Gauge Theory of Gravitation: Electro-Gravity Mixing

© E. Sánchez-Sastre [1,2], V. Aldaya [1,3]

[1] Instituto de Astrofisica de Andalucía, Granada, Spain
[2] Email: sastre@iaa.es, es-sastre@hotmail.com
[3] Email: valdaya@iaa.es

Abstract: In this work we present a generalization of the compensating method introduced by Utiyama for the formulation of the fundamental interactions of Nature. After revising the main ideas of Utiyama's theory, we focus on the case of space-time symmetry Lie groups. The geometrical interpretation of the theory allows for the construction of gauge theories of gravity that generalize Einstein's theory of General Relativity by also incorporating the torsion field. This framework permits the formulation of gravity and the rest of internal interactions on the same footing. Therefore, it is natural to study the problem of the unification of gravitation and the other fundamental interactions within this context. In particular, we present a non-trivial mixing of electromagnetism and gravitation based on the gauge symmetry associated with the central extension of the Poincaré group by the quantum mechanical phase invariance group U(1). As a consequence, a new electromagnetic force of pure gravitational origin arises.

1. Introduction

Fundamental internal interactions (electromagnetic, weak and strong) are described within the framework of gauge theory, the essence of which was introduced by Utiyama [1] in 1956. By resorting to the principle of gauge invariance, Grand Unification Models have been developed too.
On the other hand, gravity is described by the Theory of General Relativity, which is based on the principle of relativity and the principle of equivalence. In order to face the unification of gravitation and the rest of interactions firstly one should formulate gravity as some sort of gauge theory. Likewise, another additional difficulty is that, according to no-go theorems, there is no finite-dimensional group containing any internal SU(n) symmetry Lie group and the Poincaré group, acting as diffeomorphisms of Minkowski space-time, except for the direct product.

Accordingly, the goals of the present work are the following: a) Clarify the prescription that generalizes the compensating method of Utiyama for the case in which the symmetry Lie group also acts on the space-time; b) Apply the previous generalization to formulate gravity on the basis of a principle of local invariance; c) Construct a model accounting for the mixing between gravity and electromagnetism. To this end we will bypass no-go theorems by considering the central extension of the Poincaré group by U(1) (versus the supersymmetry viewpoint).

2. Gauge Theory for Internal Symmetries

One of the most relevant principles of modern Physics is the principle of gauge (or local) invariance, which states that if an elementary material physical system is invariant under a certain group G of internal transformations with n parameters $f^{(a)}$ (a = 1 , ... , n = dimG) then, this system must also be invariant under the same group of transformations but with parameters $f^{(a)}(x)$ depending on the position. The price to be paid is the introduction of compensating or gauge fields $A^{(a)}{}_{\mu}$, with the following transformation law:

$$\delta A^{(a)}{}_{\mu} = f^{(b)} C^{a}{}_{bc} A^{(c)}{}_{\mu} - \partial_{\mu} f^{(a)},$$

where $C^a{}_{bc}$ are the structure constants of the symmetry group. Then, the new Lagrangian describing the matter fields φ^α ($\alpha = 1, \ldots, N$) as well as their interaction with the compensating fields $A^{(a)}{}_\mu$, $L_{matt}(\varphi^\alpha, D_\mu \varphi^\alpha)$, with

$$D_\mu \varphi^\alpha = \partial_\mu \varphi^\alpha + A^{(a)}{}_\mu \, X^\alpha{}_{(a)\beta} \, \varphi^\beta ,$$

is invariant under the local (or gauge) group G(M). Note that $X^\alpha{}_{(a)\beta}$ are constant matrices satisfying the commutation relations of the symmetry group G.
In other words, the gauge invariant matter Lagrangian incorporating the interaction terms is obtained from the original one by replacing all ordinary derivatives of the matter fields with compensating derivatives:

$$\partial_\mu \varphi^\alpha \rightarrow D_\mu \varphi^\alpha .$$

On the other hand, the Lagrangian density L_0 describing the dynamics of the free gauge fields must be a scalar function of the specific combination

$$F^{(a)}{}_{\mu\nu} \equiv \partial_\nu A^{(a)}{}_\mu - \partial_\mu A^{(a)}{}_\nu - 1/2\, C^a{}_{bc}(A^{(b)}{}_\mu A^{(c)}{}_\nu - A^{(b)}{}_\nu A^{(c)}{}_\mu),$$

called the field strength of the field $A^{(a)}{}_\mu$.

The following geometrical interpretation can be derived:

-Covariant derivative: $D_\mu \varphi^\alpha \equiv \partial_\mu \varphi^\alpha + A^{(a)}{}_\mu X^\alpha{}_{(a)\beta} \varphi^\beta$.

-Connection: $\Gamma^\alpha{}_{\mu\beta} \equiv A^{(a)}{}_\mu X^\alpha{}_{(a)\beta}$.

-Curvature tensor: $R^\alpha{}_{\mu\nu\beta} \equiv F^{(a)}{}_{\mu\nu} X^\alpha{}_{(a)\beta}$.

3. Gauge Theory for Space-Time Symmetries

When the symmetry group also acts on the points of the space-time manifold M, the gauge algebra can be seen as the semi-direct product of the diffeomorphism algebra of M, diff(M), and the vertical gauge algebra (which accounts for the action of the group on the internal components of matter fields). As a consequence, the set of compensating fields is enlarged with respect to the pure internal symmetry case, that is, apart from fields of the type $A^{(a)}{}_\mu$ we have to introduce new fields $h^{(a)\nu}{}_{\mu\rho}$ and their transformation rules read respectively:

$$\delta A^{(a)}{}_\mu = f^{(b)} C^a{}_{bc} A^{(c)}{}_\mu - \partial_\mu f^{(a)} - A^{(a)}{}_\nu \partial_\mu (f^{(b)}\, \delta_{(b)} x^\nu),$$

$$\delta h^{(a)\nu}{}_{\mu\rho} = \partial_\mu f^{(a)} \delta^\nu{}_\rho + h^{(a)\sigma}{}_{\mu\rho} \partial_\sigma (f^{(b)}\, \delta_{(b)} x^\nu) - f^{(b)}\, \partial_\mu (\delta_{(b)} x^\sigma) h^{(a)\nu}{}_{\sigma\rho} ,$$

where $\delta_{(a)} x^\mu$ is a linear realization of the symmetry group on the space-time points, that is, the infinitesimal variation of the space-time points in the direction of the group generator (a).
In this case the new Lagrangian density invariant under the local algebra that describes the dynamics of the matter fields and their interaction with the compensating fields presents the following structure: $l_{matt} \equiv \Lambda\, L_{matt}(\varphi^\alpha, \Delta_\mu \varphi^\alpha)$, with

$$\Lambda \equiv \det(q^\mu{}_\nu) ,$$

$\Delta_\mu\varphi^\alpha \equiv k_\mu{}^\nu D_\nu\, \varphi^\alpha \equiv k_\mu{}^\nu\, (\partial_\nu\, \varphi^\alpha + A^{(a)}{}_\nu X^\alpha{}_{(a)\beta}\, \varphi^\beta)$,

$k_\mu{}^\nu \equiv \delta^\nu{}_\mu + h^{(a)\nu}{}_{\mu\rho}\delta_{(a)}x^\rho$,

$k_\mu{}^\nu\, q^\mu{}_\sigma = \delta^\nu{}_\sigma$, $k_\mu{}^\nu\, q^\sigma{}_\nu = \delta^\sigma{}_\mu$.

It is worth remarking that due to the structure of the generalized compensating derivative $\Delta_\mu\varphi^\alpha$, it is natural to consider the set $\{A^{(a)}{}_\mu, k_\mu{}^\nu\}$as compensating fields, assuming the following transformation law for $k_\mu{}^\nu$:

$\delta k_\mu{}^\nu = k_\mu{}^\sigma\partial_\sigma(f^{(a)}\, \delta_{(a)}x^\nu) - f^{(a)}\, \partial_\mu(\delta_{(a)}x^\sigma)k_\sigma{}^\nu$.

Formulations of the theory in terms of the fields $\{A^{(a)}{}_\mu, h^{(a)\nu}{}_{\mu\rho}\}$ and $\{A^{(a)}{}_\mu, k_\mu{}^\nu\}$ are equivalent [2].

The principle of minimal coupling can be generalized by saying that the Lagrangian density invariant under the local space-time transformations that contains the interaction terms can be obtained from the original matter Lagrangian density by replacing all the ordinary derivatives of matter fields by generalized compensating derivatives:

$\partial_\mu\, \varphi^\alpha \rightarrow \Delta_\mu\varphi^\alpha$

and multiplying the result by the factor $\Lambda \equiv \det(q^\mu{}_\nu)$ to compensate the variation of the integration volume due to the non-null divergence of the generators of the local space-time symmetry group.

On the other hand, the Lagrangian density $L_0(A^{(a)}{}_\mu, \partial_\nu A^{(a)}{}_\mu, k_\mu{}^\nu, \partial_\sigma k_\mu{}^\nu)$ of the free compensating fields invariant under the local space-time algebra has to be Λ ($\equiv \det(q^\mu{}_\nu)$) times a scalar function l_0 of the objects $t^\sigma{}_{\mu\nu}$ and $f^{(a)}{}_{\mu\nu}$, i.e.

$L_0(A^{(a)}{}_\mu, \partial_\nu A^{(a)}{}_\mu, k_\mu{}^\nu, \partial_\sigma k_\mu{}^\nu) = \Lambda\, l_0(\, t^\sigma{}_{\mu\nu}\, , f^{(a)}{}_{\mu\nu})$,

where

$t^\sigma{}_{\mu\nu} \equiv q^\sigma{}_\rho(\partial_\tau k_\nu{}^\rho k_\mu{}^\tau - \partial_\tau k_\mu{}^\rho k_\nu{}^\tau) - A^{(a)}{}_\rho\, [k_\mu{}^\rho\, \partial_\nu(\delta_{(a)}x^\sigma) - k_\nu{}^\rho\, \partial_\mu\, (\delta_{(a)}x^\sigma)]$,

$f^{(a)}{}_{\mu\nu} \equiv k_\mu{}^\rho k_\nu{}^\sigma F^{(a)}{}_{\rho\sigma} \equiv k_\mu{}^\rho k_\nu{}^\sigma\, [\partial_\sigma A^{(a)}{}_\rho - \partial_\rho\, A^{(a)}{}_\sigma - 1/2\, C^a{}_{bc}(A^{(b)}{}_\rho\, A^{(c)}{}_\sigma - A^{(b)}{}_\sigma\, A^{(c)}{}_\rho)]$.

The gauge theory of space-time symmetries just presented leads to the following geometrical interpretation. Fields $k_\upsilon{}^\rho$ and their inverse $q^\sigma{}_\rho$ can be interpreted as tetradic fields or tetrads, in terms of which we can write the metric tensor:

$g_{\mu\nu} \equiv q^\sigma{}_\mu\, q^\rho{}_\nu\, \eta_{\sigma\rho}$

and the inverse $g^{\mu\nu} \equiv k_\sigma{}^\mu\, k_\rho{}^\nu\, \eta^{\sigma\rho}$. We also arrive at the equality:

$\Lambda \equiv \det(q^\mu{}_\nu)=[-\det(g_{\mu\nu})]^{1/2}$.

Apart from the metric, we can construct another important geometrical object: a space-time connection $\Gamma^\sigma{}_{\mu\upsilon}$ given by the expression:

$\Gamma^\sigma{}_{\mu\nu} \equiv q^\rho{}_\mu\, [A^{(a)}{}_\nu\, \partial_\rho(\delta_{(a)}x^\tau)\, k_\tau{}^\sigma - \partial_\nu\, k_\rho{}^\sigma]$.

The covariant derivative of the metric tensor with respect to the connection $\Gamma^{\sigma}_{\mu\nu}$ vanishes, i.e. $g_{\mu\nu;\sigma} = 0$. We can also introduce the corresponding curvature and torsion tensors, $R^{\rho}_{\sigma\mu\nu}$ and $\Theta^{\sigma}_{\mu\nu}$, associated with $\Gamma^{\sigma}_{\mu\nu}$, given respectively by:

$$R^{\rho}_{\sigma\mu\nu} \equiv \partial_{\nu}\Gamma^{\rho}_{\sigma\mu} - \partial_{\mu}\Gamma^{\rho}_{\sigma\nu} - \Gamma^{\rho}_{\tau\mu}\Gamma^{\tau}_{\sigma\nu} + \Gamma^{\rho}_{\tau\nu}\Gamma^{\tau}_{\sigma\mu} ,$$

$$\Theta^{\sigma}_{\mu\nu} \equiv \Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu} .$$

The following relations can be obtained too:

$$R^{\rho}_{\sigma\mu\nu} = k_{\theta}^{\ \rho}\, q^{\lambda}_{\ \sigma}\, q^{\tau}_{\ \mu}\, q^{\omega}_{\ \nu} f^{(a)}_{\ \ \tau\omega}\, \partial_{\lambda}(\delta_{(a)}x^{\theta}) ,$$

$$\Theta^{\sigma}_{\mu\nu} = k_{\theta}^{\ \sigma}\, q^{\rho}_{\ \mu}\, q^{\lambda}_{\ \nu}\, t^{\ \theta}_{\ \rho\lambda} .$$

4. Gauge Theories of Gravity

Depending on the chosen space-time symmetry group the structure of $\Gamma^{\sigma}_{\mu\nu}$ incorporates different compensating fields and the resulting connection defines different geometries. Likewise, within each geometry it is possible to construct a gauge gravitational theory.
For example:
- Weitzenbock geometry is related to the gauge theory of the space-time translation group (e.g. [3]).
- Riemann- Cartan geometry is associated with the gauge theory of the Poincaré group (e.g. [4,5]).
- Weyl- Cartan geometry arises when considering the gauge theory of the Weyl group (e.g. [6]).

It is remarkable that the formulation of gravity as a gauge theory allows for the interpretation of the principle of equivalence as a principle of local invariance. Moreover, the gauge theory of gravitation leads to generalizations of the theory of General Relativity by incorporating also the torsion tensor. A relevant example is the Einstein-Cartan theory, which is characterized by a Lagrangian density linear in the curvature and quadratic in the torsion.

The experimental detection of the torsion could be related to phenomena such as the additional splitting of the spectral lines of the electron in the atom or the CP violation in particles decays, but if the coupling constant of the torsion were equal to the gravitational constant then all the laboratory effects would be very small and could not be verified experimentally. However, the relevance of the torsion is more patent in certain cosmological models in the sense that gravitational collapses can be avoided. In fact, in Einstein-Cartan theory and in some models with dynamical torsion it is possible to obtain regular cosmological solutions with respect to the metric.

5. Mixing of Gravitation and Electromagnetism

The present electro-gravity mixing (see [7]) is based on two important notions: the local invariance and the central extension of a group. Let us introduce this last concept. Working at the infinitesimal level, given the Lie algebra of a Lie group G, characterized by the commutation relations

$$[X_{(a)}, X_{(b)}] = C^{c}{}_{ab} X_{(c)} ,$$

the new algebra of the central extension of the group G by the group U(1) is defined by modifying the previous commutators by means of the incorporation of a new central generator (which, therefore, commutes with the rest of generators) Ξ associated with the parameter ζ of U(1) and the introduction of new constant parameters $C^{\zeta}{}_{ab}$:

$$[X_{(a)}, X_{(b)}] = C^{c}{}_{ab} X_{(c)} + C^{\zeta}{}_{ab} \Xi,$$

and $[X_{\zeta}, X_{(a)}] = 0$ for all generators $X_{(a)}$ of the original algebra.

We are interested in the central extension of the Poincaré group by U(1). In this case, the commutator of the Lorentz and translation generators is modified according to

$$[M_{\mu\nu}, P_{\rho}] = C^{\sigma}{}_{\mu\nu,\rho} P_{\sigma} + C^{\zeta}{}_{\mu\nu,\rho} \Xi ,$$

with

$$C^{\sigma}{}_{\mu\nu,\rho} \equiv \eta_{\nu\rho}\delta^{\sigma}{}_{\mu} - \eta_{\mu\rho}\delta^{\sigma}{}_{\nu} ,$$

$$C^{\zeta}{}_{\mu\nu,\rho} \equiv \lambda_{\nu}\eta_{\mu\rho} - \lambda_{\mu}\eta_{\nu\rho} ,$$

where λ_{μ} is a vector in the Poincaré co-algebra belonging to a given co-adjoint orbit, and can be related to the coupling constant, κ , of the mixing.

When considering the gauge theory of the centrally extended Poincaré group, according to Section 3, we remark the presence of the coupling constant κ through the structure constant $C^{\zeta}{}_{\mu\nu,\rho}$ in the generalized field strength $f^{(\zeta)}{}_{\mu\upsilon}$ associated with the parameter of U(1). Without loss of generality we can select a preferred direction for λ_{μ} ,

$$\lambda_{\mu} = -\kappa\, \delta^{0}{}_{\mu} ,$$

so that we arrive at

$$C^{\zeta}{}_{\sigma\rho,\mu} \equiv \kappa\, (\eta_{\rho\mu}\, \delta^{0}{}_{\sigma} - \eta_{\sigma\mu}\, \delta^{0}{}_{\rho}) .$$

In order to construct an electro-gravity theory in the most economical way, and bearing in mind the results of the gauge theory of the Poincaré group, it is enough to consider only the Lorentz and U(1) generalized curvatures, which respectively read:

$$F^{(\lambda\rho)}{}_{\mu\nu} = \partial_{\nu}A^{(\lambda\rho)}{}_{\mu} - \partial_{\mu}A^{(\lambda\rho)}{}_{\nu} - \eta_{\theta\sigma} (A^{(\lambda\theta)}{}_{\mu} A^{(\sigma\rho)}{}_{\nu} - A^{(\lambda\theta)}{}_{\nu} A^{(\sigma\rho)}{}_{\mu}) ,$$

$$F^{(\zeta)}{}_{\mu\nu} = \partial_{\nu}A^{(\zeta)}{}_{\mu} - \partial_{\mu}A^{(\zeta)}{}_{\nu} - 1/2\; C^{\zeta}{}_{\lambda,\theta\rho} (A^{(\lambda)}{}_{\mu} A^{(\theta\rho)}{}_{\nu} - A^{(\lambda)}{}_{\nu} A^{(\theta\rho)}{}_{\mu})$$

$$= \partial_{\nu}A^{(\zeta)}{}_{\mu} - \partial_{\mu}A^{(\zeta)}{}_{\nu} + \kappa\, \eta_{ij} (A^{(j)}{}_{\mu} A^{(0i)}{}_{\nu} - A^{(j)}{}_{\nu} A^{(0i)}{}_{\mu}) ,$$

where η_{ij} is the Minkowski metric and the Latin indices i, j run from 1 to 3. Likewise, we assume the standard relation between translational and tetradic fields:

$$A^{(\nu)}{}_{\mu} = \delta^{\nu}{}_{\mu} - q^{\nu}{}_{\mu} .$$

The potentials $A^{(\zeta)}{}_{\mu}$ can be decomposed as follows:

$A^{(\zeta)}{}_{\mu} = A^{(ELEC)}{}_{\mu} + \kappa\, B^{(GRAV)}{}_{\mu}$,

where $B^{(GRAV)}{}_{\mu}$ is an ''electromagnetic'' contribution of pure gravitational origin and $A^{(ELEC)}{}_{\mu}$ coincides with the electromagnetic potential when $\kappa\to 0$, thus recovering the Einstein-Maxwell theory associated with the gauging of the direct product of the Poincaré group and the electromagnetic U(1).
Note that $B^{(GRAV)}{}_{\mu}$ must be a function of the gravitational potentials. For simplicity we will assume the simplest case, that is, $B^{(GRAV)}{}_{\mu}$ as a function of the tetrads (or the metric). It is worth remarking that the theory can be developed working on first order in κ. In fact, it is expected that

$|\kappa\, e| \leq m_{\text{electron}} \rightarrow \kappa \leq 6 \times 10^{-12}$ Kg/C .

Therefore, the maximum supposed value for κ would correspond to the mass-charge relation of the electron. In such a case, the physical content of the module of λ_{μ} would be essentially the quotient of coupling constants (gravitational and electromagnetic ones). This is in fact a feature of unified (gauge) theories, for example, in the electro-weak theory the tangent of the Weinberg angle gives precisely the relation between the isospin and hypercharge coupling constants.

Taking into account that $A^{(\zeta)}{}_{\mu} = A^{(ELEC)}{}_{\mu} + \kappa\, B^{(GRAV)}{}_{\mu}$, the field strength $F^{(\zeta)}{}_{\mu\nu}$ admits the following decomposition:

$$F^{(\zeta)}{}_{\mu\nu} = F^{(ELEC)}{}_{\mu\nu} + \kappa\, F^{(GRAV)}{}_{\mu\nu} ,$$

With

$$F^{(ELEC)}{}_{\mu\nu} = \partial_{\nu} A^{(ELEC)}{}_{\mu} - \partial_{\mu} A^{(ELEC)}{}_{\nu} ,$$

$$F^{(GRAV)}{}_{\mu\nu} = \partial_{\nu} B^{(GRAV)}{}_{\mu} - \partial_{\mu} B^{(GRAV)}{}_{\nu} + \eta_{ij}\, (A^{(j)}{}_{\mu}\, A^{(0i)}{}_{\nu} - A^{(j)}{}_{\nu}\, A^{(0i)}{}_{\mu}) .$$

As a consequence, the field $B^{(GRAV)}{}_{\mu}$ could be responsible for some electromagnetic force associated with very massive rotating systems, as $A^{(0i)}{}_{\mu}$ is somehow related to Coriolis-like forces. A similar effect is the so-called Blackett effect or gravitational magnetism, which consists in the generation of magnetic fields from electrically neutral rotating objects.

The simplest electro-gravitational gauge invariant Lagrangian density for the free compensating fields in our model has the form:

$$L_0 = -1/4\, \Lambda f^{(\zeta)}{}_{\mu\nu} f^{(\zeta)\,\mu\nu} + 1/2\, f^{(\mu\nu)}{}_{\mu\nu} = -1/4\, \Lambda\, g^{\mu\sigma} g^{\nu\rho}\, F^{(\zeta)}{}_{\mu\nu} F^{(\zeta)}{}_{\sigma\rho} + 1/2\, \Lambda\, k_{\mu}{}^{\sigma} k_{\nu}{}^{\rho} F^{(\mu\nu)}{}_{\sigma\rho} ,$$

Where $f^{(\zeta)\,\mu\nu} \equiv f^{(\zeta)}{}_{\sigma\rho}\, \eta^{\sigma\mu} \eta^{\rho\nu}$ and we recall that $g^{\sigma\rho} = k_{\mu}{}^{\sigma} k_{\nu}{}^{\rho} \eta^{\mu\nu}$, $\Lambda \equiv \det(q^{\mu}{}_{\nu})$.

The corresponding Euler-Lagrange equations for the fields $k_{\mu}{}^{\nu}$, $A^{(\sigma\rho)}{}_{\mu}$ and $A^{(\zeta)}{}_{\mu}$, up to first order in the mixing constant κ , respectively read:

a) $\Lambda\, (f^{(\mu\sigma)}{}_{\nu\sigma} - 1/2\, \delta^{\mu}{}_{\nu} f^{(\rho\sigma)}{}_{\rho\sigma}) = -\, \Phi^{\mu}{}_{\sigma}\, k_{\nu}{}^{\sigma}$,

b) $$\Lambda\, [\, k_{\theta}{}^{\mu}\, T^{\theta}{}_{\rho\sigma} - k_{\rho}{}^{\mu}\, T^{\theta}{}_{\theta\sigma} + k_{\sigma}{}^{\mu}\, T^{\theta}{}_{\theta\rho} + (k_{\theta}{}^{\mu} k_{\rho}{}^{\nu} - k_{\rho}{}^{\mu} k_{\theta}{}^{\nu})\, A^{(\theta}{}_{\sigma)\,\nu} - (k_{\sigma}{}^{\mu} k_{\theta}{}^{\nu} + k_{\theta}{}^{\mu} k_{\sigma}{}^{\nu})\, A^{(\theta}{}_{\rho)\,\nu}\,] = -\, 2\, \kappa\, \Sigma^{\mu}{}_{(\sigma\rho)} ,$$

c) $\partial_{\sigma}\, (\Lambda\, F^{(\zeta)}{}_{\mu\sigma}) = 0$,

where

$\Phi^{\mu}{}_{\sigma} \equiv \Phi^{\mu\,(ELEC)}{}_{\sigma} + \kappa\, \Phi^{\mu\,(MIXING)}{}_{\sigma}$,

$\Phi^{\mu\,(ELEC)}{}_{\sigma} \equiv 1/4\ \Lambda\ q^{\mu}{}_{\sigma} f^{(ELEC)}{}_{\lambda\theta}\, f^{(ELEC)\,\lambda\theta} - \Lambda\ q^{\theta}{}_{\sigma} f^{(ELEC)}{}_{\theta\lambda}\, f^{(ELEC)\,\mu\lambda}$,

$\Phi^{\mu\,(MIXING)}{}_{\sigma} \equiv 1/2\ \Lambda\ q^{\mu}{}_{\sigma} f^{(ELEC)}{}_{\lambda\theta}\, f^{(GRAV)\,\lambda\theta} - \Lambda\ q^{\theta}{}_{\sigma} f^{(GRAV)}{}_{\theta\lambda}\, f^{(ELEC)\,\mu\lambda}$

$- \Lambda\ q^{\theta}{}_{\sigma} f^{(ELEC)}{}_{\theta\lambda}\, f^{(GRAV)\,\mu\lambda} - \Lambda\ f^{(ELEC)\,\theta\lambda}\ \eta_{\sigma j}\, k_{\lambda}{}^{\mu}\, k_{\theta}{}^{\tau}\, A^{(0j)}{}_{\tau}$

$- \quad 1/2\ \Lambda\ f^{(ELEC)\,\lambda\theta} k_{\lambda}{}^{\nu}\, k_{\theta}{}^{\rho}\, \partial\,(\partial_{\rho}B^{(GRAV)}{}_{\nu} - \partial_{\nu}B^{(GRAV)}{}_{\rho})/\partial\, k_{\mu}{}^{\sigma}$

$+ 1/2\ \ \partial_{\lambda}[\Lambda\ f^{(ELEC)\,\theta\rho} k_{\theta}{}^{\omega}\, k_{\rho}{}^{\tau}\, \partial(\partial_{\tau}B^{(GRAV)}{}_{\omega} - \partial_{\omega}B^{(GRAV)}{}_{\tau})/\partial(\partial_{\lambda}k_{\mu}{}^{\sigma})]$,

$T^{\ \sigma}{}_{\nu\mu} \equiv q^{\sigma}{}_{\rho}(\partial_{\tau}\, k_{\nu}{}^{\rho}\, k_{\mu}{}^{\tau} - \partial_{\tau}\, k_{\mu}{}^{\rho}\, k_{\nu}{}^{\tau})$,

$\Sigma^{\mu}{}_{(\sigma\rho)} \equiv 1/2\ \Lambda\ f^{(ELEC)\,\lambda\theta}\, (k_{\lambda}{}^{\tau}\, k_{\theta}{}^{\mu} - k_{\lambda}{}^{\mu}\, k_{\theta}{}^{\tau})\, \delta^{0}{}_{\sigma}\, \eta_{i\rho}\, A^{(i)}{}_{\tau}$.

Using the curvature and torsion tensors, equations a) and b) can be written respectively in the form:

$\Lambda\ (R_{\mu\nu} - 1/2\ g_{\mu\nu}\ R) = T^{\ (ELEC)}{}_{\mu\nu} + \kappa\ T^{\ (MIXING)}{}_{\mu\nu}$,

$\Lambda\ \Theta^{\lambda}{}_{\mu\nu} = \kappa\ S^{\lambda\,(MIXING)}{}_{\mu\nu}$,

with

$T^{\ (ELEC)}{}_{\mu\nu} \equiv - q^{\sigma}{}_{\mu}\, \eta_{\rho\sigma}\, \Phi^{\rho\,(ELEC)}{}_{\nu}$,

$T^{\ (MIXING)}{}_{\mu\nu} \equiv - q^{\sigma}{}_{\mu}\, \eta_{\rho\sigma}\, \Phi^{\rho\,(MIXING)}{}_{\nu}$,

$S^{\lambda\,(MIXING)}{}_{\mu\nu} \equiv 2\ q^{\sigma}{}_{\mu}\, q^{\rho}{}_{\nu}\, \Sigma^{\lambda}{}_{(\sigma\rho)} - \delta^{\lambda}{}_{\mu}\, q^{\theta}{}_{\rho}\, q^{\tau}{}_{\nu}\, \Sigma^{\rho}{}_{(\theta\tau)} - \delta^{\lambda}{}_{\nu}\, q^{\theta}{}_{\mu}\, q^{\tau}{}_{\rho}\, \Sigma^{\rho}{}_{(\theta\tau)}$.

Note that in the $\kappa \rightarrow 0$ limit the previous motion equations reproduce those of the gauge theory of the direct product of Poincaré and U(1), thus recovering the standard Einstein-Maxwell theory (without mixing).

Likewise, one can easily evaluate the geodesics with mixing terms for a spinless particle of mass m, momentum p_{μ} (= m u_{μ} = m $dx_{\mu}/d\tau$) and charge q:

$g_{\mu\sigma}\ du^{\mu}/d\tau = - u^{\mu}\ u^{\nu}\Gamma_{\mu\nu\sigma}{}^{(Levi\text{-}Civita)} - q/m\ u^{\mu}\ F^{\ (ELEC)}{}_{\mu\sigma} - \kappa q/m\ u^{\mu}\ (\partial_{\sigma}B^{(GRAV)}{}_{\mu} - \partial_{\mu}B^{(GRAV)}{}_{\sigma})$.

This is the equation for a particle in the presence of both gravitational and electromagnetic fields, with an additional Lorentz-like force (proportional to κq) generated by the gravitational compensating potentials.

6. Conclusions

The generalization of Utiyama's theory to space-time symmetry Lie groups has been revisited. The precise form of the Lagrangian density of the free compensating fields as well as the Lagrangian density describing the interaction between matter fields and compensating fields have been presented. In this process notions such as 'compensating derivative' and 'strength tensor' have also been generalized to the case of external symmetries. In our approach we introduce, in contrast to the standard literature, compensating fields $h^{(a)\,\nu}{}_{\mu\rho}$, thus accounting for the presence of the local group index (a) instead of resorting from the starting point to the usual tetradic fields $k_\nu{}^\mu$. Likewise, our formulation allows for the introduction of compensating fields associated with local space-time translations in spite of their trivial realization on matter fields.

With respect to the mixing of interactions, a simple model of mixing between gravitation and electromagnetism has been developed, accounting for electromagnetic forces of pure gravitational origin. This model is associated with the gauging of a central extension of the Poincaré group by U(1). An important point to obtain the electromagnetic curvature with mixing terms, $F^{(\zeta)}{}_{\mu\nu}$, has been the possibility of introducing translational compensating fields $A^{(\nu)}{}_{\mu}$, since the structure constant which contains the electro-gravity mixing constant involves the translational indices. It is remarkable the appearance of an electromagnetic field of pure gravitational origin, thus, in some sense, this model revives ideas such as the gravitational magnetism. The field equations result to be of the Einstein-Maxwell type but with additional terms involving torsion and the mixing constant. We have also pointed out the modification of the geodesic equations of a spinless particle, obtaining an analog of the Lorentz force but of gravitational origin.

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