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Kerr–Schild Tetrads and the Nijenhuis Tensor

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Abstract: We write the Kerr–Schild tetrads in terms of the flat space–time tetrads and of a (1, 1) tensor S^λ_μ. This tensor can be considered as a projection operator, since it transforms (i) flat space–time tetrads into non-flat tetrads, and vice-versa, and (ii) the Minkowski space–time metric tensor into a non-flat metric tensor, and vice-versa. The S^λ_μ tensor and its inverse are constructed in terms of the standard null vector field l_μ that defines the Kerr–Schild form of the metric tensor in general relativity, and that yields black holes and non-linear gravitational waves as solutions of the vacuum Einstein's field equations. We demonstrate that the condition for the vanishing of the Ricci tensor obtained by Kerr and Schild, in empty space–time, is also a condition for the vanishing of the Nijenhuis tensor constructed out of S^λ_μ. Thus, a theory based on the Nijenhuis tensor yields an important class of solutions of the Einstein's field equations, namely, black holes and non-linear gravitational waves. We also demonstrate that the present mathematical framework can easily admit modifications of the Newtonian potential that may explain the long range gravitational effects related to galaxy rotation curves.

Keywords: Kerr–Schild metric tensor; Kerr–Schild tetrads; Nijenhuis tensor

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1. The Kerr–Schild form of the Metric Tensor

The Kerr–Schild form of the metric tensor [1] is a quite interesting construction that allows for obtaining certain solutions of the vacuum Einstein's field equations. According to the book by Stephani et al. [2], the Kerr–Schild ansatz was previously studied by Trautman [3]. It is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + l_\mu l_\nu \,, \tag{1}$$

where $\eta_{\mu\nu}$ is the metric for the Minkowski space–time in any coordinate system, and l_μ is a null vector field. In general, the Minkowski metric tensor is taken by $\eta_{\mu\nu} = (-1,+1,+1,+1)$ in a Cartesian coordinate system (which, however, is not necessarily rectangular). The form of the metric tensor (1) allows one to easily transform the covariant into contravariant components, and vice-versa, by means of the flat Minkowski metric tensor. The contravariant components of the metric tensor are given by [1]:

$$g^{\lambda\mu} = \eta^{\lambda\mu} - l^\lambda l^\mu \,. \tag{2}$$

The vector l_μ is null with respect to both the metric tensors $g_{\mu\nu}$ and $\eta_{\mu\nu}$ [1], i.e.,

$$g^{\mu\nu} l_\mu l_\nu = \eta^{\mu\nu} l_\mu l_\nu = l^\mu l_\mu = 0 \,, \tag{3}$$

where $l^\lambda = g^{\lambda\mu} l_\mu = \eta^{\lambda\mu} l_\mu$. In addition, the determinant of the metric tensor satisfies [1] $(-g) = -\det(g_{\mu\nu}) = 1$. In the context of Riemannian geometry, the Kerr–Schild decomposition can be made only for type D and type O space–times.

Two gravitational field configurations that are of great relevance in general relativity, black holes (Schwarzschild and Kerr) and non-linear gravitational waves (pp-waves), may be described by means of the Kerr–Schild ansatz [4–8]. These are vacuum solutions of Einstein's field equations. The conditions under which matter fields may be included in the framework of the Kerr–Schild ansatz (as for instance, the electromagnetic field) are discussed in Ref. [2] (see also Ref. [9]). In particular, the metric tensor in the form of Equations (1) and (2) may be treated as *exact linear perturbations* of the Minkowski space–time [10,11]. Some recent investigations on the Kerr–Schild form are given in Refs. [12,13].

As noted by Kerr and Schild, the vacuum field equations $R_{\mu\nu}l^\mu l^\nu = 0$ yield [1]

$$(l^\nu \partial_\nu l_\mu)(l^\lambda \partial_\lambda l^\mu) = 0\,. \tag{4}$$

However, from the derivative of the null condition $l^\mu l_\mu = 0$, we obtain $l^\mu \partial_\lambda l_\mu = 0$, and consequently

$$l^\mu (l^\lambda \partial_\lambda l_\mu) = 0\,. \tag{5}$$

Thus, the vector $l^\lambda \partial_\lambda l_\mu$ is null, from Equation (4), and is itself orthogonal to the null vector l^μ. Therefore, we conclude that the vector $l^\lambda \partial_\lambda l_\mu$, besides being null, is collinear to the vector l_μ, i.e.,

$$l^\lambda \partial_\lambda l_\mu = \sigma l_\mu\,. \tag{6}$$

where σ is a multiplicative factor. We are using mostly the notation of Ref. [1], with partial derivatives and adopting Cartesian coordinates, but the passage to arbitrary coordinates (and the corresponding covariant derivatives in the flat space–time) is straightforward [14]. At this point, all indices may be raised and lowered by means of the flat metric tensor $\eta_{\mu\nu}$.

The equations above express the basics of the Kerr–Schild ansatz. In the following section, we will construct the Kerr–Schild tetrads, and arrive at the (1, 1) tensor S^λ_μ. Then, in Section 3, we address the Nijenhuis tensor, and conclude that the latter vanishes, provided Equation (6) is satisfied, i.e., as long as the Ricci tensor vanishes, according to Equation (4). In Section 4, we demonstrate that a theory based on the Nijenhuis tensor has freedom enough to admit solutions that describe deviations of the Newtonian potential that may explain the phenomenology related to galaxy rotation curves. Presently, the investigation of this issue requires the existence of some sort of dark matter, which so far has not been detected. We conclude that the geometrical framework considered in this article is quite rich and should be explored beyond the results presented here.

2. Kerr–Schild Tetrads

The Kerr–Schild tetrads $e^a{}_\mu(x)$ that yield the metric tensors (1) and (2) are constructed by observing all issues of consistency. We first denote by $E^a{}_\mu$ the flat space–time tetrads (i.e., tetrads for the flat Minkowski space–time) in an arbitrary inertial state. The latin index a is a SO(3, 1) tangent space index. Given some matrix representation $\Lambda^a{}_b$ of the Lorentz Group, the tetrad fields $E^a{}_\mu$ transform as $\tilde{E}^a{}_\mu = \Lambda^a{}_b E^b{}_\mu$. We also denote the flat tangent space–time metric tensor as $\eta_{ab} = (-1, +1, +1, +1)$. Therefore, the starting point for the construction of the Kerr–Schild tetrads is the expression

$$e^a{}_\mu = E^a{}_\mu + \frac{1}{2} l^a l_\mu\,, \tag{7}$$

where

$$l^a = E^a{}_\mu l^\mu\,. \tag{8}$$

Tetrad fields of this type were recently considered in Ref. [15], in the context of the teleparallel gravity. The flat space–time tetrad fields are required to satisfy

$$\eta_{ab} E^a{}_\mu E^b{}_\nu = \eta_{\mu\nu}\,. \tag{9}$$

Thus, it follows from Equations (8) and (9) that

$$\eta_{ab} l^a l^b = \eta_{\mu\nu} l^\mu l^\nu \,. \tag{10}$$

As a consequence of the expressions above, we have

$$\eta_{ab} e^a{}_\mu e^b{}_\nu = \eta_{\mu\nu} + l_\mu l_\nu = g_{\mu\nu} \,. \tag{11}$$

The inverse tetrads are defined by

$$e_b{}^\lambda = E_b{}^\lambda - \frac{1}{2} l_b \, l^\lambda \,. \tag{12}$$

where $l_b = E_b{}^\mu \, l_\mu$. It is important to note that all indices of both $E^a{}_\mu$ and $E_b{}^\lambda$ are raised and lowered by $\eta_{\mu\nu}$, η_{ab} and their inverses. The null vectors l^λ and l^a are related by $l^\lambda = E_a{}^\lambda l^a$. The vector l^a is also a null vector, $l^a l_a = 0$. This condition can be obtained directly from Equation (10). The expressions so far obtained allow one to verify the following orthogonality properties,

$$E_b{}^\lambda E^b{}_\mu = \delta^\lambda_\mu \,, \tag{13}$$

$$e_b{}^\lambda e^b{}_\mu = \delta^\lambda_\mu \,. \tag{14}$$

The relations below can be verified by simple, direct calculations and ensure the consistency and validity of the whole formulation presented here:

$$\eta^{ab} E_a{}^\mu E_b{}^\nu = \eta^{\mu\nu} \,, \tag{15}$$

$$\eta^{ab} e_a{}^\mu e_b{}^\nu = \eta^{\mu\nu} - l^\mu l^\nu = g^{\mu\nu} \,, \tag{16}$$

$$e^a{}_\mu e^b{}_\nu g^{\mu\nu} = \eta^{ab} \,. \tag{17}$$

Presently, we return to Equations (7) and (8) and observe that after simple rearrangements, the tetrad fields (7) may be first rewritten as

$$e^a{}_\mu = \frac{1}{2} E^a{}_\lambda (\delta^\lambda_\mu + \eta^{\lambda\sigma} g_{\sigma\mu}) \,. \tag{18}$$

It is interesting to note, already at this point, (i) that the transformations properties of $e^a{}_\mu$ under Lorentz transformations are determined by the flat space–time tetrads $E^a{}_\mu$, and that (ii) the tetrad fields are written in terms of the metric tensors $\eta^{\lambda\mu}$ and $g_{\lambda\mu}$. The inverse tetrads may be rewritten in a similar form,

$$e_a{}^\mu = \frac{1}{2} E_a{}^\lambda (\delta^\mu_\lambda + \eta_{\lambda\rho} g^{\rho\mu}) \,. \tag{19}$$

By means of straightforward manipulations, we again rewrite the tetrad field $e^a{}_\mu$ given by Equation (18) in the form

$$e^a{}_\mu = E^a{}_\lambda S^\lambda_\mu \,, \tag{20}$$

where the (1, 1) tensor S^λ_μ is defined by

$$S^\lambda_\mu = \delta^\lambda_\mu + \frac{1}{2} l^\lambda l_\mu \,. \tag{21}$$

It can be easily verified that the inverse of the tensor above is

$$(S^{-1})^\mu_\beta = \delta^\mu_\beta - \frac{1}{2} l^\mu l_\beta \,, \tag{22}$$

i.e., $S^{\lambda}_{\mu}(S^{-1})^{\mu}_{\beta} = \delta^{\lambda}_{\beta}$. In terms of the inverse tensor $(S^{-1})^{\mu}_{\beta}$, we may rewrite the inverse tetrads (19) in the form

$$e_a{}^{\mu} = E_a{}^{\lambda}(S^{-1})^{\mu}_{\lambda}. \quad (23)$$

The (1, 1) tensors S^{λ}_{μ} and $(S^{-1})^{\mu}_{\beta}$ exhibit very interesting properties for both the metric tensor and for the tetrad fields. First, for the metric tensor, the following relations can be verified by simple calculations:

$$\begin{aligned} S^{\mu}_{\rho}\, S^{\nu}_{\sigma}\, \eta_{\mu\nu} &= g_{\rho\sigma}\,, & (24)\\ S^{\mu}_{\rho}\, S^{\nu}_{\sigma}\, g^{\rho\sigma} &= \eta^{\mu\nu}\,, & (25)\\ (S^{-1})^{\mu}_{\rho}\,(S^{-1})^{\nu}_{\sigma}\,\eta^{\rho\sigma} &= g^{\mu\nu}\,, & (26)\\ (S^{-1})^{\mu}_{\rho}\,(S^{-1})^{\nu}_{\sigma}\, g_{\mu\nu} &= \eta_{\rho\sigma}\,. & (27) \end{aligned}$$

As for the tetrad fields, besides Equations (20) and (23) above, we have

$$\begin{aligned} E^a{}_{\lambda} &= e^a{}_{\mu}\,(S^{-1})^{\mu}_{\lambda}\,, & (28)\\ E_a{}^{\rho} &= e_a{}^{\mu}\, S^{\rho}_{\mu}\,. & (29) \end{aligned}$$

In conclusion, we have the following rules, at least for the metric tensor and for the tetrad fields. For a given tensor S^{μ}_{λ},

- The contravariant index μ converts flat space–time quantities into non-flat quantities;
- The covariant index λ converts non-flat space–time quantities into flat quantities.

As for the inverse tensor $(S^{-1})^{\alpha}_{\beta}$, the opposite takes place,

- The contravariant index α converts non-flat quantities into flat quantities;
- The covariant index β converts flat space–time quantities into non-flat quantities.

The general conclusion of the present analysis is that a (1, 1) tensor of the type S^{μ}_{λ} may play some role in gravity theories. On the other hand, such a (1, 1) tensor plays a relevant role in the construction of the Nijenhuis tensor. In the following section, we will investigate the implications of the S^{μ}_{λ} tensor given by Equation (21) in the context of the Nijenhuis tensor.

3. The Nijenhuis Tensor

The Nijenhuis tensor is a very interesting geometrical quantity, since it is entirely independent of any affine connection. It is defined by [16,17]

$$N^{\lambda}_{\mu\nu} = S^{\alpha}_{\mu}\partial_{\alpha}S^{\lambda}_{\nu} - S^{\alpha}_{\nu}\partial_{\alpha}S^{\lambda}_{\mu} - S^{\lambda}_{\alpha}(\partial_{\mu}S^{\alpha}_{\nu} - \partial_{\nu}S^{\alpha}_{\mu})\,. \quad (30)$$

This tensor may be established in any differentiable manifold M of arbitrary dimension D. To our knowledge, the Nijenhuis tensor has been considered in physics in two different contexts. First, it was considered in the study of dynamical integral models [18,19]. In this context, the vanishing of the Nijenhuis tensor yields interesting properties in manifolds with dual symplectic structures. One of these properties is the emergence of a number of conserved quantities in involution, that are necessary for the complete integrability of certain dynamical systems [18,19]. The Nijenhuis tensor is constructed out of a (1, 1) tensor S^{λ}_{μ} not related to Equation (21), but to a dual symplectic structure of the manifold.

In the context of complex manifolds, however, the vanishing of the Nijenhuis tensor is related to the existence of integrable almost complex structures [17]. Some formulations of string theory demand that the internal six extra dimensions correspond to a complex manifold, endowed with a complex structure for which the Nijenhuis tensor vanishes [20], while the ordinary four dimensional space–time is identified with the Minkowski space–time. Here, we present an alternative point of view and show that the Nijenhuis tensor may play some role in gravity.

Let us analyse the expression of the Nijenhuis tensor constructed out of the tensor S_μ^λ given by Equation (21). By just using the null conditions $l^\mu l_\mu = 0$ and $l^\mu \partial_\alpha l_\mu = 0$, we obtain

$$N_{\mu\nu}^\lambda = \frac{1}{2}[l^\lambda l_\mu (l^\alpha \partial_\alpha l_\nu) - l^\lambda l_\nu (l^\alpha \partial_\alpha l_\mu)] \,. \tag{31}$$

Now considering the validity of Equation (6), which follows from the vanishing of the Ricci tensor in empty space–times, we observe that the Nijenhuis tensor vanishes,

$$N_{\mu\nu}^\lambda = 0 \,. \tag{32}$$

Therefore, some solutions of Einstein's field equations in a vacuum may be obtained from a theory based on the Nijenhuis tensor, provided the tensor S_μ^λ is given by Equation (21). The metric tensor is determined by Equation (24), in which case the geometrization of the space–time is achieved after solving some field equations in a flat space–time. Note that $R_{\mu\nu} l^\mu l^\nu = 0$ does represent only one field equation, i.e., it does not represent the entirety of the vanishing of the Ricci tensor in a vacuum, which is given by 10 field equations. By just counting the number of field equations (again, in vacuum), it is clear that the vanishing of the Nijenhuis tensor does not, in general, lead to the vanishing of the Ricci tensor. The point of contact between the vanishing of both the Ricci and Nijenhuis tensor seems to be given by Equation (6) only.

It is not the purpose of the present article to establish a complete theory for gravity entirely based on the Nijenhuis tensor, because of the limitation to vacuum solutions, or to solutions for which the energy-momentum tensor for the matter fields satisfies $T_{\mu\nu} l^\mu l^\nu = 0$ [2]. Of course, this limitation takes place in the context of Einstein's general relativity. The issue of the interaction of the tensor field S_μ^λ with matter fields requires a quite deep analysis that will not be carried out in this article. The possibility of a non-ordinary coupling of the Nijenhuis and/or the S_μ^λ tensors with the matter fields might indicate a further departure from the standard formulation of general relativity, which deserves a very careful investigation. In any case, we will easily address two theories constructed out of $N_{\mu\nu}^\lambda$ in flat space–time. In these two theories, the action integral is varied with respect to the tensor field S_μ^λ, and both theories lead to the vanishing of the Nijenhuis tensor. The Minkowski metric tensor $\eta^{\mu\nu}$ is held fixed. The first one is determined by the Lagrangian density

$$L_1 = k\sqrt{-g}\, N^\mu N_\mu \,, \tag{33}$$

where k is a constant, $\sqrt{-g}$ refers to the flat space–time in arbitrary coordinates, $N_\mu = N_{\lambda\mu}^\lambda = S_\mu^\rho \partial_\rho S_\lambda^\lambda - S_\rho^\lambda \partial_\mu S_\lambda^\rho$ and $N^\mu = \eta^{\mu\lambda} N_\lambda$. By neglecting the surface terms that arise in the variation of L_1, we find that this Lagrangian density yields the field equations

$$\sqrt{-g}\, S_\rho^\mu \,\partial_\lambda N^\lambda + \sqrt{-g}\, N^\mu \,\partial_\rho S_\lambda^\lambda - \delta_\rho^\mu \,\partial_\lambda(\sqrt{-g} N^\nu S_\nu^\lambda) = 0 \,. \tag{34}$$

The second theory is the Yang–Mills type theory, also in flat space–time, defined by the Lagrangian density

$$L_2 = k\sqrt{-g}\, N_{\beta\gamma}^\alpha N_{\mu\nu}^\lambda \,\eta_{\alpha\lambda}\eta^{\beta\mu}\eta^{\gamma\mu} \equiv k\sqrt{-g}\, N_\lambda^{\mu\nu} N_{\mu\nu}^\lambda \,, \tag{35}$$

where we have defined $N_\lambda^{\mu\nu} = N_{\beta\gamma}^\alpha \,\eta_{\alpha\lambda}\eta^{\beta\mu}\eta^{\gamma\mu}$; k and $\sqrt{-g}$ are the same as in L_1. Again, we neglect the surface terms that arise from the variation of the Lagrangian density above. The field equations that follow from L_2 are

$$\begin{aligned} &\sqrt{-g} N_\lambda^{\mu\nu} \,\partial_\alpha S_\nu^\lambda - \sqrt{-g} N_\alpha^{\lambda\nu} \partial_\lambda S_\nu^\mu \\ &-\partial_\lambda(\sqrt{-g} N_\alpha^{\nu\mu}\, S_\nu^\lambda) + \partial_\lambda(\sqrt{-g} N_\nu^{\lambda\mu}\, S_\alpha^\nu) = 0 \,. \end{aligned} \tag{36}$$

Obviously, $N_{\mu\nu}^\lambda = 0$ is a solution of both Equations (34) and (36).

Unfortunately, it is not possible to establish a comparison of the Lagrangian densities (33) and (35) with the Hilbert–Einstein Lagrangian density. Likewise, it is not possible to relate Equations (34) and (36) to Einstein's equations in vacuum. Of course, there will always exist a relationship between Einstein's equation in terms of the null vector l_μ with the field equations above, again in terms of the null vector l_μ, but this relationship is too much restricted, and perhaps not useful. A relationship between Einstein's equations with the field equations above, Equations (34) and (36), for the basic field variable S^λ_μ, apparently does not exist.

One interesting feature of a theory based on the Nijenhuis tensor is that the theory cannot be linearised, although, of course, the solutions may be linearised in the sense of weak field limits, for instance. Let us consider an expression for S^λ_μ in the form

$$S^\lambda_\mu = \delta^\lambda_\mu + \varepsilon\, h^\lambda_\mu \tag{37}$$

where ε is an infinitesimal parameter. It is easy to verify that

$$N^\lambda_{\mu\nu}(S) = \varepsilon^2 N^\lambda_{\mu\nu}(h)\,. \tag{38}$$

Therefore, a theory based on the Nijenhuis tensor is definitely a non-linear theory.

One straightforward consequence of expression (21) for the tensor S^λ_μ is the following. The null vector l^μ may be considered an eigenvalue of the tensor S^λ_μ in the sense that $S^\lambda_\mu\, l^\mu = l^\lambda$. In fact, this relation may be generalised to several products of tensor S^λ_μ, as for instance $S^\lambda_\sigma\, S^\sigma_\rho\, S^\rho_\mu\, l^\mu = l^\lambda$. An arbitrary product of the tensor S^λ_μ yields

$$S^\lambda_{\alpha_1}\, S^{\alpha_1}_{\alpha_2} \cdots S^{\alpha_{n-1}}_\mu = \delta^\lambda_\mu + \frac{n}{2}\, l^\lambda l_\mu\,. \tag{39}$$

Before closing this section, we present an alternative expression for the Nijenhuis tensor. We consider an arbitrary (1, 1) tensor S^λ_μ and define the quantity $\mathcal{T}^\lambda_{\mu\nu}$ according to

$$\mathcal{T}^\lambda_{\mu\nu} = \partial_\mu\, S^\lambda_\nu - \partial_\nu\, S^\lambda_\mu\,. \tag{40}$$

This quantity was considered in Ref. [21], in the analysis of a consistent theory for massless spin 2 fields. In terms of the expression above, it is possible to rewrite the Nijenhuis tensor as

$$N^\lambda_{\mu\nu} = -S^\lambda_\sigma[\mathcal{T}^\sigma_{\mu\nu}(S) + S^\alpha_\mu\, S^\beta_\nu\, \mathcal{T}^\sigma_{\alpha\beta}(S^{-1})]\,. \tag{41}$$

We observe that $\mathcal{T}^\lambda_{\mu\nu}$ may be understood as a building block for the Nijenhuis tensor. Indeed, in Ref. [22], a very clear exposition of complex manifolds is presented, which supports this interpretation. In summary, a complex structure is determined by a real mixed tensor J. In a $2n$ dimensional manifold, this mixed tensor satisfies (in the notation of Ref. [22]) $J^P_N J^N_M = -\delta^P_M$ and, in this context, the Nijenhuis tensor is defined by

$$N^P_{MN} = J^Q_M(\partial_Q J^P_N - \partial_N J^P_Q) - J^Q_N(\partial_Q J^P_M - \partial_M J^P_Q)\,. \tag{42}$$

The almost complex structure J defines a complex structure, if and only if the associated Nijenhuis tensor vanishes [22].

If we restrict the tensor S^λ_μ to expression (21), namely, $S^\lambda_\mu = \delta^\lambda_\mu + \frac{1}{2} l^\lambda l_\mu$, then it is not difficult to demonstrate that

$$N^\sigma_{\mu\nu} = -(\delta^\alpha_\mu \delta^\beta_\nu - S^\alpha_\mu S^\beta_\nu) S^\sigma_\rho \mathcal{T}^\rho_{\alpha\beta}. \tag{43}$$

Thus, we observe that if $\mathcal{T}^\rho_{\alpha\beta}$ vanishes, then $N^\sigma_{\mu\nu}$ vanishes, but the opposite is not true. It seems to be impossible to invert the expression above and write $\mathcal{T}^\rho_{\alpha\beta}$ in terms of $N^\sigma_{\mu\nu}$.

4. Modifications of the Newtonian Potential

One interesting development of the mathematical framework determined by both the Nijenhuis tensor and the Kerr–Schild tetrads is the freedom in modifying the Newtonian potential. We will demonstrate in this section that this freedom accommodates modified long range Newtonian potentials that may explain certain models of dark matter. Let us start by considering the line element for the Kerr space–time given in both Refs. [10,14]. More specifically, we will consider Equation (20.46) of Ref. [14]. This is the form in which the Kerr metric first appeared in the literature. By making the angular momentum parameter $a = 0$ in this equation, we obtain the line element for the Schwarzschild space–time,

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 + \frac{2m}{r}(dt + \frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz)^2 \\ &= (\eta_{\alpha\beta} + l_\alpha l_\beta)dx^\alpha dx^\beta\,, \end{aligned} \tag{44}$$

where $r^2 = x^2 + y^2 + z^2$ and $m = GM/c^2$. Thus, the null vector l_α is identified as

$$l_\alpha = \left(\frac{2m}{r}\right)^{1/2}(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r})\,. \tag{45}$$

One possible modification of the Newtonian potential was proposed in Refs. [23,24], based on the analyses of galactic rotation curves investigated in Ref. [25]. According to Refs. [23,24], the standard Newtonian potential may be modified by means of the prescription

$$\Phi_g = -\frac{GM}{r} + \frac{GM}{\lambda}\ln\left(\frac{r}{\lambda}\right), \tag{46}$$

where the constant length λ may be taken to be $\lambda \approx 1$ kpc = 3260 light years. Different values of λ are discussed in Ref. [25]. Thus, the modification of the Newtonian potential is achieved by replacing

$$\frac{2m}{r} \to \frac{2m}{r} - \frac{2m}{\lambda}\ln\left(\frac{r}{\lambda}\right), \tag{47}$$

in Equation (45). Of course, the value of λ depends on the nature and/or model of the galaxies in consideration, but, as we will show below, the result to be presented here does not depend on the particular value of λ.

By defining a function $f(r)$ according to

$$f(r) = \frac{2m}{r} - \frac{2m}{\lambda}\ln\left(\frac{r}{\lambda}\right), \tag{48}$$

we may define a modified form of the null vector l_α as

$$l_\alpha = \sqrt{f(r)}\left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right). \tag{49}$$

The crucial property of the null vector l_α that leads to the vanishing of the Nijenhuis tensor is Equation (6). Equation (45) above satisfies this property. However, it is possible to show by simple calculations that Equation (49) also satisfies this property. Let us define the function $a(r)$ according to

$$a(r) = x\partial_x f + y\partial_y f + z\partial_z f \equiv x^i\partial_i f\,. \tag{50}$$

It is easy to show that the equation $l^\lambda\partial_\lambda l_\mu = \sigma l_\mu$ holds true, where the multiplicative factor $\sigma(r)$ is given by

$$\sigma(r) = \frac{a(r)}{2r\sqrt{f}}\,. \tag{51}$$

Therefore, the equation $l^\lambda \partial_\lambda l_\mu = \sigma l_\mu$, which leads to the vanishing of the Nijenhuis tensor, is verified irrespective of the expression of the function $f(r)$, which means that the standard Newtonian potential is not fixed by a theory constructed out of the Nijenhuis tensor. Obviously, the vanishing of the 10 components of the Ricci tensor does not lead to the result above; rather, it leads the well known textbook result given by the standard Newtonian potential characterized by the constant mass parameter m.

In Section 3 of Ref. [23], the various forms that the modified Newtonian potential may assume are discussed. It is also discussed, based on previous studies (see references therein), that the Newtonian potential (better saying, the actual gravitational potential) should be deducted on empirical grounds. This is an ongoing debate, to which the present analysis might contribute, since the function $f(r)$ is arbitrary, but restricted to satisfy certain asymptotic boundary conditions fixed by the actual physical configuration. However, it must be noted that the arbitrariness in the expression of the function $f(r)$ may be related to the fact that in nature we do not have inertial reference frames. All physical frames in nature are accelerated. Inertial reference frames are idealisations, which are relevant of course, but these idealisations require a drastic simplification of the space–time manifold.

The freedom in the determination of the Newtonian gravitational potential in the present formalism (which is expressed by the g_{00} component of the metric tensor in general relativity) may be related to the determination of the classical electromagnetic scalar potential $V(x) = A_0(x)$. In the Hamiltonian formulation of the classical electromagnetic field, in terms of the 4-vector potential A_μ, one finds that A_0 arises as a Lagrange multiplier for the Gauss law $\nabla \cdot \vec{E} = 0$, which is an elliptic differential equation and a first class constraint of the theory. The determination of the scalar potential A_0 is made only after the scalar charge density (source) $\rho(x)$ is established, after which A_0 is given by the well known integral of $\rho(x)$. Likewise, it could be that the Newtonian potential in the present mathematical framework is fixed only after the determination of both the matter distribution and the coupling of the tensor field S^λ_μ with the matter fields, i.e., the coupling of S^λ_μ with the matter energy momentum 4 vector P^μ_{matter} and/or with the matter energy-momentum tensor $T_{\mu\nu}$. As we mentioned before, this issue deserves further investigation.

5. Comments

In this article, we have established tetrads for the Kerr–Schild form of the metric tensor in general relativity. The Kerr–Schild ansatz has been considered in the literature as a manifestation of the gravitational field on a flat Minkowski background. In spite of the limitations of this formulation, the Kerr–Schild ansatz and its consequences are very interesting. The tetrad fields obtained in the present analysis, Equations (20) and (23), are given by the multiplication of the tetrad fields for the flat space–time with a (1, 1) tensor S^λ_μ. The emergence of this tensor immediately leads us to address the Nijenhuis tensor. This tensor dispenses the consideration of any space–time affine connection, and this feature simplifies the formulation of a gravitational theory. The main conclusion of the present article is that the mathematical structure determined by the Nijenhuis tensor and the associated Kerr–Schild tetrads constitutes a rich geometrical framework that deserves further investigation.

We have observed that the condition for the vanishing of the Ricci tensor in the context of general relativity, as obtained by Kerr and Schild, and which leads to important gravitational field configurations, is also a condition for the vanishing of the Nijenhuis tensor. Thus, the connection between gravitational field configurations and the Nijenhuis tensor should be explored. In principle, the latter tensor would lead to theories that yield field equations in a flat space–time. By means of Equation (24), we cast the results obtained from S^λ_μ in a geometrical form. The analysis of field equations in flat space–time, and the a posteriori geometrization by means of Equation (24), would greatly simplify the analysis of gravitational field configurations. It must be noted that in the context of a theory defined in flat space–time, the problem of the existence of essential singularities in general relativity may be put in a different perspective. The singularities would be a consequence of the system of differential equations in flat space–time, just like the singularities in Maxwell's theory, and not a feature of curved space–times, or of spaces with curvature and/or torsion, i.e., they are not singularities of the space–time itself. Thus, the problem of the existence of singularities in the standard metrical formulation of general relativity would be shifted from a conceptual problem to a mathematical, albeit important problem.

Finally, we concluded that the mathematical framework considered in this article may address the so called "dark matter problem", in fact with no dark matter at all. We have observed in Section 4 that the vanishing of the Nijenhuis tensor also admits modifications of the Newtonian potential that might explain the unresolved problem of the galaxies rotation curves. This is an open problem, to which the present analysis could contribute.

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