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Vector bundles on fuzzy Kähler manifolds

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We propose a matrix regularization of vector bundles over a general closed Kähler manifold. This matrix regularization is given as a natural generalization of the Berezin–Toeplitz quantization and gives a map from sections of a vector bundle to matrices. We examine the asymptotic behaviors of the map in the large-N limit. For vector bundles with algebraic structure, we derive a beautiful correspondence of the algebra of sections and the algebra of corresponding matrices in the large-N limit. We give two explicit examples for monopole bundles over a complex projective space CP^n and a torus T^{2n}.

Subject Index B25

1. Introduction

The notion of noncommutative geometry appears in various studies of superstring theory and M-theory [1–3] and it suggests that noncommutative geometry might be suitable to describe space-time on the Planck scale rather than a smooth manifold. In noncommutative geometry, we consider the space-time coordinates as a set of noncommutative operators on some Hilbert space. A particular family of noncommutative geometry is called fuzzy geometry, which is the case when the Hilbert space is finite-dimensional and the space-time coordinates are finite-dimensional square matrices. This fuzzy geometry plays an important role in matrix models of superstring theory and M-theory.

In order to describe physics on such fuzzy geometry, it is necessary to formulate various fields on this geometry. For example, to describe the low-energy effective theories of D-branes, we need a fuzzy description of the field theories in the matrix models. For this purpose, it is important to find a description of a fuzzy version of vector bundles, since ordinary fields are described as sections of some vector bundles. The motivation of this paper is to generalize a matrix regularization [4], which is a map from functions on a smooth manifold to corresponding matrices on a fuzzy geometry. More specifically, we establish matrix regularization of vector bundles over a connected closed Kähler manifold.

Conventionally, the matrix regularization of functions on a closed symplectic manifold is described in the following manner. Let us consider a closed $2n$-dimensional symplectic manifold

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(M, ω). From the symplectic structure ω, one can define a volume form $\mu := \omega^{\wedge n}/n!$ and a Poisson bracket

$$\{f, g\} := W^{\mu\nu}\partial_\mu f \partial_\nu g, \tag{1}$$

where f, g are smooth functions and $W^{\mu\nu}$ is the Poisson tensor defined by $\omega_{\mu\nu}W^{\mu\rho} = \delta^\rho_\nu$. Let $\{N_p\}$ be a sequence of strictly increasing integers satisfying $N_p \to \infty$ as $p \to \infty$. The matrix regularization is defined as a sequence of linear maps $T_p : C^\infty(M) \to M_{N_p}(\mathbb{C})$ that satisfies [5]

$$\begin{aligned} &\lim_{p\to\infty} |T_p(f)T_p(g) - T_p(fg)| = 0, \\ &\lim_{p\to\infty} |i\hbar_p^{-1}[T_p(f), T_p(g)] - T_p(\{f, g\})| = 0, \\ &\lim_{p\to\infty} (2\pi\hbar_p)^n \mathrm{Tr}\, T_p(f) = \int_M \mu f. \end{aligned} \tag{2}$$

Here, $\hbar_p = (kp)^{-1}$ for some constant k and $|\cdot|$ is a matrix norm. These conditions can be seen as an analogue of the canonical quantization of classical mechanics where the phase space is $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$. These relations are essential in deriving the action of the matrix model from the worldvolume action of a membrane [4].

For a symplectic manifold M, it is known that there indeed exists a map T_p satisfying Eq. (2). A systematic and beautiful construction of such a map is given by the Berezin–Toeplitz quantization [6,7]. In this quantization, we first consider a suitable Dirac operator with N_p zero modes [7]. Then, one defines $T_p(f)$ by $T_p(f) = \Pi f \Pi$, where Π is the projection operator onto the Dirac zero modes. The map T_p, sometimes referred to as the Toeplitz operator, indeed satisfies all the properties of Eq. (2).

The Toeplitz operators for more general fields than functions were proposed in Refs. [8–12]. In more recent studies [13,14], it is shown that the Toeplitz operator of general fields on a closed Riemann surface enjoys beautiful properties, which are a natural generalization of Eq. (2).

In this paper, we investigate the Berezin–Toeplitz quantization of vector bundles over a general closed Kähler manifold. We show that the asymptotic properties of the Toeplitz operator given in Refs. [13,14] also exist in higher-dimensional manifolds. We derive a large-p asymptotic expansion of the product $T_p(\varphi)T_p(\chi)$ for arbitrary sections of vector bundles (general fields) φ, χ, up to the second order in $1/p$. From this asymptotic expansion, we obtain important relations of the Toeplitz operator including generalization of Eq. (2). We also give explicit examples of monopole bundles over a fuzzy CP^n [15,16] and fuzzy T^{2n} [11], where the Dirac operator zero modes have relatively simple representations.[1]

This paper is organized as follows. In Sect. 2, we propose the Berezin–Toeplitz quantization for general vector bundles and derive the asymptotic expansion. In Sects. 3 and 4, we consider the Berezin–Toeplitz quantization of monopole bundles over CP^n and T^{2n}, respectively. In Sect. 5, we give a summary and a discussion.

2. Berezin–Toeplitz quantization

In this section, we consider the Berezin–Toeplitz quantization for vector bundles and derive an asymptotic expansion of the quantization map. In Sect. 2.1, we define the Toeplitz operator for vector bundles. In Sect. 2.2, we derive the asymptotic behaviors of the Toeplitz operators. In

[1] See Ref. [17] for the analysis of Dirac operator zero modes of Riemann surfaces with higher genera, where the zero modes have more complex representations than those of $CP^1 = S^2$ and T^2.

Sect. 2.3, we show the relation between the trace of the Toeplitz operator and the integral of the corresponding field in the large-N limit. In Sect. 2.4, we construct the matrix Laplacian.

2.1 *Berezin–Toeplitz quantization for vector bundles*

We consider a closed connected $2n$-dimensional Kähler manifold M with a Kähler structure (g, J, ω), where g is a Riemannian metric, J is a complex structure, and ω is a symplectic form satisfying the compatibility condition:

$$\omega(\cdot, \cdot) = g(J\cdot, \cdot). \tag{3}$$

The Kähler potential K is a function defined by the local relation $\omega = i\partial\bar{\partial}K$ where ∂, $\bar{\partial}$ are Dolbeault differential operators. A natural volume form is defined by $\mu := \omega^{\wedge n}/n!$. In terms of the local real coordinates $\{x^\mu\}_{\mu=1}^{2n}$, we have $\mu = \sqrt{g}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{2n}$. To define the quantization map, we will introduce three Hermitian vector bundles L, S_c, and E. L is a prequantum line bundle, S_c is a spin-c bundle, and E is the target bundle that we want to quantize. L can be defined for a quantizable manifold, which we will discuss below, and S_c is known to exist for any Kähler manifold. For any vector bundle F, we will denote the connection and the curvature of F by $\nabla^F = d + A^F$ and $R^F := (\nabla^F)^2 = dA^F + A^F \wedge A^F$, respectively, where A^F is the connection one-form of F.

A prequantum line bundle L is a complex line bundle with a connection ∇^L such that its curvature (field strength) R^L is proportional to the symplectic form:

$$R^L = -ik\omega. \tag{4}$$

Here, the constant factor k is chosen such that $\frac{i}{2\pi}\int_\Sigma R^L \in \mathbb{Z}$, where $\Sigma \subseteq M$ is any two-cycle of M. This condition is equivalent to the condition that the symplectic form $\frac{k}{2\pi}\omega$ is in the second integer cohomology $H^2(M, \mathbb{Z})$. Manifolds that allow the existence of this prequantum line bundle are called quantizable manifolds. For a 2D manifold $M = \Sigma$, we can take $k = 2\pi/\int_M \omega$. The connection one-form A^L is defined by the local expression of the connection $\nabla^L = d + A^L$. Using the Kähler potential K, one can choose a connection one-form by

$$A^L = -\frac{k}{2}(\partial - \bar{\partial})K. \tag{5}$$

Let $\Gamma(\,\cdot\,)$ be a set of all the smooth sections of the vector bundle. Then, an element of $\Gamma(L)$ is a smooth complex scalar field coupling to a U(1) background gauge field A^L. For the 2D case, the curvature is proportional to the volume form, which means that sections of L are complex scalar fields coupling to uniform magnetic flux.

Next, we consider the spin-c structure (see Refs. [18,19] for a more rigorous mathematical treatment). The canonical spin-c bundle is defined by $S_c := \bigoplus_{p=0}^n \Lambda^{0,p}(T^*M)$; i.e., its fiber is a sum of $(0, p)$-forms. This bundle is formally equal to $S \otimes L_c^{1/2}$, where S is the canonical spin bundle and L_c is the determinant line bundle of the holomorphic tangent bundle $L_c := \det T^{(1,0)}M$. In the case of a nonspin manifold, S and the square root bundle $L_c^{1/2}$ themselves are not well defined and only the tensor product $S_c = S \otimes L_c^{1/2}$ is well defined.[2] A connection of S_c is locally given by $\nabla^{S_c} = d + A^S + \frac{1}{2}A^{L_c}$. The connection one-form of the canonical spin bundle S

[2]Precisely speaking, though both S and $L_c^{1/2}$ can be locally defined, the cocycle conditions of the transition functions are not satisfied for nonspin manifolds. However, the violations of the cocycle conditions cancel out for the formal tensor product $S_c = S \otimes L_c^{1/2}$, so that S_c is globally well defined. CP^{2m} $(m \in \mathbb{N})$ is an example of nonspin manifold with the spin-c structure.

is defined by

$$A^S = \frac{1}{4}\gamma^a_{(2n)}\gamma^b_{(2n)}\Omega_{ab} \tag{6}$$

where $\{\gamma^a_{(2n)}\}^{2n}_{a=1}$ is a set of gamma matrices satisfying the Clifford algebra $\{\gamma^a_{(2n)}, \gamma^b_{(2n)}\} = 2\delta_{ab}I_{2^n}$ discussed in Appendix A2 and $\Omega_{ab} = \Omega_{ab\mu}dx^\mu$ is the spin connection one-form

$$\Omega_{ab\mu} = e_a{}^\nu g_{\nu\rho}\left(\partial_\mu e_b{}^\rho + \Gamma^\rho_{\mu\sigma}e_b{}^\sigma\right). \tag{7}$$

Here, $\{e_a\}^{2n}_{a=1}$ is a set of the local orthonormal frame fields (vielbeins) satisfying $g(e_a, e_b) = \delta_{ab}$. The connection one-form of L_c is given by $A^{L_c} = -\sum_{m=1}^{n}\Omega_{m\bar{m}}$, where m and $\bar{m}$ are indices of complexified orthonormal frame vector fields introduced in Eq. (A1). We can interpret the sections of S_c as complex spinor fields coupling to $\frac{1}{2}A^{L_c}$.

Now, we consider the target bundle E. We assume that E is a finite-rank Hermitian vector bundle. We express E as a homomorphism bundle (Hom-bundle) $\mathrm{Hom}(E_2, E_1)$, where E_i $(i = 1, 2)$ are some Hermitian vector bundles. Here, $\mathrm{Hom}(E_2, E_1)$ is a vector bundle whose fiber at a point $x \in M$ is a vector space of linear maps from the fiber of E_2 at x to the fiber of E_1 at x. Note that any vector bundles can always be written as the Hom-bundle. The reason why we introduce the Hom-bundle is to introduce an algebraic structure that we will quantize. Namely, there is a natural product structure between $\Gamma(\mathrm{Hom}(E_2, E_1)) \times \Gamma(\mathrm{Hom}(E_3, E_2)) \to \Gamma(\mathrm{Hom}(E_3, E_1))$, following from the pointwise composition of the linear maps. This product is mapped to the product of matrices in the quantization that we discuss below.

The description using the Hom-bundle is applicable to most fields appearing in physics. For example, let $\tilde{L}$ be a complex line bundle with a connection one-form $A^{\tilde{L}}$. Then, $\tilde{L}^{\otimes q}$ can be written as $\mathrm{Hom}(\tilde{L}^{\otimes r}, \tilde{L}^{\otimes q+r})$ for any integers q, r. This means that a section of $\tilde{L}^{\otimes q}$, which is a complex scalar field coupling to $A^{\tilde{L}}$ with charge q, can also be regarded as a linear map from fields with charge r to those with charge $q + r$. Another example is that adjoint matter fields are regarded as linear maps from fundamental matter fields to themselves. Finally, tensor fields can also be viewed as linear maps between tensor fields with various ranks. For instance, a section of $\mathrm{Hom}((TM)^{\otimes r}, (TM)^{\otimes q})$ is a tensor field of (q, r) type:

$$(\varphi_1)^{\mu_1\mu_2\cdots\mu_q} = (\varphi)^{\mu_1\mu_2\cdots\mu_q}{}_{\nu_1\nu_2\cdots\nu_r}(\varphi_2)^{\nu_1\nu_2\cdots\nu_r}, \tag{8}$$

which corresponds to $(TM)^{\otimes q}\otimes(T^*M)^{\otimes r} \simeq \mathrm{Hom}((TM)^{\otimes r}, (TM)^{\otimes q})$.

As we have discussed, $\Gamma(\mathrm{Hom}(E_2, E_1))$ can be thought of a linear map $\Gamma(E_2) \to \Gamma(E_1)$. We can extend this linear structure to a map $\Gamma(S_c\otimes L^{\otimes p}\otimes E_2) \to \Gamma(S_c\otimes L^{\otimes p}\otimes E_1)$ by just acting as an identity on fibers of the auxiliary bundle $S_c\otimes L^{\otimes p}$ at each point $x \in M$. Here, p is an integer. Note that $\Gamma(S_c\otimes L^{\otimes p}\otimes E_i)$ are infinite-dimensional vector spaces. If we can restrict this linear map to be a map between finite-dimensional subspaces, such a map can be regarded as a finite-dimensional matrix. This is the main idea of the Berezin–Toeplitz quantization. In order to realize such a scenario, let us consider Dirac operators on $\Gamma(S_c\otimes L^{\otimes p}\otimes E_i)$ by

$$D_i = i\gamma^a_{(2n)}\nabla^{S_c\otimes L^{\otimes p}\otimes E_i}_{e_a} = ie_a{}^\mu\gamma^a_{(2n)}\left(\partial_\mu + \frac{1}{4}\Omega_{ab\mu}\gamma^a_{(2n)}\gamma^b_{(2n)} - \frac{1}{2}\sum_{m=1}^{n}\Omega_{m\bar{m}\mu} + pA^L_\mu + A^E_\mu\right). \tag{9}$$

We equip an inner product on $\Gamma(S_c\otimes L^{\otimes p}\otimes E_i)$ by

$$(\psi', \psi) := \int_M \mu\,(\psi')^\dagger \cdot \psi \quad (\psi, \psi' \in \Gamma(S_c \otimes L^{\otimes p} \otimes E_i)) \tag{10}$$

where $(\psi')^\dagger \cdot \psi$ is a Hermitian inner product of a fiber $S_c\otimes L^{\otimes p}\otimes E_i$ at point $x \in M$, which is defined by a combination of Hermitian metrics of S_c, L, and E_i. In the language of physicists, † and · simply mean the Hermitian conjugation and the contractions of indices, respectively. The norm is defined by $|\psi| = \sqrt{(\psi, \psi)}$. The space of normalizable zero modes Ker D_i is finite-dimensional. With the particular choice of the gamma matrices in Appendix A2, one can compute its dimension $N_i := \dim \mathrm{Ker} D_i$ for sufficiently large p using the Atiyah–Singer index theorem and the vanishing theorem as shown in Appendix A3. Here, p controls the dimension N_i, where N_i plays the role of the matrix size of the matrix regularization map. Now, let Π_i be a projection from $\Gamma(S_c \otimes L^{\otimes p} \otimes E_i)$ to $\mathrm{Ker} D_i$. We define the Berezin–Toeplitz quantization of $\Gamma(\mathrm{Hom}(E_2, E_1))$ by

$$T_p^{(E_1,E_2)}(\varphi) = \Pi_1 \varphi \Pi_2 \quad (\varphi \in \Gamma(\mathrm{Hom}(E_2, E_1))). \tag{11}$$

Here, $T_p^{(E_1,E_2)}(\varphi)$ is a map from $\mathrm{Ker}D_2$ to $\mathrm{Ker}D_1$ and therefore it can be represented as an $N_1 \times N_2$ matrix. As we will see below, the Toeplitz operator (11) enjoys a nice asymptotic behavior, which gives a generalization of Eq. (2).

2.2 *Asymptotic expansion of Toeplitz operators*

We can also consider another bundle $\mathrm{Hom}(E_3, E_2)$ and define a Toeplitz operator $T_p^{(E_2,E_3)}(\chi) = \Pi_2 \chi \Pi_3$ for $\chi \in \Gamma(\mathrm{Hom}(E_3, E_2))$. Then, we can consider a product $T_p^{(E_1,E_2)}(\varphi) T_p^{(E_2,E_3)}(\chi)$. As shown in Appendix A4, the Toeplitz operator (11) has the following asymptotic expansion in $\hbar_p = (kp)^{-1}$:

$$T_p^{(E_1,E_2)}(\varphi) T_p^{(E_2,E_3)}(\chi) = \sum_{i=0}^{\infty} \hbar_p^i T_p^{(E_1,E_3)}(C_i(\varphi, \chi)), \tag{12}$$

where the symbols C_i on the right-hand side are maps from $\Gamma(\mathrm{Hom}(E_2, E_1)) \times \Gamma(\mathrm{Hom}(E_3, E_2))$ to $\Gamma(\mathrm{Hom}(E_3, E_1))$. We find that the first three C_i are explicitly given by

$$\begin{aligned} C_0(\varphi, \chi) &= \varphi \chi, \\ C_1(\varphi, \chi) &= -\frac{1}{2} G^{\alpha\beta} (\nabla_\alpha \varphi)(\nabla_\beta \chi), \\ C_2(\varphi, \chi) &= \frac{1}{8} G^{\alpha\beta} G^{\gamma\delta} \left[(\nabla_\alpha \varphi) \left(i R_{\beta\gamma\mu\nu} W^{\mu\nu} - 2 R^{E_2}_{\beta\gamma} \right) (\nabla_\delta \chi) + (\nabla_\alpha \nabla_\gamma \varphi)(\nabla_\beta \nabla_\delta \chi) \right]. \end{aligned} \tag{13}$$

Here, we have introduced a tensor $G^{\alpha\beta} := g^{\alpha\beta} + i W^{\alpha\beta}$, where $g^{\alpha\beta}$ is the inverse of the metric tensor and $W^{\alpha\beta}$ is a Poisson tensor defined by $\omega_{\mu\nu} W^{\mu\rho} = \delta_\nu^\rho$. In Eq. (13), $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature tensor for the metric g and $R^{E_2}_{\alpha\beta} := R^{E_2}(\partial_\alpha, \partial_\beta)$ is a component of the curvature of E_2. The operator ∇_α is the covariant derivative on each field. For example, it acts on $\varphi \in \Gamma(\mathrm{Hom}(E_2, E_1))$ as $\nabla_\alpha \varphi = \partial_\alpha \varphi + A_\alpha^{E_1} \varphi - \varphi A_\alpha^{E_2}$, where A^{E_i} is a connection one-form of E_i. In Appendix A5, we check that Eq. (13) is consistent with the associativity of the operator product.

We leave the proof of Eq. (12) to Appendix A4 and discuss here some important corollaries of Eq. (13). From Eq. (13), it is easy to show the following relation:

$$\lim_{p\to\infty} \left| T_p^{(E_1,E_2)}(\varphi) T_p^{(E_2,E_3)}(\chi) - T_p^{(E_1,E_3)}(\varphi\chi) \right| = 0. \tag{14}$$

Moreover, let us consider a function $f \in C^\infty(M)$ and identity operator $\mathbf{1}_{E_i} \in \Gamma(\mathrm{End}(E_i))$. Then, we can consider the following commutator-like operation:

$$[T(f\mathbf{1}), T_p^{(E_1,E_2)}(\varphi)] := T_p^{(E_1,E_1)}(f\mathbf{1}_{E_1}) T_p^{(E_1,E_2)}(\varphi) - T_p^{(E_1,E_2)}(\varphi) T_p^{(E_2,E_2)}(f\mathbf{1}_{E_2}). \tag{15}$$

Using the asymptotic expansion to Eq. (15), one finds

$$\lim_{p\to\infty} \left| i\hbar_p^{-1} \left[T(f\mathbf{1}), T_p^{(E_1,E_2)}(\varphi) \right] - T_p^{(E_1,E_2)}(\{f, \varphi\}) \right| = 0, \tag{16}$$

where the generalized (covariantized) Poisson bracket { , } is defined by

$$\{f, \varphi\} := W^{\alpha\beta}(\partial_\alpha f)(\nabla_\beta \varphi). \tag{17}$$

From this correspondence, one can express the covariant derivative on φ by this commutator-like operation in matrix models.

For the trivial line bundle $E_i = M \times \mathbb{C}$, i.e., for ordinary complex valued functions and for simple pointwise products, the relations (14) and (16) reduce to the first two in Eq. (2).

2.3 *Trace of the Toeplitz operator*

Let us consider the case for an endomorphism bundle $\mathrm{End}(E_1) = \mathrm{Hom}(E_1, E_1)$. Then, we can consider the Toeplitz operator of $\varphi \in \Gamma(\mathrm{End}(E_1))$ given by

$$T_p^{(E_1,E_1)}(\varphi) = \Pi_1 \varphi \Pi_1. \tag{18}$$

In this case, we can define a trace of the Toeplitz operator. As shown in Appendix A6, we obtained the following property:

$$\lim_{p\to\infty} (2\pi\hbar_p)^n \mathrm{Tr}\, T_p^{(E_1,E_1)}(\varphi) = \int_M \mu\, \mathrm{tr}_{E_1}\varphi. \tag{19}$$

Here, tr_{E_1} is a trace in terms of vector space of the fiber of E_1. This result is a generalization of the third equation in Eq. (2).

2.4 *Bochner Laplacian and its matrix regularization*

Let E be a Hermitian vector bundle over M and let $\nabla^E\colon \Gamma(E) \to \Gamma(E\otimes T^*M)$ be a Hermitian connection of E. Let us also consider the adjoint of the connection $(\nabla^E)^*\colon \Gamma(E\otimes T^*M) \to \Gamma(E)$. Then, the Bochner Laplacian Δ^E is defined by

$$\Delta^E \varphi := (\nabla^E)^* \nabla^E \varphi. \tag{20}$$

In terms of the local coordinate, we write

$$\Delta^E \varphi = -g^{\mu\nu}\nabla_\mu \nabla_\nu \varphi, \tag{21}$$

where the first covariant derivative is simply equal to $\nabla_\nu\varphi = (\partial_\nu + A^E_\nu)\varphi$ but the second covariant derivative acts on $\nabla_\nu\varphi$ as $\nabla_\mu\nabla_\nu\varphi = (\partial_\mu + A^E_\mu)\nabla_\nu\varphi - \Gamma^\rho_{\mu\nu}\nabla_\rho\varphi$. If a section of E has an additional orthonormal index, the covariant derivative is assumed to be $\nabla_\mu\varphi_a = (\partial_\mu + A^E_\mu)\varphi_a + \Omega_{ab\mu}\varphi_b$. In this notation, we have $\nabla_\mu e_a^\nu = 0$ and $\nabla_\mu \gamma^a_{(2n)} = 0$. Also, let us introduce $\nabla_a := \nabla_{e_a} = e_a{}^\mu\nabla_\mu$. Then, we have useful identities $\Delta^E = -\nabla_a\nabla_a$ and $[\nabla_a, \nabla_b]\varphi = R^E(e_a, e_b)\varphi$.[3]

In order to construct the matrix Laplacian, let us consider the following trick. Let $\{X^A\}_{A=1,2,\ldots,d}$ be isometric embedding coordinate functions satisfying

$$(\partial_\mu X^A)(\partial_\nu X^A) = g_{\mu\nu}, \tag{22}$$

where the existence of such an embedding is ensured by the Nash embedding theorem for sufficiently large d. As shown in Appendix A7, the Laplacian can be written by using the isometric embedding functions and covariant Poisson bracket:

$$\Delta^E\varphi = -\{X^A, \{X^A, \varphi\}\}. \tag{23}$$

This expression is given in terms of the generalized Poisson bracket; it is easy to find the corresponding matrix Laplacian.

[3]There is also another expression $\Delta^E = -(\nabla^E_{e_a})^2 + \nabla^E_{\nabla^{TM}_{e_a} e_a}$ and $([\nabla^E_{e_a}, \nabla^E_{e_b}] - \nabla^E_{[e_a,e_b]})\varphi = R^E(e_a, e_b)\varphi$, which we can find in the mathematical literature.

From Eq. (23), it is natural to define the matrix Laplacian $\hat{\Delta}$ by

$$\hat{\Delta} T_p^{(E_1,E_2)}(\varphi) := \hbar_p^{-2} \left[T(X^A \mathbf{1}), \left[T(X^A \mathbf{1}), T_p^{(E_1,E_2)}(\varphi) \right] \right], \tag{24}$$

for $\varphi \in \Gamma(\mathrm{Hom}(E_2, E_1))$. Here, [,] is the generalized commutator defined in Eq. (15). We can see that $\hat{\Delta}$ is a Hermitian operator that is positive semidefinite in terms of the Frobenius inner product. In Ref. [13], it is shown that the spectra of the Bochner and the matrix Laplacians agree in the large-p limit.[4]

3. Fuzzy CP^n

In this section, we consider a Berezin–Toeplitz quantization of a monopole bundle over a complex projective space CP^n. Other constructions of such quantization maps are given in Refs. [15,16]. In Sect. 3.1, we define a complex projective space CP^n and describe its basic properties. In Sect. 3.2, we explicitly construct a complete orthonormal basis of the kernel of the Dirac operator. In Sect. 3.3, we calculate Toeplitz operators of embedding functions. In Sects. 3.4 and 3.5, we discuss the continuum Laplacian and the matrix Laplacian, respectively, for a monopole bundle.[5]

3.1 *Geometry of CP^n*

Firstly, let us define CP^n, which is a closed connected $2n$-dimensional Kähler manifold. For $Z, Z' \in \mathbb{C}^{n+1} \setminus \{0\}$, we will define a equivalence relation $\sim$ by

$$Z \sim Z' \quad :\Leftrightarrow \quad {}^{\exists} c \in \mathbb{C} \setminus \{0\} : \ Z = cZ'. \tag{25}$$

Then, CP^n is defined by

$$CP^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \sim . \tag{26}$$

This space can be covered by a set of $n+1$ patches $\{U_\alpha\}_{\alpha=1}^{n+1}$ where $U_\alpha := \{[Z] \in CP^n | Z^\alpha \neq 0\}$. Here, $[Z] = [Z^1, Z^2, \ldots, Z^{n+1}]$ is a representative class with respect to the relation $\sim$ and is called the homogeneous coordinates. For a patch U_α, one can define the inhomogeneous coordinates $(z^1_{(\alpha)}, z^2_{(\alpha)}, \cdots, z^n_{(\alpha)})$ such that

$$z^\mu_{(\alpha)} = \begin{cases} Z^\mu / Z^\alpha & (\mu = 1, 2, \ldots, \alpha - 1) \\ Z^{\mu+1} / Z^\alpha & (\mu = \alpha, \alpha + 1, \ldots, n) \end{cases}. \tag{27}$$

In order to define a Kähler structure of CP^n, let us consider a local function K_α on a patch U_α as

$$K_\alpha(p) := \log\left(1 + \sum_{\mu=1}^{n} |z^\mu_{(\alpha)}(p)|^2 \right) = \log\left(\sum_{\mu=1}^{n+1} |Z^\mu / Z^\alpha|^2 \right). \tag{28}$$

For $x \in U_\alpha \cap U_\beta$, we have

$$K_\alpha(x) = K_\beta(x) + \log\left(Z^\beta / Z^\alpha \right) + \log\left(\overline{Z^\beta / Z^\alpha} \right). \tag{29}$$

[4] This is explicitly shown for the case dim $M = 2$ [13] and the proof can be easily generalized in the case of the general Kähler manifold that we are considering in this paper.

[5] The correspondence of matrices and (charged) fields was studied in Refs. [15,16], where they use the projective module construction and the Fock space construction. In particular, the correspondence of Laplacians is extensively studied in Ref. [16]. In our formalism, the underlying mechanism of these correspondences is revealed based on the asymptotic expansion of the Toeplitz operators. Furthermore, our formalism can be applied to any general Kähler manifolds and any vector bundles.

By acting the Dolbeault differentials $\partial, \bar{\partial}$, we have $\partial\bar{\partial}K_\alpha = \partial\bar{\partial}K_\beta$. Thus, we can define a closed two-form ω locally written as

$$\omega = i\partial\bar{\partial}K. \tag{30}$$

From now on, we will omit the subscripts of the patch. By using the local complex coordinates z^μ, ω is written as

$$\omega = i\frac{(1+|z|^2)\delta_{\mu\nu} - \bar{z}^\mu z^\nu}{(1+|z|^2)^2} dz^\mu \wedge d\bar{z}^\nu. \tag{31}$$

Here and hereafter, the Einstein sum convention is assumed. Also, we have defined $|z|^2 := z^\mu \bar{z}^\mu$. We can see that ω is a nondegenerate form. Thus, ω is a symplectic structure on CP^n and the local function $K = \log(1+|z|^2)$ satisfying Eq. (30) is called the Kähler potential. We now define a standard almost complex structure J by $J(\partial_\mu) = i\partial_\mu,\ J(\partial_{\bar{\mu}}) = -i\partial_{\bar{\mu}}$, where $\partial_\mu = \partial/\partial z^\mu$ and $\partial_{\bar{\mu}} = \partial/\partial\bar{z}^\mu$. Then, the compatible metric $g(\,\cdot,\cdot) := \omega(\,\cdot, J\,\cdot\,)$ is of the form

$$g = g_{\mu\bar{\nu}} dz^\mu \otimes d\bar{z}^\nu + g_{\bar{\nu}\mu} d\bar{z}^\nu \otimes dz^\mu. \tag{32}$$

The components of the metric are given by

$$g_{\mu\bar{\nu}} = g_{\bar{\nu}\mu} = \frac{(1+|z|^2)\delta_{\mu\nu} - \bar{z}^\mu z^\nu}{(1+|z|^2)^2}. \tag{33}$$

This metric is called the Fubini–Study metric. The volume element is given by

$$\sqrt{\det g} = (1+|z|^2)^{-n-1} \tag{34}$$

and the inverse metric is given by

$$g^{\mu\bar{\nu}} = g^{\bar{\nu}\mu} = (1+|z|^2)(\delta_{\mu\nu} + z^\mu \bar{z}^\nu). \tag{35}$$

The triple (ω, g, J) gives the Kähler structure of CP^n.

Let us discuss the isometric embedding of CP^n into $\mathbb{R}^{n^2+2n}$. Let us consider a particular representative of the homogeneous coordinate $\zeta = (\zeta^1, \zeta^2, \ldots, \zeta^{n+1})$ such that $|\zeta|^2 = 1$. On the patch U_{n+1}, for instance, it is related to the inhomogeneous coordinate z by

$$\zeta = \frac{(z^1, z^2, \ldots, z^n, 1)^{\mathrm{T}}}{\sqrt{1+|z|^2}} \in \mathbb{C}^{n+1}, \tag{36}$$

where we fix the phase of ζ so that ζ^{n+1} is a positive real number. The rank-1 Hermitian projection $P_\zeta = \zeta\zeta^\dagger$ can be expanded as

$$P_\zeta = \frac{1}{n+1} I_{n+1} - \sqrt{2} X^A T_A. \tag{37}$$

Here, $\{T_A\}_{I=1}^{n^2+2n}$ are Hermitian generators of SU$(n+1)$ in a fundamental representation satisfying

$$T_A T_B = \frac{1}{2(n+1)} \delta_{AB} I_{n+1} + \frac{1}{2}(d_{ABC} + i f_{ABC}) T_C. \tag{38}$$

d_{ABC} and f_{ABC} are completely symmetric and antisymmetric structure constants, respectively. From the fact that P_ζ is a projector, the real coefficients $\{X^A\}_{A=1}^{n^2+2n}$ satisfy

$$X^A X^A = \frac{n}{n+1}, \quad d_{ABC} X^A X^B + \sqrt{2}\left(\frac{n-1}{n+1}\right) X^C = 0. \tag{39}$$

A straightforward calculation shows that the Fubini–Study metric (32) can be written as

$$ds^2 = \mathrm{tr}(dP_\zeta dP_\zeta) = 2\mathrm{tr}(dX^A T_A dX^B T_B) = dX^A dX^A. \tag{40}$$

Therefore, $\{X^A\}_{A=1}^{n^2+2n}$ are isometric embedding functions. X^A can also be written as

$$X^A = -\sqrt{2}\zeta^\dagger T_A \zeta. \tag{41}$$

Let us consider an action

$$\zeta \mapsto U\zeta, \tag{42}$$

for $U \in \mathrm{SU}(n+1)$. This transformation leaves the metric invariant and hence is an isometry of CP^n. Since T_A is an invariant tensor of $\mathrm{SU}(n+1)$, the embedding functions X^A transform as the adjoint representation of $\mathrm{SU}(n+1)$.

Finally, let us consider the prequantum line bundle over CP^n. One can construct L as a dual bundle of the tautological line bundle over CP^n. The curvature of L is Eq. (4) with $k = 1$. One can check that the integral of $iR^L/2\pi$ over any two-cycle is equal to 1 as follows. Since the rank of the second homology group of CP^n is 1, there is only one independent two-cycle. Let us take a particular two-cycle $CP^1 = \{[Z^1, Z^2, 0,\ldots,0]\} \subset CP^n$. The symplectic form in this two-cycle is $\omega = i\frac{dz\wedge d\bar{z}}{1+|z|^2}$, where $z = Z^1/Z^2$. Then, it is easy to show that

$$\frac{i}{2\pi}\int_{CP^1} R^L = \frac{1}{2\pi}\int_{CP^1}\omega = 1. \tag{43}$$

3.2 *Zero modes of the Dirac operator on CP^n*

In this subsection, we construct a complete orthonormal basis of the Dirac zero modes on CP^n.

Let $D^{(p)}$ be a twisted Dirac operator on $\Gamma(S_c \otimes L^{\otimes p})$. We take a specific representation of the gamma matrices given in Eq. (A6). As shown in Appendix A3, the Dirac operator zero mode $\psi^{(p)} \in \Gamma(S_c \otimes L^{\otimes p})$ has only one spinor component $\psi^{(p)} = f^{(p)}\,|+\rangle^{\otimes n}$. Here, $f^{(p)} \in \Gamma(L^{\otimes p})$ and $|+\rangle$ is a 2D spinor $(1, 0)^{\mathrm{T}}$. As shown in Appendix A8, the zero-mode equation $D^{(p)}\psi^{(p)} = 0$ is simplified to

$$\left(\partial_{\bar{\mu}} + pA^L_{\bar{\mu}}\right) f^{(p)} = 0. \tag{44}$$

Plugging $K = \log(1 + |z|^2)$ and $k = 1$ into Eq. (5), one finds

$$pA^L_{\bar{\mu}} = \frac{pz^\mu}{2(1+|z|^2)}. \tag{45}$$

Thus, the zero-mode equation becomes

$$\left(\partial_{\bar{\mu}} + \frac{pz^\mu}{2(1+|z|^2)}\right) f^{(p)} = 0, \tag{46}$$

and the general solution to this equation is

$$f^{(p)} = (1+|z|^2)^{-p/2}\phi(z), \tag{47}$$

where $\phi(z)$ is an arbitrary holomorphic function.

Now, let us consider the norm of the zero modes. Since any holomorphic function can be expanded in Taylor series around $z = 0$, let us consider a function $\phi_s(z) := (z^1)^{s_1}(z^2)^{s_2}\cdots(z^n)^{s_n}$, where $s = (s_1, s_2, \ldots, s_n) \in (\mathbb{Z}_{\geq 0})^n$, and check whether the zero mode $\psi_s^{(p)} = (1+|z|^2)^{-p/2}\phi_s\,|+\rangle^{\otimes n}$ is normalizable or not. In Appendix B1, we show that the norm

$$\left|\psi_s^{(p)}\right|^2 = \int_{CP^n}\mu\,\frac{|z^1|^{2s_1}|z^2|^{2s_2}\cdots|z^n|^{2s_n}}{(1+|z|^2)^p} \tag{48}$$

is convergent if and only if $\sum_{i=1}^n s_i < p+1$ is satisfied. It is shown in Appendix B1 that a complete orthonormal basis of $\mathrm{Ker}\,D^{(p)}$ can be chosen as

$$\psi_s^{(p)} = (I_{s,p})^{-1/2}(1+|z|^2)^{-p/2}(z^1)^{s_1}(z^2)^{s_2}\cdots(z^n)^{s_n}\,|+\rangle^{\otimes n},$$
$$^\forall i \in \{1, 2, \ldots, n\}:\ s_i \in \mathbb{Z}_{\geq 0}\quad \text{s.t.}\quad \sum_{i=1}^n s_i \leq p, \tag{49}$$

where $I_{s,\,p}$ is given in Eq. (B2).

There is another expression of Eq. (49) in terms of the normalized inhomogeneous coordinate ζ given in Eq. (36). The orthonormal basis (49) can be written as

$$\psi^{(p)}_{\boldsymbol{\alpha}_p} = c^{(p)}_{\boldsymbol{\alpha}_p} f^{(p)}_{\boldsymbol{\alpha}_p} \left|+\right\rangle^{\otimes n}, \tag{50}$$

where

$$f^{(p)}_{\boldsymbol{\alpha}_p} = \zeta^{\alpha_1}\zeta^{\alpha_2}\cdots\zeta^{\alpha_p}, \tag{51}$$

and the collective index $\boldsymbol{\alpha}_p = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ is an element of

$$\Sigma_p = \{1, 2, \ldots, n+1\}^p/\text{permutation}. \tag{52}$$

The normalization factor $c^{(p)}_{\boldsymbol{\alpha}_p}$ is given by

$$c^{(p)}_{\boldsymbol{\alpha}_p} = \sqrt{\frac{(p+n)!}{(2\pi)^n \prod_{i=1}^{n+1} n_i(\boldsymbol{\alpha}_p)!}}, \tag{53}$$

where $n_i(\boldsymbol{\alpha}_p)$ is the number of components of $\boldsymbol{\alpha}_p$ equal to i.

The dimension of Ker $D^{(p)}$ is

$$\dim \operatorname{Ker} D^{(p)} = \frac{(n+p)!}{n!p!}, \tag{54}$$

which is the number of independent symmetric polynomials of degree p with n variables. Equation (54) can also be understood from the representation theory of $\mathfrak{su}(n+1)$. Let $V_{(d_1,d_2,\ldots,d_n)}$ be an irreducible representation of $\mathfrak{su}(n+1)$ with Dynkin index $(d_1, d_2,\ldots,d_n)$. From Eq. (42), one can see that ζ is in the representation space $V_{(1,\,0,\ldots,0)}$, which implies that the set of all symmetric polynomials of ζ^i of degree p is isomorphic to the representation space $V_{(p,\,0,\ldots,0)}$. Thus, we have

$$\operatorname{Ker} D^{(p)} = V_{(p,0,\ldots,0)}. \tag{55}$$

According to the hook length formula, the dimension of $V_{(p,\,0,\ldots,0)}$ is indeed equal to $\frac{(n+p)!}{n!p!}$. This viewpoint in terms of representation theory will also play a very important role in the following discussions.

As calculated in Ref. [20], one can also obtain dim Ker $D^{(p)}$ from the index theorem. Since the vanishing theorem holds, we have $\dim \operatorname{Ker} D^{(p)} = \operatorname{Ind} D^{(p)}$. Then, from the index theorem, we obtain

$$\dim \operatorname{Ker} D^{(p)} = \int_{CP^n} \operatorname{Td}(T^{(1,0)}CP^n) \wedge \operatorname{ch}(L^{\otimes p}), \tag{56}$$

where Td and ch stand for the Todd class and Chern character, respectively. For CP^n, we have[6]

$$\operatorname{Td}(T^{(1,0)}CP^n) = \left(\frac{\omega/2\pi}{1-e^{-\omega/2\pi}}\right)^{n+1}, \quad \operatorname{ch}(L^{\otimes p}) = e^{p\omega/2\pi}. \tag{57}$$

The coefficient of the term proportional to $(\omega/2\pi)^{\wedge n}$ in the integrand of Eq. (56) can be evaluated using the residue theorem:

$$C_{p,n} := \frac{1}{2\pi i}\oint \frac{dz}{z^{n+1}}\left(\frac{z}{1-e^{-z}}\right)^{n+1} e^{pz} = \frac{1}{2\pi i}\oint dz \frac{e^{pz}}{(1-e^{-z})^{n+1}}, \tag{58}$$

where the integration contour is a counterclockwise loop enclosing the origin $z = 0$. By integrating by parts, one can verify

$$C_{p,n} = \frac{p+1}{n} C_{p+1,n-1} = \cdots = \frac{(n+p)!}{n!p!} C_{n+p,0} = \frac{(n+p)!}{n!p!}. \tag{59}$$

[6] We sometimes write α^n for $\alpha^{\wedge n}$ for any differential form α. The exponential of a differential form α is defined as $e^\alpha = \sum_{k=0}^{\infty} \frac{\alpha^{\wedge k}}{k!}$.

To obtain the last equality, we use

$$C_{n+p,0}=\frac{1}{2\pi i}\oint dz\frac{e^{(n+p)z}}{1-e^{-z}}=\frac{1}{2\pi i}\oint dz\frac{e^{(n+p)z}}{z}\left(\sum_{l=0}^{\infty}\frac{(-z)^l}{(l+1)!}\right)^{-1}=1. \tag{60}$$

Using the result of Appendix B1, we have $\int_{CP^n}\left(\frac{\omega}{2\pi}\right)^{\wedge n}=1$ and we therefore obtain

$$\dim\mathrm{Ker}\,D^{(p)}=\int_{CP^n}C_{p,n}\left(\frac{\omega}{2\pi}\right)^{\wedge n}=C_{p,n}=\frac{(n+p)!}{n!p!}. \tag{61}$$

3.3 *Matrix regularization of embedding functions*

We will show that the embedding functions $\{X^A\}_{A=1}^{n^2+2n}$ defined in Eq. (41) are mapped to

$$T_p(X^A)=\frac{\sqrt{2}}{p+n+1}L_A^{(p)}. \tag{62}$$

Here, $\{L_A^{(p)}\}_{A=1}^{n^2+2n}$ are generators of SU($n+1$) in the irreducible representation with Dynkin index (p, 0,…,0) satisfying

$$\left[L_A^{(p)},L_B^{(p)}\right]=if_{ABC}L_C^{(p)},\quad \left(L_A^{(p)}\right)^2=\frac{np(p+n+1)}{2(n+1)}I. \tag{63}$$

Let $\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p\in\Sigma_p$ be collective indices labeling the orthonormal basis of $\mathrm{Ker}\,D^{(p)}$. From Eq. (41), the Toeplitz operator $T_p(X^A)$ is given by

$$\begin{aligned}T_p(X^A)_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p}&:=\int_{CP^n}\mu\left(\psi^{(p)}_{\boldsymbol{\alpha}_p}\right)^\dagger X^A\psi^{(p)}_{\boldsymbol{\beta}_p}\\&=-\sqrt{2}\sum_{i,j=1}^{n+1}(T_A)_{ij}c^{(p)}_{\boldsymbol{\alpha}_p}c^{(p)}_{\boldsymbol{\beta}_p}\int_{CP^n}\mu\left(f^{(p)}_{\boldsymbol{\alpha}_p}\right)^*f^{(p)}_{\boldsymbol{\beta}_p}\zeta^j\bar{\zeta}^i\\&=-\sqrt{2}\sum_{i,j=1}^{n+1}(T_A)_{ij}c^{(p)}_{\boldsymbol{\alpha}_p}c^{(p)}_{\boldsymbol{\beta}_p}\int_{CP^n}\mu\left(f^{(p+1)}_{\boldsymbol{\alpha}_p\oplus i}\right)^*f^{(p+1)}_{\boldsymbol{\beta}_p\oplus j}.\end{aligned} \tag{64}$$

Here, we have introduced the notation $\boldsymbol{\alpha}_p\oplus\boldsymbol{\gamma}_l=(\alpha_1,\alpha_2,\dots,\alpha_p,\gamma_1,\gamma_2,\dots,\gamma_l)\in\Sigma_{p+l}$ for $\boldsymbol{\alpha}_p=(\alpha_1,\alpha_2,\dots,\alpha_p)\in\Sigma_p$ and $\boldsymbol{\gamma}_l=(\gamma_1,\gamma_2,\dots,\gamma_l)\in\Sigma_l$. Using the orthonormality condition, we have

$$\begin{aligned}T_p(X^A)_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p}&=-\sqrt{2}\sum_{i,j=1}^{n+1}(T_A)_{ij}\left(\frac{c^{(p)}_{\boldsymbol{\alpha}_p}}{c^{(p+1)}_{\boldsymbol{\alpha}_p\oplus i}}\right)^2\frac{c^{(p)}_{\boldsymbol{\beta}_p}}{c^{(p)}_{\boldsymbol{\alpha}_p}}\delta_{\boldsymbol{\alpha}_p\oplus i,\boldsymbol{\beta}_p\oplus j}\\&=-\frac{\sqrt{2}}{p+n+1}\frac{c^{(p)}_{\boldsymbol{\beta}_p}}{c^{(p)}_{\boldsymbol{\alpha}_p}}\sum_{i,j=1}^{n+1}(T_A)_{ij}(n_i(\boldsymbol{\alpha}_p)+1)\delta_{\boldsymbol{\alpha}_p\oplus i,\boldsymbol{\beta}_p\oplus j}.\end{aligned} \tag{65}$$

The Kronecker delta $\delta_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p}$ is defined by

$$\delta_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p}=\begin{cases}1 & (\boldsymbol{\alpha}_p=\boldsymbol{\beta}_p)\\ 0 & (\boldsymbol{\alpha}_p\neq\boldsymbol{\beta}_p)\end{cases}, \tag{66}$$

and we have used

$$\frac{c^{(p+1)}_{\boldsymbol{\alpha}_p\oplus i}}{c^{(p)}_{\boldsymbol{\alpha}_p}}=\sqrt{\frac{p+n+1}{n_i(\boldsymbol{\alpha}_p)+1}} \tag{67}$$

in the second equality.

Secondly, let us define

$$L_A^{(p)} := \frac{p+n+1}{\sqrt{2}} T_p(X^A), \quad \mathcal{L}_A^{(0)} X^B := -\frac{i}{\sqrt{2}}\{X^A, X^B\}, \tag{68}$$

where the Poisson bracket $\{X^A, X^B\}$ is given in Eq. (B14). From Eq. (95), they satisfy

$$\left[L_A^{(p)}, T_p(X^B)\right] = T_p\left(\mathcal{L}_A^{(0)} X^B\right). \tag{69}$$

By using Eqs. (69) and (B18), we find

$$\left[L_A^{(p)}, L_B^{(p)}\right] = \frac{p+n+1}{\sqrt{2}} T_p\left(\mathcal{L}_A^{(0)} X^B\right) = i f_{ABC} \frac{p+n+1}{\sqrt{2}} T_p(X^C) = i f_{ABC} L_C^{(p)}. \tag{70}$$

This shows that $\{L_A^{(p)}\}_{A=1}^{n^2+2n}$ is SU$(n+1)$ generators in some representation. To identify the representation, let us calculate the quadratic Casimir. From Eq. (65), we obtain

$$\left(L_A^{(p)}\right)^2_{\boldsymbol{\alpha}_p, \boldsymbol{\beta}_p} = \frac{c^{(p)}_{\boldsymbol{\beta}_p}}{c^{(p)}_{\boldsymbol{\alpha}_p}} \sum_{i,j,i',j'=1}^{n+1} (n_i(\boldsymbol{\alpha}_p)+1)(n_{j'}(\boldsymbol{\alpha}_p)+1+\delta_{i,j'}-\delta_{j,j'})(T_A)_{ij}(T_A)_{j'i'} \delta_{\boldsymbol{\alpha}_p \oplus i \ominus j', \boldsymbol{\beta}_p \oplus i' \ominus j}. \tag{71}$$

Using the Fierz identity

$$(T_A)_{ij}(T_A)_{j'i'} = \frac{1}{2}\left(\delta_{i,i'}\delta_{j,j'} - \frac{1}{n+1}\delta_{ij}\delta_{i'j'}\right), \tag{72}$$

we obtain

$$\left(L_A^{(p)}\right)^2_{\boldsymbol{\alpha}_p, \boldsymbol{\beta}_p} = \frac{np(p+n+1)}{2(n+1)} \delta_{\boldsymbol{\alpha}_p, \boldsymbol{\beta}_p}. \tag{73}$$

This is exactly the quadratic Casimir eigenvalue of the representation $(p, 0, \ldots, 0)$ and therefore $\{L_A^{(p)}\}_{A=1}^{n^2+2n}$ is in the irreducible representation $(p, 0, \ldots, 0)$.

3.4 *Laplace operator on* $\Gamma(L^{\otimes q})$

Consider a Laplace operator on $\Gamma(L^{\otimes q})$,

$$\Delta^{(q)} = -g^{\mu\nu}\nabla_\mu \nabla_\nu = -\{X^A, \{X^A, f^{(q)}\}\}, \tag{74}$$

for $f^{(q)} \in \Gamma(L^{\otimes q})$. Here, $\{\cdot, \cdot\}$ is a generalized Poisson bracket defined in Eq. (17) and $\{X^A\}_{A=1}^{n^2+2n}$ are isometric embedding functions. Let us also define differential operators $\{\mathcal{L}_A^{(q)}\}_{A=1}^{n^2+2n}$ on $\Gamma(L^{\otimes q})$ by

$$\mathcal{L}_A^{(q)} f^{(q)} := \frac{1}{\sqrt{2}}\left(-i\{X^A, f^{(q)}\} + q X^A f^{(q)}\right). \tag{75}$$

As shown in Appendix B2, they satisfy the commutation relations of the generator of SU$(n+1)$:

$$\left[\mathcal{L}_A^{(q)}, \mathcal{L}_B^{(q)}\right] = i f_{ABC} \mathcal{L}_C^{(q)}. \tag{76}$$

By a straightforward calculation, we can derive

$$\Delta^{(q)} = 2\left(\mathcal{L}_A^{(q)}\right)^2 - \frac{q^2 n}{n+1}. \tag{77}$$

Thus, the eigenvalue of $\Delta^{(q)}$ is given by $2E - \frac{q^2 n}{n+1}$, where E is an eigenvalue of $(\mathcal{L}_A^{(q)})^2$.

Let us evaluate the eigenvalues of $(\mathcal{L}_A^{(q)})^2$. To do this, let us consider how one can write elements of $\Gamma(L^{\otimes q})$ in terms of local coordinates. We remind ourselves that, in the overlapping patch $U_\alpha \cap U_\beta$, A^L transforms as

$$A^L(z_{(\alpha)}) = A^L(z_{(\beta)}) - d\lambda(z_{(\beta)}), \tag{78}$$

where

$$\lambda(z_{(\beta)}) = -\frac{1}{2}\left[\log\left(\frac{Z^\alpha}{Z^\beta}\right) - \log\left(\frac{\bar{Z}^\alpha}{\bar{Z}^\beta}\right)\right]. \tag{79}$$

Here, $Z = (Z^1, Z^2, \ldots, Z^{n+1})$ are the homogeneous coordinates of CP^n. Hence, any element $f^{(q)} \in \Gamma(L^{\otimes q})$ should transform as

$$f^{(q)}(z_{(\alpha)}) = e^{q\lambda(z_{(\beta)})} f^{(q)}(z_{(\beta)}) = \left(\frac{Z^\alpha}{Z^\beta}\right)^{-\frac{q}{2}} \left(\frac{\bar{Z}^\alpha}{\bar{Z}^\beta}\right)^{\frac{q}{2}} f^{(q)}(z_{(\beta)}). \tag{80}$$

Thus, we can choose a basis of $\Gamma(L^{\otimes q})$ as

$$(Z^\mu \bar{Z}^\mu)^{-k-\frac{q}{2}} Z^{\sigma_1} Z^{\sigma_2} \cdots Z^{\sigma_{k+q}} \bar{Z}^{\tau_1} \bar{Z}^{\tau_2} \cdots \bar{Z}^{\tau_k}, \tag{81}$$

where $k \in \mathbb{Z}_{\geq 0}$. With the normalized homogeneous coordinates ζ given in Eq. (36), we define a basis of $\Gamma(L^{\otimes q})$ as

$$f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k}(z) := \zeta^{\sigma_1} \zeta^{\sigma_2} \cdots \zeta^{\sigma_{k+q}} \bar{\zeta}^{\tau_1} \bar{\zeta}^{\tau_2} \cdots \bar{\zeta}^{\tau_k}, \tag{82}$$

From Eq. (37), one can see that ζ and $\bar{\zeta}$ are in the representation spaces $V_{(1,0,\ldots,0)}$ and $V^*_{(1,0,\cdots,0)}$, respectively, where $V^*_{(1,0,\cdots,0)}$ is the complex conjugate representation space of $V_{(1,0,\ldots,0)}$. This implies that the set of all polynomials of $\zeta^i, \bar{\zeta}^j$ of degree $(k+q, k)$ denoted by $\mathrm{Pol}_{k+q,k}(\zeta, \bar{\zeta})$ is isomorphic to $V_{(k+q,0,\cdots,0)} \otimes V^*_{(k,0,\cdots,0)}$, because of the symmetric index structure of the polynomials. Using the irreducible decomposition $V_{(k+q,0,\cdots,0)} \otimes V^*_{(k,0,\cdots,0)} = \bigoplus_{i=0}^{k} V_{(i+q,0,\cdots,0,i)}$, we have

$$\Gamma(L^{\otimes q}) = \bigoplus_{k=0}^{\infty} \mathrm{Pol}_{k+q,k}(\zeta, \bar{\zeta}) = \bigoplus_{k=0}^{\infty} \left(V_{(k+q,0,\cdots,0)} \otimes V^*_{(k,0,\cdots,0)}\right) = \bigoplus_{k=0}^{\infty} V_{(k+q,0,\cdots,0,k)}. \tag{83}$$

The eigenvalues of $(\mathcal{L}^{(q)}_A)^2$ are those of the quadratic Casimir for the representations $(k+q, 0, \ldots, 0, k)$, which are given by

$$E_k = \frac{1}{2}\left((k+q)(k+n) + k(k+q+n) + \frac{nq^2}{n+1}\right). \tag{84}$$

We can find eigenvectors of $(\mathcal{L}^{(q)}_A)^2$ from a similar group theoretic correspondence. The eigenvectors corresponding to $V_{(k+q,0,\ldots,0,k)}$ are

$$f^{(q)}_{k,I}(z) = \sum_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k} c^{(q)}_{I,\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k} f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k}(z) = \sum_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k} c^{(q)}_{I,\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k} \zeta^{\sigma_1} \zeta^{\sigma_2} \cdots \zeta^{\sigma_{k+q}} \bar{\zeta}^{\tau_1} \bar{\zeta}^{\tau_2} \cdots \bar{\zeta}^{\tau_k}, \tag{85}$$

where $c^{(q)}_{I,\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k} := (c^{(q)}_I)_{\sigma_1\cdots\sigma_{k+q},\tau_1\cdots\tau_k}$ is a coefficient tensor that is completely symmetric in σ and τ, respectively, and traceless under any contraction between σ_a and τ_b. The index I labels different weights of $V_{(k+q,0,\ldots,0,k)}$.[7] We also choose $c^{(q)}_{I,\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k}$ such that

$$\left(f^{(q)}_{k,I}, f^{(q)}_{k',I'}\right) := \int_{CP^n} \mu \left(f^{(q)}_{k,I}\right)^* f^{(q)}_{k',I'} = \delta_{k,k'}\delta_{I,I'}. \tag{86}$$

In Appendix B3, we show a direct computation of Eq. (84).

3.5 *Matrix regularization of $\Gamma(L^{\otimes q})$ and the Laplace operator*

In this subsection, we explicitly evaluate the Toeplitz operator for a complete basis of $\Gamma(L^{\otimes q})$ given by the eigenfunctions of $\Delta^{(q)}$ and discuss the matrix Laplacian.

Let us consider a matrix regularization of $\Gamma(L^{\otimes q})$ by

$$T_p(f^{(q)}) = \Pi^{(p+q)} f^{(q)} \Pi^{(p)}, \quad (f^{(q)} \in \Gamma(L^{\otimes q})) \tag{87}$$

[7] For example, for $n = 1$, we can take eigenvalues of $\mathcal{L}^{(q)}_3$ as the index I.

where $\Pi^{(p)} : \Gamma(S_c \otimes L^{\otimes p}) \to \mathrm{Ker}\, D^{(p)}$ is the projection. As discussed in the previous subsection, we can choose a complete basis of $\Gamma(L^{\otimes q})$ by

$$f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k}(z) = \zeta^{\sigma_1}\zeta^{\sigma_2}\cdots\zeta^{\sigma_{k+q}}\bar{\zeta}^{\tau_1}\bar{\zeta}^{\tau_2}\cdots\bar{\zeta}^{\tau_k}. \tag{88}$$

Then, the matrix regularization of $f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k}$ is given by

$$T_p\left(f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k}\right)_{\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_p} := \int_{CP^n}\mu\left(\psi^{(p+q)}_{\boldsymbol{\alpha}_{p+q}}\right)^\dagger f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k}\psi^{(p)}_{\boldsymbol{\beta}_p} = \frac{c^{(p+q)}_{\boldsymbol{\alpha}_{p+q}}c^{(p)}_{\boldsymbol{\beta}_p}}{\left(c^{(p+q+k)}_{\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k}\right)^2}\delta_{\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k,\boldsymbol{\beta}_p\oplus\boldsymbol{\sigma}_{k+q}}. \tag{89}$$

From Eq. (24) and $\hbar_p = p^{-1}$, we define a Laplace operator acting on $T_p(f^{(q)})$ by

$$\hat{\Delta}(T_p(f^{(q)})) = p^2[T(X^A),[T(X^A),T_p(f^{(q)})]]. \tag{90}$$

Using Eqs. (62) and (63), we have

$$\hat{\Delta}(T_p(f^{(q)})) = \frac{2p^2}{(p+q+n+1)(p+n+1)}\left((L_A\circ)^2 - \frac{q^2 n}{2(n+1)}\right)T_p(f^{(q)}), \tag{91}$$

where we have defined $L_A \circ T_p(f^{(q)}) := L_A^{(p+q)}T_p(f^{(q)}) - T_p(f^{(q)})L_A^{(p)}$. The operation $L_A\circ$ satisfies

$$[L_A\circ, L_B\circ] = i f_{ABC}L_C\circ, \tag{92}$$

and hence they are representations of the generators of $\mathrm{SU}(n+1)$. Their representation space is

$$V_{(p+q,0,\ldots,0)} \otimes V^*_{(p,0,\ldots,0)} = \bigoplus_{k=0}^{p} V_{(k+q,0,\ldots,0,k)}. \tag{93}$$

This is a similar decomposition to Eq. (83) except for the cut-off p. From this, we see that the eigenmatrices of $\hat{\Delta}$ are in the irreducible representation $V_{(k+q,0,\ldots,0,k)}$ and the eigenvalue of $\hat{\Delta}$ is given by

$$\frac{2p^2}{(p+q+n+1)(p+n+1)}\left(E_k - \frac{q^2 n}{2(n+1)}\right) = 2E_k - \frac{q^2 n}{n+1} + O(p^{-1}) \tag{94}$$

for $k = 1, 2,\ldots,p$, where E_k is given by Eq. (84). This shows that the spectrum of the matrix Laplacian $\hat{\Delta}$ is the truncated version of the spectrum of the Bochner Laplacian Δ up to a correction of order $O(1/p)$.

More explicitly, we can show the stronger identity

$$T_p\left(\mathcal{L}^{(q)}_A f^{(q)}\right) = L_A \circ T_p(f^{(q)}) \tag{95}$$

for any $f^{(q)} \in \Gamma(L^{\otimes q})$. This is shown in Appendix B4. From this identity, we can easily derive the correspondence of eigenvalues or eigenvectors that we discussed above. Note that $T_p(f^{(q)}_{k,I})$ can be written as

$$T_p\left(f^{(q)}_{k,I}\right)_{\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_p} = \int_{CP^n}\mu\left(\psi^{(p+q)}_{\boldsymbol{\alpha}_{p+q}}\right)^\dagger f^{(q)}_{k,I}\psi^{(p)}_{\boldsymbol{\beta}_p} = c^{(p+q)}_{\boldsymbol{\alpha}_{p+q}}c^{(p)}_{\boldsymbol{\beta}_p}\left(f^{(q)}_{\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_p}, f^{(q)}_{k,I}\right), \tag{96}$$

where $(\,\cdot,\,\cdot)$ is the inner product defined in Eq. (86). Since $f^{(q)}_{\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_p}$ can be expanded by the orthonormal basis $f^{(q)}_{k',I'}$ for $k' \le p$, we find $T_p(f^{(q)}_{k,I}) = 0$ for $k > p$. For $k \le p$, Eq. (95) implies that $f^{(q)}_{k,I}$ and $T_p(f^{(q)}_{k,I})$ both have exactly the same Casimir eigenvalues and weights. For the quadratic Casimir, we have

$$(L_A\circ)^2\, T_p\left(f^{(q)}_{k,I}\right) = T_p\left(\left(\mathcal{L}^{(q)}_A\right)^2 f^{(q)}_{k,I}\right) = E_k T_p\left(f^{(q)}_{k,I}\right), \tag{97}$$

and we can see that the eigenvalues of $(L_A\circ)^2$ are $\{E_k\}_{k=0}^p$ as expected.

To see the correspondence of the trace and the integral (19), let us calculate the Frobenius inner product:

$$\left(T_p\left(f_{k,I}^{(q)}\right), T_p\left(f_{k',I'}^{(q)}\right)\right) := \mathrm{Tr}\left[T_p\left(f_{k,I}^{(q)}\right)^\dagger T_p\left(f_{k',I'}^{(q)}\right)\right]. \tag{98}$$

For k, $k' \leq p$, Eq. (98) is nonvanishing only if $T_p(f_{k,I}^{(q)})$ and $T_p(f_{k',I'}^{(q)})$ belong to the same representation having the same weights. Thus, we have

$$\left(T_p\left(f_{k,I}^{(q)}\right), T_p\left(f_{k',I'}^{(q)}\right)\right) \propto \delta_{k,k'}\delta_{I,I'}. \tag{99}$$

More explicitly, we can show

$$\left(T_p\left(f_{k,I}^{(q)}\right), T_p\left(f_{k',I'}^{(q)}\right)\right) = \frac{(p+q+n)!(p+n)!}{(2\pi)^n(p-k)!(p+q+k+n)!}\delta_{k,k'}\delta_{I,I'}. \tag{100}$$

See Appendix B5 for the proof. For finite k and k', we have the large-p expansion

$$\left(T_p\left(f_{k,I}^{(q)}\right), T_p\left(f_{k',I'}^{(q)}\right)\right) = \frac{p^n}{(2\pi)^n}\delta_{k,k'}\delta_{I,I'} + O(p^{n-1}), \tag{101}$$

which is consistent with Eq. (86) through the correspondence for the trace and integral (19).

4. Fuzzy T^{2n}

In this section, we consider a Berezin–Toeplitz quantization of a monopole bundle over a torus $T^{2n} \simeq (S^1)^{2n}$ [11]. In Sect. 4.1, we define a torus T^{2n} and describe its basic properties. In Sect. 4.2, we explicitly construct a complete orthonormal basis of the kernel of the Dirac operator. In Sect. 4.3, we calculate Toeplitz operators of embedding functions. In Sects. 4.4 and 4.5, we discuss the continuum Laplacian and the matrix Laplacian, respectively, for a monopole bundle.[8]

4.1 *Geometry of T^{2n}*

Let us consider the Euclidean space $\mathbb{R}^{2n}$ equipped with a flat metric. We introduce an equivalent relation

$$^\forall x = (x^1, x^2, \dots, x^{2n}) \in \mathbb{R}^{2n}: \quad x^a \sim x^a + 2\pi l_a \quad (a = 1, 2, \dots, 2n), \tag{102}$$

where l_a are some positive constants. Under this identification, we define a $2n$-dimensional torus T^{2n} as a quotient space

$$T^{2n} = \mathbb{R}^{2n}/\sim. \tag{103}$$

The flat metric and its associated Kähler form on T^{2n} are given by

$$g = \sum_{a=1}^{2n} dx^a \otimes dx^a, \quad \omega = \sum_{m=1}^{n} dx^{2m-1} \wedge dx^{2m} = i\, dz^\mu \wedge d\bar{z}^\mu. \tag{104}$$

Here, the real and complex coordinates are related by $z^\mu = (x^{2\mu-1} + ix^{2\mu})/\sqrt{2}$ for $\mu = 1, 2, \dots, n$. T^{2n} is isometrically embedded in $\mathbb{R}^{4n}$ such that

$$X^{2a-1} = l_a \cos(x^a/l_a), \quad X^{2a} = l_a \sin(x^a/l_a). \quad (a = 1, 2, \dots, 2n) \tag{105}$$

Now, let us consider the bundle structures on T^{2n}. Since T^{2n} is a spin manifold, we can simply use the spin bundle S. Since T^{2n} is flat, the spin connection of S is flat as well. We also introduce the prequantum line bundle L. The two-cycles of T^{2n} are simply T^2 and the curvature $R^L = -ik\omega$

[8] In Ref. [11], the 2D case is studied. In this paper, we study its higher-dimensional extension.

is nonvanishing on T^2 spanned by (x^{2m-1}, x^{2m}) for $m = 1, 2,\ldots,n$. Hence, the prequantization condition for T^{2n} is satisfied for k and l_a such that

$$^{\forall}m \in \{1, 2, \ldots, n\}: \quad q_m := \frac{i}{2\pi}\int_{T^2} R^L = 2\pi k l_{2m-1} l_{2m} \in \mathbb{N}. \tag{106}$$

The condition is satisfied if and only if the ratio of areas $\frac{l_{2m-1}l_{2m}}{l_{2m'-1}l_{2m'}}$ is rational for any m, m'.

4.2 *Zero modes of the Dirac operator on T^{2n}*

In this subsection, we construct a complete orthonormal basis of the Dirac zero modes on T^{2n} [21].

Let $D^{(p)}$ be a twisted Dirac operator on $\Gamma(S\otimes L^{\otimes p})$. By the same argument as in Sect. 3.2, the zero-mode equation $D^{(p)}\psi^{(p)} = 0$ for $\psi^{(p)} = f^{(p)}\,|+\rangle^{\otimes n}$ is simplified to

$$\left(\partial_{\bar{\mu}} + pA^L_{\bar{\mu}}\right) f^{(p)} = 0. \tag{107}$$

Here, A^L can be chosen as

$$A^L = -ik\sum_{m=1}^{n} x^{2m-1}dx^{2m} = -\frac{k}{2}(z^\mu + \bar{z}^\mu)(dz^\mu - d\bar{z}^\mu). \tag{108}$$

Thus, the zero-mode equation is

$$\left(\partial_{\bar{\mu}} + \frac{kp}{2}(z^\mu + \bar{z}^\mu)\right) f^{(p)} = 0. \tag{109}$$

We also have to pay attention to the boundary conditions. Since $f^{(p)}(x)$ is a section of the nontrivial bundle $L^{\otimes p}$, $f^{(p)}(x)$ transforms under a coordinate change. For T^{2n}, this property is described in terms of the boundary conditions as follows. Consider the coordinate change $x^{2m}\mapsto x^{2m} + 2\pi l_{2m}$. Under this change, the connection one-form $A^L(x)$ does not change and correspondingly the element of $\Gamma(L^{\otimes p})$ should be periodic under this coordinate shift for each m. Similarly, under the coordinate change $x^{2m-1}\mapsto x^{2m-1} + 2\pi l_{2m-1}$, $A^L(x)$ transforms as $A^L(x)\mapsto A^L(x) - d\lambda(x)$ where $\lambda(x) = i2\pi k l_{2m-1}x^{2m}$. Correspondingly, $f^{(p)}$ should transform as $f^{(p)}(x)\mapsto e^{p\lambda(x)}f^{(p)}(x)$ for each m. These boundary conditions and the differential equation (109) are closed on each T^2 with the coordinates (x^{2m-1}, x^{2m}). Hence, we can separate the variables and the general solution is

$$f^{(p)}(x) = \prod_{m=1}^{n}\left(e^{-\frac{kp}{2}(x^{2m-1})^2}\phi_m(x^{2m-1} + ix^{2m})\right). \tag{110}$$

The boundary conditions are now given by

$$\begin{aligned} &\phi_m(x^{2m-1} + ix^{2m} + i2\pi l_{2m}) = \phi_m(x^{2m-1} + ix^{2m}), \\ &\phi_m(x^{2m-1} + ix^{2m} + 2\pi l_{2m-1}) = e^{-ipq_m\tau_m}e^{pq_m(x^{2m-1}+ix^{2m})/l_{2m}}\phi_m(x^{2m-1} + ix^{2m}). \end{aligned} \tag{111}$$

Here, $\tau_m := \mathrm{i}l_{2m-1}/l_{2m}$ is the moduli parameter of the mth T^2. From the first condition, one can write

$$\phi_m(x^{2m-1} + ix^{2m}) = \sum_{s\in\mathbb{Z}} d_s e^{s(x^{2m-1}+ix^{2m})/l_{2m}} \tag{112}$$

for some complex constants d_s. The second condition gives

$$d_s = e^{i\pi(2s-pq_m)\tau_m}d_{s-pq_m}. \tag{113}$$

To solve this recursion equation, let us write $s = pq_m l + i_m$ for $l \in \mathbb{Z}$ and $i_m \in \{0, 1,\ldots,pq_m - 1\}$. Then, the solution is

$$d_{pq_m l+i_m} = c^{(p)}_{i_m} e^{i\pi\left(l+\frac{i_m}{pq_m}\right)^2 pq_m\tau_m} \tag{114}$$

for some complex constants $c^{(p)}_{i_m}$. Hence, there are pq_m linearly independent solutions to Eq. (111):

$$\phi_m(x^{2m-1} + ix^{2m}) = \sum_{i_m=0}^{pq_m-1} c^{(p)}_{i_m} \sum_{l\in\mathbb{Z}} e^{i\pi\left(l+\frac{i_m}{pq_m}\right)^2 pq_m\tau_m} e^{\left(l+\frac{i_m}{pq_m}\right)\frac{pq_m}{l_{2m}}(x^{2m-1}+ix^{2m})}. \tag{115}$$

Therefore, from Eq. (110), one can take a complete basis of the zero-mode solutions as

$$f^{(p)}_i(x) = \prod_{m=1}^{n} f^{(p)}_{i_m}(x^{2m-1}, x^{2m}), \tag{116}$$

where $i = (i_1, i_2,\ldots,i_n)$ and

$$f^{(p)}_{i_m}(x^{2m-1}, x^{2m}) := \left(\frac{kp}{4\pi^3 l^2_{2m}}\right)^{1/4} e^{-\frac{kp}{2}(x^{2m-1})^2} \sum_{l\in\mathbb{Z}} e^{i\pi\left(l+\frac{i_m}{pq_m}\right)^2 pq_m\tau_m} e^{\left(l+\frac{i_m}{pq_m}\right)\frac{pq_m}{l_{2m}}(x^{2m-1}+ix^{2m})}. \tag{117}$$

Here, we have fixed the constant $c^{(p)}_{i_m} = \left(\frac{kp}{4\pi^3 l^2_{2m}}\right)^{1/4}$. Note that the index $i_m \in \{0, 1,\ldots,pq_m - 1\}$ is rather considered to be an element of the additive group $\mathbb{Z}/pq_m\mathbb{Z}$ because of the cyclic structure $f^{(p)}_{i_m} = f^{(p)}_{i_m+pq_m}$. This basis is not only complete but also orthonormal. In Appendix C1, we show the relation

$$\int_0^{2\pi l_{2m-1}} dx^{2m-1} \int_0^{2\pi l_{2m}} dx^{2m} \left(f^{(p)}_{i_m}\right)^* f^{(p)}_{j_m} = \delta_{i_m, j_m}, \tag{118}$$

which implies the orthonormality

$$\int_{T^{2n}} \mu \left(f^{(p)}_i\right)^* f^{(p)}_j = \prod_{m=1}^{n} \delta_{i_m, j_m}. \tag{119}$$

Now, let us check that the number of zero modes is consistent with the index theorem and the vanishing theorem. As we obtained in Eq. (116), the number of linearly independent zero modes is

$$\dim \mathrm{Ker}\, D^{(p)} = p^n \prod_{m=1}^{n} q_m. \tag{120}$$

On the other hand, the index theorem and the vanishing theorem imply

$$\dim \mathrm{Ker}\, D^{(p)} = \mathrm{Ind}\, D^{(p)} = \int_{T^{2n}} e^{\frac{ip}{2\pi}R^L} = \frac{(kp)^n}{(2\pi)^n} \int_{T^{2n}} \mu = p^n \prod_{m=1}^{n} q_m. \tag{121}$$

4.3 *Matrix regularization of embedding functions*

Now, let us consider the following functions:

$$u_m = e^{ix^{2m-1}/l_{2m-1}}, \quad v_m = e^{ix^{2m}/l_{2m}}. \tag{122}$$

By using these functions, an isometric embedding $X^A : T^{2n} \to \mathbb{R}^{4n}$ can be written as

$$X^{4m-3} = \frac{l_{2m-1}}{2}\left(u_m + u^*_m\right), \quad X^{4m-2} = \frac{l_{2m-1}}{2i}\left(u_m - u^*_m\right),$$
$$X^{4m-1} = \frac{l_{2m}}{2}\left(v_m + v^*_m\right), \quad X^{4m} = \frac{l_{2m}}{2i}\left(v_m - v^*_m\right). \tag{123}$$

We consider the matrix regularization of these functions.

We define a matrix regularization of $C^\infty(T^{2n})$ by

$$T_p(f) = \Pi^{(p)} f \Pi^{(p)}, \quad (f \in C^\infty(T^{2n})) \tag{124}$$

where $\Pi^{(p)} : \Gamma(S \otimes L^{\otimes p}) \to \mathrm{Ker}\, D^{(p)}$ is the Hermitian projection. Using the integral in Appendix C1 (or see Ref. [11]), we have

$$\begin{aligned} U_m^{(p)} &:= T_p(u_m) = I_{pq_1} \otimes \cdots \otimes I_{pq_{m-1}} \otimes U_{pq_m} \otimes I_{pq_{m+1}} \otimes \cdots \otimes I_{pq_n}, \\ V_m^{(p)} &:= T_p(v_m) = I_{pq_1} \otimes \cdots \otimes I_{pq_{m-1}} \otimes V_{pq_m} \otimes I_{pq_{m+1}} \otimes \cdots \otimes I_{pq_n}, \end{aligned} \tag{125}$$

where

$$U_{pq_m} = e^{-\frac{1}{4kpl_{2m-1}^2}} \begin{pmatrix} 1 & & & \\ & e^{i\frac{2\pi}{pq_m}} & & \\ & & \ddots & \\ & & & e^{i\frac{2(pq_m-1)\pi}{pq_m}} \end{pmatrix},$$

$$V_{pq_m} = e^{-\frac{1}{4kpl_{2m}^2}} \begin{pmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}. \tag{126}$$

These matrices satisfy the algebra of the noncommutative torus $U_{pq_m} V_{pq_m} = e^{i\frac{2\pi}{pq_m}} V_{pq_m} U_{pq_m}$. Therefore, the matrix regularization of the embedding functions is

$$\begin{aligned} T_p(X^{4m-3}) &= \frac{l_{2m-1}}{2}\left(U_m^{(p)} + U_m^{(p)\dagger}\right), \quad & T_p(X^{4m-2}) &= \frac{l_{2m-1}}{2i}\left(U_m^{(p)} - U_m^{(p)\dagger}\right), \\ T_p(X^{4m-1}) &= \frac{l_{2m}}{2}\left(V_m^{(p)} + V_m^{(p)\dagger}\right), \quad & T_p(X^{4m}) &= \frac{l_{2m}}{2i}\left(V_m^{(p)} - V_m^{(p)\dagger}\right). \end{aligned} \tag{127}$$

4.4 *Laplace operator on* $\Gamma(L^{\otimes q})$

Consider the Laplace operator on $\Gamma(L^{\otimes q})$

$$\Delta^{(q)} = -\sum_{a=1}^{2m} \left(D_a^{(q)}\right)^2 = -\sum_{m=1}^{n} \left(D_m^{(q)} D_{\bar{m}}^{(q)} + D_{\bar{m}}^{(q)} D_m^{(q)}\right), \tag{128}$$

where $D_a^{(q)}$ is the connection of $\Gamma(L^{\otimes q})$ in the real coordinates x^a and $D_m^{(q)}$ and $D_{\bar{m}}^{(q)}$ are those in the complex coordinates. Also let us define the inner product:

$$(f^{(q)}, g^{(q)}) := \int_{T^{2n}} \mu (f^{(q)})^* g^{(q)}. \quad (f^{(q)}, g^{(q)} \in \Gamma(L^{\otimes q})) \tag{129}$$

Here, $\mu = \omega^{\wedge n}/n! = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{2n}$.

First, let us examine the spectrum of the Laplacian $\Delta^{(q)}$ for $q = 0$, i.e., the case for the ordinary functions $C^\infty(T^{2n})$. One can easily see that the normalized eigenfunctions of $\Delta^{(0)}$ are

$$f_b(x) = [(2\pi)^{2n} l_1 l_2 \cdots l_{2n}]^{-1/2} \prod_{a=1}^{2n} e^{i b_a x^a / l_a}, \tag{130}$$

and the eigenvalues are given by

$$E_b = \sum_{a=1}^{2n} \left(\frac{b_a}{l_a}\right)^2, \tag{131}$$

where $b = (b_1, b_2, \ldots, b_{2n}) \in \mathbb{Z}^{2n}$. They satisfy

$$(f_b, f_{b'}) = \prod_{a=1}^{2n} \delta_{b_a, b'_a}. \tag{132}$$

Now, let us consider the spectrum of the Laplacian $\Delta^{(q)}$ for $q \neq 0$. Since we have

$$\left[D_m^{(q)}, D_{\bar{m}'}^{(q)}\right] = kq\delta_{m,m'}, \quad \left[D_m^{(q)}, D_{m'}^{(q)}\right] = \left[D_{\bar{m}}^{(q)}, D_{\bar{m}'}^{(q)}\right] = 0, \tag{133}$$

we can define creation and annihilation operators

$$a_m^{(q)} := i\frac{D_{\bar{m}}^{(q)}}{\sqrt{kq}}, \quad a_m^{(q)\dagger} := i\frac{D_m^{(q)}}{\sqrt{kq}}, \tag{134}$$

satisfying $[a_m^{(q)}, a_{m'}^{(q)\dagger}] = \delta_{m,m'}$. Then, the Laplace operator can be written as

$$\Delta^{(q)} = 2kq\sum_{m=1}^{n}\left(N_m^{(q)} + \frac{1}{2}\right), \tag{135}$$

where $N_m^{(q)} := a_m^{(q)\dagger}a_m^{(q)}$ are the number operators. Note that the lowest eigenmodes of $\Delta^{(q)}$ should vanish under the action of $a_m^{(q)} \propto D_{\bar{m}}^{(q)}$ for all m. This means that the lowest eigenmodes are $f_j^{(q)}$ given in Eq. (117), which appeared in the discussion of the Dirac zero modes. The other eigenmodes are obtained by acting the creation operators on the lowest eigenmodes $f_j^{(q)}$. Thus, the normalized eigenfunctions of $\Delta^{(q)}$ are

$$f_{c,j}^{(q)} = \prod_{m=1}^{n} f_{c_m, j_m}^{(q)}, \quad f_{c_m, j_m}^{(q)} = \frac{\left(a_m^{(q)\dagger}\right)^{c_m}}{\sqrt{c_m!}} f_{j_m}^{(q)}(z^m), \tag{136}$$

and the corresponding eigenvalues are

$$E_c = 2kq\sum_{m=1}^{n}\left(c_m + \frac{1}{2}\right). \tag{137}$$

Here $c = (c_1, c_2, \ldots, c_n) \in (\mathbb{Z}_{\geq 0})^n$. More explicitly, the eigenfunctions are given by

$$\begin{aligned} f_{c_m, j_m}^{(q)}(z^m) = {} & \left(\frac{kq}{4\pi^3 l_{2m}^2}\right)^{1/4} \frac{(-i)^{c_m}}{\sqrt{2^{c_m} c_m!}} e^{-\frac{kq}{2}(x^{2m-1})^2} \\ & \times \sum_{l\in\mathbb{Z}} e^{i\pi\left(l+\frac{j_m}{qq_m}\right)^2 qq_m\tau_m} e^{\left(l+\frac{j_m}{qq_m}\right)\frac{qq_m}{l_{2m}}(x^{2m-1}+ix^{2m})} \\ & \times H_{c_m}\left(\sqrt{kq}\left(x^{2m-1} - 2\pi l_{2m-1}\left(l + \frac{j_m}{qq_m}\right)\right)\right). \end{aligned} \tag{138}$$

Here, $H_n(x)$ is the Hermite polynomial satisfying the recursion $H_{n+1}(x) = 2xH_n(x) - H'_n(x)$.

4.5 *Matrix regularization of $\Gamma(L^{\otimes q})$ and the Laplace operator*

In this subsection, we explicitly evaluate the Toeplitz operator for a complete basis of $\Gamma(L^{\otimes q})$ given by the eigenfunctions of $\Delta^{(q)}$ and discuss the matrix Laplacian.

The matrix regularization of $\Gamma(L^{\otimes q})$ is defined by

$$T_p(f^{(q)}) = \Pi^{(p+q)} f^{(q)} \Pi^{(p)}, \quad (f^{(q)} \in \Gamma(L^{\otimes q})) \tag{139}$$

where $\Pi^{(p)} : \Gamma(S \otimes L^{\otimes p}) \to \mathrm{Ker}\, D^{(p)}$ is the Hermitian projection.

For $q = 0$, we have the eigenfunctions f_b given in Eq. (130). Using the results of Appendix C1, we have

$$T_p(f_b) = [(2\pi)^{2n} l_1 l_2 \cdots l_{2n}]^{-1/2} e^{-\frac{1}{4kp}\sum_{m=1}^{n}\left(\frac{b_{2m-1}^2 - b_{2m-1}}{l_{2m-1}^2} + i\frac{2b_{2m-1}b_{2m}}{l_{2m-1}l_{2m}} + \frac{b_{2m}^2 - b_{2m}}{l_{2m}^2}\right)}$$
$$\times (U^{(pq_1)})^{b_1}(V^{(pq_1)})^{b_2} \otimes (U^{(pq_2)})^{b_3}(V^{(pq_2)})^{b_4} \otimes \cdots \otimes (U^{(pq_n)})^{b_{2n-1}}(V^{(pq_n)})^{b_{2n}}. \quad (140)$$

For $q \neq 0$, the Toeplitz operators of the eigenfunctions $f_{c,j}^{(q)}$ given in Eq. (138) are

$$T_p\left(f_{c,j}^{(q)}\right)_{i,i'} = \prod_{m=1}^{n} \int_0^{2\pi l_{2m-1}} dx^{2m-1} \int_0^{2\pi l_{2m}} dx^{2m} \left(f_{i_m}^{(p+q)}\right)^* f_{c_m,j_m}^{(q)} f_{i'_m}^{(p)}, \quad (141)$$

where $i = (i_1, i_2, \ldots, i_n)$ and $i' = (i'_1, i'_2, \cdots i'_n)$ are the labels of the Dirac zero modes. The integral on the right-hand side of Eq. (141) is computed in Appendix C2 and the result is

$$\int_0^{2\pi l_{2m-1}} dx^{2m-1} \int_0^{2\pi l_{2m}} dx^{2m} \left(f_{i_m}^{(p+q)}\right)^* f_{c_m,j_m}^{(q)} f_{i'_m}^{(p)}$$
$$= \frac{i^{c_m}}{\sqrt{2^{c_m} c_m!}} \left(\frac{kq}{4\pi^3 l_{2m}^2}\right)^{1/4} \left(\frac{p}{p+q}\right)^{\frac{c_m}{2}+\frac{1}{4}} \sum_{t=1}^{(p+q)q_m} \delta_{i_m, j_m + i'_m + qq_m t}^{(\mathrm{mod}\ (p+q)q_m)} \sum_{l\in\mathbb{Z}}$$
$$\times e^{i\pi\left(l + \frac{pq_m i_m - (p+q)q_m i'_m}{(p+q)pq q_m^3}\right)^2 (p+q)pq q_m^3 \tau_m}$$
$$\times H_{c_m}\left(2\pi l_{2m-1}\sqrt{k(p+q)pq q_m^2}\left(l + \frac{pq_m i_m - (p+q) q_m i'_m}{(p+q)pq q_m^3}\right)\right). \quad (142)$$

From Eq. (24), we define a Laplace operator acting on $T_p(f^{(q)})$ by

$$\hat{\Delta}^{(q)} T_p(f^{(q)}) = (kp)^2 [T(X^A), [T(X^A), T_p(f^{(q)})]]$$
$$= \frac{l_{2m-1}^2 (kp)^2}{2} \sum_{m=1}^{n} \left([T(u_m), [T(u_m)^\dagger, T_p(f^{(q)})]] + [T(u_m)^\dagger, [T(u_m), T_p(f^{(q)})]]\right)$$
$$+ \frac{l_{2m}^2 (kp)^2}{2} \sum_{m=1}^{n} \left([T(v_m), [T(v_m)^\dagger, T_p(f^{(q)})]] + [T(v_m)^\dagger, [T(v_m), T_p(f^{(q)})]]\right). \quad (143)$$

The second expression is obtained by using Eq. (127). For $q = 0$, we can easily see that the spectrum of $\hat{\Delta}^{(0)}$ approaches that of $\Delta^{(0)}$ as

$$\hat{\Delta}^{(0)} T_p(f_b) = 4(kp)^2 \sum_{m=1}^{n} \left(l_{2m-1}^2 \sin^2\left(\frac{\pi b_{2m}}{pq_m}\right) + l_{2m}^2 \sin^2\left(\frac{\pi b_{2m-1}}{pq_m}\right)\right) T_p(f_b)$$
$$= \left(\sum_{a=1}^{2n} \left(\frac{b_a}{l_a}\right)^2 + O(p^{-1})\right) T_p(f_b). \quad (144)$$

We can also see the correspondence between the trace and the integral. In fact, we have

$$(2\pi\hbar_p)^n \mathrm{Tr}[T_p(f_b)^\dagger T_p(f_{b'})] = (2\pi\hbar_p)^n [(2\pi)^{2n} l_1 l_2 \cdots l_{2n}]^{-1} e^{-\frac{1}{2kp}\sum_{m=1}^{n}\left(\frac{b_{2m-1}^2 - b_{2m-1}}{l_{2m-1}^2} + i\frac{2b_{2m-1}b_{2m}}{l_{2m-1}l_{2m}} + \frac{b_{2m}^2 - b_{2m}}{l_{2m}^2}\right)}$$
$$\times \prod_{m=1}^{n} \left(e^{-\frac{b_{2m-1}}{2kp l_{2m-1}^2}} e^{-\frac{b_{2m}}{2kp l_{2m}^2}} pq_m\right) \delta_{b_m, b'_m}^{(\mathrm{mod}\ pq_m)}$$
$$= \delta_{b_m, b'_m}^{(\mathrm{mod}\ pq_m)} + O(p^{-1}), \quad (145)$$

which is consistent with Eq. (132). For $q \neq 0$, the eigenvalue problem of the Laplace operator $\hat{\Delta}^{(q)}$ is related to the Hofstadter problem as noted in Ref. [11]. It is numerically shown in Ref. [11] that the spectrum of $\hat{\Delta}^{(q)}$ approaches that of $\Delta^{(q)}$ in the commutative limit.[9]

5. Conclusion and future problems

In this paper, we have studied the Berezin–Toeplitz quantization of vector bundles over a general closed connected Kähler manifold, which is a continuation of our previous studies of 2D cases [13,14]. In our formalism, we treat a vector bundle as a homomorphism bundle and treat its sections as some linear operator between suitable twisted spinor fields. By restricting the vector spaces of each twisted spinor field to finite-dimensional kernels of Dirac operators, we defined a quantization map from fields (sections of the vector bundle) to matrices. We obtained a large-p asymptotic behavior of the product $T_p(\varphi)T_p(\chi)$ for arbitrary sections of vector bundles φ, χ up to the second order in $1/p$. This is a natural generalization of the relation of matrix regularization (2). The matrix Laplacian acting on such matrices can be written in terms of a commutator-like operation and its spectrum in the large-p limit is shown to be equal to that of the usual Bochner Laplacian acting on continuum fields. Our result is a generalization of Refs. [15,16], where fuzzy CP^n is considered, to the general Kähler manifold. As explicit examples, we considered monopole bundles over a fuzzy CP^n and fuzzy T^{2n} and we confirmed that in the case of CP^n our formulation correctly reproduces the results in Refs. [15,16].

Our framework is applicable to a wide class of fields. For example, an (r, s) tensor field gives a homomorphism from $\Gamma(TM^{\otimes s})$ to $\Gamma(TM^{\otimes r})$, and we can apply our formulation. It is interesting to construct a fuzzy version of the higher spin theories [22,23] by using our method. It is also possible to consider a matrix regularization of spinor fields. The spinor fields on the lattice have the problems of doublers and chiral anomaly and we can consider similar problems on fuzzy spaces [24–29]. Our method will enable us to deal with similar problems on a general Kähler manifold. Our method can also be used to construct fuzzy field theories in arbitrary background fields. It is important to understand how various background field configurations such as instantons are realized on fuzzy spaces.

Let us comment on some possible generalizations of our study. Throughout this paper, we assumed that the manifold M is Kähler. In particular, we assumed that the manifold has an integrable complex structure. However, it is possible to construct a quantization of functions with almost complex structure that is not necessarily integrable (see, e.g., Ref. [7]). Moreover, it is also possible to consider noncompact manifolds and orbifolds [7]. Therefore, the Berezin–Toeplitz quantization of vector bundles might also be defined over more general manifolds than the closed Kähler case (e.g., the fuzzy S^4 [30–33]). We can also consider more challenging problems such as a quantization of odd-dimensional manifolds [34–37] or manifolds with boundaries. These studies are important to uncover the various branes of such geometries such as odd-dimensional branes and orientifold planes [38] in the framework of matrix models [36]. Yet another possible generalization is the Berezin–Toeplitz quantization of nonlocal operators such as Wilson lines. As a Wilson line sends a spinor at one point to a spinor at a different point, it gives a linear map between twisted spinor spaces. The Wilson line or loop is an essen-

[9] In Ref. [11], only the 2D case is considered, while we consider a higher-dimensional torus T^{2n}. However, T^{2n} can be decomposed to the tensor product of T^2 so that the results of Ref. [11] can also be applied to our case.

tial ingredient of gauge theories and the quantization of Wilson lines may shed light on studies of gauge theories on fuzzy geometries.

Finally, another direction for the study of fuzzy spaces is the inverse problem of quantization. While, in quantization, one constructs a quantum geometry from a given classical geometry, it is also interesting to consider the problem of finding a classical geometry from a given quantum geometry. See Refs. [39–47] for developments in this direction. We consider that the inverse problem can be generalized for the case of vector bundles. The matrix counterparts of vector bundles should contain various geometric information and finding a method for extracting such information will bring great progress to the understanding of fuzzy geometry.

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Appendix A. Proofs and formulas for general Kähler manifolds

A1. Useful choice of orthonormal frame fields

In this appendix, we will introduce particular orthonormal frame fields (vielbeins), which simplify our argument.

Let us choose an element $e_1 \in \Gamma(TM)$ such that $g(e_1, e_1) = 1$. Then, $e_2 := Je_1 \in \Gamma(TM)$ satisfies $g(e_a, e_b) = \delta_{ab}$ for $a, b = 1, 2$, which follows from the Kähler condition (3). Next, choose an arbitrary $e_3 \in \Gamma(TM)$ such that $g(e_a, e_b) = \delta_{ab}$ for $a, b = 1, 2, 3$. Then, $e_4 := Je_3 \in \Gamma(TM)$ also satisfies $g(e_a, e_b) = \delta_{ab}$ for $a, b = 1, 2, 3, 4$. By continuing the above argument, we can construct a complete orthonormal field. This choice is useful because the symplectic form can be written as

$$\omega = \sum_{m=1}^{n} \theta^{2m-1} \wedge \theta^{2m} \tag{A1}$$

where $\{\theta^a\}_{a=1,2,\ldots,2n}$ is the dual basis of $\{e_a\}_{a=1,2,\ldots,2n}$.

It is also convenient to introduce complexified fields

$$w_m := \frac{1}{\sqrt{2}}(e_{2m-1} - ie_{2m}), \quad \bar{w}_m := \frac{1}{\sqrt{2}}(e_{2m-1} + ie_{2m}), \tag{A2}$$

for $m = 1, 2,\ldots,n$. Note that the properties $Jw_m = iw_m$ and $J\bar{w}_m = -i\bar{w}_m$ imply that w_m and $\bar{w}_m$ are holomorphic and antiholomorphic vector fields, respectively. In this frame, the metric components are

$$g(w_m, \bar{w}_l) = g(\bar{w}_m, w_l) = \delta_{ml}, \quad g(w_m, w_l) = g(\bar{w}_m, \bar{w}_l) = 0. \tag{A3}$$

A2. Gamma matrices in Weyl representation

In this appendix, we will consider the gamma matrices in the Weyl representation for a $2n$-dimensional manifold.

Let $\{\gamma^a_{(2n)}\}_{a=1,2,\ldots,2n}$ be a set of square matrices with size 2^n. They are called gamma matrices when they satisfy the Clifford algebra for $\mathbb{R}^{2n}$:

$$\left\{\gamma^a_{(2n)}, \gamma^b_{(2n)}\right\} = 2\delta^{ab} I_{2^n}. \tag{A4}$$

Here, $\{\,,\}$ is the anticommutator and I_{2^n} is the identity matrix with size 2^n. We can also define a chirality matrix by

$$\gamma_{(2n)} := (-i)^n \gamma^1_{(2n)}\gamma^2_{(2n)}\cdots\gamma^{2n}_{(2n)}. \tag{A5}$$

This matrix is Hermitian and anticommutes with all of the gamma matrices $\{\gamma_{(2n)}, \gamma^a_{(2n)}\} = 0$. We can choose a representation such that $\gamma_{(2n)} = \sigma^3 \otimes I_{2^{n-1}}$, where $\{\sigma^a\}_{a=1,2,3}$ are the Pauli matrices and $\otimes$ is the Kronecker product. The Weyl (chiral) representation can be constructed by the recursion

$$\begin{aligned}
&\gamma^1_{(2)} = \sigma^1,\ \gamma^2_{(2)} = \sigma^2,\\
&\gamma^i_{(2n+2)} = \sigma^2 \otimes \gamma^i_{(2n)} \quad (i = 1, 2, \ldots, 2n),\\
&\gamma^{2n+1}_{(2n+2)} = \sigma^2 \otimes \gamma_{(2n)},\\
&\gamma^{2n+2}_{(2n+2)} = -\sigma^1 \otimes I_{2^n}.
\end{aligned} \tag{A6}$$

We extensively use these relations in proofs given in the following appendices.

Also, consider the gamma matrices in the complex orthonormal frame defined by

$$\gamma^m_{(2n)} := \frac{\gamma^{2m-1}_{(2n)} + i\gamma^{2m}_{(2n)}}{\sqrt{2}}, \quad \gamma^{\bar{m}}_{(2n)} := \frac{\gamma^{2m-1}_{(2n)} - i\gamma^{2m}_{(2n)}}{\sqrt{2}}. \tag{A7}$$

These matrices satisfy $\{\gamma^m_{(2n)}, \gamma^{\bar{l}}_{(2n)}\} = 2\delta_{ml} I_{2^n}$, $\{\gamma^m_{(2n)}, \gamma^l_{(2n)}\} = \{\gamma^{\bar{m}}_{(2n)}, \gamma^{\bar{l}}_{(2n)}\} = 0$, and $(\gamma^m_{(2n)})^\dagger = \gamma^{\bar{m}}_{(2n)}$. Let $|\pm\rangle$ be the normalized eigenvector of σ^3 with eigenvalue ± 1. Then, we can recursively show the important properties,

$$\begin{aligned}
&c_m \gamma^{\bar{m}}_{(2n)} |+\rangle^{\otimes n} = 0 \quad \Rightarrow \quad c_m = 0,\\
&\gamma^m_{(2n)} |+\rangle^{\otimes n} = 0, \quad \gamma^m_{(2n)}\gamma^{\bar{l}}_{(2n)} |+\rangle^{\otimes n} = 2\delta_{ml} |+\rangle^{\otimes n},
\end{aligned} \tag{A8}$$

where c_m is a complex number.

A3. Vanishing theorem and index theorem

Let D_i be the Dirac operator on $\Gamma(S_c \otimes L^{\otimes p} \otimes E_i)$. In this appendix, we will show that the zero modes of D_i have positive chirality and $\dim\mathrm{Ker}\, D_i = \mathrm{rank}(E_i)(2\pi\hbar_p)^{-n}\int_M \mu + O(p^{n-1})$ for sufficiently large p. The former is known as the vanishing theorem and the latter is a consequence of the index theorem. We also show that nonzero eigenvalues of D_i have a large gap of $O(\sqrt{p})$. For notational brevity, we will omit superscripts of covariant derivatives and simply write ∇; we also omit the identity operators unless required.

The chirality operator $\gamma_{(2n)} = I_{2^{n-1}} \otimes \sigma^3$ anticommutes with D_i and we find

$$D_i = \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}. \tag{A9}$$

Here, $\pm$ indicates the chirality of the space on which the operators are acting.

We first compute the square of D_i, which is needed to show $\mathrm{Ker} D_i^- = \{0\}$ for large enough p. From Eq. (A9), we have

$$(D_i)^2 = \begin{pmatrix} D_i^- D_i^+ & 0 \\ 0 & D_i^+ D_i^- \end{pmatrix}. \tag{A10}$$

We also use the Weitzenböck formula,

$$(D_i)^2 = -\nabla_a \nabla_a + \frac{i}{2} \hbar_p^{-1} \gamma^a_{(2n)} \gamma^b_{(2n)} \omega_{ab} - \frac{1}{2} \gamma^a_{(2n)} \gamma^b_{(2n)} R^{S_c \otimes E_i}_{ab}, \tag{A11}$$

where $R^{S_c \otimes E_i}_{ab} := R^{S_c \otimes E_i}(e_a, e_b)$. Let us introduce the differential operators

$$\nabla_m := \nabla_{w_m} = \frac{1}{\sqrt{2}}(\nabla_{2m-1} - i\nabla_{2m}), \quad \nabla_{\bar{m}} := \nabla_{\bar{w}_m} = \frac{1}{\sqrt{2}}(\nabla_{2m-1} + i\nabla_{2m}). \tag{A12}$$

Employing these operators, we have, for fixed m,

$$\begin{aligned} \nabla_{2m-1}\nabla_{2m-1} + \nabla_{2m}\nabla_{2m} &= 2\nabla_m \nabla_{\bar{m}} - [\nabla_m, \nabla_{\bar{m}}] \\ &= 2\nabla_m \nabla_{\bar{m}} - \hbar_p^{-1} - R^{S_c \otimes E_i}_{m\bar{m}}. \end{aligned} \tag{A13}$$

Here, we have used Eqs. () and (A2) in the last equality and $R^{S_c \otimes E_i}_{m\bar{m}} := R^{S_c \otimes E_i}(w_m, \bar{w}_m)$. Using the above equation, the first term of Eq. (A11) can be written as

$$-\nabla_a \nabla_a = -2\nabla_m \nabla_{\bar{m}} + n\hbar_p^{-1} + R^{S_c \otimes E_i}_{m\bar{m}}, \tag{A14}$$

where the repeated indices a and m are summed. Hence, we have

$$(D_i)^2 = -2\nabla_m \nabla_{\bar{m}} + \hbar_p^{-1} A_n + R_i, \tag{A15}$$

where

$$\begin{aligned} A_n &:= n + \frac{i}{2} \gamma^a_{(2n)} \gamma^b_{(2n)} \omega_{ab}, \\ R_i &:= -\frac{1}{2} \gamma^a_{(2n)} \gamma^b_{(2n)} R^{S_c \otimes E_i}_{ab} + R^{S_c \otimes E_i}_{m\bar{m}}. \end{aligned} \tag{A16}$$

More explicitly, $R^{S_c \otimes E_i}$ is given by

$$R^{S_c \otimes E_i} = R^S + \frac{1}{2} R^{L_c} + R^{E_i}, \quad R^S_{ab} = \frac{1}{4} R_{abcd} \gamma^c_{(2n)} \gamma^d_{(2n)}, \quad R^{L_c}_{ab} = -R_{abm\bar{m}}, \tag{A17}$$

where R_{abcd} is the Riemann curvature tensor. Then, we have

$$R_i = \frac{1}{2} R + \frac{1}{2} \gamma^a_{(2n)} \gamma^b_{(2n)} R_{abm\bar{m}} - \frac{1}{2} \gamma^a_{(2n)} \gamma^b_{(2n)} R^{E_i}_{ab} + R^{E_i}_{m\bar{m}}, \tag{A18}$$

where R is the scalar curvature and we have used $\gamma^a_{(2n)} \gamma^b_{(2n)} \gamma^c_{(2n)} \gamma^d_{(2n)} R_{abcd} = -2R$ and $R_{m\bar{m}l\bar{l}} = -\frac{1}{2} R$.

A $2^n \times 2^n$ matrix A_n has the following properties if we use the Weyl representation discussed in Appendix A2. The first property is that A_n is diagonal and positive semidefinite. This can be shown recursively as follows. From Eq. (A6), one obtains $A_{n+1} = I_2 \otimes A_n + I_{2^{n+1}} - \sigma^3 \otimes \gamma_{(2n)}$ and it shows that if A_n is diagonal and positive semidefinite, so is A_{n+1}. By checking $A_1 = I_2 - \sigma^3$, which is obviously diagonal and positive semidefinite, we proved the first property. The second property of A_n is that its eigenvector with eigenvalue 0 is proportional to $|+\rangle^{\otimes n}$. This can be shown by a similar recursive method.

Let us use Eq. (A15) to prove $\mathrm{Ker} D_i^- = \{0\}$ for large enough p. For any $\psi \in \Gamma(S_c \otimes L^{\otimes p} \otimes E_i) \backslash \{0\}$, we have

$$|D_i \psi|^2 = 2|\nabla_{\bar{m}} \psi|^2 + \hbar_p^{-1}(\psi, A_n \psi) + (\psi, R_i \psi) \geq \hbar_p^{-1}(\psi, A_n \psi) - |R_i||\psi|^2. \tag{A19}$$

For ψ that is not proportional to $|+\rangle^{\otimes n}$, $(\psi, A_n \psi)$ is strictly positive. Therefore, for sufficiently large p satisfying $\hbar_p^{-1} > |R_i||\psi|^2/(\psi, A_n \psi)$, the right-hand side of Eq. (A19) becomes positive, implying that $D_i \psi \neq 0$. This means that the Dirac zero modes must be proportional to $|+\rangle^{\otimes n}$ for sufficiently large p. Since $|+\rangle^{\otimes n}$ has positive chirality, we conclude that $\mathrm{Ker} D_i^- = \{0\}$ for large enough p.

We next show that $\dim \mathrm{Ker}\, D_i = \mathrm{rank}(E_i)(2\pi\hbar_p)^{-n}\int_M \mu + O(p^{n-1})$. Note that, when $\mathrm{Ker} D_i^- = \{0\}$, we have the following relations:

$$\dim \mathrm{Ker}\, D_i = \dim \mathrm{Ker}\, D_i^+ = \mathrm{Ind}\, D_i. \tag{A20}$$

On the other hand, the Atiyah–Singer index theorem states that

$$\mathrm{Ind} D_i = \int_M \mathrm{Td}(T^{(1,0)}M) \wedge \mathrm{ch}(L^{\otimes p} \otimes E_i) \tag{A21}$$

Here, $\mathrm{Td}(\cdot)$ and $\mathrm{ch}(\cdot)$ are the Todd class and the Chern character, respectively, and $T^{(1,0)}M$ is the holomorphic tangent bundle. By expanding in p, we find

$$\dim \mathrm{Ker}\, D_i = \mathrm{rank}(E_i) \int_M e^{\frac{ip}{2\pi} R^L} + O(p^{n-1}) = \frac{\mathrm{rank}(E_i)}{(2\pi\hbar_p)^n} \int_M \mu + O(p^{n-1}). \tag{A22}$$

Finally, we prove that nonzero eigenvalues of D_i have a large gap of $O(\sqrt{p})$. Let λ be a nonzero eigenvalue of D_i. Then, the eigenvalue equation for $(D_i)^2$ is equivalent to

$$\begin{cases} D_i^- D_i^+ \psi^+ = \lambda^2 \psi^+, \\ D_i^+ D_i^- \psi^- = \lambda^2 \psi^-, \end{cases} \tag{A23}$$

for $\psi \in \Gamma(S_c \otimes L^{\otimes p} \otimes E_i) \backslash \{0\}$, where $\psi^\pm$ is the positive/negative chirality mode of ψ. If $\psi^- \neq 0$, Eq. (A19) implies that $\lambda^2 \geq O(p)$. If $\psi^- = 0$, we have $\psi^+ \neq 0$ in order for ψ to be nonzero. By using the relation $D_i^+ D_i^- (D_i^+ \psi^+) = \lambda^2 (D_i^+ \psi^+)$, we again find that Eq. (A19) implies $\lambda^2 \geq O(p)$. Thus, in any case, we have $\lambda^2 \geq O(p)$. This shows that λ^2 is at least of $O(p)$ and thus the nonzero eigenvalues of D_i indeed have a gap of at least $O(\sqrt{p})$.

A4. Asymptotic expansion for Toeplitz operators

In this appendix, we compute the product $T_p^{(E_1,E_2)}(\varphi) T_p^{(E_2,E_3)}(\chi)$ for $\varphi \in \Gamma(\mathrm{Hom}(E_2, E_1))$ and $\chi \in \Gamma(\mathrm{Hom}(E_3, E_2))$ and show that it can be expanded in a power series of $\hbar_p$ for sufficiently large p. The computation technique used in this appendix is based on Ref. [10].

First, we compute

$$\begin{aligned} T_p^{(E_1,E_2)}(\varphi) T_p^{(E_2,E_3)}(\chi) &= \Pi_1 \varphi \Pi_2 \chi \Pi_3 \\ &= T_p^{(E_1,E_3)}(\varphi\chi) - \Pi_1 \varphi (1 - \Pi_2) \chi \Pi_3. \end{aligned} \tag{A24}$$

For the computation of $1 - \Pi_2$, let us consider the following Hermitian operator on $\Gamma(S_c \otimes L^{\otimes p} \otimes E_2)$:

$$P_2 := \begin{pmatrix} 0 & D_2^- \left(D_2^+ D_2^-\right)^{-1} \\ \left(D_2^+ D_2^-\right)^{-1} D_2^+ & 0 \end{pmatrix}. \tag{A25}$$

Note that, since $\mathrm{Ker} D_2^- = \mathrm{Ker} D_2^+ D_2^- = \{0\}$ for sufficiently large p as shown in Appendix A3, the inverse of $D_2^+ D_2^-$ always exists. Let us consider the following combination:

$$D_2 P_2 = P_2 D_2 = \begin{pmatrix} D_2^- \left(D_2^+ D_2^-\right)^{-1} D_2^+ & 0 \\ 0 & 1 \end{pmatrix}. \tag{A26}$$

This gives the projection onto $(\mathrm{Ker} D_2)^\perp$, which should be equivalent to $1 - \Pi_2$. Thus, we find that

$$1 - \Pi_2 = D_2 P_2 = D_2 (P_2)^2 D_2. \tag{A27}$$

By using Eqs. (A24) and (A27), for $\psi \in \mathrm{Ker} D_1$ and $\phi \in \mathrm{Ker} D_3$, we obtain

$$\left(\psi, T_p^{(E_1,E_2)}(\varphi)\, T_p^{(E_2,E_3)}(\chi)\,\phi\right) = \left(\psi, T_p^{(E_1,E_3)}(\varphi\chi)\,\phi\right) - \left(\psi, \varphi D_2\,(P_2)^2\, D_2\chi\phi\right)$$
$$= \left(\psi, T_p^{(E_1,E_3)}(\varphi\chi)\,\phi\right) + \left(\psi, \varphi'\,(P_2)^2\,\chi'\phi\right). \quad \text{(A28)}$$

Here, we have introduced the notation $\varphi' := i\gamma^a_{(2n)}\nabla_a\varphi$. We also used $D_1\psi = D_3\phi = 0$ and

$$\nabla^{E_1}(\varphi\varphi_2) = (\nabla^{\mathrm{Hom}(E_2,E_1)}\varphi)\varphi_2 + \varphi(\nabla^{E_2}\varphi_2), \quad \text{(A29)}$$

for $\varphi_2 \in \Gamma(E_2)$. Because $\gamma^b_{(2n)}\phi$ has the chirality -1, $\chi'\phi$ is in $(\mathrm{Ker}D_2)^\perp$. On $(\mathrm{Ker}D_2)^\perp$, the projection $1 - \Pi_2 = D_2P_2$ is the identity operator, which implies that P_2 is the inverse of D_2. Thus, Eq. (A28) can be written as

$$\left(\psi, T_p^{(E_1,E_2)}(\varphi)\, T_p^{(E_2,E_3)}(\chi)\,\phi\right) = \left(\psi, T_p^{(E_1,E_3)}(\varphi\chi)\,\phi\right) + \left(\psi, \varphi'\,(D_2)^{-2}\,\chi'\phi\right). \quad \text{(A30)}$$

Let us then calculate $(D_2)^{-2}$ acting on $\chi'\phi$. By using $A_n\gamma^b_{(2n)}\,|+\rangle^{\otimes n} = 2\gamma^b_{(2n)}\,|+\rangle^{\otimes n}$, which can be obtained from Eq. (A8), we have

$$(D_2)^{-2} = \left(-2\nabla_m\nabla_{\bar m} + 2\hbar_p^{-1} + R_2\right)^{-1}$$
$$= \frac{\hbar_p}{2} - \frac{\hbar_p}{2}(D_2)^{-2}R_2 + \hbar_p(D_2)^{-2}\nabla_m\nabla_{\bar m}, \quad \text{(A31)}$$

on $\chi'\phi$. From $D_3\phi = 0$, one can obtain $\nabla_{\bar m}\phi = 0$ (see also Appendix A8). Then, Eq. (A30) becomes

$$(\psi, T_p^{(E_1,E_2)}(\varphi)T_p^{(E_2,E_3)}(\chi)\phi) = (\psi, T_p^{(E_1,E_3)}(\varphi\chi)\phi) + \frac{\hbar_p}{2}(\psi, \varphi'\chi'\phi) + \epsilon, \quad \text{(A32)}$$

where

$$\epsilon = \epsilon_1 + \epsilon_2,$$
$$\epsilon_1 = -\frac{\hbar_p}{2}(\psi, \varphi'(D_2)^{-2}R_2\chi'\phi)$$
$$\epsilon_2 = \hbar_p(\psi, \varphi'(D_2)^{-2}(\nabla_m\nabla_{\bar m}\chi')\phi) + \hbar_p(\psi, \varphi'(D_2)^{-2}(\nabla_{\bar m}\chi')\nabla_m\phi). \quad \text{(A33)}$$

Let us estimate the order of ϵ with respect to $\hbar_p$. If we set ϕ, ψ, φ, and χ to $O(\hbar_p^0)$, the nontrivial p-dependences only appear in $\nabla_m\phi$ and $(D_2)^{-2}$. As we discussed in Appendix A3, all eigenvalues of $(D_2)^2$ are in the range $[C_1\hbar_p^{-1} - C_2, \infty)$, where C_1 and C_2 are p-independent constants. Hence, the eigenvalues of $(D_2)^{-2}$ are in $(0, (C_1\hbar_p^{-1} - C_2)^{-1}]$. From this property and the fact that the norm of a positive operator is equal to its maximum eigenvalues, we find that $|(D_2)^{-2}| = O(\hbar_p)$. For $\nabla_m\phi$, we can calculate

$$|\nabla_m\phi|^2 = -(\phi, \nabla_{\bar m}\nabla_m\phi) = (\phi, [\nabla_m, \nabla_{\bar m}]\phi) - (\phi, \nabla_m\nabla_{\bar m}\phi)$$
$$= \hbar_p^{-1}|\phi|^2 - \left(\phi, R^{S_c\otimes E_3}_{m\bar m}\phi\right)$$
$$= O\left(\hbar_p^{-1}\right). \quad \text{(A34)}$$

From these estimations, it follows that

$$|\epsilon_1| = O\left(\hbar_p^2\right), \quad |\epsilon_2| = O\left(\hbar_p^{3/2}\right). \quad \text{(A35)}$$

Then, we obtain

$$\left(\psi, T_p^{(E_1,E_2)}(\varphi)\, T_p^{(E_2,E_3)}(\chi)\,\phi\right)$$
$$= \left(\psi, T_p^{(E_1,E_3)}(\varphi\chi)\,\phi\right) - \frac{\hbar_p}{2}\left(\psi, (\nabla_a\varphi)(\nabla_b\chi)\,\gamma^a_{(2n)}\gamma^b_{(2n)}\phi\right) + O\left(\hbar_p^{3/2}\right). \quad \text{(A36)}$$

From Eq. (A8), we have

$$(\psi, T_p^{(E_1,E_2)}(\varphi)T_p^{(E_2,E_3)}(\chi)\phi) = (\psi, T_p^{(E_1,E_3)}(\varphi\chi)\phi) - \hbar_p(\psi, (\nabla_m\varphi)(\nabla_{\bar{m}}\chi)\phi) + O(\hbar_p^{3/2}). \tag{A37}$$

From the asymptotic expansion of the Bergmann kernel [48], the products of the Toeplitz operators also allow asymptotic expansion of integer power. Thus, the $O(\hbar_p^{3/2})$ term is actually further bounded to $O(\hbar_p^2)$. The expansion (A37) reproduces $C_0(\varphi, \chi)$ and $C_1(\varphi, \chi)$ in Eq. (13). This can be checked by noticing that the tensor $G^{ab} = g^{ab} + iW^{ab}$ has components $G^{m\bar{l}} = 2\delta_{ml},\ G^{\bar{m}l} = G^{ml} = G^{\bar{m}\bar{l}} = 0$.

One can also evaluate $C_2(\varphi, \chi)$ by recursively using Eq. (A31). Applying Eq. (A31) to ϵ_1, one finds

$$\begin{aligned}\epsilon_1 &= -\frac{\hbar_p}{4}\left(\psi, \varphi'\left(\hbar_p - \hbar_p (D_2)^{-2} R_2 + 2\hbar_p (D_2)^{-2}\nabla_m\nabla_{\bar{m}}\right) R_2\chi'\phi\right)\\ &= -\frac{\hbar_p^2}{4}\left(\psi, \varphi' R_2\chi'\phi\right) + O\left(\hbar_p^{5/2}\right).\end{aligned} \tag{A38}$$

Applying Eq. (A31) to ϵ_2, one finds

$$\begin{aligned}\epsilon_2 &= \frac{\hbar_p}{2}\left(\psi, \varphi'\left(\hbar_p - \hbar_p (D_2)^{-2} R_2 + 2\hbar_p (D_2)^{-2}\nabla_l\nabla_{\bar{l}}\right)\nabla_m\left(\nabla_{\bar{m}}\chi'\right)\phi\right)\\ &= \frac{\hbar_p^2}{2}\left(\psi, \varphi'\nabla_m\left(\nabla_{\bar{m}}\chi'\right)\phi\right) + \hbar_p^2\left(\psi, \varphi'(D_2)^{-2}\nabla_l\nabla_{\bar{l}}\nabla_m\left(\nabla_{\bar{m}}\chi'\right)\phi\right) + O\left(\hbar_p^{5/2}\right)\\ &= -\frac{\hbar_p^2}{2}\left(\psi, \left(\nabla_m\varphi'\right)\left(\nabla_{\bar{m}}\chi'\right)\phi\right) - \hbar_p\left(\psi, \varphi'(D_2)^{-2}\nabla_m\left(\nabla_{\bar{m}}\chi'\right)\phi\right)\\ &\quad + \hbar_p^2\left(\psi, \varphi'(D_2)^{-2}\nabla_l\nabla_m\left(\nabla_{\bar{l}}\nabla_{\bar{m}}\chi'\right)\phi\right) + O\left(\hbar_p^{5/2}\right).\end{aligned} \tag{A39}$$

Note that the second term of the last expression is exactly equal to $-\epsilon_2$. This implies

$$\begin{aligned}\epsilon_2 &= -\frac{\hbar_p^2}{4}\left(\psi, \left(\nabla_m\varphi'\right)\left(\nabla_{\bar{m}}\chi'\right)\phi\right) + \epsilon_2' + O\left(\hbar_p^{5/2}\right),\\ \epsilon_2' &= \frac{\hbar_p^2}{2}\left(\psi, \varphi'(D_2)^{-2}\nabla_l\nabla_m\left(\nabla_{\bar{l}}\nabla_{\bar{m}}\chi'\right)\phi\right).\end{aligned} \tag{A40}$$

Again using Eq. (A31) with ϵ_2', we have

$$\begin{aligned}\epsilon_2' &= \frac{\hbar_p^2}{4}\left(\psi, \varphi'\left(\hbar_p - \hbar_p (D_2)^{-2} R_2 + 2\hbar_p (D_2)^{-2}\nabla_k\nabla_{\bar{k}}\right)\nabla_l\nabla_m\left(\nabla_{\bar{l}}\nabla_{\bar{m}}\chi'\right)\phi\right)\\ &= \frac{\hbar_p^3}{2}\left(\psi, \varphi'(D_2)^{-2}\nabla_k\nabla_{\bar{k}}\nabla_l\nabla_m\left(\nabla_{\bar{l}}\nabla_{\bar{m}}\chi'\right)\phi\right) + O\left(\hbar_p^3\right)\\ &= \frac{\hbar_p^3}{2}\left(\psi, \varphi'(D_2)^{-2}\nabla_k\nabla_l\nabla_{\bar{k}}\nabla_m\left(\nabla_{\bar{l}}\nabla_{\bar{m}}\chi'\right)\phi\right) - \frac{\hbar_p^2}{2}\left(\psi, \varphi'(D_2)^{-2}\nabla_l\nabla_m\left(\nabla_{\bar{l}}\nabla_{\bar{m}}\chi'\right)\phi\right) + O\left(\hbar_p^3\right)\\ &= \frac{\hbar_p^3}{2}\left(\psi, \varphi'(D_2)^{-2}\nabla_k\nabla_l\nabla_m\left(\nabla_{\bar{k}}\nabla_{\bar{l}}\nabla_{\bar{m}}\chi'\right)\phi\right) - 2\epsilon_2' + O\left(\hbar_p^3\right).\end{aligned} \tag{A41}$$

Similar to Eq. (A34), we can find $|\nabla_k\nabla_l\nabla_m\phi| = O(\hbar_p^{-3/2})$ and $|\nabla_l\nabla_m\phi| = O(\hbar_p^{-1})$. This implies $\epsilon_2' = O(\hbar_p^{5/2})$. Therefore, we obtain

$$\epsilon = -\frac{\hbar_p^2}{4}\left(\psi, \varphi' R_2\chi'\phi\right) - \frac{\hbar_p^2}{4}\left(\psi, \left(\nabla_m\varphi'\right)\left(\nabla_{\bar{m}}\chi'\right)\phi\right) + O\left(\hbar_p^{5/2}\right). \tag{A42}$$

From Eqs. (A18) and (A8), one finds

$$\epsilon = \hbar_p^2 \left(\psi, (\nabla_m\varphi)\left(R_{\bar{m}lk\bar{k}} - R^{E_2}_{\bar{m}l}\right)(\nabla_{\bar{l}}\chi)\,\phi\right) + \frac{\hbar_p^2}{2}\left(\psi, (\nabla_m\nabla_l\varphi)\left(\nabla_{\bar{m}}\nabla_{\bar{l}}\chi\right)\phi\right) + O\left(\hbar_p^{5/2}\right). \tag{A43}$$

This gives the coefficient $C_2(\varphi, \chi)$ of the asymptotic expansion (13).

A5. Consistency check of the asymptotic expansion

In this appendix, we check that the asymptotic expansion (12) with Eq. (13) derived in Appendix A4 indeed satisfies the associativity of the Toeplitz operator product. For $\varphi \in \Gamma(\mathrm{Hom}(E_2, E_1))$, $\chi \in \Gamma(\mathrm{Hom}(E_3, E_2))$, and $\psi \in \Gamma(\mathrm{Hom}(E_4, E_3))$, it should be true that

$$\left(T_p^{(E_1,E_2)}(\varphi)T_p^{(E_2,E_3)}(\chi)\right)T_p^{(E_3,E_4)}(\psi) = T_p^{(E_1,E_2)}(\varphi)\left(T_p^{(E_2,E_3)}(\chi)T_p^{(E_3,E_4)}(\psi)\right). \tag{A44}$$

This imposes a condition

$$\sum_{i=0}^{j} C_{j-i}\left(C_i(\varphi, \chi), \psi\right) - C_i\left(\varphi, C_{j-i}(\chi, \psi)\right) = 0, \tag{A45}$$

for all $j \in \mathbb{Z}_{\geq 0}$.

We will check that the conditions (A45) for $j = 0, 1, 2$ are satisfied by C_0, C_1, C_2 given in Eq. (13). The condition for $j = 0$ is satisfied from the associativity of the linear maps:

$$C_0\left(C_0(\varphi, \chi), \psi\right) - C_0\left(\varphi, C_0(\chi, \psi)\right) = (\varphi\chi)\psi - \varphi(\chi\psi) = 0. \tag{A46}$$

For $j = 1$, the left-hand side of Eq. (A45) is given by

$$\begin{aligned}&\sum_{i=0}^{1} C_{1-i}\left(C_i(\varphi, \chi), \psi\right) - C_i\left(\varphi, C_{1-i}(\chi, \psi)\right)\\ &\quad= -\nabla_m(\varphi\chi)(\nabla_{\bar{m}}\psi) + \varphi(\nabla_m\chi)(\nabla_{\bar{m}}\psi) - (\nabla_m\varphi)(\nabla_{\bar{m}}\chi)\psi + (\nabla_m\varphi)\nabla_{\bar{m}}(\chi\psi).\end{aligned} \tag{A47}$$

This is vanishing because of the Leibniz rule of the covariant derivatives. Similarly, the condition for $j = 2$ is also satisfied:

$$\begin{aligned}&\sum_{i=0}^{2} C_{2-i}\left(C_i(\varphi, \chi), \psi\right) - C_i\left(\varphi, C_{2-i}(\chi, \psi)\right)\\ &\quad= -(\nabla_m\varphi)\,\chi R^{E_3}_{\bar{m}l}\left(\nabla_{\bar{l}}\psi\right) + (\nabla_m\varphi)\,R^{E_2}_{\bar{m}l}\chi\left(\nabla_{\bar{l}}\psi\right) - (\nabla_m\varphi)\left([\nabla_{\bar{m}}, \nabla_l]\chi\right)\nabla_{\bar{l}}\psi\\ &\quad= 0.\end{aligned} \tag{A48}$$

Thus, the asymptotic expansion given in Eqs. (12) and (13) is consistent with the associativity condition (A44) up to $\hbar_p^2$.

A6. Trace of Toeplitz operators

In this appendix, we will show Eq. (19).

First, by using the Schwartz kernel representation, the trace of $T_p^{(E_1,E_1)}(\varphi)$ is expressed as

$$\mathrm{Tr}\,T_p^{(E_1,E_1)}(\varphi) = \int_M \mu(x)\mathrm{tr}_{S_c\otimes E_1}\left(B(x, x)\varphi(x)\right) \tag{A49}$$

where $B(x, y)$ is the Bergman kernel defined by

$$(\Pi_1\psi)(x) = \int_M \mu(y)B(x, y)\psi(y) \tag{A50}$$

for any $\psi \in \Gamma(S_c \otimes L^{\otimes p} \otimes E_1)$. In Ref. [48], it is shown that the Bergmann kernel has the following large-p asymptotic expansion:

$$B(x,x) = (2\pi\hbar_p)^{-n} P \mathbf{1}_{E_1} + O(p^{n-1}), \tag{A51}$$

where P is the projection onto the zero-mode component $|+\rangle^{\otimes n}$ of the fiber of S. By plugging Eq. (A51) into Eq. (A49), we obtain Eq. (19).

A7. General properties of the Laplace operator on $\Gamma(E)$

In this appendix, we show that the Bochner Laplacian defined in Eq. (20) can be expressed as

$$\Delta^{(E)}\varphi = -g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = -\{X^A, \{X^A, \varphi\}\}, \tag{A52}$$

where $\{\cdot, \cdot\}$ is the generalized Poisson bracket (17) and X^A is an isometric embedding function.

From the definition of the generalized Poisson bracket (17), we have

$$\begin{aligned}
-\{X^A, \{X^A, \varphi\}\} &= -W^{\alpha\beta}W^{\gamma\delta}(\partial_\alpha X^A)\nabla_\beta[(\partial_\gamma X^A)(\nabla_\delta\varphi)] \\
&= -W^{\alpha\beta}W^{\gamma\delta}(\partial_\alpha X^A)[(\nabla_\beta\partial_\gamma X^A)(\nabla_\delta\varphi) + (\partial_\gamma X^A)(\nabla_\beta\nabla_\delta\varphi)] \\
&= -W^{\alpha\beta}W^{\gamma\delta}[\nabla_\beta((\partial_\alpha X^A)(\partial_\gamma X^A))(\nabla_\delta\varphi) - (\nabla_\beta\partial_\alpha X^A)(\partial_\gamma X^A)(\nabla_\delta\varphi) \\
&\quad + (\partial_\alpha X^A)(\partial_\gamma X^A)(\nabla_\beta\nabla_\delta\varphi)] \\
&= -W^{\alpha\beta}W^{\gamma\delta}[(\nabla_\beta g_{\alpha\gamma})(\nabla_\delta\varphi) - (\nabla_\beta\partial_\alpha X^A)(\partial_\gamma X^A)(\nabla_\delta\varphi) + g_{\alpha\gamma}(\nabla_\beta\nabla_\delta\varphi)] \\
&= -W^{\alpha\beta}W^{\gamma\delta}g_{\alpha\gamma}(\nabla_\beta\nabla_\delta\varphi) \\
&= -g^{\beta\delta}(\nabla_\beta\nabla_\delta\varphi).
\end{aligned} \tag{A53}$$

Here, we have used $\nabla W = 0$, which follows from the general properties of the Kähler structure, $\nabla g = \nabla J = \nabla\omega = 0$. In the last equality, we used $W^{\alpha\beta}W^{\gamma\delta}g_{\alpha\gamma} = g^{\beta\delta}$, which we can check using the local orthonormal frame. Therefore, Eq. (23) holds for any Kähler manifold M.

A8. Simplification of the zero-mode equation

In this appendix, we argue that the Dirac equation is reduced to a simpler differential equation of holomorphic sections.

The twisted spin-c Dirac operator $\Gamma(S_c \otimes L^{\otimes p} \otimes E)$ over M is given by Eq. (9). From $\Omega_{ml} = \Omega_{\bar{m}\bar{l}} = 0$, we have

$$\Omega_{ab}\gamma^a_{(2n)}\gamma^b_{(2n)} = \Omega_{m\bar{l}}\gamma^m_{(2n)}\gamma^{\bar{l}}_{(2n)} + \Omega_{\bar{l}m}\gamma^{\bar{l}}_{(2n)}\gamma^m_{(2n)} = 2\Omega_{m\bar{l}}\gamma^m_{(2n)}\gamma^{\bar{l}}_{(2n)} - 2\sum_{m=1}^{n}\Omega_{m\bar{m}}, \tag{A54}$$

where we have used $\Omega_{m\bar{l}} = -\Omega_{\bar{l}m}$ and $\{\gamma^m_{(2n)}, \gamma^{\bar{l}}_{(2n)}\} = 2\delta_{ml}I_{2^n}$ in the last equality. As shown in Appendix A3, the zero mode ψ is of the form $\psi = f\,|+\rangle^{\otimes n}$, where f is a section of $L^{\otimes p} \otimes E$. From Eq. (A8), we then have

$$\begin{aligned}
D\psi &= i\bar{w}_m{}^{\bar{\mu}}\gamma^{\bar{m}}_{(2n)}|+\rangle^{\otimes n}\left(\partial_{\bar{\mu}} + pA^L_{\bar{\mu}} + A^E_{\bar{\mu}}\right)f = 0 \\
&\Rightarrow \quad {}^\forall m \in \{1,\ldots,n\}: \quad \bar{w}_m{}^{\bar{\mu}}\left(\partial_{\bar{\mu}} + pA^L_{\bar{\mu}} + A^E_{\bar{\mu}}\right)f = 0 \\
&\Rightarrow \quad \left(\partial_{\bar{\mu}} + pA^L_{\bar{\mu}} + A^E_{\bar{\mu}}\right)f = 0.
\end{aligned} \tag{A55}$$

This indicates that f is a holomorphic section of $L^{\otimes p} \otimes E$.

Appendix B. Proofs and formulas for CP^n

B1. Integration formula for CP^n

In this appendix, we calculate

$$I_{s,t,p} := \int_{CP^n} \mu \, \frac{\prod_{i=1}^n (\bar{z}^i)^{s_i} (z^i)^{t_i}}{(1+|z|^2)^p}, \tag{B1}$$

which is a typical integral appearing in our discussion of CP^n. Here, $s = (s_1, s_2, \ldots, s_n)$, $t = (t_1, t_2, \ldots, t_n) \in (\mathbb{Z}_{\geq 0})^n$, and $p \in \mathbb{Z}$. The result is

$$I_{s,t,p} = I_{s,p}\delta_{s,t}, \quad I_{s,p} = \frac{(2\pi)^n (p - \sum_{i=1}^n s_i)! \prod_{i=1}^n (s_i!)}{(p+n)!}. \tag{B2}$$

Here, the Kronecker delta is defined as $\delta_{s,t} := \prod_{i=1}^n \delta_{s_i,t_i}$ and the factor $I_{s,\,p}$ is convergent if and only if

$$\sum_{i=1}^n s_i < p + 1. \tag{B3}$$

Now, let us begin the proof. First, since $CP^n \backslash U_\alpha$ has zero measure, the integral over CP^n is computed in a single patch:

$$I_{s,t,p} = \int_{\mathbb{R}^{2n}} \frac{\prod_{i=1}^n \left(\frac{x^{2i-1} - i x^{2i}}{\sqrt{2}}\right)^{s_i} \left(\frac{x^{2i-1} + i x^{2i}}{\sqrt{2}}\right)^{t_i}}{\left(1 + \frac{|x|^2}{2}\right)^{p+n+1}} dx^1 dx^2 \cdots dx^{2n}. \tag{B4}$$

Here, we are using real coordinates $x = (x^1, x^2, \ldots, x^{2n})$ defined by

$$x^{2\mu-1} = \frac{z^\mu + \bar{z}^\mu}{\sqrt{2}}, \quad x^{2\mu} = \frac{z^\mu - \bar{z}^\mu}{\sqrt{2}i}. \tag{B5}$$

We can employ the angular coordinates $(\rho_i, \theta_i) \in [0, \infty) \times [0, 2\pi)$ such that

$$x^{2i-1} = \sqrt{2}\,\rho_i \cos\theta_i, \;\; x^{2i} = \sqrt{2}\,\rho_i \sin\theta_i. \tag{B6}$$

This gives

$$I_{s,t,p} = \prod_{i=1}^n \left(2\int_0^\infty \rho_i d\rho_i \int_0^{2\pi} d\theta_i\right) \frac{\prod_{i=1}^n \left(\rho_i e^{i\theta_i}\right)^{s_i} \left(\rho_i e^{-i\theta_i}\right)^{t_i}}{\left(1 + \sum_{i=1}^n \rho_i^2\right)^{p+n+1}}. \tag{B7}$$

The angular integrals give a factor $\delta_{s,\,t}$. Then, we obtain $I_{s,\,t,\,p} = I_{s,\,p}\delta_{s,\,t}$ where

$$I_{s,p} = (4\pi)^n \int_{[0,\infty)^n} \frac{d\rho_1 d\rho_2 \cdots d\rho_n}{\left(1 + \sum_{i=1}^n \rho_i^2\right)^{p+n+1}} \prod_{i=1}^n \rho_i^{2s_i+1}. \tag{B8}$$

We can use the spherical coordinates $(\rho, \phi_1, \phi_2, \ldots, \phi_{n-1}) \in [0, \infty) \times [0, \pi/2]^{n-1}$ given by

$$\begin{aligned} &\rho_1 = \rho\cos\phi_1, \quad \rho_2 = \rho\sin\phi_1\cos\phi_2, \quad \ldots, \\ &\rho_{n-1} = \rho\left(\prod_{i=1}^{n-2}\sin\phi_i\right)\cos\phi_{n-1}, \quad \rho_n = \rho\prod_{i=1}^{n-1}\sin\phi_i, \end{aligned} \tag{B9}$$

and we obtain

$$I_{s,p} = (4\pi)^n \int_0^\infty d\rho \frac{\rho^{2\sum_{i=1}^n (s_i+1)-1}}{(1+\rho^2)^{p+n+1}} \prod_{i=1}^{n-1}\left(\int_0^{\pi/2} d\phi_i \sin^{2\sum_{j=i+1}^n (s_j+1)-1}(\phi_i)\cos^{2s_i+1}(\phi_i)\right). \tag{B10}$$

Note that the Beta function

$$B(x, y) = 2\int_0^{\pi/2} d\phi \sin^{2x-1}\phi\, \cos^{2y-1}\phi = 2\int_0^\infty d\rho \frac{\rho^{2x-1}}{(1+\rho^2)^{x+y}} \tag{B11}$$

only converges for $\mathrm{Re}\,x, \mathrm{Re}\,y > 0$. Then, we can see that $I_{r,w}$ is convergent if and only if Eq. (B3) is satisfied and the value of $I_{s,p}$ is

$$I_{s,p} = (2\pi)^n B\left(\sum_{i=1}^{n}(s_i+1),\, p+1-\sum_{i=1}^{n} s_i\right)\prod_{i=1}^{n-1} B\left(\sum_{j=i+1}^{n}(s_j+1),\, s_i+1\right). \tag{B12}$$

Using $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ and $\Gamma(x+1) = x!$, we finally obtain Eq. (B2).

B2. Proof of Eq. (76)

Here, we will show that the operator (75) satisfies the commutation relation (76).

We first show

$$\{X^A, X^B\} = -\sqrt{2} f_{ABC} X^C, \tag{B13}$$

which is needed in the proof of Eq. (76). In the complex coordinates, the Poisson tensor is given by $W^{\mu\bar{\nu}} = -W^{\bar{\nu}\mu} = -ig^{\mu\bar{\nu}}$ and it gives

$$\{X^A, X^B\} = -ig^{\mu\bar{\nu}}[(\partial_\mu X^A)(\partial_{\bar{\nu}} X^B) - (A \leftrightarrow B)]. \tag{B14}$$

From Eqs. (41) and (36), the embedding function can be written as

$$X^A = -\frac{\sqrt{2}}{1+|z|^2}\left(\bar{z}^\mu (T_A)_{\mu\nu} z^\nu + (T_A)_{\mu\, n+1}\bar{z}^\mu + (T_A)_{n+1\,\mu} z^\mu + (T_A)_{n+1\, n+1}\right). \tag{B15}$$

By using this expression, we have

$$\partial_\mu X^A = -\frac{\bar{z}^\mu X^A}{1+|z|^2} - \frac{\sqrt{2}(\zeta^\dagger T_A)_\mu}{\sqrt{1+|z|^2}}, \quad \partial_{\bar{\nu}} X^A = -\frac{z^\nu X^A}{1+|z|^2} - \frac{\sqrt{2}(T_A\zeta)_\nu}{\sqrt{1+|z|^2}}. \tag{B16}$$

Also using Eq. (35), we obtain

$$\begin{aligned} g^{\mu\bar{\nu}}\partial_\mu X^A &= \sqrt{2(1+|z|^2)}\left((\zeta^\dagger T_A)_{n+1}\bar{z}^\nu - (\zeta^\dagger T_A)_\nu\right), \\ g^{\bar{\nu}\mu}\partial_{\bar{\nu}} X^A &= \sqrt{2(1+|z|^2)}\left((T_A\zeta)_{n+1} z^\mu - (T_A\zeta)_\mu\right). \end{aligned} \tag{B17}$$

Thus, we have

$$\begin{aligned} \{X^A, X^B\} &= i\sqrt{2(1+|z|^2)}\left((\zeta^\dagger T_A)_{n+1}\bar{z}^\nu - (\zeta^\dagger T_A)_\nu\right)\left(\frac{z^\nu X^B}{1+|z|^2} + \frac{\sqrt{2}(T_B\zeta)_\nu}{\sqrt{1+|z|^2}}\right) - (A \leftrightarrow B) \\ &= i2\zeta^\dagger [T_A, T_B]\zeta. \end{aligned} \tag{B18}$$

Using Eqs. (38) and (41), we obtain Eq. (B13).

Let us prove Eq. (76). From the definition (75), we have

$$[\mathcal{L}_A^{(q)}, \mathcal{L}_B^{(q)}] f^{(q)} = -\frac{1}{2}\{X^A, \{X^B, f^{(q)}\}\} + \frac{1}{2}\{X^B, \{X^A, f^{(q)}\}\} - iq\{X^A, X^B\} f^{(q)}. \tag{B19}$$

Using the definition of the generalized Poisson bracket, we calculate as follows:

$$\begin{aligned} 2\left[\mathcal{L}_A^{(q)}, \mathcal{L}_B^{(q)}\right] f^{(q)} &= -W^{\alpha\beta} W^{\gamma\delta}(\partial_\alpha X^A)\nabla_\beta[(\partial_\gamma X^B)(\nabla_\delta f^{(q)})] \\ &\quad + W^{\alpha\beta} W^{\gamma\delta}(\partial_\alpha X^B)\nabla_\beta[(\partial_\gamma X^A)(\nabla_\delta f^{(q)})] - i2q\{X^A, X^B\} f^{(q)} \\ &= -iq\{X^A, X^B\} f^{(q)} - (W^{\alpha\beta} W^{\gamma\delta} - W^{\gamma\beta} W^{\alpha\delta}) \\ &\quad \times (\nabla_\beta[(\partial_\alpha X^A)(\partial_\gamma X^B)])(\nabla_\delta f^{(q)}). \end{aligned} \tag{B20}$$

Here, we have used $[\nabla_\beta, \nabla_\delta]f^{(q)} = -iq\omega_{\beta\delta}f^{(q)}$ and $\omega_{\mu\nu}W^{\mu\rho} = \delta^\rho_\nu$. By using $W^{\alpha\beta}W^{\gamma\delta} - W^{\gamma\beta}W^{\alpha\delta} = W^{\alpha\gamma}W^{\beta\delta}$, which we can check in the orthonormal coordinates, we obtain

$$\begin{aligned}\left[\mathcal{L}^{(q)}_A, \mathcal{L}^{(q)}_B\right] f^{(q)} &= -\frac{1}{2}W^{\alpha\gamma}W^{\beta\delta}(\nabla_\beta[(\partial_\alpha X^A)(\partial_\gamma X^B)])(\nabla_\delta f^{(q)}) - i\frac{q}{2}\{X^A, X^B\}f^{(q)} \\ &= -\frac{1}{2}\{\{X^A, X^B\}, f^{(q)}\} - i\frac{q}{2}\{X^A, X^B\}f^{(q)}. \end{aligned} \tag{B21}$$

Therefore, using Eqs. (B13) and (75), we have shown the relation (76).

B3. Direct calculation of Eq. (84)

Let us evaluate $\Delta^{(q)}f^{(q)}_{k,I}$. First, by the definition of $\Delta^{(q)}$, we have

$$\Delta^{(q)} = -g^{\mu\bar{\nu}}\left(D^{(q)}_\mu D^{(q)}_{\bar{\nu}} + D^{(q)}_{\bar{\nu}} D^{(q)}_\mu\right), \tag{B22}$$

The covariant derivatives on $f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k}$ defined in Eq. (82) are given by

$$\begin{aligned} D^{(q)}_\mu f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k} &= \left(\partial_\mu + qA^L_\mu\right) f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k} = \left(\sum_{a=1}^{k+q}\frac{\delta_{\mu,\sigma_a}}{z^{\sigma_a}} - (k+q)\frac{\bar{z}^\mu}{1+|z|^2}\right) f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k}, \\ D^{(q)}_{\bar{\nu}} f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k} &= (\partial_{\bar{\nu}} + qA^L_{\bar{\nu}}) f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k} = \left(\sum_{b=1}^{k}\frac{\delta_{\nu,\tau_b}}{\bar{z}^{\tau_b}} - k\frac{z^\nu}{1+|z|^2}\right) f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k}. \end{aligned} \tag{B23}$$

Here, we set $z^{n+1} = \bar{z}^{n+1} = 1$. Thus, we have

$$\Delta^{(q)} f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k} = -2g^{\mu\bar{\nu}}\left[\left(\sum_{a=1}^{k+q}\frac{\delta_{\mu,\sigma_a}}{z^{\sigma_a}} - (k+q)\frac{\bar{z}^\mu}{1+|z|^2}\right)\left(\sum_{b=1}^{k}\frac{\delta_{\nu,\tau_b}}{\bar{z}^{\tau_b}} - k\frac{z^\nu}{1+|z|^2}\right) - \left(k+\frac{q}{2}\right)g_{\mu\bar{\nu}}\right] f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k}. \tag{B24}$$

By using Eqs. (35) and (85), we obtain

$$\Delta^{(q)} f^{(q)}_{k,I} = 2\left(k(k+q) + n\left(k + \frac{q}{2}\right)\right) f^{(q)}_{k,I}. \tag{B25}$$

Here, we have used the traceless property $\sum_{\sigma_a,\tau_b} c^{(q)}_{I,\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k}\delta_{\sigma_a,\tau_b} = 0$. By comparing Eq. (B25) with Eq. (77), we find Eq. (84).

B4. Proof of Eq. (95)

In this appendix, we give a proof of the important identity (95).

Using Eqs. (B17) and (B23), we have

$$\begin{aligned} \mathcal{L}^{(q)}_A f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k} &= \left(-\sum_{a=1}^{p+q}\frac{(T_A\zeta)_{\sigma_a}}{\zeta^{\sigma_a}} + \sum_{b=1}^{p}\frac{(\zeta^\dagger T_A)_{\tau_b}}{\bar{\zeta}^{\tau_b}}\right) f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k} \\ &= -\sum_{i,j=1}^{n+1}(T_A)_{ij}n_i(\boldsymbol{\sigma}_{k+q}) f^{(q)}_{\boldsymbol{\sigma}_{k+q}\ominus i\oplus j,\boldsymbol{\tau}_k} + \sum_{i,j=1}^{n+1}(T_A)_{ij}n_j(\boldsymbol{\tau}_k) f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k\ominus j\oplus i}. \end{aligned} \tag{B26}$$

Here, $n_i(\boldsymbol{\alpha}_p)$ is the number of components of $\boldsymbol{\alpha}_p$ equal to i and $\ominus$ is the inverse operation of $\oplus$, namely, $\boldsymbol{\tau}_k \ominus j = (\tau_1, \ldots, \tau_{b-1}, \tau_{b+1}, \ldots, \tau_k)$ for $j = \tau_b$. We calculate the Toeplitz operator of the above object as

$$\begin{aligned}
&T_p(\mathcal{L}_A^{(q)} f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k})_{\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_p}\\
&= -\sum_{i,j=1}^{n+1} (T_A)_{ij}\, n_i\left(\boldsymbol{\sigma}_{k+q}\right) T_p\left(f^{(q)}_{\boldsymbol{\sigma}_{k+q}\ominus i\oplus j,\boldsymbol{\tau}_k}\right)_{\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_p} + \sum_{i,j=1}^{n+1} (T_A)_{ij}\, n_j\left(\boldsymbol{\tau}_k\right) T_p\left(f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k\ominus j\oplus i}\right)_{\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_p}\\
&= \frac{c^{(p+q)}_{\boldsymbol{\alpha}_{p+q}} c^{(p)}_{\boldsymbol{\beta}_p}}{\left(c^{(p+q+k)}_{\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k}\right)^2} \sum_{i,j=1}^{n+1} (T_A)_{ij}\delta_{\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k\oplus i,\boldsymbol{\beta}_p\oplus\boldsymbol{\sigma}_{k+q}\oplus j}\left[-n_i(\boldsymbol{\sigma}_{k+q}) + \frac{n_j(\boldsymbol{\tau}_k)(n_i(\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k)+1)}{n_j(\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k)+\delta_{i,j}}\right],
\end{aligned} \tag{B27}$$

where we have used Eqs. (89) and (53). On the other hand, $L_A \circ T_p(f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k})$ is given by

$$\begin{aligned}
\left(L_A \circ T_p\left(f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k}\right)\right)_{\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_p} &= \frac{c^{(p+q)}_{\boldsymbol{\alpha}_{p+q}} c^{(p)}_{\boldsymbol{\beta}_p}}{\left(c^{(p+q+k)}_{\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k}\right)^2} \sum_{i,j=1}^{n+1} (T_A)_{ij}\delta_{\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k\oplus i,\boldsymbol{\beta}_p\oplus\boldsymbol{\sigma}_{k+q}\oplus j}\\
&\quad\times\left(-\frac{(n_j(\boldsymbol{\alpha}_{p+q})+\delta_{i,j})(n_i(\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k)+1)}{n_j(\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k)+\delta_{i,j}} + n_i(\boldsymbol{\beta}_p)+\delta_{i,j}\right).
\end{aligned} \tag{B28}$$

Here, we have used Eqs. (89), (53), and the following expression of $L_A^{(p)}$:

$$\left(L_A^{(p)}\right)_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p} = -\frac{c^{(p)}_{\boldsymbol{\beta}_p}}{c^{(p)}_{\boldsymbol{\alpha}_p}} \sum_{i,j=1}^{n+1} (T_A)_{ij}(n_i(\boldsymbol{\alpha}_p)+1)\delta_{\boldsymbol{\alpha}_p\oplus i,\boldsymbol{\beta}_p\oplus j}, \tag{B29}$$

which follows from Eqs. (65) and (62). Comparing Eq. (B27) with Eq. (B28), we find $T_p(\mathcal{L}_A^{(q)} f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k}) = L_A \circ T_p(f^{(q)}_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k})$, which implies Eq. (95).

B5. Proof of Eq. (100)

In this appendix, we prove Eq. (100).

Let us start with Eq. (99) for $k, k' \leq p$. For fixed $k \leq p$, we first show that the proportional factor does not depend on I labeling the different weights of the eigenstates. Let $\{H_a\}_{a=1}^n$ be a basis of Cartan subalgebra of $\mathfrak{su}(n+1)$, i.e., a set of mutually commuting elements in $\{T_A\}_{A=1}^{n^2+2n}$. Then, there exists a complete basis of $\mathfrak{su}(n+1)$ called the Cartan–Weyl basis $\{H_a, E_\alpha\}$ satisfying

$$[H_a, H_b] = 0, \quad [H_a, E_{\pm\alpha}] = \pm\alpha_a E_{\pm\alpha}, \quad [E_\alpha, E_{-\alpha}] = \sum_a \alpha_a H_a, \quad E_\alpha^\dagger = E_{-\alpha}. \tag{B30}$$

Here, α runs over all roots of $\mathfrak{su}(n+1)$. Now, let us consider its irreducible representations $\rho_1 : \mathfrak{su}(n+1) \to \mathrm{End}(V_1)$ and $\rho_2 : \mathfrak{su}(n+1) \to \mathrm{End}(V_2)$, where

$$V_1 = \mathrm{Span}_{\mathbb{C}}(\{f_{k,I}\}), \qquad \rho_1(T_A) = \mathcal{L}_A^{(q)}, \tag{B31}$$

$$V_2 = \mathrm{Span}_{\mathbb{C}}(\{T_p(f_{k,I})\}), \quad \rho_2(T_A) = L_A \circ . \tag{B32}$$

Here, k is fixed and V_1 and V_2 shall be generated by running the subscript I over all weights. We take the label I as the n-dimensional vector $I = (I_1, I_2, \ldots, I_n)$ such that

$$\rho_1(H_a) f_{k,I} = I_a f_{k,I}. \tag{B33}$$

In this notation, the correspondence (95) implies

$$\rho_2(v) T_p(f_{k,I}) = T_p(\rho_1(v) f_{k,I}), \tag{B34}$$

for any $v \in \mathfrak{su}(n+1)$. Note that from Eq. (B30) we have

$$\rho_1(E_\alpha) f_{k,I} = N_{\alpha,I} f_{k,I+\alpha} \tag{B35}$$

for a complex constant number $N_{\alpha,\,I}$. We again act $\rho_1(E_{-\alpha})$ on both sides of Eq. (B35) and obtain

$$\rho_1(E_{-\alpha}E_\alpha) f_{k,I} = C_{\alpha,I} f_{k,I}, \tag{B36}$$

where $C_{\alpha,\,I}$ is given by

$$C_{\alpha,I} = (f_{k,I}, \rho_1(E_{-\alpha}E_\alpha) f_{k,I}) = (\rho_1(E_\alpha) f_{k,I}, \rho_1(E_\alpha) f_{k,I}) = |N_{\alpha,I}|^2. \tag{B37}$$

Here, we have assumed that $f_{k,\,I}$ and $f_{k,\,I+\alpha}$ are both normalized. From Eqs. (B34), (B35), and (B36), we have

$$\begin{aligned}(T_p(f_{k,I+\alpha}), T_p(f_{k,I+\alpha})) &= |N_{\alpha,I}|^{-2}(\rho_2(E_\alpha)T_p(f_{k,I+\alpha}), \rho_2(E_\alpha)T_p(f_{k,I+\alpha}))\\ &= |N_{\alpha,I}|^{-2}(T_p(f_{k,I+\alpha}), \rho_2(E_{-\alpha}E_\alpha)T_p(f_{k,I+\alpha}))\\ &= |N_{\alpha,I}|^{-2}(T_p(f_{k,I+\alpha}), T_p(\rho_1(E_{-\alpha}E_\alpha)f_{k,I+\alpha}))\\ &= (T_p(f_{k,I}), T_p(f_{k,I})).\end{aligned} \tag{B38}$$

Since this holds for any I and α, we find

$$(T_p(f_{k,I}), T_p(f_{k,I})) = (T_p(f_{k,I'}), T_p(f_{k,I'})), \tag{B39}$$

for general weights I, I'.

From the above argument, we only have to compute $(T_p(f_{k,\,I}), T_p(f_{k,\,I}))$ for a specific I. Let us consider a particular element

$$f^{(q)}_{k,I} := c^{(2k+q)}_{\mathbf{1}_{k+q}\oplus\mathbf{2}_k}(\zeta^1)^{k+q}(\bar\zeta^2)^k. \tag{B40}$$

Here, we have introduced $\mathbf{1}_{k+q} = (1, 1, \ldots, 1)$ and $\mathbf{2}_k = (2, 2, \ldots, 2)$. From Eq. (53), the normalization constant is given by $c^{(2k+q)}_{\mathbf{1}_{k+q}\oplus\mathbf{2}_k} := \sqrt{\frac{(2k+q+n)!}{(2\pi)^n k!(k+q)!}}$. By using Eq. (89), we have

$$T_p\left(f^{(q)}_{k,I}\right)_{\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_p} = c^{(2k+q)}_{\mathbf{1}_{k+q}\oplus\mathbf{2}_k}\frac{c^{(p+q)}_{\boldsymbol{\alpha}_{p+q}}c^{(p)}_{\boldsymbol{\beta}_p}}{\left(c^{(p+k+q)}_{\boldsymbol{\alpha}_{p+q}\oplus\mathbf{2}_k}\right)^2}\delta_{\boldsymbol{\alpha}_{p+q}\oplus\mathbf{2}_k,\boldsymbol{\beta}_p\oplus\mathbf{1}_{k+q}} \tag{B41}$$

and the only nonvanishing components are

$$T_p\left(f^{(q)}_{k,I}\right)_{\mathbf{1}_{k+q}\oplus\boldsymbol{\rho}_{p-k},\mathbf{2}_k\oplus\boldsymbol{\rho}_{p-k}} = c^{(2k+q)}_{\mathbf{1}_{k+q}\oplus\mathbf{2}_k}\frac{c^{(p+q)}_{\mathbf{1}_{k+q}\oplus\boldsymbol{\rho}_{p-k}}c^{(p)}_{\mathbf{2}_k\oplus\boldsymbol{\rho}_{p-k}}}{\left(c^{(p+k+q)}_{\mathbf{1}_{k+q}\oplus\mathbf{2}_k\oplus\boldsymbol{\rho}_{p-k}}\right)^2}. \tag{B42}$$

For $p - k < 0$, we see that such matrices should vanish. Using Eq. (53), we find

$$\begin{aligned}&\left(T_p\left(f^{(q)}_{k,I}\right), T_p\left(f^{(q)}_{k,I}\right)\right)\\ &\quad= \frac{(2k+q+n)!(p+q+n)!(p+n)!}{(2\pi)^n k!(k+q)!((p+q+k+n)!)^2}\sum_{\boldsymbol{\rho}_{p-k}}\frac{(n_1(\boldsymbol{\rho}_{p-k})+k+q)!}{n_1(\boldsymbol{\rho}_{p-k})!}\frac{(n_2(\boldsymbol{\rho}_{p-k})+k)!}{n_2(\boldsymbol{\rho}_{p-k})!}.\end{aligned} \tag{B43}$$

Let us set $a := n_1(\boldsymbol{\rho}_{p-k})$ and $b := n_2(\boldsymbol{\rho}_{p-k})$, which satisfy $0 \le a + b \le p - k$. Here, for fixed a and b, the number of possible configurations of $\boldsymbol{\rho}_{p-k}$ is $\frac{(p-k-a-b+n-2)!}{(n-2)!(p-k-a-b)!}$ for $n > 1$. Thus, we have

$$\begin{aligned}&\sum_{\boldsymbol{\rho}_{p-k}}\frac{\left(n_1\left(\boldsymbol{\rho}_{p-k}\right)+k+q\right)!}{n_1\left(\boldsymbol{\rho}_{p-k}\right)!}\frac{\left(n_2\left(\boldsymbol{\rho}_{p-k}\right)+k\right)!}{n_2\left(\boldsymbol{\rho}_{p-k}\right)!}\\ &\quad= \sum_{a=0}^{p-k}\sum_{b=0}^{p-k-a}\frac{(a+k+q)!}{a!}\frac{(b+k)!}{b!}\frac{(p-k-a-b+n-2)!}{(n-2)!\,(p-k-a-b)!},\end{aligned} \tag{B44}$$

for $n > 1$. Let us use the Chu–Vandermonde identity,

$$\sum_{a=0}^{m} \frac{(a+i)!(j+m-a)!}{a!(m-a)!} = \frac{i!j!(i+j+m+1)!}{(i+j+1)!m!}, \tag{B45}$$

for any non-negative integers m, i, and j. By applying this identity to Eq. (B44), we find

$$\sum_{\boldsymbol{\rho}_{p-k}} \frac{(n_1(\boldsymbol{\rho}_{p-k})+k+q)!}{n_1(\boldsymbol{\rho}_{p-k})!} \frac{(n_2(\boldsymbol{\rho}_{p-k})+k)!}{n_2(\boldsymbol{\rho}_{p-k})!} = \frac{k!(k+q)!(p+q+k+n)!}{(2k+q+n)!(p-k)!}. \tag{B46}$$

For $n = 1$, we have

$$\begin{aligned}\sum_{\boldsymbol{\rho}_{p-k}} \frac{(n_1(\boldsymbol{\rho}_{p-k})+k+q)!}{n_1(\boldsymbol{\rho}_{p-k})!} \frac{(n_2(\boldsymbol{\rho}_{p-k})+k)!}{n_2(\boldsymbol{\rho}_{p-k})!} &= \sum_{a=0}^{p-k} \frac{(a+k+q)!}{a!} \frac{(p-a)!}{(p-k-a)!} \\ &= \frac{k!(k+q)!(p+q+k+1)!}{(2k+q+1)!(p-k)!},\end{aligned} \tag{B47}$$

and thus Eq. (B46) holds for any $n \in \mathbb{N}$. By plugging Eq. (B46) into Eq. (B43), we obtain Eq. (100).

Appendix C. Proofs and formulas for T^{2n}

C1. Integration formula for T^{2n}

In this appendix, we explicitly calculate

$$I^{(a,b)}_{m,i_m,j_m} := \int_0^{2\pi l_{2m-1}} dx^{2m-1} \int_0^{2\pi l_{2m}} dx^{2m} \left(f^{(p)}_{i_m}\right)^* e^{i\frac{ax^{2m-1}}{l_{2m-1}}} e^{i\frac{bx^{2m}}{l_{2m}}} f^{(p)}_{j_m}. \tag{C1}$$

Here, $a, b \in \mathbb{Z}$ and $f^{(p)}_{i_m}$ is defined in Eq. (117).

By plugging Eq. (117) into Eq. (C1), we have

$$\begin{aligned} I^{(a,b)}_{m,i_m,j_m} = &\left(\frac{kp}{4\pi^3 l_{2m}^2}\right)^{1/2} \sum_{l,l' \in \mathbb{Z}} e^{i\pi\left(l+\frac{i_m}{pq_m}\right)^2 pq_m\tau_m} e^{i\pi\left(l'+\frac{j_m}{pq_m}\right)^2 pq_m\tau_m} \\ &\times \int_0^{2\pi l_{2m-1}} dx^{2m-1} e^{-kp(x^{2m-1})^2} e^{\left(l+l'+\frac{i_m+j_m}{pq_m}\right)\frac{pq_m x^{2m-1}}{l_{2m}}} e^{i\frac{ax^{2m-1}}{l_{2m-1}}} \\ &\times \int_0^{2\pi l_{2m}} dx^{2m} e^{-i\left(l-l'+\frac{i_m-j_m-b}{pq_m}\right)\frac{pq_m x^{2m}}{l_{2m}}}. \end{aligned} \tag{C2}$$

Then, performing the integral of x^{2m} and taking the summation of l', we obtain

$$\begin{aligned} I^{(a,b)}_{m,i_m,j_m} = &\left(\frac{kp}{\pi}\right)^{1/2} \delta^{(\mathrm{mod}\ pq_m)}_{i_m-j_m-b,0}\, e^{-\frac{1}{4kp}\left(\frac{a^2}{l_{2m-1}^2}+i\frac{2ab}{l_{2m-1}l_{2m}}+\frac{b^2}{l_{2m}^2}\right)} e^{i\frac{2\pi a i_m}{pq_m}} \\ &\times \sum_{l\in\mathbb{Z}} \int_0^{2\pi l_{2m-1}} dx^{2m-1} e^{-kp\left(x^{2m-1}-2\pi l_{2m-1}\left(l+\frac{i_m}{pq_m}\right)-\frac{i}{2kp}\left(\frac{a}{l_{2m-1}}+i\frac{b}{l_{2m}}\right)\right)^2}. \end{aligned} \tag{C3}$$

Here, we have defined

$$\delta^{(\mathrm{mod}\ n)}_{a,b} = \begin{cases} 1 & (a-b \in n\mathbb{Z}) \\ 0 & (\text{otherwise}) \end{cases}. \tag{C4}$$

By shifting the coordinate $x^{2m-1} \mapsto x^{2m-1} + 2\pi l_{2m-1} l$, we can convert the summation of l into extending the integration range to $\mathbb{R}$. This yields the usual Gaussian integral and we obtain

$$I^{(a,b)}_{m,i_m,j_m} = e^{-\frac{1}{4kp}\left(\frac{a^2}{l_{2m-1}^2}+i\frac{2ab}{l_{2m-1}l_{2m}}+\frac{b^2}{l_{2m}^2}\right)} e^{i\frac{2\pi a i_m}{pq_m}} \delta^{(\mathrm{mod}\ pq_m)}_{i_m-j_m-b,0}. \tag{C5}$$

For $a = b = 0$, we can see that

$$I^{(0,0)}_{m,i_m,j_m} = \delta^{(\mathrm{mod}\ pq_m)}_{i_m-j_m,0}, \tag{C6}$$

which means the orthonormality (118). For $(a, b) = (1, 0)$ and (0,1), we can see that Eq. (C5) can be written in terms of the clock and shift matrices (126) as

$$I^{(1,0)}_{m,i_m,j_m} = (U_{pq_m})_{i_m,j_m}, \quad I^{(0,1)}_{m,i_m,j_m} = (V_{pq_m})_{i_m,j_m}. \tag{C7}$$

Similarly, for general a and b, we have

$$I^{(a,b)}_{m,i_m,j_m} = e^{-\frac{1}{4kp}\left(\frac{a^2-a}{l^2_{2m-1}}+i\frac{2ab}{l_{2m-1}l_{2m}}+\frac{b^2-b}{l^2_{2m}}\right)}((U_{pq_m})^a(V_{pq_m})^b)_{i_m,j_m}. \tag{C8}$$

C2. Proof of Eq. (142)

In this appendix, we give a derivation of Eq. (142).

To show Eq. (142), we introduce the Jacobi theta function

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu,\tau) = \sum_{l\in\mathbb{Z}} e^{i\pi(l+a)^2\tau}e^{i2\pi(l+a)(\nu+b)}, \tag{C9}$$

and rewrite the zero mode (117) as

$$f^{(p)}_{i_m}(x^{2m-1}, x^{2m}) = \left(\frac{kp}{4\pi^3 l^2_{2m}}\right)^{1/4} e^{-\frac{kp}{2}(x^{2m-1})^2}\vartheta\begin{bmatrix} i_m/pq_m \\ 0 \end{bmatrix}\left(\frac{pq_m}{2\pi l_{2m}}\left(x^{2m}-ix^{2m-1}\right), pq_m\tau_m\right). \tag{C10}$$

There is the following identity of the theta function [49]:

$$\begin{aligned}
&\vartheta\begin{bmatrix} r/N_1 \\ 0 \end{bmatrix}(N_1z_1, N_1\tau)\,\vartheta\begin{bmatrix} s/N_2 \\ 0 \end{bmatrix}(N_2z_2, N_2\tau) \\
&= \sum_{t=1}^{N_1+N_2}\vartheta\begin{bmatrix} \frac{r+s+N_1t}{N_1+N_2} \\ 0 \end{bmatrix}(N_1z_1+N_2z_2, (N_1+N_2)\tau) \\
&\quad\times\vartheta\begin{bmatrix} \frac{N_2r-N_1s+N_1N_2t}{N_1N_2(N_1+N_2)} \\ 0 \end{bmatrix}(N_1N_2(z_1-z_2), N_1N_2(N_1+N_2)\tau).
\end{aligned} \tag{C11}$$

This implies

$$f^{(q)}_{j_m}(x)f^{(p)}_{i'_m}(y) = [(p+q)q_m]^{-1/2}\sum_{t=1}^{(p+q)q_m} f^{(p+q)}_{j_m+i'_m+qq_mt}(\tilde{x})f^{((p+q)pqq^2_m)}_{pq_mj_m-qq_mi'_m+pqq^2_mt}(\tilde{y}), \tag{C12}$$

where

$$\tilde{x}^a := \frac{qx^a+py^a}{p+q}, \quad \tilde{y}^a := \frac{x^a-y^a}{(p+q)q_m}. \tag{C13}$$

Now, let us calculate the combination $f^{(q)}_{c_m,j_m}(x)f^{(p)}_{i'_m}(y)$, which appears in the integrand of Eq. (142). To do this, we act $a^{(q)\dagger}_m(x)$ on Eq. (C12) c_m times. Here, $a^{(q)\dagger}_m(x)$ is the creation operator (134). From the chain rule of the covariant derivative, we have

$$a^{(q)\dagger}_m(x) = \sqrt{\frac{q}{p+q}}a^{(p+q)\dagger}_m(\tilde{x}) + \sqrt{\frac{p}{p+q}}a^{((p+q)pqq^2_m)\dagger}_m(\tilde{y}). \tag{C14}$$

By using Eqs. (136) and (C14), we find

$$f^{(q)}_{c_m,j_m}(x)f^{(p)}_{i'_m}(y) = \sum_{t=1}^{(p+q)q_m}\sum_{c'_m=0}^{c_m}\sqrt{\frac{c_m!}{(c_m-c'_m)!c'_m!}\frac{q^{c'_m}p^{c_m-c'_m}}{(p+q)^{c_m+1}q_m}}f^{(p+q)}_{c'_m,j_m+i'_m+qq_mt}(\tilde{x})f^{((p+q)pqq_m^2)}_{c_m-c'_m,pq_mj_m-qq_mi'_m+pq^2_mt}(\tilde{y}). \tag{C15}$$

By setting $x^a=y^a$, the above equation becomes

$$f^{(q)}_{c_m,j_m}(x)f^{(p)}_{i'_m}(x) = \sum_{t=1}^{(p+q)q_m}\sum_{c'_m=0}^{c_m}\sqrt{\frac{c_m!}{(c_m-c'_m)!c'_m!}\frac{q^{c'_m}p^{c_m-c'_m}}{(p+q)^{c_m+1}q_m}}f^{(p+q)}_{c'_m,j_m+i'_m+qq_mt}(x)f^{((p+q)pqq_m^2)}_{c_m-c'_m,pq_mj_m-qq_mi'_m+pq^2_mt}(0). \tag{C16}$$

By using Eqs. (C16) and (118), we find

$$\int_0^{2\pi l_{2m-1}}dx^{2m-1}\int_0^{2\pi l_{2m}}dx^{2m}(f^{(p+q)}_{i_m})^* f^{(q)}_{c_m,j_m}f^{(p)}_{i'_m} = \sqrt{\frac{p^{c_m}}{(p+q)^{c_m+1}q_m}}\sum_{t=1}^{(p+q)q_m}f^{((p+q)pqq_m^2)}_{c_m,pq_mj_m-qq_mi'_m+pq^2_mt}(0)\,\delta^{(\mathrm{mod}\ (p+q)q_m)}_{i_m,j_m+i'_m+qq_mt}. \tag{C17}$$

By plugging Eq. (138) into the above equation, we finally obtain Eq. (142).

References

[1] N. Seiberg and E. Witten, J. High Energy Phys. **9909**, 032 (1999).
[2] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, Phys. Rev. D **55**, 5112 (1997).
[3] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, Nucl. Phys. B **498**, 467 (1997).
[4] J. Hoppe, Soryushiron Kenkyu Electron. **80**, 145 (1989).
[5] J. Arnlind, J. Hoppe, and G. Huisken, J. Differ. Geom. **91**, 1 (2012).
[6] M. Bordemann, E. Meinrenken, and M. Schlichenmaier, Commun. Math. Phys. **165**, 281 (1994).
[7] X. Ma and G. Marinescu, J. Geom. Anal. **18**, 565 (2008).
[8] E. Hawkins, Commun. Math. Phys. **202**, 517 (1999).
[9] E. Hawkins, Commun. Math. Phys. **215**, 409 (2000).
[10] E. Hawkins, Commun. Math. Phys. **255**, 513 (2005).
[11] H. Adachi, G. Ishiki, T. Matsumoto, and K. Saito, Phys. Rev. D **101**, 106009 (2020).
[12] V. P. Nair, Phys. Rev. D **102**, 025015 (2020).
[13] H. Adachi, G. Ishiki, S. Kanno, and T. Matsumoto, Phys. Rev. D **103**, 126003 (2021).
[14] H. Adachi, G. Ishiki, S. Kanno, and T. Matsumoto, Progress of Theoretical and Experimental Physics (2022), doi:10.1093/ptep/ptac171.
[15] U. Carow-Watamura, H. Steinacker, and S. Watamura, J. Geom. Phys. **54**, 373 (2005).
[16] B. P. Dolan, I. Huet, S. Murray, and D. O'Connor, J. High Energy Phys. **0707**, 007 (2007).
[17] M. Honda, arXiv:2008.11461 [hep-th] [Search inSPIRE].
[18] X. Ma and G. Marinescu, Holomorphic Morse Inequalities and Bergman Kernels (Birkhäuser, Basel, 2007).
[19] H. Lawson and M. Michelsohn, Spin Geometry (Princeton University Press, Princeton, NJ, 1989).
[20] B. P. Dolan and C. Nash, J. High Energy Phys. **0210**, 041 (2002).
[21] Y. Tenjinbayashi, H. Igarashi, and T. Fujiwara, Ann. Phys. **322**, 460 (2007).
[22] C. Fronsdal, Phys. Rev. D **18**, 3624 (1978).
[23] M. A. Vasiliev, The Many Faces of the Superworld, 533 (2000)
[24] H. Grosse and P. Presnajder, Lett. Math. Phys. **33**, 171 (1995).
[25] U. Carow-Watamura and S. Watamura, Commun. Math. Phys. **183**, 365 (1997).
[26] A. P. Balachandran, T. R. Govindarajan, and B. Ydri, Mod. Phys. Lett. A **15**, 1279 (2000).

[27] H. Aoki, S. Iso, and K. Nagao, Phys. Rev. D **67**, 085005 (2003).
[28] H. Aoki, S. Iso, and K. Nagao, Phys. Rev. D **67**, 065018 (2003).
[29] A. P. Balachandran and G. Immirzi, Phys. Rev. D **68**, 065023 (2003).
[30] J. Medina and D. O'Connor, J. High Energy Phys. **0311**, 051 (2003).
[31] Y. Abe, Phys. Rev. D **70**, 126004 (2004).
[32] K. Hasebe, Nucl. Phys. B **956**, 115012 (2020).
[33] K. Hasebe, Phys. Rev. D **105**, 065010 (2022).
[34] H. Awata, M. Li, D. Minic, and T. Yoneya, J. High Energy Phys. **0102**, 013 (2001).
[35] B. P. Dolan and D. O'Connor, J. High Energy Phys. **0310**, 060 (2003).
[36] T. Yoneya, J. High Energy Phys. **1606**, 058 (2016).
[37] K. Hasebe, Nucl. Phys. B **934**, 149 (2018).
[38] H. Itoyama and A. Tokura, Phys. Rev. D **58**, 026002 (1998).
[39] H. Shimada, Nucl. Phys. B **685**, 297 (2004).
[40] D. Berenstein and E. Dzienkowski, Phys. Rev. D **86**, 086001 (2012).
[41] G. Ishiki, Phys. Rev. D **92**, 046009 (2015).
[42] L. Schneiderbauer and H. C. Steinacker, J. Phys. A **49**, 285301 (2016).
[43] G. Ishiki, T. Matsumoto, and H. Muraki, J. High Energy Phys. **1608**, 042 (2016).
[44] T. Asakawa, G. Ishiki, T. Matsumoto, S. Matsuura, and H. Muraki, Prog. Theor. Exp. Phys. **2018**, 063B04 (2018).
[45] G. Ishiki, T. Matsumoto, and H. Muraki, Phys. Rev. D **98**, 026002 (2018).
[46] S. Terashima, J. High Energy Phys. **1807**, 008 (2018).
[47] A. Sako, arXiv:2205.09019 [math-ph] [Search inSPIRE].
[48] X. Dai, K. Liu, and X. Ma, J. Differ. Geom. **72**, 1 (2006).
[49] D. Mumford, Tata Lectures on Theta I (Birkhäuser, Basel, 1983)