Seven-branes and Instantons in Type IIB Supergravity

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Introduction

The subject of this thesis will be introduced. The studies performed in the coming chapters attempt to improve on the understanding of certain aspects of string theory. Here, I will give a personal point of view on the subject of string theory after which I will briefly sketch the most elementary properties of string theory that allow me to further introduce the subject of this thesis.

The aim of theoretical physics is to describe the observable world around us through mathematically consistent theories in which physical quantities are derived from first principles. By the observable world is meant the collection of things whose existence has been experimentally established. The 'first principles' are postulated by the scientific community and reflect properties of the mathematical theories that are not disproved by experiment.

The above attempt to describe the arena of theoretical physics is necessarily imprecise, because research in theoretical physics sometimes forces one to change one's perspective on its very nature. For example the advent of quantum theory strongly influenced the way we think about observable quantities. The advent of general relativity and input from cosmological observations influenced our notion of the 'world around us'. Undoubtedly, this will go on as science progresses.

An important aspect of the above attempted description of theoretical physics is the need for a mathematically consistent theory. There are generally speaking two approaches to the construction of new theories. One method goes by modeling large amounts of experimental data obtained from measurements at a particular energy scale. Once a consistent model is found that reproduces the data to within the observed accuracy, one can extrapolate the model to higher energy scales to see what predictions it gives. This may be referred to as a bottom-up approach. This approach has the advantage of being strongly correlated to experiment and the disadvantage that it is by construction energy scale dependent making it difficult to construct a model that is valid at all energy scales. The other approach starts by postulating aspects that we (a particular scientific community in which consensus has been reached) wish to attribute to nature, because they have found to be rigorously true in experiments (possibly performed at different energy scales) and to construct a mathematically consistent theory around these postulates that is valid at all energy scales. This approach may be referred to as a top-down approach. It has the advantage of being applicable at all energy scales and the disadvantage that it does not necessarily (by construction) lead to a connection with experiment. A connection with experiment will only be possible if the theory makes a unique prediction about what happens at an experimentally accessible energy scale.

Two beautiful examples of a bottom-up approach are the standard model of elementary particles constructed by Glashow, Salam and Weinberg and the so-called ΛCDM model of cosmology that is used to describe the currently observed cosmological data such as the dark energy and dark matter components of our universe. In the abbreviation ΛCDM the Λ denotes the cosmological constant parameterizing the dark energy and CDM stands for cold dark matter.

A particularly nice example of a top-down approach (within the context of classical physics) is Einstein's theory of general relativity. In the context of quantum theory an example is provided by quantum field theory which derives from a small number of postulates and can sometimes be shown to apply at all energy scales[1]. The top-down approach is most useful in finding the mathematical arena in which a physical theory can be formulated.

Quantum field theory is not a unique theory, because it does not specify its own field content and neither does it specify the properties of the space-time on which the fields are propagating. Another drawback of quantum field theory is that it has proven, at least up to date, only possible to describe gravity at some energy scale as an effective field theory. This situation is markedly different in string theory. String theory is another top-down approach that is at present still under construction in that its complete quantum mechanical formulation is not fully known. However, there is very strong evidence that the generalization of string theory, called M-theory, is a unique theory that is valid at all energy scales, contains gravity and predicts the dimensionality of the world in which extended objects, generally called branes, are moving to be 11-dimensional space-time. Further, it has been shown that many quantum field theories have a natural interpretation as low energy approximations of string theory. The matter content of low-energy string theory is fixed provided a space-time background is chosen and the precise configuration of branes is specified. The matter content then derives from the lowest level excitations of the strings moving in such a background. String theory will be discussed in more detail below.

Let me pause here for a moment and briefly elaborate on the notion of a brane. Branes are objects in string theory that can have $p = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ spatial dimensions. There are two types of strings: open and closed strings. The endpoints of an open string are attached to a brane. Those branes on which an open string is ending

[1] Due to the uncertainty relation in energy, quantum theories, both relativistic and non-relativistic, know, in principle, about all energy scales. A quantum theory therefore has the important property of being able to predict its own regime of validity.

(not all branes are of this type) are called Dirichlet p-branes or Dp-branes for short. The endpoints of a string can be charged and from the point of view of a Dp-brane this charge is a point particle leading to a particular form of electrodynamics on the brane. When multiple Dp-branes are coinciding the theory on the set of coinciding branes is a particular form of a non-Abelian gauge theory. Branes have played a crucial role in many of the successes achieved by string theory such as the counting of black hole microstates and the AdS/CFT correspondence that relates a gauge theory (without gravity) to a string theory (containing gravity). Further, branes have proven to be of fundamental importance in string theory phenomenology such as 4-dimensional models of particle physics/cosmology obtained from string theory.

Let us go back to string theory as a top-down theory. Although the properties of string theory, that are listed in the paragraph preceding the last paragraph, may appear appealing from the point of view of a top-down theoretician they provide no guiding principles in attempts to construct out of string theory the physics of elementary particles and/or cosmology. Basically, one must, in order to describe low energy 4-dimensional physics, decide how to embed the 4-dimensional theory in the 11-dimensional framework and at the same time produce the right matter content of for example the standard model from a unique 11-dimensional theory while breaking supersymmetry somewhere along the way. Needless to say this is a very difficult task that is furthermore complicated by the high degree of non-uniqueness in which 4-dimensional physics can be obtained from string theory. This high degree of non-uniqueness in producing 4-dimensional physics from string theory is encoded in a concept that has been dubbed the string theory landscape. What the right approach is towards such a concept is the subject of much current debate.

Before discussing the subject matter of this thesis let me give a brief and by no means complete overview of string theory.

There exist five perturbatively defined string theories that go under the names: type I, type IIA, type IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$ superstring theory. The type I string theory is a theory of both open and closed unoriented (meaning no world-sheet directionality exists) strings and possesses 16 space-time supersymmetries. The type IIA and IIB string theories are theories of closed oriented strings with 32 space-time supersymmetries. The closed strings of the type IIA theory have right and left moving world-sheet fermions of the same chirality while for the type IIB string they have opposite chirality, so that the type IIA world-sheet theory is chiral and the IIB world-sheet theory is non-chiral. The two heterotic superstring theories contain oriented closed strings with a left moving sector of world-sheet fields that is identical to that of the type II theories while the right movers are obtained by compactifying the 26-dimensional bosonic string theory on a 16-dimensional torus all of whose radii are equal. The 24 bosonic coordinates (world-sheet scalars) of the 26-dimensional bosonic string are divided into two groups: 8 bosonic world-sheet scalars that describe the position of the string in the 10-dimensional (uncompactified) space-time and 16 bosonic modes that take values in the 16-dimensional torus. The groups $SO(32)$ and $E_8 \times E_8$ determine the form taken by these 16 bosonic modes and their momenta. The heterotic theories both possess 16 space-time supersymmetries.

The type IIA and type IIB string theories contain besides closed strings also a spectrum of Dp-branes. As mentioned earlier open strings have the property that their endpoints must end on a Dp-brane and hence type IIA and type IIB string theory in the presence of Dp-branes become theories of both open and closed strings.

The spectrum of excitations of the above five string theories consists of fields that are organized into supermultiplets. The coupling of the string is described by a massless dynamical field, the dilaton. There exist field theories, called supergravity theories, with 16 or 32 supersymmetries in 10-dimensional space-time, whose fields are organized in the same supermultiplets as is the case for the massless excitations of the above-mentioned string theories. It turns out that string theory and supergravity are related as will now be discussed.

Each of the above-mentioned five string theories are well-defined in a perturbative setting. When the massless fields form a background satisfying the supergravity equations of motion then the dynamics of a string on such a background can be defined perturbatively in regions where the string coupling g_s is small.

The strings all have a finite length that is inversely related to the tension of the string. In a regime in which the string length is small compared to other distance scales in the theory it is possible to describe the string dynamics (of the massless excitations) in terms of an effective field theory. To obtain a field theory description one first expands the string theory in g_s after which one expands it further in α' (the square root of the string length). By tree level string theory is meant that the expansion in g_s is truncated right after the leading contribution (tree-level string scattering diagrams). By further truncating the α' expansion of the tree-level string diagrams keeping only the leading terms one obtains supergravity as the effective description.

This may seem as a tremendous oversimplification of the physics of string theory, and to a large extent it is, but it has turned out that certain statements derived from supergravity are protected by supersymmetry and can be turned into statements that are correct at all order in α'. Further, as mentioned above, field theory solutions to supergravity are starting points of a background on which string theory can be formulated (at least when g_s is small in the supergravity background). Even more so, this simple field theory approach can be used to gain insights into some non-perturbative aspects of the theory. Branes are a good example of this. The Dp-branes of perturbative string theory have a mass that is inversely proportional to the string coupling and are therefore extremely massive objects in the perturbative regime. These Dp-branes show up in supergravity as certain p-brane solutions, i.e. solutions describing extended objects that extend in p spatial directions [1]. Therefore, solutions to the supergravity equations of motion can be used to study non-perturbative aspects of the theory.

Consistent theories of quantum gravity must be free of gauge anomalies. A gravitational anomaly would imply that at the quantum level local Lorentz invariance, or what is the same, general coordinate invariance is broken. If the theories contain furthermore besides gravity Yang–Mills fields then again consistency requires there to be no gauge anomalies present in the theory. If the theory contains space-time chiral fermions or chiral gauge fields (gauge fields whose fields strengths are self-dual) there can be associated chiral anomalies. In the years 1983 and 1984 (the first string theory revolution) it has been shown that the supergravity approximations of the five different string theories are free of anomalies. The type IIB supergravity has chiral fermions and a chiral 4-form potential leading to chiral, gravitational and mixed anomalies. It was shown in [2] that all these anomalies cancel. The type IIA supergravity theory is non-chiral (it contains two chiral fermions of opposite chirality), does not have non-Abelian gauge fields and no self-dual gauge fields. It is therefore trivially free anomalies. The fact that the type I theory and the two heterotic theories that all contain chiral fermions and non-Abelian gauge fields are free of anomalies was shown in [3].

Approximately ten years after the first string theory revolution it was realized that the five different string theories are all related via so-called duality transformations. Further, in this web of dualities an 11-dimensional theory plays an important role. The string coupling of the type IIA supergravity theory can be interpreted as coming from the circle reduction of the unique 11-dimensional supergravity with 32 supersymmetries. Weakly coupled type IIA string theory thus corresponds to a very small circle extending in the eleventh dimension. But when the IIA string coupling becomes large the circle decompactifies and an 11-dimensional theory appears [4, 5]. The type IIA superstring becomes the 11-dimensional supermembrane, that was first introduced in [6]. Without going into the details some of the most common string-string dualities are listed. Type IIA string theory on a 9-dimensional space-time times a circle of radius R is said to be T-dual to the type IIB string theory on a 9-dimensional space-time times a circle of radius α'/R [7,8]. The type IIB string theory in the strong coupling limit is dual to itself, a duality referred to as S-duality which will play an important role in this thesis. The type IIB theory can be related to the type I theory through the introduction of 32 D9-branes and one so-called orientifold O9-plane (see section 2.5). The strong coupling limit of the type I theory is dual to weakly coupled heterotic $SO(32)$. Finally M-theory on the orbifold $S^1/\mathbb{Z}_2$ which is an interval with a hyperplane at each end gives heterotic $E_8 \times E_8$ [9, 10] and the list goes on. Strongly coupled string theory is a subject that has been given the name of M-theory whose low energy approximation must be the 11-dimensional supergravity theory since this latter theory is unique.

To summarize string theory provides a perturbatively well-defined description of quantum gravity. The five different string theories are free of anomalies. At the non-perturbative level all the five string theories are related to a unique theory, called M-theory, whose low energy approximation is eleven-dimensional supergravity. Further, besides being a quantum theory of gravity string theory also naturally contains Yang–Mills sectors. Hence, the full non-perturbative version is expected to provide a unique theory of quantum gravity, in which the dimensionality of space-time is predicted by the theory itself and which naturally contains Yang–Mills gauge interactions. At the perturbative level the theory contains only one dimensionful parameter, the string length, but this may be an artifact of the perturbative description. The input on which string theory is founded is special relativity and quantum theory. The dynamical objects of the theory are strings and branes.

Besides having an understanding of the laws that govern the dynamics of the fundamental objects of the theory it is equally important to understand the structure of the vacua of the theory. In recent years it has become gradually more and more apparent that the vacuum structure of string theory/M-theory is overwhelmingly vast and complex. The vacua of string theory/M-theory are collectively referred to as the landscape of the theory [11, 12]. The notion and even the definition of the landscape is at present not well-formulated. But, it is an unsurmountable notion that needs to be understood before one can truly hope to test the theory against experiment.

A theory that in part addresses both the vacuum structure of M-theory and parts of the M-theory moduli space is F-theory that was first introduced in [13]. It was mentioned that the type IIB theory in the strong coupling limit is dual to itself, a property known as S-duality. The type IIB theory actually possesses a discrete group of duality transformations, the group $SL(2,\mathbb{Z})$ [14]. The coupling constant of the type IIB theory is not a real massless field as is the case in the other four string theories, but rather a massless complex field, denoted by τ. Besides being a coupling constant τ is at the same time a field that couples to branes, viz. 7-branes and instantons. The 7-branes are described by F-theory and the instantons can be thought of as the electric/magnetic dual partners of the 7-branes. What F-theory is will be explained in section 2.6. Compactifications of F-theory down to four dimensions provide interesting insights into the 4-dimensional vacuum structure of the IIB theory (see for example [15]). The landscape of F-theory vacua will not be discussed in this thesis.

F-theory provides furthermore a means to study the IIB theory in the non-perturbative regime where the complex coupling τ is of order unity. This should be contrasted with the perturbative regime where τ is such that the string coupling g_s is small. F-theory will be used in this thesis to argue for the existence of novel types of branes, called Q7-branes. The notion of a Q7-brane leads to the notion of Q-instantons that are related to Q7-branes by electro-magnetic duality. Understanding the world-volume theory of the Q7-branes and the role of the Q-instantons provides a means to study the IIB theory in, so far, poorly investigated corners of its moduli space. By moduli space is meant the set of inequivalent values of the complex type IIB coupling constant τ. This work aims at improving the understanding of the complete set of so-called one-half BPS branes (preserving 16 supersymmetries) of the type IIB supergravity theory.

From the set of one-half BPS objects that are present in type IIB supergravity almost all the branes have been accounted for in type IIB string theory. These are the branes that are referred as the (p', q') p-branes with p' and q' relatively prime integers. These are p-dimensional branes on which a (p', q') string is ending. The above-mentioned Q7-branes and Q-instantons are not of this type.

This thesis is organized as follows. Chapter 1 gives an overview of type IIB supergravity. In chapter 2 the branes of the IIB theory are reviewed and the notion of F-theory is introduced. Chapter 3 deals with the subject of Q7-branes and their possible F-theory interpretation. The Q-instantons are discussed in chapter 4. The chapter on 7-branes is based on [16,17] and the chapter on instantons is based on [18].

Chapter 1

Overview of type IIB supergravity

The conventions used throughout this thesis are presented in appendix A.

This chapter contains an overview of the properties of type IIB supergravity. In section 1.1 the $SU(1,1)$ covariant formulation of the theory is presented. In section 1.2 two scalars are defined T and χ' that will play an important role in this thesis. They parameterize the coset manifold $SL(2,\mathbb{R})/SO(2)$ and are related to the dilaton ϕ and RR axion χ by a field redefinition. Then in section 1.3 the local supersymmetry algebra is presented in terms of the scalars T and χ'.

1.1 The $SU(1,1)$ covariant formulation

The bosonic field content of type IIB supergravity consists of the metric, two scalars, two 2-form potentials and a chiral 4-form potential with a self-dual 5-form field strength making up a total of 128 bosonic degrees of freedom. The fermionic field content consists of two Majorana–Weyl dilatini as well as two Majorana–Weyl gravitini comprising a total of 128 fermionic degrees of freedom. These fields are organized into one 10-dimensional $N = 2$ supergravity multiplet. The two scalars parameterize the coset space $SU(1,1)/U(1)$ that is isomorphic to $SL(2,\mathbb{R})/SO(2)$. The local supersymmetry algebra for this theory has been constructed in [19–21] and shown to only close on-shell.

The two scalars in the theory transform non-linearly under $SU(1,1)$ or $SL(2,\mathbb{R})$. This transformation can be made linear by introducing a third scalar as well as a local $U(1)$ gauge symmetry. Upon fixing the $U(1)$ gauge the third scalar is eliminated as well as the local $U(1)$ gauge transformation.

The R-symmetry group of the global type IIB supersymmetry algebra acts in the local theory as a local $U(1)$ transformation on the spinors. In the coset formulation this local $U(1)$ symmetry is identified with the local $U(1)$ symmetry of the coset description. After fixing the $U(1)$ gauge of the coset formulation the spinors still transform under the $U(1)$ R-symmetry group.

The $SU(1,1)/U(1)$ coset model is described in subsection 1.1.1. The bosonic fields and their equations of motion are given in subsections 1.1.2 and 1.1.3 and the local supersymmetry algebra (up to second order fermions) is given in subsection 1.1.4.

1.1.1 The $SU(1,1)/U(1)$ coset

Consider the group $SU(1,1)$ consisting of all complex two by two matrices g with $\det g = 1$ that satisfy $g^{-1} = \eta g^{\dagger}\eta$ where $\eta = \mathrm{diag}(1,-1)$. The generators of this group are H and $T_{\pm}$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad T_{+}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \quad T_{-}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.1.1}$$

and they satisfy the algebra

$$[H,H] = 0\,, \quad [H,T_{\pm}] = \pm 2T_{\pm}\,. \tag{1.1.2}$$

The matrix H generates the maximal compact subgroup $U(1)$.

The coset $SU(1,1)/U(1)$ consists of all $SU(1,1)$ matrices V that are identified under the transformations of the compact subgroup $U(1)$. If one takes V to depend on space-time points x then the equivalence under $U(1)$ becomes a gauge symmetry. A (left-)coset representative V transforms as

$$V(x) \to gV(x)h(x)\,, \tag{1.1.3}$$

where $g \in SU(1,1)$ and $h \in U(1)$. The coset representative V is parameterized as

$$V = \begin{pmatrix} V_{-}^{1} & V_{+}^{1} \\ V_{-}^{2} & V_{+}^{2} \end{pmatrix}\,, \tag{1.1.4}$$

where $(V_{\mp}^{1})^{*} = V_{\pm}^{2}$ and

$$V_{-}^{\alpha}V_{+}^{\beta} - V_{-}^{\beta}V_{+}^{\alpha} = \epsilon^{\alpha\beta} = \epsilon_{\alpha\beta} \tag{1.1.5}$$

with $\alpha,\beta = 1,2$ denoting $SU(1,1)$ indices and with $\epsilon^{12} = +1$.

Using the matrix V a left-invariant Lie algebra element of $SU(1,1)$ can be written as

$$V^{-1}dV = \begin{pmatrix} -iQ & P \\ \bar{P} & iQ \end{pmatrix}\,. \tag{1.1.6}$$

In terms of the components of V eq. (1.1.6) reads

$$P = -\epsilon_{\alpha\beta} V_+^\alpha dV_+^\beta \,, \tag{1.1.7}$$
$$\bar{P} = \epsilon_{\alpha\beta} V_-^\alpha dV_-^\beta \,, \tag{1.1.8}$$
$$Q = -i\epsilon_{\alpha\beta} V_-^\alpha dV_+^\beta \,. \tag{1.1.9}$$

Under a local $U(1)$ transformation $h = e^{i\alpha(x)H}$ with parameter $\alpha(x)$ the 1-forms Q and P transform as

$$\begin{aligned} Q &\to Q - d\alpha \,, \\ P &\to e^{-2i\alpha} P \,. \end{aligned} \tag{1.1.10}$$

Both P and Q are invariant under global $SU(1,1)$ transformations. The 1-form Q is real-valued and transforms as a composite $U(1)$ gauge connection under local $U(1)$ transformations. The 1-form P is complex-valued and is referred to as the coset Zweibein. The $U(1)$ weight w of P is $w = 2$ [1]. The gauge-covariant derivative, denoted by D, of P is defined in the standard way as $DP = dP - 2iQ \wedge P$. The Bianchi identity for P_μ is given by

$$DP = 0 \,. \tag{1.1.12}$$

From eqs. (1.1.5), (1.1.7) and (1.1.8) it can be concluded that the covariant derivative D acting on $V_\pm^\alpha$ gives

$$DV_+^\alpha = V_-^\alpha P \,, \tag{1.1.13}$$
$$DV_-^\alpha = V_+^\alpha \bar{P} \,. \tag{1.1.14}$$

It is convenient to define the following $U(1)$ gauge-invariant (right-invariant) matrix of 1-forms p as follows

$$p = V \begin{pmatrix} 0 & P \\ \bar{P} & 0 \end{pmatrix} V^{-1} \,. \tag{1.1.15}$$

It can be shown that the components of p are the three Noether currents that derive from the global $SU(1,1)$ invariance of the scalar kinetic terms of the IIB Lagrangian. The Bianchi identity for p reads

$$dp - 2p \wedge p = 0 \,. \tag{1.1.16}$$

[1] A field X is assigned a $U(1)$-weight $w(X)$ if it transforms under a local $U(1)$ transformation $h = e^{i\alpha H}$ as

$$X \to e^{-iw\alpha} X \,. \tag{1.1.11}$$

Using this definition it follows that $w(V_\pm^\alpha) = \pm 1$ and $w(P) = +2$.

1.1.2 The non-self-dual bosonic type IIB action

In the conventions of [22] the bosonic part of the IIB supergravity action is given by

$$S = \int_{\mathcal{M}_{10}} \left(\star 1 R - 2 \star P \wedge \bar{P} - \frac{1}{2} \star G_3 \wedge \bar{G}_3 - 4 \star F_5 \wedge F_5 + \frac{i}{2} F_5 \wedge \epsilon_{\alpha\beta} A_2^\alpha \wedge F_3^\beta \right) . \tag{1.1.17}$$

In order to obtain the type IIB bosonic field equations one must additionally impose the self-duality condition $F_5 = \star F_5$ on-shell[2] which is why (1.1.17) is referred to as the non-self-dual type IIB action. The forms P (defined in 1.1.1), G_3 and F_5 are defined via the Bianchi identities,

$$DP = dP - 2iQ \wedge P = 0 \, , \tag{1.1.18}$$

$$DG_3 = dG_3 - iQ \wedge G_3 = -P \wedge \bar{G}_3 \, , \tag{1.1.19}$$

$$dF_5 = -\frac{i}{8} G_3 \wedge \bar{G}_3 \, . \tag{1.1.20}$$

The solution to the Bianchi identity for G_3 is given by

$$G_3 = -\epsilon_{\alpha\beta} V_+^\alpha F_3^\beta \qquad \text{where} \qquad F_3^\beta = dA_2^\beta \, . \tag{1.1.21}$$

The 2-forms, A_2^α, transform as a doublet under $SU(1,1)$ and transform under gauge transformations as $\delta A_2^\alpha = d\Lambda_1^\alpha$. The solution to the Bianchi identity for F_5 is

$$F_5 = dA_4 + \frac{i}{16} \epsilon_{\alpha\beta} A_2^\alpha \wedge F_3^\beta \, . \tag{1.1.22}$$

The reality properties of A_2^α and A_4 are

$$(A_2^1)^* = A_2^2 \qquad \text{and} \qquad A_4^* = A_4 \, . \tag{1.1.23}$$

The equations of motion that follow from the action (1.1.17) supplemented with the self-duality condition of the 5-form are:

$$R_{\mu\nu} = P_\mu \bar{P}_\nu + \bar{P}_\mu P_\nu + \frac{1}{6} F_{\sigma_1 \cdots \sigma_4 \mu} F^{\sigma_1 \cdots \sigma_4}{}_\nu + \frac{1}{8} \left(G_{\sigma_1\sigma_2\mu} \bar{G}^{\sigma_1\sigma_2}{}_\nu + \bar{G}_{\sigma_1\sigma_2\mu} G^{\sigma_1\sigma_2}{}_\nu - \frac{1}{6} g_{\mu\nu} \bar{G}_{\sigma_1\sigma_2\sigma_3} G^{\sigma_1\sigma_2\sigma_3} \right) \tag{1.1.24}$$

$$D \star P = \frac{1}{4} \star G_3 \wedge G_3 \tag{1.1.25}$$

$$D \star G_3 = P \wedge * \bar{G}_3 + 4i \star F_5 \wedge G_3 \tag{1.1.26}$$

$$\star F_5 = F_5 \, . \tag{1.1.27}$$

[2]It is possible to construct an action that incorporates the self-duality condition $F_5 = \star F_5$ [23]. Since in this thesis the emphasis will be on the gravity-scalar sector of the bosonic action the chiral 4-form action of [23] will not be needed.

1.1.3 The form fields

It is possible to dualize the 2-forms, A_2^α, to a doublet of 6-forms, A_6^α, with $(A_6^1)^* = A_6^2$, via the respective duality relation and Bianchi identity,

$$F_7^\alpha = i \star \left(V_-^\alpha G_3 - V_+^\alpha \bar{G}_3\right) , \tag{1.1.28}$$

$$dF_7^\alpha = 4 F_3^\alpha \wedge F_5 \, . \tag{1.1.29}$$

Solving the Bianchi identity (1.1.29) gives

$$F_7^\alpha = dA_6^\alpha + \frac{4}{3} F_5 \wedge A_2^\alpha - \frac{8}{3} A_4 \wedge F_3^\alpha \, . \tag{1.1.30}$$

In verifying that (1.1.30) satisfies (1.1.29) one can use the following 2-form identity:

$$\epsilon_{\beta\gamma} F_3^\beta \wedge F_3^\gamma \wedge A_2^\alpha = -2 \epsilon_{\beta\gamma} A_2^\beta \wedge F_3^\gamma \wedge F_3^\alpha \, . \tag{1.1.31}$$

The object G_7 that is defined by

$$G_7 = -\epsilon_{\alpha\beta} V_+^\alpha F_7^\beta \tag{1.1.32}$$

satisfies the Bianchi identity,

$$DG_7 + P \wedge \bar{G}_7 = 4 G_3 \wedge F_5 \, . \tag{1.1.33}$$

From eq. (1.1.28) it follows that $G_7 = i \star G_3$.

It is further possible to introduce a triplet of 8-forms [24, 25], $A_8^{\alpha\beta} = A_8^{\beta\alpha}$, with $(A_8^{11})^* = A_8^{22}$ and $(A_8^{12})^* = A_8^{12}$, via the respective duality relation and Bianchi identity

$$F_9^{\alpha\beta} = i \star \left(V_+^\alpha V_+^\beta \bar{P} - V_-^\alpha V_-^\beta P\right) , \tag{1.1.34}$$

$$dF_9^{\alpha\beta} = \frac{1}{4} F_3^{(\alpha} \wedge F_7^{\beta)} \, . \tag{1.1.35}$$

Solving the Bianchi identity (1.1.35), $F_9^{\alpha\beta}$ can be written as

$$F_9^{\alpha\beta} = dA_8^{\alpha\beta} + \frac{1}{16} F_7^{(\alpha} \wedge A_2^{\beta)} - \frac{3}{16} F_3^{(\alpha} \wedge A_6^{\beta)} \, . \tag{1.1.36}$$

As follows from eq. (1.1.34) the field strengths $F_9^{\alpha\beta}$ satisfy the $SU(1,1)$ invariant constraint

$$\epsilon_{\alpha\gamma} \epsilon_{\beta\delta} V_-^\alpha V_+^\beta F_9^{\gamma\delta} = 0 \, . \tag{1.1.37}$$

The p-form gauge fields ($p = 2, 4, 6, 8$) have the following gauge transformations:

$$\begin{aligned}
\delta A_2^\alpha &= d\Lambda_1^\alpha\,, && (1.1.38)\\
\delta A_4 &= d\Lambda_3 - \frac{i}{16}\epsilon_{\alpha\beta}\, F_3^\alpha \wedge \Lambda_1^\beta\,, && (1.1.39)\\
\delta A_6^\alpha &= d\Lambda_5^\alpha + \frac{4}{3}F_5 \wedge \Lambda_1^\alpha - \frac{8}{3}F_3^\alpha \wedge \Lambda_3\,, && (1.1.40)\\
\delta A_8^{\alpha\beta} &= d\Lambda_7^{\alpha\beta} + \frac{1}{16}F_7^{(\alpha} \wedge \Lambda_1^{\beta)} - \frac{3}{16}F_3^{(\alpha} \wedge \Lambda_5^{\beta)}\,. && (1.1.41)
\end{aligned}$$

This can also be written as:

$$\begin{aligned}
\delta A_2^\alpha &= d\Lambda_1^\alpha\,, && (1.1.42)\\
\delta A_4 &= d\Sigma_3 + \frac{i}{16}\epsilon_{\alpha\beta}\, A_2^\alpha \wedge \delta A_2^\beta\,, && (1.1.43)\\
\delta A_6^\alpha &= d\Sigma_5^\alpha + \frac{8}{3}A_2^\alpha \wedge \delta A_4 - \frac{4}{3}A_4 \wedge \delta A_2^\alpha + \frac{i}{12}A_2^\alpha \wedge \epsilon_{\gamma\delta}\, \delta A_2^\gamma \wedge A_2^\delta\,, && (1.1.44)\\
\delta A_8^{\alpha\beta} &= d\Sigma_7^{\alpha\beta} + \frac{3}{16}A_2^{(\alpha} \wedge \delta A_6^{\beta)} - \frac{1}{16}A_6^{(\alpha} \wedge \delta A_2^{\beta)} - \frac{1}{4}A_2^\alpha \wedge A_2^\beta \wedge \delta A_4 \\
&\quad + \frac{1}{6}A_4 \wedge A_2^{(\alpha} \wedge \delta A_2^{\beta)} - \frac{1}{12}A_2^\alpha \wedge A_2^\beta \wedge \frac{i}{16}\epsilon_{\gamma\delta}\, \delta A_2^\gamma \wedge A_2^\delta\,, && (1.1.45)
\end{aligned}$$

where the gauge transformation parameters Σ_p differ from Λ_p.

1.1.4 Supersymmetry transformation rules

Below are given the supersymmetry variations of the fields of type IIB supergravity as they are presented in [22] up to second order in fermions:

$$\begin{aligned}
\delta g_{\mu\nu} &= 2i\,\bar\epsilon\gamma_{(\mu}\psi_{\nu)} + 2i\bar\epsilon_C\gamma_{(\mu}\psi_{C\,\nu)}\,, && (1.1.46)\\
\delta V_+^\alpha &= V_-^\alpha\bar\epsilon_C\lambda\,, && (1.1.47)\\
\delta V_-^\alpha &= V_+^\alpha\bar\epsilon\lambda_C\,, && (1.1.48)\\
\delta A_{\mu\nu}^\alpha &= V_-^\alpha\bar\epsilon\gamma_{\mu\nu}\lambda + V_+^\alpha\bar\epsilon_C\gamma_{\mu\nu}\lambda_C \\
&\quad + 4iV_-^\alpha\bar\epsilon_C\gamma_{[\mu}\psi_{\nu]} + 4iV_+^\alpha\bar\epsilon\gamma_{[\mu}\psi_{C\,\nu]}\,, && (1.1.49)\\
\delta A_{\mu\nu\rho\sigma} &= \bar\epsilon\gamma_{[\mu\nu\rho}\psi_{\sigma]} - \bar\epsilon_C\gamma_{[\mu\nu\rho}\psi_{C\,\sigma]} - \frac{3i}{8}\epsilon_{\alpha\beta}A_{[\mu\nu}^\alpha\delta A_{\rho\sigma]}^\beta\,, && (1.1.50)\\
\delta A_{\mu_1\ldots\mu_6}^\alpha &= iV_-^\alpha\bar\epsilon\gamma_{\mu_1\ldots\mu_6}\lambda - iV_+^\alpha\bar\epsilon_C\gamma_{\mu_1\ldots\mu_6}\lambda_C + 12V_-^\alpha\bar\epsilon_C\gamma_{[\mu_1\ldots\mu_5}\psi_{\mu_6]} \\
&\quad - 12V_+^\alpha\bar\epsilon\gamma_{[\mu_1\ldots\mu_5}\psi_{C\,\mu_6]} + 40A_{[\mu_1\ldots\mu_4}\delta A_{\mu_5\mu_6]}^\alpha \\
&\quad - 20\delta A_{[\mu_1\ldots\mu_4}A_{\mu_5\mu_6]}^\alpha - \frac{15i}{2}A_{[\mu_1\mu_2}^\alpha\epsilon_{\beta\gamma}A_{\mu_3\mu_4}^\beta\delta A_{\mu_5\mu_6]}^\gamma\,, && (1.1.51)
\end{aligned}$$

$$\begin{aligned}
\delta A^{\alpha\beta}_{\mu_1\ldots\mu_8} &= -iV^\alpha_+ V^\beta_+ \bar\epsilon\gamma_{\mu_1\ldots\mu_8}\lambda_C + iV^\alpha_- V^\beta_- \bar\epsilon_C\gamma_{\mu_1\ldots\mu_8}\lambda \\
&\quad +8V^{(\alpha}_+ V^{\beta)}_- \bar\epsilon\gamma_{[\mu_1\ldots\mu_7}\psi_{\mu_8]} - 8V^{(\alpha}_+ V^{\beta)}_- \bar\epsilon_C\gamma_{[\mu_1\ldots\mu_7}\psi_{C\,\mu_8]} \\
&\quad +\frac{21}{4}A^{(\alpha}_{[\mu_1\ldots\mu_6}\delta A^{\beta)}_{\mu_7\mu_8]} - \frac{7}{4}A^{(\alpha}_{[\mu_1\mu_2}\delta A^{\beta)}_{\mu_3\ldots\mu_8]} \\
&\quad -35A^{(\alpha}_{[\mu_1\mu_2}A^{\beta)}_{\mu_3\mu_4}\delta A_{\mu_5\ldots\mu_8]} + 70A_{[\mu_1\ldots\mu_4}A^{(\alpha}_{\mu_5\mu_6}\delta A^{\beta)}_{\mu_7\mu_8]} \\
&\quad -\frac{105i}{8}A^{(\alpha}_{[\mu_1\mu_2}A^{\beta)}_{\mu_3\mu_4}\epsilon_{\gamma\delta}A^\gamma_{\mu_5\mu_6}\delta A^\delta_{\mu_7\mu_8]}\,, \qquad (1.1.52)
\end{aligned}$$

$$\delta\lambda = iP_\mu\gamma^\mu\epsilon_C - \frac{i}{24}G_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon\,, \qquad (1.1.53)$$

$$\begin{aligned}
\delta\psi_\mu &= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{i}{2}Q_\mu\right)\epsilon + \frac{i}{480}F_{\mu\nu_1\ldots\nu_4}\gamma^{\nu_1\ldots\nu_4}\epsilon \\
&\quad +\frac{1}{96}G^{\nu\rho\sigma}\gamma_{\mu\nu\rho\sigma}\epsilon_C - \frac{3}{32}G_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon_C\,. \qquad (1.1.54)
\end{aligned}$$

The $U(1)$ weight of the complex Weyl dilatino, λ, is $\frac{3}{2}$ and the $U(1)$ weight of the complex Weyl gravitino, ψ_μ, is $\frac{1}{2}$. Under global $SU(1,1)$ transformations λ, ψ_μ and ϵ transform as scalars.

The commutator of two supersymmetry transformations closes on the bosonic fields up to local symmetries of the theory except for the 4-form in which case closure also requires the self-duality condition (1.1.27) to be imposed. The local symmetries are general coordinate transformations (or local Lorentz transformations), gauge transformations and the local $U(1)$ transformation of the coset model. The commutator of two supersymmetries acting on the spinors closes up to local Lorentz transformations, local $U(1)$ transformations and upon imposing the self-duality condition (1.1.27) as well as the fermionic equations of motion. The fermionic equations of motion are given by (up to first order in fermions) [19]

$$\gamma^\mu D_\mu\lambda = \frac{i}{240}\gamma^{\mu_1\ldots\mu_5}F_{\mu_1\ldots\mu_5}\lambda\,, \qquad (1.1.55)$$

$$\gamma^{\mu\nu\rho}D_\nu\psi_\rho = -\frac{i}{2}\gamma^\rho\gamma^\mu P_\rho\lambda_C - \frac{i}{48}\gamma^{\nu\rho\lambda}\gamma^\mu\bar G_{\nu\rho\lambda}\lambda\,, \qquad (1.1.56)$$

where D_μ contains the $U(1)$ connection Q_μ. By applying a supersymmetry transformation on the fermionic field equations the bosonic equations of motion (1.1.24) to (1.1.26) follow. Hence, the type IIB local supersymmetry algebra is an on-shell algebra.

The type IIB local supersymmetry algebra described above is unique up to the introduction of 10-form potentials [22, 26]. These 10-forms will be briefly considered in the next subsection.

1.1.5 Inclusion of ten-form potentials

As shown in [22] the type IIB supersymmetry algebra is compatible with the introduction of two types of 10-form potentials: a doublet and quadruplet. Ten-form potentials are interesting for the inclusion of space-time filling 9-branes in the IIB theory, see section 2.5. The quadruplet consists of 10-forms, $A^{\alpha\beta\gamma}$, symmetric in α, β and γ with reality conditions

$$(A_{10}^{111})^* = A_{10}^{222}\,, \qquad (A_{10}^{112})^* = A_{10}^{122}\,. \tag{1.1.57}$$

The supersymmetry algebra closes on $A_{10}^{\alpha\beta\gamma}$ provided they have the following gauge transformation:

$$\delta A_{10}^{\alpha\beta\gamma} = d\Lambda_9^{\alpha\beta\gamma} - \frac{1}{15}F_9^{(\alpha\beta} \wedge \Lambda_1^{\gamma)} + \frac{4}{15}F_3^{(\alpha} \wedge \Lambda_7^{\beta\gamma)}\,. \tag{1.1.58}$$

Under supersymmetry the ten-form quadruplet transforms as

$$\begin{aligned}
\delta A^{\alpha\beta\gamma}_{\mu_1\ldots\mu_{10}} \;=\;& iV_+^{(\alpha}V_+^{\beta}V_-^{\gamma)}\bar\epsilon_C\gamma_{\mu_1\ldots\mu_{10}}\lambda_C - iV_-^{(\alpha}V_-^{\beta}V_+^{\gamma)}\bar\epsilon\gamma_{\mu_1\ldots\mu_{10}}\lambda \\
&+\frac{20}{3}V_+^{(\alpha}V_+^{\beta}V_-^{\gamma)}\bar\epsilon\gamma_{[\mu_1\ldots\mu_9}\psi_{C\,\mu_{10}]} - \frac{20}{3}V_-^{(\alpha}V_-^{\beta}V_+^{\gamma)}\bar\epsilon_C\gamma_{[\mu_1\ldots\mu_9}\psi_{\mu_{10}]} \\
&-12A^{(\alpha\beta}_{[\mu_1\ldots\mu_8}\delta A^{\gamma)}_{\mu_9\mu_{10}]} + 3A^{(\alpha}_{[\mu_1\mu_2}\delta A^{\beta\gamma)}_{\mu_3\ldots\mu_{10}]} \\
&-\frac{63}{4}A^{(\alpha}_{[\mu_1\ldots\mu_6}A^{\beta}_{\mu_7\mu_8}\delta A^{\gamma)}_{\mu_9\mu_{10}]} + \frac{21}{4}A^{(\alpha}_{[\mu_1\mu_2}A^{\beta}_{\mu_3\mu_4}\delta A^{\gamma)}_{\mu_5\ldots\mu_{10}]} \\
&-210A_{[\mu_1\ldots\mu_4}A^{(\alpha}_{\mu_5\mu_6}A^{\beta}_{\mu_7\mu_8}\delta A^{\gamma)}_{\mu_9\mu_{10}]} + 105A^{(\alpha}_{[\mu_1\mu_2}A^{\beta}_{\mu_3\mu_4}A^{\gamma)}_{\mu_5\mu_6}\delta A_{\mu_7\ldots\mu_{10}]} \\
&+\frac{315i}{8}A^{(\alpha}_{[\mu_1\mu_2}A^{\beta}_{\mu_3\mu_4}A^{\gamma)}_{\mu_5\mu_6}\epsilon_{\delta\tau}A^{\delta}_{\mu_7\mu_8}\delta A^{\tau}_{\mu_9\mu_{10}]}\,.
\end{aligned} \tag{1.1.59}$$

Besides the quadruplet of 10-forms there also exists a doublet of 10-form potentials A_{10}^{α} satisfying the reality condition: $(A_{10}^1)^* = A_{10}^2$. In [22] the form of the supersymmetry transformation for A_{10}^{α} is given as

$$\begin{aligned}
\delta A^{\alpha}_{\mu_1\ldots\mu_{10}} \;=\;& V_-^{\alpha}\bar\epsilon\gamma_{\mu_1\ldots\mu_{10}}\lambda + V_+^{\alpha}\bar\epsilon_C\gamma_{\mu_1\ldots\mu_{10}}\lambda_C \\
&+20iV_+^{\alpha}\bar\epsilon\gamma_{[\mu_1\ldots\mu_9}\psi_{C\,\mu_{10}]} + 20iV_-^{\alpha}\bar\epsilon_C\gamma_{[\mu_1\ldots\mu_9}\psi_{\mu_{10}]}\,.
\end{aligned} \tag{1.1.60}$$

The commutator of two supersymmetries acting on A_{10}^{α} produces the following gauge transformation

$$\delta A_{10}^{\alpha} = d\Lambda_9^{\alpha}\,. \tag{1.1.61}$$

However, in [26] it is shown, using superspace techniques, that the Bianchi identity for the 11-form field strength F_{11}^{α} is given by

$$dF_{11}^{\alpha} = a\left(i\epsilon_{\beta\gamma}F_3^{\beta} \wedge F_9^{\gamma\alpha} + 3F_5 \wedge F_7^{\alpha}\right)\,, \tag{1.1.62}$$

where a is some nonzero real number. This Bianchi identity is solved by[3]

$$\begin{aligned} F_{11}^{\alpha} &= dA_{10}^{\alpha} + \frac{ai}{5}\epsilon_{\beta\gamma}A_2^{\beta}\wedge F_9^{\gamma\alpha} - \frac{4ai}{5}\epsilon_{\beta\gamma}F_3^{\beta}\wedge A_8^{\gamma\alpha} \\ &\quad + \frac{6a}{5}A_4\wedge F_7^{\alpha} - \frac{9a}{5}F_5\wedge A_6^{\alpha}\,, \end{aligned} \tag{1.1.63}$$

and implies the following gauge transformation for A_{10}^{α}

$$\begin{aligned} \delta A_{10}^{\alpha} &= d\tilde{\Lambda}_9^{\alpha} - \frac{ai}{5}\epsilon_{\beta\gamma}\Lambda_1^{\beta}\wedge F_9^{\gamma\alpha} - \frac{4ai}{5}\epsilon_{\beta\gamma}F_3^{\beta}\wedge\Lambda_7^{\gamma\alpha} \\ &\quad - \frac{6a}{5}\Lambda_3\wedge F_7^{\alpha} - \frac{9a}{5}F_5\wedge\Lambda_5^{\alpha}\,. \end{aligned} \tag{1.1.64}$$

There exists no redefinition of $\tilde{\Lambda}_9^{\alpha}$ appearing in (1.1.64) such that eqs. (1.1.61) and (1.1.64) are equivalent. There is thus a conflict situation if the 10-forms A_{10}^{α} in [22] and in [26] are the same. It is at present unclear what the resolution of the difference between eq. (1.1.61) and eq. (1.1.64) is (see also the discussion at the end of section (2.5)).

1.2 The scalars T and χ'

An $SU(1,1)$ charge tensor $q_{\alpha\beta} = q_{\beta\alpha}$ transforming in the adjoint of $SU(1,1)$ is introduced. Its components are raised and lowered with the $SU(1,1)$ invariant metric $\epsilon_{\alpha\beta}$,

$$q^{\alpha}{}_{\beta} = \epsilon^{\alpha\gamma}q_{\gamma\beta} \qquad \text{raising with the second index on } \epsilon\,, \tag{1.2.1}$$

$$q_{\alpha\beta} = \epsilon_{\gamma\alpha}q^{\gamma}{}_{\beta} \qquad \text{lowering with the first index on } \epsilon\,. \tag{1.2.2}$$

If one exponentiates the matrix $i\,q^{\alpha}{}_{\beta}$ (left upper index for the rows and the right lower index for the columns) the resulting matrix is an element of the group $SU(1,1)$. The charge tensor $q_{\alpha\beta}$ can be used to construct scalars that are invariant under local $U(1)$ transformations. Since the manifold $SU(1,1)/U(1)$ is two dimensional it can be parameterized using two scalars. Two scalars, denoted by T and χ', are defined as

$$T = q_{\alpha\beta}V_+^{\alpha}V_-^{\beta}\,, \tag{1.2.3}$$

$$d\chi' = -i\frac{P}{q_{\alpha\beta}V_+^{\alpha}V_+^{\beta}} + i\frac{\bar{P}}{q_{\alpha\beta}V_-^{\alpha}V_-^{\beta}}\,. \tag{1.2.4}$$

It is shown in appendix B that χ' is a parity odd scalar.

[3] In ten space-time dimensions it does not make sense to talk about a gauge invariant 11-form field strength since each term in (1.1.63) vanishes identically by itself. It is however useful as a tool to obtain the gauge transformation for A_{10}^{α}.

It will prove convenient to employ the following parametrization of $q_{\alpha\beta}$:

$$q_{\alpha\beta} = q_\alpha q_\beta + 4 \det Q \, \tilde{q}_\alpha \tilde{q}_\beta \,, \tag{1.2.5}$$

where q_α and $\tilde{q}_\beta$ satisfy

$$q_{[\alpha} \tilde{q}_{\beta]} = \frac{i}{2} \epsilon_{\alpha\beta} \,. \tag{1.2.6}$$

The reality properties of q_α and $\tilde{q}_\alpha$ are $(q_1)^* = q_2$ and $(\tilde{q}_1)^* = \tilde{q}_2$. The parameter $\det Q$ is the determinant of a matrix Q defined by

$$Q = -\frac{i}{2} S^{-1} \mathbf{q} S \,, \qquad S = \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \,, \tag{1.2.7}$$

where $\mathbf{q}$ is the matrix whose components are $q^\alpha{}_\beta$. The matrix S establishes the isomorphism between $SU(1,1)$ and $SL(2,\mathbb{R})$. The matrix e^Q is an element of $SL(2,\mathbb{R})$. When $\det Q = 0$ and $\det Q > 0$ the matrices e^Q form the $\mathbb{R}$ and $SO(2)$ subgroups of $SL(2,\mathbb{R})$, respectively[4]. In eq. (1.2.5) the value of $\det Q$ is independent of the parameters q_α and $\tilde{q}_\alpha$ so that when using (1.2.5) it can be freely send to zero. The limit $\det Q \to 0$ (through positive values of $\det Q$) corresponds to going from $SO(2)$ to $\mathbb{R}$.

It is possible to express dT using (1.1.13) and (1.1.14) in terms of P and $\bar{P}$. By combining the expressions for dT and $d\chi'$ the following expression for P, that is defined in eq. (1.1.7), in terms T and χ' can be obtained

$$P = \left(\frac{q_{\alpha\beta} V_+^\alpha V_+^\beta}{q_{\gamma\delta} V_-^\gamma V_-^\delta} \right)^{1/2} \left[\frac{1}{2} \frac{dT}{(T^2 - 4 \det Q)^{1/2}} + \frac{i}{2} (T^2 - 4 \det Q)^{1/2} d\chi' \right] \,, \tag{1.2.8}$$

where the identity

$$q_{\alpha\beta} V_+^\alpha V_+^\beta \, q_{\gamma\delta} V_-^\gamma V_-^\delta = T^2 - 4 \det Q \tag{1.2.9}$$

has been used. In deriving (1.2.9) one uses the completeness relation (1.1.5) and the parametrization (1.2.5). Since the left hand side of eq. (1.2.9) forms the modulus squared of a complex number the right hand side is non-negative. Hence it must be that $T^2 \geq 4 \det Q$. Further, from the definition of T, eq. (1.2.3), and the parametrization of $q_{\alpha\beta}$, eq. (1.2.5), it follows that T can be written as the sum of two squares, so that $T \geq 0$. This and eq. (1.2.9) implies $T \geq 2\sqrt{\det Q}$. The factor in front of the square brackets in (1.2.8) is immaterial and forms a local $U(1)$ phase factor. It can thus be gauged away using (1.1.10).

The defining equation for χ', eq. (1.2.4), will be solved for the cases $\det Q = 0$ and $\det Q > 0$ separately.

[4]For the purposes of this thesis the case $\det Q < 0$ will not be interesting for reasons explained in subsection 3.8.4.

1.2.1 The $\det Q = 0$ case

When $\det Q = 0$ the solution to eq. (1.2.4) is

$$\chi' = \frac{2q_{(\alpha}\tilde{q}_{\beta)}V_+^\alpha V_-^\beta}{q_\gamma q_\delta V_+^\gamma V_-^\delta} \tag{1.2.10}$$

The $U(1)$ gauge can be fixed by taking

$$q_\alpha V_+^\alpha = q_\alpha V_-^\alpha \,. \tag{1.2.11}$$

It then follows from this gauge choice together with the completeness relation (1.1.5) and the identity (1.2.9) that

$$q_\alpha V_+^\alpha = q_\alpha V_-^\alpha = T^{1/2} \,, \tag{1.2.12}$$

$$\tilde{q}_\alpha V_+^\alpha = (\tilde{q}_\alpha V_-^\alpha)^* = \frac{T^{1/2}}{2}\left(\chi' + \frac{i}{T}\right) \,. \tag{1.2.13}$$

1.2.2 The $\det Q > 0$ case

The $U(1)$ gauge symmetry can be fixed by taking

$$\frac{q_{\alpha\beta}V_+^\alpha V_+^\beta}{q_{\gamma\delta}V_-^\gamma V_-^\delta} = e^{4i\sqrt{\det Q}\,\chi'} \,. \tag{1.2.14}$$

It is also possible to write an expression for χ' without fixing the $U(1)$ gauge. This goes by solving eq. (1.2.4) for χ'.

When $\det Q > 0$ both the parameters q_α and $\tilde{q}_\alpha$ can be parameterized in terms of $q_{\alpha\beta}$. This is not possible when $\det Q = 0$ because in that case $q_{\alpha\beta}$, as is clear from eq. (1.2.5), does not depend on $\tilde{q}_\alpha$ and eq. (1.2.6) is not sufficient to find $\tilde{q}_\alpha$ in terms of $q_{\alpha\beta}$. This fact will be of some relevance later in subsection 1.3.3 when q_α and $\tilde{q}_\alpha$ will be parameterized in terms of $q_{\alpha\beta}$. When q_α and $\tilde{q}_\alpha$ are parameterized in terms of $q_{\alpha\beta}$ the parameter $\det Q$ depends on q_α and $\tilde{q}_\alpha$. It is then no longer possible to send $\det Q$ to zero independently of q_α and $\tilde{q}_\alpha$.

Using the parametrization (1.2.5) of $q_{\alpha\beta}$ eq. (1.2.4) can be solved for χ' giving

$$e^{4i\sqrt{\det Q}\,\chi'} = e^{i\theta}\frac{\left(q_\alpha + 2i\sqrt{\det Q}\tilde{q}_\alpha\right)V_+^\alpha}{\left(q_\gamma - 2i\sqrt{\det Q}\tilde{q}_\gamma\right)V_+^\gamma}\,\frac{\left(q_\beta + 2i\sqrt{\det Q}\tilde{q}_\beta\right)V_-^\beta}{\left(q_\delta - 2i\sqrt{\det Q}\tilde{q}_\delta\right)V_-^\delta} \,, \tag{1.2.15}$$

where $\theta \in \mathbb{R}$ is an integration constant.

For notational convenience the following combinations of q_α and $\tilde{q}_\alpha$ are defined:

$$s_\alpha = e^{-i\theta/4}\left(q_\alpha - 2i\sqrt{\det Q}\tilde{q}_\alpha\right) \,, \tag{1.2.16}$$

$$t_\alpha = e^{i\theta/4}\left(q_\alpha + 2i\sqrt{\det Q}\tilde{q}_\alpha\right) \,. \tag{1.2.17}$$

These combinations satisfy the properties:

$$q_{\alpha\beta} = t_{(\alpha}s_{\beta)}\,, \tag{1.2.18}$$
$$t_{[\alpha}s_{\beta]} = 2\sqrt{\det Q}\,\epsilon_{\alpha\beta}\,, \tag{1.2.19}$$
$$(t_1)^* = s_2\,, \qquad (t_2)^* = s_1\,. \tag{1.2.20}$$

By comparing the gauge fixed expression (1.2.14) with the ungauge fixed expression (1.2.15) the gauge fixing condition (1.2.14) can be rewritten as

$$s_\alpha V_+^\alpha = t_\alpha V_-^\alpha\,. \tag{1.2.21}$$

In this gauge fixed setting $V_\pm^\alpha$ are parameterized as:

$$t_\alpha V_-^\alpha = \left(T + 2\sqrt{\det Q}\right)^{1/2}\,, \tag{1.2.22}$$
$$s_\alpha V_+^\alpha = \left(T + 2\sqrt{\det Q}\right)^{1/2}\,, \tag{1.2.23}$$
$$s_\alpha V_-^\alpha = \left(T - 2\sqrt{\det Q}\right)^{1/2} e^{-2i\sqrt{\det Q}\,\chi'}\,, \tag{1.2.24}$$
$$t_\alpha V_+^\alpha = \left(T - 2\sqrt{\det Q}\right)^{1/2} e^{2i\sqrt{\det Q}\,\chi'}\,. \tag{1.2.25}$$

When the charges $q_{\alpha\beta}$ are not quantized it is possible to imagine the parameter $\sqrt{\det Q}$ to be infinitesimally small. If one expands the formulae of this subsection in the parameter $\sqrt{\det Q}$ and keeps only terms that are of first order in $\sqrt{\det Q}$ all the expressions of subsection 1.2.1 are recovered (assuming that the integration constant θ goes to zero as $\det Q \to 0$).

The 1-form $d\chi'$ is the Hodge dual of the 9-form $q_{\alpha\beta}F_9^{\alpha\beta}$ as is clear from eqs. (1.1.34) and (1.2.9),

$$q_{\alpha\beta}F_9^{\alpha\beta} = \star(T^2 - 4\det Q)d\chi'\,. \tag{1.2.26}$$

1.3 Supersymmetry transformation rules in terms of T and χ'

It is desirable to express the supersymmetry transformation rules, eqs. (1.1.46) to (1.1.54), in terms of the scalars T and χ' for the cases $\det Q > 0$ and $\det Q = 0$ separately. As the $\det Q = 0$ case can be treated via a limit procedure from the $\det Q > 0$ case, the latter is treated first.

1.3.1 The $\det Q > 0$ case

It is not possible to apply the $U(1)$ gauge fixing condition (1.2.21) straightforwardly to the supersymmetry transformation rules (1.1.47) and (1.1.48) because the supersymmetry transformations (1.1.47) and (1.1.48) do not respect the gauge choice (1.2.21). This problem can be cured by redefining the local supersymmetry transformation δ with a local $U(1)$ transformation as

$$\tilde{\delta} = \delta + \delta_{U(1)} \,. \tag{1.3.1}$$

Then under $\tilde{\delta}$ the objects $V_\pm^\alpha$ transform as

$$\tilde{\delta} V_+^\alpha = V_-^\alpha \bar{\epsilon}_C \lambda - iS\, V_+^\alpha \,, \tag{1.3.2}$$

$$\tilde{\delta} V_-^\alpha = V_+^\alpha \bar{\epsilon} \lambda_C + iS\, V_-^\alpha \,, \tag{1.3.3}$$

where the second terms in eqs. (1.3.2) and (1.3.3) correspond to infinitesimal $U(1)$ transformations in which for S the following Ansatz is taken

$$S = a\bar{\epsilon}_C \lambda + a^* \bar{\epsilon} \lambda_C \,. \tag{1.3.4}$$

The condition that is imposed on the redefined supersymmetry transformation $\tilde{\delta}$ in order that it preserves the gauge choice (1.2.21) is

$$s_\alpha \tilde{\delta} V_+^\alpha = t_\alpha \tilde{\delta} V_-^\alpha \,. \tag{1.3.5}$$

The condition (1.3.5) requires a in (1.3.4) to be

$$a = -\frac{i}{2} \frac{s_\alpha V_-^\alpha}{s_\beta V_+^\beta} \,. \tag{1.3.6}$$

The redefined supersymmetry transformation $\tilde{\delta}$ only acts non-trivially on the scalars, i.e. on all other fields $\tilde{\delta} = \delta$ (up to second order in fermions). The reason is that the non-scalar bosonic fields do not transform under local $U(1)$ and the fermions do not see this local $U(1)$ as long as one works up to second order in fermions.

The objects P, Q and G_3 as defined in (1.1.7), (1.1.8) and (1.1.21), respectively can be expressed in terms of T and χ'. For this purpose one can use eq. (1.2.19) giving the relation between $\epsilon_{\alpha\beta}$, that appears in the definitions of P, Q and G_3, and t_α and s_α. Then one can employ the parametrization of $V_\pm^\alpha$ as given in eqs. (1.2.22) to (1.2.25) and the freedom to perform the local $U(1)$ transformation $P \to e^{-2i\alpha} P$, $Q \to Q - d\alpha$ and $G_3 \to e^{-i\alpha} G_3$ with parameter α given by

$$e^{-2i\alpha} = \left(\frac{q_{\alpha\beta} V_-^\alpha V_-^\beta}{q_{\gamma\delta} V_+^\gamma V_+^\delta} \right)^{1/2} \,, \tag{1.3.7}$$

to find

$$
\begin{aligned}
P &= \frac{1}{2}\frac{dT}{(T^2-4\det Q)^{1/2}} + \frac{i}{2}(T^2-4\det Q)^{1/2}d\chi'\,, && (1.3.8)\\
Q &= \frac{T}{2}\,d\chi'\,, && (1.3.9)\\
G_3 &= \frac{1}{4\sqrt{\det Q}}\left[\left(T+2\sqrt{\det Q}\right)^{1/2} e^{-i\sqrt{\det Q}\,\chi'} t_\alpha F_3^\alpha - \right.\\
&\quad \left.\left(T-2\sqrt{\det Q}\right)^{1/2} e^{i\sqrt{\det Q}\,\chi'} s_\alpha F_3^\alpha\right]\,, && (1.3.10)\\
F_5 &= dA_4 + \frac{i}{64\sqrt{\det Q}}\left(t_\alpha A_2^\alpha \wedge s_\beta F_3^\beta - s_\alpha A_2^\alpha \wedge t_\beta F_3^\beta\right)\,, && (1.3.11)
\end{aligned}
$$

where also the 5-form F_5 is given in terms of the 2-forms $s_\alpha A_2^\alpha$ and $t_\alpha A_2^\alpha$. Further also the spinors λ, ψ_μ and ϵ are redefined as

$$
\lambda' = e^{-3i\alpha/2}\lambda\,, \qquad \psi'_\mu = e^{-i\alpha/2}\psi_\mu\,, \qquad \epsilon' = e^{-i\alpha/2}\epsilon\,. \qquad (1.3.12)
$$

In terms of these primed spinors and the redefined supersymmetry transformation $\tilde{\delta}$ the supersymmetry transformation rules in terms of T and χ' read

$$
\begin{aligned}
\tilde{\delta} g_{\mu\nu} &= 2i\,\bar{\epsilon}'\gamma_{(\mu}\psi'_{\nu)} + 2i\,\bar{\epsilon}'_C\gamma_{(\mu}\psi'_{C\,\nu)}\,, && (1.3.13)\\
\tilde{\delta} T &= \left(T^2-4\det Q\right)^{1/2}\left(\bar{\epsilon}'_C\lambda' + \bar{\epsilon}'\lambda'_C\right)\,, && (1.3.14)\\
\tilde{\delta}\chi' &= \frac{i}{\left(T^2-4\det Q\right)^{1/2}}\left(\bar{\epsilon}'_C\lambda' - \bar{\epsilon}'\lambda'_C\right)\,, && (1.3.15)\\
t_\alpha\tilde{\delta}A^\alpha_{\mu\nu} &= e^{i\sqrt{\det Q}\,\chi'}\left[\left(T+2\sqrt{\det Q}\right)^{1/2}\bar{\epsilon}'\gamma_{\mu\nu}\lambda'\right.\\
&\quad + \left(T-2\sqrt{\det Q}\right)^{1/2}\bar{\epsilon}'_C\gamma_{\mu\nu}\lambda'_C\\
&\quad +4i\left(T+2\sqrt{\det Q}\right)^{1/2}\bar{\epsilon}'_C\gamma_{[\mu}\psi'_{\nu]}\\
&\quad \left.+4i\left(T-2\sqrt{\det Q}\right)^{1/2}\bar{\epsilon}'\gamma_{[\mu}\psi'_{C\,\nu]}\right]\,,\\
& && (1.3.16)\\
\delta A_{\mu\nu\rho\sigma} &= \bar{\epsilon}'\gamma_{[\mu\nu\rho}\psi'_{\sigma]} - \bar{\epsilon}'_C\gamma_{[\mu\nu\rho}\psi'_{C\,\sigma]} +\\
&\quad \frac{3i}{32\sqrt{\det Q}}\left(s_\alpha A^\alpha_{[\mu\nu}\,t_\beta\tilde{\delta}A^\beta_{\rho\sigma]} - t_\alpha A^\alpha_{[\mu\nu}\,s_\beta\tilde{\delta}A^\beta_{\rho\sigma]}\right)\,, && (1.3.17)
\end{aligned}
$$

$$
\begin{aligned}
t_\alpha\tilde{\delta}A^\alpha_{\mu_1\ldots\mu_6} &= e^{i\sqrt{\det Q}\,\chi'}\Big[i\left(T+2\sqrt{\det Q}\right)^{1/2}\bar{\epsilon}'\gamma_{\mu_1\ldots\mu_6}\lambda' \\
&\quad -i\left(T-2\sqrt{\det Q}\right)^{1/2}\bar{\epsilon}'_C\gamma_{\mu_1\ldots\mu_6}\lambda'_C \\
&\quad +12\left(T+2\sqrt{\det Q}\right)^{1/2}\bar{\epsilon}'_C\gamma_{[\mu_1\ldots\mu_5}\psi'_{\mu_6]} \\
&\quad -12\left(T-2\sqrt{\det Q}\right)^{1/2}\bar{\epsilon}'\gamma_{[\mu_1\ldots\mu_5}\psi'_{C\,\mu_6]}\Big] \\
&\quad +40A_{[\mu_1\ldots\mu_4}\,t_\alpha\tilde{\delta}A^\alpha_{\mu_5\mu_6]} - 20\tilde{\delta}A_{[\mu_1\ldots\mu_4}\,t_\alpha A^\alpha_{\mu_5\mu_6]} \\
&\quad -\frac{15i}{8\sqrt{\det Q}}\,t_\alpha A^\alpha_{[\mu_1\mu_2}\left(t_\beta A^\beta_{\mu_3\mu_4}\,s_\gamma\tilde{\delta}A^\gamma_{\mu_5\mu_6]} - s_\beta A^\beta_{\mu_3\mu_4}\,t_\gamma\tilde{\delta}A^\gamma_{\mu_5\mu_6]}\right)\,,
\end{aligned}
\tag{1.3.18}
$$

$$
\begin{aligned}
q_{\alpha\beta}\tilde{\delta}A^{\alpha\beta}_{\mu_1\ldots\mu_8} &= -i\left(T^2-4\det Q\right)^{1/2}\left(\bar{\epsilon}'\gamma_{\mu_1\ldots\mu_8}\lambda'_C - \bar{\epsilon}'_C\gamma_{\mu_1\ldots\mu_8}\lambda'\right) \\
&\quad +8T\left(\bar{\epsilon}'\gamma_{[\mu_1\ldots\mu_7}\psi'_{\mu_8]} - \bar{\epsilon}'_C\gamma_{[\mu_1\ldots\mu_7}\psi'_{C\,\mu_8]}\right) \\
&\quad +\frac{21}{8}\,t_\alpha A^\alpha_{[\mu_1\ldots\mu_6}\,s_\beta\tilde{\delta}A^\beta_{\mu_7\mu_8]} + \frac{21}{8}\,s_\alpha A^\alpha_{[\mu_1\ldots\mu_6}\,t_\beta\tilde{\delta}A^\beta_{\mu_7\mu_8]} \\
&\quad -\frac{7}{8}\,t_\alpha A^\alpha_{[\mu_1\mu_2}\,s_\beta\tilde{\delta}A^\beta_{\mu_3\ldots\mu_8]} - \frac{7}{8}\,s_\alpha A^\alpha_{[\mu_1\mu_2}\,t_\beta\tilde{\delta}A^\beta_{\mu_3\ldots\mu_8]} \\
&\quad +\frac{70}{2}A_{[\mu_1\ldots\mu_4}\,t_\alpha A^\alpha_{\mu_5\mu_6}\,s_\beta\tilde{\delta}A^\beta_{\mu_7\mu_8]} + \frac{70}{2}A_{[\mu_1\ldots\mu_4}\,s_\alpha A^\alpha_{\mu_5\mu_6}\,t_\beta\tilde{\delta}A^\beta_{\mu_7\mu_8]} \\
&\quad -35\,t_\alpha A^\alpha_{[\mu_1\mu_2}\,s_\beta A^\beta_{\mu_3\mu_4}\tilde{\delta}A_{\mu_5\ldots\mu_8]} \\
&\quad -\frac{105i}{64\sqrt{\det Q}}\left(t_\alpha A^\alpha_{[\mu_1\mu_2}\,s_\beta A^\beta_{\mu_3\mu_4} + s_\alpha A^\alpha_{[\mu_1\mu_2}\,t_\beta A^\beta_{\mu_3\mu_4}\right)\times \\
&\quad \times\left(t_\gamma A^\gamma_{\mu_5\mu_6}\,s_\delta\tilde{\delta}A^\delta_{\mu_7\mu_8]} - s_\gamma A^\gamma_{\mu_5\mu_6}\,t_\delta\tilde{\delta}A^\delta_{\mu_7\mu_8]}\right)\,,
\end{aligned}
\tag{1.3.19}
$$

$$
\tilde{\delta}\lambda' = iP_\mu\gamma^\mu\epsilon'_C - \frac{i}{24}G_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon'\,,
\tag{1.3.20}
$$

$$
\begin{aligned}
\tilde{\delta}\psi'_\mu &= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{i}{2}Q_\mu\right)\epsilon' + \frac{i}{480}F_{\mu\nu_1\ldots\nu_4}\gamma^{\nu_1\ldots\nu_4}\epsilon' \\
&\quad +\frac{1}{96}G^{\nu\rho\sigma}\gamma_{\mu\nu\rho\sigma}\epsilon'_C - \frac{3}{32}G_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon'_C\,,
\end{aligned}
\tag{1.3.21}
$$

with P, Q, G_3 and F_5 as given in eqs. (1.3.8) to (1.3.11). The supersymmetry transformation rules for $s_\alpha A^\alpha_{2,6}$ follow by taking the complex conjugate of eqs. (1.3.16) and (1.3.18).

1.3.2 The scalars η and φ and the p-forms $\mathcal{A}_p$

The scalar kinetic terms for T and χ' that follow from the term $2\star P\wedge\bar{P}$ in the action (1.1.17) are given by

$$2\star P\wedge\bar{P}=\frac{1}{2}\frac{\star dT\wedge dT}{T^2-4\det Q}+\frac{1}{2}\left(T^2-4\det Q\right)\star d\chi'\wedge d\chi'\,. \qquad (1.3.22)$$

The kinetic term for T is not canonically normalized. Further, below eq. (1.2.9) it is concluded that $T\geq 2\sqrt{\det Q}$. The scalar $2\sqrt{\det Q}\,\chi'$ for the case $\det Q>0$ is an angular variable with period 2π as is clear from eq. (1.2.24) or (1.2.25): $2\sqrt{\det Q}\,\chi'$ and $2\sqrt{\det Q}\,\chi'+2\pi$ are to be identified. In order to work with canonically normalized scalars the scalars η and φ are introduced via

$$\frac{T}{2\sqrt{\det Q}} = \cosh\eta\,, \qquad (1.3.23)$$

$$2\sqrt{\det Q}\,\chi' = \varphi\,, \qquad (1.3.24)$$

where $0\leq\eta<\infty$ and $0\leq\varphi<2\pi$. Eq. (1.3.22) then becomes equal to

$$\frac{1}{2}\star d\eta\wedge d\eta+\frac{1}{2}\sinh^2\eta\star d\varphi\wedge d\varphi\,. \qquad (1.3.25)$$

The geometrical interpretation of η and φ as coordinates on the moduli space will be given in section 1.4.

The supersymmetry transformation rules eqs. (1.3.13) to (1.3.21) can readily be rewritten in terms of η and φ. When doing so it is, for notational purposes, convenient to write the supersymmetry transformations rules for the form fields in terms of the following 2,6,8-forms, denoted by $\mathcal{A}_{2,6,8}$, and defined via

$$\mathcal{A}_{2,6} = \frac{t_\alpha A^\alpha_{2,6}}{2(\det Q)^{1/4}}\,, \qquad (1.3.26)$$

$$\mathcal{A}_8 = q_{\alpha\beta}A^{\alpha\beta}_8\,. \qquad (1.3.27)$$

The 4-form A_4 does require such a redefinition. The 2- and 6-forms $\mathcal{A}_{2,6}$ are complex-valued while the 8-form $\mathcal{A}_8$ is real-valued. When working with the fields η, φ, $\mathcal{A}_{2,6,8}$ it is assumed that $\det Q$ is some fixed non-zero number. In this formulation the following symmetry is manifest:

$$\eta \to \eta\,,\quad (T\to T) \qquad (1.3.28)$$

$$\varphi \to \varphi+2\sqrt{\det Q}\,,\quad (\chi'\to\chi'+1) \qquad (1.3.29)$$

$$\mathcal{A}_{2,6} \to e^{i\sqrt{\det Q}}\mathcal{A}_{2,6}\,, \qquad (1.3.30)$$

$$\mathcal{A}_8 \to \mathcal{A}_8\,, \qquad (1.3.31)$$

$$\lambda' \to \lambda'\,, \qquad (1.3.32)$$

$$\psi'_\mu \to \psi'_\mu\,,\qquad \epsilon'\to\epsilon'\,. \qquad (1.3.33)$$

This set of transformation forms the $SO(2)$ subgroup of the classical $SL(2,\mathbb{R})$ duality group.

It is straightforward to write down the IIB action or the equations of motion in terms of the scalars η and φ and the form fields $\mathcal{A}_2$ and A_4. This can be achieved by substituting the definitions (1.3.23), (1.3.24) and (1.3.26) into the expressions for P, G_3 and F_5 as given in eqs. (1.3.8), (1.3.10) and (1.3.11) in terms of which the action (1.1.17) is written. The result is:

$$P = \frac{1}{2}d\eta + \frac{i}{2}\sinh\eta\, d\varphi\,, \tag{1.3.34}$$

$$Q = \frac{1}{2}\cosh\eta\, d\varphi\,, \tag{1.3.35}$$

$$G_3 = \cosh\tfrac{\eta}{2}\, e^{-i\varphi/2}d\mathcal{A}_2 - \sinh\tfrac{\eta}{2}\, e^{i\varphi/2}d\bar{\mathcal{A}}_2\,, \tag{1.3.36}$$

$$F_5 = dA_4 + \frac{i}{16}\left(\mathcal{A}_2\wedge d\bar{\mathcal{A}}_2 - \bar{\mathcal{A}}_2\wedge d\mathcal{A}_2\right)\,. \tag{1.3.37}$$

1.3.3 The (p,q,r) parametrization

In eq. (1.2.7) the matrix Q was defined in terms of the numbers $q_{\alpha\beta}$. The traceless $SL(2,\mathbb{R})$ algebra valued charge matrix Q can be parameterized by three numbers, that will be denoted by p,q,r, as

$$Q = \begin{pmatrix} r/2 & p \\ -q & -r/2 \end{pmatrix}\,. \tag{1.3.38}$$

By writing p and q in terms of t_α and s_α using eqs. (1.3.38), (1.2.7) and (1.2.5) it can be shown that $p,q\geq 0$.

The purpose of introducing the numbers p,q,r is two-fold. On the one hand it will enable one to relate the variables η and φ to the complex axidilaton field τ in terms of which IIB supergravity is often formulated. On the other hand from the point of view of quantization the $SL(2,\mathbb{R})$ covariant formulation is more convenient than the $SU(1,1)$ covariant formulation.

Consider the complex axidilaton field $\tau=\chi+ie^{-\phi}$ with χ the RR scalar and ϕ the dilaton that transforms as

$$\Lambda\tau = \frac{a\tau+b}{c\tau+d} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Lambda = e^{Q}\,. \tag{1.3.39}$$

A point τ_0 is a fixed point under the e^Q transformation if it satisfies the equation

$$e^Q\tau_0 = \tau_0\,. \tag{1.3.40}$$

The fixed points τ_0 of e^Q with $0\leq \mathrm{Im}\,\tau_0<\infty$ for $\det Q\geq 0$ are given by

$$\tau_0 = -\frac{r}{2q} + \frac{i}{q}\sqrt{\det Q}\,. \tag{1.3.41}$$

The relation between (η, φ) (or (T, χ')) and $(\tau, \bar{\tau})$ is established by taking, within the $U(1)$ gauge choice (1.2.21),

$$\frac{t_\alpha V_+^\alpha}{s_\beta V_+^\beta} = \frac{\tau - \tau_0}{\tau - \bar{\tau}_0} . \tag{1.3.42}$$

The left hand side of eq. (1.3.42) can be related to T and χ' via eqs. (1.2.23) and (1.2.25). Transforming T and χ' as in eqs. (1.3.28) and (1.3.29) can be seen to be equivalent to transforming $\tau \to e^Q \tau$ with $\det Q > 0$ under which the right hand side of eq. (1.3.42) transforms as

$$\frac{\tau - \tau_0}{\tau - \bar{\tau}_0} \to e^{2i\sqrt{\det Q}} \frac{\tau - \tau_0}{\tau - \bar{\tau}_0} , \tag{1.3.43}$$

Next, for the 2- 6- and 8-forms, the combinations, denoted by B_2, C_2, C_6, B_6, C_8, B_8 and D_8 that are defined via

$$\begin{aligned}
B_2 &= \frac{1}{2}(A_2^1 + A_2^2) , && (1.3.44)\\
C_2 &= \frac{i}{2}(A_2^1 - A_2^2) , && (1.3.45)\\
C_6 &= \frac{1}{2}(A_6^1 + A_6^2) , && (1.3.46)\\
B_6 &= \frac{i}{2}(A_6^1 - A_6^2) , && (1.3.47)\\
C_8 &= A_8^{11} + A_8^{22} + 2A_8^{12} , && (1.3.48)\\
B_8 &= -A_8^{11} - A_8^{22} + 2A_8^{12} , && (1.3.49)\\
D_8 &= 2iA_8^{11} - 2iA_8^{22} . && (1.3.50)
\end{aligned}$$

are taken to transform under $SL(2, \mathbb{R})$ as:

$$\begin{aligned}
\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} &\to \Lambda \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} , && (1.3.51)\\
\begin{pmatrix} B_6 \\ C_6 \end{pmatrix} &\to \Lambda \begin{pmatrix} B_6 \\ C_6 \end{pmatrix} , && (1.3.52)\\
\begin{pmatrix} \frac{1}{2}D_8 & -B_8 \\ C_8 & -\frac{1}{2}D_8 \end{pmatrix} &\to \Lambda \begin{pmatrix} \frac{1}{2}D_8 & -B_8 \\ C_8 & -\frac{1}{2}D_8 \end{pmatrix} \Lambda^{-1} , && (1.3.53)
\end{aligned}$$

where

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \qquad \text{with} \qquad \det \Lambda = 1 . \tag{1.3.54}$$

The 2- and 6-forms B_2 and B_6 denote the NSNS 2- and 6-forms, respectively. While the 2-, 6- and 8-forms C_2, C_6 and C_8, represent the RR 2-, 6- and 8-forms, respectively[5]. Finally, the 8-form D_8 is needed in order to have explicit $SL(2,\mathbb{R})$ covariance. The 2- and 6-forms transform non-trivially under the full $SL(2,\mathbb{R})$ group while the 8-forms only transform non-trivially under the $PSL(2,\mathbb{R})$ group.

In terms of the NSNS and RR fields the form fields $\mathcal{A}_{2,6,8}$ can be written as:

$$\mathcal{A}_2 = \frac{-i}{(\mathrm{Im}\,\tau_0)^{1/2}}\left(-C_2 + \tau_0 B_2\right) , \tag{1.3.55}$$

$$\mathcal{A}_6 = \frac{-i}{(\mathrm{Im}\,\tau_0)^{1/2}}\left(-B_6 + \tau_0 C_6\right) , \tag{1.3.56}$$

$$\mathcal{A}_8 = pC_8 + qB_8 + \tfrac{r}{2}D_8 , \tag{1.3.57}$$

where the relation between the parameter t_α appearing in (1.3.26) and in (1.3.27) (through eqs. (1.2.18) and (1.2.20)) and the parameter τ_0 is given by

$$\frac{t_1}{2(\det Q)^{1/4}} = \frac{-i}{2(\mathrm{Im}\,\tau_0)^{1/2}}(\tau_0 - i) , \tag{1.3.58}$$

$$\frac{t_2}{2(\det Q)^{1/4}} = \frac{-i}{2(\mathrm{Im}\,\tau_0)^{1/2}}(\tau_0 + i) . \tag{1.3.59}$$

If one then transforms $\mathcal{A}_{2,6,8}$ as given in eqs. (1.3.55), (1.3.56) and (1.3.57) under the $SL(2,\mathbb{R})$ transformations (1.3.51), (1.3.52) and (1.3.53) with $\Lambda = e^Q$ the transformation rules (1.3.30) and (1.3.31) are recovered for the case $\det Q > 0$.

Associated with the matrix of 8-form potentials transforming in the adjoint of $SL(2,\mathbb{R})$, eq. (1.3.53), there is a matrix of 9-form field strengths that can be seen to be dual to $-2S^{-1}pS$, i.e.

$$-2S^{-1}pS = \star\begin{pmatrix} \frac{1}{2}dD_8 & -dB_8 \\ dC_8 & -\frac{1}{2}dD_8 \end{pmatrix} + \cdots \tag{1.3.60}$$

with S as defined in (1.2.7) and with p as defined in (1.1.15). The dots in (1.3.60) denote contributions from wedge products of lower rank potentials that can be obtained using eq. (1.1.36) and eqs. (1.3.48) to (1.3.50). The matrix $-2S^{-1}pS$ is the matrix of Noether currents that derive from the $SL(2,\mathbb{R})$ invariance of the IIB action.

1.3.4 The $\det Q = 0$ case

The supersymmetry transformation rules in terms of T and χ' for the $\det Q = 0$ case can be obtained by taking the formulae (1.3.13) to (1.3.21) and treating $\sqrt{\det Q}$ as an

[5]From the point of view of the gauge transformations the fields $C_{6,8}$ (as well as the 4-form A_4) are not yet quite the RR gauge potentials. One can redefine the potentials with wedge products of lower rank potentials such that the gauge transformations are those of the RR potentials. The same comment applies to the NSNS 6-form B_6. These redefinitions with lower rank potentials are given in [27].

infinitesimally small parameter keeping only terms that are of first order in $\sqrt{\det Q}$. In order to take the $\det Q \to 0$ limit for the 2-forms it is convenient to take the limit of the following real combinations:

$$\frac{1}{2}\left(t_\alpha \tilde{\delta} A^\alpha_{\mu\nu} + s_\alpha \tilde{\delta} A^\alpha_{\mu\nu}\right) , \tag{1.3.61}$$

$$\frac{1}{2i}\left(t_\alpha \tilde{\delta} A^\alpha_{\mu\nu} - s_\alpha \tilde{\delta} A^\alpha_{\mu\nu}\right) , \tag{1.3.62}$$

and likewise for the 6-forms. Before stating the full result of the limit procedure it is mentioned that the primed spinors as defined in (1.3.12), at zeroth order in $\sqrt{\det Q}$, are equal to the unprimed spinors and that the first order relations between the primed and unprimed spinors in taking the $\det Q \to 0$ of (1.3.13) to (1.3.21) are never needed, so that the prime on λ, ψ_μ and ϵ can be dropped. The resulting supersymmetry transformation rules are:

$$\begin{aligned}
\tilde{\delta} g_{\mu\nu} &= 2i\,\bar{\epsilon}\gamma_{(\mu}\psi_{\nu)} + 2i\,\bar{\epsilon}_C\gamma_{(\mu}\psi_{C\,\nu)} \,, && (1.3.63)\\
\tilde{\delta} T &= T\bar{\epsilon}_C\lambda + T\bar{\epsilon}\lambda_C \,, && (1.3.64)\\
\tilde{\delta}\chi' &= \frac{i}{T}\,\bar{\epsilon}_C\lambda - \frac{i}{T}\,\bar{\epsilon}\lambda_C \,, && (1.3.65)\\
q_\alpha \tilde{\delta} A^\alpha_{\mu\nu} &= T^{1/2}\bar{\epsilon}\gamma_{\mu\nu}\lambda + T^{1/2}\bar{\epsilon}_C\gamma_{\mu\nu}\lambda_C + 4iT^{1/2}\bar{\epsilon}_C\gamma_{[\mu}\psi_{\nu]} \\
&\quad +4iT^{1/2}\bar{\epsilon}\gamma_{[\mu}\psi_{C\,\nu]} \,, && (1.3.66)\\
\tilde{q}_\alpha \tilde{\delta} A^\alpha_{\mu\nu} &= -\frac{i}{2}T^{-1/2}\bar{\epsilon}\gamma_{\mu\nu}\lambda + \frac{i}{2}T^{-1/2}\bar{\epsilon}_C\gamma_{\mu\nu}\lambda_C + 2T^{-1/2}\bar{\epsilon}_C\gamma_{[\mu}\psi_{\nu]} \\
&\quad -2T^{-1/2}\bar{\epsilon}\gamma_{[\mu}\psi_{C\,\nu]} + \frac{1}{2}\chi' q_\alpha \tilde{\delta} A^\alpha_{\mu\nu} \,, && (1.3.67)\\
\delta A_{\mu\nu\rho\sigma} &= \bar{\epsilon}\gamma_{[\mu\nu\rho}\psi_{\sigma]} - \bar{\epsilon}_C\gamma_{[\mu\nu\rho}\psi_{C\,\sigma]} \\
&\quad -\frac{3}{8}\,q_\alpha A^\alpha_{[\mu\nu}\,\tilde{q}_\beta\tilde{\delta}A^\beta_{\rho\sigma]} + \frac{3}{8}\,\tilde{q}_\alpha A^\alpha_{[\mu\nu}\,q_\beta\tilde{\delta}A^\beta_{\rho\sigma]} \,, && (1.3.68)\\
q_\alpha \tilde{\delta} A^\alpha_{\mu_1\ldots\mu_6} &= iT^{1/2}\bar{\epsilon}\gamma_{\mu_1\ldots\mu_6}\lambda - iT^{1/2}\bar{\epsilon}_C\gamma_{\mu_1\ldots\mu_6}\lambda_C \\
&\quad +12T^{1/2}\bar{\epsilon}_C\gamma_{[\mu_1\ldots\mu_5}\psi_{\mu_6]} - 12T^{1/2}\bar{\epsilon}\gamma_{[\mu_1\ldots\mu_5}\psi_{C\,\mu_6]} \\
&\quad +40A_{[\mu_1\ldots\mu_4}\,q_\alpha\tilde{\delta}A^\alpha_{\mu_5\mu_6]} - 20\tilde{\delta}A_{[\mu_1\ldots\mu_4}\,q_\alpha A^\alpha_{\mu_5\mu_6]} \\
&\quad -\frac{15}{2}\,q_\alpha A^\alpha_{[\mu_1\mu_2}\left(q_\beta A^\beta_{\mu_3\mu_4}\,\tilde{q}_\gamma\tilde{\delta}A^\gamma_{\mu_5\mu_6]} - \tilde{q}_\beta A^\beta_{\mu_3\mu_4}\,q_\gamma\tilde{\delta}A^\gamma_{\mu_5\mu_6]}\right) , && (1.3.69)\\
\tilde{q}_\alpha \tilde{\delta} A^\alpha_{\mu_1\ldots\mu_6} &= \frac{i}{2}T^{1/2}(\chi' - \frac{i}{T})\bar{\epsilon}\gamma_{\mu_1\ldots\mu_6}\lambda - \frac{i}{2}T^{1/2}(\chi' + \frac{i}{T})\bar{\epsilon}_C\gamma_{\mu_1\ldots\mu_6}\lambda_C \\
&\quad +6T^{1/2}(\chi' - \frac{i}{T})\bar{\epsilon}_C\gamma_{[\mu_1\ldots\mu_5}\psi_{\mu_6]} - T^{1/2}(\chi' + \frac{i}{T})\bar{\epsilon}\gamma_{[\mu_1\ldots\mu_5}\psi_{C\,\mu_6]}
\end{aligned}$$

$$
\begin{aligned}
& +40A_{[\mu_1\ldots\mu_4}\,\tilde{q}_\alpha\tilde{\delta}A^\alpha_{\mu_5\mu_6]} - 20\tilde{\delta}A_{[\mu_1\ldots\mu_4}\,\tilde{q}_\alpha A^\alpha_{\mu_5\mu_6]} \\
& -\frac{15}{2}\,\tilde{q}_\alpha A^\alpha_{[\mu_1\mu_2}\left(q_\beta A^\beta_{\mu_3\mu_4}\,\tilde{q}_\gamma\tilde{\delta}A^\gamma_{\mu_5\mu_6]} - \tilde{q}_\beta A^\beta_{\mu_3\mu_4}\,q_\gamma\tilde{\delta}A^\gamma_{\mu_5\mu_6]}\right)\,, \quad (1.3.70)
\end{aligned}
$$

$$
\begin{aligned}
q_\alpha q_\beta\tilde{\delta}A^{\alpha\beta}_{\mu_1\ldots\mu_8} \;=\;& -iT\bar{\epsilon}\gamma_{\mu_1\ldots\mu_8}\lambda_C + iT\bar{\epsilon}_C\gamma_{\mu_1\ldots\mu_8}\lambda \\
& +8T\,\bar{\epsilon}\gamma_{[\mu_1\ldots\mu_7}\psi_{\mu_8]} - 8T\,\bar{\epsilon}_C\gamma_{[\mu_1\ldots\mu_7}\psi_{C\,\mu_8]} \\
& +\frac{21}{4}\,q_\alpha A^\alpha_{[\mu_1\ldots\mu_6}\,q_\beta\tilde{\delta}A^\beta_{\mu_7\mu_8]} - \frac{7}{4}\,q_\alpha A^\alpha_{[\mu_1\mu_2}\,q_\beta\tilde{\delta}A^\beta_{\mu_3\ldots\mu_8]} \\
& -35\,q_\alpha A^\alpha_{[\mu_1\mu_2}\,q_\beta A^\beta_{\mu_3\mu_4}\,\tilde{\delta}A_{\mu_5\ldots\mu_8]} + 70A_{[\mu_1\ldots\mu_4}\,q_\alpha A^\alpha_{\mu_5\mu_6}\,q_\beta\tilde{\delta}A^\beta_{\mu_7\mu_8]} \\
& -\frac{105}{8}\,q_\alpha A^\alpha_{[\mu_1\mu_2}\,q_\beta A^\beta_{\mu_3\mu_4}\left(q_\gamma A^\gamma_{\mu_5\mu_6}\,\tilde{q}_\delta\tilde{\delta}A^\delta_{\mu_7\mu_8]} - \tilde{q}_\gamma A^\gamma_{\mu_5\mu_6}\,q_\delta\tilde{\delta}A^\delta_{\mu_7\mu_8]}\right)\,,
\end{aligned}
\quad (1.3.71)
$$

$$
\tilde{\delta}\lambda = iP_\mu\gamma^\mu\epsilon_C - \frac{i}{24}G_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon\,, \quad (1.3.72)
$$

$$
\begin{aligned}
\tilde{\delta}\psi_\mu \;=\;& \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{i}{2}Q_\mu\right)\epsilon + \frac{i}{480}F_{\mu\nu_1\ldots\nu_4}\gamma^{\nu_1\ldots\nu_4}\epsilon \\
& +\frac{1}{96}G^{\nu\rho\sigma}\gamma_{\mu\nu\rho\sigma}\epsilon_C - \frac{3}{32}G_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon_C\,,
\end{aligned}
\quad (1.3.73)
$$

with P, Q, G_3 and F_5 given by

$$
P = \frac{1}{2}\frac{dT}{T} + \frac{i}{2}T\,d\chi'\,, \quad (1.3.74)
$$

$$
Q = \frac{T}{2}\,d\chi'\,, \quad (1.3.75)
$$

$$
G_3 = iT^{1/2}\tilde{q}_\alpha F^\alpha_3 - \frac{i}{2}T^{1/2}(\chi' + \frac{i}{T})q_\alpha F^\alpha_3\,, \quad (1.3.76)
$$

$$
F_5 = dA_4 + \frac{1}{16}\left(q_\alpha A^\alpha_2\wedge\tilde{q}_\beta F^\beta_3 - \tilde{q}_\alpha A^\alpha_2\wedge q_\beta F^\beta_3\right)\,. \quad (1.3.77)
$$

For the following pair of values of q_α and $\tilde{q}_\alpha$

$$
q_1 = q_2 = \frac{y}{2}\,, \qquad \tilde{q}_1 = -\tilde{q}_2 = -\frac{i}{y}\,, \quad (1.3.78)
$$

it is customary to write

$$
T^{1/2} = \sqrt{p}\frac{1}{(\mathrm{Im}\,\tau)^{1/2}}\,, \quad (1.3.79)
$$

$$
\chi' = \frac{1}{p}\,\chi\,, \quad (1.3.80)
$$

where p is an arbitrary real but positive number. In fact p is one of the three parameters p, q, r that parameterize Q, see eq. (1.3.38). The complex field τ is the axidilaton field: $\tau = \chi + ie^{-\phi}$ where χ is the RR axion and ϕ is the dilaton. For the choice (1.3.78) and with the definitions (1.3.79) and (1.3.80) the parametrization of $V_{\pm}^{\alpha}$, obtained via (1.2.12) and (1.2.13) is

$$V_{+}^{1} = \frac{1+i\tau}{2(\mathrm{Im}\,\tau)^{1/2}}\,, \tag{1.3.81}$$

$$V_{+}^{2} = \frac{1-i\tau}{2(\mathrm{Im}\,\tau)^{1/2}}\,, \tag{1.3.82}$$

$$V_{-}^{1} = \frac{1+i\bar{\tau}}{2(\mathrm{Im}\,\tau)^{1/2}}\,, \tag{1.3.83}$$

$$V_{-}^{2} = \frac{1-i\bar{\tau}}{2(\mathrm{Im}\,\tau)^{1/2}}\,. \tag{1.3.84}$$

Further using the definitions (1.3.44) to (1.3.48) it is clear that one has

$$q_\alpha A_2^\alpha = 2\sqrt{p}\,B_2\,, \tag{1.3.85}$$

$$\tilde{q}_\alpha A_2^\alpha = -\frac{1}{\sqrt{p}}C_2\,, \tag{1.3.86}$$

$$q_\alpha A_6^\alpha = 2\sqrt{p}\,C_6\,, \tag{1.3.87}$$

$$\tilde{q}_\alpha A_6^\alpha = -\frac{1}{\sqrt{p}}B_6\,, \tag{1.3.88}$$

$$q_\alpha q_\beta A_8^{\alpha\beta} = pC_8\,, \tag{1.3.89}$$

The supersymmetry transformation rules expressed in terms of the fields τ, B_2, C_2, B_6, C_6 and C_8 can be read off directly from eqs. (1.3.63) to (1.3.73).

In the formulation containing both the RR scalar as well as the RR 8-form the following symmetry is manifest:

$$\phi \to \phi\,, \tag{1.3.90}$$

$$\chi \to \chi + b \tag{1.3.91}$$

$$B_2 \to B_2\,, \tag{1.3.92}$$

$$C_2 \to C_2 + bB_2\,, \tag{1.3.93}$$

$$B_6 \to B_6 + bC_6\,, \tag{1.3.94}$$

$$C_6 \to C_6\,, \tag{1.3.95}$$

$$C_8 \to C_8\,, \tag{1.3.96}$$

$$\lambda \to \lambda\,, \tag{1.3.97}$$

$$\psi_\mu \to \psi_\mu\,, \qquad \epsilon \to \epsilon\,. \tag{1.3.98}$$

This set of transformation forms the $\mathbb{R}$ subgroup of the classical $SL(2,\mathbb{R})$ duality group.

Fields	Group	Order of S
τ	$PSL(2,\mathbb{R})$	2
B_2, C_2	$SL(2,\mathbb{R})$	4
$\lambda, \psi_\mu, \epsilon$	double cover of $SL(2,\mathbb{R})$	8

Table 1.3.1: Transformation groups for the IIB fields that transform non-trivially under duality.

The spinor transformation rules under $SL(2,\mathbb{R})$ in terms of τ can be obtained from the spinor supersymmetry rules (1.3.72) and (1.3.73) in which

$$P = -\frac{d\tau}{\tau - \bar{\tau}}\,, \qquad (1.3.99)$$

$$Q = i\frac{d(\tau + \bar{\tau})}{2(\tau - \bar{\tau})}\,, \qquad (1.3.100)$$

$$G_3 = \frac{-i}{(\mathrm{Im}\tau)^{1/2}}\left(-dC_2 + \tau dB_2\right)\,, \qquad (1.3.101)$$

$$F_5 = dA_4 + \frac{1}{8}\left(C_2 \wedge dB_2 - B_2 \wedge dC_2\right)\,. \qquad (1.3.102)$$

Under the full $SL(2,\mathbb{R})$ duality group the spinors λ, ψ_μ and ϵ transform as:

$$\lambda \to e^{3i\beta}\lambda\,, \quad \psi_\mu \to e^{i\beta}\psi_\mu\,, \quad \epsilon \to e^{i\beta}\epsilon\,, \qquad (1.3.103)$$

where

$$e^{2i\beta} = \frac{|c\tau + d|}{c\tau + d}\,. \qquad (1.3.104)$$

Using (1.3.103) it can concluded that one needs to perform the S-duality transformation, $S : \tau \to -\frac{1}{\tau}$, eight times on the spinors in order to return to the same configuration, that is, $S^8 = 1$. On the other hand the S-duality acting on the 2-forms B_2 and C_2 and on τ transforming as in (1.3.51) and (1.3.39) is of fourth, respectively, second order. The transformation groups for the various IIB fields and the order of S are summarized in table 1.3.1.

1.4 The classical moduli space

The classical moduli space of type IIB supergravity is the space of allowed values for the scalars. This space is the coset $SL(2,\mathbb{R})/SO(2)$ - also referred to as the upper half plane $\{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$. The coordinates $(\tau, \bar{\tau})$ are called Poincaré coordinates whereas the coordinates (η, φ) are referred to as geodesic polar coordinates [28] for reasons that will become clear later. The coordinate ranges for η and φ are given by $0 \leq \eta < \infty$ and $0 \leq \varphi < 2\pi$.

The kinetic terms for the scalars can be thought of as defining a metric on the moduli space. The scalar kinetic terms in the action (1.1.17) are determined by $2 \star P \wedge \bar{P}$. Using for P the expression (1.3.99) one finds

$$2 \star P \wedge \bar{P} = \frac{1}{2}\frac{\star d\tau \wedge d\bar{\tau}}{(\operatorname{Im}\tau)^2}\,. \tag{1.4.1}$$

If instead one were to use the expression (1.3.34) for P the result would be

$$2 \star P \wedge \bar{P} = \frac{1}{2} \star d\eta \wedge d\eta + \frac{1}{2}\sinh^2 \eta \star d\varphi \wedge d\varphi\,. \tag{1.4.2}$$

The kinetic terms (1.4.1) are invariant under the transformations (1.3.39). These are fractional linear transformations and they form the group $PSL(2,\mathbb{R}) \equiv SL(2,\mathbb{R})/\{\pm\mathbb{1}\}$.

The relation between $(\tau, \bar{\tau})$ and (η, φ) can be obtained by starting with eq. (1.3.42), substituting in it eqs. (1.2.23) and (1.2.25) and subsequently use eqs. (1.3.23) and (1.3.24). The result is

$$\frac{\tau - \tau_0}{\tau - \bar{\tau}_0} = e^{i\varphi} \tanh \tfrac{\eta}{2}\,. \tag{1.4.3}$$

The point τ_0 is defined in eq. (1.3.41) to be a fixed point of the transformation $\tau \to e^{Q}\tau$.

There are two types of geodesics on the upper half plane $\{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$. The first type consists of straight semi-infinite vertical lines: $\operatorname{Re}\tau = \text{cst}$. The second type is formed by semi-circles of radius r whose center a is on the real axis: $\operatorname{Im}\tau = \left(r^2 - (\operatorname{Re}\tau - a)^2\right)^{1/2}$.

The Poincaré coordinate axes are $\operatorname{Im}\tau = \text{cst}$ and $\operatorname{Re}\tau = \text{cst}$ of which the latter are geodesics. The coordinate axes in geodesic polar coordinates are formed by $\varphi = \varphi_0$ and $\eta = \eta_0$. These are the respective circles:

$$\varphi = \varphi_0 \;:\quad \left(\operatorname{Re}\tau - \operatorname{Re}\tau_0 + \cot\varphi_0 \operatorname{Im}\tau_0\right)^2 + (\operatorname{Im}\tau)^2 = \left(\frac{\operatorname{Im}\tau_0}{\cos\varphi_0}\right)^2\,, \tag{1.4.4}$$

$$\eta = \eta_0 \;:\quad \left(\operatorname{Re}\tau - \operatorname{Re}\tau_0\right)^2 + \left(\operatorname{Im}\tau - \cosh\eta_0 \operatorname{Im}\tau_0\right)^2 = \left(\sinh\eta_0 \operatorname{Im}\tau_0\right)^2\,. \tag{1.4.5}$$

The lines $\varphi = \varphi_0$ all go through the point τ_0 with $0 < \operatorname{Im}\tau_0 < \infty$ and form semi-circles with their center on the real axis (geodesics), see figure 1.4.1.

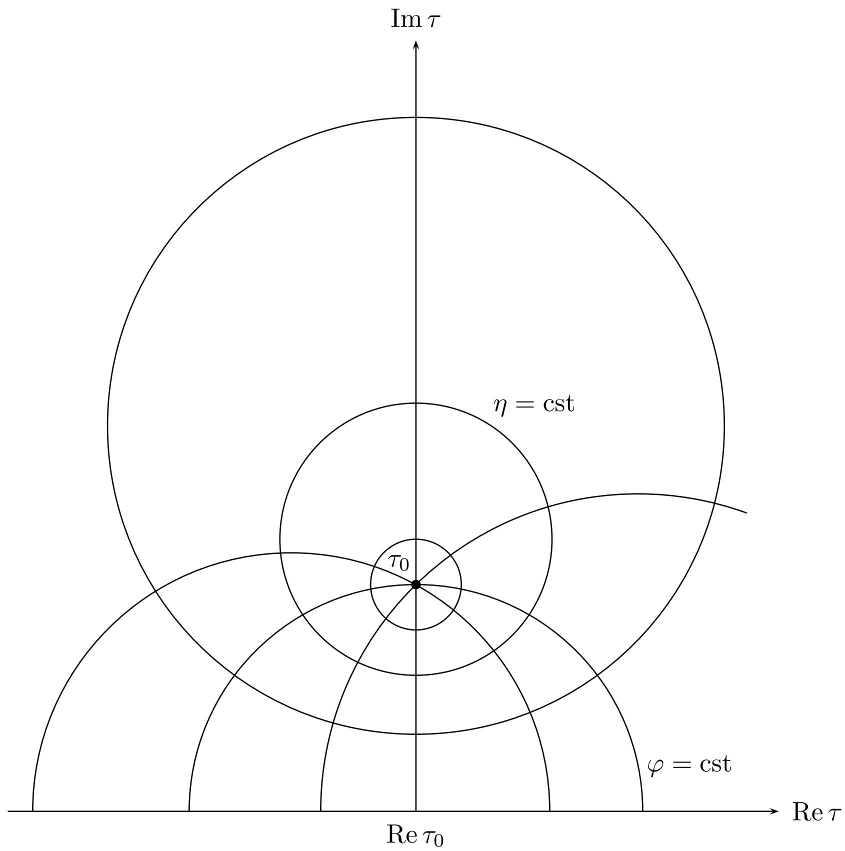

Figure 1.4.1: Geodesic polar coordinate system: the lines $\varphi = \mathrm{cst}$ are geodesics of the Poincaré metric and η measures the Poincaré distance from τ to τ_0 along $\varphi = \mathrm{cst}$.

The Poincaré distance measured along all the straight semi-infinite vertical lines that go through the point τ_0 with $\mathrm{Im}\,\tau_0 = 0$ is given by $\int d\phi$ where $\mathrm{Im}\,\tau = e^{-\phi}$. The Poincaré distance measured along all the semi-circles that go through the point τ_0 with $0 < \mathrm{Im}\,\tau_0 < \infty$ is given by $\int d\eta$. Hence, the scalars ϕ and η measure the Poincaré distance along a geodesic that goes through the point τ_0. This is the origin of the name geodesic polar coordinates for the coordinates (η, φ).

Fixed points τ_0 of $\tau \to e^Q\tau$ may be thought of as defining the origin of a coordinate system. For the fixed points with $\det Q = 0$ the origin is somewhere on the real axis $\mathrm{Im}\,\tau = 0$ (or after performing a certain $PSL(2,\mathbb{R})$ transformation at $\mathrm{Im}\,\tau = \infty$) and the coordinate axes are the semi-infinite vertical lines (geodesics) $\mathrm{Re}\,\tau = \mathrm{cst}$ together with the real line $\mathrm{Im}\,\tau = 0$. For the fixed points with $\det Q > 0$ the origin is located at a point τ_0 with $0 < \mathrm{Im}\,\tau_0 < \infty$ and the coordinate axes are formed by the lines $\eta = \mathrm{cst}$ together with all semi-circles that go through the point τ_0 that are the lines $\varphi = \mathrm{cst}$.

The matrix e^Q is equal to

$$e^Q = \cos(\sqrt{\det Q})\mathbb{1} + \frac{\sin(\sqrt{\det Q})}{\sqrt{\det Q}} Q\,. \tag{1.4.6}$$

From this expression it is clear that the set of transformations with $\det Q = 0$ forms the $\mathbb{R}$ subgroup of $SL(2,\mathbb{R})$, whereas the set of transformations with $\det Q > 0$ forms the $SO(2)$ subgroup of $SL(2,\mathbb{R})$. Any element Λ of $PSL(2,\mathbb{R})$ with $\mathrm{Tr}\,\Lambda \leq 2$ can be written as $\Lambda = e^Q$ where Q is given by (1.2.7) in which $p \geq 0$ and $q \geq 0$. The trace of Λ constitutes an $SL(2,\mathbb{R})$ conjugacy class.

Chapter 2

The one-half BPS branes of type IIB supergravity

2.1 Introduction

The closed IIB superstring theory can be extended by including Dirichlet p-branes or Dp-branes. These are objects with a $(p+1)$-dimensional world-volume on which an open fundamental string is ending. The open fundamental string, called an F-string, appears in type IIB supergravity as an infinitely long and straight string solution preserving one-half, that is, 16 supersymmetries. The fundamental string can be defined as that 1-brane that couples to the NSNS 2-form B_2 and that preserves 16 supersymmetries. The Dp-branes of type IIB string theory have $p = -1, 1, 3, 5, 7, 9$.

When the fundamental string ends on a Dp-brane the charges at the endpoints of the F-string produce a vector on the world-volume of the Dp-brane. This leads to a $U(1)$ gauge field, a Born–Infeld vector (see section 2.3), on the world-volume of the Dp-brane. When the D1-brane, also called the D-string, is placed in the region of IIB moduli space where the string coupling g_s is small and in which the RR axion is non-vanishing, it carries F-string charges on its world-sheet. Such a string will be referred to as an $(n,1)$ string [29] where n denotes the F-string charge and the one reflects the fact that there is only one D-string. Alternatively, it is possible to interpret the $(n,1)$ string as the fundamental string (with no strings ending on it[1].) of some $SL(2,\mathbb{Z})$ transformed theory.

The Dirichlet branes of the $SL(2,\mathbb{Z})$ transformed theory are objects on which a fundamental string is ending that is the $SL(2,\mathbb{Z})$ transformed version of the F-string.

[1] A string world-sheet theory with vectors is classically equivalent to a world-sheet theory without vectors because a vector in 1+1 dimensions is a non-propagating field that can be integrated out [30, 31].

For example a D-string with no F-string charge on it, i.e. a $(0,1)$ string is the S-dual of the fundamental string and ends on the S-dual of the Dp-branes. Due to the $SL(2,\mathbb{Z})$ duality of the theory there exists an infinite number of fundamental strings and associated Dirichlet branes all of which are $SL(2,\mathbb{Z})$ transformations of the F-string – Dp-brane system.

The one-half BPS objects of the type IIB theory can in most cases be characterized by saying that they form the object on which a string is ending that relates to the F-string by some $SL(2,\mathbb{Z})$ transformation. However, there are some one-half BPS objects for which such an interpretation does not exist (or is at least not very apparent). These objects arise in the cases $p=-1,7,9$. The 9-branes will be briefly discussed in section 2.5 and the case $p=7$ will be briefly discussed in section 2.4 and more elaborately in chapter 3. The instanton case that has $p=-1$ will be discussed in chapter 4.

2.2 One-half BPS projectors, tensions and gauge potentials

In order to study the one-half BPS p-branes of type IIB supergravity it is useful to first consider the p-brane as a probe brane in 10-dimensional Minkowski space-time. In this setting one can use the IIB super Poincaré algebra to classify the one-half BPS states.

A fermionic operator Q^i is introduced with $i=1,2$ an $SO(2)$ (R-symmetry group) index. The Q^i are taken to be Majorana–Weyl spinors with chirality given by $\gamma_{11}Q^i = Q^i$. A global IIB supersymmetry transformation on Minkowski space-time is then generated by $\delta = \bar{\epsilon}_1 Q^1 + \bar{\epsilon}_2 Q^2$. The anti-commutator of two fermionic generators Q^i is given by

$$\begin{aligned}\{Q^i,Q^j\} &= \delta^{ij}P_+\gamma^\mu C^{-1}P_\mu + P_+\gamma^\mu C^{-1}Z^{ij}_\mu + \frac{1}{3!}\epsilon^{ij}P_+\gamma^{\mu_1\mu_2\mu_3}C^{-1}Z_{\mu_1\mu_2\mu_3}\\ &\quad +\frac{1}{5!}\delta^{ij}P_+\gamma^{\mu_1\dots\mu_5}C^{-1}Z^{+}_{\mu_1\dots\mu_5} + \frac{1}{5!}P_+\gamma^{\mu_1\dots\mu_5}C^{-1}Z^{+\,ij}_{\mu_1\dots\mu_5}\,, \qquad (2.2.1)\end{aligned}$$

with P_μ the momentum operator and the Z's represent central charges. The central charges Z^{ij}_μ and $Z^{+\,ij}_{\mu_1\dots\mu_5}$ are symmetric in i and j and traceless, i.e. $\delta_{ij}Z^{ij}=0$ and $\delta_{ij}Z^{+\,ij}_{\mu_1\dots\mu_5}=0$. One can thus write

$$\begin{aligned}Z^{11}_\mu = -Z^{22}_\mu \equiv Z^{(1)}_\mu\,, &\quad Z^{12}_\mu = Z^{21}_\mu \equiv Z^{(2)}_\mu\,,\\ Z^{+\,11}_{\mu_1\dots\mu_5} = -Z^{+\,22}_{\mu_1\dots\mu_5} \equiv Z^{+\,(1)}_{\mu_1\dots\mu_5}\,, &\quad Z^{+\,12}_{\mu_1\dots\mu_5} = Z^{+\,21}_{\mu_1\dots\mu_5} \equiv Z^{+\,(2)}_{\mu_1\dots\mu_5}\,. \qquad (2.2.2)\end{aligned}$$

The plus on the 5-form central charges means to indicate that these 5-forms are self-dual. The operator P_+ is the chirality operator, $P_+ = \frac{1}{2}(1+\gamma_{11})$. It can be checked, by using formulae (A.2.10) and (A.2.11), that the right hand side of (2.2.1) has the same symmetry properties as the left hand side. Further, the number of independent components on both sides equals 528.

The R-symmetry group of the entire super Poincaré IIB supersymmetry algebra is $SO(2)$. The central charges in (2.2.1) form representations of the R-symmetry group. The 1-form central charges Z_{μ}^{ij} as well as the self-dual 5-form central charges $Z^{+\,ij}_{\mu_1\ldots\mu_5}$ form doublets under $SO(2)$. The remaining central charges $Z_{\mu_1\mu_2\mu_3}$ and $Z^{+}_{\mu_1\ldots\mu_5}$ form singlets under $SO(2)$.

From the anti-commutator of two supersymmetry generators defined on Minkowski space-time (2.2.1) the structure of the one-half supersymmetry projector can be obtained. An important role is played by the central charges of the IIB algebra on Minkowski space-time. In the local supersymmetry algebra these projectors take the same form with the only difference that the central charges in the projectors of the probe branes on Minkowksi space-time become functions of the two real scalars of the IIB theory. In the local theory the p-brane solutions can be constructed as fully back-reacted solutions and in this case the analysis of the central charges has been performed in [32].

Suppose the vacuum is a massive string (treated as a probe-brane in Minkowski space-time) lying in the 1-direction and preserving half of the 32 supersymmetries. In the rest frame, $P_\mu = M\delta^0_\mu$ and $Z^{ij}_\mu = Z^{ij}\delta^1_\mu$, of the string one then has

$$\{Q^i, Q^j\} = P_+\gamma^{\underline{0}}\left(\delta^{ij}M + \gamma_{\underline{0}}\gamma_{\underline{1}}Z^{ij}\right)C^{-1}\,, \tag{2.2.3}$$

in which the central charge Z^{ij} is

$$Z^{ij} = \sigma_3^{ij}Z^{(1)} + \sigma_1^{ij}Z^{(2)}\,, \tag{2.2.4}$$

where σ_3 and σ_1 are the standard Pauli matrices symmetric in i and j. The one-half BPS projector P in that $\{Q^i, Q^j\}$ annihilates the vacuum consisting of a massive string in the 1-direction is then

$$P = \frac{1}{2}\left(\mathbb{1} \pm \gamma_{\underline{01}}\frac{Z^{(1)}\sigma_3 + Z^{(2)}\sigma_1}{\sqrt{(Z^{(1)})^2 + (Z^{(2)})^2}}\right)\,, \tag{2.2.5}$$

where the mass M is related to the central charges via

$$M = \sqrt{(Z^{(1)})^2 + (Z^{(2)})^2}\,. \tag{2.2.6}$$

This relation follows from the condition that $P^2 = P$.

One could also consider the time index on the central charge Z^{ij}_μ. This component can be dualized to a 9-form central charge with nine spatial indices corresponding to a probe 9-brane in the background Minkowski space-time. This gives rise to the projector

$$P = \frac{1}{2}\left(\mathbb{1} \pm \frac{\tilde{Z}^{(1)}\sigma_3 + \tilde{Z}^{(2)}\sigma_1}{\sqrt{(\tilde{Z}^{(1)})^2 + (\tilde{Z}^{(2)})^2}}\right), \tag{2.2.7}$$

where $\tilde{Z}^{(1)}$ and $\tilde{Z}^{(2)}$ are defined via

$$\epsilon_{01\ldots9} Z_0^{ij} = Z_{1\ldots9}^{ij} = \tilde{Z}^{(1)}\sigma_3^{ij} + \tilde{Z}^{(2)}\sigma_1^{ij}\,. \tag{2.2.8}$$

It can be concluded that the projectors for the strings and the 9-branes are doublet representations of $SO(2)$. The 3-brane projector and by dualization of the 3-brane central charge, the 7-brane projectors form singlets under $SO(2)$. There are further in eq. (2.2.1) three self-dual 5-form central charges: a doublet and a singlet. The doublet provides the projectors for the 5-branes of the theory whereas the singlet forms the central charge of a Kaluza–Klein monopole [33].

Generalizing the one-half BPS projectors for the massive branes to the locally supersymmetric theory the central charges appearing in the projectors become functions of the scalars of the theory. Other than that the structure of the projectors is left unaltered. In the local theory the action for a massive brane (not being the Kaluza–Klein monopole) takes the following general form (in Einstein frame)

$$\int_{\Sigma_{p+1}} d^{p+1}\sigma\, |f|\, \sqrt{-g_{p+1}} + \int_{\Sigma_{p+1}} X_{p+1}\,, \tag{2.2.9}$$

where $|f|$ is the tension, a function of τ and $\bar{\tau}$, g_{p+1} is the determinant of the metric g_{AB} on the p-brane world-volume Σ_{p+1} and X_{p+1} is the gauge potential that couples electrically to the p-brane. Both $X_{A_1\ldots A_{p+1}}$ and g_{AB} are pull-backs of the target space-time metric $g_{\mu\nu}$ and potential $X_{\mu_1\ldots\mu_{p+1}}$. These pull-backs are

$$g_{AB} = \frac{\partial X^\mu}{\partial\sigma^A}\frac{\partial X^\nu}{\partial\sigma^B} g_{\mu\nu}\,, \tag{2.2.10}$$

$$X_{A_1\ldots A_{p+1}} = \frac{\partial X^{\mu_1}}{\partial\sigma^{A_1}}\cdots\frac{\partial X^{\mu_{p+1}}}{\partial\sigma^{A_{p+1}}} X_{\mu_1\ldots\mu_{p+1}}\,, \tag{2.2.11}$$

in which X^μ are the embedding coordinates and σ^A the world-volume coordinates. The static gauge corresponds to taking

$$\frac{\partial X^\mu}{\partial\sigma^A} = \delta^\mu_A\,. \tag{2.2.12}$$

The kinetic terms for the $9-p$ world-volume scalars X^{p+1} to X^9, describing the motion of the brane in the target space-time, is provided by $\sqrt{-g_{p+1}}$. In order for the bosonic degrees of freedom of the world-volume theory of the p-brane to correspond to the bosonic part of some supermultiplet it is necessary that besides the $9-p$ scalars there is one massless vector on the $(p+1)$-dimensional world-volume theory. The origin of the vector lies in requiring the coupling to X_{p+1} to be gauge invariant (see section 2.3). For the case $p=1$ such a vector can always be integrated out and the world-sheet theory has eight scalars. For the case $p=9$ there are no world-volume scalars and world-volume supersymmetry only requires vectors.

Requiring that (2.2.9) vanishes under a supersymmetry variation gives rise to one-half BPS projectors in the local theory. Knowing that the structure of these projectors is already fixed up to a scalar dependent function by the global algebra it is possible to deduce the $SL(2,\mathbb{R})$ or $SU(1,1)$ representations of the $(p+1)$-form potentials. The supersymmetry variation of a $(p+1)$-form potential must contain spinor bilinears that have zero $U(1)$ charge with respect to the $U(1)$ connection of the coset $SU(1,1)/U(1)$ (see chapter 1). These spinor bilinears involve either the gravitini ψ_μ with the local supersymmetry parameter ϵ or the dilatini λ together with ϵ. Since ψ_μ, λ and ϵ all carry a $U(1)$ charge the $U(1)$ charge of the spinor bilinear (when necessary) must be neutralized by scalars in the form of $V_\pm^\alpha$. From the way the $V_\pm^\alpha$ objects appear in the supersymmetry transformation of the $(p+1)$-forms (after closure of the commutator of two supersymmetry transformations acting on the $(p+1)$-forms into the universal supersymmetry algebra has been verified) the representation of that potential under $SU(1,1)$ follows. The result is that there exists a doublet A_2^α of 2-forms, a 4-form singlet, a doublet A_6^α of 6-forms, a triplet $A_8^{\alpha\beta}$ of 8-forms, a quadruplet $A_{10}^{\alpha\beta\gamma}$ of 10-forms and a doublet A_{10}^α of 10-forms. The massive branes of type IIB supergravity are listed in table 2.2.1. The brane actions can be obtained by using formula (2.2.9) together with the entries of table 2.2.1.

For completeness table 2.2.1 also includes projectors, tensions and electric couplings that apply to space-time instantons. Projectors such as $\epsilon=0$ do not make sense in classical Lorentzian type IIB supergravity as it would imply the vanishing of ϵ_C. Instantons, however should be interpreted as approximations to Euclidean path integrals in which case $\epsilon=0$ does not imply the vanishing of ϵ_C. The instanton case will be treated separately in chapter 4.

One way to deduce what type of string can end on a brane is by constructing gauge invariant Wess–Zumino terms. The gauge transformations for the potentials are given at the end of subsection (1.1.3) and in subsection 1.1.5 for the 10-forms. The Wess–Zumino (WZ) term of a p-brane is of the form $\int_{\Sigma_{p+1}} \mathrm{WZ}_{p+1}$. If the surface Σ_{p+1} has no boundary, as will always be assumed, then the gauge transformation of WZ_{p+1} is allowed to be the exterior derivative of some p-form. In most cases the WZ term can only be made gauge invariant by inclusion of a Born–Infeld (BI) vector. This will be illustrated in the next section by considering the case of the 3-brane. The introduction of a Born–Infeld vector for the 7-brane case is discussed in section 2.4.

	$\lvert f\rvert$	$X_{(p+1)}$	Projector
$p=-1$	$\frac{1}{4\sqrt{\det Q}}\log\frac{T+2\sqrt{\det Q}}{T-2\sqrt{\det Q}}$ $\det Q>0$	$\pm i\chi'$ $\det Q>0$	$\epsilon'=0$ (upper sign) $\epsilon'_C=0$ (lower sign)
$p=-1$	$\frac{1}{T}$ $\det Q=0$	$\pm i\,\chi'$ $\det Q=0$	$\epsilon=0$ (upper sign) $\epsilon_C=0$ (lower sign)
$p=1$	$\left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{1/2}$	$\pm\frac{1}{2}q_\alpha A_2^\alpha$	$\frac{q_\alpha V_-^\alpha}{\left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{1/2}}\,\epsilon\mp\gamma_{\underline{01}}\epsilon_C=0$
$p=3$	1	$\pm 4A_4$	$\frac{1}{2}\left(1\mp i\gamma_{\underline{0123}}\right)\epsilon=0$
$p=5$	$\left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{1/2}$	$\pm\frac{1}{2}q_\alpha A_6^\alpha$	$\frac{q_\alpha V_-^\alpha}{\left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{1/2}}\,\epsilon\mp i\gamma_{\underline{01}\cdots\underline{5}}\epsilon_C=0$
$p=7$	$q_{\alpha\beta}V_+^\alpha V_-^\beta$	$\pm q_{\alpha\beta}A_8^{\alpha\beta}$	$\frac{1}{2}\left(1\mp i\gamma_{\underline{01}\cdots\underline{7}}\right)\epsilon=0$
$p=9$	$\left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{3/2}$	$\pm\frac{3}{2}q_\alpha q_\beta q_\gamma A_{10}^{\alpha\beta\gamma}$	$\frac{q_\alpha V_-^\alpha}{\left(q_\beta q_\gamma V_+^\beta V_-^\gamma\right)^{1/2}}\,\epsilon\mp i\epsilon_C=0$
$p=9$	$\left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{1/2}$	$\mp\frac{1}{2}q_\alpha A_{10}^\alpha$	$\frac{q_\alpha V_-^\alpha}{\left(q_\beta q_\gamma V_+^\beta V_-^\gamma\right)^{1/2}}\,\epsilon\mp\epsilon_C=0$

Table 2.2.1: Tensions $|f|$, gauge potentials X_{p+1} and 1/2 BPS projectors for instantons (first two rows), p-branes (third to eighth row). The plus/minus signs refer to branes/anti-branes.

2.3 Three-brane

The emergence of a BI vector through the requirement of a gauge invariant WZ term will be discussed explicitly for the case $p = 3$. The 4-form gauge transformation is given in eq. (1.1.43). Using eq. (1.2.6) it follows that

$$\delta \left(A_4 - \frac{1}{16} q_\alpha A_2^\alpha \wedge \tilde{q}_\beta A_2^\beta \right) = d\Sigma_3 - \frac{1}{8} \tilde{q}_\alpha A_2^\alpha \wedge \delta q_\beta A_2^\beta \,. \tag{2.3.1}$$

To cancel the second term in (2.3.1) one must add a new degree of freedom to the 3-brane world-volume theory (there are already 6 embedding scalars in static gauge describing the position of the 3-brane in the target space-time) in the form of a BI vector $q_\beta V_1^\beta$ whose gauge transformation is such that

$$\delta q_\alpha F_2^\alpha = \delta q_\alpha \left(A_2^\alpha + dV_1^\alpha \right) = 0 \,, \tag{2.3.2}$$

where the BI field strength $F_2^\alpha = A_2^\alpha + dV_1^\alpha$ has been introduced. It follows that the gauge transformation of V_1^α is

$$\delta V_1^\alpha = -\Sigma_1^\alpha + d\Sigma^\alpha \,, \tag{2.3.3}$$

where Σ^α is a doublet of world-volume scalar gauge transformation parameters. The gauge invariant WZ term is then

$$4A_4 + \frac{1}{4} q_\alpha A_2^\alpha \wedge \tilde{q}_\beta A_2^\beta - \frac{1}{2} \tilde{q}_\alpha A_2^\alpha \wedge q_\beta F_2^\beta \,. \tag{2.3.4}$$

In terms of q_α and $\tilde{q}_\alpha$ taken as in (1.3.78) with the definitions of subsection 1.3.4 the WZ term becomes

$$4A_4 - \frac{1}{2} B_2 \wedge C_2 + C_2 \wedge F_2 \,, \tag{2.3.5}$$

where $F_2 = B_2 + dV_1$ is defined such that $q_\alpha F_2^\alpha = 2\sqrt{p}\, F_2$. In the choice (1.3.78) the BI vector is associated with the NSNS 2-form B_2. Hence, it can be interpreted as coming from the open fundamental string ending on the 3-brane, which has thus turned into a D3-brane. The WZ term (2.3.5) is however not yet the full answer. One should consider the D3-brane in the regime where the string coupling g_s is small. In the quantum IIB moduli space $SO(2)\backslash PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$ (see figure 3.9.1 on page 81) this is at $\tau_0 = i\infty$, i.e. at infinity reached along the imaginary τ axis. The point $\tau_0 = i\infty$ is a fixed point of the transformations (1.3.90) to (1.3.93). Applying these transformations to (2.3.5) one finds

$$4A_4 - \frac{1}{2} B_2 \wedge C_2 + C_2 \wedge F_2 \to 4A_4 - \frac{1}{2} B_2 \wedge C_2 + C_2 \wedge F_2 - \frac{1}{2} F_2 \wedge F_2 \,, \tag{2.3.6}$$

where an irrelevant total derivative, $-\frac{1}{2} dV_1 \wedge dV_1$, has been added to the right hand side of (2.3.6). Since the D3-brane is defined in the region near $\tau_0 = i\infty$ one must require it to be invariant under the transformations (1.3.90) to (1.3.93) that leave $i\infty$ invariant. This can be achieved by adding to (2.3.5) the gauge invariant term $+\frac{1}{2}\chi' F_2 \wedge F_2$. The resulting D3-brane WZ term is then

$$\mathrm{WZ}_{\mathrm{D3}} = C_4 + C_2 \wedge F_2 + \frac{1}{2}\chi F_2 \wedge F_2 \,, \tag{2.3.7}$$

where $C_4 = 4A_4 - \frac{1}{2}B_2 \wedge C_2$.

Instead of choosing q_α and $\tilde{q}_\alpha$ as in (1.3.78) they could have been kept arbitrary. In that case the BI vector would be $q_\alpha V_1^\alpha$ that is tied to the 2-form $q_\alpha A_2^\alpha$. Using the definitions of the NSNS and RR 2-forms, eqs. (1.3.44) and (1.3.45), one can write $q_\alpha A_2^\alpha$ as

$$q_\alpha A_2^\alpha = p' B_2 - q' C_2 \,, \tag{2.3.8}$$

where $p' = q_1 + q_2$ and $q' = i(q_1 - q_2)$. The parameters p' and q' are taken to be positive. The string charged with respect to (2.3.8) can be called a (p', q') string that can be considered to be a bound state of p' fundamental strings and q' D-strings [29]. The 3-brane with $q_\alpha V_1^\alpha$ as its BI vector can be called a (p', q') 3-brane since it has a (p', q') string ending on it. The WZ term of a (p', q') 3-brane is

$$\mathrm{WZ}_{(p',\,q')\ \text{3-brane}} = 4A_4 + \frac{1}{4} q_\alpha A_2^\alpha \wedge \tilde{q}_\beta A_2^\beta - \frac{1}{2}\tilde{q}_\alpha A_2^\alpha \wedge q_\beta F_2^\beta + \frac{1}{8}\chi' q_\alpha F_2^\alpha \wedge q_\beta F_2^\beta \,, \tag{2.3.9}$$

where χ' is defined in eq. (1.2.10).

The NSNS and RR 2-forms transform under $SL(2,\mathbb{Z})$ as in (1.3.51). This implies that q' and p' transform as

$$\begin{pmatrix} -q' \\ p' \end{pmatrix} \to (\Lambda^{-1})^T \begin{pmatrix} -q' \\ p' \end{pmatrix} . \tag{2.3.10}$$

When the numbers p' and q' are integers and relatively prime the (p', q') string is related to the fundamental $(1,0)$ string by an $SL(2,\mathbb{Z})$ transformation, say Λ_0. A (p', q') string with p' and q' relatively prime can be considered to be the fundamental string of the Λ_0 transformed IIB string theory whose coupling constant is the Λ_0 transformed version of e^ϕ and whose perturbative region in moduli space is the region near the point that is the Λ_0 transform of $i\infty$ [34].

The world-volume degrees of freedom of a (p', q') 3-brane with p' and q' relatively prime consist of six embedding scalars and one BI vector, that together form the bosonic part of an $N = 4$, $d = 4$ vector supermultiplet. The kinetic terms for the embedding scalars and for the BI vector are contained in the Dirac–Born–Infeld (DBI) part of the (p', q') 3-brane action. The DBI action and the WZ term are related by kappa symmetry [26]. The type IIB $SL(2,\mathbb{Z})$ transformations acting on the background fields to which the 3-brane couples manifest themselves, on the world-volume of the 3-brane, as electric-magnetic duality transformations.

So far, the 3-brane world-volume action, since the 3-brane was put near $\tau = i\infty$ or $\Lambda i\infty$ for some $\Lambda \in SL(2,\mathbb{Z})$, was required to be invariant under the $\mathbb{R}$ subgroup of $SL(2,\mathbb{R})$ or after quantization under integers shifts of χ' defined in (1.2.10). The 3-brane couples to a 4-form that is a singlet under $SL(2,\mathbb{Z})$ and so the 3-brane can move around in the moduli space, figure 3.9.1 on page 81, of the IIB theory. Is there are a supergravity description of the 3-brane world-volume theory when the 3-brane is placed near any of the points $\tau = i$ or $\tau = \rho$ of the quantum moduli space (3.9.1)?

Classically, it is possible to write down a 3-brane action that is invariant under the full $SL(2,\mathbb{R})$ duality group. This can be done by writing down an action that depends on two BI vectors transforming as a doublet under $SL(2,\mathbb{R})$. These two BI vectors are non-locally related in that their fields strengths are dual to each other. Such actions have been written down in [35] and [36] where in [36] the Pasti–Sorokin–Tonin (PST) formalism [37, 38] was used. These actions are kappa symmetric and unique. Because they are $SL(2,\mathbb{R})$ invariant they can be placed at any point of the moduli space, in particular at $\tau = i, \rho$. Since these points are fixed points of certain $SO(2)$ transformations that belong to the $SL(2,\mathbb{R})$ group and since $SO(2)$ rotations always rotate both the BI vectors, it is not expected that the actions in [35, 36] near $\tau = i, \rho$ can be reduced to an action containing a single BI vector. A 3-brane near $\tau = i, \rho$ will be called a Q3-brane. Near the point $\tau = i\infty$ it is possible to eliminate one of the two BI vectors with the remaining BI vector being related to the NSNS 2-form that is invariant under the shift of the RR axion of which $\tau = i\infty$ is a fixed point.

Finally, one should distinguish between the case with two coinciding $(1,0)$ 3-branes, say, that would give rise to a non-Abelian world-volume theory with gauge group $U(2)$, and a $(2,0)$ 3-brane that is defined to be one 3-brane on which is ending a $(2,0)$ string. The number of 3-branes is determined by the parameter in the WZ term multiplying the 4-form, which here was taken to be one.

2.4 Seven-branes

The 8-form potential that couples to a 7-brane has the general form $q_{\alpha\beta}A_8^{\alpha\beta}$. Consider the case $q_{\alpha\beta} = q_\alpha q_\beta$, i.e. $\det Q = 0$. As shown in [27] the gauge invariant and χ' shift symmetry invariant WZ term contains one BI vector, viz. $q_\alpha V_1^\alpha$. The full expression for the WZ term will not be needed. The 8-form potential $q_\alpha q_\beta A_8^{\alpha\beta}$ can be written as (see subsection 1.3.3) $pC_8 + qB_8 + \frac{r}{2}D_8$. Using that the string charges p' and q' are given by $p' = \frac{1}{2}(q_1 + q_2)$ and $q' = \frac{i}{2}(q_1 - q_2)$ it can be concluded that the 7-brane parameters p, q, r and the string parameters p' and q' are related via

$$p = p'^2\,, \quad q = q'^2\,, \quad r = \pm 2p'q'\,, \tag{2.4.1}$$

in which p' and q' are taken positive.

In order that one is dealing with a single 7-brane on which a fundamental string is ending it must be that p' and q' are relatively prime. Just as for the case of the 3-brane one can refer to such a 7-brane as a (p', q') 7-brane. When they are not relatively prime the 7-brane is formed out of a coincident set of single (p', q') 7-branes and for such a system of identical coincident 7-branes the world-volume theory is non-Abelian. The non-Abelian nature of the world-volume theory of two coincident identical (p', q') 7-branes comes from the presence of two additional massless vectors that are associated with open strings that have their endpoints on different branes. The gauge invariance of the 8-form WZ term, $\int_{\Sigma_8} q_\alpha q_\beta A_8^{\alpha\beta}$, when one considers only the center of mass motion of the coincident branes, does not require the non-Abelian BI vectors. If the relative motion is taken into consideration the world-volume scalars become non-Abelian and correspondingly non-Abelian BI vectors are needed to make the coupling to the 8-form $q_\alpha q_\beta A_8^{\alpha\beta}$ gauge invariant, see e.g. [39, 40].

Consider next the case $\det Q > 0$, which will be referred as a Q7-brane. In this case $q_{\alpha\beta}$ can be parameterized as in (1.2.5) and it follows that in order to make the WZ term $\int_{\Sigma_8} q_{\alpha\beta} A_8^{\alpha\beta}$ gauge invariant under the gauge transformations (1.1.45) two BI vectors are needed [17]. It will be argued in section 3.11 that the properties of Q7-brane solutions can be understood in terms of certain F-theory 7-branes becoming coincident. F-theory will be discussed in section 2.6 and F-theory 7-branes are (p', q') 7-branes with p' and q' relatively prime. This means that in the limit in which one only considers the center of mass motion of the Q7-brane one needs two Abelian BI vectors to produce a gauge invariant WZ term. More will be said about this in section 3.11 where it will be argued that a Q7-branes consists of two or more (p', q') 7-branes such that (at least) two out of the full set of 7-branes making up the Q7-brane differ in their values for p' and q'. For example a particular Q7-brane could be formed out of one $(1, 0)$ 7-brane and one $(1, 1)$ 7-brane or out of two $(1, 0)$ 7-branes and one $(1, 1)$ 7-brane.

Instead of referring to a 7-brane as a (p', q') 7-brane it will prove more convenient to refer to a generic 7-brane as a (p, q, r) 7-brane since this captures all possible 7-branes including those that are formed by taking different (p', q') 7-branes coincident.

2.5 Nine-branes

This section is meant to address the possible role of the doublet of 10-form potentials in the IIB theory. Some of the results presented below are non conclusive and could be considered to be work in progress.

The IIB supersymmetry algebra has two $\mathbb{Z}_2$ automorphisms that are often denoted by $(-1)^{F_L}$ and Ω. The action of $(-1)^{F_L}$ and Ω on the fields of the local supersymmetry algebra can be deduced from the supersymmetry transformations. The result for the fields of subsection (1.3.4) is

$$\begin{aligned}
(-1)^{F_L}:&\quad \epsilon\to\epsilon_C\,,\quad \psi_\mu\to\psi_{C\,\mu}\,,\quad \lambda\to\lambda_C\,,\\
(-1)^{F_L}:&\quad \phi\to\phi\,,\quad \chi\to-\chi\quad ,\\
(-1)^{F_L}:&\quad B_2\to B_2\,,\quad C_2\to-C_2\,,\quad A_4\to-A_4\,,\quad C_6\to-C_6\,,\\
(-1)^{F_L}:&\quad B_6\to B_6\,,\quad C_8\to-C_8\,,\quad B_8\to-B_8\,,\quad D_8\to D_8\,. \qquad (2.5.1)
\end{aligned}$$

and

$$\begin{aligned}
\Omega:&\quad \epsilon\to i\epsilon_C\,,\quad \psi_\mu\to i\psi_{C\,\mu}\,,\quad \lambda\to i\lambda_C\,,\\
\Omega:&\quad \phi\to\phi\,,\quad \chi\to-\chi\quad ,\\
\Omega:&\quad B_2\to-B_2\,,\quad C_2\to C_2\,,\quad A_4\to-A_4\,,\quad C_6\to C_6\,,\\
\Omega:&\quad B_6\to-B_6\,,\quad C_8\to-C_8\,,\quad B_8\to-B_8\,,\quad D_8\to D_8\,. \qquad (2.5.2)
\end{aligned}$$

Using the transformations (1.3.103), (1.3.39), (1.3.51), (1.3.52), (1.3.53) it can be shown that $S(-1)^{F_L}S^{-1}=\Omega$ where S is the S-duality transformation with $a=d=0$ and $b=-c=1$.

Nine-branes couple to 10-form potentials. As shown in [22] there are two sets of 10-form potentials, a quadruplet $A_{10}^{\alpha\beta\gamma}$ and a doublet A_{10}^{α} representation of $SU(1,1)$ whose properties are summarized in subsection 1.1.5. Consider first the quadruplet. A generic 10-form would take the form $q_{\alpha\beta\gamma}A^{\alpha\beta\gamma}$ where $q_{\alpha\beta\gamma}$ is a rank 3 symmetric tensor (a generalization of $q_{\alpha\beta}$ introduced in section 1.2). One could then ask the question: does the potential $q_{\alpha\beta\gamma}A^{\alpha\beta\gamma}$ couple to a 9-brane? As shown in [27, 41] the answer is affirmative if $q_{\alpha\beta\gamma}=q_\alpha q_\beta q_\gamma$. At leading order, i.e. for zero Born–Infeld field strengths the 1/2 BPS 9-brane action in Einstein frame is given by

$$S=\int_{\mathcal{M}_{10}} d^{10}x\,\left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{3/2}\sqrt{-g}\pm\frac{3}{2}\int_{\mathcal{M}_{10}} q_\alpha q_\beta q_\gamma A_{10}^{\alpha\beta\gamma}\,, \qquad (2.5.3)$$

where the integration is over the entire 10-dimensional space-time. The 1/2 BPS supersymmetry projector is

$$\frac{q_\alpha V_-^\alpha}{\left(q_\beta q_\gamma V_+^\beta V_-^\gamma\right)^{1/2}}\,\epsilon\mp i\epsilon_C=0\,. \qquad (2.5.4)$$

Employing the $U(1)$ gauge choice, eq. (1.2.11), the projector reduces to

$$\epsilon\mp i\epsilon_C=0\,. \qquad (2.5.5)$$

This projector can be interpreted as

$$\epsilon=\mp\Omega\epsilon\,, \qquad (2.5.6)$$

where the action of Ω on the supersymmetry transformation parameter ϵ is given in the first line of eq. (2.5.2). The projector (2.5.6) is the starting point of the type I truncation of type IIB supergravity down to the $N=1$ supergravity multiplet of type I supergravity.

Next, consider the doublet of 10-form potentials. Here, one can again ask the question does $q_\alpha A^\alpha_{10}$, i.e. the generic member of the 10-form doublet, couple to a 9-brane? The answer is also again yes it does. The 1/2 BPS 9-brane action is given by

$$S = \int_{\mathcal{M}_{10}} d^{10}x \left(q_\alpha q_\beta V^\alpha_+ V^\beta_-\right)^{1/2} \sqrt{-g} \pm \frac{1}{2}\int_{\mathcal{M}_{10}} q_\alpha A^\alpha_{10}\,. \tag{2.5.7}$$

The 1/2 BPS supersymmetry projector is

$$\frac{q_\alpha V^\alpha_-}{\left(q_\beta q_\gamma V^\beta_+ V^\gamma_-\right)^{1/2}}\,\epsilon \pm \epsilon_C = 0\,. \tag{2.5.8}$$

Using the same $U(1)$ gauge as in the quadruplet case the projector can be written as

$$\epsilon = \mp(-1)^{F_L}\epsilon\,, \tag{2.5.9}$$

where the transformation $(-1)^{F_L}$ is given in (2.5.1). The projector (2.5.9) is the starting point for the truncation of type IIB supergravity down to the $N=1$ supergravity multiplet of heterotic supergravity.

In the $U(1)$ gauge choice with the parameter q_α as taken in (1.3.78) the brane action (2.5.3) corresponds to the D9-brane. The S-dual of the D9-brane has a 1/2 BPS projector that depends on τ. This is because S-duality acts on spinors as a local $U(1)$ transformation depending on the axidilaton field τ, see eq. (1.3.103). Another way to see this is to replace in formulae (2.5.3) and (2.5.4) q_α with $\tilde{q}_\alpha$. This can be done as the calculations do not depend on the properties of q_α. The supersymmetry projector is then (2.5.4) with q_α replaced by $\tilde{q}_\alpha$. Using the parametrization (1.3.78) the projector for the S-dual D9-brane is obtained. If in the projector of the S-dual D9-brane the axion is put to zero then the resulting projector agrees with (2.5.9), but there is a priori no reason to put the axion equal to zero. The projector of the S-dual D9-brane can be written as

$$iS\epsilon \mp S\epsilon_C = 0\,, \tag{2.5.10}$$

where $S\epsilon$ is the S-dual transformed version of ϵ. Using that $S\Omega S^{-1} = (-1)^{F_L}$ eq. (2.5.10) is also

$$S\epsilon = \pm(-1)^{F_L}S\epsilon\,. \tag{2.5.11}$$

Hence, in the S-dual transformed basis Ω acts as $(-1)^{F_L}$. Comparing eq. (2.5.11) with eq. (2.5.9) it is clear that the two projectors are inequivalent. As it will be argued next, this inequivalence is lifted when one truncates the type IIB theory (or divides out the IIB theory) using Ω or $(-1)^{F_L}$.

The full truncations are obtained by working out what (2.5.6) and (2.5.9) imply for the fields of IIB supergravity by requiring that the projectors are consistent with the supersymmetry rules. This must be done since the projectors are statements about supersymmetry in the entire 10-dimensional space-time. The answer can be read off from the transformations of Ω and $(-1)^{F_L}$ given in (2.5.2) and (2.5.1). The projectors (2.5.6) and (2.5.9) can be interpreted as saying that a given type IIB configuration of fields must be identified with its Ω, respectively, $(-1)^{F_L}$ transformed versions. Hence, fields that are odd under Ω, respectively, $(-1)^{F_L}$ will be truncated from the spectrum. It can be observed that the set of bosonic fields that survive the Ω truncation are the S-dual partners of the bosonic fields that survive the $(-1)^{F_L}$ truncation. For the fermions this is also the case when the axion χ is zero, which it is in both truncations.

The process of dividing out the IIB theory by Ω is known as orientifolding the theory. It corresponds not only to the insertion of a D9-brane, but at the same time of an O9-plane. Tadpole cancelation conditions, that is requiring that the force between the O9-plane and the D9-brane cancel, forces the D9-brane charges to be 32 in units of 1 for each D9-brane in order to cancel the charge of a single O9-plane. The open string sector for a theory with 32 D9-branes and one O9-plane has gauge group $SO(32)$. In [33] the idea is advocated that the SO(32) heterotic string theory can be obtained from the IIB theory by applying an S-duality transformation on the system of 32 D9-branes plus one O9-plane. The S-duality transforms Ω into $S\Omega S^{-1}$ which should be interpreted as world-sheet parity on the world-sheet of the D1-string that is the fundamental string ending on the S-dual D9-brane.

Further, it might be relevant to remark that the 1/2 BPS supersymmetry projector of the system of 32 S-dual D9-branes orientifolded by $S\Omega S^{-1}$ preserves the same supersymmetries as the 9-brane charged under $q_\alpha A_{10}^\alpha$. Therefore from the point of view of supersymmetry it is possible to add such a 9-brane to a system of 32 S-dual D9-branes orientifolded by $S\Omega S^{-1}$ since $S\Omega S^{-1} = (-1)^{F_L}$. Similarly, if one adds to a system of 32 D9-branes orientifolded by Ω the 9-brane that couples to $\tilde{q}_\alpha A_{10}^\alpha$ supersymmetry is preserved. Could it be that the combined systems of 9-branes coupled to $q_\alpha q_\beta q_\gamma A_{10}^{\alpha\beta\gamma}$ and $\tilde{q}_\alpha A_{10}^\alpha$ and of 9-branes coupled to $\tilde{q}_\alpha \tilde{q}_\beta \tilde{q}_\gamma A_{10}^{\alpha\beta\gamma}$ and $q_\alpha A_{10}^\alpha$ has an effective orientifold interpretation[2]?

[2] The gauge transformations for A_{10}^α, when its supersymmetry transformation is given by (1.1.60), are trivial, eq. (1.1.61), the 9-branes that couple to A_{10}^α are not expected to interfere with the Yang–Mills fields coming from the open string sectors. It is mentioned that results from superspace, obtained in [26] suggest a rather different form for the gauge transformation of A_{10}^α. If a 10-form with such a gauge structure exist it would imply that the supersymmetry variation for them differs from (1.1.60) and so they would not lead to the 1/2 BPS 9-branes (2.5.7) (see also the discussion in subsection 1.1.5).

2.6 F-theory

2.6.1 Duality between type I on T^2 and type IIB on $T^2/\mathbb{Z}_2$

Consider type I string theory as a system of 32 D9-branes plus one O9-plane. This configuration is T-dual to type IIA string theory on $S^1/\mathbb{Z}_2$. The orbifold $S^1/\mathbb{Z}_2$ is defined as $x^9 \sim x^9 + 1$ and $x^9 \sim -x^9$ where x^9 is the coordinate along S^1. The background $\mathcal{M}_{1,8} \times S^1/\mathbb{Z}_2$ with $\mathcal{M}_{1,8}$ some 9-dimensional space-time can be turned into an orientifold background of the IIA theory by taking $\mathbb{Z}_2 = \{1, I_9\Omega\}$ where I_9 is the inversion operation of the 9th coordinate, i.e. $I_9 : x^9 \rightarrow -x^9$ [42]. The orbifold $S^1/\mathbb{Z}_2$ has two fixed points and at each one of them an orientifold 8-plane is placed. With each O8-plane there are eight D8-branes coinciding. In total there are thus 16 D8-branes. The number of D8-branes is 16 and not 32 because the action I_9 forces the D8-branes at the fixed points of $S^1/\mathbb{Z}_2$ to be paired. Because of this pairing one O8-plane with 8 D8-branes is effectively a stack of 16 D8-branes whose world-volume theory is truncated by Ω leading to the gauge group $SO(16)$. Gauge groups for stacks of D-branes plus an orientifold were studied in [43]. Therefore, type IIA on $S^1/\mathbb{Z}_2$, also known as type I' string theory, has gauge group $SO(16)\times SO(16)$. Performing a second T-duality transformation (reduction over a world-volume direction and uplifting over a transverse space direction) relates the above-mentioned IIA configuration on $S^1/\mathbb{Z}_2$ with two O8-planes to a IIB configuration on $T^2/\mathbb{Z}_2$ which is an orbifold with four fixed points. At each fixed point there is an O7-plane coincident with four D7-branes. The system of four D7-branes and one O7-plane has gauge group $SO(8)$.

The relation between type I on T^2 and type IIB on T^2/Z_2 can be made more explicit by considering the reductions of these theories. The field content of the type I supergravity multiplet is: a metric $g_{\mu\nu}$, a 2-form $C_{\mu\nu}$ (the type IIB RR 2-form) and the dilaton ϕ. This is the set of IIB fields that are even under Ω, see eq. (2.5.2). Taking $\mu = (a, 8, 9)$ with $a = 0, 1, \ldots, 7$ and reducing the 10-dimensional metric on T^2 gives in eight dimensions a metric g_{ab}, two Kaluza–Klein vectors A^1_a and A^2_a and three real scalars that can be organized into one complex scalar τ, the complex structure of the 2-torus and $\tilde{\varphi}$, describing the size of the 2-torus. The reduction of the 2-form $C_{\mu\nu}$ on T^2 gives in eight dimensions a 2-form C_{ab}, two vectors A^3_a and A^4_a and one real scalar C_{89}. The two real scalars $\tilde{\varphi}$ and C_{89} can be organized into one complex scalar $\sigma = C_{89} + ie^{-\tilde{\varphi}}$, which is the complex Kähler modulus of the 2-torus. Both τ and σ transform under $PSL(2,\mathbb{Z})$. Of course, there is also the real 8-dimensional dilaton denoted by φ.

Consider next the reduction of the type IIB theory over $T^2/\mathbb{Z}_2$. The 2-torus T^2 is described by the complex coordinate $z = x^8 + ix^9$ that is such that $z \sim z + 1$ and $z \sim z + \sigma$ for some complex structure modulus σ. The orbifold $T^2/\mathbb{Z}_2$ is obtained by identifying z with $-z$. The space $\mathcal{M}_{1,7} \times T^2/\mathbb{Z}_2$ in which $\mathcal{M}_{1,7}$ is some 8-dimensional space-time, can be turned into an orientifold background of type IIB by combing the $\mathbb{Z}_2$ symmetry $z \rightarrow -z$ with the perturbative world-sheet symmetry $(-1)^{F_L}\Omega$. One has $(-1)^{F_L}\Omega = -\mathbb{1}$ with $-\mathbb{1}$ an element of $SL(2,\mathbb{Z})$ as follows from eqs. (2.5.1) and (2.5.2). Hence, when reducing type IIB over $T^2/\mathbb{Z}_2$ the $\mathbb{Z}_2$ group is taken to be $\{1, I_{89}(-1)^{F_L}\Omega\}$, where I_{89} means $I_{89} : z \to -z$. When reducing the 10-dimensional metric over $T^2/\mathbb{Z}_2$ one obtains an 8-dimensional metric, g_{ab}, together with three real scalars, a complex structure scalar σ and one real scalar φ describing the size of $T^2/\mathbb{Z}_2$. The two Kaluza–Klein vectors are truncated since they are even under $(-1)^{F_L}\Omega$ but odd under I_{89}. The NSNS and RR 2-forms $B_{\mu\nu}$ and $C_{\mu\nu}$, respectively are both odd under $(-1)^{F_L}\Omega$, therefore only those components of $B_{\mu\nu}$ and $C_{\mu\nu}$ that are odd under I_{89} remain. These are the components with one leg in the space $T^2/\mathbb{Z}_2$ and one leg in the space $\mathcal{M}_{1,7}$. This leads to a total of four vectors denoted by A^1_a, A^2_a, A^3_a and A^4_a. The self-dual 4-form is even under $(-1)^{F_L}\Omega = -\mathbb{1}$ and hence only those components that are even under I_{89} remain. These are the components with two or no legs in $T^2/\mathbb{Z}_2$, i.e. A_{ab89} and A_{abcd}. The 4-form A_{abcd} can be dualized in eight dimensions to a 2-form and this 2-form together with A_{ab89}, due to the self-duality constraint in ten dimensions, produce one 2-form in eight dimensions, denoted by C_{ab}. The complex axidilaton τ reduces to a complex axidilaton in eight dimensions that is again denoted by τ.

The identification of the type I supergravity multiplet reduced over T^2 and the type IIB supergravity multiplet reduced over $T^2/\mathbb{Z}_2$ is obtained by identifying the 8-dimensional fields. The dilaton in type I on T^2 was denoted by φ and this field is identified with the size of the orbifold $T^2/\mathbb{Z}_2$, that is equally denoted by φ, such that weakly coupled type I on T^2 corresponds to a large $T^2/\mathbb{Z}_2$. The complex structure and Kähler modulus of the 2-torus on the type I side is identified with the complex axidilaton field and the complex structure modulus of $T^2/\mathbb{Z}_2$, respectively, on the IIB side. The type I 2-form C_{ab} on the type IIB side originates from the reduction of the 4-form. Finally, the metric and the four 1-forms A^1_a to A^4_a coming from the type I and IIB sides are identified. The reduction of the supergravity multiplets thus leads to a $(U(1))^4$ gauge group in eight dimensions. The total number of degrees of freedom in eight dimensions form an $N = 1$, $d = 8$ supergravity multiplet containing 48 bosonic and 48 fermionic degrees of freedom plus two $N = 1$, $d = 8$ vector supermultiplets each containing 8 bosonic and 8 fermionic degrees of freedom[3].

The open string sector of type I string theory has gauge group $SO(32)$. On the other hand the orientifold background $\mathcal{M}_{1,7} \times T^2/\mathbb{Z}_2$ contains four O7-planes placed at the fixed points of the orbifold $T^2/\mathbb{Z}_2$ with $\mathbb{Z}_2$ denoting $z \to -z$. These fixed points are $z = 0, \frac{1}{2}, \frac{\sigma}{2}, \frac{\sigma+1}{2}$. Each of the O7-planes is coincident with a stack of four D7-branes. Hence, on the IIB side the open string gauge group is $(SO(8))^4$. The reduction of type I over T^2 needs to employ Wilson lines that break $SO(32)$ down to $(SO(8))^4$.

[3]The number N counts the number of irreducible supercharges. In eight dimensions an irreducible spinor is Majorana and has 16 real components. The 48+48 N=1, $d = 8$ supergravity multiplet contains the graviton, a 2-form, two vectors and one real scalar (see for example [44]).

2.6.2 Type IIB on $T^2/\mathbb{Z}_2$ is F-theory on $T^4/\mathbb{Z}_2$

The orientifolding obtained by taking $T^2/\{1, I_{89}(-1)^{F_L}\Omega\}$ can be given a purely geometrical interpretation. The action $(-1)^{F_L}\Omega$ is equal to the $-\mathbb{1}$ element of $SL(2,\mathbb{Z})$. This means that when going around any of the four fixed points of $T^2/\{1, I_{89}\}$ the type IIB fields transform under the $-\mathbb{1}$ element of $SL(2,\mathbb{Z})$. The $-\mathbb{1}$ element of $SL(2,\mathbb{Z})$ can be geometrically interpreted as follows. Introduce a complex coordinate w describing a 2-torus with modular parameter τ, so that $w \sim w+1$ and $w \sim w+\tau$ with τ the axidilaton field. To each point of the transverse space of the 7-brane configuration one adds the 2-torus described by w. Identifying the IIB theory when going around a fixed point of $T^2/\{1, I_{89}\}$ means that one considers w and $-w$ equivalent as the transformation $w \to -w$ corresponds to the $-\mathbb{1}$ element of $SL(2,\mathbb{Z})$. Hence, the 2-torus $\{w \in \mathbb{C} \mid w \sim w+1, w \sim w+\tau, w \sim -w\}$ is the orbifold $T^2/\mathbb{Z}_2$. In this way one engineers a description in which both the metric and τ are treated fully geometrically. The orientifold background $T^2/\{1, I_{89}(-1)^{F_L}\Omega\}$ can now be described as $T^4/\mathbb{Z}_2$ where

$$T^4/\mathbb{Z}_2 = \{(z,w) \in \mathbb{C}^2 \mid z \sim z+1, z \sim z+\sigma, z \sim -z, w \sim w+1, w \sim w+\tau, w \sim -1\}\,. \tag{2.6.1}$$

The modular parameters σ and τ are arbitrary complex constants. The orbifold $T^4/\mathbb{Z}_2$ forms the starting point of the orbifold definition of the surface K3 (see for example [45]). The interpretation of the type IIB orientifold background $T^2/\{1, I_{89}(-1)^{F_L}\Omega\}$ as the orbifold limit of the K3 surface was first considered in [46].

F-theory is a 12-dimensional interpretation of certain type IIB backgrounds that was first introduced in [13]. The complex axidilaton field can be interpreted as the complex modulus of a 2-torus that is added to the 10-dimensional IIB theory. Solutions describing 7-branes can in this language be constructed by fibering the 2-torus over a 2-dimensional base manifold that is the transverse space of the 7-branes. The F-theory approach to 7-brane solutions does not try to answer the question whether or not there really exists a 12-dimensional origin of the IIB theory, instead it is just a convenient mathematical tool for analyzing 7-brane solutions. $N = 1$ supersymmetry in eight dimensions is realized by requiring the T^2 fibration over the 7-brane transverse space to form a Calabi–Yau 2-fold, i.e. a K3 surface. The K3 surface is the unique compact Calabi–Yau 2-fold.

One of the main arguments[4] in support of a 12-dimensional theory is the statement that when a $U(1)$ gauge field is introduced on the world-sheet of some (p', q') string with p' and q' relatively prime (so that it can be considered the fundamental string of the (p', q') transformed perturbative IIB string theory) the dynamics of such a string appears to have a 12-dimensional character [13, 30]. In [13] it is argued that (off-shell) the quantization of such a string would require a background space-time with signature $(10, 2)$ ([13] also gives arguments why the second timelike direction is

[4] For more arguments in support of a 12-dimensional F-theory see [47].

not visible in the construction of 7-brane solutions) and in [30] an action for a (p', q') string with two Born–Infeld vectors is constructed that seems to have a 12-dimensional interpretation. Both the arguments of [13, 30] are valid off-shell.

2.6.3 F-theory on K3

In the previous subsection it was shown that it is possible to interpret the 7-brane orientifold background geometrically using a 12-dimensional approach in which τ becomes the complex structure of a 2-torus. The orbifold $T^4/\mathbb{Z}_2$ can be obtained as a special limit of the K3 surface.

The K3 surface is the unique compact Calabi–Yau 2-fold and can be described as an elliptic fibration of T^2 over a base manifold. For the moment this will be just some 2-dimensional manifold. Later when 7-branes are included the base manifold is identified with the transverse space of the 7-branes. The axidilaton is identified with the complex structure modulus or the modular parameter of the elliptically fibered T^2. For the K3 surface the base manifold is a 2-sphere.

An elliptically fibered 2-torus is described by the following elliptic curve

$$y^2 = x^3 + P(z)x + Q(z) \,, \tag{2.6.2}$$

in which x, y, z are complex variables defined on the Riemann sphere and in which P and Q are polynomials in z. The z coordinate is the complex coordinate of the base manifold. As will be shown in section 3.11 in order for (2.6.2) to describe the K3 surface the polynomials P and Q must be of order 8 and 12, respectively. The axidilaton field τ, the modular parameter of the 2-torus, defined via (2.6.2), is given through

$$j(\tau) = \frac{4P^3}{4P^3 + 27Q^2} \,, \tag{2.6.3}$$

where $j(\tau)$ is Klein's modular j-function. Conversely, any $\tau(z)$ that does not satisfy (2.6.3) is not the modular parameter of an elliptically fibered 2-torus. Hence, functions $\tau(z)$ not satisfying (2.6.3) do not correspond to a Calabi–Yau 2-fold. The elliptic curve and τ have the scale symmetry

$$P \to \lambda^4 P \,, \quad Q \to \lambda^6 Q \,, \quad x \to \lambda^2 x \,, \quad y \to \lambda^3 y \,. \tag{2.6.4}$$

The K3 surface is without any singularities provided that $4P^3 + 27Q^2 \neq 0$. For more explicit details about K3 see for example [45, 48].

The F-theory 7-branes are located at those points on the base manifold where $4P^3 + 27Q^2$ vanishes making the fibre singular at that point. The polynomial $4P^3 + 27Q^2$ is of order 24, so that there are in total 24 F-theory 7-branes. When going around a zero of $4P^3 + 27Q^2$ the axidilaton τ will undergo the $PSL(2, \mathbb{Z})$ transformation corresponding to a particular F-theory 7-brane. These and related issues will be discussed in full detail in the next chapter. The type of singularity depends on the details of the zeros of $4P^3 + 27Q^2$, i.e. whether or not the zero of $4P^3 + 27Q^2$ is also a zero of either P and/or Q and what the orders of the zeros of P, Q and $4P^3+27Q^2$ are. The singularities of an elliptically fibered 2-torus have been classified by Kodaira (see for example [49]) and the relation between the singularity type of the singular fibre with the order of the zeros of P, Q and $4P^3+27Q^2$ follows from applying Tate's algorithm [50]. The possible singularities are listed in table 2.6.1 which has been adopted from [51]. Table 2.6.1 is useful in determining the non-Abelian parts of the 7-brane gauge groups. An A_{n-1} singularity for $n \geq 2$ leads to a gauge group $SU(n)$, a D_{n+4} singularity to gauge group $SO(2(n+4))$ and the E_6, E_7 and E_8 sigularities lead to the exceptional gauge groups E_6, E_7 and E_8. The third to fifth rows of table 2.6.1 correspond to the Argyres–Douglas singularities [52,53] denoted by H_0, H_1 and H_2. When the singularity type in table 2.6.1 is referred to as 'none' then the 7-brane gauge group is trivial when it concerns the first row (there is no 7-brane since the order of the zero of $4P^3 + 27Q^2$ is zero) and Abelian when it concerns the third row of table 2.6.1. The non-Abelian gauge groups may be enlarged by $U(1)$ factors.

In [13] it has been argued that the above-mentioned F-theory on K3 is dual to heterotic string theory compactified on T^2 with appropriately chosen Wilson lines to break the heterotic gauge groups $SO(32)$ and $E_8 \times E_8$ down to the 7-brane gauge groups. These 7-brane gauge groups depend on the positions of the 7-branes, i.e. whether they are coincident with other 7-branes or not.

Here the duality with heterotic string theory will be discussed for the generic situation in which none of the 7-branes are coincident. In order to argue for the duality the type IIB theory will be reduced over a 2-sphere with 24 7-branes and the result will be compared with the heterotic theory reduced over T^2 with Wilson lines. The reduction is not as explicit as in subsection 2.6.1 but can still be done qualitatively. The following discussion is similar to the one in [13].

The number of complex parameters describing the polynomials $P(z)$ and $Q(z)$ is 22 since P is an order 8 and Q is an order 12 polynomial. One complex parameter can be fixed using the scale symmetry (2.6.4) and three more complex parameters can be fixed using the $SL(2,\mathbb{C})$ reparametrization invariance of the Riemann sphere on which z is defined. Thus the number of free parameters characterizing F-theory on K3 with 24 7-branes is 18. These 18 moduli will appear as complex scalars in the 8-dimensional reduced theory. The 18 moduli describe the relative positions of the 24 7-branes and thus there will be a $(U(1))^{18}$ 8-dimensional gauge group. Since the reduction is over a 2-sphere not all the components of the 10-dimensional metric can be consistently turned on. The reduction of the 10-dimensional metric over S^2 will give, besides an 8-dimensional metric, two Kaluza–Klein vectors and one real scalar describing the size of the 2-sphere. There is no complex structure scalar. The 2-forms are truncated from the spectrum due to the fact that the j-function is invariant under $PSL(2,\mathbb{Z})$ and so τ and $PSL(2,\mathbb{Z})$ transformations of τ are declared equivalent. This implies

Order zero P	Order zero Q	Order zero $4P^3 + 27Q^2$	Singularity Type
≥ 0	≥ 0	0	none
0	0	n	A_{n-1}
≥ 1	1	2	none (H_0)
1	≥ 2	3	A_1 (H_1)
≥ 2	2	4	A_2 (H_2)
2	≥ 3	$n + 6$	D_{n+4}
≥ 2	3	$n + 6$	D_{n+4}
≥ 3	4	8	E_6
3	≥ 5	9	E_7
≥ 4	5	10	E_8

Table 2.6.1: The Kodaira classification of singular fibres of an elliptically fibered 2-torus. The relation between the orders of the zeros of P, Q, $4P^3 + 27Q^2$ and the singularity type follows from Tate's algorithm. When the singularity in the last column is called 'none' it means that the group contains no non-Abelian part.

that one must in fact declare equivalent all IIB field configurations that are related by $SL(2, \mathbb{Z})$. This truncates all the p-forms with $p \neq 0$ from the IIB spectrum[5]. The reduction of the 4-form that can either have two or no legs in the 2-sphere gives rise to one 2-form in eight dimensions. The resulting 8-dimensional fields comprise one $N = 1$, $d = 8$ supergravity multiplet and 18 $N = 1$, $d = 8$ vector multiplets. There are in total 20 vectors (there are two vectors in the $N = 1$, $d = 8$ supergravity multiplet) giving rise to the gauge group $(U(1))^{20}$.

The reduction of the $N = 1$, $d = 10$ heterotic supergravity multiplet over a 2-

[5]One may wonder about the 18 $U(1)$'s coming from the 7-branes since these are associated with Born–Infeld vectors that arise as the zero modes of the 2-forms. The statement is that one cannot define any 2-form globally, but since the 2-form zero modes will be formed out of functions of the solution they can be globally well-defined.

torus gives the $N=1$, $d=8$ supergravity multiplet coupled to two $N=1$, $d=8$ vector multiplets (the analysis is analogous to the reduction of the type I supergravity multiplet over T^2). The heterotic gauge groups $SO(32)$ and $E_8 \times E_8$ are of rank 16 and thus have a $(U(1))^{16}$ subgroup. Hence, one must add Wilson lines on the T^2 that break $SO(32)$ or $E_8 \times E_8$ down to $(U(1))^{16}$. Therefore, also the reduction of heterotic supergravity over T^2 can give rise to one $N=1$, $d=8$ supergravity multiplet and 18 $N=1$, $d=8$ vector multiplets. In the duality the heterotic dilaton is identified with the real Kähler modulus describing the size of the 2-sphere. Weakly coupled heterotic string theory corresponds to a small S^2.

The orbifold limit in which the K3 with 24 7-branes reduces to $T^4/\mathbb{Z}_2$ of the previous subsection is obtained by taking $P^3 = cQ^2$ in which c is some nonzero complex number [46]. This implies that $P = R^2$ and $Q = c^{-1/2}R^3$ where R is an arbitrary polynomial of order four. The scaling symmetry (2.6.4) and the freedom to perform $SL(2,\mathbb{C})$ transformations imply that R has one unfixed complex parameter. The one free complex parameter is the complex structure modulus σ of the base manifold $T^2/\mathbb{Z}_2$. The 7-branes are located at the zeros of $4P^3 + 27Q^2$ which in the limit becomes equal to R^6. Hence, there are in total four zeros at each of which six $\det Q = 0$ (p,q,r) 7-branes have come together. This admits an interpretation in terms of 4 D7-branes plus one O7-plane as will be explicitly shown in section 3.11. The F-theory gauge symmetry $(U(1))^{20}$ in the orbifold limit is enhanced to $(SO(8))^4 \times (U(1))^4$ which is T-dual to type I as argued 2.6.1. This can be understood from the heterotic side as follows. The orbifold limit corresponds to an enhancement of the fully broken $(U(1))^{16}$ gauge group, coming from the Yang–Mills sector reduced over the 2-torus, to $(SO(8))^4$ (which is both a subgroup of $SO(32)$ and $E_8 \times E_8$). If on the F-theory side one considers a large sized base manifold $T^2/\mathbb{Z}_2$ then the heterotic theory becomes strongly coupled. Applying an S-duality to the strongly coupled heterotic theory then gives weakly coupled type I on T^2 with gauge group $(SO(8))^4$. The emergence of the gauge group $(SO(8))^4$ on the F-theory side can be understood by taking $P = R^2$ and $Q = c^{-1/2}R^3$ and looking in table 2.6.1. Since R has four different zeros, and each zero of R is a second order zero of P, a third order zero of Q and sixth order zero of $4P^3 + 27Q^2$ the gauge group, according to table 2.6.1, at each zero of R should be $SO(8)$ and hence a gauge group $(SO(8))^4$ appears.

In [54] it is shown that there exist more orbifold limits of $K3$ that give rise to new gauge symmetry enhancements involving the exceptional groups E_8, E_7 and E_6. These F-theory orbifolds are $T^4/\mathbb{Z}_n$ with $n = 3, 4, 6$ and require that either the polynomial P or the polynomial Q are equal to zero, so that τ is fixed to be either ρ or i, respectively.

Finally, it is mentioned here that there also exists a fiberwise T-duality between F-theory on K3 and M-theory on a T^2 fibered over some 9-dimensional manifold. This is explained for example in [15]. This last point is mentioned to underline the richness of the F-theory moduli space. One can construct dualities between F-theory on K3 with type I on T^2, with both the heterotic theories on T^2 and with M-theory on a fibered T^2.

Chapter 3

Seven-branes

3.1 Introduction

This chapter will start by discussing the local properties of 7-branes. By local is meant the properties of a single 7-brane for which, as will be shown, there do not exist any globally well-defined solutions[1]. Starting in section 3.8 globally well-defined solutions that necessarily contain multiple 7-branes will be discussed. The discussion presented in this chapter is based on [16, 17].

3.2 Seven-brane sources

In subsection 1.1.3 it is shown that it is possible to introduce an $SU(1,1)$ triplet of 8-form potentials, $A_8^{\alpha\beta}$. A generic 8-form can be written as $q_{\alpha\beta}A_8^{\alpha\beta}$, where $q_{\alpha\beta}$ is the $SU(1,1)$ charge tensor defined in section 1.2. Knowing that there exists such a triplet of 8-forms a natural question to ask is: does $q_{\alpha\beta}A_8^{\alpha\beta}$ couple electrically to some 7-brane? The answer will prove to be yes as long as $\det Q \geq 0$, where the matrix Q is related to $q_{\alpha\beta}$ via eqs. (1.2.7) and (1.3.38). To establish the existence of these 7-branes, supergravity solutions for them will be constructed. In principle one can then from the solutions derive statements about dynamics through a study of the zero modes. In the following it will be assumed, unless stated otherwise, that the 2- and 4-form field strengths are zero.

The 7-brane world-volume is denoted by $\mathcal{M}_8$ in (3.2.3). The world-volume metric g_{AB} (whose determinant is denoted by $g_{(8)}$) and 8-form $q_{\alpha\beta}A^{\alpha\beta}_{A_1\ldots A_8}$ are the pull-backs

[1] The term 'single 7-brane' here means a 7-brane solution that has an electric description in terms of a single 8-form.

of the target space-time metric $g_{\mu\nu}$ and 8-form $q_{\alpha\beta}A^{\alpha\beta}_{\mu_1\ldots\mu_8}$,

$$g_{AB} = \frac{\partial X^\mu}{\partial\sigma^A}\frac{\partial X^\nu}{\partial\sigma^B}g_{\mu\nu}\,, \qquad (3.2.1)$$

$$q_{\alpha\beta}A^{\alpha\beta}_{A_1\ldots A_8} = \frac{\partial X^{\mu_1}}{\partial\sigma^{A_1}}\cdots\frac{\partial X^{\mu_8}}{\partial\sigma^{A_8}}q_{\alpha\beta}A^{\alpha\beta}_{\mu_1\ldots\mu_8}\,, \qquad (3.2.2)$$

where the $X^\mu(\sigma)$ are the embedding coordinates of the 7-brane. In section 2.2 it is shown that the following static 7-brane action

$$S = -\int_{\mathcal{M}_8} d^8\sigma\, T\sqrt{-g_{(8)}} + \int_{\mathcal{M}_8} q_{\alpha\beta}A_8^{\alpha\beta}\,, \qquad (3.2.3)$$

where T is defined in eq. (1.2.3), preserves half of the IIB supersymmetries with a projector given by

$$P\epsilon = \frac{1}{2}\left(1 + i\gamma_{\underline{0}\cdots\underline{7}}\right)\epsilon = \frac{1}{2}\left(1 - i\gamma_{\underline{8}}\gamma_{\underline{9}}\right)\epsilon = 0\,, \qquad (3.2.4)$$

for a 7-brane extended in the directions $x^1,\ldots,x^7$. Here the static gauge,

$$\frac{\partial X^\mu}{\partial\sigma^A} = \delta^\mu_A\,, \qquad (3.2.5)$$

has been employed.

3.3 Electric coupling of 7-branes

The field strengths $F_9^{\alpha\beta}$ of the 8-forms $A_8^{\alpha\beta}$ are defined in eqs. (1.1.34) and (1.1.35). In section 1.2 it is shown that the 8-form $q_{\alpha\beta}A_8^{\alpha\beta}$ is dual to the scalar χ' defined in eq. (1.2.4). The duality relation between $q_{\alpha\beta}A_8^{\alpha\beta}$ and χ' is given in eq. (1.2.26). The equation of motion for $q_{\alpha\beta}F_9^{\alpha\beta}$ immediately follows from eq. (1.2.26) and reads

$$d\star\left[(T^2 - 4\det Q)^{-1}q_{\alpha\beta}F_9^{\alpha\beta}\right] = 0\,. \qquad (3.3.1)$$

The equations of motion for T and χ' follow from the IIB action (1.1.17) in which G_3 and F_5 are put to zero and in which P is given by eq. (1.2.8).

The equations of motion for T and $q_{\alpha\beta}A_8^{\alpha\beta}$ can be obtained from the following action

$$S = \int_{\mathcal{M}_{10}}\left(\star 1\, R - \frac{1}{2}\frac{\star dT\wedge dT}{T^2 - 4\det Q} - \frac{1}{2}\frac{\star q_{\alpha\beta}F_9^{\alpha\beta}\wedge q_{\alpha\beta}F_9^{\alpha\beta}}{T^2 - 4\det Q}\right)\,. \qquad (3.3.2)$$

In order to couple the action to the 7-brane source action (3.2.3) one simply adds (3.2.3) to (3.3.2). This coupled system provides an electric coupling description. It will prove useful to write the source action (3.2.3) as a 10-dimensional bulk integral. To this end a 7-brane current 8-form with delta function support on the 7-brane world-volume, is introduced. This current is denoted by J_8 and it can be written as

$$J^{\mu_1\cdots\mu_8} = \frac{1}{\sqrt{-g}}\int_{\mathcal{M}_8} dX^{\mu_1}\wedge\cdots\wedge dX^{\mu_8}\,\delta(x - X(\sigma))\,. \tag{3.3.3}$$

The 2-form dual of the current is a closed form

$$d\star J_8 = 0\,. \tag{3.3.4}$$

The electrically coupled system is then described by the action

$$\begin{aligned} S \;=\; & \int_{\mathcal{M}_{10}}\left(\star 1\,R - \frac{1}{2}\frac{\star dT\wedge dT}{T^2 - 4\det Q} - \frac{1}{2}\frac{\star q_{\alpha\beta}F_9^{\alpha\beta}\wedge q_{\alpha\beta}F_9^{\alpha\beta}}{T^2 - 4\det Q}\right) \\ & - \int_{\mathcal{M}_{10}} d^{10}x\int_{\mathcal{M}_8} d^8\sigma\,\delta(x - X(\sigma))\,T\sqrt{-g_{(8)}} \\ & + \int_{\mathcal{M}_{10}} q_{\alpha\beta}A_8^{\alpha\beta}\wedge\star J_8\,. \end{aligned} \tag{3.3.5}$$

The sourced version of eq. (3.3.1) is

$$d\star\left[(T^2 - 4\det Q)^{-1}q_{\alpha\beta}F_9^{\alpha\beta}\right] = \star J_8\,. \tag{3.3.6}$$

3.4 Magnetic coupling of 7-branes

In order to describe the 7-brane coupling magnetically a first order action is constructed for a 1-form F_1 that satisfies the Bianchi identity

$$dF_1 = 0\,, \tag{3.4.1}$$

so that locally $F_1 = d\chi'$. This Bianchi identity can be obtained from an action principle by introducing an 8-form $q_{\alpha\beta}A_8^{\alpha\beta}$ as a Lagrange multiplier field whose equation of motion is (3.4.1). The first order action for F_1 is

$$\begin{aligned} S \;=\; & \int_{\mathcal{M}_{10}}\left(\star 1\,R - \frac{1}{2}\frac{\star dT\wedge dT}{T^2 - 4\det Q} - \frac{1}{2}\left(T^2 - 4\det Q\right)\star F_1\wedge F_1\right. \\ & \left. - F_1\wedge q_{\alpha\beta}dA_8^{\alpha\beta}\right)\,. \end{aligned} \tag{3.4.2}$$

The equation of motion for $q_{\alpha\beta}A_8^{\alpha\beta}$ gives (3.4.1). Substituting the solution, $F_1 = d\chi'$, to (3.4.1) back into the action (3.4.2) one finds the action for T and χ'. The equation of motion for F_1 gives the duality relation between F_1 and $q_{\alpha\beta}dA_8^{\alpha\beta}$. Substituting this back into the action (3.4.2) the action for $q_{\alpha\beta}A_8^{\alpha\beta}$, eq. (3.3.2), is found.

Once again, in order to couple this action to the 7-brane source term (3.2.3) one simply adds (3.2.3) to (3.4.2). The magnetically coupled system is described by the action

$$\begin{aligned} S &= \int_{\mathcal{M}_{10}} \Big(\star 1\, R - \frac{1}{2}\frac{\star dT \wedge dT}{T^2 - 4\det Q} - \frac{1}{2}\left(T^2 - 4\det Q\right) \star F_1 \wedge F_1 \\ &\quad - F_1 \wedge q_{\alpha\beta} dA_8^{\alpha\beta}\Big) - \int_{\mathcal{M}_{10}} d^{10}x \int_{\mathcal{M}_8} d^8\sigma\, \delta(x - X(\sigma))\, T\sqrt{-g_{(8)}} \\ &\quad + \int_{\mathcal{M}_{10}} q_{\alpha\beta}A_8^{\alpha\beta} \wedge \star J_8 \,. \end{aligned} \tag{3.4.3}$$

The equations of motion for $q_{\alpha\beta}A_8^{\alpha\beta}$ and F_1 are

$$dF_1 = \star J_8 = d\star G_9 \quad \Rightarrow \quad F_1 = d\chi' + \star G_9 \,, \tag{3.4.4}$$

and

$$\star q_{\alpha\beta} dA_8^{\alpha\beta} = \left(T^2 - 4\det Q\right) F_1 \,, \tag{3.4.5}$$

respectively. These two equations implicitly define the axion χ'. In eq. (3.4.4) the object G_9 has been introduced. Since $\star J_8$ satisfies $d \star J_8 = 0$ it can, at least locally, be represented as the differential of a 1-form that will be called $\star G_9$. One has

$$\star J_8 = d\star G_9 \tag{3.4.6}$$

in which

$$G^{\mu_1\cdots\mu_9} = \frac{1}{\sqrt{-g}} \int_{\mathcal{M}_9} dX^{\mu_1} \wedge \cdots \wedge dX^{\mu_9}\, \delta(x - X(\xi)) \,. \tag{3.4.7}$$

In the last equation the delta function has support on a 9-dimensional surface $\mathcal{M}_9$, parameterized by coordinates ξ, whose boundary is the world-volume $\mathcal{M}_8$ of the 7-brane. The 9-dimensional surface is associated with the world-volume of a Dirac 8-brane, which is a brane generalization of the Dirac string stemming from a monopole. In the present case the Dirac 8-brane is stemming from the 7-brane. It is by means of the Dirac 8-brane that the 7-brane will couple magnetically to the axion field strength $F_1 = d\chi'$. Substituting the solution (3.4.4) for F_1 back into the action (3.4.2) coupled to (3.2.3) one obtains

$$\begin{aligned} S &= \int_{\mathcal{M}_{10}} \left(\star 1\, R - \frac{1}{2}\frac{\star dT \wedge dT}{T^2 - 4\det Q} - \frac{1}{2}\left(T^2 - 4\det Q\right) \star F_1 \wedge F_1 \right) \\ &\quad - \int_{\mathcal{M}_{10}} d^{10}x \int_{\mathcal{M}_8} d^8\sigma\, \delta(x - X(\sigma))\, T\, \sqrt{-g_{(8)}} \,, \end{aligned} \tag{3.4.8}$$

where $F_1 = d\chi' + \star G_9$ and the Wess–Zumino term has disappeared.

Using eqs. (1.2.8) and (1.2.9) the result for the magnetic coupling can be written as

$$\begin{aligned} S &= \int_{\mathcal{M}_{10}} \left(\star 1\, R - 2 \star \hat{P} \wedge \hat{P}^*\right) \\ &\quad - \int_{\mathcal{M}_{10}} d^{10}x \int_{\mathcal{M}_8} d^8\sigma\, \delta(x - X(\sigma))\, q_{\alpha\beta} V_-^\alpha V_+^\beta \sqrt{-g_{(8)}}\,, \end{aligned} \tag{3.4.9}$$

where $\hat{P}$ is

$$\hat{P} = P + \frac{i}{2} q_{\alpha\beta} V_+^\alpha V_+^\beta \star G_9\,. \tag{3.4.10}$$

In terms of $\hat{P}$ the duality relation between the 8-form $q_{\alpha\beta}A_8^{\alpha\beta}$ and the axion χ' in the presence of sources takes the form

$$q_{\alpha\beta} F_9^{\alpha\beta} = -i \star \left(q_{\alpha\beta} V_-^\alpha V_-^\beta \hat{P} - q_{\alpha\beta} V_+^\alpha V_+^\beta \hat{P}^*\right)\,. \tag{3.4.11}$$

This relates the equations of motion and the Bianchi identity of the 8-form $q_{\alpha\beta}A_8^{\alpha\beta}$ to the Bianchi identity and the equations of motion of the axion χ'. The Bianchi identity for $\hat{P}$ can be written as

$$\hat{D}\hat{P} \equiv d\hat{P} - 2i\hat{Q} \wedge \hat{P} = \frac{i}{2} q_{\alpha\beta} V_+^\alpha V_+^\beta \star J_8\,, \tag{3.4.12}$$

in which

$$\hat{Q} = Q - \frac{1}{2} q_{\alpha\beta} V_+^\alpha V_-^\beta \star G_9\,, \tag{3.4.13}$$

with Q the composite $U(1)$ gauge field defined in (1.1.9). Eq. (3.4.12) is the 'sourced' version of the Bianchi identity (1.1.12).

The magnetic coupling can also be described with respect to the axidilaton field $(\tau, \bar{\tau})$. To achieve this one chooses a $U(1)$ gauge for $V_\pm^\alpha$ appearing in (3.4.9). One such $U(1)$ gauge is given in eqs. (1.3.81) to (1.3.84). Then writing $q_{\alpha\beta}$ in terms of the real numbers p, q, r via eqs. (1.2.7) and (1.3.38) one finds

$$\begin{aligned} S &= \int_{\mathcal{M}_{10}} d^{10}x \sqrt{-g} \left(R - \frac{1}{2(\mathrm{Im}\,\tau)^2} \mid \partial_\mu \tau + (p + q\tau^2 + r\tau)\, (\star G_9)_\mu \mid^2\right) \\ &\quad - \int_{\mathcal{M}_{10}} d^{10}x \int_{\mathcal{M}_8} d^8\sigma\, \delta(x - X(\sigma))\, \frac{1}{\mathrm{Im}\tau} (p + q|\tau|^2 + r\mathrm{Re}\,\tau) \sqrt{-g_{(8)}}\,. \end{aligned} \tag{3.4.14}$$

If one takes $p = 1$ and $q = r = 0$ the action (3.4.14) describes the coupling of a D7-brane. The units used in (3.4.14) are

$$16\pi G_N^{(10)} = 1\,, \tag{3.4.15}$$

where $G_N^{(10)}$ is the 10-dimensional Newton's constant. To restore the factors of $G_N^{(10)}$ in the action (3.4.14) it is noted that the bulk part gets multiplied by a factor of $(16\pi G_N^{(10)})^{-1}$ and the tension of a D7-brane is normalized such that it is given by

$$\frac{e^{\phi}}{(2\pi)^7\, l_s^8\, g_s}\,. \tag{3.4.16}$$

Here l_s is the string length, the square root of α' and g_s is the string coupling constant, the vacuum expectation value of e^{ϕ}. Customarily, the 10-dimensional Newton's constant is related to l_s via

$$16\pi G_N^{(10)} = (2\pi)^7\, l_s^8\, g_s^2\,. \tag{3.4.17}$$

Hence, for the coupling of a D7-brane one would write

$$\begin{aligned} S \;\; = \;\; & \frac{1}{16\pi G_N^{(10)}}\int_{\mathcal{M}_{10}} d^{10}x\sqrt{-g}\left(R - \frac{1}{2(\mathrm{Im}\,\tau)^2}\mid \partial_\mu\tau + (\star G_9)_\mu \mid^2\right) \\ & -\frac{g_s}{16\pi G_N^{(10)}}\int_{\mathcal{M}_{10}} d^{10}x\int_{\mathcal{M}_8} d^8\sigma\,\delta(x - X(\sigma))\,\frac{1}{\mathrm{Im}\tau}\sqrt{-g_{(8)}}\,. \end{aligned} \tag{3.4.18}$$

It follows that the relative coefficient between bulk and source terms does not depend on l_s or α'. The factor of g_s in front of the source term can be rescaled away by shifting the value of the dilaton ϕ while at the same time rescaling the value of the RR axion.

3.4.1 Unobservability of the Dirac 8-brane

To describe the magnetic coupling of the 7-brane to the axion the Dirac 8-brane, eqs. (3.4.6) and (3.4.7), has been introduced into the action (3.4.8). As in the classical Dirac monopole problem, the Dirac brane is not a physical object, i.e. the dynamics of the system should not depend on the orientation of the Dirac 8-brane in the bulk. This is reflected in the fact that the 8-brane equations of motion are not independent. They are satisfied provided the axion field equations hold. To see this, let us derive the axion field equation and the equation of motion of the embedding coordinates of the Dirac 8-brane.

The field equation of χ' is

$$\partial_\mu\left[\sqrt{-g}\,\left(T^2 - 4\det Q\right)\,F^\mu\right] = 0\,, \tag{3.4.19}$$

where F_1 satisfies the Bianchi identity (3.4.4). The variation of (3.4.8) with respect to the Dirac 8-brane world-volume coordinates $X^\mu(\xi)$, appearing in F_1 produces the equation

$$\partial_\mu\left[\sqrt{-g}\,\left(T^2 - 4\det Q\right)\,F^\mu\right]\mid_{\mathcal{M}_9} = 0\,, \tag{3.4.20}$$

which is nothing but the χ' field equation, eq. (3.4.19), pulled-back on the Dirac 8-brane world-volume. Therefore, the Dirac brane is not physical. Its position can be anywhere in space-time and it is invisible provided the Dirac veto holds, which does not allow the Dirac brane to intersect the world-volumes of the objects coupled to χ' in an electric way. If such objects (which would be instantons) are present, their 'currents' contribute to the right hand side of eq. (3.4.19), while eq. (3.4.20) remains sourceless. The two equations are then consistent provided the world-volumes of the Dirac brane and the electrically charged objects never intersect[2]. In quantum theory, as is well known, the unobservability of the Dirac branes is guaranteed by the Dirac quantization condition which results in the quantization of corresponding fluxes.

3.5 Dirac strings and monodromy

The hatted 1-form fields $\hat{P}$ and $\hat{\mathcal{Q}}$ defined in (3.4.10) and (3.4.13) can be collected into the matrix-valued 1-form

$$V \begin{pmatrix} -i\hat{Q} & \hat{P} \\ \hat{P}^* & i\hat{Q} \end{pmatrix} V^{-1} = dVV^{-1} - SQS^{-1} \star G_9 \,, \tag{3.5.1}$$

with S defined in eq. (1.2.7) and where V is the matrix defined in equation (1.1.4). Let us define

$$\hat{p} = p - SQS^{-1} \star G_9 \,, \tag{3.5.2}$$

where p is defined in eq. (1.1.15). Then $\hat{p}$ satisfies the Bianchi identity

$$d\hat{p} - 2p \wedge p = -SQS^{-1} \star J_8 \,. \tag{3.5.3}$$

Outside the source the Bianchi identity (3.5.3) is solved by

$$p = \frac{1}{2}(dC)C^{-1} \,, \tag{3.5.4}$$

where $C = VV^\dagger$. Alternatively, this solution can be written as

$$DC = 0 \qquad \text{with} \qquad D = d - 2p \,. \tag{3.5.5}$$

This equation can be interpreted as saying that C is parallel transported with respect to the flat connection p. Let $\gamma(\lambda)$ be some path parameterized by λ which runs from

[2] Additional complications and subtleties regarding the Dirac branes and corresponding singular terms in the action and equations of motion arise when the action contains Wess–Zumino terms with 'bare' electric and/or magnetic potentials. In such cases it becomes much less trivial to reconcile the Dirac veto with the physical field equations. This happens for example in the case of the M5-brane [55]. In [56–59] a consistent method was developed to resolve these problems and related problems of anomalies.

0 to 1. Then one has

$$C(\lambda = 1) = \mathcal{P}\exp[2\int_\gamma p]\ C(\lambda = 0)\,, \tag{3.5.6}$$

where $\mathcal{P}$ denotes the path ordering symbol. Since the connection is flat the quantity $\mathcal{P}\exp[2\int_\gamma p]$ for closed γ will only depend on the base point of the closed path. The location of the base point can be changed by a similarity transformation,

$$\mathcal{P}\exp[2\oint_\gamma p] \;\rightarrow\; H\,\mathcal{P}\exp[2\oint_\gamma p]\,H^{-1} \quad \text{where} \quad H = \mathcal{P}\exp[2\int_{\tilde\gamma} p]\,, \tag{3.5.7}$$

with the path $\tilde\gamma$ connecting the initial to the final base point. This means that the eigenvalues of the monodromy matrix, $\mathcal{P}\exp[2\oint_\gamma p]$, are preserved under shifting the position of the base point. Therefore a physical quantity that can be associated with the Bianchi identity $dp - 2p \wedge p = 0$ is the Wilson line[3]

$$\mathrm{Tr}\,\mathcal{P}\exp[2\oint_\gamma p]\,. \tag{3.5.8}$$

For 7-brane solutions the matrix p only depends on the two coordinates transverse to the brane. In that case the quantity $\mathcal{P}\exp[2\oint_\gamma p]$ will determine the monodromy of C and thus of the scalars that parameterize it.

The monodromy of the matrix $C = VV^\dagger$ is given by

$$C(\lambda = 1) = \mathcal{P}\exp[2\oint_\gamma p]\ C(\lambda = 0)\,. \tag{3.5.9}$$

Let us consider a path γ which encircles the 7-brane (point) source in the transverse space. Further it is assumed that γ encloses an area of infinitesimal size, denoted by D. Expanding the path-ordered expression up to second order it is found that

$$C(\lambda = 1) = \left(1 + \int_D (d\hat p - 2p \wedge p) + \cdots\right) C(\lambda = 0) \left(1 + \int_D (d\hat p - 2p \wedge p)^\dagger\right)\,, \tag{3.5.10}$$

where use has been made of the fact that $pC = Cp^\dagger$ and $\oint_\gamma p = \int_D d\hat p$. According to eq. (3.5.3) this can be written as

$$C(\lambda = 1) = \left(1 - SQS^{-1} + \cdots\right) C(\lambda = 0) \left(1 - (SQS^{-1})^\dagger + \cdots\right)\,. \tag{3.5.11}$$

Since the monodromy of $C = VV^\dagger$ when going at an infinitesimal distance around a 7-brane is known and since the parametrization of V in terms of τ, see eqs. (1.3.81) to

[3] The terminology is borrowed from Yang–Mills theory. Here p is not a gauge field. It is because of a mathematical similarity that this quantity is referred to as a Wilson line.

(1.3.84), is known the monodromy of τ can be obtained. It follows that τ transforms as

$$\tau \to e^{Q}\tau\,. \tag{3.5.12}$$

The Wilson line (3.5.8) when evaluated around the contour γ encircling a 7-brane at an infinitesimal distance can be evaluated and is equal to $\mathrm{Tr}\,e^{Q}$. Hence, the Wilson line computes what is called the $SL(2,\mathbb{R})$ conjugacy class.

In order to evaluate the Wilson line (3.5.8) up to the first nontrivial term the following general result [60] is used. Let $A(\epsilon)$ be an arbitrary $SU(1,1)$ matrix which tends to 1 as $\epsilon \to 0$ then one can write A as

$$A(\epsilon) = \exp\left[\epsilon\alpha_1^i T^i + \epsilon^2\alpha_2^i T^i + \cdots\right]\,, \tag{3.5.13}$$

where the T^i are the generators of the $SU(1,1)$ group and where the α_1^i etc. are arbitrary functions. Expanding $A(\epsilon)$ up to second order the trace of A (up to this order in ϵ) is given by

$$\mathrm{Tr}A(\epsilon) = 2 + \frac{1}{2}\mathrm{Tr}\left(\epsilon\alpha_1^i T^i\right)^2 + \cdots\,. \tag{3.5.14}$$

Applying this formula to eq. (3.5.8) with

$$\epsilon\alpha_1^i T^i = 2\int_D (d\hat{p} - 2p\wedge p) \tag{3.5.15}$$

the following result is found

$$\mathrm{Tr}\mathcal{P}\exp[2\oint_\gamma p] = 2 + \frac{1}{2}\mathrm{Tr}\,4(SQS^{-1})^2 + \cdots = 2 - 4\det Q + \cdots\,. \tag{3.5.16}$$

When the globally well-defined 7-brane solutions will be discussed in section 3.10 and onwards the base point dependence discussed in this section will play a role in characterizing the various 7-branes in the solution. It will turn out that one can only meaningfully speak of 7-branes as representative elements of $SL(2,\mathbb{Z})$ conjugacy classes, a result that is related to the base point independence of the Wilson line (3.5.16).

3.6 Supersymmetry and holonomy of the Killing spinor

In order to describe 7-brane solutions sourced by eq. (3.2.3) the Killing spinor equations must be solved imposing the supersymmetry projector (3.2.4). The Killing spinor equations are

$$\delta\lambda \;=\; -\frac{i}{\tau-\bar{\tau}}\,(\gamma^{\mu}\partial_{\mu}\tau)\,\epsilon_C = 0\,, \tag{3.6.1}$$

$$\delta\psi_{\mu} \;=\; \left(\partial_{\mu} + \tfrac{1}{4}\omega_{\mu}{}^{ab}\gamma_{ab} + \frac{1}{4(\tau-\bar{\tau})}\partial_{\mu}(\tau+\bar{\tau})\right)\epsilon = 0\,. \tag{3.6.2}$$

These Killing spinor equations are obtained from (1.3.72) and (1.3.73) in which P and Q are taken to be (1.3.74) and (1.3.75).

Eqs. (3.6.1) and (3.6.2) transform covariantly under the $SL(2,\mathbb{R})$ transformations (1.3.39) and (1.3.103). Observe that, unlike τ, the Killing spinor ϵ does transform under $S^2 = -\mathbb{1}$ as $\epsilon \to i\,\epsilon$. Under $S^4 = \mathbb{1}$ one has $\epsilon \to -\epsilon$. Only S^8 acts as the identity on ϵ. This means that ϵ transforms under the double cover of $SL(2,\mathbb{Z})$.

From eq. (3.2.4) it follows that $(\gamma_{\underline{8}} - i\gamma_{\underline{9}})\,\epsilon = 0 = (\gamma_{\underline{8}} + i\gamma_{\underline{9}})\,\epsilon_C$. If it is assumed that τ and the metric do not depend on the world-volume coordinates $x^0,\ldots,x^7$ and a conformally flat transverse metric is chosen, then eq. (3.6.1) implies $(\partial_8 - i\partial_9)\,\bar{\tau} = 0$. The complex transverse coordinate $z = x^8 + ix^9$ is introduced so that now $\partial_z\bar{\tau} = 0$, i.e. τ is a holomorphic function. In complex coordinates the condition on ϵ can be written as $\gamma_{\underline{z}}\epsilon = 0$. Under these conditions, the most general 7-brane solution to eqs. (3.6.1) and (3.6.2) is given by [24, 61–63]

$$ds^2 \;=\; -dt^2 + d\vec{x}_7^{\,2} + (\mathrm{Im}\,\tau)|f|^2 dz d\bar{z}\,, \tag{3.6.3}$$

$$\tau \;=\; \tau(z)\,, \tag{3.6.4}$$

$$f \;=\; f(z)\,, \tag{3.6.5}$$

$$\epsilon \;=\; \left(\frac{\bar{f}}{f}\right)^{1/4}\epsilon_0\,, \tag{3.6.6}$$

where ϵ_0 is a constant spinor satisfying $\gamma_{\underline{z}}\epsilon_0 = 0$.

The functions τ and f are assumed to be defined on the Riemann sphere. The form of the solution is therefore fixed up to $SL(2,\mathbb{C})$ transformations

$$z \to \frac{az+b}{cz+d}\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})\,. \tag{3.6.7}$$

These are the most general global coordinate transformations that do not change the structure of the branch cuts and singularities of τ and f in the complex z-plane. Note that locally (but not globally) one can always choose a basis in which $f(z) = 1$. This, however, has the disadvantage of introducing multi-valued coordinates w defined via $dw = f(z)dz$.

Although the configurations (3.6.3)–(3.6.6) are locally supersymmetric, they must satisfy further conditions to be globally well-defined and supersymmetric. The main issue here will be the possible multi-valuedness of $\tau(z)$ and $f(z)$, due to the presence of branch cuts.

Whenever a branch cut of the function τ is crossed τ and ϵ must transform as in eqs. (1.3.39) and (1.3.103). The transformation rule for the function f when crossing a branch cut can be obtained by requiring that the holonomy of ϵ be well-defined. The holonomy of the Killing spinor is computed with respect to the generalized connection in (3.6.2), which is the sum of the Lorentz connection and the $U(1)$ connection of the $SU(1,1)/U(1)$ coset model. The integrability condition of (3.6.2) requires that the total curvature vanishes but the Riemann curvature of the transverse space and the $U(1)$ curvature are, separately, non-trivial.

If the Killing spinor ϵ is parallel-transported using the connection in (3.6.2), evaluated on the solution, eqs. (3.6.3) and (3.6.4), from a base point b around a closed loop γ_b it can be shown that the holonomy (with respect to the Lorentz group) of ϵ is given by

$$\epsilon(b) \to \exp\left(-\frac{i}{2}\,\mathrm{Im}\oint_{\gamma_b}(\log f)'dz\right)\epsilon(b)\,, \tag{3.6.8}$$

where the prime denotes differentiation with respect to z.

The holonomy phase factor will depend on the base point b but only through the homotopy class of γ_b due to the vanishing total curvature. The holonomy with respect to the generalized connection will be trivial if eq. (3.6.8) equals[4] an $SL(2,\mathbb{R})$ transformation as given in eq. (1.3.103)

$$\exp\left(-\frac{i}{2}\,\mathrm{Im}\oint_{\gamma_b}(\log f)'dz\right) = e^{i\beta}\,. \tag{3.6.9}$$

Let γ_b be parameterized by $\lambda \in [0,1]$. Then

$$\exp\left(\frac{i}{2}\,\mathrm{Im}\oint_{\gamma_b}(\log f)'dz\right) = \left(\frac{f(\lambda=1)}{|f(\lambda=1)|)}\right)^{1/2}\left(\frac{|f(\lambda=0)|}{f(\lambda=0))}\right)^{1/2}\,. \tag{3.6.10}$$

The requirement (3.6.9) then leads to the following condition for the function f

$$f(\lambda=1) = (c\tau+d)f(\lambda=0)\,. \tag{3.6.11}$$

Thus, when crossing a branch cut it must be that

$$f \to (c\tau+d)f\,. \tag{3.6.12}$$

The $SL(2,\mathbb{R})$ transformation properties of τ, f and ϵ are summarized in table 3.6.1.

The metric $g_{\mu\nu}$ does not transforms under $SL(2,\mathbb{R})$ and must be single-valued modulo coordinate transformations. On the other hand, $\mathrm{Im}\,\tau$ appears explicitly in

[4] In general one can also allow for nontrivial spin structures.

Fields	Group	Order of S
τ	$PSL(2,\mathbb{R})$	2
f	$SL(2,\mathbb{R})$	4
ϵ	double cover of $SL(2,\mathbb{R})$	8

Table 3.6.1: Some $SL(2,\mathbb{R})$ transformation properties of τ, f and ϵ.

the expression (3.6.3) for $g_{\mu\nu}$ and it may transform into $|c\tau+d|^{-2}\mathrm{Im}\,\tau$ when crossing a branch cut. In general, the extra factor $|c\tau+d|^{-2}$ cannot be eliminated by an $SL(2,\mathbb{C})$ transformation and, thus, it must be compensated by $f(z)$. From eq. (3.6.12) it is clear that the metric remains invariant when crossing a branch cut of the function τ.

3.7 Field equations and local 7-brane solutions

3.7.1 The equation of motion of X^μ

The 7-brane equation of motion obtained by varying (3.4.8) or (3.2.3) with respect to the world-volume field $X^\mu(\sigma)$ is

$$\begin{aligned} & g_{\mu\nu}\nabla_A\left(T\sqrt{-g_8}\,g^{AB}\tfrac{\partial X^\mu}{\partial\sigma^B}\right)-\sqrt{-g_8}\,\tfrac{\partial T}{\partial X^\mu}= \\ & \tfrac{1}{8!}e^{A_1\ldots A_8}\,\tfrac{\partial X^{\mu_1}}{\partial\sigma^{A_1}}\cdots\tfrac{\partial X^{\mu_8}}{\partial\sigma^{A_8}}\,\epsilon_{\mu_1\ldots\mu_8\mu\rho}\,\left(T^2-4\det Q\right)\,F^\rho\,, \end{aligned} \tag{3.7.1}$$

where F_1 is defined in eq. (3.4.4) and where

$$\nabla_A X^\mu_B=\partial_A X^\mu_B-\Gamma^C_{AB}\,X^\mu_C+\frac{\partial X^\rho}{\partial\sigma^A}\Gamma^\mu_{\rho\nu}\,X^\nu_B\,, \tag{3.7.2}$$

for some tensor X^μ_B in which Γ^C_{AB} is the Levi–Civitá connection of the pulled-back metric g_{AB} and in which $\Gamma^\mu_{\rho\nu}$ is the Levi–Civitá connection of the target space-time metric $g_{\mu\nu}$. The objects $e^{A_1\ldots A_8}$ and $\epsilon_{\mu_1\ldots\mu_8\mu\rho}$ are defined in appendix A. When varying eq. (3.2.3) with respect to $X^\mu(\sigma)$ one employs eq. (3.4.5) to bring the result in the form (3.7.1).

Employing the static gauge (3.2.5) the supersymmetric configuration (3.6.3) and (3.6.4) corresponds to a 7-brane that is static and does not fluctuate in the transverse directions, i.e. $X^{8,9}=\mathrm{cst}$. From the point of view of the 2-dimensional transverse space the Dirac 8-brane has reduced to a Dirac string ending on a magnetic point source. The equation of motion for X^μ in static gauge (3.2.5) with $X^{8,9} = \text{cst}$ reduces to

$$\partial_i T = \left(T^2 - 4\det Q\right)\sqrt{-g}\,\epsilon_{01\ldots 7ij}\left(\partial^j\chi' - (\star G_9)^j\right)\,, \tag{3.7.3}$$

with $i, j = 8, 9$. Away from the source and the Dirac string one finds

$$\partial_8 T = (T^2 - 4\det \mathrm{Q})\partial_9\chi'\,, \quad \partial_9 \mathrm{T} = -(\mathrm{T}^2 - 4\det \mathrm{Q})\partial_8\chi'\,. \tag{3.7.4}$$

Specializing eqs. (3.7.4) to the cases $\det Q = 0$ and $\det Q > 0$ it is found that

$$\det Q = 0\ : \quad \frac{\partial}{\partial\bar{z}}\left(\chi' + \frac{i}{T}\right) = 0\,, \tag{3.7.5}$$

$$\det Q > 0\ : \quad \frac{\partial}{\partial\bar{z}}\left(\chi' + \frac{i}{4\sqrt{\det Q}}\log\frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}}\right) = 0\,, \tag{3.7.6}$$

with $z = x^8 + ix^9$. Expressing the conditions (3.7.5) and (3.7.6) in terms of the axidilaton field τ via eqs. (1.3.79) and (1.3.80) for the $\det Q = 0$ case and via eq. (1.3.42) for the $\det Q > 0$ case it is found that the τ must be holomorphic in agreement with eq. (3.6.4).

The conditions (3.7.5) and (3.7.6) under which the 7-brane can be considered static coincides with the condition that τ is a holomorphic function. As it follows from equation (3.7.3), along the Dirac string the holomorphicity fails, so the Dirac string plays the role of a branch cut of τ. Crossing of these branch cuts is related to the nontrivial monodromy of the function τ in agreement with the results of section 3.5.

3.7.2 The axidilaton equation of motion

Consider eq. (3.4.4). Let D denote a disk in the transverse space whose center is the location of a 7-brane. Integrating (3.4.4) over D gives

$$\int_D dF_1 = \int_D \star J_8 = \oint_{\partial D}\left(d\chi' + \star G_9\right)\,. \tag{3.7.7}$$

In eq. (3.7.7) $d\chi'$ is an exact 1-form and does not contribute to the integral. The non-zero value of the integral in (3.7.7) is due to the Dirac string. Alternatively, the Dirac string can be taken out of (3.7.7) granted the differential $d\chi'$ fails to be exact everywhere on the boundary ∂D. Taking the boundary to be a circle of radius r, the integral (3.7.7) can be written as

$$\int_D dF_1 = \oint_{\partial D} d\chi' = \int_0^{2\pi}\frac{d\chi'}{d\theta}d\theta = \int_D \star J_8 = 1\,. \tag{3.7.8}$$

Here θ is an angular variable parameterizing the circle ∂D. Its range is $0 \le \theta < 2\pi$. The solution for χ' that satisfies (3.7.8) is

$$\chi' = \frac{\theta}{2\pi}\,. \tag{3.7.9}$$

The function χ' is single-valued on the cut plane where the cut is formed by the line $\theta = 0$. The solution to the equation of motion for the scalar T, that follows from the action (3.4.8), is constrained to satisfy the condition (3.7.5) or (3.7.6). It is concluded that in order to solve for the equations of motion for T and χ', or equivalently, for the equations of motion for τ, the Dirac strings $\star G_9$ in the actions (3.4.8) and (3.4.14) can be ignored at the price of working with functions τ that have branch cuts. The behavior of τ when crossing a branch cut is given in eq. (3.5.12).

In this subsection the equation of motion for τ that follows from varying the action (3.4.14) with respect to $\bar\tau$ will be solved with the understanding that the Dirac string will be replaced by branch cuts. Performing a variation of the action (3.4.14) with respect to $\bar\tau$ and substituting in the resulting equation of motion the metric (3.6.3) leads to the following equation of motion for τ

$$\partial\bar\partial\tau - 2\frac{\partial\tau\bar\partial\tau}{\tau-\bar\tau} = -\frac{i}{4}\delta(z-z_0,\bar z-\bar z_0)\left(p+q\tau^2+r\tau\right)\,, \tag{3.7.10}$$

where $\partial = \frac{\partial}{\partial z}$. Due to the presence of the delta function[5] it is not possible, at this stage, to assume that τ is a holomorphic function. Eq. (3.7.10) can be integrated as follows. Let D be an infinitesimal disk $|z-z_0| \le \delta$ whose boundary is denoted by γ_δ. Integrating eq. (3.7.10) over D gives

$$\lim_{\delta\to 0}\int_D \left(\partial\bar\partial\tau - 2\frac{\partial\tau\bar\partial\tau}{\tau-\bar\tau}\right)\frac{i}{2}dz\wedge d\bar z = -\frac{i}{4}\lim_{\delta\to 0}\oint_{\gamma_\delta}\tau' dz = -\frac{i}{4}\left(p+q\tau^2+r\tau\right)_{z=z_0}\,, \tag{3.7.12}$$

where the prime denotes differentiation with respect to z. Green's theorem[6] has been used to relate the integral over D to the integral over the boundary γ_δ together with the fact that $\bar\partial\tau = 0$ along γ_δ (except on a set of measure zero).

Assuming that when $q, r \neq 0$ the limit $\lim_{z\to z_0}\tau$ exists one may write

$$2\pi i\tau(z_0) = \lim_{\delta\to 0}\oint_{\gamma_\delta}\frac{\tau}{z-z_0}dz\,. \tag{3.7.14}$$

[5] The delta function in (3.7.10) is defined via

$$\int \frac{i}{2}dz\wedge d\bar z\,\delta(z,\bar z) = 1\,. \tag{3.7.11}$$

[6] In complex notation Green's theorem for any real-analytic function F defined on $D/\{z_0\}$ reads

$$\int_D \partial\bar\partial F\frac{i}{2}dz\wedge d\bar z = \frac{i}{4}\left(\oint_{\partial D}\bar\partial F d\bar z - \oint_{\partial D}\partial F dz\right)\,. \tag{3.7.13}$$

Therefore one has

$$\lim_{\delta\to 0}\oint_{\gamma_\delta}\left(2\pi i\tau' - p\frac{1}{z-z_0} - q\frac{\tau^2}{z-z_0} - r\frac{\tau}{z-z_0}\right)dz = 0\,. \qquad (3.7.15)$$

This form of the axidilaton equation of motion is convenient to derive an approximation of the solution close to the source terms at z_0. This derivation goes as follows. Assume that the integrand of (3.7.15) is an analytic function without any poles in the interior of γ_δ. Then it admits in D a Taylor expansion

$$2\pi i\tau' - p\frac{1}{z-z_0} - q\frac{\tau^2}{z-z_0} - r\frac{\tau}{z-z_0} = \sum_{n=0}^{\infty} a_n (z-z_0)^n\,. \qquad (3.7.16)$$

In the limit $|z-z_0| \to 0$ the poles on the left hand side will dominate all the terms on the right hand side. In this approximation the right hand side of (3.7.16) can be put to zero, and one is left with the homogeneous version of eq. (3.7.16), i.e.

$$2\pi i\tau' - p\frac{1}{z-z_0} - q\frac{\tau^2}{z-z_0} - r\frac{\tau}{z-z_0} = 0\,. \qquad (3.7.17)$$

The solutions to (3.7.17) are

$$e^{2\pi i\tau/p} = z - z_0 \qquad \text{for } \det Q = 0 \text{ with } q = r = 0\,, \qquad (3.7.18)$$

$$c\left(\frac{\tau-\tau_0}{\tau-\bar\tau_0}\right)^{\frac{\pi}{\sqrt{\det Q}}} = z - z_0 \qquad \text{for } \det Q > 0\,, \qquad (3.7.19)$$

where $c \neq 0$ is a constant. All these solutions have a non-trivial monodromy that is in agreement with eq. (3.5.12). Solutions with $\det Q = 0$ and nonzero values for q and r can be obtained by performing $SL(2,\mathbb{R})$ transformations of (3.7.18).

With respect to the field χ' both solutions (3.7.18) and (3.7.19) have axion charge equal to one. The axion charge is defined as

$$\int_0^{2\pi}\frac{d\chi'}{d\theta}d\theta\,, \qquad (3.7.20)$$

where $z - z_0 = re^{i\theta}$. This can be verified by rewriting the solutions (3.7.18) and (3.7.19) in terms of T and χ' for the cases $\det Q = 0$ and $\det Q > 0$ separately.

3.7.3 The Einstein equations

Varying the action (3.4.14) with respect to the metric (ignoring the Dirac string by working with a multi-valued τ) and substituting equations (3.6.3) and (3.6.4) one finds that the $z\bar z$ component of the Einstein equations is given by

$$\partial\bar\partial\log|f|^2 = -\frac{1}{2}\delta(z-z_0,\bar z-\bar z_0)\frac{i}{\tau-\bar\tau}\left(p + q|\tau|^2 + r\mathrm{Re}\,\tau\right)\,. \qquad (3.7.21)$$

All other components of the Einstein equations are identically zero. The local expression for f must be such that when crossing a branch cut of the function τ, f transforms as in eq. (3.6.12).

Integrating eq. (3.7.21) over a disk D that is bounded by γ and using Green's theorem (see footnote 6) and that $\bar{\partial} f = 0$ along γ (except on a set of measure zero) it is found that

$$\mathrm{Im}\oint_\gamma (\log f)' dz = -\frac{i}{\tau - \bar{\tau}}\left(p + q|\tau|^2 + r\mathrm{Re}\,\tau\right)_{z=z_0}. \tag{3.7.22}$$

Substituting eqs. (3.7.18) and (3.7.19) on the right hand side of (3.7.22) it follows that

$$\mathrm{Im}\oint_\gamma (\log f)' dz = -\sqrt{\det Q}\,, \tag{3.7.23}$$

where e^Q is the monodromy matrix of τ measured when going around the contour γ.

The solutions for τ, eqs. (3.7.18) and (3.7.19) can be written as[7]

$$\tau = \frac{p}{2\pi i}\log(z - z_0) \quad \text{for } \det Q = 0 \text{ with } q = r = 0\,, \tag{3.7.24}$$

$$\tau = \mathrm{Re}\,\tau_0 + \mathrm{Im}\,\tau_0\,\tan\left(\sqrt{\det Q}\,w + \frac{\pi}{2}\right) \quad \text{for } \det Q > 0\,, \tag{3.7.25}$$

where w is

$$w = \frac{1}{2\pi i}\log\frac{z - z_0}{c}\,. \tag{3.7.26}$$

When going around z_0 by sending $z - z_0 \to e^{2\pi i}(z - z_0)$, or what is the same $w \to w+1$, the axidilaton τ transforms as (1.3.39) with a, b, c and d given by e^Q, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^Q = \cos\sqrt{\det Q}\,\mathbb{1} + \frac{\sin\sqrt{\det Q}}{\sqrt{\det Q}}Q\,. \tag{3.7.27}$$

Since the function f transforms as in (3.6.12) with c and d as given in (3.7.27) it is concluded that when crossing the branch cuts of τ, eqs. (3.7.24) and (3.7.25), f transforms as

$$f \to f \quad \text{for } \det Q = 0 \text{ with } q = r = 0\,, \tag{3.7.28}$$

$$f \to \left[\cos\sqrt{\det Q} - \sin\sqrt{\det Q}\,\tan\left(\sqrt{\det Q}\,w + \frac{\pi}{2}\right)\right] f \quad \text{for } \det Q > 0\,. \tag{3.7.29}$$

The unique functions f that transform as in (3.7.28) and (3.7.29) while satisfying eq. (3.7.23) are

$$f = 1 \quad \text{for } \det Q = 0 \text{ with } q = r = 0\,, \tag{3.7.30}$$

$$f = \cos\left(\sqrt{\det Q}\,w + \frac{\pi}{2}\right) \quad \text{for } \det Q > 0\,. \tag{3.7.31}$$

[7] This form of τ very closely represents the local 7-brane solutions discussed in [63].

The expression for f for the $\det Q > 0$ case can be rewritten in terms of τ as

$$f = \frac{\operatorname{Im}\tau_0}{(\tau-\tau_0)^{1/2}(\tau-\bar{\tau}_0)^{1/2}}\,. \tag{3.7.32}$$

3.7.4 Local geometry of a 7-brane

At this stage in the analysis the local characteristics of a (p, q, r) 7-brane are known. For the case $\det Q = 0$ with $q = r = 0$, i.e. a D7-brane of charge p the local geometry can be inferred from expressions (3.6.3), (3.6.6), (3.7.24) and (3.7.30). Putting all this together one finds

$$\begin{aligned} ds^2 &= -dt^2 + d\vec{x}_7^{\,2} + (\operatorname{Im}\tau)dzd\bar{z}\,, && (3.7.33)\\ \tau &= \frac{p}{2\pi i}\log(z-z_0)\,, && (3.7.34)\\ \epsilon &= \epsilon_0\,. && (3.7.35) \end{aligned}$$

For the case $\det Q > 0$ the 7-brane is referred to as a Q7-brane and the local geometry follows from (3.6.3), (3.6.6), (3.7.19) and (3.7.32). The resulting local Q7-brane solution is

$$\begin{aligned} ds^2 &= -dt^2 + d\vec{x}_7^{\,2} + (\operatorname{Im}\tau)|f|^2 dzd\bar{z}\,, && (3.7.36)\\ c\left(\frac{\tau-\tau_0}{\tau-\bar{\tau}_0}\right)^{\frac{\pi}{\sqrt{\det Q}}} &= z - z_0\,, && (3.7.37)\\ f &= \frac{\operatorname{Im}\tau_0}{(\tau-\tau_0)^{1/2}(\tau-\bar{\tau}_0)^{1/2}}\,, && (3.7.38)\\ \epsilon &= \left(\frac{(\tau-\tau_0)(\tau-\bar{\tau}_0)}{(\bar{\tau}-\bar{\tau}_0)(\bar{\tau}-\tau_0)}\right)^{1/8}\epsilon_0\,. && (3.7.39) \end{aligned}$$

The local Q7-brane solution (3.7.36) to (3.7.39) has been written down in terms of the fields as defined in subsection 1.3.4. This will prove a useful way of writing the Q7-brane solution when combining the local geometry of Q7-branes with that of D7-branes later in section 3.10. However, it is insightful to rewrite the local Q7-brane in terms of the fields T, χ' and ϵ' that are used in subsection 1.3.1. The relation between τ and T and χ' can be obtained using eq. (1.3.1) together with eqs. (1.2.22) to (1.2.25). The relation is

$$\frac{\tau-\tau_0}{\tau-\bar{\tau}_0} = \left(\frac{T-2\sqrt{\det Q}}{T+2\sqrt{\det Q}}\right)^{1/2} e^{2i\sqrt{\det Q}\,\chi'}\,. \tag{3.7.40}$$

From which it follows that (3.7.37) becomes

$$\chi' + \frac{i}{4\sqrt{\det Q}}\log\frac{T+2\sqrt{\det Q}}{T-2\sqrt{\det Q}} = \frac{1}{2\pi i}\log\frac{z-z_0}{c}\,. \tag{3.7.41}$$

Further, the metric (3.7.36) becomes

$$ds^2 = -dt^2 + d\vec{x}_7^2 + q\,\left(T^2 - 4\det Q\right)^{-1/2} dz d\bar{z}\,. \tag{3.7.42}$$

In order to find the expression for the Killing spinor ϵ' defined in eq. (1.3.12) one must perform the local $U(1)$ transformation (1.3.7) and (1.3.12). The Killing spinor ϵ, given in eq. (3.7.39), is written in the $U(1)$ gauge of subsection 1.3.1 in which $V_\pm^\alpha$ is given in eqs. (1.3.81) to (1.3.84). Using this $U(1)$ gauge choice together with the relation between $q_{\alpha\beta}$ and p, q, r, eqs. (1.2.7) and (1.3.38), one recognizes

$$\frac{q_{\alpha\beta}V_+^\alpha V_+^\beta}{q_{\gamma\delta}V_-^\gamma V_-^\delta} = \frac{(\tau-\tau_0)(\tau-\bar{\tau}_0)}{(\bar{\tau}-\bar{\tau}_0)(\bar{\tau}-\tau_0)}\,, \tag{3.7.43}$$

so that via eqs. (1.3.7) and (1.3.12) the Killing spinor ϵ' is given by

$$\epsilon' = \epsilon_0\,. \tag{3.7.44}$$

In terms of T, χ' and ϵ' the local geometry of a Q7-brane shows a lot of similarity with the local geometry of a D7-brane, eqs. (3.7.33) to (3.7.35). The only property that does not straightforwardly map into each other is the behavior of $\mathrm{Im}\,\tau$ for the D7-brane and that of T for the Q7-brane. Other then that the axions and Killing spinors $\mathrm{Re}\,\tau$ and ϵ for the D7-brane and χ' and ϵ' for the Q7-brane behave in the same way.

In order to further study the local geometry of a Q7-brane it is convenient to introduce polar coordinates r and θ via

$$\frac{z-z_0}{c} = re^{i\theta}\,, \tag{3.7.45}$$

where $0 < r < 1$. From eqs. (3.7.41) and (3.7.42) it can be concluded that the metric can be written as

$$ds^2 = -dt^2 + d\vec{x}_7^2 + \frac{|c|^2}{4\mathrm{Im}\,\tau_0}\left(r^{-\frac{\sqrt{\det Q}}{\pi}} - r^{\frac{\sqrt{\det Q}}{\pi}}\right)\left(dr^2 + r^2 d\theta^2\right)\,. \tag{3.7.46}$$

Consider the Q7-brane metric near $r = 0$. Then (3.7.46) behaves as

$$ds^2 \sim -dt^2 + d\vec{x}_7^2 + \frac{|c|^2}{4\mathrm{Im}\,\tau_0} r^{-\frac{\sqrt{\det Q}}{\pi}}\left(dr^2 + r^2 d\theta^2\right)\,. \tag{3.7.47}$$

This is the metric of a cone whose apex is at $r = 0$ with a deficit angle δ equal to $\sqrt{\det Q}$. This result is to be contrasted with the local geometry of a D7-brane, eq. (3.7.33), where the metric behaves logarithmically as one approaches the location of the D7-brane.

Instead of looking at the local form of the metric it is possible to calculate the deficit angle using the function f. This is because the orders of the poles of the function $f(z)$ at $z = z_0$ determine the deficit angle δ at the location of the source. Let γ be a closed circular contour that encircles the point z_0. Then one has

$$\delta = i \oint_\gamma (\log f)' dz \,. \qquad (3.7.48)$$

Combining eqs. (3.7.23) and (3.7.48) the following expression for the deficit angle at the location of the source is found

$$\delta = \sqrt{\det Q} \,, \qquad (3.7.49)$$

in agreement with eq. (3.7.47). There is no deficit angle at the position of a (p, q, r) 7-brane for which $\det Q = 0$.

Eqs. (3.7.41) and (3.7.42) can be used to construct the solution for the 8-form $q_{\alpha\beta} A_8^{\alpha\beta}$. The duality relation (1.2.26) expressed in terms of the polar coordinates (3.7.45) reads

$$q_{\alpha\beta} F^{\alpha\beta}_{01\cdots 7r} = \frac{1}{2\pi r} \left(T^2 - 4\det Q \right) = \frac{dT}{dr} \,, \qquad (3.7.50)$$

where the last equality follows from eq. (3.7.41). Assuming that $q_{\alpha\beta} A_8^{\alpha\beta}$ only depends on r it is found that

$$q_{\alpha\beta} A^{\alpha\beta}_{01\cdots 7} = T + k \,, \qquad (3.7.51)$$

where k is an integration constant. Specializing to the case $\det Q = 0$ with $q = r = 0$ using the definitions (1.3.79) and (1.3.89) it follows that the RR 8-form C_8 for the case of a charge p D7-brane is given by

$$C_{01\cdots 7} = \frac{1}{\operatorname{Im} \tau} + k \,. \qquad (3.7.52)$$

At the level of the solution the electric dual of χ', i.e. the 8-form $q_{\alpha\beta} A^{\alpha\beta}_{01\cdots 7}$, is equal to the scalar T.

In general the function f, as indicated in table 3.6.1 on page 68, transforms under $SL(2, \mathbb{R})$. Since, however, use has been made of the relation between $q_{\alpha\beta}$ and p, q, r, eqs. (1.2.7) and (1.3.38), implying that q and p are both always positive, the explicit form of f, eq. (3.7.38), transforms under $PSL(2, \mathbb{R})$.

3.8 Mass and the BPS equation

From the analysis of the previous section it is clear that a single 7-brane can never produce a metric as well as a function τ that is defined everywhere. The local geometry of a D7-brane and of a Q7-brane leads to a curvature singularity at a finite distance from the brane. This occurs at the value of the radial coordinate r for which $\mathrm{Im}\,\tau$ goes through zero.

The monodromy transformations of the functions τ and f have been obtained. They transform as $\tau \to e^{Q}\tau = \frac{a\tau+b}{c\tau+d}$ and $f \to (c\tau+d)f$ whenever τ crosses a Dirac string or equivalently a branch cut. The question is how to interpolate between the local solutions. This problem is essentially a problem of how to globally organize the positioning of the Dirac strings. When this has been determined the global transformation properties of τ and f are fixed and since they are analytic functions they are fixed uniquely by specifying their monodromy transformations.

Granted that globally well-defined solutions exist, the total mass for them can be computed. This will be the subject of this section. The explicit construction of the globally well-defined solutions will be discussed in section 3.10.

3.8.1 Mass of a 7-brane solution

There are two contributions the total mass of a 7-brane solution. The first contribution comes from the deficit angles at the locations of the Q7-branes (see subsection 3.7.4). The second contribution comes from the energy contained in the bulk field τ.

The action describing the coupling to the IIB supergravity axidilaton sector of n 7-branes is given by

$$\begin{aligned} S &= \int_{\mathcal{M}_{10}} d^{10}x \left(\star 1\, R - 2\star \hat{P}\wedge \hat{P}^*\right) \\ &\quad - \int_{\mathcal{M}_{10}} d^{10}x \sum_{j=1}^{n} \int_{\mathcal{M}_8^j} d^8\sigma_j\, \delta(x - X_j(\sigma_j)) q^j_{\alpha\beta} V_-^{\alpha} V_+^{\beta} \sqrt{-g^j_{(8)}}\,, \end{aligned} \tag{3.8.1}$$

with

$$\hat{P} = P + \frac{i}{2}\sum_{j=1}^{n} q^j_{\alpha\beta} V_+^{\alpha} V_+^{\beta} \star G_9^j\,. \tag{3.8.2}$$

The world-volume $\mathcal{M}_8^j$ of each 7-brane, carrying a charge $q^j_{\alpha\beta}$, is parameterized by σ_j^A and is located in target space at the point $X_j(\sigma_j)$. The world-volume metric is g^j_{AB} and the Dirac 8-brane stemming from the 7-brane is described by G_9^j.

Consider the 7-branes to be wrapped on a T^7 with radii $R^1, \cdots, R^7$ so they can be viewed as point-particles moving in a $1+2$-dimensional space-time. The total energy of a massive particle in $1+2$ dimensions is measured by the deficit angle at infinity via the formula [64]

$$m = \frac{1}{16\pi G_N^{(3)}} \int d^2x \sqrt{|\gamma|}\, R(\gamma)\,, \tag{3.8.3}$$

where $G_N^{(3)}$ is the (2+1)-dimensional Newton's constant, related to the 10-dimensional one by

$$G_N^{(3)} = \frac{G_N^{(10)}}{(2\pi)^7 R^1 \cdots R^7}\,, \tag{3.8.4}$$

and γ is the metric of the transverse space.

For static solutions in 2+1 dimensions one has $R(\gamma) = -T^0{}_0$, where $R(\gamma)$ is the Ricci scalar of the metric γ and where $T^0{}_0$ is the time-time component of the energy-momentum tensor. Hence the energy is given by

$$m = \frac{1}{16\pi G_N^{(3)}} \int d^2x \sqrt{|\gamma|}\, R(\gamma) = -\frac{1}{16\pi G_N^{(3)}} \int \frac{i}{2} dz \wedge d\bar{z} \sqrt{|\gamma|} T^0{}_0\,. \tag{3.8.5}$$

Using action (3.8.1) it can be seen that

$$T^0{}_0 = -\frac{1}{\sqrt{|\gamma|}} \frac{1}{(\mathrm{Im}\tau)^2} \partial\tau\bar{\partial}\bar{\tau} - \sum_j \frac{1}{\sqrt{|\gamma|}} \delta(z - z_j, \bar{z} - \bar{z}_j) \frac{1}{\mathrm{Im}\tau} \left(p + q|\tau|^2 + r\mathrm{Re}\,\tau\right)\,, \tag{3.8.6}$$

where j labels the points z_j where the particles are located, the mass m can be written as

$$m = \frac{1}{16\pi G_N^{(3)}} \left(\int \frac{i}{2} dz \wedge d\bar{z} \frac{\partial\tau\bar{\partial}\bar{\tau}}{(\mathrm{Im}\tau)^2} + 2\sum_j \delta_j \right)\,, \tag{3.8.7}$$

where δ_j is the deficit angle at the location of the jth particle at the point z_j. In obtaining eq. (3.8.7) use has been made of eqs. (3.7.21), (3.7.23) and (3.7.49).

3.8.2 BPS formula

In this subsection a BPS equation for 7-brane solutions relating the energy m to a suitably defined $U(1)$ charge will be constructed.

The asymptotic region of the transverse space, the region $|z| \to \infty$, corresponds to a single point on the Riemann sphere, e.g. the point $z = \infty$. The location of infinity is arbitrary. One could have chosen to place it at any other point, z_0 say, of the Riemann sphere. Generally speaking a point z_0 on the Riemann sphere is mapped to an asymptotic region of the transverse space when the physical distance from z_0 to any other point z diverges as $|z - z_0|^{1-4G_N^{(3)}m}$ while $m < 1/4G_N^{(3)}$ (that is for non-compact transverse spaces) [64].

Eqs. (3.7.23) and (3.7.49) show that the function f can be used to compute deficit angles. The total mass of the solution is measured by the deficit angle at infinity. The energy (3.8.7) can thus be alternatively computed as follows

$$m = \frac{1}{8\pi G_N^{(3)}} \mathrm{Im} \oint_{z=\infty} (\log f)' dz = \frac{\delta_\infty}{8\pi G_N^{(3)}}\,, \tag{3.8.8}$$

where the contour integral encircles the point $z = \infty$ (in a clockwise direction) and δ_∞ is the deficit angle at infinity. Equating expressions (3.10.11) and (3.8.8) gives

$$\int_{TS} \frac{i}{2} dz \wedge d\bar{z} \frac{\partial\tau\bar{\partial}\bar{\tau}}{(\mathrm{Im}\tau)^2} = -2 \sum_j \delta_j + 2\delta_\infty \,. \tag{3.8.9}$$

The subscript 'TS' below the integral sign means transverse space. The left hand side of eq. (3.8.9) can be written as an integral over the 2-form dQ with Q the 1-form given in (1.3.100). This implies that the deficit angles at the locations of the 7-branes in the transverse space, including the deficit angle at infinity, can also be computed via line integrals of the 1-form Q, the $U(1)$ connection of the coset $SL(2,\mathbb{R})/SO(2)$, around the locations of the 7-branes or infinity in transverse space

$$\delta = \frac{1}{2} \oint Q \,. \tag{3.8.10}$$

This expression can be taken as the definition of $U(1)$ charge (see [65]). Applying this formula to compute the $U(1)$ charge at infinity, i.e. $\oint_{z=\infty} Q$ going around $z = \infty$ in a clockwise direction, becomes a relation between the mass and the $U(1)$ charge with the characteristic form of a (saturated) BPS bound,

$$16\pi G_N^{(3)}\, m = \oint_{z=\infty} Q \,. \tag{3.8.11}$$

As discussed in [65], all the solutions of the system under consideration that have the asymptotic behavior allowing to define mass and $U(1)$ charge are automatically supersymmetric and such that the mass and charge satisfy (3.8.10). In other words, there are no 7-brane solutions for which the mass exceeds the bound (3.8.10). This excludes, for example, 7-brane solutions with horizons.

3.8.3 Dividing out by $SL(2,\mathbb{Z})$

The total mass of a 7-brane solution will be finite provided the integral in expression (3.8.7) is finite. The integral in (3.8.7) is the pull-back of an integration over the moduli space, i.e. the set of inequivalent values for τ. If one considers the classical moduli space,

$$\frac{PSL(2,\mathbb{R})}{SO(2)} \,, \tag{3.8.12}$$

then the energy (3.8.7) will be infinite. It is necessary to divide out (3.8.12) by discrete subgroups of $PSL(2,\mathbb{R})$ in order to make the integral finite. The largest possible discrete subgroup of $PSL(2,\mathbb{R})$ is $PSL(2,\mathbb{Z})$. Subgroups of $PSL(2,\mathbb{Z})$ can also be used as was shown explicitly in [16]. In this thesis the possibility to work with subgroups of $PSL(2,\mathbb{Z})$ will not be discussed. Thus, τ and $PSL(2,\mathbb{Z})$ transformations of τ must be considered physically equivalent.

The metric (3.6.3) is by construction a singlet under $SL(2,\mathbb{Z})$. The Killing spinor ϵ, eq. (3.6.6), transforms under local $U(1)$ transformations and can therefore naturally exist everywhere on a 7-brane background. The pair (τ, f) transforms under $SL(2,\mathbb{Z})$ and values of (τ, f) and $SL(2,\mathbb{Z})$ transforms of (τ, f) are equivalent. This means that the IIB supergravity theory on a background of 7-branes is divided out by $SL(2,\mathbb{Z})$. The 2-forms of the IIB theory do not transform under a local $U(1)$, they transform as a doublet under $SL(2,\mathbb{Z})$ and thus have no way of surviving the truncation of dividing the IIB theory out by $SL(2,\mathbb{Z})$. The 6-forms that are dual to the 2-forms suffer the same fate as the 2-forms.

Perhaps, more unexpectedly, also the 8-forms are truncated from the spectrum of fields of IIB supergravity. This is not in contradiction with the fact that one can still have nontrivial axidilaton fields. It means that after dividing out by $SL(2,\mathbb{Z})$ the axidilaton field cannot be dualized into an 8-form anymore. Another way to see that 8-forms must drop out, is by noting that the three 8-forms transform in the adjoint of $SL(2,\mathbb{Z})$ (see eq. (1.3.53)) and hence by declaring the three 8-forms and all their $SL(2,\mathbb{Z})$ transformed versions equivalent necessarily forces them to be zero. It follows from this that there is no electric coupling description possible for systems with multiple 7-branes. They form inherently magnetic systems.

3.8.4 What about $\det Q < 0$?

The $SL(2,\mathbb{R})$ duality group has three subgroups: $\mathbb{R}$, $SO(1,1)$ and $SO(2)$. The transformations e^Q with $\det Q = 0$, $\det Q < 0$ and $\det Q > 0$ belong to these respective subgroups. In this subsection it will be argued that there are no 7-branes that correspond to the $SO(1,1)$ subgroup. One could, in principle include the case $\det Q < 0$ in the discussion of 7-brane solutions. The solution for the local form for τ whose monodromy around a point $z = z_0$ is of the form $\tau \to e^Q\tau$ with $\det Q < 0$ is given by

$$\left(\frac{\tau - \tau_0^+}{\tau - \tau_0^-}\right)^{\frac{i\pi}{\sqrt{-\det Q}}} = z - z_0\,, \tag{3.8.13}$$

where

$$\tau_0^\pm = -\frac{r}{2q} \pm \frac{1}{q}\sqrt{-\det Q}\,, \tag{3.8.14}$$

which is such that $e^Q\tau_0^\pm = \tau_0^\pm$.

There are two problems with this possibility. The first problem is that the solution (3.8.13) is ill-defined at the point $z = z_0$. It is not possible to add a source term to the IIB action for the case $\det Q < 0$. One cannot, for example, substitute the solution (3.8.13) into the right hand side of the sourced Einstein equation, eq. (3.7.21). The second and perhaps more severe problem is that when one considers the quantum moduli space

$$\frac{PSL(2,\mathbb{R})}{SO(2)\times PSL(2,\mathbb{Z})} \tag{3.8.15}$$

none of the orbifold points corresponds to $\tau_0^{\pm}$. It can be shown that after quantization from $SL(2,\mathbb{R})$ to $SL(2,\mathbb{Z})$ the points $\tau_0^{\pm}$ are irrational points on the real line $\operatorname{Im}\tau = 0$ and such points lie outside the quantum moduli space. There is therefore no way that one can ever construct a globally well-defined solution in which a $\det Q < 0$ 7-brane occurs[8].

3.9 The quantum moduli space

If one assumes that two τ's differing by a $PSL(2,\mathbb{Z})$ transformation are equivalent, then τ takes values in the following set

$$\{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}/PSL(2,\mathbb{Z})\,. \tag{3.9.1}$$

The equivalence relation,

$$\tau \sim \tau' \quad \text{where} \quad \tau' = \frac{a\tau+b}{c\tau+d} \quad \text{with} \quad a,b,c,d \in \mathbb{Z} \quad \text{and} \quad ad-bc=1\,, \tag{3.9.2}$$

defines a space, an orbifold, that can be parameterized by the following region in the upper half-plane [68],

$$F = \{\tau \in \mathbb{C} \mid -\tfrac{1}{2} \leq \operatorname{Re}\tau < \tfrac{1}{2},\ \ |\tau| > 1,\ \ \text{circle segment } |\tau| = 1 \text{ from } \rho \text{ to } i\}\,,$$

where $\rho = e^{2\pi i/3}$. This set is called the fundamental domain of the modular group $PSL(2,\mathbb{Z})$ and will be denoted by F. It will also be referred to as the type IIB quantum moduli space. This region is depicted in figure 3.9.1. The orbifold points are $i\infty$, i and ρ. The circle segments from ρ to i and from i to $\rho+1$ are identified under the transformation $S(\tau) = -\frac{1}{\tau}$. The lines $\operatorname{Re}\tau = \pm\frac{1}{2}$ above the points ρ and $\rho+1$ are identified under the transformation $T(\tau) = \tau + 1$. In matrix notation

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,. \tag{3.9.3}$$

Any element of $PSL(2,\mathbb{Z})$ can be written as some product of S and T matrices [68].

[8] In the context of F-theory on K3 this statement can be refined by saying that it is not possible to force F-theory 7-branes to coincide at a point z_0, say, such that the monodromy around z_0 is of the form e^Q with $\det Q < 0$. There do exist F-theory 7-brane configurations that cannot be collapsed to one point around which the monodromy is of the type e^Q with $\det Q < 0$ [66, 67].

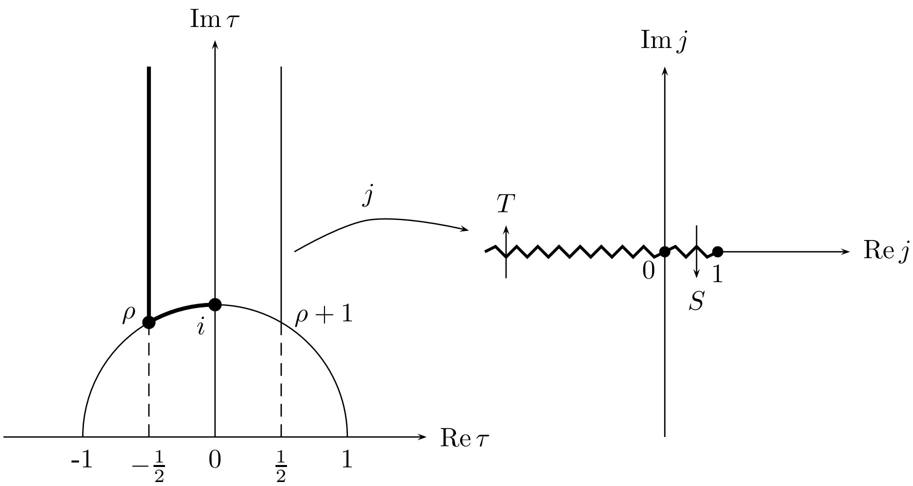

Figure 3.9.1: The fundamental domain and the mapping properties of j. The circle segments from ρ to i and from i to $\rho+1$ are identified as well as the lines $\mathrm{Re}\,\tau = \pm\frac{1}{2}$ above the points ρ and $\rho+1$.

Klein's modular function j maps F onto the Riemann sphere. It maps the line $\mathrm{Re}\,\tau = -\frac{1}{2}$ above the point $\tau = \rho$ onto the negative real axis, $(-\infty, 0)$, and the circle segment from $\tau = \rho$ to $\tau = i$ along $|\tau| = 1$ onto the interval $(0, 1)$. The points ρ and i are mapped to the points 0 and 1, respectively. The orbifold point at $\tau = i\infty$ gets mapped to the point ∞. Further, all points that are on the left/right of the line $\mathrm{Re}\,\tau = 0$ get mapped to points that lie above/below the real axis. The line $\mathrm{Re}\,\tau = 0$ itself maps to $(1, \infty)$ on the positive real axis. This mapping is conformal for all points strictly inside F. Further, the j-function is modular invariant, i.e. $j\left(\frac{a\tau+b}{c\tau+d}\right) = j(\tau)$. The inverse function j^{-1} maps the complex plane onto F.

The function $j(\tau)$ for $\mathrm{Im}\,\tau \to \infty$, i.e. near $\tau_0 = i\infty$, behaves as

$$j(\tau) \sim e^{-2\pi i\tau}\,. \tag{3.9.4}$$

Near $\tau_0 = i$ the behavior of $j(\tau)$ is

$$j(\tau) = 1 + c\left(\frac{\tau - i}{\tau + i}\right)^2 + \cdots\,, \tag{3.9.5}$$

with $c \neq 0$, that is $j(i) - 1$ is a second order zero point. Similarly, it is shown that

τ_0	(p,q,r)	$\pi/\sqrt{\det Q}$	$SL(2,\mathbb{Z})$
$i\infty$	$(1,0,0)$	∞	T
i	$(\frac{\pi}{2},\frac{\pi}{2},0)$	2	S
ρ	$(\frac{2\pi}{3\sqrt{3}},\frac{2\pi}{3\sqrt{3}},\frac{2\pi}{3\sqrt{3}})$	3	$T^{-1}S$

Table 3.9.1: Properties of the orbifold points $\tau_0 = i\infty, i, \rho$. The value ∞ for $\pi/\sqrt{\det Q}$ in the case $\tau_0 = i\infty$ means that this is a point of infinite order, i.e. there is no power of T such that it equals the identity. The last column indicates the transformations e^Q of which τ_0 is a fixed point. The entries of Q, the numbers p,q,r, given in the second column, are the smallest possible numbers such that τ_0 is a fixed point of e^Q.

near the point $\tau_0 = \rho$, which is a third order zero point, $j(\tau)$ behaves as

$$j(\tau) = c\left(\frac{\tau-\rho}{\tau-\bar{\rho}}\right)^3 + \cdots\,, \tag{3.9.6}$$

with $c \neq 0$. By the modular invariance of the j-function the expansions (3.9.4), (3.9.5) and (3.9.6) also provide expansions of $j(\Lambda\tau)$ around $\Lambda i\infty$, Λi and $\Lambda\rho$, respectively.

The orbifold points $\tau_0 = i\infty, i, \rho$ are fixed points of, respectively, T, S and $T^{-1}S$. By writing these transformations as e^Q with Q as given in (1.2.7) the numbers p,q,r that correspond to τ_0 can be obtained. Some data regarding the orbifold points τ_0 is summarized in table 3.9.1.

The branch cut of the function j^{-1} runs from $-\infty$ to 1 over the real axis. The branch cut is depicted by the zigzag lines of figure 3.9.1. The behavior of j^{-1} when crossing the branch cut is indicated by the arrows in figure 3.9.1. Crossing the branch cut between $-\infty$ and 0 from a point below the branch cut to a point above the branch cut corresponds to a T monodromy. Crossing the branch between 0 and 1 corresponds to an S monodromy. If one encircles the point $-\infty$ in the complex j-plane then the monodromy is T while when one encircles the point 1 the monodromy is S. Encircling the point 0 either gives $T^{-1}S$ when the base point lies below the real axis or ST^{-1} when the base point lies above the real axis. For the monodromy $ST^{-1} = S(T^{-1}S)S^{-1}$ the fixed point is not ρ but $\rho + 1 = S\rho$. One should consider the points $i\infty$, i and ρ merely as representative elements of the set of points that are fixed points under $\Lambda T\Lambda^{-1}$, $\Lambda S\Lambda^{-1}$ and $\Lambda T^{-1}S\Lambda^{-1}$, respectively, for arbitrary $\Lambda \in SL(2,\mathbb{Z})$, i.e. $i\infty$, i and ρ represent fixed point sets of the $SL(2,\mathbb{Z})$ conjugacy classes T, S and $T^{-1}S$. The $SL(2,\mathbb{Z})$ conjugacy classes are for $\det Q = 0$ given by

$\pm T^n$ (with $n = 0, 1, 2, \ldots$), while for $\det Q > 0$ they are given by $\pm S$, $\pm T^{-1}S$ and $\pm(T^{-1}S)^2$.

3.10 Constructing globally well-defined 7-brane solutions

This section discusses how to construct globally well-defined solutions. The focus will be on solutions with τ monodromy group $PSL(2,\mathbb{Z})$. The group $PSL(2,\mathbb{Z})$ is generated by T and S, defined in section 3.9. To show that one can also work with subgroups of $PSL(2,\mathbb{Z})$ the construction for the monodromy group $\Gamma_0(2)$ whose generators are T and ST^2S was included in [16]. Having chosen a monodromy group one can specify the functions τ and f by choosing them to transform as $\tau \to e^Q\tau = \frac{a\tau+b}{c\tau+d}$ and $f \to (c\tau+d)f$ whenever τ crosses a Dirac string or equivalently a branch cut.

3.10.1 The function τ

Since τ and $\Lambda\tau$ are identified there must exist a function, $j(\tau)$, that is monodromy neutral, i.e. $j(\tau)$ must be an automorphic function of the monodromy group,

$$j(\Lambda\tau) = j(\tau)\,, \tag{3.10.1}$$

where Λ is any element of the monodromy group. The local expansions of the function j around the fixed points τ_0 of Λ must reduce to the expressions (3.7.18) and (3.7.19).

A region of the complex upper half plane containing values of τ that are inequivalent under the monodromy group and that are related to all the points in the upper half plane is a fundamental domain of the monodromy group. For the monodromy group $PSL(2,\mathbb{Z})$ the moduli space (the space of inequivalent values of τ) is

$$\frac{PSL(2,\mathbb{R})}{SO(2)\cdot PSL(2,\mathbb{Z})}\,. \tag{3.10.2}$$

The space 3.10.2 is referred to as the quantum moduli space as this is the conjectured moduli space of the full quantum IIB theory [14]. Properties of this space and of the modular function $j(\tau)$, Klein's modular j-function, can be found in section 3.9. The space (3.10.2) is an orbifold whose orbifold points $i\infty$, i and ρ form representative elements of fixed point sets that are invariant under the $SL(2,\mathbb{Z})$ conjugacy classes formed by taking all the conjugated elements of the transformations T, S and $T^{-1}S$, respectively.

The function $j(\tau)$ maps the fundamental domain onto the Riemann sphere $\hat{\mathbb{C}}$ in a one-to-one fashion, so that the inverse function j^{-1} exists. The function $\tau(z)$ is then given by $\tau(z) = j^{-1}(z)$. To describe systems with many 7-branes an additional map from the Riemann sphere to N copies of itself is introduced. This map is given by the N to 1 automorphism $z \to P(z)/Q(z)$ for polynomials $P(z)$ and $Q(z)$. For $N = 1$ these polynomials are fixed by the requirement that the three orbifold points of the fundamental domain are mapped to three given points in the z-plane. For $N > 1$ the polynomials $P(z)$ and $Q(z)$ are fixed by the further requirement of how many branes are placed at the three points $z_{i\infty}, z_\rho$ and z_i where the subscript indicates the value of τ at that point (up to an $SL(2,\mathbb{Z})$ transformation). In the next section the explicit realizations of $P(z)$ and $Q(z)$ will be given. Summarizing, one has the sequence of maps

$$z \quad \stackrel{N\to 1}{\longrightarrow} \quad \frac{P(z)}{Q(z)} \quad \stackrel{j^{-1}}{\longrightarrow} \quad \tau(z) = j^{-1}\left(\frac{P(z)}{Q(z)}\right) . \tag{3.10.3}$$

The inverse mapping j^{-1} which maps from the Riemann sphere $\hat{\mathbb{C}}$ onto the fundamental domain has branch cuts connecting the points $z_{i\infty}$ to z_ρ and z_ρ to z_i.

monodromy group	generators	orbifold points	area	modular function	$F(\tau)$
$PSL(2,\mathbb{Z})$	T, S	$i\infty,\ \rho,\ i$	$\pi/3$	$j(\tau)$	$\eta^2(\tau)$

Table 3.10.1: Properties of the groups $PSL(2,\mathbb{Z})$ and realizations of the functions τ and F. The role of the function $F(\tau)$ is explained in subsection 3.10.2. The column headed ‘area’ refers to the area of the fundamental domain of $PSL(2,\mathbb{Z})$.

3.10.2 The function f

The function f must be such that when crossing a branch cut of τ with τ transforming as $\tau \to e^Q\tau = \frac{a\tau+b}{c\tau+d}$ it transforms as $f \to (c\tau + d)f$. In table 3.10.2 the monodromies of τ and f are presented. The monodromies are measured when going around the points $z_{i\infty}$, z_ρ, z_i in a counter clockwise direction.

The function $f(z)$ can be written in the form

$$f(z) = F(\tau)h(z)\,, \tag{3.10.4}$$

location	$SL(2,\mathbb{Z})$	(p,q,r)	monodromy f	$\sqrt{\det Q}$
$z_{i\infty}$	T	$(1,0,0)$	$f\to f$	0
z_i	S	$(\pi/2,\pi/2,0)$	$f\to -\tau f$	$\pi/2$
z_ρ	$T^{-1}S$	$(\frac{2\pi}{3\sqrt{3}},\frac{2\pi}{3\sqrt{3}},\frac{2\pi}{3\sqrt{3}})$	$f\to -\tau f$	$\pi/3$

Table 3.10.2: The monodromies of τ and f, the p,q,r values and the value for $\sqrt{\det Q}$ for $\tau_0 = i\infty$, ρ, i.

where $F(\tau)$ will be a cusp form that transforms under $PSL(2,\mathbb{Z})$ as

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = e^{i\beta(a,b,c,d,\tau)}(c\tau+d)F(\tau)\,, \tag{3.10.5}$$

and $h(z)$ is a function of z that is chosen such that when going around a 7-brane it transforms as

$$h(z)\to e^{-i\beta(a,b,c,d,\tau(z))+\pi i\,k+2\pi i\,l}h(z)\,, \tag{3.10.6}$$

with $k,l\in\mathbb{Z}$. The term πk in (3.10.6) determines whether f transforms under $PSL(2,\mathbb{Z})$ (k = even) or under $-PSL(2,\mathbb{Z})$ (k = odd). The term $2\pi l$ in (3.10.6) is not seen by f, but is visible from the point of view of the Killing spinor ϵ, (3.6.6), because it transforms, a.o. with a factor $e^{-\pi i\,l}=(-1)^l$. Hence, the even or oddness of l relates to spinstructure.

The Dedekind eta function squared, $\eta^2(\tau)$, transforms as (3.10.5) for specific values of the phase β. The Dedekind eta function is a cusp form, i.e. it goes to zero as $\tau\to i\infty$ and is nonzero for values of τ that satisfy $0<\operatorname{Im}\tau<\infty$. For $\operatorname{Im}\tau\to\infty$ one has

$$\eta(\tau)\to e^{2\pi i\tau/24}\,. \tag{3.10.7}$$

The transformations of the Dedekind η-function under the $PSL(2,\mathbb{Z})$-transformations T, S and $T^{-1}S$ are given by

$$T\quad:\quad \eta^2(\tau+1)=e^{\pi i/6}\eta^2(\tau)\,, \tag{3.10.8}$$

$$S\quad:\quad \eta^2\left(-\frac{1}{\tau}\right)=e^{-\pi i/2}\tau\,\eta^2(\tau)\,, \tag{3.10.9}$$

$$T^{-1}S\quad:\quad \eta^2\left(-\frac{\tau+1}{\tau}\right)=e^{-2\pi i/3}\tau\,\eta^2(\tau)\,. \tag{3.10.10}$$

Using the monodromies of this cusp form and the required monodromies of $f(z)$, given in table 3.10.2, the explicit form of the function $h(z)$ can be derived. In subsections 3.10.3 and 3.10.4 explicit expressions for $h(z)$ will be given.

For general N, defined in eq. (3.10.3), the mass formula (3.8.7) becomes

$$m = \frac{1}{16\pi G_N^{(3)}} \left(N \times \text{area fundamental domain} + \ 2\sum_j \delta_j \right) , \tag{3.10.11}$$

where the area is measured with the area element

$$\frac{i}{2}\frac{d\tau \wedge d\bar{\tau}}{(\mathrm{Im}\,\tau)^2} , \tag{3.10.12}$$

and is given in table 3.10.1.

This completes the general discussion of the construction of globally well-defined 7-brane solutions. Explicit examples will be discussed in the next subsection.

3.10.3 $N = 1$ solutions

Consider the following choice for $\tau(z)$

$$\text{I.a}: \quad j(\tau) = \frac{z_i - z_{i\infty}}{z_i - z_\rho}\frac{z - z_\rho}{z - z_{i\infty}} . \tag{3.10.13}$$

At the points $z_{i\infty}$, z_i, z_ρ the j-functions takes the values ∞, 1 and 0, respectively. The monodromies of τ around the points $z_{i\infty}$, z_i and z_ρ can be found from the local expressions of the j-function as given in eqs. (3.9.4), (3.9.5) and (3.9.6). Since $j(\tau)$ is $PSL(2,\mathbb{Z})$ invariant the local monodromies of τ are determined up to $PSL(2,\mathbb{Z})$ transformations. Working with representative elements of $SL(2,\mathbb{Z})$ conjugacy classes one can take the monodromies around $z_{i\infty}$, z_i and z_ρ to be the respective transformations T, S and $T^{-1}S$ whose fixed points are $i\infty$, i and ρ, respectively. The monodromies of τ are elements of the group $PSL(2,\mathbb{Z})$. The function f transforms under $SL(2,\mathbb{Z})$ and therefore sees the difference between, e.g. an S or a $-S$ transformation. In this section the focus will be on those $SL(2,\mathbb{Z})$ transformations that when written as e^Q have $q > 0$. This guarantees that all contributions to the energy (3.8.7) are positive. In section 3.10.4 the analysis will be extended to the inclusion of objects with a negative tension. It will be shown that such objects play an important role in the construction of the so-called F-theory solutions.

Now that τ has been given in (3.10.13) and the $SL(2,\mathbb{Z})$ monodromies around $z_{i\infty}$, z_i and z_ρ have been chosen to be T, S and $T^{-1}S$ one is in a position to construct the function f. The required transformations for f around the points $z_{i\infty}$, z_i and z_ρ are given in table 3.10.2. The general form of f has been given in eq. (3.10.4). Using that $\eta^2(\tau)$ transforms as in eqs. (3.10.8), (3.10.9) and (3.10.10), it can be seen that f must be given by

$$\text{I.a}: \quad f(\tau) = \eta^2(\tau)(z - z_{i\infty})^{-1/12}(z - z_i)^{-1/4}(z - z_\rho)^{-1/6}\,. \tag{3.10.14}$$

For example, going around $z_{i\infty}$, i.e. sending $z - z_{i\infty} \to e^{2\pi i}(z - z_{i\infty})$ gives a factor $e^{\pi i/6}$ from the Dedekind eta function squared and a factor $e^{-\pi i/6}$ from the factor $(z - z_{i\infty})^{-1/12}$ appearing in (3.10.14), so that $f \to f$ as required. Going around z_i, i.e. sending $z - z_i \to e^{2\pi i}(z - z_i)$ gives $e^{-\pi i/2}\tau$ coming from η^2 and a factor $e^{-\pi i/2}$ coming from the factor $(z - z_i)^{-1/4}$ in (3.10.14), together this gives $-\tau$ as required. A similar calculation confirms the factor $(z - z_\rho)^{-1/6}$ in (3.10.14) for the point z_ρ. If one consider the function τ, eq. (3.10.13), near infinity, i.e. near $z = \infty$, it is found that the monodromy of τ around infinity is trivial. The functions f near $z = \infty$ behaves as $f \sim z^{-1/2}$. Going around $z = \infty$ is the same as going counter clockwise around $1/z$ near $z = 0$, i.e. $z \to e^{-2\pi i}z$ when going around $z = \infty$. Doing so leads to the monodromy $f \to -f$, so that the $SL(2, \mathbb{Z})$ monodromy around $z = \infty$ is equal to $-\mathbb{1}$.

The solution for τ and f, eqs. (3.10.13) and (3.10.14), can be pictorially represented as in figure 3.10.1. The branch cuts are those of the function j^{-1}, see figure 3.9.1 with the exception of the $-\mathbb{1}$ branch cut which is not a branch cut from the point of view of τ only.

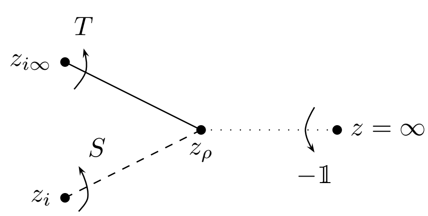

Figure 3.10.1: Pictorial representation of the solution I.a.

Consider next the following $N = 1$ choice for $\tau(z)$

$$\text{I.b}: \quad j(\tau) = \frac{z - z_\rho}{z - z_{i\infty}}\,. \tag{3.10.15}$$

This form for τ can be obtained from (3.10.13) by applying an $SL(2, \mathbb{C})$ transformation. The $SL(2, \mathbb{C})$ transformations are reparametrizations of the Riemann sphere.

The monodromies of τ and f are preserved by such transformations. Hence, in order to construct a second $N = 1$ solution, independent of the I.a solution, eqs. (3.10.13) and (3.10.14), the function f must be chosen such that its monodromies cannot be obtained from (3.10.14) via the above-mentioned $SL(2,\mathbb{C})$ transformation. The monodromies around $z_{i\infty}$ and z_ρ are taken to be the $SL(2,\mathbb{Z})$ transformations T and $T^{-1}S$, respectively, just as is the case in the I.a solution. However, taking the monodromy around $z_i = \infty$ to be $-S$ leads to a new solution. The function f is given by

$$\text{I.b}: \quad f(\tau) = \eta^2(\tau)(z - z_{i\infty})^{-1/12}(z - z_\rho)^{-1/6}\,. \tag{3.10.16}$$

Going around $z_i = \infty$ the Dedekind eta function squared transforms with a factor $e^{-\pi i/2}\tau$ while the factor $(z - z_{i\infty})^{-1/12}(z - z_\rho)^{-1/6}$ behaves as $z^{-1/4}$ leading to an additional factor of $e^{\pi i/2}$ when going around $z_i = \infty$, so that in total f transforms as $f \to \tau f$ implying that the $SL(2,\mathbb{Z})$ monodromy is $-S$. This can be verified by taking S as defined in (3.9.3) from which one reads off that $c = 1$ and $d = 0$ in the parametrization (3.7.27), so that indeed $f \to \tau f$. The I.b solution can be pictorially represented as in figure 3.10.2.

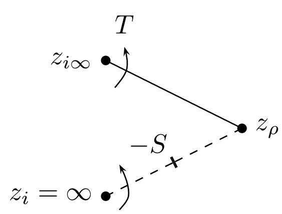

Figure 3.10.2: Pictorial representation of the solution I.b.

In a similar fashion two more independent solutions can be constructed for the $N = 1$ case these will be referred to as the I.c and I.d solutions and are given by

$$\text{I.c}: \quad j(\tau) = \frac{z_i - z_{i\infty}}{z - z_{i\infty}}\,, \tag{3.10.17}$$

$$\text{I.c}: \quad f(\tau) = \eta^2(\tau)(z - z_{i\infty})^{-1/12}(z - z_i)^{-1/4}\,, \tag{3.10.18}$$

$$\text{I.d}: \quad j(\tau) = \frac{z - z_\rho}{z_i - z_\rho}\,, \tag{3.10.19}$$

$$\text{I.d}: \quad f(\tau) = \eta^2(\tau)(z - z_i)^{-1/4}(z - z_\rho)^{-1/6}\,. \tag{3.10.20}$$

For the I.c solution the monodromy around $z_\rho = \infty$ is $-T^{-1}S$ and for the I.d solution the monodromy around $z_{i\infty} = \infty$ is $-T$. These solutions have the pictorial representations given in figures 3.10.3 and 3.10.4.

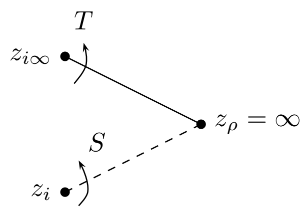

Figure 3.10.3: Pictorial representation of the solution I.c.

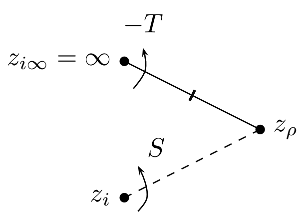

Figure 3.10.4: Pictorial representation of the solution I.d.

In drawing the figures representing the I.a to I.d solutions a certain notation has been introduced. The rules for drawing such figures are summarized in figure 3.10.5.

The I.a solution is one containing three 7-branes whereas the I.b, I.c and I.d solutions all contain two 7-branes. The I.b and I.c solutions both contain one D7-brane and one particular Q7-brane. The I.d solution contains no D7-branes, but two different Q7-branes.

The τ functions for the solutions I.a to I.d are related via $SL(2,\mathbb{C})$ transformations. This, however, is not true for the functions f. The monodromies of f differ from case

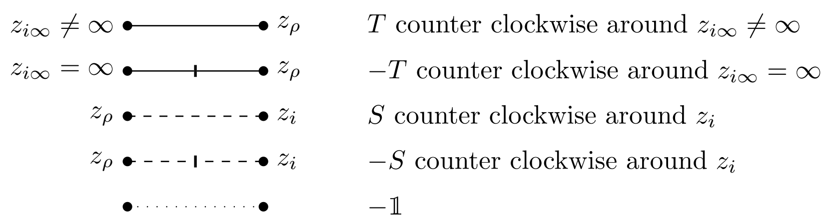

Figure 3.10.5: Notation for the pictorial representations of $SL(2,\mathbb{Z})$ monodromies of the pair (τ, f). Each point $z_{i\infty} \neq \infty$ is the endpoint of a T branch cut. A $-T$ monodromy is only allowed around the point $z_{i\infty} = \infty$. Points z_i (equal to ∞ or not) are the endpoints of either an S or a $-S$ monodromy. Points z_ρ (equal to ∞ or not) are the endpoints of two branch cuts one with T or $-T$ and one with S or $-S$ monodromy. A $-\mathbb{1}$ branch cut can go from any point $z_{i\infty}, z_i, z_\rho$ or $z = \infty$ (where τ takes any value not equal to $i\infty, i, \rho$) to any other such point.

to case by essential minus signs.

The pictorial representation of the I.a to I.d solutions used the rules given in figure 3.10.5. These rules will also be used later when solutions with $N > 1$ are discussed.

In principle one could next consider solutions with $N > 1$, but for this purpose there is still one piece of information missing. The point is that the primary information comes from τ which, from the $SL(2,\mathbb{Z})$ point of view, is the most insensitive function. For example when the τ monodromy is trivial the function f can transform under $\pm\mathbb{1}$. When it transforms as $-\mathbb{1}$ this could be because it so happens that the monodromy is measured around a point where two S branch cuts meet. When the monodromy is $+\mathbb{1}$ this could be because the monodromy is measured around a point where a S and a $-S$ branch cut meet. There are no values of $p, q \geq 0$ such that e^Q equals $-S$. Therefore, this is not a Q7-brane. If the monodromy of f is $+\mathbb{1}$ branch cuts are still needed and this is due to the essential path dependence of monodromies. Even though around a point where an S and a $-S$ branch cut meet the monodromy is trivial, the S and $-S$ branch cuts may have a role to play in the monodromies around other points.

It will be shown in the next subsection that points with $-S$ monodromy and also those with $-T^{-1}S$ monodromy correspond to objects with a negative tension. In order not to have explicit negative energy sources in the solution they will always be taken coincident with positive tension Q7-branes so that the local deficit angles are always non-negative. This is the most conservative viewpoint that one can adopt. Less conservative would be to allow for the presence of explicit negative energy sources in the solution giving rise to negative deficit angles. Although this in principle can be done and is not ruled out by supersymmetry it will not be considered here.

3.10.4 Including negative tension objects

When writing S or $T^{-1}S$ as e^Q in both cases one finds that $q > 0$ (implying $p > 0$ since $\det Q = qp - r^2/4 > 0$). This means that it is possible to use the formulae derived in section 3.7. The reason is that the results are based on the relation between $q_{\alpha\beta}$ and p, q, r, a relation implying $p, q > 0$ as explained below eq. (1.3.38).

When writing $-S$ as e^Q one would find $p = q = -\frac{\pi}{2}$ and $r = 0$ while for $-T^{-1}S$ one would find $p = q = r = -\frac{4\pi}{3\sqrt{3}}$. This implies that for $-T^{-1}S$ one would have $\sqrt{\det Q} = \frac{2\pi}{3}$. This should be contrasted with the case $T^{-1}S$ for which $\sqrt{\det Q} = \frac{\pi}{3}$. The axidilaton field τ should not see the difference between $T^{-1}S$ or $-T^{-1}S$. It is clear that applying, for example, formula (3.7.19) to the case $-T^{-1}S$ with $\sqrt{\det Q} = \frac{2\pi}{3}$ would give the wrong answer. Hence, there is no source term interpretation for points with monodromies $-S$ or $-T^{-1}S$.

On the other hand it is not too difficult to construct solutions containing points z_i or z_ρ around which the monodromy is $-S$ or $-T^{-1}S$, respectively[9]. The local behavior for τ near z_i and z_ρ will be the same as for 7-branes with S and $T^{-1}S$ monodromy. The function f, does see the difference between, e.g. $T^{-1}S$ and $-T^{-1}S$. Under $T^{-1}S$ the function f transforms as $f \to -\tau f$, a transformation that is fully induced by the transformation of τ under $T^{-1}S$ as is shown in 3.7.3, while under $-T^{-1}S$ the function f transforms as $f \to \tau f$. The latter transformation of f cannot be interpreted as being fully generated by its $PSL(2,\mathbb{Z})$ part. In this case instead of inserting a factor $(z - z_\rho)^{-1/6}$ as is done for the $T^{-1}S$ case one inserts the factor $(z - z_\rho)^{-1/6}(z - z_\rho)^{1/2} = (z - z_\rho)^{1/3}$ into the function f whenever a $-T^{-1}S$ object is placed at z_ρ. Similarly, a factor $(z - z_i)^{-1/4}(z - z_i)^{1/2} = (z - z_i)^{1/4}$ is inserted for each $-S$ object placed at z_i. The fact that factors $(z - z_\rho)^{1/3}$ and $(z - z_i)^{1/4}$ have a positive power of 1/3 and 1/4, respectively, means that the metric at the location of the $-T^{-1}S$ and $-S$ objects has a negative deficit angle. Hence, these objects have a negative tension and contribute negatively to the total energy of the solution. As mentioned before, in order not to have explicit negative energy sources in the solution they will always be taken coincident with positive tension Q7-branes so that the local deficit angles are always positive.

To summarize, the behavior of τ is the same for a $-T^{-1}S$ or a $-S$ negative tension object as it is for a $T^{-1}S$ or an S (positive tension) Q7-brane. The behavior of f due

[9] Points $z_{i\infty}$ around which the monodromy is $-T$ will not be considered except when $z_{i\infty} = \infty$ in which case there is no object whose monodromy is $-T$, but only an overall monodromy that happens to be $-T$.

to the presence of a $-T^{-1}S$ or a $-S$ object is given by

$$-T^{-1}S: \quad f(\tau) = \eta^2(\tau)(z - z_\rho)^{1/3} \cdots , \tag{3.10.21}$$

$$-S: \quad f(\tau) = \eta^2(\tau)(z - z_i)^{1/4} \cdots . \tag{3.10.22}$$

The nature of the negative tension objects with $-S$ or $-T^{-1}S$ monodromy is not clear, but it will be shown that they have a role to play in the construction of, e.g. the F-theory solutions.

The results of this subsection can be used to derive the asymptotic form of the metric for the I.b, I.c and I.d solutions discussed in subsection 3.10.3. In these three cases the monodromy around $z = \infty$ is of the form $-e^Q$ where Q is the usual matrix with $q, p \geq 0$. The minus sign in the monodromy follows from a square root branch cut in the function f so that near $z = \infty$ one has for the function f

$$f_{-e^Q} = f_{e^Q}\, z^{-1/2} , \tag{3.10.23}$$

where f_{e^Q} is the form f takes when the monodromy is e^Q, which has been derived in subsection 3.7.3. For a T monodromy $f = 1$ and for an S or a $T^{-1}S$ monodromy f is given in terms of τ in eq. (3.7.32). Therefore one has

$$f_{-T} = z^{-1/2} , \tag{3.10.24}$$

$$f_{-S} = \frac{1}{(\tau - i)^{1/2}(\tau + i)^{1/2}} z^{-1/2} , \tag{3.10.25}$$

$$f_{-T^{-1}S} = \frac{\frac{1}{2}\sqrt{3}}{(\tau - \rho)^{1/2}(\tau - \bar{\rho})^{1/2}} z^{-1/2} , \tag{3.10.26}$$

in which the dependence of $\tau(z)$ near $z = \infty$ follows from expressions (3.10.15), (3.10.17) and (3.10.19) depending on which solution, I.b to I.d, one is considering. The asymptotic form for the metric then follows from combining this result for f with the asymptotic form for τ and eq. (3.6.3). It follows that for a $-T$ monodromy around $z = \infty$ the metric behaves for $|z| \to \infty$ as $\frac{\log |z|}{|z|}$ while for $-S$ and $-T^{-1}S$ the metric goes as $|z^{-1/4}|^2$ and $|z^{-1/3}|^2$, respectively.

The rules for representing the pair (τ, f) are given in figure 3.10.5. These rules follow from the $N = 1$ solutions of the previous subsection together with the negative tension objects of this subsection.

3.10.5 Seven-branes and $SL(2, \mathbb{Z})$ conjugacy classes

The $SL(2, \mathbb{Z})$ conjugacy classes are for $\det Q = 0$ given by $\pm T^n$ (with $n = 0, 1, 2, \ldots$), while for $\det Q > 0$ they are given by $\pm S$, $\pm T^{-1}S$ and $\pm(T^{-1}S)^2$ [66, 67]. These $SL(2, \mathbb{Z})$ conjugacy classes can be interpreted as follows. The conjugacy class T^n is formed by n coincident D7-branes. The conjugacy class $-T^n$ does not correspond to any object, but can arise as the monodromy measured around $z = \infty$ (see for example the I.d solution shown in figure 3.10.4). The conjugacy classes S and $T^{-1}S$ correspond to Q7-branes associated with the orbifold points $\tau_0 = i$ and $\tau_0 = \rho$, respectively. The $SL(2,\mathbb{Z})$ conjugacy classes $-S$ and $-T^{-1}S$ correspond to certain negative tension objects. The conjugacy class $(T^{-1}S)^2$ can be interpreted as formed out of two coincident $T^{-1}S$ Q7-branes. Finally, the conjugacy class $-(T^{-1}S)^2$ can be read as the result of putting a $T^{-1}S$ Q7-brane on top of a negative tension $-T^{-1}S$ object.

Instead of interpreting $-T^{-1}S$ as a negative tension object one could also say that it corresponds to a set of four coincident $T^{-1}S$ Q7-branes, based on the $SL(2,\mathbb{Z})$ identity $-T^{-1}S = (T^{-1}S)^4$. The two interpretations are not physically equivalent. For the case of four coincident $T^{-1}S$ Q7-branes one must insert a factor $(z - z_\rho)^{-4/6}$ into the function f. It follows that under $(T^{-1}S)^4$ the function f transforms as $f \to e^{-2\pi i}\tau f$ whereas under $-T^{-1}S$ was shown to transform as $f \to \tau f$. The difference between these two interpretations lies in the behavior of the Killing spinor ϵ, eq. (3.6.6). For $(T^{-1}S)^4$ and for $-T^{-1}S$ the transformation of ϵ differs a sign, i.e. the spinstructure for a $-T^{-1}S$ object differs from the spinstructure of four coincident $T^{-1}S$ Q7-branes. It is also clear that, unless there are negative tension objects, one cannot have more than eight $T^{-1}S$ Q7-branes in a solution (coincident or not) because they make up a total deficit angle of 4π as follows from eq. (3.8.7) (see also section 3.11). Similar statements apply to S Q7-branes. A $-S$ monodromy can also be read as coming from three coincident S Q7-branes, based on the $SL(2,\mathbb{Z})$ identity $-S = S^3$. Also in this case the difference between the two interpretations lies in the behavior of the Killing spinor. Further, unless there are negative tension objects, one cannot have more than six S Q7-branes in a solution (see section 3.11).

3.10.6 General solution with N and $\tau(z = \infty)$ arbitrary

For the $N = 1$ case four distinct solutions could be constructed. It will be clear that the number of distinct solutions grows fast as N becomes bigger. A considerable limitation would be to consider only those configurations for which the asymptotic value of τ can be arbitrary but not equal to $i\infty, i$ or ρ. In the $N = 1$ case there is only one solution that has this property, namely the $N = 1a$ solution. There is good physical motivation to make this restriction. The asymptotic value for τ is the coupling constant and when it can be arbitrary it is possible to consider the asymptotic regime as the perturbative starting point for some underlying theory. This could be ordinary type IIB superstring theory whose coupling constant is the asymptotic value of $(\mathrm{Im}\,\tau)^{-1}$ or it could be perturbative with respect to another coupling constant (see for example the hypothetical Q-string tension of section 3.12). Hence, from now on only those solutions for which τ at $z = \infty$ is a free parameter of the solution, will be considered.

The form of τ for arbitrary N and $\tau(z=\infty)$ is

$$j(\tau) = \frac{P_N}{P_N + Q_N}\,, \tag{3.10.27}$$

where P_N and Q_N are arbitrary polynomials of degree N. Writing P_N and Q_N as

$$P_N = c_P(z - z_\rho^1)\cdots(z - z_\rho^N)\,, \tag{3.10.28}$$
$$Q_N = c_Q(z - z_i^1)\cdots(z - z_i^N)\,, \tag{3.10.29}$$

with c_P and c_Q nonzero complex constants, the general form taken by the function f is

$$f(\tau) = \eta^2(\tau)\,(P_N + Q_N)^{-1/12}\,(z - z_\rho^1)^{-s_1}\cdots(z - z_\rho^N)^{-s_N}\,(z - z_i^1)^{-t_1}\cdots(z - z_i^N)^{-t_N}\,, \tag{3.10.30}$$

in which s_j and t_j, $j = 1, \ldots, N$, are as given in table 3.10.3.

location	monodromy of ϵ	function h	(k, l)
z_i^j	S	$s_j = -1/4$	$(k = \text{even}, l = \text{even})$
z_i^j	$-S$	$s_j = 1/4$	$(k = \text{odd}, l = \text{even})$
z_ρ^j	$T^{-1}S$	$t_j = -1/6$	$(k = \text{even}, l = \text{even})$
z_ρ^j	$-T^{-1}S$	$t_j = 1/3$	$(k = \text{odd}, l = \text{even})$

Table 3.10.3: Form of the function h defined in eq. (3.10.4) for Q7-branes and negative tension objects. The integers k and l are defined in eq. (3.10.6).

Any other monodromy such as S^3 or $-(T^{-1}S)^4$ can be obtained by combining the monodromies given in table 3.10.3. As explained at the end of the previous subsection monodromies such as $-S$ or S^3 are inequivalent within the double cover of $SL(2,\mathbb{Z})$.

The positive values for s_j and t_j in (3.10.30) lead to negative deficit angles. In order not to have negative energy sources in the solution one must take sufficient Q7-branes coincident with the negative tension objects so that the deficit angle becomes positive. It is not necessary to include negative tension objects in order to write down fully well-defined solutions, however, they can be used to undo the presence of certain Q7-branes if desired.

3.11 F-theory and Q7-branes

The well-known 7-brane configurations of F-theory have the properties that the monodromy of τ close to the points z_i, z_ρ is the identity in $PSL(2,\mathbb{Z})$ and T around $z_{i\infty}$. Further, the function f has no poles or zeros, which means that there are no local sources of energy. The function f does have a zero at infinity and the order of the zero determines the energy through eq. (3.8.8). This implies that the monodromy of f around any point not being infinity is $f \to f$. The above conditions are satisfied if and only if τ and f are of the following form

$$j(\tau) = \frac{P^3}{P^3+Q^2}, \tag{3.11.1}$$

$$f(\tau) = \eta^2(\tau)\left(P^3+Q^2\right)^{-1/12}, \tag{3.11.2}$$

where P^3+Q^2 is a polynomial of order N whose zeros are the locations of the $\det Q = 0$ (p,q,r) 7-branes.

Solutions of this type exist whenever N can be divided by either 2 or 3. Thus they exist for $N = 2, 3, 4, 6, 8, 9, 10, 12, 24$. The deficit angle at infinity is given by $2\pi N/12$. For $N = 12$ this corresponds to a cylinder and for $N = 24$ to a 2-sphere. Values of $12 < N < 24$ do not correspond to a regular space. Requiring additionally that the value τ at infinity is arbitrary singles out $N = 6, 12, 24$ as the only possible solutions. Solutions with $N = 2, 3, \ldots$ also exist but for these the asymptotic value of τ is equal to i or ρ and so for these solutions there does not exist a perturbative regime at infinity making these solutions inherently non-perturbative. The number of free complex parameters for the $N = 6, 12, 24$ cases is $3, 8, 18$, respectively. These numbers can be found by counting the number of free complex parameters in P and Q and subtracting four complex parameters that are associated to the $SL(2,\mathbb{C})$ coordinate freedom and to the scale symmetry $P \to \lambda^4 P$ and $Q \to \lambda^6 Q$ that leaves invariant τ and scales f with a factor λ^{-1}. The scaling of f implies that the transverse space metric scales with $|\lambda|^{-2}$ which is harmless in that the overall scale of the transverse space metric is a free real parameter. Further the scaling of f implies a shift of the overall phase of the Killing spinor that can be absorbed in a redefinition of the constant spinor ϵ_0 in (3.6.6).

The case $N = 6$ is the first instance in which solutions with only $\det Q = 0$ (p,q,r) 7-branes and an arbitrary value for τ at infinity are possible. The $N = 6$ case of eqs. (3.11.1) and (3.11.2) has the pictorial representation given in figure 3.11.1 using the rules given in figure 3.10.5.

As is clear from figure 3.11.1 the $SL(2,\mathbb{Z})$ monodromy around small loops encircling only z_i^1, z_i^2, z_i^3, z_ρ^1 or z_ρ^2 is $+\mathbb{1}$. At the points z_i^j with $j = 1, 2, 3$ there is an S Q7-brane coincident with a $-S$ negative tension object such that locally the mass and charge cancel. At the points z_ρ^j with $j = 1, 2$ there are two $T^{-1}S$ Q7-branes

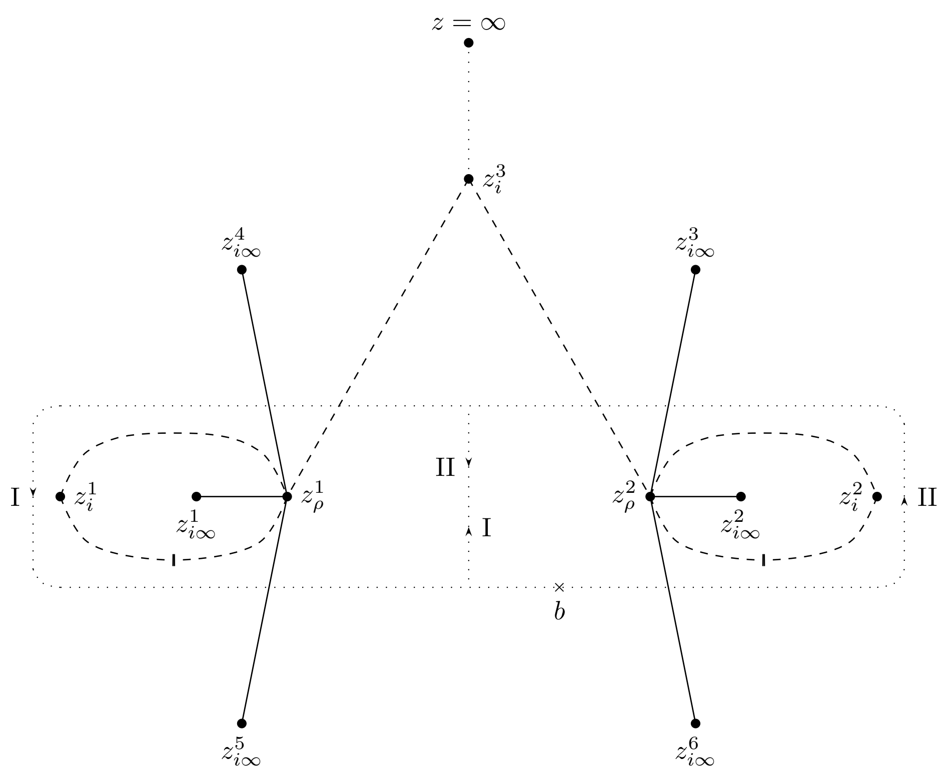

Figure 3.11.1: Pictorial representation of the F-theory $N = 6$ solution. The monodromy of the O7-plane and of the split O7-plane is measured with respect to the base point b. The O7-plane monodromy, $-T^{-4}$, is measured along I+II. The monodromy around II starting in b is ST^{-2}. To measure the monodromy around loop I starting in b one must first go around I+II and then around II in opposite direction. The monodromy measured going around loop I starting in b is T^2ST^{-4}.

coincident with one $-T^{-1}S$ negative tension object so that the masses and charges cancel.

Of the six zeros of $P^3 + Q^2$ two, the points $z_{i\infty}^1$ and $z_{i\infty}^2$, are non-perturbatively related to the point $z = \infty$ due to the fact that $z_{i\infty}^1$ and $z_{i\infty}^2$ are enclosed by S branch cuts. Suppose that the arbitrary value of τ at infinity is taken to be close to the point $\tau = i\infty$, so that the $(1,0)$ string coupling constant $(\operatorname{Im}\tau)^{-1}$ is close to zero, then there exist loops that start at $z = \infty$, encircle any of the points $z^j_{i\infty}$ for $j = 3, 4, 5, 6$ once and return to $z = \infty$. Since these loops only cross a T branch cut, the 4-four 7-branes at $z^j_{i\infty}$ for $j = 3, 4, 5, 6$ can be considered to be $(1, 0)$ 7-branes, and they will have this interpretation as long as $\tau(z = \infty)$ is close to $i\infty$ and a restricted set of loops is considered. This solution can thus describe at most four D7-branes.

Consider again figure 3.11.1. There is another loop around which the monodromy has a perturbative interpretation. This is the loop indicated by $I+II$ in the caption of figure 3.11.1. The $SL(2,\mathbb{Z})$ monodromy measured around this loop is $-T^{-4}$, which is the monodromy of a single O7-plane. The figure also shows that if one leaves the perturbative regime the notion of an O7-plane breaks down. The O7-plane monodromy can be decomposed into the monodromy measured around loop I and loop II [10]. Loop I and II encircle the points $z^5_{i\infty}$ and $z^6_{i\infty}$, but always in a way that involves crossing an S branch cut. The monodromy measured around loop I and II is T^2ST^{-4} and ST^{-2}, respectively. These monodromies can be written as $T^2ST^{-4} = M_1TM_1^{-1}$ and $ST^{-2} = M_2TM_2^{-1}$ with

$$M_1 = \pm \begin{pmatrix} 3 & 3\lambda_1 - 1 \\ 1 & \lambda_1 \end{pmatrix} \quad \text{and} \quad M_2 = \pm \begin{pmatrix} 1 & \lambda_2 \\ 1 & 1+\lambda_2 \end{pmatrix} . \tag{3.11.3}$$

Due to the above relations these branes can be viewed as $SL(2,\mathbb{Z})$-transformed versions of D7-branes. Instead of $\tau \to i\infty$ the complex scalar goes to the real line in their vicinity and henceforth IIB perturbative string theory is not valid there. The D7-brane has a $(1, 0)$ string ending on it. Using the transformation (2.3.10) with Λ equal to either M_1 or M_2 it follows that the 7-brane with monodromy ST^{-2} has a $(1, 1)$ string ending on it while the 7-brane with monodromy T^2ST^{-4} has a $(3, 1)$ string ending on it.

The $N = 24$ solution can be constructed by taking four copies of the $N = 6$ solution. For the $N = 24$ case the form of $j(\tau)$, eq. (3.11.1), is identical to (2.6.3) (the numbers 4 and 27 appearing in (2.6.3) have in (3.11.1) been absorbed into the definitions of P and Q). In fact F-theory on K3, discussed in subsection 2.6.3, is precisely described by eqs. (3.11.1) and (3.11.2), where $fdzd\tau$ is the holomorphic $(2, 0)$-form of the K3 manifold. The base manifold over which the 2-torus with modular parameter τ is elliptically fibered is given by eq. (3.6.3). The metric behaves logarithmically near an F-theory 7-brane (a zero of $P^3 + Q^2$ of order one that is not also a zero of P and/or Q) where the fiber becomes singular. Since the total deficit angle at infinity is 4π the base manifold is a 2-sphere with 24 F-theory 7-branes. Out of the 24 7-branes 16 can be taken to be of type $(1, 0)$ with monodromy T, 4 to be of type $(1, 1)$ with monodromy ST^{-2} and another 4 to be of type $(3, 1)$ with monodromy T^2ST^{-4}.

[10] This splitting of the O7-plane was predicted in [46] by using arguments from field theory whose validity can be understood by using 3-branes as probes and by studying what happens on the worldvolume of the 3-brane [69].

In the orientifold limit, $P^3 = cQ^2$ with $c \neq 0$, so that τ is an arbitrary constant and $P = R^2$ and $Q = R^3$ in which R is a 4th order polynomial, this $N = 24$ solution has the metric

$$ds^2 = -dt^2 + d\vec{x}_7^{\,2} + |R^{-1/2}|^2 dz d\bar{z}\,. \tag{3.11.4}$$

The polynomial R has four zeros at each of which the metric has deficit angle π. This is the metric of $T^2/\mathbb{Z}_2$. The function f transforms as $f \to -f$ when going around an orbifold point of $T^2/\mathbb{Z}_2$, which is the transformation of f under the $-\mathbb{1}$ element of $SL(2,\mathbb{Z})$. The solution is interpreted as modding out $\text{Mink}_{1,7} \times T^2$ by $\mathbb{Z}_2 = \{1, I_{89}(-1)^{F_L}\Omega\}$ [46] or equivalently as F-theory on $T^4/\mathbb{Z}_2$ (see also subsections 2.6.2 and 2.6.1).

It was realized in [54] that apart from the $T^4/\mathbb{Z}_2$ orbifold limit of the K3 with 24 7-branes one can further consider the orbifold limits $T^4/\mathbb{Z}_n$ for $n = 3, 4, 6$ in which τ is not an arbitrary constant but equal to either i or ρ[11]. Consider the case $\tau = i$ in (3.11.1) which means that $Q = 0$ and $f \propto P^{-1/4}$. The polynomial P has eight different zeros. Since $P^{-1/4} = P^{-3/12}$ with each factor $(z - z_i)^{-1/12}$ coming from an F-theory 7-brane there are at each zero of P three coincident F-theory 7-branes. The number of complex parameters in P is 9 and 4 of them can be fixed using the $SL(2,\mathbb{C})$ coordinate freedom plus the scale symmetry mentioned above, so that there are 9-4=5 free complex parameters. For the case $\tau = \rho$ it must be that $P = 0$ and so $f \propto Q^{-1/6}$ in which case Q has twelve zeros and at each zero of Q two F-theory 7-branes are coinciding. In this case the number of free complex parameters is 13-4=9.

The five free complex parameters in the case $\tau = i$ can be fixed by grouping the 8 different zeros of P into three groups of zeros of order 3, 3 and 2, i.e. $f \propto (z - z_i^1)^{-3/4}(z - z_i^2)^{-3/4}(z - z_i^3)^{-1/2}$. The three deficit angles are $3\pi/2$, $3\pi/2$ and π which are the three deficit angles of the metric for $T^2/\mathbb{Z}_4$. The orbifold $T^2/\mathbb{Z}_4$ is defined by a complex parameter z which is such that $z \sim z + 1$, $z \sim z + i$ and $z \sim iz$. The complex structure modulus of the base manifold is fixed to be equal to i. The size of the base is still one free real modulus. The fixed points are $z = 0, \frac{1}{2}, \frac{1+i}{2}$. The group $\mathbb{Z}_4$ consists of four generators that are of the form $(R(\frac{\pi}{2}))^n$ with $n = 0, 1, 2, 3$ and in which $R(\frac{\pi}{2})$ denotes a rotation over $\frac{\pi}{2}$ (acting on z as multiplication by i). The fixed points 0 and $\frac{1+i}{2}$ are fixed points of order four because they are invariant under the action of $R(\frac{\pi}{2})$ (up to the torus identifications $z \sim z + 1$ and $z \sim z + i$) which is an order four element of $\mathbb{Z}_4$. The point $\frac{1}{2}$ is order 2 since it is invariant under $R(\frac{\pi}{2})^2 = -1$ (up to the torus identifications $z \sim z + 1$ and $z \sim z + i$) which is of order two in $\mathbb{Z}_4$. Knowing the orders of the fixed points the deficit angles (that must add up to 4π) $3\pi/2$, $3\pi/2$ and π follow. When going around the order four fixed points the function f transforms under $S^3 = -S$ while when going around the orbifold point of order two f transforms under $S^2 = -\mathbb{1}$. The complex structure modulus τ of the fibre is fixed to be equal to i. Given the $SL(2,\mathbb{Z})$ monodromies when going around

[11] See for example [70] for details on 2-dimensional orbifolds.

the three fixed points of the base it is concluded that the fibre is the orbifold $T^2/\mathbb{Z}_4$ in which $\mathbb{Z}_4$ has as its elements $(-S)^n$ for $n = 0, 1, 2, 3$. The F-theory description is thus $T^4/\mathbb{Z}_4$ and this should be equal to type IIB on the orientifold $T^2/\mathbb{Z}_4$ in which the orientifold actions form the group $\mathbb{Z}_4$ given by [54]

$$\mathbb{Z}_4 = \{1, -SR(\tfrac{\pi}{2}), (-SR(\tfrac{\pi}{2}))^2, (-SR(\tfrac{\pi}{2}))^3\}\,, \tag{3.11.5}$$

in which $(-SR(\frac{\pi}{2}))^2 = (-1)^{F_L}\Omega I_{89}$.

Reducing IIB over a point of order 4 gives the following 8-dimensional fields: a metric g_{ab}, one real scalar (the size of $T^2/\mathbb{Z}_4$), one 2-form (coming from the 4-form) and one complex vector $A_a = (B + iC)_{az}$. There are no Kaluza–Klein vectors and no free complex structure modulus of $T^2/\mathbb{Z}_4$ since this is fixed to be i. These fields form an $N = 1$, $d = 8$ supergravity multiplet. It is shown in [54] that near a point of order 4 the 7-brane gauge group is E_7 [12]. The orbifold $T^2/\mathbb{Z}_4$ has three fixed points of order 4, 4 and 2, so that the complete gauge group is $E_7 \times E_7 \times SO(8)$ which is of rank 18. The number 18 is also the number of free complex parameters in F-theory on K3, and therefore forms the highest possible rank of a 7-brane gauge group. Hence, the gauge symmetry enhancement has involved all the free 7-brane moduli of F-theory on K3 with 24 non-coincident 7-branes and preserved the rank of the gauge group. This should be contrasted with the case of IIB on the orientifold $T^2/\mathbb{Z}_2$ discussed earlier. In the latter case the 7-brane gauge group was $(SO(8))^4 \times (U(1))^2$ (times $(U(1))^2$ coming from the Kaluza–Klein vectors). It was mentioned that there can be at most 4 D7-branes in the $N = 6$ solution and so there can be at most 16 D7-branes in the $N = 24$ solution. In the realization of the gauge group $(SO(8))^4$ all the D7-branes (plus the four orientifold planes) play a role. The T^2ST^{-4} and ST^{-2} 7-branes that are shown in figure 3.11.1 on page 96, thus did not play a role in the symmetry enhancement for the case of IIB on $T^2/\mathbb{Z}_2$. For the case of IIB on $T^2/\mathbb{Z}_4$ this is different since the 16 D7-branes alone could never produce a gauge group of rank 18. Therefore the T^2ST^{-4} and ST^{-2} 7-branes must have played a role in the enhancement. In fact in [71] (see [72] for an account of the $SO(8)$ gauge group via open strings) it is shown how exceptional gauge groups can arise by considering not only open strings with two endpoints but also multi-pronged strings that have more than two endpoints in particular 3-string junctions play an important role in the analysis of [71]. The orbifold limits $T^4/\mathbb{Z}_3$ and $T^4/\mathbb{Z}_6$ of F-theory on K3 can be discussed analogously to the $T^4/\mathbb{Z}_4$ case [54].

So far two extreme cases have been considered. The first extreme case is the situation in which none of the 24 F-theory 7-branes are coincident and the second

[12] This also follows by consulting table 2.6.1, page 53, using that at an orbifold point of order 4 (a third order zero of P) nine F-theory 7-branes are coincident so that the third order zero of P becomes a nineth order zero of $P^3 + Q^2$ with $Q = 0$ by construction. Since the polynomial Q does not play any role one can assume that it has a fifth order zero at the same point where P has a third order zero. With this data table 2.6.1 yields the gauge group E_7.

extreme case is the situation in which large numbers of F-theory 7-branes coincide to the extent that τ must be a constant. An intermediate case will be to consider coincident F-theory 7-branes with a non-constant τ. It will be shown that these solutions have exactly the properties of non-constant τ solutions involving Q7-branes. In fact the Q7-branes such as the S or $T^{-1}S$ Q7-branes will have a direct interpretation in terms of coincident F-theory 7-branes.

Before continuing I would like to stress that the remaining part of this section is based on preliminary results and ideas that are not contained in [16, 17] and which can be regarded as work in progress.

Consider the following 7-brane solution containing eight D7-branes (these are all $(1,0)$ 7-branes as can be seen be drawing the pictorial representation of the solution given in (3.11.6) and (3.11.7)) and eight $T^{-1}S$ Q7-branes and which furthermore has $\tau(z=\infty)$ arbitrary

$$j(\tau) = \frac{P_8}{P_8 + Q_4^2}\,, \tag{3.11.6}$$

$$f(\tau) = \eta^2(\tau)\left(P_8 + Q_4^2\right)^{-1/12} P_8^{-1/6}\,. \tag{3.11.7}$$

The total deficit angle at $z=\infty$ is equal to 4π and hence the transverse space of the 7-branes is compact. The polynomials P and Q are of order eight and four, respectively, as indicated by the subscript 8 and 4. The eight zeros of P_8 produce eight places where the metric has a deficit angle. Around each of the zeros of P_8 the pair (τ, f) transforms under $T^{-1}S$. These are thus the locations of the eight $T^{-1}S$ Q7-branes. Setting $P_8 = cQ_4^2$ for some nonzero c forces τ to be constant, but further arbitrary. The resulting solution is the orbifold $T^4/\mathbb{Z}_2$ discussed earlier. Setting τ equal to i in eqs. (3.11.6) and (3.11.7) implies that $Q_4 = 0$ while $f \propto P_8^{-1/4}$ just as in the case discussed above. The limit $\tau = \rho$ however does not exist since it is not possible to set P_8 equal to zero in f. The solution (3.11.6) and (3.11.7) contains the two orbifold limits $T^4/\mathbb{Z}_n$ with $n = 2, 4$ suggesting that (3.11.6) and (3.11.7) is a special case of the generic configuration: F-theory on K3 with 24 non-coincident 7-branes. For convenience the situation with 24 non-coincident 7-branes is repeated here as

$$j(\tau) = \frac{P_8^3}{P_8^3 + Q_{12}^2}\,, \tag{3.11.8}$$

$$f(\tau) = \eta^2(\tau)\left(P_8^3 + Q_{12}^2\right)^{-1/12}\,, \tag{3.11.9}$$

with the orders of the polynomials P and Q written as subscripts. Next, consider a situation in which eight of the twelve zeros of Q_{12} coincide with the eight zeros of P_8, i.e. take

$$Q_{12} = P_8 Q_4\,, \tag{3.11.10}$$

where Q_4 is an arbitrary polynomial of order four. Substituting eq. (3.11.10) into eqs. (3.11.8) and (3.11.9) gives eqs. (3.11.6) and (3.11.7). The choice (3.11.10) implies that $P_8^3 + Q_{12}^2 = P_8^2(P_8 + Q_4^2)$ so that out of the originally 24 non-coincident F-theory 7-branes 8 are coincident with one other 7-brane and 8 remain non-coincident. Since the solution (3.11.6) and (3.11.7) is interpreted as consisting of 8 D7-branes and 8 $T^{-1}S$ Q7-branes it follows that each of the $T^{-1}S$ Q7-branes is formed out of two coinciding F-theory 7-branes. Further, it must be that the coinciding F-theory 7-branes forming a $T^{-1}S$ Q7-brane have different strings ending on them or what is the same their monodromies must have been different[13]. In the F-theory literature one speaks of mutually non-local F-theory 7-branes as being F-theory 7-branes that are separated by branch cuts in the sense that they cannot, with respect to the same base point, have the same monodromy. It is precisely this type of F-theory 7-branes that are forced to coincide by taking Q_{12} as in eq. (3.11.10). When writing (3.11.8) it is implicitly assumed that the polynomials in the numerator and denumerator have no common factors. One can still allow for common factors with the understanding that then the fibre becomes singular. The fact that (3.11.10) (or any of the orbifold constructions for that matter) forms a singular limit makes it ambiguous to state precisely which of the 24 F-theory 7-branes are coinciding and which are not - the ambiguity resides in the fact that one can only properly view a 7-brane as a representative element of an $SL(2,\mathbb{Z})$ conjugacy class.

In terms of the 24 F-theory 7-branes, that can be thought of as having monodromies T (16 times), ST^{-2} (4 times) and T^2ST^{-4} (4 times) the $T^{-1}S$ Q7-brane (as a representative element of the $T^{-1}S$ conjugacy class) can be thought of as consisting of one T and one $T^{-2}S$ 7-brane (giving $TST^{-2} = T^2(T^{-1}S)T^{-2}$ or, depending on the order, $T(T^{-1}S)T^{-1}$ monodromy) or as one T and one T^2ST^{-4} 7-brane (giving $T^4(T^{-1}S)T^{-4}$ or $T^3(T^{-1}S)T^{-3}$). Further, the mass of a $T^{-1}S$ Q7-brane, contributing $\frac{\pi}{6}$ to the total deficit angle, is equal to the sum of the masses of the F-theory 7-branes each of which contributes $\frac{\pi}{12}$ to the total deficit angle.

Now, that it is known how to embed (3.11.6) and (3.11.7) into the context of F-theory on K3, it becomes possible to use table 2.6.1 on page 53 to find out about the Q7-brane gauge group. The location of any of the eight $T^{-1}S$ Q7-branes forms a first order zero of Q_{12}, a first order zero of P_8 and a second order zero of $P_8^3 + Q_{12}^2$. From table 2.6.1 it is concluded that this gives rise to the Argyres–Douglas singularity H_0 and that in this case there is no singularity type, i.e. the gauge group must be

[13] One may wonder if the statement that a $T^{-1}S$ Q7-brane consist of two F-theory 7-branes that when separated have a different monodromy has an interpretation in terms of the 8-forms to which the different 7-branes couple. However, due to the identifications under $SL(2,\mathbb{Z})$ there are no 8-forms in the solution that can be different from zero. The electric description only works locally and was worked out in section 3.7. Since the F-theory 7-branes forming a $T^{-1}S$ Q7-brane when separated from each other must be separated by a branch cut there is no electric picture of a $T^{-1}S$ Q7-brane as consisting of two other 7-branes simply because the electric description breaks down at the global level.

Abelian. The Tate algorithm only provides information about the non-Abelian part of the gauge group or the non-existence thereof. Therefore, at the moment it is not clear whether the $T^{-1}S$ Q7-brane gauge group is $U(1)$ or e.g. $(U(1))^2$ (after all it is formed out of two coinciding F-theory 7-branes each with a $U(1)$ gauge group). In order to find the number of $U(1)$ factors it may be instructive to study the zero modes of the solution (3.11.6) and (3.11.7). In particular the 2-form zero modes that are vectors may give information about the number of $U(1)$ factors. Even though the 2-forms themselves must be zero since the solution identifies $SL(2,\mathbb{Z})$ transformed IIB fields, the 2-form zero modes that will be functions of τ and f can still exist.

In the case of F-theory with 24 non-coincident 7-branes the 7-brane gauge group was found to be $(U(1))^{18}$ a result that followed from the fact that there are in total 18 free complex parameters describing the relative positions of the 7-branes. In the case of the solution (3.11.6) and (3.11.7) the number of free complex parameters is 10. So in going from (3.11.8) and (3.11.9) to (3.11.6) and (3.11.7) 8 parameters have been identified with 8 others. The embedding of the solution (3.11.6) and (3.11.7) into (3.11.8) and (3.11.9) through (3.11.10) implies that the eight fixed parameters describe the relative positions of the two F-theory 7-branes that make up a $T^{-1}S$ Q7-brane. These parameters should therefore show up as world-volume degrees of freedom on the world-volume of the $T^{-1}S$ Q7-brane. However, since they are not free parameters of the solution (3.11.6) and (3.11.7) it is difficult to formulate precisely how these scalar moduli show up in the world-volume theory of the Q7-brane[14]. For example the source term (3.2.3) presupposes that the scalars describing the relative position of the two F-theory 7-branes are fixed to be equal to some constant. There are furthermore two world-volume scalar degrees of freedom that describe the position of the $T^{-1}S$ Q7-brane in the transverse space and these are free parameters of the solution. There are thus in total 4 scalar degrees of freedom on the world-volume of the $T^{-1}S$ Q7-brane. World-volume supersymmetry then requires there to be two BI vectors. This latter fact is natural if one considers the problem of how to assign the 18 $U(1)$'s of the F-theory solution to the 7-branes of (3.11.6) and (3.11.7). At most 8 of the total of 18 $U(1)$'s can be assigned to the 8 D7-branes of (3.11.6) and (3.11.7). This leaves one with 10 $U(1)$'s that must be assigned to 8 $T^{-1}S$ Q7-branes. Since each $T^{-1}S$ Q7-brane is identical to any other $T^{-1}S$ Q7-brane they must all have the same number of $U(1)$'s. Obviously, assigning to each of them a single $U(1)$ is insufficient, and so the next possibility is to attribute a $(U(1))^2$ to each of them. The condition is that the rank of the 7-brane gauge group is not allowed to exceed 18 as this is the highest rank in the case of the 24 F-theory 7-branes in (3.11.8) and (3.11.9). The highest rank in the solution (3.11.6) and (3.11.7) is obtained by assigning 8 of the 10 free parameters to the 8 $T^{-1}S$ Q7-branes and the remaining two to D7-branes. This

[14]This is not a problem that is special to the case at hand but is generic in supergravity. There does not, so far, exist a technique to obtain the world-volume action of coinciding branes directly from the properties of the solution.

clearly gives a gauge group of rank 18.

In the following arguments will be given that support, but not prove the above mentioned assertion that the $T^{-1}S$ Q7-brane gauge group is $(U(1))^2$. In section 2.4 it was mentioned that the gauge invariant coupling to an 8-form that couples to a Q7-brane requires the introduction of two BI vectors. It is expected that when the 2-form zero modes are considered in the local neighborhood of a $T^{-1}S$ Q7-brane that two 2-form zero modes (BI vectors) will be present simultaneously. At the linearized level the two BI vectors are independent. Assume that their Lagrangian is required to be invariant under the Q7-brane monodromy. Let $\mathcal{V}$ denote a column vector of two BI vectors that transforms under $SL(2,\mathbb{Z})$ just as the 2-forms do in eq. (1.3.51), i.e. as $\mathcal{V} \to \Lambda\mathcal{V}$. An $SL(2,\mathbb{Z})$ invariant and quadratic action for $\mathcal{V}$ is given by $\star(d\mathcal{V})^T \wedge SQd\mathcal{V}$, which is easily seen to be unique. The $T^{-1}S$ Q7-brane monodromy can be written as the product of single F-theory 7-brane monodromies. Writing the F-theory 7-brane monodromies as e^{Q_1} and e^{Q_2} with Q_1 and Q_2 two zero determinant traceless matrices it follows that the charge matrix Q of the Q7-brane can be written in terms of Q_1 and Q_2 as follows

$$e^{Q_1}e^{Q_2} = e^{Q} \quad \text{with} \quad Q = c\left(Q_1 + Q_2 + \frac{1}{2}[Q_1, Q_2]\right) , \tag{3.11.11}$$

where c is some real constant. For the case of a $T^{-1}S$ Q7-brane Q_2 is related to Q_1 by an $SL(2,\mathbb{Z})$ transformation. The expression (3.11.11) can be derived using the Baker–Campbell–Hausdorff formula. The matrices Q_1 and Q_2 can be written in terms of the string charges p' and q' that were introduced in eq. (2.3.8) and below as

$$Q_1 = S\begin{pmatrix} -q_1' \\ p_1' \end{pmatrix}\begin{pmatrix} -q_1' \\ p_1' \end{pmatrix}^T , \tag{3.11.12}$$

and similarly for Q_2. Using Q as in eq. (3.11.11) the proposed quadratic zero mode kinetic term becomes

$$\star(d\mathcal{V})^T \wedge SQd\mathcal{V} = c\star(d\mathcal{V})^T \wedge SQ_1 d\mathcal{V} + c\star(d\mathcal{V})^T \wedge SQ_2 d\mathcal{V} . \tag{3.11.13}$$

The commutator $[Q_1, Q_2]$ cancels at this order in the BI vectors. The leading term in the WZ term can be written as

$$\int_{\Sigma_8} \mathrm{Tr}\, Q \begin{pmatrix} \frac{1}{2}D_8 & -B_8 \\ C_8 & -\frac{1}{2}D_8 \end{pmatrix} , \tag{3.11.14}$$

and here the commutator $[Q_1, Q_2]$ remains. Therefore gauge invariance of (3.11.14) under the 8-form gauge transformation (1.1.45) will introduce interactions between the two BI vectors and the form of these interactions will be dictated by the commutator $[Q_1, Q_2]$.

As mentioned earlier the compact solution containing eight $T^{-1}S$ Q7-branes, eqs. (3.11.6) and (3.11.7), was obtained by fixing eight parameters in the compact F-theory solution containing 24 7-branes. These parameters can be interpreted as being the moduli describing the relative positions of the two F-theory 7-branes that make up a $T^{-1}S$ Q7-brane[15]. The moduli describing the position of the two coinciding F-theory 7-branes are still free. The bosonic world-volume degrees of freedom of a $T^{-1}S$ Q7-brane are thus expected to consist of 4 scalars (two for the relative motion of the two F-theory 7-branes and two for their center of mass motion) and two (interacting) BI vectors, so that the world-volume action should be formed out of two interacting $N = 1$, $d = 8$ vector multiplets and the gauge group is expected to be $(U(1))^2$ [16]. That is to say, if the assumptions about the independence of the two BI vectors and the invariance of the zero mode action under the Q7-brane monodromy are correct[17].

So far for the $T^{-1}S$ Q7-branes. Consider next a solution containing only S Q7-branes that is compact and has $\tau(z = \infty)$ arbitrary. Such a solution is given by

$$j(\tau) = \frac{P_2^3}{P_2^3 + Q_6}\,, \tag{3.11.15}$$

$$f(\tau) = \eta^2(\tau)\left(P_2^3 + Q_6\right)^{-1/12} Q_6^{-1/4}\,. \tag{3.11.16}$$

The solution, eqs. (3.11.15) and (3.11.16), follows from eqs. (3.11.8) and (3.11.9) by taking

$$P_8 = P_2 Q_6\,, \qquad Q_{12} = Q_6^2\,. \tag{3.11.17}$$

The six S Q7-branes are located at the zeros of Q_6. Since $P_8^3 + Q_{12}^2 = Q_6^3(P_2^3 + Q_6)$ the S Q7-brane consists of three coincident F-theory 7-branes. At any of the zeros of Q_6 the orders of the zeros of P_8, Q_{12} and $P_8^3 + Q_{12}^2$ are 1, 2 and 3, so that according to table 2.6.1 one is dealing with the Argyres–Douglas singularity H_1 in which case the gauge group is $SU(2)$. This means that an S Q7-brane must have $SU(2)$ gauge group. In terms of the 24 F-theory 7-branes, that can be thought of as having monodromies T (16 times), ST^{-2} (4 times) and T^2ST^{-4} (4 times) the S Q7-brane (as a representative elements of the S conjugacy class) can be thought of as consisting of two T and one $T^{-2}S$ 7-brane (giving $TTT^{-2}S = S$ or, depending on the order, $T^{-2}(S)T^2$, $T^{-1}(S)T$ monodromy) or as two T and one T^2ST^{-4} 7-brane (giving $T^4(S)T^{-4}$, $T^3(S)T^{-3}$ or T^2ST^{-2}). The non-Abelian gauge group $SU(2)$ can be understood to result from

[15] The Wess–Zumino term (3.11.14) as well as the action (3.2.3) should be interpreted in the limit in which the two scalars describing the relative positions of the two branes are fixed to be equal to some constant.

[16] From the point of view of a probe Q3-brane the two vector zero modes may be related by an electro-magnetic duality transformation, so that on a single probe Q3-brane world-volume they are not independent degrees of freedom.

[17] It is stressed here once again that the statements regarding the number of $U(1)$'s in the $T^{-1}S$ Q7-brane gauge group rest on the embedding of the solution (3.11.6) and (3.11.7) into (3.11.8) and (3.11.9) through (3.11.10) as providing the correct interpretation of the $T^{-1}S$ Q7-brane.

the fact that there are two coincident D7-branes involved. Again, it is expected that there will be additional $U(1)$ factors coming from the local properties of the 2-form zero modes near an S Q7-brane. The argument will be the same as for the case of a $T^{-1}S$ Q7-brane leading to a local gauge group $SU(2) \times (U(1))^2$. The rank of the S Q7-brane gauge group is three. Since the solution (3.11.15) and (3.11.16) has 6 D7-branes and 6 S Q7-branes and has 6 free complex parameters the maximal rank is 18 which occurs when all the 6 free parameters are assigned to the positions of the 6 S Q7-branes. The maximal rank agrees with the rank of the 7-brane gauge group in the case of the solution (3.11.8) and (3.11.9).

It will be clear that one can by suitably choosing P_8 and Q_{12} construct many more solutions containing for example both S and $T^{-1}S$ Q7-branes and/or coincident $T^{-1}S$ Q7-branes, etc.

Apart from the compact solutions with total deficit angle equal to 4π there exist also non-compact solutions containing Q7-branes such as the I.a solution of subsection 3.10.3. It can be shown that the I.a solution is a special case of the F-theory $N = 6$ solution. In fact any solution with no explicit negative energy sources and for which $\tau(z) \neq \text{cst}$ and with $\tau(z = \infty)$ is arbitrary (but not equal to $i\infty, i$ or ρ) containing Q7-branes can be considered a special case of the F-theory solutions with either $N = 6, 12$ or 24 in which the numerator and denumerator in (3.11.1) are allowed to have common factors.

3.12 Open strings stretched between 7-branes

The solution near a D7-brane is singular, the metric behaves logarithmically. Near a Q7-brane there appears a deficit angle. Often when singularities appear in the solution in the Einstein frame they disappear when considering the solution from the point of view of the frame of the closed IIB superstring. However, if one considers the metric near a D7-brane in the string frame it is still singular. This may well be related to the fact that a single D7-brane in string theory is an inconsistent background with uncancelled tadpoles. There exists another frame in which the solution appears regular. For a general (p, q, r) 7-brane define the frame $d\tilde{s}^2$ by

$$d\tilde{s}^2 = dwd\bar{w} \quad \text{with} \quad dw = q^{1/2}(\tau - \tau_0)^{1/2}(\tau - \bar{\tau}_0)^{1/2} f dz\,. \tag{3.12.1}$$

Setting $\det Q = 0$ (3.12.1) one obtains

$$d\tilde{s}^2 = \frac{|\sqrt{q}\tau + \sqrt{p}|^2}{\text{Im}\,\tau}\, ds_E^2\,, \tag{3.12.2}$$

where ds_E^2 is the Einstein frame metric and the prefactor is the square of the tension of a (p', q') string with $p'^2 = p$ and $q'^2 = q$. It is clear that

$$\int_\gamma |dw| \tag{3.12.3}$$

measures the mass of a string stretched between two identical (p', q') 7-branes or stretched between a probe 3-brane, parallel with the 7-branes and a (p', q') 7-brane along a contour γ that lies in the complex z-plane not crossing any branch cuts [73]. The mass of the string then translates into certain BPS mass formulae (realized for straight contours in the w-plane) for the world-volume theory of the 3-brane.

The natural question is what about the Q7-branes viewed from the point of view of the frame (3.12.1)? The metric near a Q7-brane is fully regular in the frame (3.12.1). The relation with the Einstein frame for $\det Q > 0$ is

$$d\tilde{s}^2 = \frac{|q^{1/2}(\tau - \tau_0)^{1/2}(\tau - \bar{\tau}_0)^{1/2}|^2}{\operatorname{Im}\tau} ds_E^2 \,. \tag{3.12.4}$$

The prefactor in (3.12.4) can be rewritten as

$$\frac{|q^{1/2}(\tau - \tau_0)^{1/2}(\tau - \bar{\tau}_0)^{1/2}|^2}{\operatorname{Im}\tau} = \left(T^2 - 4\det Q\right)^{1/2} = 2\sqrt{\det Q}\sinh\eta \,. \tag{3.12.5}$$

The geometrical interpretation of η is given in section 1.4. It measures the Poincaré distance along a geodesic (semi-circles whose center is on the real axis) connecting τ and τ_0. By comparison the dilaton ϕ measures the Poincaré distance along a geodesic (straight semi-infinite vertical lines) connecting τ and $\tau = i\infty$ or τ_0 with $\operatorname{Im}\tau_0 = 0$.

Purely hypothetically, in analogy with the $\det Q = 0$ case, one could consider

$$\int_\gamma |dw| \tag{3.12.6}$$

for the case $\det Q > 0$ to compute the mass a new type of string whose tension is proportional to $\left(T^2 - 4\det Q\right)^{1/4}$. Since the metric $d\tilde{s}^2 = dwd\bar{w}$ is flat such a string can be BPS by choosing a straight contour in the w-plane. Since the background cannot have any nonzero 2-forms, due to the identifications under $SL(2,\mathbb{Z})$, these strings as well as the massive (p', q') strings are chargeless. One could refer to such a string as a Q-string since it ends on a Q7-brane.

3.13 Discussion

The most general 7-brane configuration that is globally well-defined with finite energy is necessarily supersymmetric, see subsection 3.8.2. The value of τ at infinity, $z = \infty$, can be fixed to be equal to either $i\infty, i$ or ρ or it is taken to be a free parameter not equal to either $i\infty, i$ or ρ. Besides that one can choose to allow for negative energy sources (only then the number of 7-branes can exceed 24), or negative energy sources that are (locally) cancelled by a Q7-brane. Those solutions for which $\tau(z = \infty)$ is a free parameter not equal to either $i\infty, i$ or ρ and in which any negative energy source is locally cancelled, containing some number of Q7-branes can be obtained from the F-theory $N = 6, 12$ or 24 solutions assuming that the numerator and denumerator in (3.11.1) are allowed to have common factors.

The embedding of compact Q7-brane solutions in the context of F-theory on K3 makes it possible to interpret the Q7-branes as follows. The $T^{-1}S$ Q7-brane is formed by forcing together two mutually non-local F-theory 7-branes. The S Q7-brane is formed by taking three F-theory 7-branes coincident of which two must be mutually non-local. The $T^{-1}S$ Q7-brane gauge group is suggested to be $(U(1))^2$ and similarly the gauge group of an S Q7-brane is suggested to be $SU(2) \times (U(1))^2$.

The Q7-branes, from the point of view of the local analysis, i.e. in the limit in which the scalars describing the relative positions of the various F-theory 7-branes that make up a Q7-brane are constant, seem to be living in their own world, so to speak, with their own 8-form to which they couple and a metric (3.7.42) that is hinting towards the existence of an open stretched BPS Q-string that stretches from one Q7-brane to another (see section 3.12).

Chapter 4

Instantons

4.1 Introduction

The objects that are electrically charged under χ' defined in (1.2.4) and that are dual to a Q7-brane are called Q-instantons [18]. These are BPS solutions of the Wick rotated Euclidean IIB supergravity. It is well-known that the object that is dual to the D7-brane is the D-instanton [61]. In this chapter a path integral analysis of the Q-instantons is presented providing one with their tunneling interpretation and their basic physical properties such as their charge and on-shell Euclidean action.

At first sight the existence of new half-supersymmetric instanton solutions to Euclidean IIB supergravity might be surprising since a simple analysis of the Killing spinor equations and field equations seems to lead to the unique D-instanton solution of [61], up to an $SL(2,\mathbb{Z})$ transformation. There remains however the possibility that the Q-instantons preserve the same supersymmetries as the D-instanton (just as Q7-branes and D7-branes preserve the same supersymmetries) but that they differ from the D-instanton in the source and boundary terms. This is in fact what happens and it leads to different on-shell actions for the D- and Q-instantons.

In [74] it was shown that the D-instanton contributes to higher order corrections to the string effective action in the form of $\mathcal{R}^4$ terms. Since Q-instantons preserve the same supersymmetries as the D-instanton, they are expected to contribute to the same $\mathcal{R}^4$ terms as well. This will be argued to be the case.

This chapter is organized as follows. In section 4.2 the construction and properties of the D-instanton are reviewed. These results are compared in section 4.3 with the Q-instanton source and boundary terms and its on-shell action. In section 4.4 a path integral analysis of the Q-instanton is performed and in section 4.5 the Q-instanton is argued to contribute to the $\mathcal{R}^4$ terms near the points $\tau_0 = i, \rho$ (and their $SL(2,\mathbb{Z})$ transforms) of the type IIB quantum moduli space.

When D-branes are added to the type IIB supergravity theory the duality group $SL(2,\mathbb{R})$ is broken down to the subgroup that is generated by the shift symmetry of the RR axion, i.e. the $\mathbb{R}$ subgroup of $SL(2,\mathbb{R})$. This for example implies that all the D-brane actions are invariant under the shift of the RR axion (see e.g. the discussion on the 3-brane in section 2.3). Likewise, when a Q-brane is added to the type IIB supergravity theory the duality group $SL(2,\mathbb{R})$ is broken down to the subgroup that is generated by the shift symmetry of the χ', i.e. the $SO(2)$ subgroup of $SL(2,\mathbb{R})$. Hence, all the brane solutions of IIB supergravity are associated to fixed points of e^Q with either $\det Q = 0$ or $\det Q > 0$. The case $\det Q = 0$ corresponds to the (p', q') branes[1] with p' and q' relatively prime integers corresponding to the charges of the string that is ending on the brane. The case $\det Q > 0$ corresponds to Q-branes. The case $\det Q < 0$ does not arise because there are no fixed points of e^Q with $\det Q < 0$ that are part of the quantum moduli space (3.10.2) as explained in subsection 3.8.4.

As shown in subsection 3.7.4 the Q7-brane configurations can be described in terms of the variables T and χ' that are defined by the following relations

$$\frac{\tau - \tau_0}{\tau - \bar{\tau}_0} = e^{2i\sqrt{\det Q}\,\mathcal{T}}\,, \tag{4.1.1}$$

where $\mathcal{T}$ is given by

$$\mathcal{T} = \chi' + \frac{i}{4\sqrt{\det \mathrm{Q}}} \log \frac{T + 2\sqrt{\det \mathrm{Q}}}{T - 2\sqrt{\det \mathrm{Q}}}\,, \qquad \chi' \sim \chi' + \frac{\pi}{\sqrt{\det Q}}\,. \tag{4.1.2}$$

As follows from eq. (4.1.1) the values χ' and $\chi' + \pi/\sqrt{\det Q}$ are to be identified.

Table 3.9.1 on page 82, shows the values of p, q, r for each of the orbifold points τ_0 of figure 3.9.1, depicted on page 81 and the related $SL(2,\mathbb{Z})$ conjugacy classes. The periodicity of the axion χ' is determined by the value of $\pi/\sqrt{\det Q}$. The discrete isometries are the $PSL(2,\mathbb{Z})$ transformations generated by T and S. The periodicity of χ' follows from the fact that the transformations S and $T^{-1}S$ are, respectively, of order 2 and 3 in $PSL(2,\mathbb{Z})$.

In terms of the fields T and χ' the scalar kinetic terms of the IIB action (1.1.17) take the form of eq. (1.3.22). Following subsection 1.3.2 a 9-form field strength that is dual to $d\chi'$ is introduced

$$(T^2 - 4\mathrm{det}Q)\, d\chi' = \star(pF_9 + qH_9 + \frac{r}{2}G_9) \equiv \star\mathcal{F}_9\,, \tag{4.1.3}$$

where the 9-forms F_9, H_9, G_9 are organized in a triplet transforming in the adjoint of $SL(2,\mathbb{R})$. From the axion χ' equation of motion (when ignoring its coupling to the 2-forms and 6-forms) it follows that

$$d\mathcal{F}_9 = 0\,, \tag{4.1.4}$$

[1] The $SL(2,R)$ covariant actions which describe the (p', q') branes have been constructed in [26,27]. They can be regarded as the $SL(2,R)$ transformed Dp-brane actions.

so that locally

$$\mathcal{F}_9 = d\mathcal{A}_8 \,, \tag{4.1.5}$$

with $\mathcal{A}_8$ given in eq. (1.3.27). The Q7-brane minimally couples[2] to $\mathcal{A}_8$ via the Wess–Zumino term

$$S^{Q7}_{min} = m \int \mathcal{A}_8 \,, \tag{4.1.6}$$

where m is the Q7-brane electric charge with respect to $\mathcal{A}_8$, or magnetic charge associated with its axion dual χ'.

The dynamics of the 9-form $\mathcal{F}_9$ can be described by the following first order action

$$\begin{aligned} S\left[g_{\mu\nu}, \chi', \mathcal{F}_9, T\right] \quad = \quad & \int_{\mathcal{M}_{9,1}} \left(*1R - \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star dT \wedge dT - \right. \\ & \left. \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star \mathcal{F}_9 \wedge \mathcal{F}_9 - \chi' d\mathcal{F}_9 \right) \,. \end{aligned} \tag{4.1.7}$$

In the action (4.1.7) the axion χ' appears (in a shift symmetry invariant way) as a Lagrange multiplier. Using the results of appendix B it can be seen that the term $d\mathcal{F}_9$ is parity odd as is χ'. The variation of (4.1.7) with respect to $\mathcal{F}_9$ gives the duality relation (4.1.3). If this relation is substituted back into the action the action for χ' is obtained. When (4.1.7) is varied with respect to χ' the Bianchi identity for $\mathcal{F}_9$ (4.1.4) is found. Substituting its solution (4.1.5) back into the action (4.1.7) a second order action for $\mathcal{A}_8$ results. The action (4.1.7) will be the starting point of the discussion on the Q-instantons.

4.2 D-instantons

Before discussing the new Q-instanton solution of type IIB supergravity, the derivation of the D-instanton solution [61] is briefly discussed. The D-instanton is a solution of the equations of motion of the axion and dilaton coupled to gravity in Euclidean space. The Wick rotation of the axion kinetic term is carried out by taking into account that the axion is an axial scalar and hence gets replaced with $i\chi$[3]. The Wick rotation thus changes the sign of the Einstein term and the dilaton kinetic term leaving intact the sign of the axion kinetic term. So the Euclidean action is

$$S = \int_{\mathcal{M}_{10}} \left(- \star 1R + \frac{1}{2} \star d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} \star d\chi \wedge d\chi \right) \,. \tag{4.2.1}$$

[2]For an early discussion on instantons and monopole-like configurations related to $(d-2)$-form gauge fields in d-dimensional space-time see [75].

[3]It is anticipated that in section 4.4 it will be shown that from the path integral point of view it is not allowed to send χ to $i\chi$ (or χ' to $i\chi'$ when it concerns the Q-instanton) under a Wick rotation. Since, in this and the next section only classical Euclidean field theory is discussed that only provides on-shell information about the saddle point approximation sending χ to $i\chi$ is harmless.

Note that the action is invariant under the axion shift symmetry $\chi \rightarrow \chi + b$ with b a constant real parameter. The Einstein equations and the equations of motion of the axion and the dilaton, which follow from (4.2.1), have the form

$$R_{mn} - \frac{1}{2}\left(\partial_m\phi\partial_n\phi - e^{2\phi}\partial_m\chi\partial_n\chi\right) = 0\,, \tag{4.2.2}$$

$$\nabla_m(e^{2\phi}\nabla^m\chi) = 0\,, \tag{4.2.3}$$

$$\nabla_m\nabla^m\,\phi + e^{2\phi}\,(\partial\chi)^2 = 0\,. \tag{4.2.4}$$

The Ansatz imposed on the fields to get the D-instanton solution of (4.2.2) to (4.2.4) is

$$g_{mn} = \delta_{mn}, \quad d\chi = \pm e^{-\phi}\,d\phi = \mp de^{-\phi}. \tag{4.2.5}$$

Eq. (4.2.5) is nothing but the Bogomol'nyi bound saturation condition imposed on the axidilaton system in flat space. The upper and lower signs in (4.2.5) correspond to the D-instanton and anti-D-instanton, respectively. When (4.2.5) is imposed eqs. (4.2.2) to (4.2.4) reduce to

$$\partial_m(e^{2\phi}\partial^m\chi) = 0 \quad \rightarrow \quad \partial^2\,e^{\phi} = 0\,, \tag{4.2.6}$$

$$\partial_m\partial^m\phi + (\partial\phi)^2 = 0 \quad \rightarrow \quad e^{-\phi}\,\partial^2\,e^{\phi} = 0. \tag{4.2.7}$$

A spherically symmetric solution to the above equations that describes a single (anti-)instanton is

$$e^{\phi} = e^{\phi_\infty} + \frac{c}{r^8}, \quad \chi - \chi_\infty = \mp(e^{-\phi} - e^{\phi_\infty})\,, \tag{4.2.8}$$

where the upper sign stands for the instanton and the lower sign corresponds to the anti-instanton, ϕ_∞ and χ_∞ are the values of the dilaton and axion at $r = \sqrt{x^m x_m} = \infty$ and $c > 0$ is (roughly speaking) the instanton charge, namely,

$$c = \frac{2\pi|n|}{8\mathrm{Vol}(S^9)}\,, \tag{4.2.9}$$

with $\mathrm{Vol}(S^9)$ being the volume of a 9-sphere of a unit radius and n being an integer which manifests the instanton charge quantization [61]. Note that from (4.2.8) it follows that for the D-instanton $\chi + e^{-\phi}$ is constant and for the anti-D-instanton $\chi - e^{-\phi}$ is constant everywhere in the 10-dimensional Euclidean space.

The solution (4.2.8) is singular at $r = 0$ which implies that it is sourced by a point-like object (the instanton) sitting at $r = 0$. The (anti-)instanton contribution to the right hand side of the axion and dilaton field equations (4.2.6) to (4.2.7) is as follows

$$\partial_m(e^{2\phi}\partial^m\chi) = \mp 2\pi|n|\,\delta^{(10)}(\vec{x})\,, \qquad e^{-\phi}\,\partial^2\,e^{\phi} = -2\pi|n|\,e^{-\phi}\,\delta^{(10)}(\vec{x})\,. \tag{4.2.10}$$

Eqs. (4.2.10) can be obtained by varying the supergravity action (4.2.1) coupled to the instanton source

$$\begin{aligned} S &= \int_{\mathcal{M}_{10}} \left(- \star 1 R + \frac{1}{2} \star d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} \star d\chi \wedge d\chi \right) \\ &\quad + 2\pi |n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \left(e^{-\phi} \pm \chi \right) \star 1 \,, \end{aligned} \tag{4.2.11}$$

and imposing the Ansatz (4.2.5).

The presence of the instanton source term breaks the invariance of the action (4.2.11) under the shift symmetry $\chi \rightarrow \chi + b$. The invariance can be restored by adding to eq. (4.2.11) the boundary term

$$- \int_{\partial \mathcal{M}_{10}} \chi \, e^{2\phi} \star d\chi = - \int_{\mathcal{M}_{10}} d(\chi \, e^{2\phi} \star d\chi) = \int_{\mathcal{M}_{10}} d^{10}x \, \partial_m (\chi \, e^{2\phi} \, \partial^m \chi) \,, \tag{4.2.12}$$

such that

$$\int_{\partial \mathcal{M}_{10}} e^{2\phi} \star d\chi = \pm 2\pi |n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \star 1 = \pm 2\pi |n| \,. \tag{4.2.13}$$

Note that this boundary condition is compatible with eqs. (4.2.10).

The appearance of the boundary term (4.2.12) in the supergravity action can be understood best if one starts from the action that includes the field strength $F_9 = dA_8$ of the 8-form gauge field A_8 and then dualizes it into the axion action by adding the term $\int_{\mathcal{M}_{10}} \chi dF_9$ (compare with (4.1.7))

$$S = \int_{\mathcal{M}_{10}} \left(- \star 1 R + \frac{1}{2} \star d\phi \wedge d\phi + \frac{1}{2} e^{-2\phi} \star F_9 \wedge F_9 \right) + \int_{\mathcal{M}_{10}} \chi \, dF_9 \,. \tag{4.2.14}$$

If in (4.2.14) the field F_9 is considered as the independent one (i.e. not a curl of A_8), the variation with respect to this field gives the duality relation $F_9 = e^{2\phi} \star d\chi$ that can be substituted back into the action (4.2.14) thus reducing it to

$$S = \int_{\mathcal{M}_{10}} \left(- \star 1 R + \frac{1}{2} \star d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} \star d\chi \wedge d\chi \right) - \int_{\partial \mathcal{M}_{10}} \chi \, e^{2\phi} \star d\chi \,. \tag{4.2.15}$$

The boundary term appeared as a result of the integration by parts of the last term in (4.2.14). To summarize, the shift symmetry invariant action for the IIB supergravity - D-instanton system is

$$\begin{aligned} S &= \int_{\mathcal{M}_{10}} \left(- \star 1 R + \frac{1}{2} \star d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} \star d\chi \wedge d\chi \right) - \int_{\partial \mathcal{M}_{10}} \chi \, e^{2\phi} \star d\chi \\ &\quad + 2\pi |n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \left(e^{-\phi} \pm \chi \right) \star 1 \,. \end{aligned} \tag{4.2.16}$$

The on-shell value of this action can be computed by substituting into (4.2.16) the instanton solution (4.2.5) and (4.2.8). Then the bulk part of the action vanishes because of the Bogomol'nyi bound saturation, the contribution from the boundary term gets cancelled by the χ part of the source term, and one is left with

$$S_D|_{\text{on-shell}} = 2\pi|n|\, e^{-\phi_\infty}\,, \tag{4.2.17}$$

where e^{ϕ_∞} is the string coupling constant.

In section 4.4 it will be shown that the result (4.2.17) corresponds to a saddle point approximation of a path integral that computes the transition amplitude between axion conjugate momentum eigenstates or, what is the same, between Noether charge eigenstates of the Noether current, associated to the shift symmetry $\chi \to \chi + b$, that differ by n units. As shown in section 4.4 (see also [74]), in order to obtain a saddle point approximation between axion χ eigenstates, one must add to (4.2.17) the imaginary term

$$-2\pi n i \chi_\infty\,, \tag{4.2.18}$$

with $n > 0$ for the D-instanton and $n < 0$ for the anti-D-instanton. The axion that appears in (4.2.18) is the RR axion χ of the Lorentzian IIB theory (and not the Wick rotated one of this section). Thus the D-instanton action takes the form

$$S_D = -2\pi i|n|\tau_\infty\,. \tag{4.2.19}$$

4.3 Q-instantons

The analog of eq. (4.2.14), that should provide the relevant boundary term in the Q-instanton action, is the Euclidean version of the action (4.1.7), namely

$$S = \int_{\mathcal{M}_{10}} \left(-\star 1 R + \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star dT \wedge dT + \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star \mathcal{F}_9 \wedge \mathcal{F}_9 + \chi' d\mathcal{F}_9 \right)\,, \tag{4.3.1}$$

where χ' has been replaced by $i\chi'$. The $\mathcal{F}_9$ equation of motion gives the duality relation between $\mathcal{F}_9$ and the Wick rotated χ' (similar to (4.1.3)). Substituting the duality relation back into the action it is found that

$$\begin{aligned} S \;\; = \;\; & \int_{\mathcal{M}_{10}} \left(-\star 1 R + \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star dT \wedge dT - \frac{1}{2}(T^2 - 4\det Q) \star d\chi' \wedge d\chi' \right) \\ & - \int_{\partial\mathcal{M}_{10}} (T^2 - 4\det Q)\, \chi' \star d\chi'\,. \end{aligned} \tag{4.3.2}$$

To this action the one-half BPS Q-instanton source action

$$2\pi|n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \left(\frac{1}{4\sqrt{\det Q}} \log\frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}} \pm \chi' \right) \star 1 \tag{4.3.3}$$

(see table 2.2.1 on page 40) is coupled. The source term guarantees that the Q-instanton solution will be defined on the entire 10-dimensional Euclidean space. As in the D-instanton case, the Q-instanton charge is quantized ($|n| = 1, 2, 3, \cdots$) as will be demonstrated in the next subsection.

As in the D-instanton case it will be assumed that for the solution under consideration the 10-dimensional Euclidean space is flat and, as follows from the Einstein equation, T and χ' are related by the following Bogomol'nyi bound saturation condition

$$d\chi' = \pm (T^2 - 4\det Q)^{-1}\, dT\,. \tag{4.3.4}$$

Upon adding (4.3.3) to eq. (4.3.2) the following action in flat Euclidean space is obtained

$$\begin{aligned} S \;=\; & \int_{\mathcal{M}_{10}} \left(\frac{1}{2} \frac{1}{T^2 - 4\det Q} dT \wedge \star dT - \frac{1}{2}(T^2 - 4\det Q) d\chi' \wedge \star d\chi' \right) \\ & - \int_{\partial\mathcal{M}_{10}} (T^2 - 4\det Q)\, \chi' \star d\chi' \\ & + 2\pi|n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \left(\frac{1}{4\sqrt{\det Q}} \log \frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}} \pm \chi' \right) \star 1\,. \end{aligned} \tag{4.3.5}$$

As in the D-instanton case, due to the presence of the source term there is only one boundary, $\partial\mathcal{M}_{10}$, that is located at $r = \sqrt{x^m x_m} = \infty$. The action (4.3.2) is invariant under arbitrary shifts of the axion, $\chi' \to \chi' + b$ (where b is any real number) provided that

$$\int_{\partial\mathcal{M}_{10}} (T^2 - 4\det Q) \star d\chi' = \pm\, 2\pi|n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \star 1 \;=\; \pm\, 2\pi|n|. \tag{4.3.6}$$

The equations of motion of the fields χ' and T that follow from eq. (4.3.5) acquire the contribution of the instanton source term and take the following form, after imposing the Bogomol'nyi bound (4.3.4),

$$\partial_m((T^2 - 4\det Q)\, \partial^m \chi') = \mp\, 2\pi|n|\delta^{(10)}(\vec{x})\,, \tag{4.3.7}$$

$$\partial_m \partial^m T = -2\pi|n|\delta^{(10)}(\vec{x})\,. \tag{4.3.8}$$

The Q-instanton solution to eqs. (4.3.4) and (4.3.8) is

$$\begin{aligned} \chi' - \chi'_\infty \;&=\; \mp \left[\frac{1}{4\sqrt{\det Q}} \log \left(\frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}} \right) - \frac{1}{4\sqrt{\det Q}} \log \left(\frac{T_\infty + 2\sqrt{\det Q}}{T_\infty - 2\sqrt{\det Q}} \right) \right]\,, \\ T \;&=\; T_\infty + \frac{c}{r^8}\,, \end{aligned} \tag{4.3.9}$$

where c has been given in eq. (4.2.9). Substituting this solution into the action (4.3.5) the on-shell value of the Q-instanton action can be seen to be

$$S_Q \mid_{\text{on-shell}} = \frac{\pi|n|}{2\sqrt{\det \mathrm{Q}}} \log \frac{T_\infty + 2\sqrt{\det \mathrm{Q}}}{T_\infty - 2\sqrt{\det \mathrm{Q}}}\,. \tag{4.3.10}$$

In section 4.4 it will be shown that the result (4.3.10) corresponds to a saddle point approximation of a path integral that computes the transition amplitude between Noether charge eigenstates of the Noether current, associated to the shift symmetry $\chi' \to \chi' + b$, that differ by n units. As will also be shown in section 4.4, in order to obtain a saddle point approximation between axion χ' eigenstates one must add to (4.2.17) the imaginary term

$$-2\pi n i \chi'_\infty\,, \tag{4.3.11}$$

with $n > 0$ for a Q-instanton and $n < 0$ for an anti-Q-instanton. The axion that appears in (4.3.11) is the axion χ' of the Lorentzian IIB theory (and not the Wick rotated one of this section). The Q-instanton action thus acquires the form

$$S_Q = -2\pi i |n| \mathcal{T}_\infty\,. \tag{4.3.12}$$

The Q-instanton solution, eq. (4.3.9), has a singularity at the point $r = 0$. The singularity is similar to the one found in the case of the D-instanton, see eq. (4.2.8). In the D-instanton case, when going to the string frame the solution becomes a wormhole [61] in which the point $r = 0$ represents the asymptotic region of one end of the wormhole. In section 3.12 the notion of a Q-string whose tension is proportional to $\left(T^2 - 4\det Q\right)^{1/4}$ has been introduced. In the frame $ds_Q^2 = \left(T^2 - 4\det Q\right)^{1/4} ds_E^2$ the solution again becomes a wormhole with the point $r = 0$ representing the asymptotic region of one end of the wormhole.

4.3.1 Q-instanton charge quantization

The quantization of the Q-instanton charge, eq. (4.3.6), follows from the standard Dirac–Nepomechie–Teitelboim quantization condition [76–78] applied to the Q(-1)-brane (Q-instanton) and a Euclidean Q7-brane in a way similar to the D-instanton case [61]. Assume that the spatial volume of the 7-brane is compact with the topology of S^7. If one keeps one point on the S^7 surface fixed and transports the 7-brane along closed paths its world-volume will have the topology of S^8. The wave function of this compact 7-brane will acquire, due to its minimal coupling to the axion dual 8–form (4.1.6) (with $m = 1$), the following phase factor

$$e^{i\int_{\Sigma_8} \mathcal{A}_8}\,, \tag{4.3.13}$$

where Σ_8 is the world-volume of the compact 7-brane. Using Stokes' theorem it is possible to write

$$\int_{\Sigma_8} \mathcal{A}_8 = \int_S \mathcal{F}_9 = -\int_{S'} \mathcal{F}_9\,, \tag{4.3.14}$$

where S and S' are the two capping surfaces of the world-volume $\Sigma_8 = S^8$. The single-valuedness of the wave function (4.3.13) requires that

$$\int_{S^9} \mathcal{F}_9 = 2\pi n\,, \tag{4.3.15}$$

where $S^9 = S \bigcup S'$. Taking now into account the duality relation between $\mathcal{F}_9$ and the Wick rotated axion χ', as follows by varying eq. (4.3.1) with respect to $\mathcal{F}_9$, eq. (4.3.6) is found that relates the value of the Q–instanton boundary term to its quantized charge.

4.3.2 The half-BPS condition

Here it will be shown that the Bogomol'nyi bound (4.3.4) also follows by analyzing the Killing spinor equations. Using the results from subsection 1.3.1, in the Lorentzian IIB theory with vanishing 3- and 5-form field strengths the Killing spinor equations can be written as

$$\begin{aligned} \delta\Psi'_m &= (\nabla_m - \frac{i}{2}Q_m)\epsilon'\,, & (4.3.16)\\ \delta\lambda' &= iP_m\gamma^m\epsilon'_C\,, & (4.3.17)\end{aligned}$$

where the tilde on δ appearing in (1.3.21) and (1.3.20) has been dropped and in which

$$\begin{aligned} P_m &= \frac{1}{2}\frac{\partial_m T}{(T^2-4\det Q)^{1/2}} + \frac{i}{2}(T^2-4\det Q)^{1/2}\partial_m\chi'\,, & (4.3.18)\\ Q_m &= \frac{T}{2}\partial_m\chi'\,. & (4.3.19)\end{aligned}$$

Wick rotating eqs. (4.3.16) and (4.3.17) by sending χ' to $i\chi'$. Treating eqs. (4.3.16) and (4.3.17) and their complex conjugates separately one obtains

$$\begin{aligned} 0 &= \left(\nabla_m + \frac{T}{4}\partial_m\chi'\right)\epsilon'\,, & (4.3.20)\\ 0 &= \left(\nabla_m - \frac{T}{4}\partial_m\chi'\right)\epsilon'_C\,, & (4.3.21)\\ 0 &= \left(\frac{\partial_m T}{(T^2-4\det Q)^{1/2}} - (T^2-4\det Q)^{1/2}\partial_m\chi'\right)\gamma^m\epsilon'_C\,, & (4.3.22)\\ 0 &= \left(\frac{\partial_m T}{(T^2-4\det Q)^{1/2}} + (T^2-4\det Q)^{1/2}\partial_m\chi'\right)\gamma^m\epsilon'\,, & (4.3.23)\end{aligned}$$

where ϵ' and ϵ'_C are Wick rotated spinors.

The 1/2 BPS condition for the Q-instanton is

$$\partial_m \chi' = (T^2 - 4\det Q)^{-1} \partial_m T \,, \qquad \epsilon' = 0 \,, \tag{4.3.24}$$

and for the anti-Q-instanton

$$\partial_m \chi' = -(T^2 - 4\det Q)^{-1} \partial_m T \,, \qquad \epsilon'_C = 0 \,. \tag{4.3.25}$$

When either (4.3.24) or (4.3.25) holds the Q- or anti-Q-instanton source term (4.3.3) is half BPS. Using $g_{mn} = \delta_{mn}$ it follows that for the anti-Q-instanton the Killing spinor ϵ is given by

$$\epsilon' = (T^2 - 4\det Q)^{1/8} \epsilon'_0 \,, \tag{4.3.26}$$

where ϵ'_0 is a constant spinor.

One should consider the Wick rotation as taking place in the context of a path integral. Schematically the path integral is

$$\int \mathcal{D}\psi \cdots e^{i \int_{\mathcal{M}_{9,1}} d^{10}x \mathcal{L}[\psi, \ldots]} \,, \tag{4.3.27}$$

where ψ is any irreducible IIB spinor. Since ψ is a Majorana–Weyl spinor it is possible to write the Lagrangian in terms of ψ only without using its complex conjugate. Further, one can explicitly use the Lorentzian signature gamma matrices writing everywhere explicitly the Zehnbeins. Then under Wick rotation the following happens. The path integral becomes

$$\int \mathcal{D}\psi \cdots e^{- \int_{\mathcal{M}_{10}} d^{10}x \mathcal{L}[\psi, \ldots]} \,, \tag{4.3.28}$$

in which the Lagrangian takes exactly the same form as in the Lorentzian case but with all the fields in principle complex-valued. The Lagrangian is holomorphic in all the IIB fields. The background $\mathcal{M}_{10}$ is taken to be flat Euclidean space which requires the component e_0^0 to be purely imaginary. In the complexified Euclidean action the Majorana condition on the spinors can no longer be imposed so that the spinors ϵ' and ϵ'_C are not related under charge conjugation. The one-half BPS condition $\epsilon' = 0$ or $\epsilon'_C = 0$ should thus be interpreted as a one-half BPS condition in the context of the complexified IIB action on a Euclidean space.

4.4 Path integral approach to Q-instantons

In this section the approach taken in section 4.3 will be justified by deriving the saddle point approximation of transition amplitudes between axion conjugate momentum eigenstates. Further, the imaginary part that, as was mentioned, should be added to the on-shell action, eq. (4.3.11), will be shown to follow from a Fourier transformation relating axion conjugate momentum eigenstates and axion field eigenstates. The discussions and arguments presented in this section are inspired by [79,80]. See [81–84] for related work in four dimensions.

4.4.1 Wick rotated path integrals and axions

In classical field theory when going from the Lorentzian IIB supergravity to Wick rotated Euclidean IIB supergravity χ' is replaced by $i\chi'$. Here, it will be shown that at the level of the path integral χ' does not get replaced by $i\chi'$ when Wick rotating the path integral.

Consider the path integral

$$\int \mathcal{D}T\,\mathcal{D}\mathcal{F}_9\,\mathcal{D}\chi'\,e^{iS\,[T,\mathcal{F}_9,\chi']}\,, \tag{4.4.1}$$

with $S\,[T,\mathcal{F}_9,\chi']$ as given in (4.1.7). The metric is not included in the discussion concerning the path integral since the metric for the instanton solutions is flat. The axion χ' in (4.4.1) can be integrated over using the identity

$$\int \mathcal{D}\chi'\,e^{-i\chi' d\mathcal{F}_9} = \delta[d\mathcal{F}_9]\,, \tag{4.4.2}$$

where $\delta[d\mathcal{F}_9]$ is a delta-functional, implying that $d\mathcal{F}_9 = 0$.

The Wick rotated version of (4.4.1) is

$$\int \mathcal{D}T\,\mathcal{D}\mathcal{F}_9\,\mathcal{D}\chi'\,e^{-S_E\,[T,\mathcal{F}_9,\chi']}\,, \tag{4.4.3}$$

where $S_E = -iS$(Wick rotated) is given by (leaving out the metric)

$$\begin{aligned} S_E\,[T,\mathcal{F}_9,\chi'] \quad &= \quad \int_{\mathcal{M}_{10}} \left(\frac{1}{2}\frac{1}{T^2 - 4\det Q} \star dT \wedge dT \right. \\ &\quad \left. + \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star \mathcal{F}_9 \wedge \mathcal{F}_9 + i\chi' d\mathcal{F}_9\right)\,. \end{aligned} \tag{4.4.4}$$

In the path integral (4.4.3) the integrations are over paths of field configurations with Dirichlet boundary conditions for the fields T and $\mathcal{F}_9$ while free or no boundary conditions are imposed for the field χ'. These boundary conditions are the same as those imposed on the variations of the action (4.1.7) with respect to T, $\mathcal{F}_9$ and χ'. The variation of χ' is entirely free without any boundary conditions because it appears in (4.4.4) without a derivative.

Notice that χ' in (4.4.4) has not been replaced by $i\chi'$. Now in the Euclidean path integral χ' can be again integrated out using the identity (4.4.2) which allows one to go to a second order formalism. If instead χ' had been replaced by $i\chi'$ this would have no longer been possible and the first order action in the Euclidean path integral would not have been equivalent to an 8-form gauge theory anymore since the Bianchi identity $d\mathcal{F}_9 = 0$ and its consequence $\mathcal{F}_9 = d\mathcal{A}_8$ would not arise.

4.4.2 The role of the moduli space

Consider the last term in (4.4.4)

$$i\int_{\mathcal{M}_{10}} \chi' d\mathcal{F}_9 = -i\int_{\mathcal{M}_{10}} d\chi' \wedge \mathcal{F}_9 + i\int_{\partial\mathcal{M}_{10}} \chi' \mathcal{F}_9 \,. \tag{4.4.5}$$

If it is required that the Euclidean path integral respects the standard IIB symmetry $\chi' \to \chi' + b$ where b is any real number then it is found that $\mathcal{F}_9$ should satisfy the following boundary condition

$$b\int_{\partial\mathcal{M}_{10}} \mathcal{F}_9 = 2\pi n \qquad \text{with } n \in \mathbb{Z} \,. \tag{4.4.6}$$

Since b is arbitrary this means that $\int_{\partial\mathcal{M}_{10}} \mathcal{F}_9$ has to vanish. This would mean that there is no instanton present. If instead one requires that the axion can undergo integer, in particular, unit shifts $\chi' \to \chi' + 1$ then it is found that

$$\int_{\partial\mathcal{M}_{10}} \mathcal{F}_9 = 2\pi n \qquad \text{with } n \in \mathbb{Z} \,. \tag{4.4.7}$$

It is concluded from this that instantons can only exist in axidilaton theories whose moduli space is given by (3.10.2). The situation with the 7-brane solutions is in this respect entirely analogous. There the arguments to use (3.10.2) are based on the requirement of having 7-brane solutions with finite energy [85] (see also subsection 3.8.3). The conclusion that one must factor the moduli space $SL(2,\mathbb{R})/SO(2)$ by $SL(2,\mathbb{Z})$ in order to even speak about instantons is clear from the path integral point of view and does not follow from the classical field theory approach of the previous two sections.

4.4.3 Integrating over $\mathcal{F}_9$

Instead of integrating out χ' it is also possible to integrate (4.4.3) over $\mathcal{F}_9$. This is achieved by defining a new 9-form $\mathcal{F}'_9$

$$\mathcal{F}'_9 = \mathcal{F}_9 + i(T^2 - 4\det Q) \star d\chi' \,. \tag{4.4.8}$$

A shift of $\mathcal{F}_9$ in the imaginary direction does not affect the integration in (4.4.3). The action (4.4.4) now becomes

$$S_E\,[T,\mathcal{F}_9',\chi'] \;=\; \int_{\mathcal{M}_{10}}\left(\frac{1}{2}\frac{1}{T^2-4\mathrm{det}\,Q}\star dT\wedge dT+\frac{1}{2}\frac{1}{T^2-4\mathrm{det}\,Q}\star\mathcal{F}_9'\wedge\mathcal{F}_9' + \frac{1}{2}(T^2-4\mathrm{det}\,Q)\star d\chi'\wedge d\chi'\right)+i\int_{\partial\mathcal{M}_{10}}\chi'\mathcal{F}_9\,. \qquad (4.4.9)$$

Even though $\mathcal{F}_9$ appears in the boundary term of (4.4.9) the $\mathcal{F}_9'$ integral is a Gaussian as the integration is over $\mathcal{F}_9'$ with Dirichlet boundary conditions. The $\mathcal{F}_9$ in the boundary term is not integrated over, but is fixed by the identification $\chi'\sim\chi'+1$, see eq. (4.4.7). Integrating over $\mathcal{F}_9'$ leads to the following path integral

$$\int_F (T^2-4\mathrm{det}\,Q)^{1/2}\mathcal{D}T\,\mathcal{D}\chi'\,e^{-\tilde{S}_E\,[T,\chi']}\,, \qquad (4.4.10)$$

where F below the integral sign means to indicate that only integrations over the paths of field configurations that are within the fundamental domain of the quantum moduli space (3.10.2) are to be performed. From now on this will always be assumed and the label F will be suppressed. The integration measure[4] now contains the factor $(T^2-4\mathrm{det}\,Q)^{1/2}$ and the Euclidean action $\tilde{S}_E\,[T,\chi']$ is given by

$$\tilde{S}_E\,[T,\chi'] \;=\; \int_{\mathcal{M}_{10}}\left(\frac{1}{2}\frac{1}{T^2-4\mathrm{det}\,Q}\star dT\wedge dT+\frac{1}{2}(T^2-4\mathrm{det}\,Q)\star d\chi'\wedge d\chi'\right) + i\int_{\partial\mathcal{M}_{10}}\chi'\mathcal{F}_9\,. \qquad (4.4.11)$$

4.4.4 Splitting the χ' integration into bulk and boundary integrations

The integration over χ' can be split up into two pieces: the integration over bulk χ' field configurations and the integration over boundary χ'_∂ field configurations. The bulk χ' field configurations will be denoted by the same symbol as was used in the previous subsections. Since now χ'_∂ is written for the boundary values this should cause no confusion. This split is most easily done using Dirichlet boundary conditions for the paths appearing in the path integral over the bulk χ' field configurations. When this is done (4.4.10) can be written as

$$\int (T^2-4\mathrm{det}\,Q)^{1/2}\mathcal{D}T\,\mathcal{D}\chi'\mathcal{D}\chi'_\partial\,e^{-S_E\,[T,\chi',\chi'_\partial]}\,, \qquad (4.4.12)$$

[4]In terms of τ and $\bar{\tau}$ the integration measure would be $(\mathrm{Im}\,\tau)^{-2}\mathcal{D}\tau\mathcal{D}\bar{\tau}$, which is $PSL(2,\mathbb{Z})$ invariant.

with Dirichlet boundary conditions on the integrations over T and χ'. The action appearing in (4.4.12) is given by

$$S_E\left[T,\chi',\chi'_\partial\right] = \int_{\mathcal{M}_{10}}\left(\frac{1}{2}\frac{1}{T^2-4\det Q}\star dT\wedge dT+\frac{1}{2}(T^2-4\det Q)\star d\chi'\wedge d\chi'\right) +i\int_{\partial\mathcal{M}_{10}}\chi'_\partial\,\mathcal{F}_9\,. \qquad (4.4.13)$$

The variation of the bulk part of (4.4.13) with respect to T and χ' satisfying Dirichlet boundary conditions produces the standard (non-Wick rotated) IIB axidilaton equations of motion.

4.4.5 Tunneling interpretation

In this subsection it will be discussed what is precisely computed by the Euclidean path integral (4.4.3), i.e. by (4.4.12). One would like to interpret (4.4.12) in terms of matrix elements describing a tunneling process from an initial ($t=-\infty$) time-like hypersurface Σ_i to a final ($t=+\infty$) time-like hypersurface Σ_f. The time-like hypersurfaces Σ_i and Σ_f constitute surfaces on which field operator states exist. In order to describe this within the space-time $\mathcal{M}_{9,1}$ spatial infinity is added to it as a point giving rise to the manifold $\mathcal{M}_{9,1}\cup\{r=\infty\}$, where r is a radial coordinate. The topology of this one-point compactified space-time is given by $\mathbb{R}\times S^9$ whose boundary $\partial\left(\mathcal{M}_{9,1}\cup\{r=\infty\}\right)$ is given by the disjoint union $\Sigma_i\cup\Sigma_f$ where the initial and final time-like hypersurfaces have the topology of S^9.

The instanton charge $\int_{\partial\mathcal{M}_{10}}\mathcal{F}_9$ that appears in the imaginary part of eq. (4.4.13) is equal to $\int_{\Sigma_f}\mathcal{F}_9^f-\int_{\Sigma_i}\mathcal{F}_9^i$. Multiplying this equality by χ'_∂ gives

$$i\int_{\partial\mathcal{M}_{10}}\chi'_\partial\,\mathcal{F}_9=i\int_{\Sigma_f}\chi'_\partial\,\mathcal{F}_9^f-i\int_{\Sigma_i}\chi'_\partial\,\mathcal{F}_9^i\,, \qquad (4.4.14)$$

where the values of the axion χ' on the initial and final timelike hypersurfaces Σ_i and Σ_f are the same: $\chi'_i=\chi'_f=\chi'_\partial$. In the following χ'_∂ and χ'_∞ will be identified. Further, it is possible to write

$$\int\mathcal{D}\chi'_\partial e^{-i\int_{\partial\mathcal{M}_{10}}\chi'_\partial\,\mathcal{F}_9}=\int\mathcal{D}\chi'_i\mathcal{D}\chi'_f\delta(\chi'_i-\chi'_f)e^{-i\int_{\Sigma_f}\chi'_f\,\mathcal{F}_9^f+i\int_{\Sigma_i}\chi'_i\,\mathcal{F}_9^i}\,. \qquad (4.4.15)$$

The boundary states in (4.4.12) at $\Sigma_{i,f}$ satisfy (4.4.7) and (4.4.14) and so the $\mathcal{F}_9^{i,f}$ boundary data are on-shell.

Next, it will be shown that one can use the duality relation (4.1.3) restricted to the surfaces $\Sigma_{i,f}$ to interpret the boundary data $\mathcal{F}_9^{i,f}$ of eq. (4.4.15) in terms of the axion momentum, or equivalently, in terms of the Noether charge density associated with the axion shift symmetry. Note that the time component of the Noether current (charge density) is equal to the axion χ' canonical momentum π' obtained by varying the Lagrangian containing the χ' kinetic term with respect to $\partial_0\chi'$

$$\pi' = \frac{\delta \mathcal{L}}{\delta(\partial_0 \chi')} = (T^2 - 4\det Q)\, \partial_0 \chi' = J_N^0 \,. \tag{4.4.16}$$

Consider tunneling between canonical momentum eigenstates of the axion χ' (or equivalently between its Noether charge eigenstates) from the initial surface Σ_i to the final surface Σ_f. These are described by the following matrix element

$$\lim_{\Delta T \to \infty} \langle \pi'_f \mid e^{-H\Delta T} \mid \pi'_i \rangle \,, \tag{4.4.17}$$

where ΔT is the Wick rotated time interval between Σ_i and Σ_f, H is the axidilaton Hamiltonian and $\pi'_{i,f}$ are the initial and final momenta of the axion.

The matrix element (4.4.17) is related by a Fourier transformation to the matrix element describing the transition between two boundary eigenstates χ'_i and χ'_f of the axion. Namely, (for $\Delta T \to \infty$) one has

$$\langle \pi'_f \mid e^{-H\Delta T} \mid \pi'_i \rangle = \int \mathcal{D}\chi'_i \mathcal{D}\chi'_f e^{-i\int_{\Sigma_f} \chi'_f\, \pi'_f + i \int_{\Sigma_i} \chi'_i\, \pi'_i} \langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_i \rangle \,, \tag{4.4.18}$$

where

$$\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_i \rangle = \langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_f \rangle \delta(\chi'_i - \chi'_f) \,. \tag{4.4.19}$$

Hence, no tunneling takes place between vacua for which $\chi'_i \neq \chi'_f$. This means that the value of χ' at, say, $t = +\infty$ acts as a superselection parameter, like the theta parameter in Yang–Mills theory. Therefore, physical processes in vacua with different values of χ'_f are not correlated.

The matrix element appearing on the right hand side of eq. (4.4.19) is given by

$$\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_f \rangle = \int (T^2 - 4\det Q)^{1/2} \mathcal{D}T\, \mathcal{D}\chi'\, e^{-S_E\,[T,\chi']} \,, \tag{4.4.20}$$

with Dirichlet boundary conditions on the integrations over T and χ' and

$$S_E\,[T,\chi'] = \int_{\mathcal{M}_{10}} \left(\frac{1}{2} \frac{1}{T^2 - 4\det Q} \star dT \wedge dT + \frac{1}{2}(T^2 - 4\det Q) \star d\chi' \wedge d\chi' \right) . \tag{4.4.21}$$

Compare eqs. (4.4.18) to (4.4.21) with (4.4.12) to (4.4.15). Eqs. (4.4.18) to (4.4.21) taken together provide a closed expression for the matrix element on the left hand side of eq. (4.4.18). On the other hand eqs. (4.4.12) to (4.4.15) provide an expression for the path integral in (4.4.12). In order that (4.4.12) computes a physical quantity, namely the matrix element of (4.4.18), the boundary values of $\mathcal{F}_9$ are taken to be associated with the boundary values of the χ' canonical momentum (4.4.16) via the duality relation 4.1.3

$$\int_{\Sigma_{i,f}} \mathcal{F}_9^{i,f} = \int_{\Sigma_{i,f}} \star(T^2 - 4\det Q)d\chi' = \int_{\Sigma_{i,f}} J_N^0 \, d\Omega_9 = \int_{\Sigma_{i,f}} \pi'_{i,f} \, d\Omega_9 \,, \tag{4.4.22}$$

where $d\Omega_9$ denotes the integration measure of the unit 9-sphere.

Using the inverse Fourier transform gives

$$\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_i \rangle = \int \mathcal{D}\pi'_i \mathcal{D}\pi'_f e^{i\int_{\Sigma_f} \chi'_f \, \pi'_f - i\int_{\Sigma_i} \chi'_i \, \pi'_i} \langle \pi'_f \mid e^{-H\Delta T} \mid \pi'_i \rangle \,, \tag{4.4.23}$$

where now

$$\langle \pi'_f \mid e^{-H\Delta T} \mid \pi'_i \rangle = \int \mathcal{D}T \, \mathcal{D}\mathcal{F}_9 \, \delta[d\mathcal{F}_9] \, e^{-S_E\,[T,\mathcal{F}_9]} \,, \tag{4.4.24}$$

with Dirichlet boundary conditions imposed on the integrations over T and $\mathcal{F}_9$ and where

$$S_E\,[T,\mathcal{F}_9] = \int_{\mathcal{M}_{10}} \left(\frac{1}{2} \frac{1}{T^2 - 4\det Q} \star dT \wedge dT + \frac{1}{2} \frac{1}{T^2 - 4\det Q} \star \mathcal{F}_9 \wedge \mathcal{F}_9 \right) . \tag{4.4.25}$$

4.4.6 Saddle point approximation

The saddle point approximation of $\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_f \rangle$ can be obtained using eqs. (4.4.23), (4.4.24) and (4.4.25). For a single instanton of charge n the saddle point approximation gives

$$\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_f \rangle \simeq N e^{i\int_{\Sigma_f} \chi'_f \, \pi'_f - i\int_{\Sigma_i} \chi'_f \, \pi'_i} e^{-S_E\,[T,\mathcal{F}_9]} \,|_{\text{on-shell}} \,, \tag{4.4.26}$$

where N is a prefactor that will not be evaluated. On the mass shell it holds true that

$$\int_{\Sigma_f} \chi'_f \, \pi'_f d\Omega_9 - \int_{\Sigma_i} \chi'_f \, \pi'_i d\Omega_9 = 2\pi n \chi'_\infty \,, \tag{4.4.27}$$

which follows from eqs. (4.4.7) and (4.4.14). Further, on-shell and outside the Q-instanton source $S_E\,[T,\mathcal{F}_9] = S_E\,[T,d\mathcal{A}_8]$. The on-shell action can be written as the sum of a quadratic term and a rest term as

$$S_E\,[T,d\mathcal{A}_8] = \frac{1}{2} \int_{\mathcal{M}_{10}} \frac{1}{T^2 - 4\det Q} \star (dT \mp \star\mathcal{F}_9) \wedge (dT \mp \star\mathcal{F}_9) \pm G \,, \tag{4.4.28}$$

with G given by

$$G = \int_{\mathcal{M}_{10}} \frac{1}{T^2 - 4\det Q} dT \wedge \mathcal{F}_9 = -\int_{\partial\mathcal{M}_{10}} \frac{1}{4\sqrt{\det Q}} \log\left(\frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}} \right) \mathcal{F}_9 \,, \tag{4.4.29}$$

where $\partial\mathcal{M}_{10} = \partial\mathcal{M}_\infty + \partial\mathcal{M}_0$. The boundaries $\partial\mathcal{M}_\infty$ and $\partial\mathcal{M}_0$ are, respectively, the 9-sphere at infinity and around the origin where the field strength F_9 fails to be exact (the location of its magnetic source). However because at $|\vec{x}| = 0$ the field T blows up, the value of G is zero at this point and only the boundary at infinity contributes. The first term in the action (4.4.28) is positive definite. One thus has the following Bogomol'nyi bound for field configurations respecting the symmetries of the Q(-1)-brane solution

$$S_I \geq \pm G\,. \tag{4.4.30}$$

Solutions that satisfy the Bogomol'nyi bound must have the property that

$$dT = \pm \star \mathcal{F}_9\,. \tag{4.4.31}$$

For such configurations the on-shell value of the action is given by

$$S_E\left[T, d\mathcal{A}_8\right]|_{\text{on-shell}} = -G = \frac{\pi|n|}{2\sqrt{\det Q}} \log\left(\frac{T_\infty + 2\sqrt{\det Q}}{T_\infty - 2\sqrt{\det Q}}\right)\,, \tag{4.4.32}$$

where $T_\infty > 2\sqrt{\det Q}$ is the asymptotic value of T. The result (4.4.32) agrees with (4.3.10) and provides a saddle point approximation of the matrix element of a transition between axion charge eigenstates (or conjugate momentum eigenstates).

Using eqs. (4.4.27) and (4.4.32) the saddle point approximation (4.4.26) becomes,

$$\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_f \rangle \simeq N e^{2\pi n i \chi'_\infty - 2\pi|n|\text{Im}\mathcal{T}_\infty} = \begin{cases} N e^{2\pi n i \mathcal{T}_\infty} & \text{for } n > 0\,, \\ N e^{2\pi n i \bar{\mathcal{T}}_\infty} & \text{for } n < 0\,. \end{cases} \tag{4.4.33}$$

The case $n > 0$ corresponds to the Q-instanton whereas $n < 0$ corresponds to the anti-Q-instanton. Thus, adding the term (4.3.11) to the action (4.3.10) leads to a saddle point approximation of the matrix element of the transition between axion eigenstates $\chi'_i = \chi'_f = \chi'_\infty$. The result (4.4.33) will be used in the next section to argue that the $\mathcal{R}^4$ terms near the points i and ρ of figure 3.9.1 on page 81, receive contributions from Q-instantons.

4.5 Q-instanton contributions to the $\mathcal{R}^4$ terms

The $\mathcal{R}^4$ terms are those terms in the effective action that are of order $(\alpha')^3$ relative to the Einstein–Hilbert term. In [74] it is argued that the part of the $\mathcal{R}^4$ terms that only contains derivatives of the metric is multiplied by a $PSL(2,\mathbb{Z})$ invariant real-analytic modular form, a generalized Eisenstein series. Such functions are eigenfunctions of the Laplace operator on the hyperbolic plane. In [86] it is shown that this picture is confirmed by requiring supersymmetry at the order $(\alpha')^3$ relative to the Einstein–Hilbert term. The $\mathcal{R}^4$ terms contain besides derivatives of the metric also contributions involving terms with derivatives of the other bosonic fields of the type IIB theory. For the NSNS fields and the RR 0-form a conjectured $SL(2,\mathbb{Z})$ invariant $\mathcal{R}^4$ term is proposed in [87]. Here only the part of the $\mathcal{R}^4$ terms that involves derivatives of the metric and that can be obtained by considering on-shell amplitudes for four graviton scattering will be considered. Consider [87]

$$\begin{aligned}\mathcal{R}^4 &= f(\tau,\bar\tau)\left(t_8^{abcdefgh}t_8^{mnpqrstu}+\frac{1}{8}\epsilon_{10}^{abcdefghij}\epsilon_{10}^{mnpqrstu}{}_{ij}\right)\times \\ &\quad \times R_{abmn}R_{cdpq}R_{efrs}R_{ghtu}+\ldots\,,\end{aligned}\tag{4.5.1}$$

where t_8 is defined in [88], ϵ_{10} is the 10-dimensional Levi-Cività tensor and R_{abmn} is the Riemann tensor. The dots indicate that there are more contributions to $\mathcal{R}^4$.

The function $f(\tau,\bar\tau)$, a generalized Eisenstein series, has the form

$$f(\tau,\bar\tau)=\sum_{(p,n)\neq(0,0)}\frac{\tau_2^{3/2}}{|p+n\tau|^3}\,,\tag{4.5.2}$$

where $\tau=\tau_1+i\tau_2$ and the sum is over all integers $p,n\in\mathbb{Z}$ except when both p and n are zero. In order to see the contributions coming from single multiply charged D- and anti-D-instantons one writes f as a Fourier series in $\tau_1=\chi$. One has [28]

$$\begin{aligned}f(\tau,\bar\tau) &= 2\zeta(3)\tau_2^{3/2}+\frac{2\pi^2}{3}\tau_2^{-1/2}+8\pi\tau_2^{1/2}\sum_{m\neq 0}\sum_{n=1}^{\infty}\left|\frac{m}{n}\right|e^{2\pi i m n\tau_1}\,K_1(2\pi|mn|\tau_2) \\ &= 2\zeta(3)\tau_2^{3/2}+\frac{2\pi^2}{3}\tau_2^{-1/2} \\ &\quad +8\pi\tau_2^{1/2}\sum_{k=1}^{\infty}k\,\sigma_{-2}(k)\left(e^{2\pi i k\tau_1}+e^{-2\pi i k\tau_1}\right)\,K_1(2\pi k\tau_2)\,,\end{aligned}\tag{4.5.3}$$

with $\sigma_{-2}(k)$ given by

$$\sigma_{-2}(k)=\sum_{d|k}d^{-2}\,,\tag{4.5.4}$$

where the sum is over all positive divisors d of k. The expression (4.5.3) is a cosine series with coefficients $16\pi\tau_2^{1/2}\,k\,\sigma_{-2}(k)\,K_1(2\pi k\tau_2)$, where K_1 is the modified Bessel function of the second kind. The τ_1 independent terms in (4.5.3) do not come from D-instantons, instead they come from an $(\alpha')^3$ tree level and a one-loop effect in the four graviton amplitude [74].

In order to see the contribution from single multiply charged D- and anti-D-instantons one considers (4.5.3) close to $\tau_0=i\infty$, i.e. in the limit $\tau_2\to\infty$. Using that for $x\to\infty$ the function $K_1(x)$ behaves as $K_1(x)=\sqrt{\frac{\pi}{2x}}e^{-x}(1+\ldots)$ it follows that at the leading order in the limit $\tau_2 \to \infty$

$$f(\tau, \bar{\tau}) \approx 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3}\tau_2^{-1/2} + 4\pi \sum_{k=1}^{\infty} k^{1/2}\, \sigma_{-2}(k) \left(e^{2\pi i k\tau} + e^{-2\pi i k\bar{\tau}}\right) . \qquad (4.5.5)$$

In the exponents of (4.5.5) one recognizes the D-instanton action (4.2.19).

The Q-instantons preserve the same supersymmetries as the D-instanton. It is therefore expected that they will also contribute to the generalized Eisenstein series (4.5.2). To justify this argument, in the remainder of this section the generalized Eisenstein series will be Fourier expanded in terms of χ' and the Fourier coefficients that are functions of T will be computed. This will result in an exact expression for f that is analogous to eq. (4.5.3). Schematically one can write

$$f(T, \chi') = \sum_{n=-\infty}^{\infty} c_n(T) e^{2\pi i n\chi'} , \qquad (4.5.6)$$

where $c_n(T)$ are the Fourier coefficients. This series is manifestly invariant under $\chi' \to \chi' + 1$ and describes the behavior of f near $\tau = \tau_0 = i, \rho$. It will be shown that f consists of a χ' independent part and of a cosine series that corresponds to an infinite sum of single multiply charged Q- and anti-Q-instantons.

The function f will be expanded around the fixed points $\tau_0 = i, \rho$ of the axidilaton moduli space. To this end it will prove convenient to work with the (η, φ) variables that are related to τ via eq. (1.4.3) and to χ' and $\mathrm{Im}\,\mathcal{T}$ via

$$\varphi = 2\sqrt{\det Q}\,\chi' \quad \text{where} \quad 0 \le \varphi < 2\pi\,, \qquad (4.5.7)$$

$$\tanh\frac{\eta}{2} = e^{-2\sqrt{\det Q}\,\mathrm{Im}\,\mathcal{T}} \quad \text{where} \quad 0 < \eta < \infty\,. \qquad (4.5.8)$$

Substituting (1.4.3) into (4.5.2) gives

$$f(\eta, \varphi) = \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\tau_0)^{3/2}}{|p + n\tau_0|^3} \frac{1}{\left(\cosh\eta + \sinh\eta\cos(\varphi + \beta(p, n; \tau_0))\right)^{3/2}} , \qquad (4.5.9)$$

where $\beta(p, n; \tau_0)$ is defined by

$$\cos\beta(p, n; \tau_0) = \frac{n^2(\mathrm{Im}\,\tau_0)^2 - (p + n\mathrm{Re}\,\tau_0)^2}{n^2(\mathrm{Im}\,\tau_0)^2 + (p + n\mathrm{Re}\,\tau_0)^2} , \qquad (4.5.10)$$

$$\sin\beta(p, n; \tau_0) = \frac{2n\,\mathrm{Im}\,\tau_0\,(p + n\mathrm{Re}\,\tau_0)}{n^2(\mathrm{Im}\,\tau_0)^2 + (p + n\mathrm{Re}\,\tau_0)^2} . \qquad (4.5.11)$$

From the definition of φ in terms of χ' it follows that the invariance of $f(T, \chi')$ under $\chi' \to \chi' + 1$ implies the invariance of $f(\eta, \varphi)$ under $\varphi \to \varphi + 2\sqrt{\det Q}$. Hence, the following Fourier series decomposition of $f(\eta,\varphi)$ is made

$$f(\eta,\varphi)=\sum_{m=-\infty}^{\infty} a_{\frac{\pi m}{\sqrt{\det Q}}}(\eta)e^{\frac{\pi}{\sqrt{\det Q}}mi\varphi}\,. \tag{4.5.12}$$

The Fourier coefficients $a_{\frac{\pi m}{\sqrt{\det Q}}}$ are given by

$$a_{\frac{\pi m}{\sqrt{\det Q}}}(\eta)=\frac{1}{2\sqrt{\det Q}}\int_0^{2\sqrt{\det Q}} d\varphi\, f(\eta,\varphi)e^{-\frac{\pi}{\sqrt{\det Q}}mi\varphi}\,. \tag{4.5.13}$$

By using (4.5.9) and by shifting the integration over φ in (4.5.13) to an integration over $\theta=\varphi+\beta(p,n;\tau_0)$ the Fourier coefficients $a_{\frac{\pi m}{\sqrt{\det Q}}}$ can be written as

$$\begin{aligned} a_{\frac{\pi m}{\sqrt{\det Q}}}(\eta) \;&=\; \frac{1}{2\sqrt{\det Q}}\sum_{(p,n)\neq(0,0)}\frac{(\operatorname{Im}\tau_0)^{3/2}}{|p+n\tau_0|^3}\,e^{\frac{\pi}{\sqrt{\det Q}}mi\beta(p,n;\tau_0)}\times \\ &\times\int_{\beta(p,n;\tau_0)}^{2\sqrt{\det Q}+\beta(p,n;\tau_0)} d\theta\frac{e^{-\frac{\pi}{\sqrt{\det Q}}mi\theta}}{(\cosh\eta+\sinh\eta\cos\theta)^{3/2}}\,. \end{aligned} \tag{4.5.14}$$

Eq. (4.5.14) will be further evaluated for the cases $\tau_0=i$ and $\tau_0=\rho$ separately.

The case $\tau_0=i$ is treated first. In table 3.9.1 on page 82 some data regarding the orbifold points $\tau_0=i,\rho$ is presented. For $\tau_0=i$ one has $2\sqrt{\det Q}=\pi$. From eqs. (4.5.10) and (4.5.11) specified to the case $\tau_0=i$ the following two identities can be derived

$$\beta(-n,p;i) \;=\; \pi+\beta(p,n;i)\,, \tag{4.5.15}$$

$$\beta(n,p;i) \;=\; \pi-\beta(p,n;i)\,. \tag{4.5.16}$$

Using the identity (4.5.15) one can write

$$\begin{aligned} &\sum_{(p,n)\neq(0,0)}\frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}}\int_{\beta(p,n;i)}^{\pi+\beta(p,n;i)} d\theta\frac{e^{-2mi\theta}}{(\cosh\eta+\sinh\eta\cos\theta)^{3/2}} = \\ &\sum_{(p,n)\neq(0,0)}\frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}}\int_{\pi+\beta(p,n;i)}^{2\pi+\beta(p,n;i)} d\theta\frac{e^{-2mi\theta}}{(\cosh\eta+\sinh\eta\cos\theta)^{3/2}} = \\ &\frac{1}{2}\sum_{(p,n)\neq(0,0)}\frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}}\int_0^{2\pi} d\theta\frac{e^{-2mi\theta}}{(\cosh\eta+\sinh\eta\cos\theta)^{3/2}}\,, \end{aligned} \tag{4.5.17}$$

where in the last equality the average of the first two lines has been taken and the property $\int_\beta^{2\pi+\beta}=\int_\beta^0+\int_0^{2\pi}+\int_{2\pi}^{2\pi+\beta}=\int_0^{2\pi}$, that follows from the fact that the integrand is periodic with the period 2π, has been used. Thus, the Fourier coefficients

(4.5.14) take the form

$$a_{2m}(\eta) = \frac{1}{2\pi} \sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} \int_0^{2\pi} d\theta \frac{\cos(2m\theta)}{(\cosh\eta + \sinh\eta\cos\theta)^{3/2}}\,, \qquad (4.5.18)$$

where the integral from 0 to 2π that involves $\sin(2m\theta)$ vanished.

The identity (4.5.16) can be used to show that the sum preceding the integral in (4.5.18) satisfies

$$\sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} = \sum_{(p,n)\neq(0,0)} \frac{e^{-2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}}\,, \qquad (4.5.19)$$

so that $a_{2m}(\eta) = a_{-2m}(\eta)$. The latter property implies that the Fourier expansion (4.5.12) becomes the following cosine series

$$f(\eta,\varphi) = a_0(\eta) + \sum_{m=1}^{\infty} a_{2m}(\eta)\left(e^{2mi\varphi} + e^{-2mi\varphi}\right)\,. \qquad (4.5.20)$$

The integral in (4.5.18) is the integral representation (up to a factor) of a toroidal function, denoted by $P^{2m}_{1/2}(\cosh\eta)$. Toroidal or ring functions are special cases of the associated Legendre functions. One has [89]

$$\int_0^{2\pi} d\theta \frac{\cos(n\theta)}{(\cosh\eta + \sinh\eta\cos\theta)^{3/2}} = 2\pi(-1)^n \frac{\Gamma(\frac{3}{2}-n)}{\Gamma(\frac{3}{2})} P^n_{1/2}(\cosh\eta)\,. \qquad (4.5.21)$$

The functions $P^n_{1/2}(\cosh\eta)$ for $n = 0,1,2,\ldots$ can be written in terms of a hypergeometric function[5] as follows [89]

$$P^n_{1/2}(\cosh\eta) = \frac{1}{2^n} \frac{\Gamma(\frac{3}{2}+n)}{\Gamma(n+1)\Gamma(\frac{3}{2}-n)} \sinh^n\eta\, F(\tfrac{n}{2}+\tfrac{3}{4}, \tfrac{n}{2}-\tfrac{1}{4}; n+1; -\sinh^2\eta)\,. \quad (4.5.23)$$

Substituting eqs. (4.5.21) and (4.5.23) into (4.5.18) the function $f(\eta,\varphi)$, eq.

[5] The hypergeometric function $F(a,b;c;z)$ is defined for $c \neq 0,-1,-2,\cdots$ as

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)} \frac{z^n}{n!}\,. \qquad (4.5.22)$$

If $a,b \neq 0,-1,2,\cdots$ then (4.5.22) converges absolutely for $|z|<1$, see e.g. [89] for more details.

(4.5.20), around the point $\tau_0 = i$ can be written as the following Fourier series

$$\begin{aligned} f(\eta,\varphi) &= \sum_{(p,n)\neq(0,0)} \frac{1}{(p^2+n^2)^{3/2}}\, F(\tfrac{3}{4},-\tfrac{1}{4};1;-\sinh^2\eta) \\ &+ \sum_{m=1}^{\infty}\sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}}\frac{1}{2^{2m}}\frac{\Gamma(\frac{3}{2}+2m)}{\Gamma(2m+1)\Gamma(\frac{3}{2})}\sinh^{2m}\eta\,\times \\ &\times F(m+\tfrac{3}{4},m-\tfrac{1}{4};2m+1;-\sinh^2\eta)\left(e^{2mi\varphi}+e^{-2mi\varphi}\right). \end{aligned} \tag{4.5.24}$$

From eqs. (4.5.7) and (4.5.8) it is known that

$$\varphi = \pi\chi' \qquad \text{and} \qquad \sinh^2\eta = \frac{T^2 - 4\det Q}{4\det Q} \qquad \text{with} \qquad \sqrt{\det Q} = \frac{\pi}{2}. \tag{4.5.25}$$

The Fourier series (4.5.24) in terms of χ' and $T^2 - 4\det Q$ associated with the fixed point $\tau_0 = i$ is analogous to the Fourier series expansion (4.5.3) around the point $\tau_0 = i\infty$ in terms of $\tau_1 = \chi$ and $\tau_2 = \mathrm{Im}\,\tau = e^{-\phi}$.

In order to make manifest the Q- and anti-Q-instanton contributions to the function f consider the expansion (4.5.24) at the leading order around the point $\eta = 0$ (that corresponds to a singular point of the associated Legendre function $P^{2m}_{1/2}(\cosh\eta)$). Note that, by virtue of the relation (1.4.3), the point $\eta = 0$ corresponds to $\tau = i$. Using that at leading order

$$\sinh^2\eta \approx 4\, e^{-2\pi \mathrm{Im}\,\mathcal{T}}, \tag{4.5.26}$$

it follows that at this order[6]

$$\begin{aligned} f(\mathcal{T},\bar{\mathcal{T}}) &\approx \sum_{(p,n)\neq(0,0)} \frac{1}{(p^2+n^2)^{3/2}} + \sum_{m=1}^{\infty}\sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}}\,\times \\ &\times \frac{\Gamma(\frac{3}{2}+2m)}{\Gamma(2m+1)\Gamma(\frac{3}{2})}\left(e^{2\pi mi\mathcal{T}}+e^{-2\pi mi\bar{\mathcal{T}}}\right). \end{aligned} \tag{4.5.28}$$

The form of the sum over $m = 1, 2, \ldots$ in eq. (4.5.28), by its analogy to the D-instanton case, supports the assertion that it reproduces the contribution of single multiply charged Q- and anti-Q-instantons as one can see by comparing (4.5.28) with eq. (4.4.33). The first term in (4.5.28) does not correspond to an instanton contribution. Its origin is yet to be understood.

[6] This can alternatively be derived by using that $P^n_{1/2}(\cosh\eta)$ can also be written as

$$P^n_{1/2}(\cosh\eta) = \frac{\Gamma(\frac{3}{2}+n)}{\Gamma(n+1)\Gamma(\frac{3}{2}-n)}\tanh^n\frac{\eta}{2}\, F(-\tfrac{1}{2},\tfrac{3}{2};1+n;-\sinh^2\tfrac{\eta}{2}). \tag{4.5.27}$$

Then using that for $\tau_0 = i$ one has $\tanh^2\frac{\eta}{2} = e^{-2\pi\mathrm{Im}\,\mathcal{T}}$ and $n = 2m$ the result eq. (4.5.28) follows.

This section is ended by briefly discussing the Fourier series expansion of f around $\tau_0 = \rho$. The starting point is eq. (4.5.14) with $\tau_0 = \rho$ and $\sqrt{\det Q} = \frac{\pi}{3}$ (see table 3.9.1 on page 82). Using eqs. (4.5.10) and (4.5.11) the following three identities are obtained

$$\begin{aligned} \beta(n, n-p; \rho) &= \frac{2\pi}{3} + \beta(p, n; \rho)\,, & (4.5.29)\\ \beta(p-n, p; \rho) &= \frac{4\pi}{3} + \beta(p, n; \rho) & (4.5.30)\\ -\beta(n, p; \rho) &= \frac{4\pi}{3} + \beta(p, n; \rho)\,. & (4.5.31) \end{aligned}$$

Using (4.5.29) and (4.5.30) one can show, in a way which is very similar to the derivation of eq. (4.5.18) for $\tau_0 = i$, that the Fourier coefficients $a_{3m}(\eta)$ are given by

$$a_{3m}(\eta) = \frac{1}{2\pi} \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} e^{3mi\beta(p,n;\rho)} \int_0^{2\pi} d\theta \frac{\cos(3m\theta)}{(\cosh\eta + \sinh\eta\cos\theta)^{3/2}}\,. \tag{4.5.32}$$

It follows by employing eq. (4.5.31) that $a_{3m}(\eta) = a_{-3m}(\eta)$. Hence, using the Fourier decomposition (4.5.12) and eqs. (4.5.32), (4.5.21) and (4.5.23) it is found that for $\tau_0 = \rho$ the function f is expanded as

$$\begin{aligned} f(\eta,\varphi) \;=\; & \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} F(\tfrac{3}{4}, -\tfrac{1}{4}; 1; -\sinh^2\eta) \\ & + \sum_{m=1}^{\infty} \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} e^{3mi\beta(p,n;\rho)} (-1)^m \frac{1}{2^{3m}} \frac{\Gamma(\frac{3}{2}+3m)}{\Gamma(3m+1)\Gamma(\frac{3}{2})} \times \\ & \times \sinh^{3m}\eta\, F(\tfrac{3m}{2}+\tfrac{3}{4}, \tfrac{3m}{2}-\tfrac{1}{4}; 3m+1; -\sinh^2\eta) \left(e^{3mi\varphi} + e^{-3mi\varphi}\right)\,. \end{aligned} \tag{4.5.33}$$

At leading order it holds true that

$$\sinh^3\eta \approx 8\, e^{-2\pi \mathrm{Im}\,\mathcal{T}}\,, \tag{4.5.34}$$

so that at this order near $\tau_0 = \rho$ eq. (4.5.33) becomes

$$\begin{aligned} f(\mathcal{T}, \bar{\mathcal{T}}) \;\approx\; & \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} + \sum_{m=1}^{\infty} \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} e^{3mi\beta(p,n;\rho)} \times \\ & \times (-1)^m \frac{\Gamma(\frac{3}{2}+3m)}{\Gamma(3m+1)\Gamma(\frac{3}{2})} \left(e^{2\pi mi\mathcal{T}} + e^{-2\pi mi\bar{\mathcal{T}}}\right)\,, \end{aligned} \tag{4.5.35}$$

where $\varphi = \frac{2\pi}{3}\chi'$ has been used.

The expressions (4.5.28) for $\tau_0 = i$ and (4.5.35) for $\tau_0 = \rho$ can be contrasted with the leading order result for $\tau_0 = i\infty$, eq. (4.5.5). The results (4.5.28) and (4.5.35) differ from (4.5.5) most notably in the axion-independent parts. At this moment it is not clear what kind of processes would account for for the χ' independent pieces of (4.5.28) and (4.5.35).

4.6 Discussion

In this chapter new 1/2 BPS instanton solutions to the Wick rotated Euclidean IIB supergravity theory have been constructed. These new instantons are referred to as Q-instantons. It was shown that Q-instantons form the electric partners of the Q7-branes of the previous chapter.

The path integral approach to the Q-instantons shows the existence of new vacua and a new superselection parameter χ'_∞. Further, it was argued that the Q-instantons contribute to the $\mathcal{R}^4$ terms near the points $\tau_0 = i, \rho$ of the quantum moduli space $SO(2)\backslash PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$. The expansion of the generalized Eisenstein series around the points $\tau_0 = i, \rho$ contains terms that do not depend on χ' and for which an interpretation is yet to be found.

The results of this and the previous chapter lend support to the idea that type IIB supergravity provides, at least in the case of the 7-branes and instantons, a valid field theory approximation of some underlying quantum theory near each of the orbifold points of the quantum axidilaton moduli space depicted in figure 3.9.1 on page 81.

The expansion of the generalized Eisenstein series around the points $\tau_0 = i, \rho$ can be given in terms of expansions in the function $\sinh\eta$ (or $(T^2 - 4\det Q)^{1/2}$). In section 3.12 this function was suggested to form the square of the tension of an open Q-string. A better understanding of the origin of the Q7-brane world-volume dynamics in terms of strings may also shed some light on the role of $\sinh\eta$ in the case of the Q-instantons and potentially on the axion χ' independent parts of the $\mathcal{R}^4$ term expanded around $\tau_0 = i, \rho$.

Conclusions and discussion

This thesis aimed at providing a better understanding of the full set of one-half BPS branes that are present in the type IIB supergravity theory. The one-half BPS objects were listed in table 2.2.1 on page 40. Of all the branes contained in that table the (p', q') p-branes with p' and q' relatively prime integers are well-established objects of the theory for which a microscopic interpretation is given in string theory as Dirichlet branes on which a (p', q') string is ending. A (p', q') string is an $SL(2, \mathbb{Z})$ transformation of the fundamental $(1, 0)$ string and so these branes can be considered to be the Dirichlet branes of the $SL(2, \mathbb{Z})$ transformed IIB string theory. Also coincident sets of identical (p', q') p-branes can be understood using string theory. The location of a $(1, 0)$ p-brane in moduli space is at the point $\tau_0 = i\infty$ where the string coupling g_s is small. The (p', q') p-branes are then located at $SL(2, \mathbb{Z})$ transformed images of $\tau_0 = i\infty$, that are rational points on the real line $\operatorname{Im}\tau = 0$ where g_s goes to infinity.

Table 2.2.1 also shows that there can be more than only (p', q') branes. For 7-branes it was found to be possible to refer to them as (p, q, r) 7-branes where the numbers (p, q, r) parameterize the monodromy of τ when going around the 7-brane. The (p', q') 7-branes as well as coincident identical (p', q') 7-branes fall into the class of (p, q, r) 7-branes for which $r^2 = 4pq$ (or $\det Q = 0$) with $p = p'^2$ and $q = q'^2$. What the analysis of chapter 3 has shown is that in type IIB supergravity one can introduce the notion of a Q7-brane, i.e. a 7-brane located at one of the orbifold points $\tau_0 = i, \rho$ (or any of their $SL(2, \mathbb{Z})$ transformed images) of the quantum moduli space $SO(2)\backslash PSL(2, \mathbb{R})/PSL(2, \mathbb{Z})$ for which $r^2 > 4pq$ (or $\det Q > 0$) and where g_s is of order unity.

The properties of the Q7-branes such as their monodromy and mass have a natural interpretation in terms of coincident F-theory 7-branes that have monodromies such as T, ST^{-2} and T^2ST^{-4}, i.e. coinciding (p', q') 7-branes with different values for p' and q'. The $T^{-1}S$ Q7-brane is formed by taking two mutually non-local F-theory 7-branes coincident whereas the case of an S Q7-brane is formed by taking three F-theory 7-branes coincident of which two must be mutually non-local (so that two are identical). It was suggested that the gauge group of a $T^{-1}S$ Q7-brane is $(U(1))^2$ and that the gauge group of an S Q7-brane is $SU(2) \times (U(1))^2$.

If the relative positions of the F-theory 7-branes that make up a Q7-brane are kept fixed so that they remain coincident and only the fluctuations associated to the center of mass motion are considered then the Q7-brane behaves effectively as a single brane that couples to an 8-form. By electro-magnetic duality it can be argued that there should exist a Q-instanton, i.e. a Q(-1)-brane that couples magnetically to the same 8-form to which a Q7-brane couples electrically. Indeed it has been shown in chapter 4 that such instantons exist.

The path integral approach to the Q-instantons shows the existence of new vacua and a new superselection parameter χ'_∞. Further, it was argued that the Q-instantons contribute to the $\mathcal{R}^4$ terms of the string effective action near the points $\tau_0 = i, \rho$ of the quantum moduli space $SO(2)\backslash PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$. The expansion of the generalized Eisenstein series around the points $\tau_0 = i, \rho$ contains terms that do not depend on χ' and for which an interpretation is yet to be found.

Further, it was mentioned in section 2.3 that there should exist such a thing as a Q3-brane that lives near the orbifold points $\tau_0 = i, \rho$. To make the list of possible Q-branes complete it was argued in section 3.12 that there may exist an open Q-string ending on a Q7-brane. The tension of a Q-string was suggested to be proportional to $(T^2 - 4\det Q)^{1/4}$ a quantity that also showed up in the Q-instanton contributions to the $\mathcal{R}^4$ terms around the points $\tau_0 = i, \rho$.

Even though the string coupling g_s is only small near $\tau_0 = i\infty$ it is believed that the results on the Q7-branes and Q-instantons allows one to consider IIB supergravity (perhaps in a restricted sense) as providing a valid field theory approximation of some underlying quantum theory near each of the orbifold points of the quantum axidilaton moduli space of figure 3.9.1 (see page 81). In fact for each orbifold point of the moduli space one can define a tension that goes to zero at that particular orbifold point. For $\tau_0 = i\infty$ this is g_s and for $SL(2,\mathbb{Z})$ transformations of $\tau_0 = i\infty$ the tension is the $SL(2,\mathbb{Z})$ transformed version of g_s while for $\tau_0 = i, \rho$ (and their $SL(2,\mathbb{Z})$ transformed images) the tension is suggested to be proportional to $(T^2 - 4\det Q)^{1/4}$. In all cases the tensions have a geometrical interpretation as measuring the geodesic distance of a geodesic that goes through the point τ_0. The geodesic along which g_s measures the geodesic distance is the line $\chi = \text{cst}$, with χ the RR axion, that goes through the point $\tau_0 = i\infty$ and similarly for the $SL(2,\mathbb{Z})$ transformed images of $\tau_0 = i\infty$. The geodesic along which $(T^2 - 4\det Q)^{1/4}$ measures the geodesic distance is the line $\chi' = \text{cst}$ (or $\varphi = \text{cst}$, see figure 1.4.1 on page 33) that goes through the points $\tau_0 = i, \rho$ (or through one of their $SL(2,\mathbb{Z})$ transformed images).

Finally, for the case of the doublet of 10-form potentials there are some open questions as to what their status in the theory is. It was argued in section 2.5 that there could be an effective notion of an O9-plane present in the combined system of 9-branes that couple to the quadruplet and the doublet of 10-forms. In the case of the 7-branes something similar has been shown to be true. The O7-plane, figure 3.11.1 on page 96, can be understood as a special limit of different coinciding F-theory 7-branes.

Appendix A

Conventions

A.1 Conventions for the bosonic sector of IIB supergravity

Lower case Greek indices refer to space-time indices. Space-time coordinates are denoted by x^μ with $\mu = 0, 1, \ldots, 9$ in which x^0 is a time-like coordinate (sometimes denoted by t) and in which the x^i for $i = 1, \ldots, 9$ are space-like coordinates. Early lower case Latin indices, such as $a, b, \ldots$ refer to flat tangent space-time indices [1]. The values taken by tangent space-time indices will be underlined, e.g. $a = \underline{0}, \underline{1}, \ldots, \underline{9}$. When space-time coordinates are given a specific symbol such as t to denote x^0 the flat index $\underline{0}$ will be denoted by $\underline{t}$. Early upper case Latin indices, such as $A, B, \ldots$ refer to world-volume indices of a p-brane. World-volume coordinates are denoted by σ^A with $A = 0, 1, \ldots, p$ in which σ^0 is a time-like coordinate and σ^I for $I = 1, \ldots, p$ are space-like coordinates. Underlined early upper case Latin indices, such as $\underline{A}, \underline{B}, \ldots$, refer to world-volume tangent space indices. Their values will also be underlined, so $\underline{A} = \underline{0}, \underline{1} \ldots, \underline{p}$. The conventions for the various indices are summarized on the next page in table A.1.1.

The tangent space to space-time has a metric that will be denoted by η_{ab}. The signature of this metric is chosen to be mostly plus, i.e.

$$\eta_{\underline{00}} = -1 \,, \qquad \eta_{\underline{ii}} = +1 \,. \tag{A.1.1}$$

The tangent space Levi–Cività tensor will be denoted by $e_{a_1 \ldots a_{10}}$. It is defined such that

$$e_{\underline{01}\ldots\underline{9}} = +1 \,. \tag{A.1.2}$$

[1] In sections 2.6 and 3.11 early lower case Latin indices will be used to denote 8-dimensional space-time indices.

Index	Values
space-time indices $\mu, \nu, \ldots$	$(0,i) = (0,1,\ldots,9)$
tangent space indices $a, b, \ldots$	$(\underline{0},\underline{i}) = (\underline{0},\underline{1},\ldots,\underline{9})$
world-volume (WV) indices $A, B, \ldots$	$(0,I) = (0,1,\ldots,p)$
tangent space WV indices $\underline{A}, \underline{B}, \ldots$	$(\underline{0},\underline{I}) = (\underline{0},\underline{1},\ldots,\underline{p})$

Table A.1.1: Summary of index notation.

The indices of $e_{a_1 \ldots a_{10}}$ are raised with the inverse tangent space metric η^{ab}, so that one has

$$e^{\underline{01}\ldots\underline{9}} = -1\,. \tag{A.1.3}$$

The Zehnbein e_μ^a is introduced via

$$e_\mu^a e_\nu^b \eta_{ab} = g_{\mu\nu}\,, \tag{A.1.4}$$

where $g_{\mu\nu}$ denotes the space-time metric. The determinant of the Zehnbein, $\det(e_\mu^a)$, will be denoted by e and the inverse Zehnbein will be denoted by e_a^μ.

The curved Levi–Cività tensor $\epsilon_{\mu_1\ldots\mu_{10}}$ is defined through

$$e_{\mu_1}^{a_1} \cdots e_{\mu_{10}}^{a_{10}} e_{a_1\ldots a_{10}} = e\, e_{\mu_1\ldots\mu_{10}} \equiv \epsilon_{\mu_1\ldots\mu_{10}}\,, \tag{A.1.5}$$

where $e_{\mu_1\ldots\mu_{10}}$ is the Levi–Cività symbol defined as

$$e_{01\ldots 9} = -e^{01\ldots 9} = +1\,. \tag{A.1.6}$$

The components of the curved Levi–Cività tensor $\epsilon_{\mu_1\ldots\mu_{10}}$ are raised with the inverse metric $g^{\mu\nu}$. The result is

$$\epsilon^{\mu_1\cdots\mu_{10}} = \frac{1}{e} e^{\mu_1\cdots\mu_{10}}\,. \tag{A.1.7}$$

The covariant derivative with respect to general coordinate transformations and local Lorentz transformations is denoted by ∇_μ, acting on tensors ξ and spinors χ as

$$\nabla_\mu \xi = \partial_\mu \xi\,, \tag{A.1.8}$$
$$\nabla_\mu \xi^\nu = \partial_\mu \xi^\nu + \Gamma^\nu_{\mu\rho}\xi^\rho\,, \tag{A.1.9}$$
$$\nabla_\mu \chi = \partial_\mu \chi + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\chi\,, \tag{A.1.10}$$
$$\nabla_\mu \chi^\nu = \partial_\mu \chi^\nu + \Gamma^\nu_{\mu\rho}\chi^\rho + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\,\chi^\nu\,. \tag{A.1.11}$$

Here $\Gamma^{\nu}_{\mu\rho}$ is the Levi–Cività connection and ${\omega_\mu}^{ab}$ is the spin connection defined by

$$ {\omega_\mu}^{a}{}_{b} = e^a_\nu\, e^\rho_b \Gamma^\nu_{\mu\rho} - e^\rho_b \partial_\mu e^a_\rho \,. \tag{A.1.12} $$

The gamma matrices $\gamma_{ab} = \gamma_{[a}\gamma_{b]}$ will be discussed in section A.2. Symmetrization and anti-symmetrization are with weight one. The Riemann tensor is defined by

$$ {R^\rho}_{\mu\nu\sigma} = 2\partial_{[\nu}\Gamma^\rho_{\sigma]\mu} + 2\Gamma^\rho_{\lambda[\nu}\Gamma^\lambda_{\sigma]\mu} \,. \tag{A.1.13} $$

The following form notation is used:

$$
\begin{aligned}
F_p &= \frac{1}{p!} F_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \,, && \text{(A.1.14)}\\
\star F_p &= \frac{1}{(10-p)!\, p!} \epsilon_{\mu_1\ldots\mu_{10-p}\nu_1\ldots\nu_p} F^{\nu_1\ldots\nu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{10-p}} \,, && \text{(A.1.15)}\\
\star 1 &= \frac{1}{10!} \epsilon_{\mu_1\ldots\mu_{10}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{10}} \,, && \text{(A.1.16)}\\
\star\star F_p &= (-1)^{p+1} F_p \,, && \text{(A.1.17)}\\
F_p \wedge G_q &= \frac{1}{p!\, q!} F_{\mu_1\ldots\mu_p} G_{\mu_{p+1}\ldots\mu_{p+q}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+q}} \,, && \text{(A.1.18)}\\
\star F_p \wedge F_p &= \frac{1}{p!} F_{\mu_1\ldots\mu_p} F^{\mu_1\ldots\mu_p} \star 1 \,. && \text{(A.1.19)}
\end{aligned}
$$

A.2 Conventions for the fermionic sector of IIB supergravity

The gamma matrices γ_a satisfy the Clifford algebra defined by

$$ \{\gamma_a, \gamma_b\} \equiv \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab} \,. \tag{A.2.1} $$

The representations of this Clifford algebra are taken to be unitary. This implies that Hermitean conjugation is given by

$$ \gamma_a^\dagger = \gamma_{\underline{0}}\gamma_a\gamma_{\underline{0}} \,. \tag{A.2.2} $$

If γ_a is a unitary representation of the Clifford algebra then so is γ_a^T. Therefore there exists a unitary matrix C, the charge conjugation matrix, such that

$$ \gamma_a^T = C\gamma_a C^{-1} \,. \tag{A.2.3} $$

It can be shown that the charge conjugation matrix C, defined through (A.2.3), in 10-dimensional Minkowski space-time can be taken to be symmetric. The matrix

γ_a^* is also a unitary representation of the Clifford algebra, but since $\gamma_a^* = (\gamma_a^\dagger)^T$ this representation follows from (A.2.2) and (A.2.3). Taking the transpose of (A.2.2) using (A.2.3) one has

$$\gamma_a^* = -B\gamma_a B^{-1}\,, \tag{A.2.4}$$

where B is a unitary matrix given by

$$B = -C\gamma_{\underline{0}}\,. \tag{A.2.5}$$

Actually, one finds that $-\gamma_a^*$ is equivalent to γ_a via a similarity transformation with a unitary matrix B. The Majorana representation[2] in which all the gamma matrices are purely imaginary can be obtained by taking the charge conjugation matrix C to be equal to $\gamma_{\underline{0}}$. It is straightforward to show that the complex conjugate of B satisfies $B^* = \gamma_{\underline{0}} C^{-1}$ so that one has:

$$BB^* = 1\,. \tag{A.2.6}$$

The chirality matrix γ_{11} is defined by

$$\gamma_{11} = \gamma_{\underline{0}}\gamma_{\underline{1}}\cdots\gamma_{\underline{9}}\,. \tag{A.2.7}$$

One defines $\gamma_{a_1\ldots a_n}$ to denote the anti-symmetrized products of gamma matrices:

$$\gamma_{a_1\ldots a_n} \equiv \gamma_{[a_1}\cdots\gamma_{a_n]}\,. \tag{A.2.8}$$

The objects $\gamma_{a_1\ldots a_n}$ and $\gamma_{b_1\ldots b_{10-n}}$ are related by duality,

$$\gamma_{a_1\ldots a_n} = -(-1)^{\frac{1}{2}n(n-1)}\frac{1}{(10-n)!}\epsilon_{a_1\ldots a_n b_1\ldots b_{10-n}}\gamma^{b_1\ldots b_{10-n}}\gamma_{11}\,. \tag{A.2.9}$$

The matrices $\gamma_{a_1\ldots a_n}C^{-1}$ and $\gamma_{11}\gamma_{a_1\ldots a_n}C^{-1}$ have the following symmetry properties:

$$(\gamma_{a_1\ldots a_n}C^{-1})^T = (-1)^{n(n-1)/2}\gamma_{a_1\ldots a_n}C^{-1}\,, \tag{A.2.10}$$

$$(\gamma_{11}\gamma_{a_1\ldots a_n}C^{-1})^T = (-1)^{n+1}(-1)^{n(n-1)/2}\gamma_{11}\gamma_{a_1\ldots a_n}C^{-1}\,. \tag{A.2.11}$$

Irreducible spinor representations of the Lorentz group $SO(9,1)$ are Majorana–Weyl spinors. On a spinor χ the Majorana condition reads

$$\chi^* = \alpha_\chi B\chi\,, \tag{A.2.12}$$

where α_χ is a yet to be determined phase factor. In order that χ satisfying the Majorana condition (A.2.12) is a consistent spinor representation of the Lorentz group in ten dimensions it must be that

$$\delta\chi^* = (\delta\chi)^* \qquad \text{with} \qquad \delta\chi = \frac{1}{2}\omega^{ab}\gamma_{ab}\chi\,, \tag{A.2.13}$$

[2]Had the charge conjugation matrix C be defined via $\gamma_a^T = -C\gamma_a C^{-1}$, which would have equally been possible since both γ_a^T and $-\gamma_a^T$ form representations of the Clifford algebra, then C would have been anti-symmetric and (A.2.4) would have been replaced by $\gamma_a^* = B\gamma_a B^{-1}$ with $B = -C\gamma_{\underline{0}}$. In this case the Majorana representation, $C = \gamma_{\underline{0}}$, implies that the gamma matrices are real.

with $\frac{1}{2}\gamma_{ab}$ the generators of $SO(9,1)$ and with ω^{ab} a set of infinitesimal parameters. It can be verified that the Majorana condition (A.2.12) is consistent with eq. (A.2.13). Further, requiring that $(\chi^*)^* = \chi$ forces α_χ to be a phase factor. Besides a Majorana condition one can, in even space-time dimensions, impose a chirality condition. The chirality operator will be denoted by $P_\pm = \frac{1}{2}(1 \pm \gamma_{11})$. The Hermitean operator $P_\pm$ is a projector, satisfying $P_\pm P_\pm = P_\pm$, $P_\pm P_\mp = 0$ and $P_+ + P_- = 1$. The latter property allows one to decompose an arbitrary spinor χ into two pieces each of which forms an eigenspinor of $P_\pm$, i.e. one has

$$\chi = P_+\chi + P_-\chi \qquad \text{with} \qquad \gamma_{11}P_\pm\chi = \pm P_\pm\chi\,. \tag{A.2.14}$$

In order for $P_\pm\chi$ to form representations of the Lorentz group it must be that

$$\delta(P_\pm\chi) = P_\pm\delta\chi \qquad \text{with} \qquad \delta\chi = \frac{1}{2}\omega^{ab}\gamma_{ab}\chi\,. \tag{A.2.15}$$

Since $P_\pm$ commutes with γ_{ab} the condition (A.2.15) is satisfied. In ten-dimensional Minkowski space-time one can have spinors that are both Majorana and Weyl. A Majorana–Weyl spinor is a spinor $P_\pm\chi$ satisfying the Majorana condition (A.2.12), i.e.

$$(P_\pm\chi)^* = \alpha_\chi B P_\pm\chi\,. \tag{A.2.16}$$

Given a spinor χ the charge conjugated spinor, χ_C, is defined by

$$\chi_C = \alpha_\chi^{-1}B^{-1}\chi^*\,, \tag{A.2.17}$$

so that a Majorana spinor is equal to its charge conjugated spinor. The Dirac conjugated spinor, $\bar\chi$, is defined by

$$\bar\chi = \chi^\dagger\gamma_{\underline{0}}\alpha_\chi^{-1}\,. \tag{A.2.18}$$

The type IIB supergravity theory contains two Majorana–Weyl dilatini, λ_1 and λ_2, of the same chirality that is taken to be

$$\gamma_{11}\lambda_1 = \lambda_1\,, \qquad \gamma_{11}\lambda_2 = \lambda_2\,, \tag{A.2.19}$$

and it contains two Majorana–Weyl gravitini, $\psi_{1\,\mu}$ and $\psi_{2\,\mu}$ of the same chirality, but opposite to that of the dilatini, i.e.

$$\gamma_{11}\psi_{1\,\mu} = -\psi_{1\,\mu}\,, \qquad \gamma_{11}\psi_{2\,\mu} = -\psi_{2\,\mu}\,. \tag{A.2.20}$$

Because λ_1 and λ_2 as well as $\psi_{1\,\mu}$ and $\psi_{2\,\mu}$ have the same chirality it is possible to consider the (reducible) complex Weyl spinors, λ and ψ_μ, defined by

$$\lambda = \lambda_1 + i\lambda_2\,, \qquad \psi_\mu = \psi_{1\,\mu} + i\psi_{2\,\mu}\,, \tag{A.2.21}$$

together with the charge conjugated spinors, λ_C and $\psi_{C\,\mu}$,

$$\lambda_C = \lambda_1 - i\lambda_2 \,, \qquad \psi_{C\,\mu} = \psi_{1\,\mu} - i\psi_{2\,\mu} \,. \tag{A.2.22}$$

Because the theory possesses $N = 2$ supersymmetry, there are also two Majorana–Weyl supersymmetry transformation parameters ϵ_1 and ϵ_2. These two parameters have the same chirality and reality properties as the gravitini $\psi_{1\,\mu}$ and $\psi_{2\,\mu}$. The reality properties of the spinors λ_1 and λ_2 as well as $\psi_{1\,\mu}$ and $\psi_{2\,\mu}$, according to eq. (A.2.12), depend on an as yet undetermined phase factors α_{λ_1}, α_{λ_2}, α_{ψ_1} and α_{ψ_2}, that in principle could all be different. For the supersymmetry parameters ϵ_1 and ϵ_2, the phase factors α_{ϵ_1} and α_{ϵ_2} are taken to be equal to α_{ψ_1} and α_{ψ_2}, respectively. The reality properties of the spinors must be consistent with the reality properties of the bosons because the two are related via supersymmetry (see e.g. eqs. (1.1.46) to (1.1.54)). The following phases are chosen:

$$\alpha_{\lambda_1} = \alpha_{\lambda_2} = \alpha_{\psi_1} = \alpha_{\psi_2} = \alpha_{\epsilon_1} = \alpha_{\epsilon_2} \equiv \alpha = i \,. \tag{A.2.23}$$

This section is ended with some comments regarding spinor bilinears. Under complex conjugation the product of two spinors χ_1 and χ_2 is taken to behave as

$$(\chi_1\chi_2)^* = \chi_2^*\chi_1^* \,. \tag{A.2.24}$$

The spinors are anticommuting Grassmann variables. One then has the following two properties satisfied by arbitrary spinors χ_1 and χ_2:

$$(\bar{\chi}_1\gamma_{a_1\ldots a_n}\chi_2)^* = (-1)^n\bar{\chi}_{1C}\gamma_{a_1\ldots a_n}\chi_{2C} \,, \tag{A.2.25}$$

$$\bar{\chi}_1\gamma_{a_1\ldots a_n}\chi_2 = (-1)^{n(n+1)/2}\bar{\chi}_{2C}\gamma_{a_1\ldots a_n}\chi_{1C} \,. \tag{A.2.26}$$

Appendix B

Parity properties

In this appendix the parity properties of the bosonic and fermionic fields are discussed. It is assumed that the background metric is Minkowski space-time. The behavior of the fields of IIB supergravity under parity can be inferred from the supersymmetry transformation rules provided the behavior of the supersymmetry parameter ϵ under parity is specified. Under parity ϵ is taken to transform as:

$$P\,\epsilon(x^0, x^i) = i\gamma_{\underline{0}}\,\epsilon_C(x^0, -x^i)\,, \tag{B.1}$$

where the coordinates x^i form a Cartesian coordinate system, so that the space inversion means $x^i \to -x^i$ for $i = 1, \ldots 9$. The supersymmetry parameter ϵ in (B.1) is the one of subsection 1.1.4. Next, it is required that the local $U(1)$ transformation, $\epsilon \to e^{-i\alpha/2}\epsilon$, is consistent with the definition (B.1) in that the $U(1)$ transformed parameter $\epsilon' = e^{-i\alpha/2}\epsilon$ transforms under parity as

$$P\,\epsilon'(x^0, x^i) = i\gamma_{\underline{0}}\,\epsilon'_C(x^0, -x^i)\,. \tag{B.2}$$

Eq. (B.2) is consistent with eq. (B.1) provided the parameter α is odd under parity. This in turn implies the following parity properties of P and Q:

$$Q_0 \to -Q_0\,, \quad Q_i \to Q_i\,, \quad P_0 \to \bar{P}_0\,, \quad P_i \to -\bar{P}_i\,. \tag{B.3}$$

This follows from the behavior of P and Q under local $U(1)$ transformations (1.1.10). In the $U(1)$ fixed frames (1.2.14) and (1.2.21) this implies that the scalar χ' in both cases must be parity odd while T is parity even[1]. This fact in turn implies that λ'

[1]The scalars T and χ' for $\det Q > 0$ are related to the dilaton ϕ and RR axion χ via eq. (1.3.42) in which $\tau = \chi + ie^{-\phi}$ and eqs. (1.2.25) and (1.2.23). It can be checked that this relation between (T, χ') and (ϕ, χ) is such that parities of one pair of scalars implies the parities of the other pair in accordance with the findings of this subsection.

Field	Parity	Field	Parity
T	even	A_{ijkl}	odd
χ'	odd	$q_\alpha A^\alpha_{0i_1\ldots i_5}$	odd
$q_\alpha A^\alpha_{0i}$	even	$q_\alpha A^\alpha_{i_1\ldots i_6}$	even
$q_\alpha A^\alpha_{ij}$	odd	$\tilde{q}_\alpha A^\alpha_{0i_1\ldots i_5}$	even
$\tilde{q}_\alpha A^\alpha_{0i}$	odd	$\tilde{q}_\alpha A^\alpha_{i_1\ldots i_6}$	odd
$\tilde{q}_\alpha A^\alpha_{ij}$	even	$q_{\alpha\beta} A^{\alpha\beta}_{0i_1\ldots i_7}$	even
A_{0ijk}	even	$q_{\alpha\beta} A^{\alpha\beta}_{i_1\ldots i_8}$	odd

Table B.1: Parity properties of the bosonic fields of type IIB supergravity separated into electric (one time index) and magnetic components (no time index) on Minkowski space-time.

transforms under parity as follows:

$$P\,\lambda'(x^0,x^i) = i\gamma_{\underline{0}}\,\lambda'_C(x^0,-x^i)\,. \tag{B.4}$$

This can be obtained from the supersymmetry transformation rules for T and χ'. Requiring that the spinor bilinears in the supersymmetry transformations of the 2-form fields, eq. (1.3.16), that are multiplied by the same scalar function, e.g. $\bar{\epsilon}'\gamma_{\mu\nu}\lambda'$ and $\bar{\epsilon}'_C\gamma_{[\mu}\psi'_{\nu]}$ in (1.3.16), transform in the same way forces the gravitino to transform under parity as

$$P\,\psi'_\mu(x^0,x^i) = i\gamma_{\underline{0}}\,\psi'_{C\,\mu}(x^0,-x^i)\,, \tag{B.5}$$

It is now straightforward to deduce the parity properties of the form fields using eqs. (1.3.16) to (1.3.19). The results are summarized in table B.1 distinguishing electric (one time index) and magnetic components (no time index).

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Nederlandse Samenvatting

In de natuur komen deeltjes voor zoals electronen, quarks en neutrino's. De quarks zijn de bouwstenen van protonen en neutronen welke op hun beurt terug te vinden zijn in de kernen van atomen. Om deze kernen heen bevindt zich een wolk van electronen. Neutrino's zijn deeltjes die voorkomen in bijvoorbeeld radioactieve processen wanneer kernen vervallen of wanneer er sprake is van kernfusie (het versmelten van twee kernen). Feitelijk bestaat alle ons bekende materie uit deze drie deeltjes: electronen, quarks en neutrino's. De enige variatie die de natuur ons hierin heeft gegeven (voor zover bekend), is dat voor elk van deze drie deeltjes er twee zwaardere partners bestaan met, afgezien van hun massa, dezelfde eigenschappen.

Deeltjes worden in de natuurkunde beschreven op wiskunde wijze middels het begrip veld. Een veld is simpel gezegd een mathematisch object dat op elk punt in de ruimte bestaat en met behulp waarvan eigenschappen van deeltjes, zoals impuls, energie, electrische lading en spin (min of meer rotatie van het deeltje) beschreven kunnen worden. Ook interacties tussen deeltjes onderling worden beschreven met behulp van velden. Bijvoorbeeld, twee electronen kunnen interactie met elkaar hebben doordat er een zogenaamd foton, een interactiedeeltje, van het ene electron naar het andere electron gaat. Zo hebben quarks onderling interactie door uitwisseling van zogenaamde gluonen en neutrino's door het uitwisselen van weer andere deeltjes, de zogenaamde W- en Z-bosonen. De precieze wijze waarop electronen, quarks en neutrino's (tezamen met hun iets zwaardere partners) interacties met elkaar hebben, wordt beschreven door het zogenaamde standaard model dat eind jaren zestig, begin jaren zeventig werd opgesteld door Glashow, Salam en Weinberg.

Naast deeltjes en hun interacties is er in de natuur ook sprake van, laat ik zeggen, 'omgeving'. Hiermee bedoel ik de achtergrond waarin deze deeltjes bewegen. Deze achtergrond is het universum waarin we leven of iets abstracter geformuleerd de ruimte-tijd. Einstein heeft rond 1915 voor het eerst duidelijk gemaakt dat deze achtergrond, de ruimte-tijd waarin we leven, dynamisch is. Sterker nog, wat wij ervaren als zwaartekracht, een interactie tussen massa's die wederom wiskundig beschreven wordt door een veld, is niets anders dan vervorming van de ruimte-tijd. Ter vergelijking, een knikker die over een vlak stuk papier rolt, verandert van bewegingsrichting en snelheid zodra het papier vervormd wordt. Zo ook bewegen deeltjes in ons heelal ten gevolge van vervormingen van de ruimte-tijd. Mathematisch worden deze vervormingen beschreven door wat het zwaartekrachtsveld genoemd wordt.

Hoeveel weten we over ons heelal? Welnu dat is niet bijster veel, maar het is mogelijk gebleken om experimenteel te bepalen wat voor soort energieën er aanwezig zijn. De totale hoeveelheid materie beslaat ongeveer 27% van de totale energie-inhoud. Echter hiervan is slechts 4% standaard modelmaterie. De overige 23% wordt donkere materie genoemd omdat deze vooralsnog nooit direct is waargenomen. Maar misschien nog opzienbarender is de resterende 73% energie die donkere energie wordt genoemd en dus het overgrote deel van de energie-inhoud vormt. De fysische oorsprong voor deze energievorm is nog niet bekend, maar een mogelijke kandidaat ervoor is de zogenaamde kosmologische constante. De aanwezigheid van donkere energie uit zich in het feit dat ons universum versnellend uitdijt.

Een belangrijke eigenschap van de natuur zoals die waargenomen wordt, is dat deze zich gedraagt volgens de wetten van de kwantummechanica. Echter, alhoewel het standaard model een kwantumtheorie is, geldt dit niet voor de theorie van Einstein, de algemene relativiteitstheorie, welke de zwaartekracht beschrijft. Laten we zeggen dat met een kwantumtheorie wordt bedoeld dat de theorie op hele klein schaal, of wat hetzelfde is, bij hele hoge energieën geldig is.

De wetten van de zwaartekracht zoals we die nu kennen (in het bijzonder de wet van Newton), zijn getest tot op ongeveer enkele tientallen micrometers. Tegelijkertijd zijn niet-gravitationele experimenten met behulp van deeltjesversnellers op afstanden van ongeveer 10^{-18} meter (ook wel een attometer genoemd) uitgevoerd.

Het feit dat zwaartekracht nooit rigoreus getest is op afstanden kleiner een micrometer betekent dat we niet weten hoeveel dimensies er zijn op dergelijke afstanden. Aangezien zwaartekracht een manifestie van de ruimte-tijd is, verandert deze als er letterlijk meer ruimte is, d.w.z. meer dimensies zijn. Het is dus mogelijk dat er extra dimensies bestaan die kleiner zijn dan een micrometer. Het is wel zo dat extra dimensies van deze orde grootte zo moeten zijn dat alle andere niet-gravitationele interacties tussen deeltjes zo is dat deze 'blind' zijn voor de extra dimensies. Dit kan gerealizeerd worden in zogenaamde braanwereld scenario's. Dit zijn modellen waarin we verondersteld worden op een kosmologisch grote braan te leven dat beweegt door een ruimte die meer dan drie ruimtelijke dimensies heeft. Een andere mogelijkheid waarin dergelijke branen niet nodig zijn, is dat de extra dimensies simpelweg kleiner zijn dan een attometer. Een eenvoudig voorbeeld is een extra dimensie in de vorm van een cirkel met een omtrek kleiner dan een attometer. Dit klinkt misschien uiterst speculatief. Toch is het belangrijk te beseffen dat er zowel theoretisch als experimenteel geen gronden zijn waarop beweerd kan worden dat de natuur 3 ruimtelijk dimensies heeft (en één tijd dimensie) net zo goed als gebleken is dat de ons bekende materie slechts 4% van een totaal van 27% aan materie beslaat.

Laten we ons afvragen op welke schaal we kunnen verwachten dat kwantumeffecten van invloed zijn op de zwaartekracht. In de kwantummechanica kunnen massive deeltjes zich gedragen als iets dat golft. Terwijl in de algemene relativiteitstheorie massieve deeltjes een zwart gat kunnen vormen. Een zwart gat is een gebied in de ruimte-tijd waaruit niets kan ontsnappen. Als nu de grootte van zo'n zwart gat in de ene theorie vergelijkbaar is met de golflengte van hetzelfde deeltje in de andere theorie dan zou het deeltje zich soms binnen en soms buiten het zwarte gat bevinden; iets dat niet kan en dus leidt tot een tegenspraak. Dit blijkt te gebeuren bij afstanden van ongeveer 10^{-35} meter. Dit wordt de Planck schaal genoemd en betekent dat we de ons bekende theorieën niet kunnen gebruiken op afstanden kleiner dan de Planck schaal en misschien al daarvoor.

Het is niet bepaald eenvoudig gebleken om een adequate kwantumtheorie voor de zwaartekracht op te schrijven. Tot op heden bestaat er geen kwantumtheorie voor de zwaartekracht waarmee voorspellingen gedaan kunnen worden over het voor ons waarneembare deel van het heelal. Echter, een wiskundig consistentie theorie die kwantumzwaartekracht beschrijft op afstanden kleiner dan de Planck schaal bestaat wel degelijk en deze theorie heet supersnaartheorie of snaartheorie in het kort.

Supersnaartheorie gaat uit van de gedachte dat er snaren bestaan met een typische lengte die gelijk is aan de Planck lengte. Er bestaan twee soorten snaren: open en gesloten snaren. Elke trillingstoestand van een snaar komt overeen met een veld. De laagste trillingstoestanden komen overeen met massaloze velden en de hogere aangeslagen toestanden met massieve velden. De massa's van de massieve velden zijn dermate hoog dat ze geen rol van betekenis spelen bij energieën die waargenomen worden in aardse experimenten of in de kosmologie. We zullen ons dus beperken tot snaren waarvan enkel de laagste trillingstoestanden aangeslagen zijn. De eigenschappen van de velden die door de snaren gegenereerd worden, hangen sterk af van de achtergrond, de ruimte-tijd, waarin de snaren zich bevinden. Door het kiezen van geschikte achtergronden is het mogelijk om deeltjes als electronen, quarks en neutrino's te produceren[2]. Echter, moet hier wel bij opgemerkt worden dat het nog niet is gelukt om electronen, quarks en neutrino's met precies dezelfde eigenschappen als in het standaard model te produceren.

Een andere belangrijke eigenschap van supersnaartheorie is dat er negen ruimtelijke dimensies (en één tijd dimensie) nodig zijn om de theorie consistent te maken. Het is dan ook van belang om snaartheorie op achtergronden te bestuderen die drie ruimtelijke dimensies hebben zoals wij die waarnemen en daarnaast zes ruimtelijke dimensies met de eigenschap dat deze klein zijn, waarbij klein betekent klein genoeg om vooralsnog niet waargenomen te zijn.

Zoals vermeld bestaan er open en gesloten snaren. De open snaren hebben twee eindpunten. Deze eindpunten eindigen altijd op een braan. Een braan is een object in snaartheorie dat een bepaald aantal dimensies heeft. Dit kan variëren van 0, 1 tot

[2]Ondanks het feit dat velden in snaartheorie massaloos zijn, kunnen velden (lees deeltjes) door interacties wel degelijk massa krijgen.

9 ruimtelijke dimensies waarbij nul dimensies een punt voorstelt. Men spreekt dan van 0-, 1- tot 9-branen. Negen-branen zijn speciaal omdat ze de hele ruimte vullen aangezien er negen ruimtelijke dimensies zijn in snaartheorie. Branen zijn eigenlijk hoger-dimensionale generalizaties van het idee snaar, wat een 1-braan genoemd kan worden. Snaartheorie is dus geenszin een theorie van enkel snaren, maar een theorie van branen. Niet alle branen in snaartheorie hebben de eigenschap dat er een open snaar op eindigt, maar binnen de context van deze thesis is het voldoende om te zeggen dat branen objecten zijn waar open snaren op eindigen. De theorie die de dynamica van de branen beschrijft, wordt bepaald door de laagst energetisch aangeslagen (massaloze) toestanden van de open snaar die op de braan eindigt. Branen spelen een cruciale rol in snaartheoretische modellen voor de beschrijving van materie, interacties en kosmologie.

Er bestaan vijf verschillende versies van snaartheorie welke alle vijf verschillende uitingsvormen zijn van één enkele onderliggende theorie die M-theorie genoemd wordt. Om redenen die hier niet nader toegelicht worden is M-theorie gedefinieerd op een achtergrond met tien ruimtelijke dimensies. In vijf speciale limietgevallen laat deze theorie zich beschrijven door één van de vijf snaartheorieën. In deze thesis staat de zogenaamde type IIB snaartheorie centraal. Type IIB snaartheorie is een theorie voor een bepaald type gesloten snaren waaraan open snaren toegevoegd kunnen worden middels het toevoegen van branen aan deze theorie. De eigenschappen van de gesloten snaren van de type IIB theorie houden nauw verband met de wijze waarop supersymmetrie zich in deze theorie manifesteerd.

De massaloze velden die door snaartheorie voorspeld worden, worden beschreven door zogenaamde supergravitatietheorieën waarbij de eigenschappen van deze theorieën bepaald worden door de achtergrond (inclusief branen) waarop de snaren bewegen. Het woord supergravitatie ontleend zijn naam aan het feit dat één van de velden altijd het gravitatieveld is en dat de theorie over een bepaald soort symmetrie beschikt die supersymmetrie genoemd wordt. Supersymmetrie houdt in dat er een symmetrie bestaat tussen wat bosonen genoemd wordt en wat fermionen genoemd wordt. Bosonen zijn deeltjes zoals fotonen en gluonen die eerder genoemd werden en interactiedeeltjes zijn. Fermionen zijn materiedeeltjes zoals electronen en quarks. De massaloze velden van de type IIB snaartheorie worden beschreven door type IIB supergravitatie.

In deze thesis staan de branen die in de type IIB snaartheorie voorkomen centraal en worden bestudeerd met behulp van type IIB supergravitatie. In de type IIB snaartheorie komen 1-, 3-, 5-, 7- en 9-branen voor. Zoals in alle vijf de snaartheorieën wordt de interactie tussen de snaren beschreven door een massaloos scalair veld, d.w.z. een veld zonder spin dat de koppelingssterkte van de interacties bepaald. De type IIB theorie onderscheidt zich van de andere vier theorieën onder andere doordat er oneindig veel equivalente (maar verschillende) keuzes gemaakt kunnen worden voor het veld dat de koppeling beschrijft. Voor elke keuze voor de koppeling bestaan er twee verschillende 1-branen: te weten één die zwak en één die sterk gekoppeld is. De eigenschappen van de overige 3-, 5-, 7- en 9-branen worden vervolgens bepaald door het type 1-braan (sterk of zwak gekoppeld) dat op deze 3-, 5-, 7- en 9-branen eindigt. Voor het geval van de 3-braan blijkt de theorie verkregen met de ene dan wel de andere 1-braan equivalent te zijn en zodoende bestaat er maar één 3-braan. Voor het geval van de 7-branen geldt juist dat er meer 7-branen bestaan dan er op deze wijze geklassificeerd worden. Dit heeft te maken met een speciale eigenschap van 7-branen, namelijk dat het mogelijk is om 7-branen waarop verschillende 1-branen (lees open snaren) eindigen op elkaar te leggen. Dit leidt tot een veel grotere klasse van 7-branen, een klasse die aangeduid wordt met de term: Q7-branen. De precieze wijze waarop dit gedaan kan worden, vormt een groot deel van wat er in deze thesis beschreven staat.

In zekere zin beschrijven 7-branen overgangen tussen verschillende toestanden van de theorie. Dit heeft te maken met het gedrag van de koppeling van de theorie wanneer er 7-branen in de achtergrond aanwezig zijn. Het blijkt dat op een achtergrond van 7-branen de koppeling overgangen vertoont tussen verschillende waarden die het kan aannemen. Een onderwerp dat nauw verband houdt met dat van 7-branen is de studie van zogenaamde instantonen. Instantonen beschrijven kwantumprocessen die klassiek gezien 'verboden' zijn. Dergelijke processen worden tunnelingsprocessen genoemd en er is dan sprake van een overgang tussen twee verschillende fysische toestanden en ook hier speelt de koppeling een belangrijke rol. Het wordt aangetoond dat het bestaan van meerdere 7-branen impliceert dat er ook meerdere instantonen bestaan welke aangeduid worden als Q-instantonen.

Ik hoop hiermee de titel van mijn thesis enigszins te hebben uitgelegd en de fysische context van het in mijn thesis beschreven werk te hebben geduid. De thesis is als volgt ingedeeld. In hoofdstuk één bespreek ik type IIB supergravitatie met de nadruk op zijn verschillende uitingsvormen aangaande de verschillende keuzes die men kan maken voor het veld dat de koppeling beschrijft. Hoofdtuk twee geeft een overzicht van de branen van de type IIB theorie zonder de Q7-branen te bespreken. Zeven-branen inclusief de Q7-branen zijn het onderwerp van hoofdstuk drie. Het laatste hoofdstuk vier bespreekt de Q-instantonen en hun eigenschappen.

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