^2(1+\alpha)^2(45+324\alpha+621\alpha^2+370\alpha^3+60\alpha^4) + \right. \\
& + 2\mu^6\nu^2\alpha(1+\alpha)(53+481\alpha+1083\alpha^2+815\alpha^3+192\alpha^4) + \\
& \left. \left. + \mu^4\nu^4(15+218\alpha+603\alpha^2+556\alpha^3+164\alpha^4) + \mu^2\nu^6(15+44\alpha+25\alpha^2) + \nu^8 \right) \right), \quad (931)
\end{aligned}$$

$$\gamma_5^{(1)} = \frac{16}{3\pi}\mu^3\alpha^3(1+\alpha)^3(3+8\alpha). \quad (932)$$

An important observation is that we have both even and odd powers of η here, with an annotation that the odd ones (here: $O(\eta^3)$, $O(\eta^5)$) originate exclusively from the Hernández–López phase, and thus, provide a means for its testing.

3. The Large-Mode-Number, Fixed-Winding-Number Limit of the Energy $E^{(1)}$

The large-mode-number, fixed-winding-number limit of the constant $\gamma^{(1)}$ is found by adding the contributions $\gamma^{(1), \text{from } \mathcal{P}^{(1)}(z)}$ (843), $\gamma^{(1), \text{from } \mathcal{A}^{(1), \text{coth}=1}(z)}$ (910), $\gamma^{(1), \text{from } \mathcal{A}^{(1), \delta}(z)}$ (918),

$$\gamma^{(1)} = \gamma_0^{(1)} + \frac{1}{\eta^2}\gamma_2^{(1)} + O\left(\frac{1}{\eta^3}\right) + O(\exp), \quad (933)$$

where

$$\gamma_0^{(1)} = \frac{1}{16\pi\omega^3} \left(-2\kappa(1-\kappa) + 4\log(1+\kappa) + 2(1-\kappa^2)\log(1-\kappa) + (1+\kappa^2)\log(1+\kappa^2) \right), \quad (934)$$

$$\gamma_2^{(1)} = \frac{\kappa^2}{128\pi\omega^3} \left(-2\kappa(1-\kappa) - 4\kappa^2\log(1+\kappa) + 2(1-\kappa^2)\log(1-\kappa) - (1+\kappa^2)\log(1+\kappa^2) \right) + \frac{\pi\kappa}{24\omega}. \quad (935)$$

We see that due to the Dirac delta contribution to the anomaly, which we have managed to determine only up to the leading order, and which appears as the last term in (935), we are able to find only the two leading even orders of the constant $\gamma^{(1)}$. Notice that the Hernández–López phase affects both these orders. The leading piece (934) can also be checked to agree with the result (4.18) of [185] upon the substitution (738) and replacing $m \rightarrow -m$ (which originates from the fact that we have a different sign of μ than this paper), or more precisely, there is $\delta E^{\text{Bethe}} = -2\omega^2\gamma_0^{(1)} = E_0^{(1)}$.

4. *The Weak-Coupling Limit of the Density $\rho^{(1),\text{from } \gamma^{(1)}}(y)$*

It is clear from the solution for $G^{(1)}(z)$ (920) that the constant $\gamma^{(1)}$ develops the following input to the density $\rho^{(1)}(y)$,

$$\rho^{(1),\text{from } \gamma^{(1)}}(y) = -\frac{\omega^2 \gamma^{(1)}}{\pi} \frac{y}{(y-c)\sqrt{(y-a)(b-y)}}. \quad (936)$$

Changing the variable y to t (539), as usual, and using the weak-coupling series of the roots a, b, c (770), (768), and the constant $\gamma^{(1)}$ (926), we obtain

$$\rho^{(1),\text{from } \gamma^{(1)}}(t) = \eta^2 \rho_2^{(1),\text{from } \gamma^{(1)}}(t) + O(\eta^4), \quad (937)$$

where

$$\begin{aligned} \rho_2^{(1),\text{from } \gamma^{(1)}}(t) &= -\frac{\gamma_0^{(1)}}{4\pi\mu\sqrt{\alpha(1+\alpha)}} \frac{1}{\sqrt{t(1-t)}} = \\ &= -\frac{\mu^2\alpha(1+\alpha) + \frac{1}{2}\sum_{n\geq 1} \left(\nu - \sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}\right)^2}{4\pi\mu\sqrt{\alpha(1+\alpha)}} \frac{1}{\sqrt{t(1-t)}}. \end{aligned} \quad (938)$$

Remark that the expansion begins only at the order $O(\eta^2)$, and that there is no order $O(\eta^3)$.

5. *The Large-Mode-Number, Fixed-Winding-Number Limit of the Density $\rho^{(1),\text{from } \gamma^{(1)}}(y)$*

Analogously, the large-mode-number, fixed-winding-number limit of the contribution $\rho^{(1),\text{from } \gamma^{(1)}}(t)$ is derived from the series (794)–(796), (933),

$$\rho^{(1),\text{from } \gamma^{(1)}}(t) = \eta \rho_{-1}^{(1),\text{from } \gamma^{(1)}}(t) + O(\eta^0) + O(\exp), \quad (939)$$

where

$$\begin{aligned} \rho_{-1}^{(1),\text{from } \gamma^{(1)}}(t) &= -\frac{\omega \gamma_0^{(1)}}{4\pi\kappa} \frac{1}{\sqrt{t(1-t)}} = \\ &= -\frac{1}{64\pi^2\omega^2\kappa} \left(-2\kappa(1-\kappa) + 4\log(1+\kappa) + 2(1-\kappa^2)\log(1-\kappa) + (1+\kappa^2)\log(1+\kappa^2) \right) \frac{1}{\sqrt{t(1-t)}}. \end{aligned} \quad (940)$$

Similarly as at weak coupling (937), this is the next-to-next-to-leading order.

D. The Density $\rho^{(1)}(y)$

1. The Boundary Resolvent $G^{(1),\text{boundary}}(z)$

Let us start from using the linear equation (502) obeyed by the resolvent $G^{(1)}(z)$ to derive this resolvent's boundary part, just as we have done in paragraphs III B 4 and III C 4. Recall it,

$$2G^{(1)}(y) = -\frac{y^2}{y^2 - \omega^2} \pi \rho^{(0)'}(y) \coth\left(\pi \rho^{(0)}(y)\right) + 2\tilde{\mathcal{P}}^{(1)}(y), \quad (941)$$

where $y \in \mathcal{C}^y$. In this paragraph, we proceed without any approximation.

The Hernández–López phase piece on the r.h.s. can be computed exactly, in a very similar way the integral (833) has been found,

$$\tilde{\mathcal{P}}^{(1)}(y) = -\frac{\mu}{2\pi^2} \frac{y^2}{y^2 - \omega^2}.$$

$$\begin{aligned} & \cdot \left(\frac{2\pi}{\omega(y^2 - \omega^2)^2} \left((y + \omega)^2(c - \omega)\sqrt{\omega - a}\sqrt{\omega - b} + (y - \omega)^2(c + \omega)\sqrt{-\omega - a}\sqrt{-\omega - b} - \right. \right. \\ & \quad \left. \left. - 2y(cy - \omega^2) \sqrt{\frac{\omega^2}{y} - a} \sqrt{\frac{\omega^2}{y} - b} \right) + \right. \\ & + \frac{2\pi}{\omega(y^2 - \omega^2)^2} \log\left(\frac{y - \omega}{y + \omega}\right) \left(- (y + \omega)^2(c - \omega)\sqrt{\omega - a}\sqrt{\omega - b} + (y - \omega)^2(c + \omega)\sqrt{-\omega - a}\sqrt{-\omega - b} + \right. \\ & \quad \left. + \frac{\omega}{2y\sqrt{\frac{\omega^2}{y} - a}\sqrt{\frac{\omega^2}{y} - b}} \left(- (2ab + (a + b)c)y^3 + (4abc + \omega^2(3(a + b) + 2c))y^2 - \right. \right. \\ & \quad \left. \left. - \omega^2(2ab + 3(a + b)c + 4\omega^2)y + \omega^4(a + b + 2c) \right) \right) + \\ & \quad + \frac{\pi(\tilde{y} - 1)^2}{\omega(\tilde{c} - 1)\sqrt{(\tilde{a} - 1)(\tilde{b} - 1)}\tilde{y}^2} \left(2\left(\sqrt{\tilde{a}\tilde{b}\tilde{c}} + \tilde{y}^2\right) - (\tilde{y} + \tilde{c})\sqrt{(\tilde{y} + \tilde{a})(\tilde{y} + \tilde{b})} + \right. \\ & + 4\left(\sqrt{\tilde{a}\tilde{b}\tilde{c}} + \tilde{y}^2\right) \log\left(\frac{\sqrt{\tilde{a}} + \sqrt{\tilde{b}}}{2}\right) - 2\sqrt{\tilde{a}\tilde{b}\tilde{c}} \log(\tilde{a}\tilde{b}) + 2\tilde{c}\sqrt{(\tilde{y} + \tilde{a})(\tilde{y} + \tilde{b})} \log\left(\frac{\sqrt{\tilde{a}\tilde{b}} - \tilde{y} + \sqrt{(\tilde{y} + \tilde{a})(\tilde{y} + \tilde{b})}}{\sqrt{\tilde{a}} + \sqrt{\tilde{b}}}\right) + \\ & \quad + \frac{\tilde{y}(\tilde{y} + \tilde{c})(\tilde{a} + \tilde{b} + 2\tilde{y})}{2\sqrt{(\tilde{y} + \tilde{a})(\tilde{y} + \tilde{b})}} \log\left(\frac{(\sqrt{\tilde{y} + \tilde{a}} + \sqrt{\tilde{y} + \tilde{b}})^2}{2\tilde{a}\tilde{b} + (\tilde{a} + \tilde{b})\tilde{y} + 2\sqrt{\tilde{a}\tilde{b}}\sqrt{(\tilde{y} + \tilde{a})(\tilde{y} + \tilde{b})}} \right) - \\ & \quad \left. - 2\tilde{c}\sqrt{(\tilde{y} - \tilde{b})(\tilde{a} - \tilde{y})} \arctan\left(\frac{\sqrt{(\tilde{y} - \tilde{b})(\tilde{a} - \tilde{y})}}{\sqrt{\tilde{a}\tilde{b}} + \tilde{y}}\right) - \frac{\tilde{y}(\tilde{y} - \tilde{c})(\tilde{a} + \tilde{b} - 2\tilde{y})}{2\sqrt{(\tilde{y} - \tilde{b})(\tilde{a} - \tilde{y})}} \arctan\left(\frac{\sqrt{(\tilde{y} - \tilde{b})(\tilde{a} - \tilde{y})}(\sqrt{\tilde{a}\tilde{b}} + \tilde{y})}{\tilde{a}\tilde{b} - \frac{1}{2}(\sqrt{\tilde{a}} - \sqrt{\tilde{b}})^2\tilde{y} + \tilde{y}^2}\right) \right) \Bigg), \quad (942) \end{aligned}$$

where the “checked” variables are defined in (831). Importantly, this quantity is finite for $y \rightarrow a$ and $y \rightarrow b$, and hence, will not contribute to the boundary resolvent.

The piece with the hyperbolic cotangent on the r.h.s. of (941) can be handled analogously as at one loop (595), and it is exclusively it that finally yields a very simple result for the boundary resolvent,

$$G^{(1),\text{boundary}}(z) = -\frac{1}{4} \left(\frac{a^2}{a^2 - \omega^2} \frac{1}{z - a} + \frac{b^2}{b^2 - \omega^2} \frac{1}{z - b} \right). \quad (943)$$

Similarly to what we have done at the end of paragraph III B 4, let us use (943) to translate the three conditions satisfied by the resolvent $G^{(1)}(z)$, namely

$$G^{(1)}(z) = -\omega^2 \gamma^{(1)} \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \quad \text{for } z \rightarrow \infty, \quad (944)$$

$$G^{(1)}(0) = 0, \quad (945)$$

$$G^{(1)'}(0) = \gamma^{(1)}, \quad (946)$$

to conditions that the density $\rho^{(1)}(y)$ must fulfill. We get exactly,

$$\int_{\mathcal{C}_y} dy \rho^{(1)}(y) = -\omega^2 \gamma^{(1)} + \frac{1}{4} \left(\frac{a^2}{a^2 - \omega^2} + \frac{b^2}{b^2 - \omega^2} \right), \quad (947)$$

$$\int_{\mathcal{C}_y} dy \frac{\rho^{(1)}(y)}{y} = \frac{1}{4} \left(\frac{a}{a^2 - \omega^2} + \frac{b}{b^2 - \omega^2} \right), \quad (948)$$

$$\int_{\mathcal{C}_y} dy \frac{\rho^{(1)}(y)}{y^2} = -\gamma^{(1)} + \frac{1}{4} \left(\frac{1}{a^2 - \omega^2} + \frac{1}{b^2 - \omega^2} \right). \quad (949)$$

We will use them as a cross-check for our density.

2. The Weak-Coupling Limit

The weak-coupling expansion of the complete density at the level $\mathcal{O}(1/J)$ is found by composing the contributions (841), (858), (893), (937), and its general structure is

$$\rho^{(1)}(t) = \rho_0^{(1)}(t) + \eta^2 \rho_2^{(1)}(t) + \eta^3 \rho_3^{(1)}(t) + \mathcal{O}(\eta^4), \quad (950)$$

where

$$\rho_0^{(1)}(t) = \rho_0^{(1),\text{from } \mathcal{A}^{(1)}(z)}(t) + \rho_0^{(1),\text{from } G^{(0)'}(z)}(t), \quad (951)$$

$$\rho_2^{(1)}(t) = \rho_2^{(1),\text{from } \mathcal{A}^{(1)}(z)}(t) + \rho_2^{(1),\text{from } G^{(0)'}(z)}(t) + \rho_2^{(1),\text{from } \gamma^{(1)}}(t), \quad (952)$$

$$\rho_3^{(1)}(t) = \rho_3^{(1),\text{from } \mathcal{P}^{(1)}(z)}(t). \quad (953)$$

It is unnecessary to print these terms explicitly here. Notice that there is no order $\mathcal{O}(\eta)$, and that the Hernández-López phase begins only at the order $\mathcal{O}(\eta^3)$. The conditions (947)–(949), expanded at small η up to the order $\mathcal{O}(\eta^3)$, can be tested to be fulfilled.

3. The Large-Mode-Number, Fixed-Winding-Number Limit

The full level- $\mathcal{O}(1/J)$ density in the large-mode-number, fixed-winding-number approximation is built from the pieces (844), (911), (919), (939), and its general structure reads

$$\rho^{(1)}(t) = \eta^3 \rho_{-3}^{(1)}(t) + \eta^2 \rho_{-2}^{(1)}(t) + \eta \rho_{-1}^{(1)}(t) + \mathcal{O}(\eta^0) + \mathcal{O}(\exp), \quad (954)$$

where

$$\rho_{-3}^{(1)}(t) = \frac{1}{\omega^2} \rho_{-3}^{(1), \text{from } \mathcal{P}^{(1)}(z)}(t) + \frac{1}{\omega^2} \rho_{-3}^{(1), \text{from } \mathcal{A}^{(1), \text{coth}=1}(z)}(t), \quad (955)$$

$$\rho_{-2}^{(1)}(t) = \frac{1}{\omega^2} \rho_{-2}^{(1), \text{from } \mathcal{P}^{(1)}(z)}(t) + \frac{1}{\omega^2} \rho_{-2}^{(1), \text{from } \mathcal{A}^{(1), \text{coth}=1}(z)}(t), \quad (956)$$

$$\rho_{-1}^{(1)}(t) = \frac{1}{\omega^2} \rho_{-1}^{(1), \text{from } \mathcal{P}^{(1)}(z)}(t) + \frac{1}{\omega^2} \rho_{-1}^{(1), \text{from } \mathcal{A}^{(1), \text{coth}=1}(z)}(t) + \rho_{-1}^{(1), \text{from } \mathcal{A}^{(1), \delta}(z)}(t) + \rho_{-1}^{(1), \text{from } \gamma^{(1)}}(t). \quad (957)$$

It explodes as η^3 at large η , as compared to η in the case of $\rho^{(0)}(t)$ (803).

Importantly, the Hernández-López phase is visible already at the leading order. As extensively explained in paragraph III B 5, it is only the leading term (955) of the density $\rho^{(1)}(t)$ that is reliable for the computation of the energy $E^{(2)}$, for example due to the violation of the three conditions (947)–(949) by the subleading terms (which we have discovered in (956), (957) to our dismay), or in other words, due to problems with $G^{(1), \text{boundary}}(z)$.

VI. SOLVING THE ALL-LOOP ONE-CUT QUADRATIC EQUATION AT THE ORDER $O(1/J^2)$ — THE ENERGY $E^{(2)}$

In this final section, we attack the all-loop one-cut quadratic equation (459), but we do not aim at anything more than finding the weak-coupling (in subsection VI A) or large-mode-number, fixed-winding-number (in subsection VI B) expansions of the constant $\gamma^{(2)}$, *i.e.*, also the energy $E^{(2)}$. Appropriate orders of these series will provide a testing opportunity for the Hernández-López phase.

A. The Weak-Coupling Limit

1. The Holomorphicity Consistency Condition for the Resolvent $G^{(2)}(z)$

If we recall the solution for the resolvent $G^{(2)}(z)$ (459),

$$G^{(2)}(z) = \frac{\mathcal{F}(z) + \left(1 - \frac{\omega^2}{z^2}\right) G^{(1)}(z)^2 + G^{(1)'}(z) - \mathcal{P}^{(2)}(z) - \mathcal{A}^{(2)}(z) - \frac{\omega^2 \mu \gamma^{(2)}}{z}}{\frac{\mu}{z^2}(z-c)\sqrt{z-a}\sqrt{z-b}}, \quad (958)$$

we see that it includes the unknown and arbitrary constant $\gamma^{(2)}$. The way to determine it has been described in paragraph V C 1, and is based on requiring the resolvent to be holomorphic everywhere outside the cut $[a, b]$, which in particular means that it should not have a pole at $z = c$. The numerator of the r.h.s. of (958) must thus vanish at this point, which leads to the expression

$$\gamma^{(2)} = \frac{c}{\omega^2 \mu} \left(\mathcal{F}(c) + \left(1 - \frac{\omega^2}{c^2}\right) G^{(1)}(c)^2 + G^{(1)'}(c) - \mathcal{P}^{(2)}(c) - \mathcal{A}^{(2)}(c) \right). \quad (959)$$

In this subsection, we will compute the weak-coupling expansions of all these contributions, up to the order $O(\eta^3)$, which is the order where the Hernández-López phase starts appearing. Also, we will derive some exact results, to be used in subsection VI B.

2. The Contribution from the Resolvent $G^{(1)}(z)$

Let us begin with the contribution

$$\gamma^{(2), \text{from } G^{(1)}(z)} \equiv \frac{c}{\omega^2 \mu} \left(\left(1 - \frac{\omega^2}{c^2}\right) G^{(1)}(c)^2 + G^{(1)'}(c) \right). \quad (960)$$

The way to compute $G^{(1)}(c)$ and $G^{(1)'}(c)$ is to expand the numerator of (920) in z around $z = c$ up to two or three leading orders, respectively. The first one, or the first and second ones, respectively, will be zero, thus canceling $(z - c)$ or $(z - c)^2$ from the denominator, and leaving us with finite results,

$$G^{(1)}(c) = \frac{-\mathcal{P}^{(1)'}(c) - \mathcal{A}^{(1)'}(c) + G^{(0)''}(c) + \frac{\omega^2 \mu \gamma^{(1)}}{c^2}}{\frac{\mu}{c^2} \sqrt{c-a} \sqrt{c-b}}, \quad (961)$$

$$\begin{aligned} G^{(1)'}(c) = & \frac{1}{2\mu(c-a)^{3/2}(c-b)^{3/2}} \left(-\omega^2 \mu (a+b-6c) \gamma^{(1)} + \right. \\ & + 2(ab-3c^2) \left(-\mathcal{P}^{(1)}(c) - \mathcal{A}^{(1)}(c) + G^{(0)'}(c) \right) + \\ & \left. + c(4ab-3(a+b)c+2c^2) \left(-\mathcal{P}^{(1)'}(c) - \mathcal{A}^{(1)'}(c) + G^{(0)''}(c) \right) + \right. \end{aligned}$$

$$+ c^2(c-a)(c-b) \left(-\mathcal{P}^{(1)''}(c) - \mathcal{A}^{(1)''}(c) + G^{(0)'''}(c) \right). \quad (962)$$

Furthermore, the values of the resolvent $G^{(0)}(z)$ and its derivatives at $z = c$ are straightforward to find. The values of the phase $\mathcal{P}^{(1)}(z)$ and its derivatives at this point are calculated from the exact formula (833). The anomaly $\mathcal{A}^{(1)}(z)$ and its derivatives there are derived by tackling the integral (848) with the method of residues.

Putting all these components together, we finally obtain the following weak-coupling series,

$$\gamma_0^{(2), \text{from } G^{(1)}(z)} = \gamma_0^{(2), \text{from } G^{(1)}(z)} + \eta^2 \gamma_2^{(2), \text{from } G^{(1)}(z)} + \eta^3 \gamma_3^{(2), \text{from } G^{(1)}(z)} + \mathcal{O}(\eta^4), \quad (963)$$

where the leading term,

$$\gamma_0^{(2), \text{from } G^{(1)}(z)} = -\mu^2 \alpha(1 + \alpha) - \frac{1}{2} \sum_{n \geq 1} \left(\nu - \sqrt{4\mu^2 \alpha(1 + \alpha) + \nu^2} \right)^2, \quad (964)$$

the next-to-leading term,

$$\begin{aligned} \gamma_2^{(2), \text{from } G^{(1)}(z)} &= \mu^2 \alpha(1 + \alpha) (11 + 28\alpha + 6\alpha^2) + \\ &+ \frac{1}{\mu^2} \sum_{n \geq 1} \left(2\mu^4 \alpha(1 + \alpha) (11 + 29\alpha + 7\alpha^2) + \mu^2 \nu^2 (13 + 38\alpha + 14\alpha^2) + \nu^4 - \right. \\ &- \left. \frac{\nu}{\sqrt{4\mu^2 \alpha(1 + \alpha) + \nu^2}} \left(4\mu^4 \alpha(1 + \alpha) (12 + 33\alpha + 10\alpha^2) + \mu^2 \nu^2 (13 + 40\alpha + 16\alpha^2) + \nu^4 \right) \right) + \\ &+ \frac{1}{4\mu^2} \left(\sum_{n \geq 1} \left(\nu - \sqrt{4\mu^2 \alpha(1 + \alpha) + \nu^2} \right)^2 \right)^2, \end{aligned} \quad (965)$$

the next-to-next-to-leading term,

$$\gamma_3^{(2), \text{from } G^{(1)}(z)} = \frac{8\mu^3 \alpha^3 (1 + \alpha)^3}{3\pi}. \quad (966)$$

The presence of the Hernández-López phase starts being visible only in this last term. Notice that there is no order $\mathcal{O}(\eta)$.

3. The Contribution from the Boundary Phase $\mathcal{P}^{(2), \text{boundary}}(z)$

In order to approach the contribution from the boundary part of the phase,

$$\gamma^{(2), \text{from } \mathcal{P}^{(2), \text{boundary}}(z)} \equiv -\frac{c}{\omega^2 \mu} \mathcal{P}^{(2), \text{boundary}}(c), \quad (967)$$

we take the expression for its bulk part (444), in the non-symmetrized form, and replace the integration with the density $\rho^{(1)}(y_2)$ by contour integration with the boundary resolvent $G^{(1), \text{boundary}}(z')$ (943), just as described in paragraph III C 1,

$$\mathcal{P}^{(2), \text{boundary}}(z) = \frac{1}{\pi} \int_a^b \oint_{\odot[a, b]} dy_1 dz' \rho^{(0)}(y_1) \frac{G^{(1), \text{boundary}}(z')}{2\pi i} \frac{1}{(z - y_1)(z - z')} \mathcal{H}(y_1, z'), \quad (968)$$

where $\mathcal{H}(y_1, z')$ is defined in (439). The contour integration is easy, as the only non-zero residues come from the poles at $z = a$ and $z = b$ inside the contour,

$$\oint_{\odot[a, b]} dz' \frac{G^{(1), \text{boundary}}(z')}{2\pi i} \frac{1}{z - z'} \mathcal{H}(y_1, z') =$$

$$\begin{aligned}
&= -\frac{\omega}{2} \left(\frac{a^2}{a^2 - \omega^2} \frac{1}{z - a} \frac{1}{y_1 a - \omega^2} + \frac{b^2}{b^2 - \omega^2} \frac{1}{z - b} \frac{1}{y_1 b - \omega^2} \right) - \\
&\quad - \frac{1}{4} \frac{a^2}{a^2 - \omega^2} \frac{1}{z - a} \left(\frac{1}{y_1 - a} + \frac{\omega^2 (y_1 - a)}{(y_1 a - \omega^2)^2} \right) \log \left(\frac{(y_1 + \omega)(a - \omega)}{(y_1 - \omega)(a + \omega)} \right) - \\
&\quad - \frac{1}{4} \frac{b^2}{b^2 - \omega^2} \frac{1}{z - b} \left(\frac{1}{y_1 - b} + \frac{\omega^2 (y_1 - b)}{(y_1 b - \omega^2)^2} \right) \log \left(\frac{(y_1 + \omega)(b - \omega)}{(y_1 - \omega)(b + \omega)} \right). \tag{969}
\end{aligned}$$

This should now be integrated over y_1 with the density $\rho^{(0)}(y_1)$, which is doable exactly in a very similar manner to (833),

$$\begin{aligned}
\mathcal{P}^{(2),\text{boundary}}(z) &= \frac{\mu}{4\pi} \left(\frac{(c - z)(ab - \omega^2)(ab(a + b)z^2 - (a^2 + b^2)(ab + \omega^2)z + \omega^2 ab(a + b))}{2(z - a)(z - b)(a^2 - \omega^2)(b^2 - \omega^2)(az - \omega^2)(bz - \omega^2)} \right. \\
&\quad \cdot \left(\frac{\sqrt{\omega + a}\sqrt{\omega + b}}{z + \omega} + \frac{\sqrt{\omega - a}\sqrt{\omega - b}}{z - \omega} - \frac{2\omega\sqrt{z - a}\sqrt{z - b}}{z^2 - \omega^2} \right) - \\
&\quad - \frac{1}{\omega(z - a)(z - b)(a^2 - \omega^2)(b^2 - \omega^2)(z^2 - \omega^2)} \cdot \\
&\quad \cdot \left(\frac{(c + \omega)(z - \omega)}{\sqrt{a + \omega}\sqrt{b + \omega}} \left(a^2 b^2 (a^2 + b^2) + \omega a^2 b^2 (a + b) - 2\omega^2 a^2 b^2 - \omega^3 ab(a + b) + \right. \right. \\
&\quad \left. \left. + (-a^2 b^2 (a + b) - 2\omega a^2 b^2 + \omega^2 ab(a + b) + \omega^3 (a^2 + b^2)) z \right) + \right. \\
&\quad \left. + \frac{(c - \omega)(z + \omega)}{\sqrt{a - \omega}\sqrt{b - \omega}} \left(a^2 b^2 (a^2 + b^2) - \omega a^2 b^2 (a + b) - 2\omega^2 a^2 b^2 + \omega^3 ab(a + b) + \right. \right. \\
&\quad \left. \left. + (-a^2 b^2 (a + b) + 2\omega a^2 b^2 + \omega^2 ab(a + b) - \omega^3 (a^2 + b^2)) z \right) \right) \log \left(\frac{\sqrt{a} + \sqrt{b}}{2} \right) + \\
&\quad + \frac{1}{\omega} \left(\frac{a^2 (\omega a^2 bc - 2\omega^2 abc + \omega^4 (b + c) - \omega^5 + (a^2 bc - \omega a^2 (b + c) + 2\omega^3 a - \omega^4) z)}{(z - a)(\omega - a)(az - \omega^2)^2 \sqrt{(a^2 - \omega^2)(ab - \omega^2)}} \right) + \\
&\quad + \frac{b^2 (bc - \omega^2) (2ab + \omega(a - b) - 2\omega^2)}{2(b - z)(b^2 - \omega^2)(bz - \omega^2) \sqrt{(b^2 - \omega^2)(ab - \omega^2)}} \log \left(\frac{\sqrt{a}\sqrt{b} - \check{a} + \sqrt{2\check{a}(\check{a} + \check{b})}}{\sqrt{a} + \sqrt{b}} \right) + \\
&\quad + \frac{1}{\omega} \left(\frac{b^2 (\omega b^2 ac - 2\omega^2 abc + \omega^4 (a + c) - \omega^5 + (b^2 ac - \omega b^2 (a + c) + 2\omega^3 b - \omega^4) z)}{(z - b)(\omega - b)(bz - \omega^2)^2 \sqrt{(b^2 - \omega^2)(ab - \omega^2)}} \right) + \\
&\quad + \frac{a^2 (ac - \omega^2) (2ab - \omega(a - b) - 2\omega^2)}{2(a - z)(a^2 - \omega^2)(az - \omega^2) \sqrt{(a^2 - \omega^2)(ab - \omega^2)}} \log \left(\frac{\sqrt{a}\sqrt{b} - \check{b} + \sqrt{2\check{b}(\check{a} + \check{b})}}{\sqrt{a} + \sqrt{b}} \right) +
\end{aligned}$$

$$\begin{aligned}
& + \frac{(z-c)(ab-\omega^2)}{(z-a)^{3/2}(z-b)^{3/2}(a^2-\omega^2)(b^2-\omega^2)(z^2-\omega^2)(az-\omega^2)^2(bz-\omega^2)^2} \\
& \cdot \left(2a^2b^2(ab+\omega^2)z^6 - 2ab(a+b) \left(a^2b^2 + \omega^2(a+b)^2 + \omega^4 \right) z^5 + \right. \\
& + (ab+\omega^2) \left(a^2b^2(a^2+b^2) + \omega^2(a^4+8a^3b+8a^2b^2+8ab^3+b^4) + \omega^4(a^2+b^2) \right) z^4 - \\
& - 4\omega^2(a+b) \left(a^2b^2(a^2+b^2) + 2\omega^2ab(a^2+ab+b^2) + \omega^4(a^2+b^2) \right) z^3 + \\
& + \omega^2(ab+\omega^2) \left(a^2b^2(a^2+b^2) + \omega^2(a^4+8a^3b+8a^2b^2+8ab^3+b^4) + \omega^4(a^2+b^2) \right) z^2 - \\
& - 2\omega^4ab(a+b) \left(a^2b^2 + \omega^2(a+b)^2 + \omega^4 \right) z + 2\omega^6a^2b^2(ab+\omega^2) \Big) \log \left(\frac{\sqrt{a}\bar{b} + \bar{z} + \sqrt{a-\omega}\bar{z}\sqrt{b-\omega}}{\sqrt{a} + \sqrt{b}} \right) + \\
& + \frac{1}{2} \left(\frac{b^2(c+\omega)\sqrt{a+\omega}}{\omega(z-b)(z+\omega)(b-\omega)(b+\omega)^{3/2}} + \frac{a^2(c-\omega)\sqrt{b-\omega}}{\omega(z-a)(z-\omega)(a+\omega)(a-\omega)^{3/2}} + \right. \\
& + \frac{a^2(\omega a^2bc - 2\omega^2abc + \omega^4(b+c) - \omega^5 + (a^2bc - \omega a^2(b+c) + 2\omega^3a - \omega^4)z)}{\omega(a-z)(a-\omega)(az-\omega^2)^2\sqrt{(a^2-\omega^2)(ab-\omega^2)}} + \\
& + \frac{b^2(bc-\omega^2)(2ab+\omega(a-b)-2\omega^2)}{2\omega(b-z)(b^2-\omega^2)(bz-\omega^2)\sqrt{(b^2-\omega^2)(ab-\omega^2)}} - \\
& - \frac{a^2\sqrt{z-b}(c-z)((a^2+\omega^2)z^2 - 4\omega^2az + \omega^2(a^2+\omega^2))}{(z-a)^{3/2}(a^2-\omega^2)(z^2-\omega^2)(az-\omega^2)^2} \Big) \log \left(\frac{a-\omega}{a+\omega} \right) + \\
& + \frac{1}{2} \left(\frac{a^2(c+\omega)\sqrt{b+\omega}}{\omega(z-a)(z+\omega)(a-\omega)(a+\omega)^{3/2}} + \frac{b^2(c-\omega)\sqrt{a-\omega}}{\omega(z-b)(z-\omega)(b+\omega)(b-\omega)^{3/2}} + \right. \\
& + \frac{b^2(\omega b^2ac - 2\omega^2abc + \omega^4(a+c) - \omega^5 + (b^2ac - \omega b^2(a+c) + 2\omega^3b - \omega^4)z)}{\omega(b-z)(b-\omega)(bz-\omega^2)^2\sqrt{(b^2-\omega^2)(ab-\omega^2)}} + \\
& + \frac{a^2(ac-\omega^2)(2ab-\omega(a-b)-2\omega^2)}{2\omega(a-z)(a^2-\omega^2)(az-\omega^2)\sqrt{(a^2-\omega^2)(ab-\omega^2)}} - \\
& - \frac{b^2\sqrt{z-a}(c-z)((b^2+\omega^2)z^2 - 4\omega^2bz + \omega^2(b^2+\omega^2))}{(z-b)^{3/2}(b^2-\omega^2)(z^2-\omega^2)(bz-\omega^2)^2} \Big) \log \left(\frac{b-\omega}{b+\omega} \right) \Big). \tag{970}
\end{aligned}$$

Finally, it remains to set $z = c$ here, and expand at small η ; the series starts at the order $O(\eta^3)$, as expected,

$$\gamma^{(2),\text{from } \mathcal{P}^{(2),\text{boundary}}(z)} = \eta^3 \gamma_3^{(2),\text{from } \mathcal{P}^{(2),\text{boundary}}(z)} + O(\eta^4), \tag{971}$$

where

$$\gamma_3^{(2),\text{from } \mathcal{P}^{(2),\text{boundary}}(z)} = \frac{20\mu^3\alpha^2(1+\alpha)^2(1+8\alpha+8\alpha^2)}{3\pi}. \tag{972}$$

4. The Contribution from the Bulk Phase $\mathcal{P}^{(2),\text{bulk}}(z)$

To proceed with the weak-coupling series of the bulk part of the phase,

$$\gamma^{(2),\text{from } \mathcal{P}^{(2),\text{bulk}}(z)} \equiv -\frac{c}{\omega^2 \mu} \mathcal{P}^{(2),\text{bulk}}(c), \quad (973)$$

it is easiest to change the integration variables y_1, y_2 in (444) to t_1, t_2 according to (539). Then, we exploit the expansions of the densities (774), (950), which have the following structure,

$$\rho^{(0)}(t) = \rho_0^{(0)}(t) + \eta^2 \rho_2^{(0)}(t) + \mathcal{O}(\eta^4), \quad (974)$$

$$\rho^{(1)}(t) = \rho_0^{(1)}(t) + \eta^2 \rho_2^{(1)}(t) + \eta^3 \rho_3^{(1)}(t) + \mathcal{O}(\eta^4). \quad (975)$$

This yields an expansion that commences at $\mathcal{O}(\eta^3)$, as expected,

$$\gamma^{(2),\text{from } \mathcal{P}^{(2),\text{bulk}}(z)} = \eta^3 \gamma_3^{(2),\text{from } \mathcal{P}^{(2),\text{bulk}}(z)} + \mathcal{O}(\eta^4), \quad (976)$$

where the leading term is given by the following integral,

$$\gamma_3^{(2),\text{from } \mathcal{P}^{(2),\text{bulk}}(z)} = -\frac{\mu}{96\pi\alpha^2(1+\alpha)^2} \int_0^1 \int_0^1 dt_1 dt_2 \frac{(t_1 - t_2)^2 \rho_0^{(0)}(t_1) \rho_0^{(1)}(t_2)}{\left(t_1 - \frac{-(\sqrt{\alpha}-\sqrt{1+\alpha})^2}{4\sqrt{\alpha(1+\alpha)}}\right)^4 \left(t_2 - \frac{-(\sqrt{\alpha}-\sqrt{1+\alpha})^2}{4\sqrt{\alpha(1+\alpha)}}\right)^4}. \quad (977)$$

A good news is that only the leading terms of both densities appear at this order, as the subleading ones of especially $\rho^{(1)}(t)$ are much more complicated. This integral can be done by the method of residues, and gives

$$\begin{aligned} \gamma_3^{(2),\text{from } \mathcal{P}^{(2),\text{bulk}}(z)} &= -\frac{4\mu^3\alpha^2(1+\alpha)^2(3+20\alpha+20\alpha^2)}{3\pi} + \\ &+ \frac{8\alpha(1+\alpha)}{3\pi\mu} \sum_{n \geq 1} \left(2\mu^4\alpha(1+\alpha)(1+10\alpha+10\alpha^2) + \mu^2\nu^2(1+13\alpha+13\alpha^2) + \nu^4 - \right. \\ &\quad \left. - \nu\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2} \left(\mu^2(1+11\alpha+11\alpha^2) + \nu^2 \right) \right). \end{aligned} \quad (978)$$

5. The Contribution from the Boundary Anomaly $\mathcal{A}^{(2),\text{boundary}}(z)$

As discussed in the one-loop case (see paragraph III C 1), the boundary contribution to the quantity in question comes only from $\mathcal{A}^{(2,1)}(z)$ (662); proceeding analogously, this time with the exact all-loop boundary resolvent (943), we find

$$\begin{aligned} \mathcal{A}^{(2),\text{boundary}}(z) &= \mathcal{A}^{(2,1),\text{boundary}}(z) = \\ &= -\pi \oint_{\odot \mathcal{C}_y} dz' \frac{G^{(1),\text{boundary}}(z')}{2\pi i} \frac{1}{z-z'} \rho^{(0)'}(z') \left(\coth\left(\pi \rho^{(0)}(z')\right) - \frac{1}{\pi \rho^{(0)}(z')} \right) = \\ &= -2 \sum_{n \geq 1} \oint_{\odot \mathcal{C}_y} dz' \frac{G^{(1),\text{boundary}}(z')}{2\pi i} \frac{1}{z-z'} \rho^{(0)}(z') \rho^{(0)'}(z') \frac{1}{\rho^{(0)}(z')^2 + n^2} = \end{aligned}$$

$$= \frac{\mu^2(b-a)}{96} \left(\frac{a^2(a-c)^2}{(a^2-\omega^2)^3} \frac{1}{z-a} - \frac{b^2(b-c)^2}{(b^2-\omega^2)^3} \frac{1}{z-b} \right). \quad (979)$$

This is just slightly more complicated than its one-loop counterpart (662), and reduces to it for $\omega = 0$. Also, it can be interpreted analogously to (663),

$$\mathcal{A}^{(2),\text{boundary}}(z) = \frac{\pi^2}{24} \left(\frac{a^2}{a^2-\omega^2} \left(\lim_{y \rightarrow a} \frac{\rho^{(0)}(y)^2}{y-a} \right) \frac{1}{z-a} + \frac{b^2}{b^2-\omega^2} \left(\lim_{y \rightarrow b} \frac{\rho^{(0)}(y)^2}{y-b} \right) \frac{1}{z-b} \right). \quad (980)$$

We then have exactly,

$$\gamma^{(2),\text{from } \mathcal{A}^{(2),\text{boundary}}(z)} \equiv -\frac{c}{\omega^2 \mu} \mathcal{A}^{(2),\text{boundary}}(c) = \frac{\mu c(b-a)}{96\omega^2} \left(\frac{a^2(a-c)}{(a^2-\omega^2)^3} - \frac{b^2(b-c)}{(b^2-\omega^2)^3} \right), \quad (981)$$

which expanded at small η reads

$$\gamma^{(2),\text{from } \mathcal{A}^{(2),\text{boundary}}(z)} = \gamma_0^{(2),\text{from } \mathcal{A}^{(2),\text{boundary}}(z)} + \eta^2 \gamma_2^{(2),\text{from } \mathcal{A}^{(2),\text{boundary}}(z)} + \mathcal{O}(\eta^4), \quad (982)$$

where the leading term,

$$\gamma_0^{(2),\text{from } \mathcal{A}^{(2),\text{boundary}}(z)} = -\frac{1}{6} \mu^4 \alpha(1+\alpha)(1+4\alpha)(3+4\alpha), \quad (983)$$

the next-to-leading term,

$$\gamma_2^{(2),\text{from } \mathcal{A}^{(2),\text{boundary}}(z)} = -\frac{1}{3} \mu^4 \alpha(1+\alpha) (2 + 57\alpha + 289\alpha^2 + 496\alpha^3 + 272\alpha^4). \quad (984)$$

Remark only even powers of η here.

6. The Contribution from the Bulk Anomaly $\mathcal{A}^{(2),\text{bulk}}(z)$

We handle the bulk part of the pertinent anomaly (452) by integrating it by parts, as its form suggests doing, expanding the hyperbolic cotangent into the sum over n (548), and substituting the exact density $\rho^{(0)}(y)$ (742), which yields

$$\mathcal{A}^{(2),\text{bulk}}(z) = -2\mu^2 \sum_{n_1 \geq 1} \int_a^b dy \rho^{(1)}(y) \frac{(y-c)^2(y-a)(b-y)}{(z-y)^2} \frac{1}{Q_{n_1}(y)} \quad (985)$$

(the summation index is n_1 as there will be a summation over n_2 in $\rho^{(1)}(y)$), where the quartic polynomial $Q_{n_1}(y)$ is defined and factorized in (850).

Since the density $\rho^{(1)}(y)$, which appears here, is known only as a weak-coupling series (950), we should change the integration variable from y to t (539), obtaining

$$\gamma^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)} \equiv -\frac{c}{\omega^2 \mu} \mathcal{A}^{(2),\text{bulk}}(c) = \sum_{n_1 \geq 1} \int_0^1 dt \rho^{(1)}(t) t(1-t) \mathcal{Q}_{n_1}(t), \quad (986)$$

where

$$\mathcal{Q}_{n_1}(t) \equiv \frac{2\mu c(b-a)^3}{\omega^2 Q_{n_1}(y)} \Big|_{y=a+(b-a)t}. \quad (987)$$

Now, we expand this quantity (987) at small η ,

$$\mathcal{Q}_{n_1}(t) = \mathcal{Q}_{0,n_1}(t) + \eta^2 \mathcal{Q}_{2,n_1}(t) + \mathcal{O}(\eta^4), \quad (988)$$

where

$$\mathcal{Q}_{0,n_1}(t) = -\frac{\mu^3}{2\sqrt{\alpha(1+\alpha)}(\mu^2 - \nu_1^2)} \frac{1}{\left(t - \frac{-(\sqrt{\alpha}-\sqrt{1+\alpha})^2}{4\sqrt{\alpha(1+\alpha)}}\right)^2} \cdot \frac{1}{t - \left(\frac{1}{2} + \frac{\nu_1^2(1+2\alpha) - \mu\sqrt{4\mu^2\alpha(1+\alpha) + \nu_1^2}}{4(\mu^2 - \nu_1^2)\sqrt{\alpha(1+\alpha)}}\right)} \frac{1}{t - \left(\frac{1}{2} + \frac{\nu_1^2(1+2\alpha) + \mu\sqrt{4\mu^2\alpha(1+\alpha) + \nu_1^2}}{4(\mu^2 - \nu_1^2)\sqrt{\alpha(1+\alpha)}}\right)}, \quad (989)$$

and

$$\begin{aligned} \mathcal{Q}_{2,n_1}(t) = & -\frac{\mu^3}{128\alpha^2(1+\alpha)^2(\mu^2 - \nu_1^2)^2} \frac{1}{\left(t - \frac{-(\sqrt{\alpha}-\sqrt{1+\alpha})^2}{4\sqrt{\alpha(1+\alpha)}}\right)^3} \cdot \\ & \frac{1}{\left(t - \left(\frac{1}{2} + \frac{\nu_1^2(1+2\alpha) - \mu\sqrt{4\mu^2\alpha(1+\alpha) + \nu_1^2}}{4(\mu^2 - \nu_1^2)\sqrt{\alpha(1+\alpha)}}\right)\right)^2} \frac{1}{\left(t - \left(\frac{1}{2} + \frac{\nu_1^2(1+2\alpha) + \mu\sqrt{4\mu^2\alpha(1+\alpha) + \nu_1^2}}{4(\mu^2 - \nu_1^2)\sqrt{\alpha(1+\alpha)}}\right)\right)^2} \cdot \\ & \cdot \left(16t(1-t)\alpha(1+\alpha)(\mu^2(5+9\alpha+12\alpha^2+4\alpha^3) - \nu_1^2(7+25\alpha+36\alpha^2+12\alpha^3)) + \right. \\ & \quad \left. + \nu_1^2(3+55\alpha+204\alpha^2+340\alpha^3+256\alpha^4+64\alpha^5) + \right. \\ & \quad \left. + 2(-1+2t)\sqrt{\alpha(1+\alpha)} \left(16t(1-t)(\mu^2 - \nu_1^2)\alpha(1+\alpha)(1+5\alpha+2\alpha^2) + \right. \right. \\ & \quad \left. \left. + \nu_1^2(9+55\alpha+118\alpha^2+112\alpha^3+32\alpha^4) \right) \right). \end{aligned} \quad (990)$$

Substituting this expansion (988), as well as the weak-coupling series of the density $\rho^{(1)}(t)$ (950), into (986), and splitting moreover the terms of the density into their no-sum and one-sum pieces — we arrive at

$$\gamma^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)} = \gamma_0^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)} + \eta^2 \gamma_2^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)} + \eta^3 \gamma_3^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)} + \mathcal{O}(\eta^4), \quad (991)$$

where these terms are to be computed through the following integrals,

$$\gamma_0^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)} = \sum_{n_1 \geq 1} \int_0^1 dt \rho_0^{(1)}(t) \Big|_{\text{no-sum}} t(1-t) \mathcal{Q}_{0,n_1}(t) + \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \int_0^1 dt \rho_0^{(1)}(t) \Big|_{\text{one-sum}, n_2} t(1-t) \mathcal{Q}_{0,n_1}(t), \quad (992)$$

and

$$\begin{aligned} \gamma_2^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)} = & \sum_{n_1 \geq 1} \left(\int_0^1 dt \rho_0^{(1)}(t) \Big|_{\text{no-sum}} t(1-t) \mathcal{Q}_{2,n_1}(t) + \int_0^1 dt \rho_2^{(1)}(t) \Big|_{\text{no-sum}} t(1-t) \mathcal{Q}_{0,n_1}(t) \right) + \\ & + \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \left(\int_0^1 dt \rho_0^{(1)}(t) \Big|_{\text{one-sum}, n_2} t(1-t) \mathcal{Q}_{2,n_1}(t) + \int_0^1 dt \rho_2^{(1)}(t) \Big|_{\text{one-sum}, n_2} t(1-t) \mathcal{Q}_{0,n_1}(t) \right), \end{aligned} \quad (993)$$

and

$$\gamma_3^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)} = \sum_{n_1 \geq 1} \int_0^1 dt \rho_3^{(1)}(t) \Big|_{\text{no-sum}} t(1-t) \mathcal{Q}_{0,n_1}(t). \quad (994)$$

All the seven integrals here can be approached with the method of residues, with less or more effort. In particular, the last integral in (993) is especially tiresome, since $\rho_2^{(1)}(t)|_{\text{one-sum}, n_2}$ is a very complicated function (for example, one contribution to it is given by (860)–(889)). Even though the end result is surprisingly simple, it has taken us several dozens of hours to complete this integration, which is one indication of how lengthy the calculations are in this project. Let us print the final findings for these integrals, where the double sums have additionally been symmetrized — we get for the leading order,

$$\begin{aligned} \gamma_0^{(2), \text{from } \mathcal{A}^{(2), \text{bulk}}(z)} &= \sum_{n \geq 1} \left(\mu^2 (1 + 6\alpha + 6\alpha^2) + \nu^2 - \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}} (\mu^2 (1 + 8\alpha + 8\alpha^2) + \nu^2) \right) + \\ &\quad + \mu^2 (1 + 2\alpha)^2 \left(\sum_{n \geq 1} \left(1 - \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha) + \nu^2}} \right) \right)^2, \end{aligned} \quad (995)$$

the next-to-leading order,

$$\begin{aligned} \gamma_2^{(2), \text{from } \mathcal{A}^{(2), \text{bulk}}(z)} &= \frac{1}{\mu^2} \sum_{n \geq 1} \left(2\mu^4 (-2 - 17\alpha - 25\alpha^2 + 16\alpha^3 + 29\alpha^4) - \mu^2 \nu^2 (13 + 28\alpha + 2\alpha^2) - \nu^4 + \right. \\ &\quad + \frac{\nu}{(4\mu^2\alpha(1+\alpha) + \nu^2)^{3/2}} \left(4\mu^6\alpha(1+\alpha)(1+2\alpha) (5 + 57\alpha + 34\alpha^2 - 24\alpha^3) \right. \\ &\quad \left. \left. + 2\mu^4\nu^2 (2 + 56\alpha + 151\alpha^2 + 80\alpha^3 - 20\alpha^4) + \mu^2\nu^4 (13 + 34\alpha + 8\alpha^2) + \nu^6 \right) \right) + \\ &\quad + \frac{1}{\mu^2} \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \left(2\mu^4 (-2 - 6\alpha + 13\alpha^2 + 48\alpha^3 + 34\alpha^4) + 2\mu^2\alpha(4 + 5\alpha) (\nu_1^2 + \nu_2^2) + 3\nu_1^2\nu_2^2 + \right. \\ &\quad + \frac{\nu_1}{(4\mu^2\alpha(1+\alpha) + \nu_1^2)^{3/2}} \left(-4\mu^6\alpha(1+\alpha) (-5 - 19\alpha + 24\alpha^2 + 112\alpha^3 + 80\alpha^4) + \right. \\ &\quad + 2\mu^4\nu_1^2 (2 + 6\alpha - 37\alpha^2 - 102\alpha^3 - 64\alpha^4) - 2\mu^2\nu_1^4\alpha(4 + 5\alpha) - \\ &\quad \left. \left. - \nu_2^2 (4\mu^2\alpha(1+\alpha) + \nu_1^2) (2\mu^2\alpha(7 + 8\alpha) + 3\nu_1^2) \right) \right) + \\ &\quad + \frac{\nu_2}{(4\mu^2\alpha(1+\alpha) + \nu_2^2)^{3/2}} \left(-4\mu^6\alpha(1+\alpha) (-5 - 19\alpha + 24\alpha^2 + 112\alpha^3 + 80\alpha^4) + \right. \\ &\quad + 2\mu^4\nu_2^2 (2 + 6\alpha - 37\alpha^2 - 102\alpha^3 - 64\alpha^4) - 2\mu^2\nu_2^4\alpha(4 + 5\alpha) - \\ &\quad \left. \left. - \nu_1^2 (4\mu^2\alpha(1+\alpha) + \nu_2^2) (2\mu^2\alpha(7 + 8\alpha) + 3\nu_2^2) \right) \right) + \\ &\quad + \frac{\nu_1}{(4\mu^2\alpha(1+\alpha) + \nu_1^2)^{3/2}} \frac{\nu_2}{(4\mu^2\alpha(1+\alpha) + \nu_2^2)^{3/2}} \left(32\mu^8\alpha^2(1+\alpha)^2 (-3 - 13\alpha + 17\alpha^2 + 76\alpha^3 + 52\alpha^4) + \right. \end{aligned}$$

$$\begin{aligned}
& +4\mu^6\alpha(1+\alpha)(-5-19\alpha+108\alpha^2+292\alpha^3+176\alpha^4)(\nu_1^2+\nu_2^2)+ \\
& +8\mu^4\alpha^2(1+\alpha)(7+8\alpha)(\nu_1^4+\nu_2^4)+2\mu^4(-2-6\alpha+115\alpha^2+264\alpha^3+148\alpha^4)\nu_1^2\nu_2^2+ \\
& +2\mu^2\alpha(13+14\alpha)\nu_1^2\nu_2^2(\nu_1^2+\nu_2^2)+3\nu_1^4\nu_2^4 \Big) \Big), \tag{996}
\end{aligned}$$

and the next-to-next-to-leading order,

$$\begin{aligned}
\gamma_3^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)} &= \frac{8\alpha(1+\alpha)}{3\pi\mu} \sum_{n \geq 1} \left(2\mu^4\alpha^2(1+\alpha)^2 + \mu^2\nu^2(1+8\alpha+8\alpha^2) + \nu^4 - \right. \\
& \left. - \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2}} (2\mu^2\alpha(1+\alpha)+\nu^2) (\mu^2(1+8\alpha+8\alpha^2)+\nu^2) \right). \tag{997}
\end{aligned}$$

The order $\mathcal{O}(\eta)$ is obviously absent in the above, while the order $\mathcal{O}(\eta^3)$ is present and originates from the phase, even though this is an anomaly contribution.

7. Summary: The Weak-Coupling Expansion of the Energy $E^{(2)}$

Collecting all the above contributions, as well as the one from the term $\mathcal{F}(z)$, which we have considered earlier (782),

$$\begin{aligned}
\gamma^{(2)} &= \gamma^{(2),\text{from } \mathcal{F}(z)} + \gamma^{(2),\text{from } G^{(1)}(z)} + \\
&+ \gamma^{(2),\text{from } \mathcal{P}^{(2),\text{boundary}}(z)} + \gamma^{(2),\text{from } \mathcal{P}^{(2),\text{bulk}}(z)} + \\
&+ \gamma^{(2),\text{from } \mathcal{A}^{(2),\text{boundary}}(z)} + \gamma^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)}, \tag{998}
\end{aligned}$$

we arrive at the final result of this entire computation — the weak-coupling expansion of the constant $\gamma^{(2)}$ up to the order $\mathcal{O}(\eta^3)$, which is the order where the Hernández–López phase appears for the first time, thus enabling its testing when the corresponding string theory computation is done,

$$\gamma^{(2)} = \gamma_0^{(2)} + \eta^2\gamma_2^{(2)} + \eta^3\gamma_3^{(2)} + \mathcal{O}(\eta^4), \tag{999}$$

where the leading term,

$$\begin{aligned}
\gamma_0^{(2)} &= -\mu^2\alpha(1+\alpha) \left(1 + \frac{1}{12}\mu^2(4+21\alpha+21\alpha^2) \right) + \\
&+ \mu^2(1+2\alpha)^2 \sum_{n \geq 1} \left(1 - \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2}} \right) + \mu^2(1+2\alpha)^2 \left(\sum_{n \geq 1} \left(1 - \frac{\nu}{\sqrt{4\mu^2\alpha(1+\alpha)+\nu^2}} \right) \right)^2, \tag{1000}
\end{aligned}$$

the next-to-leading term,

$$\begin{aligned}
\gamma_2^{(2)} &= \mu^2\alpha(1+\alpha) \left(11+28\alpha+6\alpha^2 - \frac{1}{6}\mu^2(2+57\alpha+308\alpha^2+560\alpha^3+322\alpha^4) \right) + \\
&+ 2 \sum_{n \geq 1} \left(\mu^2(-2-6\alpha+15\alpha^2+52\alpha^3+36\alpha^4) + \nu^2\alpha(5+6\alpha) - \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{\nu}{(4\mu^2\alpha(1+\alpha)+\nu^2)^{3/2}} \left(2\mu^4\alpha(1+\alpha) (-5-19\alpha+32\alpha^2+128\alpha^3+88\alpha^4) + \right. \\
& \quad \left. + \mu^2\nu^2 (-2-6\alpha+45\alpha^2+118\alpha^3+72\alpha^4) + \nu^4\alpha(5+6\alpha) \right) + \\
& + \frac{2}{\mu^2} \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \left(\mu^4 (-2-6\alpha+15\alpha^2+52\alpha^3+36\alpha^4) + \mu^2\alpha(5+6\alpha) (\nu_1^2 + \nu_2^2) + 2\nu_1^2\nu_2^2 - \right. \\
& - \frac{\nu_1}{(4\mu^2\alpha(1+\alpha)+\nu_1^2)^{3/2}} \left(2\mu^6\alpha(1+\alpha) (-5-19\alpha+32\alpha^2+128\alpha^3+88\alpha^4) + \right. \\
& \quad + \mu^4\nu_1^2 (-2-6\alpha+45\alpha^2+118\alpha^3+72\alpha^4) + \mu^2\nu_1^4\alpha(5+6\alpha) + \\
& \quad \left. + \nu_2^2 (4\mu^2\alpha(1+\alpha)+\nu_1^2) (\mu^2\alpha(9+10\alpha)+2\nu_1^2) \right) - \\
& - \frac{\nu_2}{(4\mu^2\alpha(1+\alpha)+\nu_2^2)^{3/2}} \left(2\mu^6\alpha(1+\alpha) (-5-19\alpha+32\alpha^2+128\alpha^3+88\alpha^4) + \right. \\
& \quad + \mu^4\nu_2^2 (-2-6\alpha+45\alpha^2+118\alpha^3+72\alpha^4) + \mu^2\nu_2^4\alpha(5+6\alpha) + \\
& \quad \left. + \nu_1^2 (4\mu^2\alpha(1+\alpha)+\nu_2^2) (\mu^2\alpha(9+10\alpha)+2\nu_2^2) \right) - \\
& + \frac{\nu_1}{(4\mu^2\alpha(1+\alpha)+\nu_1^2)^{3/2}} \frac{\nu_2}{(4\mu^2\alpha(1+\alpha)+\nu_2^2)^{3/2}} \left(16\mu^8\alpha^2(1+\alpha)^2 (-3-13\alpha+25\alpha^2+92\alpha^3+60\alpha^4) + \right. \\
& \quad + 2\mu^6\alpha(1+\alpha) (-5-19\alpha+140\alpha^2+356\alpha^3+208\alpha^4) (\nu_1^2 + \nu_2^2) + \\
& \quad + 4\mu^4\alpha^2(1+\alpha)(9+10\alpha) (\nu_1^4 + \nu_2^4) + \mu^4 (-2-6\alpha+147\alpha^2+328\alpha^3+180\alpha^4) \nu_1^2\nu_2^2 + \\
& \quad \left. + \mu^2\alpha(17+18\alpha)\nu_1^2\nu_2^2 (\nu_1^2 + \nu_2^2) + 2\nu_1^4\nu_2^4 \right), \tag{1001}
\end{aligned}$$

and the next-to-next-to-leading term,

$$\begin{aligned}
\gamma_3^{(2)} &= \frac{8\mu^3\alpha^2(1+\alpha)^2 (1+11\alpha+11\alpha^2)}{3\pi} + \\
& + \frac{8\alpha(1+\alpha)}{3\pi\mu} \sum_{n \geq 1} \left(2\mu^4\alpha(1+\alpha) (1+11\alpha+11\alpha^2) + \mu^2\nu^2 (2+21\alpha+21\alpha^2) + 2\nu^4 - \right. \\
& - \frac{\nu}{(4\mu^2\alpha(1+\alpha)+\nu^2)^{3/2}} \left(6\mu^4\alpha(1+\alpha) (1+10\alpha+10\alpha^2) + \mu^2\nu^2 (2+25\alpha+25\alpha^2) + 2\nu^4 \right) \left. \right). \tag{1002}
\end{aligned}$$

Observe that there is no order $O(\eta)$ here. The leading term precisely agrees with our one-loop expression (674). As mentioned, it is this last formula (1002) that provides quite a sophisticated means for testing the validity of the Hernández-López phase.

If we rather prefer to work with the second charge $Q_2|_{\text{next-to-next-to-leading}}$ (338) or the energy $E^{(2)}$ (352), we should take minus the constant $\gamma^{(2)}$, supplemented by the following combination of values of the resolvent $G^{(0)}(z)$,

$$\begin{aligned} & \frac{1}{192\omega} \left(\omega \left(G^{(0)'''}(\omega) + G^{(0)'''(-\omega)} \right) + 3 \left(G^{(0)''}(\omega) - G^{(0)''(-\omega)} \right) \right) = \\ & = -\frac{1}{4}\mu^4\alpha(1+\alpha)(1+5\alpha+5\alpha^2) - \eta^2\frac{1}{2}\mu^4\alpha(1+\alpha)(1+20\alpha+96\alpha^2+164\alpha^3+91\alpha^4) + O(\eta^4). \end{aligned} \quad (1003)$$

This concludes our computation in the weak-coupling limit.

B. The Large-Mode-Number, Fixed-Winding-Number Limit

In this subsection, the exact formula for the constant $\gamma^{(2)}$ (959) is exploited to find the three leading orders, *i.e.*, $O(\eta^4)$, $O(\eta^3)$ and $O(\eta^2)$, of its large-mode-number, fixed-winding-number series. The next-to-leading term happens to be zero, due to a non-trivial cancelation between the contributions from $\mathcal{F}(z)$ and $\mathcal{A}^{(2),\text{boundary}}(z)$. The same is true for $O(\eta^4)$ when we move to the second charge $Q_2|_{\text{next-to-next-to-leading}}$. The next-to-next-to-leading order, which in this way becomes the leading term of the second charge, is the one at which the Hernández-López phase enters for the first time; its knowledge, thus, will deliver a very non-trivial tool for testing of the phase in this non-perturbative regime. Unfortunately, it includes some integrals which we give but have not been able to solve, and therefore, we resort to an additional approximation of $\kappa \approx 0$.

1. The Contribution from the Resolvent $G^{(1)}(z)$

This input is given exactly by the formulae (960)–(962). It is enough to discover that all the phase pieces, *i.e.*, $\mathcal{P}^{(1)}(c)$, $\mathcal{P}^{(1)'}(c)$, $\mathcal{P}^{(1)''}(c)$, and all the anomaly pieces with the hyperbolic cotangent replaced by 1, *i.e.*, $\mathcal{A}^{(1),\text{coth}=1}(c)$, $\mathcal{A}^{(1),\text{coth}=1'}(c)$, $\mathcal{A}^{(1),\text{coth}=1''}(c)$ — have their large- η series beginning at the order $O(\eta)$; while the Dirac delta contributions to these anomalies, *i.e.*, $\mathcal{A}^{(1),\delta}(c)$, $\mathcal{A}^{(1),\delta'}(c)$, $\mathcal{A}^{(1),\delta''}(c)$ — start at $O(1/\eta)$. Using moreover the expansion of the constant $\gamma^{(1)}$ (933), we finally find that the resolvent $G^{(1)}(z)$ appears only far into the series,

$$\gamma^{(2),\text{from } G^{(1)}(z)} = O\left(\frac{1}{\eta}\right). \quad (1004)$$

This will be irrelevant for our discussion.

2. The Contribution from the Boundary Phase $\mathcal{P}^{(2),\text{boundary}}(z)$

Since $\mathcal{P}^{(2),\text{boundary}}(z)$ is known exactly (970), its contribution to the constant $\gamma^{(2)}$ (967) can be straightforwardly calculated,

$$\gamma^{(2),\text{from } \mathcal{P}^{(2),\text{boundary}}(z)} = \eta \gamma_{-1}^{(2),\text{from } \mathcal{P}^{(2),\text{boundary}}(z)} + O(\eta^0), \quad (1005)$$

where

$$\gamma_{-1}^{(2),\text{from } \mathcal{P}^{(2),\text{boundary}}(z)} = \frac{1}{4\pi\omega^3} \frac{1+3\kappa^2}{(1-\kappa^2)^3} \left((1-\kappa) \log(1-\kappa) + (1+\kappa) \log(1+\kappa) - \frac{\kappa}{\sqrt{1+\kappa^2}} \log\left(\kappa + \sqrt{1+\kappa^2}\right) \right). \quad (1006)$$

We have shown this expression to give the reader a feeling of the kind of functions we are dealing with here, but this order $O(\eta)$ will, too, be irrelevant for us.

3. The Contribution from the Bulk Phase $\mathcal{P}^{(2),\text{bulk}}(z)$

To compute this input (973), we need to exploit the non-symmetrized double integral (444), where we set $z = c$, change the integration variables y_1, y_2 to t_1, t_2 (539), and use the leading orders of the large- η expansions of the densities (803), (954),

$$\rho^{(0)}(t) = \eta \frac{1}{\omega} \rho_{-1}^{(0)}(t) + O(\eta^0), \quad (1007)$$

$$\rho^{(1)}(t) = \eta^3 \rho_{-3}^{(1)}(t) + O(\eta^2) + O(\exp). \quad (1008)$$

(As argued, only the leading order of the latter is fully meaningful in the “ $\coth \approx 1 + \delta$ ” approximation.) Let us print the product of these two leading terms, to show the reader the type of integrals we will be facing,

$$\begin{aligned} \rho_{-1}^{(0)}(t_1) \rho_{-3}^{(1)}(t_2) &= \frac{\kappa \sqrt{t_1(1-t_1)}}{2\pi^3 ((1-\kappa)^2 + 4\kappa t_1) ((1-\kappa)^2 + 4\kappa t_2)^3} \\ &\cdot \left(\frac{2(1-\kappa) ((1-\kappa)^2 + 4\kappa t_2)}{\sqrt{t_2(1-t_2)}} - \frac{(1-\kappa)^2 - 2(1+\kappa^2)t_2}{\sqrt{t_2(1-t_2)}} \log \left(\frac{(1-t_2) ((1-\kappa)^2 + 4\kappa t_2)}{t_2(1+\kappa)^4 (1+\kappa^2)} \right) + \right. \\ &\quad \left. + 8(1-\kappa^2) \arctan \left(\frac{2\kappa \sqrt{t_2(1-t_2)}}{1-\kappa + 2\kappa t_2} \right) - \right. \\ &\quad \left. - \frac{2((1-\kappa)^2(1+\kappa+\kappa^2) + 2\kappa(1+\kappa^2)t_2)}{\sqrt{(1+\kappa^2 + 2\kappa t_2)((1-\kappa)^2 + 2\kappa t_2)}} \arctan \left(\frac{4\kappa \sqrt{t_2(1-t_2)(1+\kappa^2 + 2\kappa t_2)((1-\kappa)^2 + 2\kappa t_2)}}{(1-\kappa)^2(1+\kappa^2) + 4\kappa(1-\kappa)^2 t_2 + 8\kappa^2 t_2^2} \right) \right). \end{aligned} \quad (1009)$$

All this leads to

$$\gamma_{-2}^{(2), \text{from } \mathcal{P}^{(2), \text{bulk}}(z)} = -\frac{c(b-a)^2}{\pi\omega^2\mu} \int_0^1 \int_0^1 dt_1 dt_2 \rho_{-1}^{(0)}(t_1) \rho_{-3}^{(1)}(t_2) \frac{1}{(c-y_1)(c-y_2)} \mathcal{H}(y_1, y_2) \Big|_{\substack{y_1 = a + (b-a)t_1 \\ y_2 = a + (b-a)t_2}}, \quad (1010)$$

where we recall that $\mathcal{H}(y_1, y_2)$ is defined in (439). Expanded at large η , this acquires the form

$$\gamma_{-2}^{(2), \text{from } \mathcal{P}^{(2), \text{bulk}}(z)} = \eta^2 \gamma_{-2}^{(2), \text{from } \mathcal{P}^{(2), \text{bulk}}(z)} + \mathcal{O}(\eta), \quad (1011)$$

where

$$\begin{aligned} \gamma_{-2}^{(2), \text{from } \mathcal{P}^{(2), \text{bulk}}(z)} &= \frac{\kappa^2}{2\pi\omega^4} \int_0^1 \int_0^1 dt_1 dt_2 \rho_{-1}^{(0)}(t_1) \rho_{-3}^{(1)}(t_2) \frac{1}{(1-\kappa)^2 + 2\kappa(t_1+t_2)} \\ &\cdot \left(4 + \frac{(1-\kappa)^4 + 4\kappa(1-\kappa)^2(t_1+t_2) + 8\kappa^2(t_1^2+t_2^2)}{\kappa(t_1-t_2)((1-\kappa)^2 + 2\kappa(t_1+t_2))} \log \left(\frac{(1-\kappa)^2 + 4\kappa t_2}{(1-\kappa)^2 + 4\kappa t_1} \right) \right). \end{aligned} \quad (1012)$$

The order at which this contributions begins is $\mathcal{O}(\eta^2)$, which is next-to-next-to-leading in $\gamma^{(2)}$, and which is the maximum accuracy we are aiming at in this paper. The double integral (1012), unfortunately, seems to be out of reach.

Being unable to solve the above integral, let us at least give its some approximate value. Consider now, therefore, a new limit,

$$\kappa \rightarrow 0, \quad (1013)$$

and expand in it the integrand in (1012). The integral obtained in this way is doable,

$$\begin{aligned} \gamma_{-2}^{(2), \text{from } \mathcal{P}^{(2), \text{bulk}}(z)} &= \kappa^5 \frac{-16}{3\pi^4\omega^4} \int_0^1 \int_0^1 dt_1 dt_2 \frac{\sqrt{t_1(1-t_1)}}{\sqrt{t_2(1-t_2)}} (t_1-t_2)^2 \left(2 + (-1+2t_2) \log \left(\frac{1-t_2}{t_2} \right) \right) + \mathcal{O}(\kappa^6) = \\ &= \kappa^5 \frac{1}{9\pi^2\omega^4} + \mathcal{O}(\kappa^6). \end{aligned} \quad (1014)$$

It starts only at the order $\mathcal{O}(\kappa^5)$. We will observe the same phenomenon in the case of the bulk anomaly (1019), namely, the Hernández-López phase will be visible there only from $\mathcal{O}(\kappa^5)$, as an addition to two more leading terms.

4. The Contribution from the Boundary Anomaly $\mathcal{A}^{(2),\text{boundary}}(z)$

The input to $\gamma^{(2)}$ from the boundary anomaly $\mathcal{A}^{(2),\text{boundary}}(z)$ is known exactly (981), and so, it is easy to expand it at large η ,

$$\gamma^{(2),\text{from } \mathcal{A}^{(2),\text{boundary}}(z)} = -\eta^3 \frac{\kappa^2 (3 + \kappa^2) (1 + 3\kappa^2)}{6\omega^4 (1 - \kappa^2)^6} + \eta^2 \frac{\kappa^2 (1 + \kappa^2) (5 + 6\kappa^2 + 5\kappa^4)}{24\omega^4 (1 - \kappa^2)^6} + \mathcal{O}(\eta). \quad (1015)$$

The problem here is that we badly need also the next-to-leading term, but we admit that we are not sure whether it is correct, due to the problems with the approximation “ $\coth \approx 1 + \delta$ ” described at the end of paragraph III B 5. It should not affect our testing of the Hernández–López phase in the small- κ limit (1013), though, as the phase appears only at the order $\mathcal{O}(\kappa^5)$, and we do not expect any alternative $\mathcal{A}^{(2),\text{boundary}}(z)$ to modify it. An argument in favor of the validity of the $\mathcal{O}(\eta^2)$ -term of (1015) is that we indeed use the exact expression for the boundary resolvent $G^{(1),\text{boundary}}(z)$ (943) to derive it, but we do not use the subleading terms of the density $\rho^{(1)}(t)$, which do have a wrong endpoint behavior.

5. The Contribution from the Bulk Anomaly $\mathcal{A}^{(2),\text{bulk}}(z)$

In the computation of the contribution of $\mathcal{A}^{(2),\text{bulk}}(z)$ (986), we repeat the steps from the one-loop counterpart in paragraph III C 5. Namely, we integrate by parts and use the “ $\coth \approx 1 + 1/\pi\rho$ ” approximation (632), which is enough to find the leading large- η order,

$$\begin{aligned} \gamma^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)} &= -\frac{c}{\omega^2 \mu} \mathcal{A}^{(2),\text{bulk}}(c) = \frac{\pi c}{\omega^2 \mu} \int_a^b dy \frac{1}{c-y} \frac{d}{dy} \left(\rho^{(0)}(y) \rho^{(1)}(y) \left(\coth \left(\pi \rho^{(0)}(y) \right) - \frac{1}{\pi \rho^{(0)}(y)} \right) \right) = \\ &= -\frac{\pi c}{\omega^2 \mu} \int_a^b dy \frac{1}{(c-y)^2} \rho^{(0)}(y) \rho^{(1)}(y) + \mathcal{O}(\text{subleading}) = \\ &= -\frac{\pi c(b-a)}{\omega^2 \mu} \int_0^1 dt \frac{1}{(c-a-(b-a)t)^2} \rho^{(0)}(t) \rho^{(1)}(t) + \mathcal{O}(\text{subleading}). \end{aligned} \quad (1016)$$

Using the large- η expansions of the densities (803), (954), we finally arrive at

$$\gamma^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)} = \eta^2 \gamma_{-2}^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)} + \mathcal{O}(\eta), \quad (1017)$$

where

$$\gamma_{-2}^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)} = \frac{\pi \kappa}{2\omega^4} \int_0^1 dt \rho_{-1}^{(0)}(t) \rho_{-3}^{(1)}(t). \quad (1018)$$

This integral, where its integrand is explicitly printed in (1009) for $t_1 = t_2 = t$, does not appear doable, and we will abandon it for now.

Because of this obstacle, let us just investigate how this term (1018) behaves at small κ (1013). Expanding the integrand, we obtain integrals involving rational functions of t and the logarithm $\log((1-t)/t)$, everything doable. In total,

$$\gamma_{-2}^{(2),\text{from } \mathcal{A}^{(2),\text{bulk}}(z)} = \kappa^2 \frac{1}{4\pi^2 \omega^4} + \kappa^4 \frac{5}{4\pi^2 \omega^4} - \kappa^5 \frac{1}{18\pi^2 \omega^4} + \mathcal{O}(\kappa^6). \quad (1019)$$

To give more insight into this series, let us underline that the even orders $\mathcal{O}(\kappa^2)$ and $\mathcal{O}(\kappa^4)$ originate exclusively from the anomaly piece in the density $\rho_{-3}^{(1)}(t)$ (955), (912), while the Hernández–López phase’s part in this density (845), and only it, is responsible for the order $\mathcal{O}(\kappa^5)$. The same has been observed in the contribution from $\mathcal{P}^{(2),\text{bulk}}(z)$ (1014). Hence, if we resort to the use of the small- κ approximation, we must probe it as far as this fifth power, which exists exclusively thanks to the Hernández–López phase, and therefore, delivers a non-trivial way of testing it.

6. *Summary: The Large-Mode-Number, Fixed-Winding-Number Expansion of the Energy $E^{(2)}$*

Collecting the contribution from the term $\mathcal{F}(z)$, which we have found earlier (821), plus all the pieces investigated in this subsection, we obtain the second most important result of this article, which is the large-mode-number, fixed-winding-number series of the constant $\gamma^{(2)}$,

$$\gamma^{(2)} = \eta^4 \gamma_{-4}^{(2)} + \eta^2 \gamma_{-2}^{(2)} + \mathcal{O}(\eta) + \mathcal{O}(\exp), \quad (1020)$$

where the leading term reads,

$$\omega^4 \gamma_{-4}^{(2)} = -\frac{\kappa^2 (1 + 3\kappa^2 + \kappa^4)}{4(1 - \kappa^2)^7}, \quad (1021)$$

while the next-to-leading one is,

$$\omega^4 \gamma_{-2}^{(2)} = -\frac{\kappa^2 (5 + 16\kappa^2 + 12\kappa^4 + 28\kappa^6 - \kappa^8)}{96(1 - \kappa^2)^7} + \mathcal{I}(\kappa), \quad (1022)$$

where

$$\begin{aligned} \mathcal{I}(\kappa) \equiv & \frac{\pi\kappa}{2} \int_0^1 dt \rho_{-1}^{(0)}(t) \rho_{-3}^{(1)}(t) + \frac{\kappa^2}{2\pi} \int_0^1 \int_0^1 dt_1 dt_2 \rho_{-1}^{(0)}(t_1) \rho_{-3}^{(1)}(t_2) \frac{1}{(1 - \kappa)^2 + 2\kappa(t_1 + t_2)} \\ & \cdot \left(4 + \frac{(1 - \kappa)^4 + 4\kappa(1 - \kappa)^2(t_1 + t_2) + 8\kappa^2(t_1^2 + t_2^2)}{\kappa(t_1 - t_2)((1 - \kappa)^2 + 2\kappa(t_1 + t_2))} \log \left(\frac{(1 - \kappa)^2 + 4\kappa t_2}{(1 - \kappa)^2 + 4\kappa t_1} \right) \right), \end{aligned} \quad (1023)$$

where the product of the leading-order densities is given in (1009).

Comments: • It all explodes at large η as η^4 . This is a shortcoming of the approximation we have chosen, that it fails to deliver a finite approximate $\gamma^{(2)}$, even though both $\gamma^{(0)}$ (802) and $\gamma^{(1)}$ (933) are finite in this limit. Recall that at one loop, all the three levels of the second charge (546), (631), (705) explode in the corresponding small- α limit; each one stronger than the previous one. There is no contradiction between these two statements since the large-mode-number, fixed-winding-number limit and $\omega \rightarrow 0$ do not commute. • The leading order comes only from $\mathcal{F}(z)$ (821), which is quite a complicated expression composed out of certain values of the resolvent $G^{(0)}(z)$ and its derivatives (461); neither the phase, nor the anomaly affect it. • There is no order $\mathcal{O}(\eta^3)$, which surprisingly canceled between $\mathcal{F}(z)$ and $\mathcal{A}^{(2),\text{boundary}}(z)$. This may be taken as an opportunity to test our general technique of dealing with boundary contributions, once the corresponding string theory computation has been completed. • The order $\mathcal{O}(\eta^2)$ is critical for us, as the Hernández-López phase makes its first appearing precisely there. Sadly, in the current limit, this term could not be derived in a closed form, but only given through some integrals. It constitutes a very non-trivial probe into the strong-coupling regime. • If one would like to dig deeper into the series (1020), there is just one obstacle on the way, besides increased technical complication, of course, which already has made the calculations extremely tedious — and it is to find how to correctly handle the hyperbolic cotangent $\coth(\pi\rho^{(0)}(y))$ at large η , *i.e.*, to overcome problems indicated in paragraph III B 5.

In order to translate (1020) into the language of the second charge $Q_2|_{\text{next-to-next-to-leading}}$ (338) or the energy $E^{(2)}$ (352), we should negate it and add to it the following quantity,

$$\begin{aligned} & \frac{1}{192\omega} \left(\omega \left(G^{(0)'''}(\omega) + G^{(0)'''}(-\omega) \right) + 3 \left(G^{(0)''}(\omega) - G^{(0)''}(-\omega) \right) \right) = \\ & = -\eta^4 \frac{\kappa^2 (1 + 3\kappa^2 + \kappa^4)}{4\omega^4 (1 - \kappa^2)^7} - \eta^2 \frac{\kappa^2 (1 + 6\kappa^2 + 6\kappa^4 + 6\kappa^6 + \kappa^8)}{32\omega^4 (1 - \kappa^2)^7} + \mathcal{O}(\eta^0). \end{aligned} \quad (1024)$$

Notice another surprise here — the leading term of (1024) precisely cancels the leading one of minus $\gamma^{(2)}$ (1021), and therefore, it is $\mathcal{O}(\eta^2)$ that becomes the leading order of the charge

$$Q_2|_{\text{next-to-next-to-leading}} = \eta^2 \frac{1}{\omega^4} \left(\frac{\kappa^2 (1 + 2\kappa^2)}{48(1 - \kappa^2)^4} - \mathcal{I}(\kappa) \right) + \mathcal{O}(\eta). \quad (1025)$$

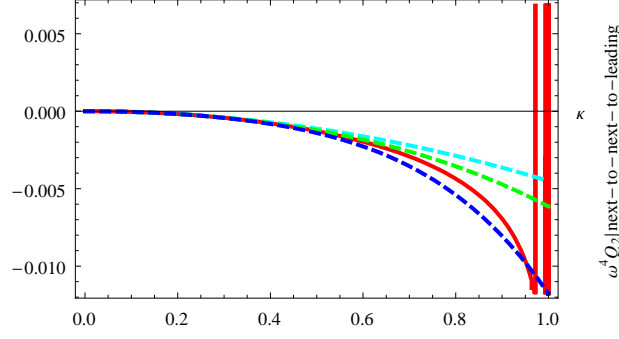


FIG. 48: A graph of $[0, 1] \ni \kappa \mapsto \omega^4(Q_2|_{\text{next-to-next-to-leading}})_{-2}$. In solid red, we plot the exact value (1025), and we compare it to its small- κ series (1026), truncated at one (dashed cyan), two (dashed green), or three (dashed blue) leading orders. The approximation appears adequate even for quite large values of κ . In particular, the $O(\kappa^5)$ term seems to work very well. The numerical noise in the solid red line for κ close to 1 stems from inaccuracies of the numerical integration.

As mentioned, all the contributions, including the phase, are responsible for its value.

We can give the reader some feeling about the integral $\mathcal{I}(\kappa)$ (1023) by computing it in the small- κ limit (1013); we obtain for the $O(\eta^2)$ term of the second charge (1025),

$$\omega^4 \left(Q_2|_{\text{next-to-next-to-leading}} \right)_{-2} = \kappa^2 \frac{1}{48} \left(1 - \frac{12}{\pi^2} \right) + \kappa^4 \frac{1}{8} \left(1 - \frac{10}{\pi^2} \right) \underbrace{- \kappa^5 \frac{1}{18\pi^2}}_{\substack{\text{the implication of} \\ \text{the presence of} \\ \text{the HL phase}}} + O(\kappa^6). \quad (1026)$$

Figure 48 shows that this is quite an accurate description of the exact object. We emphasize once more time that if there was no Hernández–López phase, these expansions would contain only even powers of κ . Again, thus, we see that the phase appears at the next-to-next-to-leading order in (1026), and at an odd power of the expansion parameter, exactly as it has been for other series, the weak-coupling and the large-mode-number, fixed-winding-number ones, which is a remarkable property. If string theory reproduces the expression (1025), or at least this number $-1/(18\pi^2)$ — it will constitute yet another very strong support for the form of the Hernández–López phase (262). ■

PROJECT II

VII. QUASI-LOCALITY OF JOINING/SPLITTING STRINGS FROM COHERENT STATES

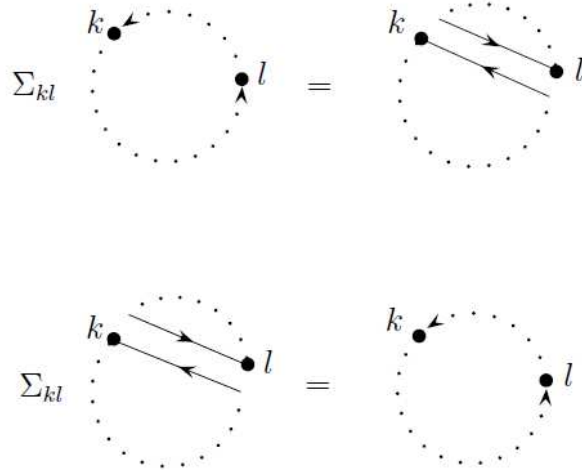
The previous five sections have been devoted to a solution of a single problem. Lacking time, we have not been able to describe the other project we had dealt with in a similar detail. In the current section, we just include our article on the subject [2].

Briefly, using the coherent state formalism, we calculate matrix elements of the one-loop non-planar dilatation operator of $\mathcal{N} = 4$ SYM theory between operators dual to folded Frolov–Tseytlin strings, and observe a curious scaling behavior. We comment on the qualitative similarity of our matrix elements to the interaction vertex of a string field theory. In addition, we present a solvable toy model for string splitting and joining. The scaling behavior of the matrix elements suggests that the contribution to the genus-one energy shift coming from semi-classical string splitting and joining is small.

A. Introduction

Integrability has played a key role in recent years exploration of planar $\mathcal{N} = 4$ SYM [79, 138, 140] as well as non-interacting type IIB string theory on $\text{AdS}_5 \times S^5$ [101, 107], tied together by the AdS/CFT correspondence [36]. Whereas integrability is expected to break down beyond the planar/non-interacting limit — most clearly demonstrated by the lift of degeneracies of anomalous dimensions in the gauge theory [79] — the AdS/CFT correspondence could still be valid [27, 36]. Lacking the framework of integrability, tests of the AdS/CFT correspondence beyond the planar limit have proved difficult. Even in the Berenstein–Maldacena–Nastase limit [56], where the free string theory can actually be quantized, no conclusive tests exist. For an up to date review, see [76]. The gauge theory calculations, although described efficiently by a quantum-mechanical Hamiltonian [85], are plagued by huge degeneracy problems [91]. The string theory computations on their side suffer from the existence of several competing proposals for the three-string vertex of light-cone string field theory, and from the necessity of truncating the vertex to a subset of decay channels. Although the BMN limit seems to be the most tractable one as far as the analysis of the non-planar sector of the theories is concerned, it might be instructive to perform the analysis in other limits as well. A limit which has been instrumental in the investigation of the planar/non-interacting case is the Frolov–Tseytlin limit [100]. A first step in the direction of extending the investigation of this limit to the non-planar/interacting situation has been taken in [196] where the decay of a folded Frolov–Tseytlin string [103] has been described using semi-classical methods. It has been argued that the integrability observed for the free string may survive in certain decay channels. In the present paper, we attack the non-planar Frolov–Tseytlin limit from the gauge theory side. Using the coherent state approach, we calculate matrix elements of the one-loop non-planar dilatation generator of $\mathcal{N} = 4$ SYM between operators dual to the folded Frolov–Tseytlin strings rotating in $S^3 \subset S^5 \subset \text{AdS}_5 \times S^5$.

We begin in subsection VII B by presenting the form of the one-loop non-planar dilatation operator in the $\text{SU}(2)$ sector of $\mathcal{N} = 4$ SYM theory. Subsequently, in subsection VII C, we review the coherent-state description of the operator dual to the folded Frolov–Tseytlin string. Subsection VII D deals with the calculation of matrix elements for the gauge-theory equivalent of string joining and string splitting. In subsection VII E, we describe a solvable toy model for the decay of the folded string which unfortunately is only a very crude approximation to the actual model. Finally, subsection VII F contains a discussion.

FIG. 49: Splitting and joining of chains by $\Sigma_{k,l}$.

B. The One-Loop Non-Planar Dilatation Operator

We consider the $SU(2)$ sector of $\mathcal{N} = 4$ SYM theory, consisting of multi-trace operators built from the two complex scalar fields Φ_1 and Φ_2 . In this subsector, the complete one-loop dilatation operator can be expressed as [79, 84]

$$H = -\frac{g_{\text{YM}}^2}{8\pi^2} \text{Tr} [\Phi_1, \Phi_2] [\check{\Phi}_1, \check{\Phi}_2], \quad \text{where} \quad \check{\Phi}_{1,2} \equiv \frac{\delta}{\delta \Phi_{1,2}}, \quad (1027)$$

or equivalently [196, 197],

$$H = H_{\text{pl}} + H_{\text{npl}}, \quad (1028)$$

where the planar part,

$$H_{\text{pl}} \equiv \tilde{\lambda} \sum_k (1 - P_{k,k+1}), \quad (1029)$$

and the non-planar one,

$$H_{\text{npl}} \equiv \frac{\tilde{\lambda}}{N} \sum_{k,l \neq k+1} (1 - P_{k,l}) \Sigma_{k+1,l}, \quad (1030)$$

In this section, we work with the coupling $\tilde{\lambda} \equiv 2g^2$. The indices refer to the position of the fields inside the operator on which H acts. The indices are periodically identified as dictated by the trace structure of the operator. The operator $P_{k,l}$ simply interchanges the indices k and l . Furthermore, if one represents an operator as a set of fields plus a permutation element giving the ordering of the fields, then $\Sigma_{k,l}$ is just the transposition $\sigma_{k,l}$ applied to this permutation [197] (see figure 49).

C. Folded String Duals via Coherent States

1. The Frolov–Tseytlin Folded String

We aim at working with operators dual to the folded Frolov–Tseytlin string spinning in $S^3 \subset S^5 \subset \text{AdS}_5 \times S^5$, with two large angular momenta J_1, J_2 ; $J \equiv J_1 + J_2$. More precisely, we consider the limit $J_1, J_2 \rightarrow \infty$ with the ratio J_1/J_2 finite. A semi-classical analysis of the string in question yields that its energy has the following expansion [103],

$$E = J \left(1 + \frac{\tilde{\lambda}}{J^2} \mathcal{E}_0 + \frac{\tilde{\lambda}^2}{J^4} \mathcal{E}_0^{(1)} + \dots \right), \quad (1031)$$

with the gauge coupling constant $\tilde{\lambda}$ appearing via the AdS/CFT dictionary $R^2/\alpha' = \sqrt{\tilde{\lambda}}$ [36], and where we also assume that $\tilde{\lambda}/J^2$ is finite. The term of linear order in $\tilde{\lambda}$ is found to be

$$\mathcal{E}_0 = 16K(m) (E(m) - (1-m)K(m)), \quad (1032)$$

where $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and the second kind, respectively. (We everywhere use the `Mathematica`'s definitions of the elliptic functions and integrals.) The parameter m is determined by

$$\frac{J_2}{J} = 1 - \frac{E(m)}{K(m)}. \quad (1033)$$

The gauge theory dual to the folded Frolov–Tseytlin string is a complicated linear combination of single-trace operators, each one containing J_1 Φ_1 's and J_2 Φ_2 's [103, 141]. It is characterized by being an eigenstate of the one-loop planar dilatation operator, H_{pl} (1029), with the eigenvalue given by $(\tilde{\lambda}/J)\mathcal{E}_0$. A more efficient way of describing the dual is by the means of the $\text{SU}(2)$ spin- $(1/2)$ coherent states. To introduce them, let us denote the two normalized eigenstates of S_z by $|\uparrow\rangle$ and $|\downarrow\rangle$. These states have the inner product

$$\langle \uparrow | \uparrow \rangle = \langle \downarrow | \downarrow \rangle = 1, \quad \langle \uparrow | \downarrow \rangle = \langle \downarrow | \uparrow \rangle = 0. \quad (1034)$$

The relevant coherent states then take the form

$$|\vec{n}\rangle \equiv \cos(\theta) |\uparrow\rangle + e^{-i\varphi} \sin(\theta) |\downarrow\rangle, \quad (1035)$$

where the angles $\theta \in [0, \pi/2]$ and $\varphi \in [0, 2\pi]$ parametrize a unit three-vector \vec{n} through

$$\vec{n} = (\cos(2\theta) \sin(\varphi), \sin(2\theta) \sin(\varphi), \cos(\varphi)). \quad (1036)$$

The folded string dual can now be described as a state of a $\text{SU}(2)$ spin chain of length J having a coherent state vector at each site [111]. Without loss of generality, we will take J to be a multiple of 4, in order for the spin chain to reflect as closely as possible the symmetries of the folded string (the entire string profile follows from its definition on a quarter period). The state representing the string reads thus

$$|\mathbf{n}\rangle = |\vec{n}_{-J/2}\rangle \otimes |\vec{n}_{-J/2+1}\rangle \otimes \dots \otimes |\vec{n}_{J/2}\rangle, \quad (1037)$$

where obviously

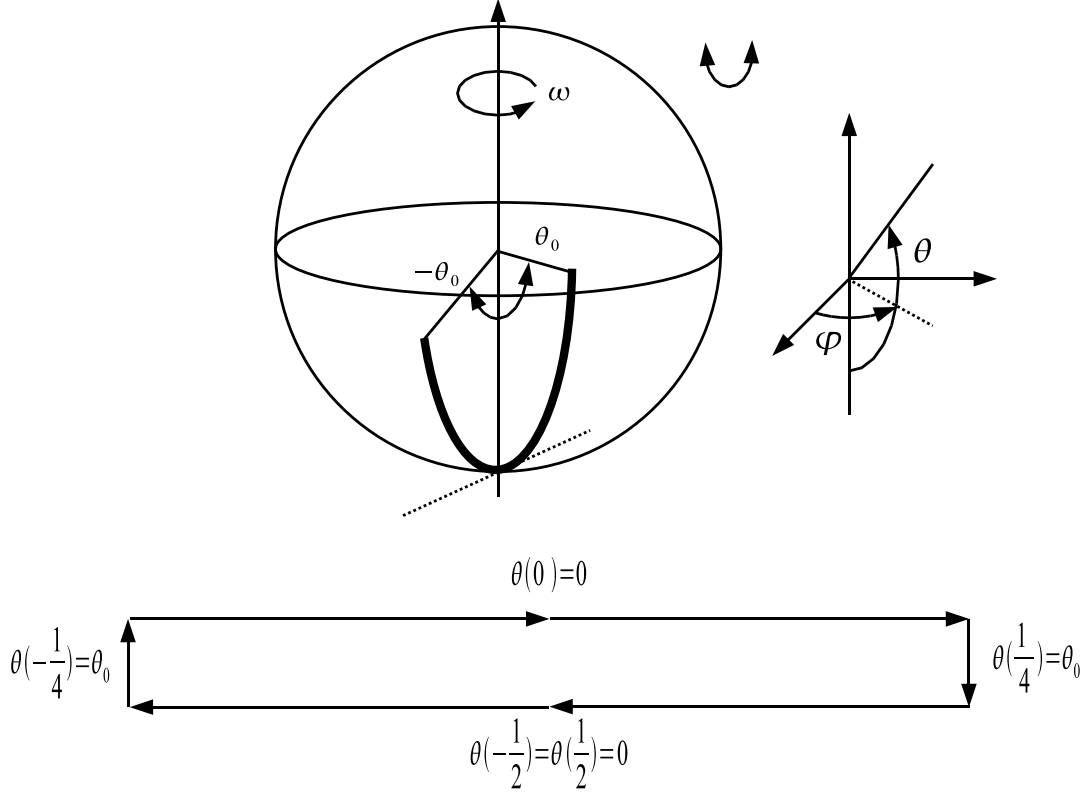
$$|\vec{n}_k\rangle = \cos(\theta_k) |\uparrow\rangle + e^{-i\varphi_k} \sin(\theta_k) |\downarrow\rangle. \quad (1038)$$

In this language, the planar energy of the string is obtained as the inner product

$$\frac{\tilde{\lambda}}{J} \mathcal{E}_0 = \langle \mathbf{n} | H_{\text{pl}} | \mathbf{n} \rangle. \quad (1039)$$

In the long-wavelength limit, where θ_k and φ_k vary only slowly, and where $J \rightarrow \infty$, which exactly corresponds to the Frolov–Tseytlin limit, one can replace these angles by continuous functions

$$\theta_k \rightarrow \theta \left(\sigma = \frac{k}{J} \right), \quad \varphi_k \rightarrow \varphi \left(\sigma = \frac{k}{J} \right), \quad (1040)$$

FIG. 50: Different values of $\theta(\sigma)$ along the string.

and one can derive an effective sigma-model action describing the model. The cyclicity property of the gauge-theory operator translates into the requirement of the vanishing of the momenta in the σ -direction, which reads

$$\mathcal{P}_\sigma = -\frac{1}{2} \int_{-1/2}^{1/2} d\sigma \cos(2\theta(\sigma)) \partial_\sigma \varphi(\sigma) = 0. \quad (1041)$$

The equations of motion following from the action mentioned above permit a solution which exactly describes the folded Frolov–Tseytlin string dual. For this solution, one has

$$\theta'(\sigma)^2 - \frac{\omega}{2\lambda} (\cos(2\theta(\sigma)) - \cos(2\theta_0)) = 0, \quad \varphi = \omega t, \quad (1042)$$

which in particular is seen to fulfill the relation (1041). The angle θ can be expressed in terms of Jacobi sine,

$$\sin(\theta(\sigma)) = \sin(\theta_0) \operatorname{sn} \left(J \sqrt{\frac{\omega}{\lambda}} \sigma \middle| \sin^2(\theta_0) \right), \quad (1043)$$

where the following relation between θ_0 and ω must hold for the string to be closed and folded exactly once

$$J \sqrt{\frac{\omega}{\lambda}} = 4K(m), \quad m = \sin^2(\theta_0). \quad (1044)$$

The angular variable $\theta(\sigma)$ obviously varies in the interval $[-\theta_0, \theta_0]$. For any given θ_0 , one has (or can impose) the following identifications, see figure 50,

$$\theta(z+n) = \theta(z), \quad \theta(1/2-z) = \theta(z), \quad \text{for all } n \in \mathbb{N}, z \in \mathbb{R}. \quad (1045)$$

In this formulation, the one-loop anomalous dimension of the gauge-theory operator is given by [111]

$$\mathcal{E}_0 = \int_{-1/2}^{1/2} d\sigma \theta'(\sigma)^2, \quad (1046)$$

and

$$\frac{J_2}{J} = \int_{-1/2}^{1/2} d\sigma \sin^2(\theta(\sigma)), \quad (1047)$$

which are easily seen to reproduce (1032) and (1033).

2. Coherent-State Strings

The coherent state vectors $|\mathbf{n}\rangle$ single out the endpoint of the folded string — a property which is not natural from the dual gauge-theory perspective as the dual operator must be cyclically symmetric. (As mentioned above, in the coherent-state framework, cyclicity manifests itself via equation (1041).) This, in particular, becomes an issue when we want to derive matrix elements between multi-cut states, cf. subsection VII D.

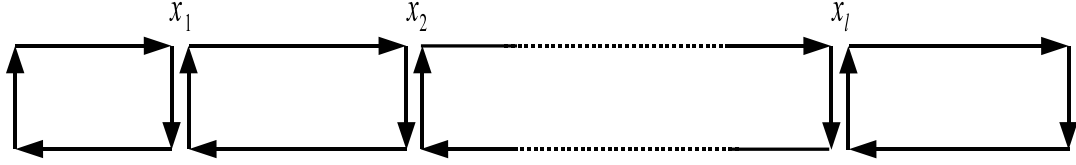
We can ensure cyclicity of the state by averaging over cyclic translations,

$$|\mathbf{n}\rangle = \sum_{k=1}^{L_n} \prod_{i=1}^{L_m} |\overrightarrow{n_{i+k}}\rangle. \quad (1048)$$

These averaged states, properly normalized, will now represent our string states. The inner product is defined as follows: Given two vectors $|\mathbf{n}\rangle = \prod_{i=1}^{L_n} |\overrightarrow{n_i}\rangle$ and $|\mathbf{m}\rangle = \prod_{j=1}^{L_m} |\overrightarrow{m_j}\rangle$, one has

$$\langle \mathbf{m} | \mathbf{n} \rangle = \delta_{L_m, L_n} \prod_{i=1}^{L_m} \langle \overrightarrow{m_i} | \overrightarrow{n_i} \rangle, \quad (1049)$$

from which the definition of $\langle\langle m|n \rangle\rangle$ proceeds.

FIG. 51: A cut state $|x_1, \dots, x_l\rangle$.

D. Matrix Elements of H_{npl}

With our new states, we have

$$\frac{\tilde{\lambda}}{J} \mathcal{E}_0 = \frac{\langle \langle \mathbf{n} | H_{\text{pl}} | \mathbf{n} \rangle \rangle}{\langle \langle \mathbf{n} | \mathbf{n} \rangle \rangle}. \quad (1050)$$

We would now like to compute matrix elements of the one-loop non-planar dilatation operator between coherent-state vectors representing the folded Frolov–Tseytlin strings. It is obvious that acting on a coherent state vector $|\mathbf{n}\rangle$ with H_{npl} gives rise to a splitting of a one-string dual into a two-string dual. Similarly, acting with H_{npl} on a direct product of two coherent state vectors $|\mathbf{n}\rangle$ and $|\mathbf{m}\rangle$ can produce a one-string dual from a two-string dual. In the more traditional gauge theory language, H_{npl} triggers trace splitting and trace joining. The matrix elements of the non-planar dilatation operator contain information about the genus-one correction to the energy of the Frolov–Tseytlin strings. It is obvious, however, that if we tried to determine this energy correction by considering H_{npl} to be a perturbation of H_{pl} , we would have to make use of degenerate perturbation theory. For instance, if we start from a coherent state vector $|\mathbf{n}\rangle$ of energy \mathcal{E}_0 as given by (1050), cut it vertically once and close the open ends — we obtain another state which up to $1/J$ corrections is an eigenstate with the same energy. The same is true if we make l vertical cuts, where $l \ll J$, see figure 51. We could also cut with some, not too large, skewness, and still obtain a degenerate state. However, we will restrict ourselves to straight-cut states since in the continuum limit, small skewness should not matter and large skewness takes us out of the sub-space of degenerate states. We notice that since $\varphi = \omega t$ is constant along the string, the inner product between two coherent states reduces to

$$\langle \vec{n}_1 | \vec{n}_2 \rangle = \cos(\theta_1 - \theta_2), \quad (1051)$$

which implies that we do not need to worry about $\varphi_{1,2}$ at all and can consistently set $\varphi_{1,2} = 0$.

1. Normalization of States

Let us denote by $|\emptyset\rangle$ the complete (uncut) folded string dual, *i.e.*,

$$|\emptyset\rangle \equiv \prod_{i=-J/2}^{J/2} |\vec{n}_i\rangle, \quad (1052)$$

with

$$|\vec{n}_i\rangle = \cos(\theta(i/J)) |\uparrow\rangle + \sin(\theta(i/J)) |\downarrow\rangle, \quad \text{where} \quad -\frac{J}{2} < i < \frac{J}{2}, \quad (1053)$$

where $\theta(x)$ is the function given in formula (1043). Furthermore, let us denote by $|x_1, \dots, x_l\rangle$ the state obtained from (1052) by cutting it vertically at the points x_1, x_2, \dots, x_l (see figure 51),

$$\begin{aligned} |x_1, \dots, x_l\rangle \equiv & \left| \prod_{i=-J/4}^{x_1 J} \vec{n}_i \prod_{i=-J/4}^{x_1 J} \vec{n}_{(x_1-1/4)J-i} \right\rangle \otimes \left| \prod_{k=x_1 J+1}^{x_2 J} \vec{n}_i \prod_{k=x_1 J+1}^{x_2 J} \vec{n}_{(x_1+x_2)J+1-i} \right\rangle \otimes \dots \\ & \dots \otimes \left| \prod_{k=x_l J+1}^{J/4} \vec{n}_i \prod_{k=x_l J+1}^{J/4} \vec{n}_{(x_l+1/4)J+1-i} \right\rangle, \end{aligned} \quad (1054)$$

where

$$-\frac{1}{4} < x_i < \frac{1}{4}, \quad l \ll J, \quad x_{j+1} - x_j \sim O(J). \quad (1055)$$

In order to determine the norm of such a state, we first consider a single piece of string, extending between the points x and y , and compute the inner product $\langle \cdot | \cdot \rangle$ between this piece and the piece which originates from it by shifting each of its coherent-state vectors by the distance δ ,

$$\mathcal{A}_{x,y,\delta} \equiv \left\langle \prod_{i=xJ}^{yJ} \vec{n}_{i-\delta J} \middle| \prod_{i=xJ}^{yJ} \vec{n}_i \right\rangle = \prod_{i=0}^{(y-x)J} \langle \vec{n}_{(x-\delta)J+i} | \vec{n}_{xJ+i} \rangle. \quad (1056)$$

For fixed δ , it is clear that $\mathcal{A}_{x,y,\delta}$ goes exponentially to zero as J increases to infinity. It is, therefore, sufficient to study the behavior of this quantity for small δ ,

$$\begin{aligned} \mathcal{A}_{x,y,\delta} &\approx \exp \left(J \int_x^y dz \log (\cos (\theta(z-\delta) - \theta(z))) \right) \approx \\ &\approx \exp \left(-J \frac{\delta^2}{2} \int_x^y dz \theta'(z)^2 \right) \approx \exp \left(-J \frac{\delta^2}{2} \mathcal{E}_{x,y} \right), \end{aligned} \quad (1057)$$

where $\mathcal{E}_{x,y}$ is given by

$$\mathcal{E}_{x,y} \equiv \int_x^y dz \theta'(z)^2 = 4K(m) (E(\text{am}(4K(m)y|m)) - E(\text{am}(4K(m)x|m)) - 4K(m)(1-m)(y-x)). \quad (1058)$$

Notice that the planar energy of the folded string stretching between x and y is $2\mathcal{E}_{x,y}$, and in particular, by definition, $\mathcal{E}_0 = \mathcal{E}_{-1/2,1/2}$. It is then easy to find the squared norm of the string with no cuts at the leading order in J , by integrating over all possible δ ,

$$\langle \langle \emptyset | \emptyset \rangle \rangle = J^2 \int_{-1/2}^{1/2} d\delta \exp \left(-J \frac{\mathcal{E}_0}{2} \delta^2 \right) = J \sqrt{\frac{2\pi J}{\mathcal{E}_0}}. \quad (1059)$$

(Since we assume $x_{j+1} - x_j \sim O(J)$, the integration range of such a Gaussian integral can always be taken to be $(-\infty, +\infty)$ when $J \rightarrow \infty$.) One of the factors of J comes from the fact that one can simultaneously make the same cyclic translation of the bra and the ket without changing anything. The second factor of J comes from the summation over nontrivial *relative* translations, and the substitution of a continuous integral for the discrete sum in the large J limit. For each smaller string in (1054), one will get a similar factor, so that

$$\langle \langle x_1, x_2, \dots, x_l | x_1, x_2, \dots, x_l \rangle \rangle = \prod_{i=0}^l \frac{l_i J \sqrt{\pi J}}{\sqrt{\mathcal{E}_{x_i, x_{i+1}}}}, \quad (1060)$$

where $x_0 \equiv -1/4$ and $x_{l+1} \equiv 1/4$, and where $l_i \equiv 2(x_{i+1} - x_i)$ is the total length of the string piece between x_i and x_{i+1} . Here, we have neglected the contributions coming from the “corners” of the string pieces, where the overlap is not anymore between $\theta(z-\delta)$ and $\theta(z)$ as in (1057). This is justified because the relevant shifts δJ are much smaller than the length of the pieces we consider.

2. Matrix Elements for String Joining

In this paragraph, we aim at computing the matrix element $\langle \langle \emptyset | H_{\text{np}} | x \rangle \rangle$. To begin with, we consider non-cyclic states.

There are in total four ways to join a two-piece state, giving rise to four different states, $|a\rangle$, $|b\rangle$, $|c\rangle$ and $|d\rangle$, as shown in figure 52. The eventual use of the cyclic states $|\cdot\rangle$ is essential here since the notion of the endpoint of the string becomes ambiguous. By reflection symmetry, states $|a\rangle$ and $|c\rangle$ yield the same expectation values, and so do the states $|b\rangle$ and $|d\rangle$. We will start with the state $|a\rangle$. The corresponding overlaps are shown in figure 53.

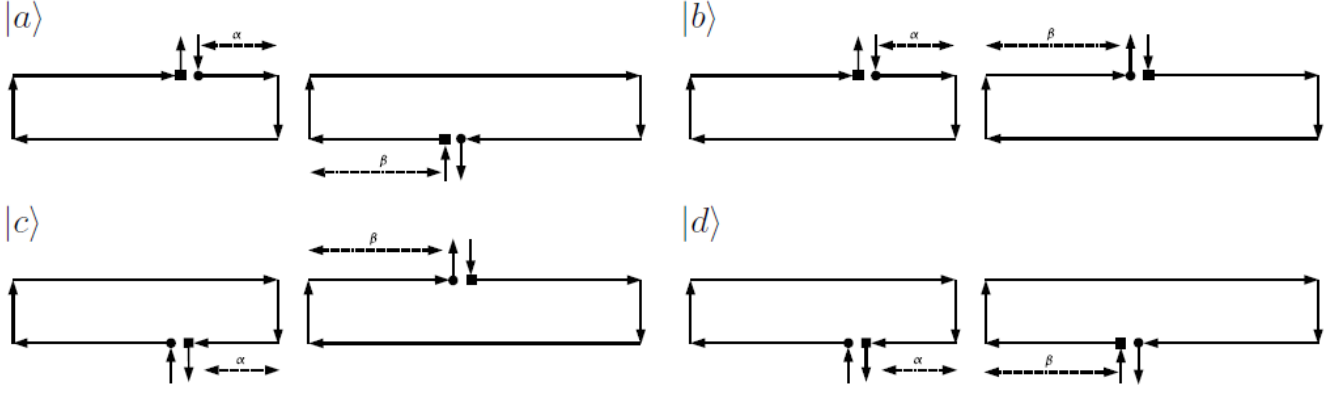


FIG. 52: Possible joinings of two bits. Sites at the squares (circles) are linked after joining, and then anti-symmetrized.

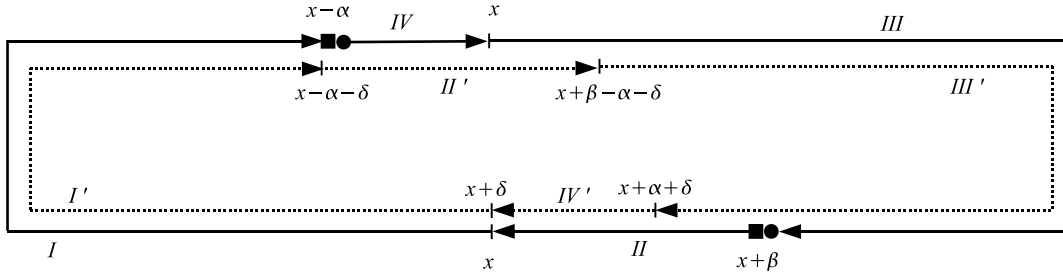


FIG. 53: Overlaps between the bra $\langle \emptyset |$ (dotted lines) and the ket $|a\rangle$ (continuous lines). Arguments for $\theta(x)$ are given at the relevant points. More precisely, the bra reads $|I', II', III', IV'\rangle$ and the ket $|I, II, III, IV\rangle$; in this figure, it is the function $\theta(x)$ which is continuous along the loop, while the sequence inside the ket is discontinuous.

As in the previous subsection, we denote by δ the shift given to $\langle \emptyset |$, and by $\langle I'_\delta |$, $\langle II'_\delta |$, $\langle III'_\delta |$, $\langle IV'_\delta |$ its corresponding δ -shifted pieces (see figure 53). We also define the planar energies of the first and second spin-chain bits respectively by

$$\mathcal{E}_1 \equiv \mathcal{E}_{-1/2-x, x} = 2\mathcal{E}_{-1/4, x}, \quad \text{and} \quad \mathcal{E}_2 \equiv \mathcal{E}_{x, 1/2-x} = 2\mathcal{E}_{x, 1/4}. \quad (1061)$$

The identity $\mathcal{E}_0 = \mathcal{E}_1 + \mathcal{E}_2$ is satisfied by construction. First, let us assume that $\beta \geq \alpha$. We have

$$\langle \langle \emptyset | a \rangle \rangle = \sum_{\delta} \mathcal{F}_{\alpha, \beta, \delta} \langle I'_\delta | I \rangle \langle II'_\delta | II \rangle \langle III'_\delta | III \rangle \langle IV'_\delta | IV \rangle, \quad (1062)$$

where the anti-symmetrization effects at the joining sites are taken into account through the factor $\mathcal{F}_{\alpha, \beta, \delta}$.

In order to do the computation, we expand as follows

$$\log(\cos(\theta(z - \epsilon) - \theta(z))) = -\frac{\epsilon^2}{2} \theta'(z)^2 + \frac{\epsilon^3}{2} \theta'(z) \theta''(z) + \mathcal{O}(\epsilon^4), \quad (1063)$$

and make use of the identities (1045) for $\theta(z)$. It is important to stress that the expansion we will use for the integrands strongly depends on the range of integration. For long-range integrations, *e.g.* $\int_{-1/2-x}^x dz f(z, x, \alpha, \beta, \delta)$, we expand in small α, β, δ only. For short-range integrations, *e.g.* $\int_0^\beta dz f(z, x, \alpha, \beta, \delta)$, we expand in small z , too.

One then gets

$$\begin{aligned} \langle I'_\delta | I \rangle &= \left\langle \prod_{i=(-1/2-x)J}^{(x-\alpha)J} \vec{n}_{i-\delta J} \middle| \prod_{i=(-1/2-x)J}^{(x-\alpha)J} \vec{n}_i \right\rangle \approx \\ &\approx \exp \left(J \int_{-1/2-x}^{x-\alpha} dz \log(\cos(\theta(z - \delta) - \theta(z))) \right) \approx \end{aligned}$$

$$\begin{aligned}
&\approx \exp \left(J \int_{-1/2-x}^x dz \left(-\frac{\delta^2}{2} \theta'(z)^2 + \frac{\delta^3}{2} \theta'(z) \theta''(z) \right) + J \frac{\delta^2 \alpha}{2} \theta'(x)^2 \right) \approx \\
&\approx \exp \left(-\frac{1}{2} \mathcal{E}_1 J \delta^2 + J \frac{\delta^2 \alpha}{2} \theta'(x)^2 \right), \tag{1064}
\end{aligned}$$

and

$$\begin{aligned}
\langle \Pi'_\delta | \Pi \rangle &= \left\langle \prod_{i=0}^{\beta J} \vec{n}_{(x+\beta-\alpha-\delta)J-i} \left| \prod_{i=0}^{\beta J} \vec{n}_{xJ+i} \right. \right\rangle \approx \\
&\approx \exp \left(-\frac{1}{2} \theta'(x)^2 J \int_0^\beta dz (2z + \alpha - \beta + \delta)^2 \right) \approx \\
&\approx \exp \left(-\frac{1}{6} J \beta (\beta^2 + 3(\alpha + \delta)^2) \theta'(x)^2 \right), \tag{1065}
\end{aligned}$$

and

$$\begin{aligned}
\langle \Pi'_\delta | \Pi \rangle &= \left\langle \prod_{i=xJ}^{(1/2-x-\beta)J} \vec{n}_{(\beta-\alpha-\delta)J+i} \left| \prod_{i=xJ}^{(1/2-x-\beta)J} \vec{n}_i \right. \right\rangle \approx \\
&\approx \exp \left(J \int_x^{1/2-x} dz \left(-\frac{(\beta-\alpha-\delta)^2}{2} \theta'(z)^2 + \frac{(\beta-\alpha-\delta)^3}{2} \theta'(z) \theta''(z) \right) + J \frac{(\beta-\alpha-\delta)^2 \beta}{2} \theta'(1/2-x)^2 \right) \approx \\
&\approx \exp \left(-\frac{1}{2} J (\beta-\alpha-\delta)^2 \mathcal{E}_2 + J \frac{(\beta-\alpha-\delta)^2 \beta}{2} \theta'(x)^2 \right), \tag{1066}
\end{aligned}$$

and

$$\begin{aligned}
\langle \text{IV}'_\delta | \text{IV} \rangle &= \left\langle \prod_{i=0}^{\alpha J} \vec{n}_{(x+\delta)J+i} \left| \prod_{i=0}^{\alpha J} \vec{n}_{xJ-i} \right. \right\rangle \approx \\
&\approx \exp \left(-\frac{1}{2} \theta'(x)^2 J \int_0^\alpha dz (2z + \delta)^2 \right) \approx \\
&\approx \exp \left(-\frac{1}{6} J \alpha (4\alpha^2 + 6\alpha\delta + 3\delta^2) \theta'(x)^2 \right). \tag{1067}
\end{aligned}$$

The four overlaps give in total the contribution

$$\exp \left(-\frac{1}{2} \mathcal{E}_1 J \delta^2 - \frac{1}{2} \mathcal{E}_2 J (\beta - \alpha - \delta)^2 - \frac{1}{3} J \theta'(x)^2 (-\alpha^3 + 3\alpha^2 \beta + 2\beta^3 - 3(\beta^2 + \alpha^2)(\beta - \alpha - \delta)) \right), \tag{1068}$$

and one perceives that the dominant region will be around $\delta \approx 0$ and $\beta \approx \alpha$, so that the leading term in $1/J$ will be obtained by taking the following approximation for the exponential,

$$\exp \left(-\frac{1}{2} \mathcal{E}_1 J \delta^2 - \frac{1}{2} \mathcal{E}_2 J (\beta - \alpha)^2 - \frac{4}{3} J \alpha^3 \theta'(x)^2 \right). \tag{1069}$$

We should now compute $\mathcal{F}_{\alpha,\beta,\delta}$ near these values of α , β and δ . One gets

$$\begin{aligned} \mathcal{F}_{\alpha,\alpha,0} &= \langle \vec{n}_{(x-\alpha-\delta)J}, \vec{n}_{(x-\alpha)J} \rangle \langle \vec{n}_{(x-\alpha-\delta)J+1}, \vec{n}_{(x+\beta)J} \rangle \cdot \\ &\quad \cdot \langle \vec{n}_{(x+\alpha+\delta)J+1}, \vec{n}_{(x+\beta)J+1} \rangle \langle \vec{n}_{(x+\alpha+\delta)J}, \vec{n}_{(x-\alpha)J+1} \rangle \Big|_{\delta=0, \beta=\alpha} \approx \frac{4}{J} \alpha \theta'(x)^2. \end{aligned} \quad (1070)$$

(We use the notation $f(\underline{A}, \underline{B})g(\underline{C}, \underline{D}) = (f(\underline{A}, \underline{B}) - f(\underline{B}, \underline{A}))g(\underline{C}, \underline{D}) + f(\underline{A}, \underline{B})(g(\underline{C}, \underline{D}) - g(\underline{D}, \underline{C}))$.) The case $\alpha > \beta$ gives the same result up to the exchange $\alpha \leftrightarrow \beta$. Furthermore, translating the finding to cyclic states implies multiplying by $L_1 L_2 J^2$. Finally, using the normalization factor

$$\mathcal{N} = \left(\sqrt{\frac{2\pi J}{\mathcal{E}_0}} L_0 J \right)^{1/2} \left(\sqrt{\frac{2\pi J}{\mathcal{E}_1}} L_1 J \right)^{1/2} \left(\sqrt{\frac{2\pi J}{\mathcal{E}_2}} L_2 J \right)^{1/2}, \quad (1071)$$

one then gets at the leading order in $1/J$,

$$\begin{aligned} \sum_{\alpha,\beta} \langle \langle \emptyset | a \rangle \rangle &\approx \frac{2}{N} \frac{4}{J} \theta'(x)^2 J^5 L_1 L_2 \int_0^\infty d\beta \int_0^\beta d\alpha \int_{-\infty}^\infty d\delta e^{-\frac{1}{2}\mathcal{E}_1 J \delta^2 - \frac{1}{2}\mathcal{E}_2 J (\beta-\alpha)^2 - \frac{4}{3} J \alpha^3 \theta'(x)^2} \alpha \approx \\ &\approx \frac{4\Gamma(\frac{2}{3})}{3^{1/3}} K(m)^{2/3} m^{1/3} \text{cn}(4K(m)x|m)^{2/3} \left(\frac{L_1 L_2}{L_0} \right)^{1/2} \left(\frac{2\pi \mathcal{E}_0}{\mathcal{E}_1 \mathcal{E}_2} \right)^{1/4} J^{1/12}. \end{aligned} \quad (1072)$$

Note that although β should live in the interval $[0, 1/4 - x]$ and δ in the interval $[-1/2, 1/2]$, integrating in both cases till infinity will not change the leading $(1/J)$ -behavior, as the integrand converges exponentially to zero for $\alpha J^{1/3} \gg 1$, $\beta J^{1/3} \gg 1$, and $\delta J^{1/2} \gg 1$.

A similar computation shows that $\langle \langle \emptyset | b \rangle \rangle$ and $\langle \langle \emptyset | d \rangle \rangle$ are of order $J^{-1/4}$, and therefore, can be neglected as compared to the $J^{1/12}$ -behavior found here. Thus, one obtains at the leading order in $1/J$,

$$\langle \langle \emptyset | H_{\text{npl}} | x \rangle \rangle = \frac{8\Gamma(\frac{2}{3})}{3^{1/3}} K(m)^{2/3} m^{1/3} \text{cn}(4K(m)x|m)^{2/3} \left(\frac{L_1 L_2}{L_0} \right)^{1/2} \left(\frac{2\pi \mathcal{E}_0}{\mathcal{E}_1 \mathcal{E}_2} \right)^{1/4} J^{1/12}. \quad (1073)$$

It is straightforward to generalize this result to an arbitrary number of cuts, where the joining takes place at position x_i . It is in order to facilitate this generalization that we have explicitly kept the parameter L_0 , although in our case we have $L_0 = 1$. We observe the occurrence of the factor $(\mathcal{E}_1 \mathcal{E}_2)^{-1/4}$, which diverges when x approaches the endpoints of the string. In this situation, thus, we cannot trust the semi-classical analysis (and hence, the overall J -scaling).

3. Matrix Elements for String Splitting

From the calculations in the last paragraph, we learn which approximations we are allowed to do in order to keep only the leading order in $1/J$. First, the terms that arise from the cyclicity of the traces are long-range terms — they appear through δ -shifts over a whole piece of spin chain, and consequently, will give in the exponential a square term times minus the planar energy of the considered piece, times J . This is what happened in equations (1064) and (1066). Conversely, terms which are integrated over short intervals will appear in the exponential starting at the cubic order (see equations (1065) and (1067)). This allows for the following approximations that will not change the leading $1/J$ term after all the integrations: • When computing overlaps over long-range parts, it is not necessary to take into account small parameters at the endpoints of the integration. For example, taking $\int_{-1/2-x}^x dz$ instead of $\int_{-1/2-x}^{x-\alpha} dz$ in (1064) would not have changed the final result. • When computing overlaps over short-range parts, one can do as if the shifts appearing in the long-range terms were equal to zero.

We are now poised to derive expectation values such as $\langle \langle x | H_{\text{npl}} | \emptyset \rangle \rangle$. $H_{\text{npl}} | \emptyset \rangle$ will yield a lot of possible double-chain states. Only the ones with lengths equal to those of $|x\rangle$, *i.e.*, the states with length $(1/2 + 2x)J$ and length $(1/2 - 2x)J$, will contribute. All these contributing states can be characterized by a value γJ expressing how far the cut took place from the straight cut between sites xJ and $(1/2 - x)J$ (see figure 54). Let us denote them by $|\{x, \gamma\}\rangle$. The following identity holds,

$$\langle \langle x | H_{\text{npl}} | \emptyset \rangle \rangle = \sum_{i=-J/2}^{J/2} \langle \langle x | \{x, i/J\} \rangle \rangle. \quad (1074)$$

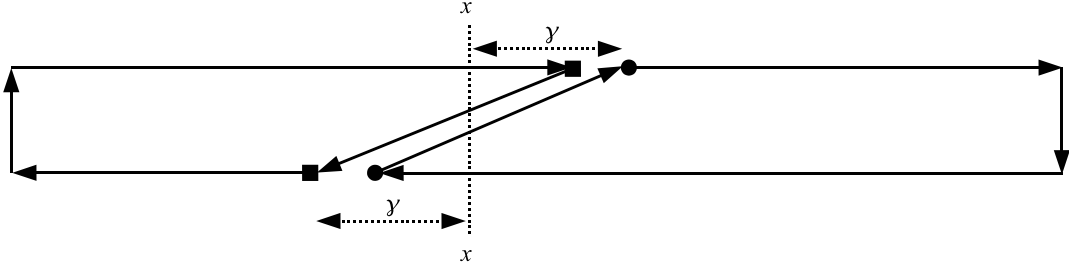


FIG. 54: A state $|\{x, \gamma\}\rangle$. The sites at the squares are anti-symmetrized, as are sites at the circles. The spin chain has been cut between the sites where θ takes the value $\theta(x + \gamma)$ and $\theta(x - \gamma)$.

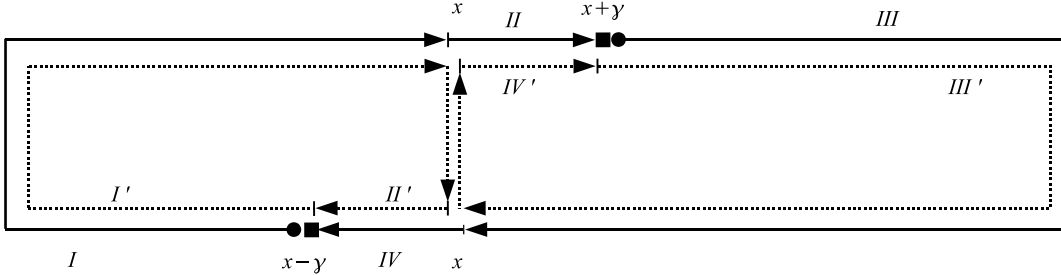


FIG. 55: Overlaps between $\langle x|$ (dotted lines) and $|\{x, \gamma\}\rangle$ (continuous lines). Arguments for $\theta(x)$ are given at the relevant points. More precisely, $\langle x|$ reads $\langle I', II' | \langle III', IV' |$, while $|\{x, \gamma\}\rangle$ is equal to $|I, II\rangle |III, IV\rangle$. As in figure 53, it is the function $\theta(x)$ which is continuous along the loop. Possible shifts δ and δ' for each piece of $\langle x|$ were put to 0 for simplicity.

Overlaps for $\langle x | \{x, i/J\} \rangle$ are shown in figure 55. In order to proceed to the full cyclic scalar product, one should then add two arbitrary shifts, δ and δ' for each piece of $\langle x|$, as well as one for $|\emptyset\rangle$. However, the effect of the latter is simply the multiplication by the factor $L_0 J$.

We thus have

$$\langle \langle x | H_{\text{np}} | \emptyset \rangle \rangle = L_0 J \sum_{\gamma, \delta, \delta'} \mathcal{F}_{\gamma, \delta, \delta'} \langle I'_\delta | I \rangle \langle II'_\delta | II \rangle \langle III'_{\delta'} | III \rangle \langle IV'_{\delta'} | IV \rangle, \quad (1075)$$

where $\mathcal{F}_{\gamma, \delta, \delta'}$ is the anti-symmetrization factor and $\langle I'_\delta |$, $\langle II'_\delta |$, $\langle III'_{\delta'} |$, $\langle IV'_{\delta'} |$ are the δ - (δ' -) shifted pieces of $\langle x|$.

Using the approximations we presented in the beginning of this paragraph, we have, for $\gamma > 0$,

$$\begin{aligned} \langle I'_\delta | I \rangle &\approx \left\langle \prod_{i=(-1/2-x)J}^{xJ} \vec{n}_{i-\delta} \right| \prod_{i=(-1/2-x)J}^J \vec{n}_i \rangle \approx \\ &\approx \exp \left(-\frac{1}{2} \mathcal{E}_1 J \delta^2 \right), \end{aligned} \quad (1076)$$

and

$$\begin{aligned} \langle II'_\delta | II \rangle &\approx \left\langle \prod_{i=0}^{\gamma J} \vec{n}_{xJ-i} \right| \prod_{i=0}^{\gamma J} \vec{n}_{xJ+i} \rangle \approx \\ &\approx \exp \left(-2\theta'(x)^2 J \int_0^\gamma dz z^2 \right) \approx \exp \left(-\frac{2}{3} J \gamma^3 \theta'(x)^2 \right), \end{aligned} \quad (1077)$$

and

$$\langle III'_{\delta'} | III \rangle \approx \left\langle \prod_{i=xJ}^{(1/2-x)J} \vec{n}_{i-\delta'} \right| \prod_{i=xJ}^{(1/2-x)J} \vec{n}_i \rangle \approx$$

$$\approx \exp\left(-\frac{1}{2}J\delta'^2\mathcal{E}_2\right), \quad (1078)$$

and

$$\begin{aligned} \langle \text{IV}'_\delta | \text{IV} \rangle &\approx \left\langle \prod_{i=0}^{\gamma J} \vec{n}_{xJ+i} \middle| \prod_{i=0}^{\gamma J} \vec{n}_{xJ-i} \right\rangle \approx \\ &\approx \exp\left(-2\theta'(x)^2 J \int_0^\gamma dz z^2\right) \approx \exp\left(-\frac{2}{3}J\gamma^3\theta'(x)^2\right). \end{aligned} \quad (1079)$$

The overlaps give, therefore, the contribution

$$\exp\left(-\frac{1}{2}\mathcal{E}_1 J\delta^2 - \frac{1}{2}\mathcal{E}_2 J(\delta')^2 - \frac{4}{3}\theta'(x)^2 J\gamma^3\right). \quad (1080)$$

Computing $\mathcal{F}_{\gamma,\delta,\delta'}$ around $\delta = \delta' = \gamma = 0$, one gets

$$\begin{aligned} \mathcal{F}_{\gamma,0,0} &= \langle \vec{n}_{(x-\gamma)J+1}, \vec{n}_{(x+\gamma)J} \rangle \underbrace{\langle \vec{n}_{(x-\gamma)J}, \vec{n}_{(x-\gamma)J} \rangle}_{\cdot} \cdot \\ &\quad \cdot \underbrace{\langle \vec{n}_{(x+\gamma)J}, \vec{n}_{(x-\gamma)J+1} \rangle}_{\cdot} \langle \vec{n}_{(x+\gamma)J+1}, \vec{n}_{(x+\gamma)J+1} \rangle \approx \frac{4}{J}\gamma\theta'(x)^2. \end{aligned} \quad (1081)$$

In the case $\gamma < 0$, extra minus signs appear, so that one can use the same results by taking the absolute value of γ instead. Using as normalization the factor

$$\mathcal{N} = \left(\sqrt{\frac{2\pi J}{\mathcal{E}_0}}L_0J\right)^{1/2} \left(\sqrt{\frac{2\pi J}{\mathcal{E}_1}}L_1J\right)^{1/2} \left(\sqrt{\frac{2\pi J}{\mathcal{E}_2}}L_2J\right)^{1/2}, \quad (1082)$$

this leads to

$$\begin{aligned} \langle \langle x | H_{\text{np}} | \emptyset \rangle \rangle &\approx \frac{1}{\mathcal{N}} \frac{4}{J} \theta'(x)^2 L_0 J^4 \int_{-\infty}^{\infty} d\gamma \int_{-\infty}^{\infty} d\delta \int_{-\infty}^{\infty} d\delta' e^{-\frac{1}{2}\mathcal{E}_1 J\delta^2 - \frac{1}{2}\mathcal{E}_2 J\delta'^2 - \frac{4}{3}J|\gamma|^3\theta'(x)^2} |\gamma| \approx \\ &\approx \frac{8\Gamma(\frac{2}{3})}{3^{1/3}} K(m)^{2/3} m^{1/3} \text{cn}(4K(m)x|m)^{2/3} \left(\frac{L_0}{L_1 L_2}\right)^{1/2} \left(\frac{2\pi\mathcal{E}_0}{\mathcal{E}_1 \mathcal{E}_2}\right)^{1/4} J^{-11/12}. \end{aligned} \quad (1083)$$

This result can be immediately extended to states which have already been cut before the action of the Hamiltonian. We note that the non-planar dilatation operator is non-Hermitian. A similar situation has been encountered in the previous investigation of the non-planar corrections to energies of the BMN states [73, 85]. There, the non-planar dilatation operator has been related to its Hermitian conjugate by a similarity transformation (see the discussion below (183)). The same is true here.

E. A Solvable Toy Model

By construction, the vertically cut multi-string states studied above are degenerate in planar energy with the complete Frolov–Tseytlin string. Let us now consider a toy model of a folded string for which the vertically cut states exhaust the space of states degenerate in energy with the uncut string. Furthermore, let us assume that the matrix elements of H_{npI} for string splitting and string joining depend only on the point of splitting and joining. Determining the first non-planar correction to the string energy under these assumptions amounts to diagonalizing the non-planar dilatation operator in the subspace of vertically cut states, which of course implies diagonalizing an infinite-dimensional matrix in the limit $J \rightarrow \infty$. This problem can easily be solved, though. Let us denote by $|i, j, k, \dots\rangle$ the state corresponding to the string cut at positions i, j, k, \dots , and by \mathcal{X}_l the matrix element corresponding to an additional cutting or joining taking place at position l . To illustrate the solution, we consider as an example only three possible sites where a cutting/joining can take place. Then in the base $\{|\emptyset\rangle, |1\rangle, |2\rangle, |1, 2\rangle, |3\rangle, |1, 3\rangle, |2, 3\rangle, |1, 2, 3\rangle\}$, the matrix we have to diagonalize is given by

$$\mathcal{M} = \begin{pmatrix} 0 & \mathcal{X}_1 & \mathcal{X}_2 & 0 & \mathcal{X}_3 & 0 & 0 & 0 \\ \mathcal{X}_1 & 0 & 0 & \mathcal{X}_2 & 0 & \mathcal{X}_3 & 0 & 0 \\ \mathcal{X}_2 & 0 & 0 & \mathcal{X}_1 & 0 & 0 & \mathcal{X}_3 & 0 \\ 0 & \mathcal{X}_2 & \mathcal{X}_1 & 0 & 0 & 0 & 0 & \mathcal{X}_3 \\ \mathcal{X}_3 & 0 & 0 & 0 & 0 & \mathcal{X}_1 & \mathcal{X}_2 & 0 \\ 0 & \mathcal{X}_3 & 0 & 0 & \mathcal{X}_1 & 0 & 0 & \mathcal{X}_2 \\ 0 & 0 & \mathcal{X}_3 & 0 & \mathcal{X}_2 & 0 & 0 & \mathcal{X}_1 \\ 0 & 0 & 0 & \mathcal{X}_3 & 0 & \mathcal{X}_2 & \mathcal{X}_1 & 0 \end{pmatrix}, \quad (1084)$$

whose eigenvalues μ are simply all the possible sum and differences between the $\mathcal{X}_{1,2,3}$'s,

$$\mu = \pm\mathcal{X}_1 \pm \mathcal{X}_2 \pm \mathcal{X}_3. \quad (1085)$$

For J different sites, the eigenvalues are distributed in a quasi-continuum between the energies

$$\pm J \int_{-1/4}^{1/4} dx \mathcal{X}_x. \quad (1086)$$

In our case, we can arrange by the means of a similarity transformation that all our matrix elements should scale as $J^{-5/12}$. Therefore, a rough scaling argument gives

$$\Delta E \approx \frac{\tilde{\lambda}}{N} \frac{J}{2} \mathcal{X}_0 \sim \tilde{\lambda} \frac{J^{7/12}}{N}. \quad (1087)$$

Now if one, again naively, assumes a BMN-like scaling for the energy of spinning strings, one needs that the genus-one contribution compared to the genus-zero one has an additional factor of J^2/N , which leads to the expectation $\Delta E \sim J/N$. It is of course not known to which extent the BMN scaling beyond the planar limit should hold for spinning strings. One knows from the analysis of [176, 186] and the field-theoretical computations of [184] that the BMN scaling for few-impurity operators breaks down already at the planar level but only at order four in $\tilde{\lambda}$. In the true picture of string splitting, we cannot claim that the straight-cut states exhaust the space of eigenstates degenerate in energy with the folded string. (As mentioned earlier, the straight-cut states are also not exact eigenstates but only eigenstates up to terms of order $1/J$.) One could argue that one should in fact replace \mathcal{X}_x of the toy model by some integral over matrix elements involving skew-cut states close to the vertically-cut ones and that this could give rise to additional factors of J . We have not been able to make a quantitative estimate of this effect, but we find it unlikely that such an integration could provide the “missing” factor $J^{5/12}$. Rather, the low power of J in (1087) seems to suggest that the process of semi-classical string splitting and joining is not of importance for the genus-one energy shift, cf. subsection VII F.

F. Discussion

Our computation shows that for long strings, a non-zero contribution to the splitting matrix element comes only from strings which are almost on top of each other (cf. formula (1063) and the subsequent calculations). This is somewhat reminiscent of the interaction vertex between strings in light cone string field theory,

$$V(X_0^i(\sigma), X_1^i(\sigma), X_2^i(\sigma)) = \int ds_0 ds_1 ds_2 \delta(J_0 - J_1 - J_2) \cdot \prod \Delta(X_1^i(\sigma + s_1) - X_0^i(\sigma + s_0)) \Delta\left(X_2^i(\sigma + s_2) - X_0^i\left(\sigma + s_0 + \frac{J_1}{J_0}\right)\right). \quad (1088)$$

In the above formula, the $s_{1,2,3}$ are direct analogs of cyclic translations in our definition of states, while the functional Dirac delta functions Δ are analogs of the property that we have found, namely, that in order for the matrix element to be non-zero, the angles defining the coherent states have to be within $J^{-1/2}$. However, the detailed calculations in paragraphs VIID 2 and VIID 3 show that more non-trivial $J^{-1/3}$ factors may also appear. Furthermore, we have seen that H_{npl} gives an effective additional operator inserted at the interaction point (1083). This is not unexpected since such operators appear generically in superstring light-cone SFT (see *e.g.* [72]). However, due to the fact that we are really able to deal only with classical states, we refrain from making any more quantitative comparison.

Our crude estimate for the order of magnitude of the genus-one energy shift due to semi-classical string joining and splitting leads to the energy scaling with an unexpectedly small power of J . An interpretation of this result may be that the contribution to the energy shift coming from such semi-classical string processes is simply quite small. In fact, for generic macroscopic rotating strings (*i.e.*, not “folded” ones), the contribution of string splitting into classical states would be very strongly suppressed. It is much more probable that the dominant non-planar contribution would come from *small* strings which would split off from the rotating string and would be reabsorbed shortly after. Unfortunately, the process of small strings splitting off is beyond the reach of the semi-classical coherent-state methods that we are exploiting. ■

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