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CP violation of quarks in A_4 modular invariance

Hiroshi Okada [a,*], Morimitsu Tanimoto [b]

[a] Asia Pacific Center for Theoretical Physics (APCTP), Headquarters San 31, Hyoja-dong, Nam-gu, Pohang 790-784, Republic of Korea
[b] Department of Physics, Niigata University, Niigata 950-2181, Japan

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ABSTRACT

We discuss the quark mass matrices in the A_4 modular symmetry, where the A_4 triplet of Higgs is introduced for each up-quark and down-quark sectors, respectively. The model has six real parameters and two complex parameters in addition to the modulus τ. By inputting six quark masses and three CKM mixing angles, we can predict the CP violation phase δ and the Jarlskog invariant J_{CP}. The predicted ranges of δ and J_{CP} are consistent with the observed values. The absolute value of V_{ub} is smaller than 0.0043, while V_{cb} is larger than 0.0436. In conclusion, our quark mass matrices with the A_4 modular symmetry can reproduce the CKM mixing matrix completely with observed quark masses.

1. Introduction

The origin of three families of quarks and leptons remains most important problems of Standard model (SM). In order to understand the flavor structure of quarks and leptons, considerable interests in the discrete flavor symmetry [1–9] have been developed by the early models of quark masses and mixing angles [10,11], more recently, the large flavor mixing angles of the leptons.

Many models have been proposed by using S_3, A_4, S_4, A_5 and other groups with larger orders to explain the large neutrino mixing angles. Among them, the A_4 flavor model is attractive one because the A_4 group is the minimal one including a triplet irreducible representation, which allows for a natural explanation of the existence of three families of leptons [12–17]. However, variety of models is so wide that it is difficult to obtain clear clues of the A_4 flavor symmetry. Indeed, symmetry breakings are required to reproduce realistic mixing angles [18]. The effective Lagrangian of a typical flavor model is given by introducing the gauge singlet scalars which are so-called flavons. Their vacuum expectation values (VEVs) determine the flavor structure of quarks and leptons. As a consequence, the breaking sector of flavor symmetry typically produces many unknown parameters.

Recently, new approach to the lepton flavor problem based on the invariance under the modular group [19], where the model of the finite modular group $\Gamma_3 \simeq A_4$ has been presented. This work inspired further studies of the modular invariance approach to the lepton flavor problem. It should be emphasized that there is a significant difference between the models based on the A_4 modular symmetry and those based on the usual non-Abelian discrete A_4 flavor symmetry. Yukawa couplings transform non-trivially under the modular symmetry and are written in terms of modular forms which are holomorphic functions of a complex parameter, the modulus τ.

It is interesting that the modular group includes S_3, A_4, S_4, and A_5 as its finite subgroups [20]. Along the work of the A_4 modular group [19], models of $\Gamma_2 \simeq S_3$ [21], $\Gamma_4 \simeq S_4$ [22] and $\Gamma_5 \simeq A_5$ [23] have been proposed. Also numerical discussions of the neutrino flavor mixing have been done based on A_4 [24,25] and S_4 [26] modular groups respectively. In particular, the comprehensive analysis of the A_4 modular group has provided a clear prediction of the neutrino mixing angles and the CP violating phase [25]. On the other hand, the A_4 modular symmetry has been applied to the $SU(5)$ grand unified theory of quarks and leptons [27], and also the residual symmetry of the A_4 modular symmetry has been investigated [28]. Furthermore, modular forms for $\Delta(96)$ and $\Delta(384)$ were constructed [29], and the extension of the traditional flavor group is discussed with modular symmetries [30].

In this work, we discuss the quark mixing angles and the CP violating phase, which were a main target of the early challenge for flavors [10,11]. Since the quark masses and mixing angles are remarkably distinguished from the leptonic ones, that is the hierarchical structure of masses and mixing angles, it is challenging to

* Corresponding author.
E-mail addresses: hiroshi.okada@apctp.org (H. Okada), tanimoto@muse.sc.niigata-u.ac.jp (M. Tanimoto).

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reproduce observed hierarchical three CKM mixing angles and the CP violating phase in the A_4 modular symmetry.[1]

We can easily construct quark mass matrices by using the A_4 modular symmetry. The up-quark and down-quark mass matrices have the same structure as the charged lepton mass matrix in Ref. [25]. Then, parameters, apart from the modulus τ, are determined by the observed quark masses. The remained parameter is only the modulus τ. However, it is very difficult to reproduce observed three CKM mixing angles by fixing τ since the observed mixing angles are considerably hierarchical angles, and moreover, precisely measured.

Therefore, we extend the Higgs sector in the A_4 modular symmetry by introducing the A_4 triplet for Higgs doublets in up-quark and down-quark sectors, respectively. Then, one complex parameter related with the A_4 tensor product appears in each quark mass matrix of the up- and down-quarks. The model has six real parameters and two complex parameters in addition to the modulus τ. It is remarked that those quark mass matrices can predict the magnitude of the CP violation of the CKM mixing by inputting quark masses and three mixing angles.

The paper is organized as follows. In section 2, we give a brief review on the modular symmetry. In section 3, we present the model for quark mass matrices. In section 4, we present numerical results. Section 5 is devoted to a summary. In Appendix A, the relevant multiplication rules of the A_4 group is presented. In Appendix B, we show how to determine the coupling coefficients of quarks. In Appendix C, we discuss the Higgs potential in our model.

2. Modular group and modular forms

The modular group $\bar{\Gamma}$ is the group of linear fractional transformation γ acting on the complex variable τ, so called modulus, belonging to the upper-half complex plane as:

$$\tau \longrightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d}\,, \quad \text{where } a,b,c,d \in \mathbb{Z} \text{ and } ad - bc = 1, \ \mathrm{Im}[\tau] > 0\,, \tag{1}$$

which is isomorphic to $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{I,-I\}$ transformation. This modular transformation is generated by S and T,

$$S : \tau \longrightarrow -\frac{1}{\tau}\,, \qquad T : \tau \longrightarrow \tau + 1\,, \tag{2}$$

which satisfy the following algebraic relations,

$$S^2 = \mathbb{I}\,, \qquad (ST)^3 = \mathbb{I}\,. \tag{3}$$

We introduce the series of groups $\Gamma(N)$ $(N = 1, 2, 3, \dots)$ defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ (\mathrm{mod} N) \right\}. \tag{4}$$

For $N = 2$, we define $\bar{\Gamma}(2) \equiv \Gamma(2)/\{I,-I\}$, while, since the element $-I$ does not belong to $\Gamma(N)$, for $N > 2$, we have $\bar{\Gamma}(N) = \Gamma(N)$, which are infinite normal subgroup of $\bar{\Gamma}$, called principal congruence subgroups. The quotient groups defined as $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$ are finite modular groups. In this finite groups Γ_N, $T^N = \mathbb{I}$ is imposed. The groups Γ_N with $N = 2, 3, 4, 5$ are isomorphic to S_3, A_4, S_4 and A_5, respectively [20].

[1] Recently, the S_3 modular symmetry is also applied to the quark sector [31].

Modular forms of level N are holomorphic functions $f(\tau)$ transforming under the action of $\Gamma(N)$ as:

$$f(\gamma\tau) = (c\tau + d)^k f(\tau)\,, \quad \gamma \in \Gamma(N), \tag{5}$$

where k is the so-called as the modular weight.

Superstring theory on the torus T^2 or orbifold T^2/Z_N has the modular symmetry [32–37]. Its low-energy effective field theory is described in terms of supergravity theory, and string-derived supergravity theory has also the modular symmetry. Under the modular transformation of Eq. (1), chiral superfields $\phi^{(I)}$ transform as [38],

$$\phi^{(I)} \to (c\tau + d)^{-k_I} \rho^{(I)}(\gamma)\phi^{(I)}, \tag{6}$$

where $-k_I$ is the modular weight and $\rho^{(I)}(\gamma)$ denotes an unitary representation matrix of $\gamma \in \Gamma(N)$.

The kinetic terms of their scalar components are written by

$$\sum_I \frac{|\partial_\mu \phi^{(I)}|^2}{(-i\tau + i\bar{\tau})^{k_I}}\,, \tag{7}$$

which is invariant under the modular transformation. Here, we use the convention that the superfield and its scalar component are denoted by the same letter. Also, the superpotential should be invariant under the modular symmetry. That is, the superpotential should have vanishing modular weight in global supersymmetric models, while the superpotential in supergravity should be invariant under the modular symmetry up to the Kähler transformation. In the following sections, we study global supersymmetric models, e.g. minimal supersymmetric standard model (MSSM) and its extension with Higgs A_4 triplet. Thus, the superpotential has vanishing modular weight. The modular symmetry is broken by the vacuum expectation value of τ, i.e. at the compactification scale, which is of order of the planck scale or slightly lower scale.

For $\Gamma_3 \simeq A_4$, the dimension of the linear space $\mathcal{M}_k(\Gamma_3)$ of modular forms of weight k is $k + 1$ [39–41], i.e., there are three linearly independent modular forms of the lowest non-trivial weight 2. These forms have been explicitly obtained [19] in terms of the Dedekind eta-function $\eta(\tau)$:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)\,, \tag{8}$$

where $q = e^{2\pi i\tau}$ and $\eta(\tau)$ is a modular form of weight 1/2. In what follows we will use the following basis of the A_4 generators S and T in the triplet representation:

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{9}$$

where $\omega = e^{i\frac{2}{3}\pi}$. The modular forms of weight 2 $(Y_1(\tau), Y_2(\tau), Y_3(\tau))$ transforming as a triplet of A_4 can be written in terms of $\eta(\tau)$ and its derivative [19]:

$$\begin{aligned} Y_1(\tau) &= \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} \right. \\ &\qquad \left. + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right), \\ Y_2(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2 \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} \right), \\ Y_3(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2 \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} \right). \end{aligned} \tag{10}$$

The overall coefficient in Eq. (11) is one possible choice; it cannot be uniquely determined. The triplet modular forms of weight 2 have the following q-expansions:

$$Y = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1+12q+36q^2+12q^3+\dots \\ -6q^{1/3}(1+7q+8q^2+\dots) \\ -18q^{2/3}(1+2q+5q^2+\dots) \end{pmatrix}. \tag{11}$$

They satisfy also the constraint [19]:

$$(Y_2(\tau))^2 + 2Y_1(\tau)Y_3(\tau) = 0\,. \tag{12}$$

3. Quark mass matrices in the A_4 triplet Higgs model

Let us consider a A_4 modular invariant flavor model for quarks. In order to construct models with minimal number of parameters, we introduce no flavons. There are freedoms for the assignments of irreducible representations and modular weights to quarks and Higgs doublets. We take similar assignments of the left-handed quarks and right-handed one as seen in the charged lepton sector [25]: that is, three left-handed quark doublets are of a triplet of A_4, and (u^c, c^c, t^c) and (d^c, s^c, b^c) are of three different singlets $(\mathbf{1}, \mathbf{1}'', \mathbf{1}')$ of A_4, respectively. For both left-handed quarks and right-handed quarks, the modular weights are assigned to be -1, while the modular weight is 0 for Higgs doublets. Then, there appear three independent couplings in the superpotential of the up-quark sector and down-quark sector, respectively:

$$w_u = \alpha_u u^c H_u Y Q + \beta_u c^c H_u Y Q + \gamma_u t^c H_u Y Q\ , \tag{13}$$

$$w_d = \alpha_d d^c H_d Y Q + \beta_d s^c H_d Y Q + \gamma_d b^c H_d Y Q\ , \tag{14}$$

where Q is the left-handed A_4 triplet quarks, and H_q is the Higgs doublets. The parameters α_q, β_q, γ_q $(q = u, d)$ are constant coefficients. If the Higgs doublets H_q are singlet of A_4, the quark mass matrices are simple form. By using the decomposition of the A_4 tensor product in Appendix A, the superpotential in Eqs. (13) and (14) gives the mass matrix of quarks, which is written in terms of modular forms of weight 2:

$$M_q = \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL}\ , \qquad (q = u, d)\ , \tag{15}$$

where τ in the modular forms $Y_i(\tau)$ is omitted. Unknown couplings α_q, β_q, γ_q can be adjusted to the observed quark masses. The remained parameter is only the modulus, τ. The numerical study of the quark mass matrix in Eq. (15) is rather easy. However, it is very difficult to reproduce observed three CKM mixing angles by fixing one complex parameter τ because the CKM mixing angles are hierarchical ones and they have been precisely measured.

Therefore, we enlarge the Higgs sector. Let us consider the Higgs doublets to be one component of a A_4 triplet [42–46] for each up-quark and down-quark, respectively as follows: We introduce A_4 triplets Higgs H_u and H_d, which are gauge doublets, as follows:

$$H_u = \begin{pmatrix} H_{u1} \\ H_{u2} \\ H_{u3} \end{pmatrix}\ , \qquad H_d = \begin{pmatrix} H_{d1} \\ H_{d2} \\ H_{d3} \end{pmatrix}. \tag{16}$$

Including these A_4 triplet Higgs, we summarize the assignments of representations and modular weights $-k_I$ to the relevant fields in Table 1.

Now, the quark mass matrices are obtained by the tensor products among the A_4 singlet right-handed quarks, the A_4 triplet modular forms $Y(\tau)$, the A_4 triplet Higgs H_q and the A_4 triplet left-handed quarks Q. Since the tensor product of $3 \otimes 3$ is decomposed into the symmetric triplet and the antisymmetric triplet as seen in Appendix A, the A_4 invariant superpotential in Eq. (13) is expressed by introducing additional two parameters g_{u1} and g_{u2} as:

Table 1
The assignments of representations and modular weights $-k_I$ to the MSSM fields, where Higgs sector is extended to the non-trivial representation of A_4, **3**.

	Q	$(u^c(d^c), c^c(s^c), t^c(b^c))$	H_u	H_d	Y
$SU(2)$	**2**	**1**	**2**	**2**	**1**
A_4	**3**	$(\mathbf{1}, \mathbf{1}'', \mathbf{1}')$	**3**	**3**	**3**
$-k_I$	-1	$(-1, -1, -1)$	0	0	$k = 2$

$$w_u = (\alpha_u u^c(1) + \beta_u c^c(1'') + \gamma_u t^c(1')) \otimes \left[g_{u1} \begin{pmatrix} 2H_{u1}Y_1 - H_{u2}Y_3 - H_{u3}Y_2 \\ 2H_{u3}Y_3 - H_{u1}Y_2 - H_{u2}Y_1 \\ 2H_{u2}Y_2 - H_{u3}Y_1 - H_{u1}Y_3 \end{pmatrix} \oplus g_{u2} \begin{pmatrix} H_{u2}Y_3 - H_{u3}Y_2 \\ H_{u1}Y_2 - H_{u2}Y_1 \\ H_{u3}Y_1 - H_{u1}Y_3 \end{pmatrix} \right] \otimes \begin{pmatrix} u \\ c \\ t \end{pmatrix}, \tag{17}$$

where the neutral component of H_{qi} is taken, and the A_4 singlet component should be extracted in the tensor product. The up-quark mass matrix is given in terms of VEV's of H_{ui}, v_{ui} in Appendix C and modular forms Y_i $(i = 1, 2, 3)$ as follows:

$$M_u = \begin{pmatrix} \alpha_u & 0 & 0 \\ 0 & \beta_u & 0 \\ 0 & 0 & \gamma_u \end{pmatrix} \times \left[\frac{g_{u1}}{\sqrt{2}} \begin{pmatrix} 2v_{u1}Y_1 - v_{u2}Y_3 - v_{u3}Y_2 & 2v_{u2}Y_2 - v_{u3}Y_1 - v_{u1}Y_3 & 2v_{u3}Y_3 - v_{u1}Y_2 - v_{u2}Y_1 \\ 2v_{u3}Y_3 - v_{u1}Y_2 - v_{u2}Y_1 & 2v_{u1}Y_1 - v_{u2}Y_3 - v_{u3}Y_2 & 2v_{u2}Y_2 - v_{u3}Y_1 - v_{u1}Y_3 \\ 2v_{u2}Y_2 - v_{u3}Y_1 - v_{u1}Y_3 & 2v_{u3}Y_3 - v_{u1}Y_2 - v_{u2}Y_1 & 2v_{u1}Y_1 - v_{u2}Y_3 - v_{u3}Y_2 \end{pmatrix} + \frac{g_{u2}}{\sqrt{2}} \begin{pmatrix} v_{u2}Y_3 - v_{u3}Y_2 & v_{u3}Y_1 - v_{u1}Y_3 & v_{u1}Y_2 - v_{u2}Y_1 \\ v_{u1}Y_2 - v_{u2}Y_1 & v_{u2}Y_3 - v_{u3}Y_2 & v_{u3}Y_1 - v_{u1}Y_3 \\ v_{u3}Y_1 - v_{u1}Y_3 & v_{u1}Y_2 - v_{u2}Y_1 & v_{u2}Y_3 - v_{u3}Y_2 \end{pmatrix} \right], \tag{18}$$

where α_u, β_u, and γ_u are taken to be real positive by rephasing right-handed quark fields without loss of generality. The down-quark mass matrix is also given by replacing u with d in Eq. (18).

The vacuum structure of our model is determined by the scalar potential $V(H_u, H_d)$, which is presented in Appendix C. Since the modular forms Y_i's do not couple to the scalar potential due to the modular weight of 0 for the Higgs doublets, the vacuum structure of the scalar potential is independent of VEV of τ. Therefore, the scalar potential is similar to the one in MSSM. As discussed in the non-SUSY model with the A_4 triplet Higgs, there are some choices of v_q's to realize the vacuum [42–46], which is the global minimum.[2] In our work, we take the simplest one of $\langle H_q \rangle$ in our SUSY framework as follows:

$$\langle H_u \rangle = \frac{1}{\sqrt{2}}(v_{u1}, 0, 0)\ , \qquad \langle H_d \rangle = \frac{1}{\sqrt{2}}(v_{d1}, 0, 0)\ , \tag{19}$$

in the basis of S and T in Eq. (9). Here v_{u1} and v_{d1} are taken to be real and $v_{u1}^2 + v_{d1}^2 = 2v_H^2$ where $v_H = 174.1$ GeV. The vacuum

[2] Other different types of the global minima coexist and are degenerate. For example, $\langle H_d \rangle = \frac{1}{\sqrt{2}}(v_d, v_d, v_d)$ and $\langle H_u \rangle = \frac{1}{\sqrt{2}}(v_u, v_u, v_u)$ lead to the global minimum. Upon small variation of the parameters around this special point, one minimum point becomes the global minimum while the other turns into a local one, and it is clearly possible to make either of them the global minimum [43].

alignment in Eq. (19) easily realizes the minimum of the scalar potential by taking the condition

$$\frac{\partial V(H_u, H_d)}{\partial H_{qk}} = 0\,, \quad (q = u, d\,;\; k = 1, 2, 3)\,, \tag{20}$$

while the Hessian

$$\frac{\partial^2 V(H_u, H_d)}{\partial H_{qk}\partial H_{qj}}\,, \quad (q = u, d\,;\; k, j = 1, 2, 3)\,, \tag{21}$$

is required to have non-negative eigenvalues, which correspond to that all physical masses being positive except for vanishing masses of the Goldstone bosons as seen in Appendix C.

Indeed, we have checked numerically for $\tan\beta = v_{u1}/v_{d1} = 10$ that the extra scalars and pseudo-scalars could be $\mathcal{O}(10)$ TeV keeping the light SM Higgs mass. This situation is achieved due to some fine-tuning and rather large scalar self-couplings by taking account of the radiative corrections of SUSY and $\tilde{m}_{H_q} = \mathcal{O}(10)$ TeV, $B = \mathcal{O}(10)$ TeV and $\mu = \mathcal{O}(10)$ TeV. However, loop corrections to the scalar masses become important as shown in two Higgs doublet model [47,48]. Therefore, such high splittings of scalar masses should be carefully examined in the context of the phenomenology. Moreover, there could be unsuppressed flavor changing neutral current (FCNC) of quarks, which was discussed in the A_4 triplet Higgs model [42]. Indeed, the study of FCNC in Kaon and B meson systems is important. However, we do not discussed the phenomenology, which is out of scope in the present work.

In our model, only H_{u1} and H_{d1} have VEVs, therefore, it is easy to find that the couplings to the observed 125 GeV Higgs boson are expected to be proportional to quark masses. This situation is understandable since H_{q1} do not mix with H_{q2} and H_{q3} in the Higgs potential as seen in Appendix C. The electromagnetism is not broken: a minimum of the potential satisfying $\partial V/\partial H^{\pm}_{qk} = 0$ gives $\langle H^{\pm}_{qk}\rangle = 0$.

It is also noticed that the VEV in Eq. (19) has a residual Z_2 symmetry of A_4. However, this Z_2 symmetry of the Higgs sector is accidental since an obtained τ of our result breaks completely A_4 symmetry. The choice of $(v_q, 0, 0)$ should be considered to reduce the number of free parameters. Indeed, the numerical fit of experimental data of the CKM matrix is improved by using another alignment of $(v_q, v_q', 0)$, which has not the Z_2 symmetry.

Finally, we obtain the up-quark and down-quark mass matrices:

$$M_q = \frac{1}{\sqrt{2}} v_{q1}\, g_{q1} \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \times \begin{pmatrix} 2Y_1 & -(1+g_q)Y_3 & -(1-g_q)Y_2 \\ -(1-g_q)Y_2 & 2Y_1 & -(1+g_q)Y_3 \\ -(1+g_q)Y_3 & -(1-g_q)Y_2 & 2Y_1 \end{pmatrix}_{RL}, \quad (q = u, d), \tag{22}$$

where $g_q \equiv g_{q2}/g_{q1}$ $(q = u, d)$. There are six real parameters α_q, β_q, γ_q $(q = u, d)$, and the VEV of the modulus, τ. In addition, we have two complex parameters g_u and g_d. It is noted that the factor $v_{q1} g_{q1}$ in front of the right hand side of Eq. (22) is absorbed into α_q, β_q and γ_q. Thus, we have six real parameters and three complex ones. That is to say, there are twelve free real parameters in our mass matrices. It is also noticed that v_{q1} does not appear explicitly in our calculations because it is absorbed in α_q, β_q and γ_q. Therefore, our numerical result is independent of $\tan\beta = v_{u1}/v_{d1}$.

The quark mass matrix in Eq. (22) has a specific flavor structure due to the A_4 symmetry. It is easily found relations among matrix elements as follows:

$$\frac{M_q(1,1)}{M_q(2,2)} = \frac{M_q(1,2)}{M_q(2,3)} = \frac{M_q(1,3)}{M_q(2,1)}\,, \qquad \frac{M_q(2,2)}{M_q(3,3)} = \frac{M_q(2,1)}{M_q(3,2)} = \frac{M_q(2,3)}{M_q(3,1)}\,. \tag{23}$$

Moreover, a constraint among Y_1, Y_2 and Y_3 in Eq. (12) provide a relation

$$\frac{M_q(2,1)}{M_q(2,2)} = \frac{(g_q-1)^2}{g_q+1}\frac{M_q(3,1)}{M_q(3,2)}\,. \tag{24}$$

These relations correlate CKM mixing angles each other. Thus, the three CKM mixing angles are not independent in our quark mass matrix. Indeed, parameter region of τ, g_u and g_d are restricted to be in rather narrow regions in order to reproduce the three CKM mixing angles, as seen in numerical result. Then, the CP violating phase is predicted in the restricted region in spite of the excess of parameters compared with observed ones.

4. Numerical results

Let us begin with explaining how to get our prediction of the CP violation in terms of twelve real parameters. At first, we take a random point of τ and g_u, g_d, which are scanned in the complex plane by generating random numbers. The scanned ranges of $\mathrm{Im}[\tau]$ are $[0.5, 10]$, in which the lower-cut 0.5 comes from the accuracy of calculating modular functions, and the upper-cut 10 is enough large for estimating Y_i in practice. On the other hand, $\mathrm{Re}[\tau]$ is scanned in the fundamental region of $[-3/2, 3/2]$ in Eq. (11) because the modular function Y_i is given in terms of $\eta(\tau/3)$. We also scan in $|g_u| \in [0, 1000]$ and $|g_d| \in [0, 1000]$ while these phases are scanned in $[-\pi, \pi]$.

Then, parameters α_q, β_q, γ_q $(q = u, d)$ are determined by computing functions $C^q_i (i = 1-3)$ in Appendix B after inputting six quark masses (see Appendix B). We use the six quark masses at the M_Z scale [49].

Finally, we can calculate three CKM mixing angles in terms of the model parameters τ, g_u and g_d, while keeping the parameter sets leading to values allowed by the experimental data of the CKM mixing angles. We continue this procedure to obtain enough points for plotting allowed region.

We adopt the data of quark Yukawa couplings at the M_Z scale as input in order to constraint the model parameters [49]:

$$y_d = (1.58^{+0.23}_{-0.10}) \times 10^{-5}, \quad y_s = (3.12^{+0.17}_{-0.16}) \times 10^{-4},$$
$$y_b = (1.639 \pm 0.015) \times 10^{-2}, \quad y_u = (7.4^{+1.5}_{-3.0}) \times 10^{-6},$$
$$y_c = (3.60 \pm 0.11) \times 10^{-3}, \quad y_t = 0.9861^{+0.0086}_{-0.0087}\,, \tag{25}$$

which give quark masses as $m_q = y_q v_H$ with $v_H = 174.1$ GeV. We also take the absolute values of CKM elements V_{us}, V_{cb} and V_{ub} for input as follows [50]:

$$|V_{us}| = 0.2243 \pm 0.0005\,, \quad |V_{cb}| = 0.0422 \pm 0.0008\,,$$
$$|V_{ub}| = (3.94 \pm 0.36) \times 10^{-3}\,. \tag{26}$$

In Eqs. (25) and (26), the error-bars denote interval of 1σ, and 3σ error-bars are used as input.

The obtained parameter region of τ, g_u and g_d are as follows:

$$\mathrm{Re}[\tau] = -(1.49 - 1.50)\,,$$
$$\mathrm{Im}[\tau] = 2.01 - 2.02\,,$$
$$\mathrm{Re}[g_u] = 0.70 - 0.93\,,$$
$$\mathrm{Im}[g_u] = \pm(0.002 - 0.022)\,,$$

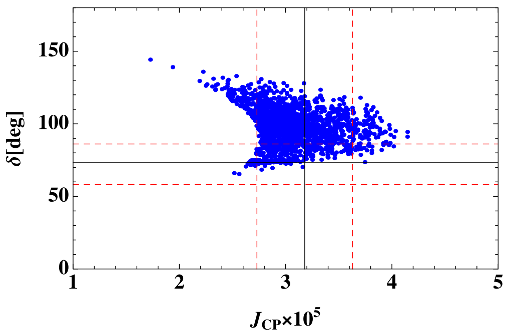

Fig. 1. Prediction of the magnitude of the CP violation on J_{CP}–δ plane, where black lines denote observed central values of J_{CP} and δ, and red dashed-lines denote their upper-bounds and lower-bounds of 3σ interval.

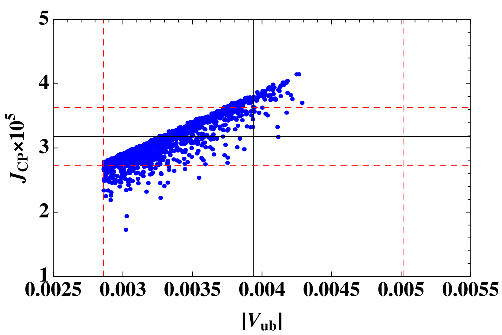

Fig. 2. Predicted J_{CP} versus $|V_{ub}|$, where black lines denote observed central values of $|V_{ub}|$ and J_{CP}, and red dashed-lines denote their upper-bounds and lower-bounds of 3σ interval.

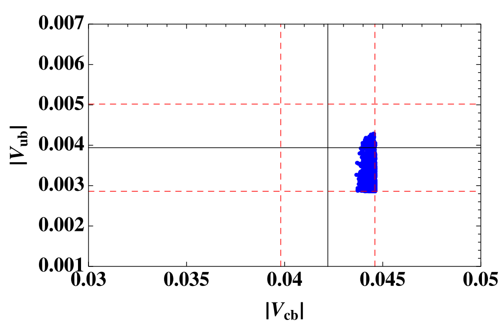

Fig. 3. The allowed region on $|V_{cb}|$–$|V_{ub}|$ plane. Notations are same in Figs. 1 and 2.

$$\mathrm{Re}[\frac{1}{g_d}] \simeq -(0.99 - 1.03) \times 10^{-3} ,$$
$$\mathrm{Im}[\frac{1}{g_d}] = -(0.052 - 0.108) , \qquad (27)$$

where the modulus τ is almost fixed. By using these values, we can predict the CP violation phase δ and the Jarlskog invariant J_{CP} [51]. Those are compared with the observed values at the electroweak scale [50]:

$$\delta = (73.5^{+4.2}_{-5.1})^\circ , \qquad J_{CP} = (3.18 \pm 0.15) \times 10^{-5} . \qquad (28)$$

Our predictions are presented in Figs. 1–3. We show the predicted CP violating phase δ versus J_{CP} in Fig. 1. Here, the observed CKM mixing elements $|V_{us}|$, $|V_{cb}|$ and $|V_{ub}|$ are input with 3σ error interval. The predicted ranges of δ and J_{CP} is (65°–140°) and (2–4)$\times 10^{-5}$, respectively. Those include the allowed regions of the experimental data in Eq. (28), which are denoted by red dashed-lines with 3σ error interval. The predicted region of δ is still broad. It is remarked that δ is more restricted if error-bars of inputting quark masses are reduced, especially, the s-quark mass and the c-quark mass are important to predict δ.

We show the $|V_{ub}|$ dependence of predicted J_{CP} in Fig. 2. Although observed $|V_{ub}|$ [0.0028, 0052] is input, our model does not allow the region larger than 0.0043. The $|V_{ub}|$ is cut below the lower-bound of experimental data. The predicted J_{CP} is approximately proportional to $|V_{ub}|$. The upper hard cut of J_{CP} is due to the maximal value of $\sin\delta = 1$.

In Fig. 3, we show the allowed region on $|V_{cb}|$–$|V_{ub}|$ plane. The $|V_{cb}|$ is restricted in the very narrow range, which is larger than 0.0436, close to the 3σ upper-bound of the observed one 0.0446. This prediction provides us a crucial test of our model.

We can also discuss the ratio of CKM matrix elements of V_{ub} and V_{cb}, which is in the range of $[0.065, 0.098]$ from Fig. 3. It should be compared with the observed values [52]:

$$\left|\frac{V_{ub}}{V_{cb}}\right| = 0.083 \pm 0.006 . \qquad (29)$$

Our prediction is inside of the observed 3σ interval in Eq. (29). This measurement was given in the semileptonic decays of Λ_b at LHCb. This prediction provides another complementary test of our model.

Finally, we show a typical set with twelve parameters as one sample, which gives us successful CKM parameters as well as J_{CP}:

$$\begin{aligned}
&\tau = -1.495 + i\,2.011 , \quad g_u = 0.918 + i\,0.0116 ,\\
&g_d = -980 - i\,18.9 ,\\
&\alpha_u/\gamma_u = 2.496 \times 10^{-5}, \quad \beta_u/\gamma_u = 5.995 \times 10^{-3},\\
&\alpha_d/\gamma_d = 2.855 \times 10^{-3},\\
&\beta_d/\gamma_d = 3.812 \times 10^{-2}, \quad \tilde{\gamma}_u \equiv \frac{1}{\sqrt{2}} v_u g_{u1} \gamma_u = 85.85\ \mathrm{GeV},\\
&\tilde{\gamma}_d \equiv \frac{1}{\sqrt{2}} v_d g_{d1} \gamma_d = 1.427\ \mathrm{GeV}.
\end{aligned} \qquad (30)$$

This set gives

$$\begin{aligned}
&|V_{us}| = 0.224 , \quad |V_{cb}| = 0.0443 , \quad |V_{ub}| = 3.20 \times 10^{-3} ,\\
&J_{CP} = 2.98 \times 10^{-5} , \quad \delta = 74.9^\circ ,
\end{aligned} \qquad (31)$$

which are remarkably consistent with the observed values. It is noticed that ratios of α_q/γ_q and β_q/γ_q $(q = u, d)$ in Eq. (30) correspond to the observed quark mass hierarchy.

In conclusion, our quark mass matrix with the A_4 modular symmetry can reproduce the CKM mixing matrix completely with observed quark masses.

5. Summary

We have discussed the quark mass matrices in the A_4 modular symmetry, where the A_4 triplet of Higgs doublets is introduced for each up-quark and down-quark sectors, respectively. The model has six real parameters and two complex parameters in addition to the modulus τ. Then, we have constrained the model parameters by inputting six quark masses and three CKM mixing angles at the electroweak scale. We have predicted the CP violation phase δ and the Jarlskog invariant J_{CP}.

The predicted ranges of δ and J_{CP} is (65°–140°) and (2–4)$\times 10^{-5}$, respectively. Those include the allowed regions of the experimental data. The absolute value of V_{ub} is smaller than 0.0043. The magnitude of V_{cb} is larger than 0.0436, which is close to the 3σ upper-bound of the observed one. Thus, our quark mass matrices with the A_4 modular symmetry can reproduce the CKM mixing matrix completely with observed quark masses.

Our mass matrices have been analyzed at the electroweak scale in this work. The renormalization-group evolution from the GUT scale to the electroweak scale have been examined in some textures of the quark mass matrix [53]. The textures of the quark mass matrix are essentially stable against the evolution. We expect that the conclusions derived in this paper do not change much even if we consider the mass matrix at the GUT scale.

We will also discuss the lepton mass matrices in the modular A_4 symmetry by introducing the A_4 triplet of Higgs doublets elsewhere.

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Appendix A. Multiplication rule of A_4 group

We take

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{32}$$

where $\omega = e^{i\frac{2}{3}\pi}$ for a triplet. In this base, the multiplication rule of the A_4 triplet is

$$\begin{aligned}
&\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} \\
&= (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}} \oplus (a_3b_3 + a_1b_2 + a_2b_1)_{\mathbf{1}'} \\
&\quad \oplus (a_2b_2 + a_1b_3 + a_3b_1)_{\mathbf{1}''} \\
&\quad \oplus \frac{1}{3}\begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}} \oplus \frac{1}{2}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix}_{\mathbf{3}}, \\
&\mathbf{1}\otimes\mathbf{1} = \mathbf{1}, \quad \mathbf{1}'\otimes\mathbf{1}' = \mathbf{1}'', \quad \mathbf{1}''\otimes\mathbf{1}'' = \mathbf{1}', \quad \mathbf{1}'\otimes\mathbf{1}'' = \mathbf{1}.
\end{aligned} \tag{33}$$

More details are shown in the review [2,3].

Appendix B. α_q/γ_q and β_q/γ_q in terms of quark masses

The coefficients α_q, β_q, and γ_q in Eq. (22) are taken to be real positive without loss of generality. These parameters are described in terms of the modulus τ and quark masses. The mass matrix is written as

$$M_q = \frac{1}{\sqrt{2}} v_q\, g_{q1}\gamma_q \begin{pmatrix} \hat{\alpha}_q & 0 & 0 \\ 0 & \hat{\beta}_q & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2Y_1 & -(1+g_q)Y_3 & -(1-g_q)Y_2 \\ -(1-g_q)Y_2 & 2Y_1 & -(1+g_q)Y_3 \\ -(1+g_q)Y_3 & -(1-g_q)Y_2 & 2Y_1 \end{pmatrix}_{RL}, \tag{34}$$

where $\hat{\alpha}_q \equiv \alpha_q/\gamma_q$ and $\hat{\beta}_q \equiv \beta_q/\gamma_q$. Then, we have three equations as:

$$\sum_{i=1}^{3} m_{q_i}^2 = \mathrm{Tr}[M_q^\dagger M_q] = \tilde{\gamma}_q^2 (1 + \hat{\alpha}_q^2 + \hat{\beta}_q^2)\, C_1^q, \tag{35}$$

$$\prod_{i=1}^{3} m_{q_i}^2 = \mathrm{Det}[M_q^\dagger M_q] = \tilde{\gamma}_q^6 \hat{\alpha}_q^2 \hat{\beta}_q^2\, C_2^q, \tag{36}$$

$$\chi = \frac{\mathrm{Tr}[M_q^\dagger M_q]^2 - \mathrm{Tr}[(M_q^\dagger M_q)^2]}{2} = \tilde{\gamma}_q^4 (\hat{\alpha}_q^2 + \hat{\alpha}_q^2\hat{\beta}_q^2 + \hat{\beta}_q^2)\, C_3^q, \tag{37}$$

where $\chi \equiv m_{q_1}^2 m_{q_2}^2 + m_{q_2}^2 m_{q_3}^2 + m_{q_3}^2 m_{q_1}^2$ and $\tilde{\gamma}_q = (v_q g_{q1}\gamma_q)/\sqrt{2}$. The coefficients C_1^q, C_2^q, and C_3^q depend only on Y_i and g_q, where Y_i's are determined if the value of modulus τ is fixed, and g_q is an arbitrary complex coefficient. Those are given explicitly as follows:

$$\begin{aligned}
C_1^q &= 4|Y_1|^2 + |g_q - 1|^2|Y_2|^2 + |g_q + 1|^2|Y_3|^2, \\
C_2^q &= 2\,\mathrm{Re}\Big[8Y_1^3 + (g_q - 1)^3 Y_2^3 - (g_q + 1)^3 Y_3^3 \\
&\quad + 6(g_q^2 - 1)Y_1Y_2Y_3\Big], \\
C_3^q &= 16|Y_1|^4 + |g_q - 1|^2|Y_2|^4 + |g_q + 1|^2|Y_3|^4 \\
&\quad + 4|g_q - 1|^2|Y_1Y_2|^2 + 4|g_q + 1|^2|Y_1Y_3|^2 \\
&\quad + |g_q^2 - 1|^2|Y_2Y_3|^2 + 4\,\mathrm{Re}\Big[(g_q - 1)^2(g_q^* + 1)Y_1^*Y_2^2Y_3^* \\
&\quad + 2(g_q^{*2} - 1)Y_1^2Y_2^*Y_3^* - (g_q + 1)^2(g_q^* - 1)Y_1^*Y_2^*Y_3^2\Big].
\end{aligned}$$

Then, we obtain two equations which describe $\hat{\alpha}$ and $\hat{\beta}$ as functions of quark masses, τ and g_q:

$$\frac{(1+s)(s+t)}{t} = \frac{(\sum m_i^2/C_1^q)(\chi/C_3^q)}{\prod m_i^2/C_2^q}, \qquad \frac{(1+s)^2}{s+t} = \frac{(\sum m_i^2/C_1^q)^2}{\chi/C_3^q}, \tag{38}$$

where we redefine the parameters $\hat{\alpha}_q^2 + \hat{\beta}_q^2 = s$ and $\hat{\alpha}_q^2\hat{\beta}_q^2 = t$. They are related as follows,

$$\hat{\alpha}_q^2 = \frac{s \pm \sqrt{s^2 - 4t}}{2}, \qquad \hat{\beta}_q^2 = \frac{s \mp \sqrt{s^2 - 4t}}{2}. \tag{39}$$

Appendix C. Scalar potential of A_4 triplet Higgs

The A_4 triplets Higgs, which are SU(2) gauge doublets, H_u and H_d are expressed as:

$$H_u = \begin{pmatrix} H_{u1} \\ H_{u2} \\ H_{u3} \end{pmatrix}, \qquad H_d = \begin{pmatrix} H_{d1} \\ H_{d2} \\ H_{d3} \end{pmatrix}. \tag{40}$$

Since each component is SU(2) doublet, it is written as:

$$H_{uk} = \begin{pmatrix} h_{uk}^+ \\ \frac{1}{\sqrt{2}}(v_{uk} + r_{uk} + iz_{uk}) \end{pmatrix}, \qquad H_{dk} = \begin{pmatrix} \frac{1}{\sqrt{2}}(v_{dk} + r_{dk} + iz_{dk}) \\ h_{dk}^- \end{pmatrix}, \tag{41}$$

where v_{uk} and v_{dk} are VEV's of H_{uk} and H_{dk}, respectively.

The A_4 invariant superpotential of Higgs sector is written by

$$w_H = \mu(H_{u_1}H_{d_1} + H_{u_2}H_{d_3} + H_{u_3}H_{d_2}). \tag{42}$$

The scalar potential of the D-term is given as

$$V_D = \frac{g_2^2}{8}(H_{u_1}^\dagger \sigma_a H_{u_1} + H_{u_2}^\dagger \sigma_a H_{u_3} + H_{u_3}^\dagger \sigma_a H_{u_2} + H_{d_1}^\dagger \sigma_a H_{d_1} + H_{d_2}^\dagger \sigma_a H_{d_3} + H_{d_3}^\dagger \sigma_a H_{d_2})^2 + \frac{g_Y^2}{8}(H_{u_1}^\dagger H_{u_1} + H_{u_2}^\dagger H_{u_3} + H_{u_3}^\dagger H_{u_2} - H_{d_1}^\dagger H_{d_1} - H_{d_2}^\dagger H_{d_3} - H_{d_3}^\dagger H_{d_2})^2 \,, \tag{43}$$

where g_2 and g_Y are gauge couplings of SU(2) and U(1), respectively, and σ_a (a=1-3) denote the Pauli matrix.

On the other hand, the soft breaking term under A_4 invariance is also given by

$$V_{soft} = \tilde{m}_{Hu}^2(H_{u_1}^\dagger H_{u_1} + H_{u_2}^\dagger H_{u_3} + H_{u_3}^\dagger H_{u_2}) + \tilde{m}_{Hd}^2(H_{d_1}^\dagger H_{d_1} + H_{d_2}^\dagger H_{d_3} + H_{d_3}^\dagger H_{d_2}) + B\mu(H_{u_1} i\sigma_2 H_{d_1} + H_{u_2} i\sigma_2 H_{d_3} + H_{u_3} i\sigma_2 H_{d_2} + \text{h.c.}). \tag{44}$$

The resulting Higgs potential is then given by:

$$\begin{aligned} V(H_u, H_d) = {} & m_{Hu}^2 H_{u_1}^\dagger H_{u_1} + |\mu|^2(|H_{u_2}|^2 + |H_{u_3}|^2) \\ & + \tilde{m}_{Hu}^2(H_{u_2}^\dagger H_{u_3} + H_{u_3}^\dagger H_{u_2}) \\ & + m_{Hd}^2 H_{d_1}^\dagger H_{d_1} + |\mu|^2(|H_{d_2}|^2 + |H_{d_3}|^2) \\ & + \tilde{m}_{Hd}^2(H_{d_2}^\dagger H_{d_3} + H_{d_3}^\dagger H_{d_2}) \\ & + \frac{g_2^2}{8}(H_{u_1}^\dagger \sigma_a H_{u_1} + H_{u_2}^\dagger \sigma_a H_{u_3} + H_{u_3}^\dagger \sigma_a H_{u_2} \\ & + H_{d_1}^\dagger \sigma_a H_{d_1} + H_{d_2}^\dagger \sigma_a H_{d_3} + H_{d_3}^\dagger \sigma_a H_{d_2})^2 \\ & + \frac{g_Y^2}{8}(H_{u_1}^\dagger H_{u_1} + H_{u_2}^\dagger H_{u_3} + H_{u_3}^\dagger H_{u_2} \\ & - H_{d_1}^\dagger H_{d_1} - H_{d_2}^\dagger H_{d_3} - H_{d_3}^\dagger H_{d_2})^2 \\ & + B\mu(H_{u_1} i\sigma_2 H_{d_1} + H_{u_2} i\sigma_2 H_{d_3} \\ & + H_{u_3} i\sigma_2 H_{d_2} + \text{h.c.}), \end{aligned} \tag{45}$$

where $m_u^2 \equiv |\mu|^2 + \tilde{m}_{Hu}^2$, $m_d^2 \equiv |\mu|^2 + \tilde{m}_{Hd}^2$.

We can study the minima in the potential $V(H_u, H_d)$ of Eq. (45) by taking the first derivative system

$$\frac{\partial V(H_u, H_d)}{\partial H_{qk}} = 0 \,, \quad (q = u, d \,;\ k = 1, 2, 3) \tag{46}$$

where H_{qk} is of the field h_{uk}^+, h_{dk}^-, r_{uk}, z_{uk}, r_{dk} and z_{dk}. Here, the Hessian

$$\frac{\partial^2 V(H_u, H_d)}{\partial H_{qk}\partial H_{qj}} \,, \quad (q = u, d \,;\ k, j = 1, 2, 3) \tag{47}$$

is required to have non-negative eigenvalues, which correspond to that all physical masses being positive except for vanishing masses of the Goldstone bosons.

Our Higgs potential analysis is same as in MSSM. Indeed, we have checked numerically by taking $\tan\beta = v_{u1}/v_{d1} = 10$ that the extra scalar and pseudo-scalar masses are larger than in $\mathcal{O}(1)$ TeV keeping the light SM Higgs mass.

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