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Axion Mass and the Ground State of Deconfining SU(2) Yang–Mills Thermodynamics

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Abstract: For the deconfinement phase of an SU(2) Yang–Mills theory, we compute the axion mass m_A by appealing to the Veneziano–Witten formula. The topological susceptibility χ arises (i) from a precisely computable thermal ground-state contribution due to a center of a relevant (anti)caloron, and (ii) from contributions due to free thermal quasi-particles in the effective theory. Both (i) and (ii) are derived by using standard Euclidean thermal field theory techniques. While contribution (i) is positive and $\propto T^4$, contribution (ii) is negative, as demanded by reflection positivity, but negligible compared to contribution (i). As a consequence, practically from the critical temperature T_c onward, a real-valued axion mass $m_A(T) = \sqrt{\frac{2}{3}\pi\frac{T^2}{M_P}}$ emerges when the Peccei–Quinn scale is assumed to be the Planck mass M_P, independently of the Yang–Mills scale that the axion associates with. We discuss why our results deviate from those found in the dilute instanton gas and interacting instanton liquid approximations, and from results obtained in lattice simulations. Assuming the universe is dark sector to be based on such ultralight axion species, which are nonrelativistic for $T \ll M_P$, we investigate the cosmological conditions for their global Bose condensation as the very early universe cooled to temperatures of the order of 10^9 eV.

Keywords: topological susceptibility; finite temperature; calorons; cosmological model; fuzzy dark matter; dark energy

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1. Introduction

The temperature (T)-dependence of the topological susceptibility χ in the deconfinement phase of an SU(2) or SU(3) Yang–Mills theory was estimated by various methods: dilute instanton gas [1], instanton liquid [2], lattice simulations [3–9], and models of the stochastic vacuum [10]. Knowing the T-dependence of χ precisely is important for understanding cosmological models whose (fuzzy) dark-matter sector is based on the quantum dynamics of ultralight axions [11–17].

Based on $\chi(T)$, the Veneziano–Witten formula can be employed to calculate the axion mass $m_A(T)$. Originally, this formula was proposed as an estimate for the mass of the pseudoscalar η'-meson in QCD. For cosmologically stable axions, however, the required smallness of axion masses and decay constants compared to particle physics scales could originate from the extreme hierarchy between the Planck mass M_P (the Peccei–Quinn scale), at which the regime gravitational attraction [18] may condense massless fermion flavors [19,20] and generate a massless, pseudoscalar, flavor-singlet field ϕ, and the Yang–Mills scales of pure SU(2) or SU(3) theories, which associate with visible matter and radiation described by the Standard Model of Particle Physics in an evolving universe [21–23]. These hierarchically smaller-than-M_P Yang–Mills scales may also, by some as-yet-unknown mechanism, be determined by physics of the Planckian regime.

As in [20], we thus consider a dynamical breaking of the global symmetry $U(1)_A$ [24,25] at the Planck scale.

Gauge field configurations that are assumed to saturate the topological susceptibility χ (see Equation (1)) are usually taken to be *instantons* in the literature, subject to a finite-temperature electric Debye screening to modify the $T = 0$ instanton density. This only allows for instanton radii $\rho < (\pi T)^{-1}$ to contribute to the model partition function [26]. The results for χ, when computed in a dilute gas of instantons (DIGA) [1], or in a liquid of instantons (IILA) [2] have, in turn, inspired fit models in lattice simulations of χ [3]. Note that these instanton-based approaches do not resemble the situation of a thermal ground state [23], which (i) is composed of spatially densely packed Harrington–Shepard (HS) *calorons* or *anticalorons* [27], exhibiting an intrinsic T-dependence of the topological charge density Q per isolated field configuration, and which (ii) is dominated by caloron radii $\rho \propto T^{1/2}$ [23]. Point (ii) implies that properly capturing the topological charge of a relevant gauge-field configuration requires enormous spatial volumes at high T, which are not available in lattice simulations [23,28]. Indeed, Figure 1 depicts the ratio of the lattice size $L = 2/T_c$, employed in an SU(3) simulation [4], and the size of the relevant caloron radius $\rho = \frac{\lambda_c^{3/2}}{2\pi T_c}\left(\frac{T}{T_c}\right)^{1/2}$ [23] as a function of T/T_c for both SU(2) and SU(3).

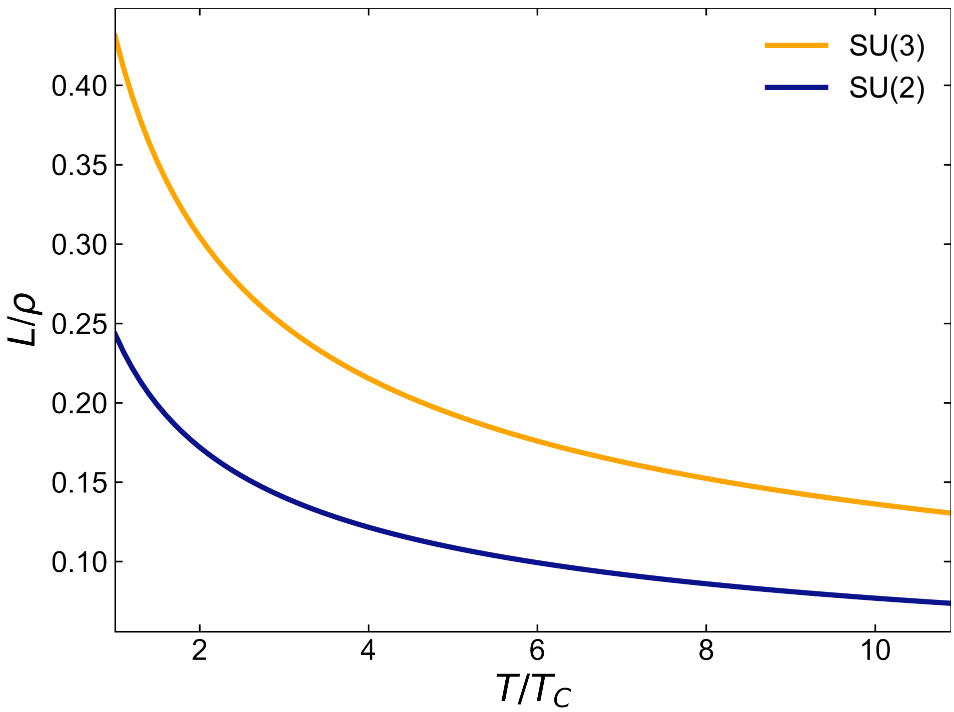

Figure 1. The ratio of lattice size $L = 2/T_c$, as employed in [4], and the caloron radius $\rho = \frac{\lambda_c^{3/2}}{2\pi T_c}\left(\frac{T}{T_c}\right)^{1/2}$ for SU(2) (ultramarine) and for SU(3) (orange). This caloron radius is relevant in the emergence of the thermal ground state [23]. For SU(2) and SU(3), the dimensionless critical temperatures ($\lambda \equiv 2\pi T/\Lambda$) are given as $\lambda_c = 13.87$ and $\lambda_c = 9.475$, respectively.

Therefore, throughout the deconfinement phase ($T \geq T_c$), the lattice size L never accommodates the center of a relevant caloron; the spatial lattice simply falls through the topological charge distribution. That is, the integral $\int d^4x\, Q(x)$ in Equation (1) below[1] is not saturated over the lattice volume.

In the rapid power-law decay of $\chi \sim a\, T^{-b}$ suggested for the gauge group SU(3) by DIGA [1], or by IILA [2] (for the SU(3) case, b ranges as $4.9 \leq b \leq 9$, depending on the number of fermion flavors in the simulation, and there is disagreement in the normalization a by a factor of ten between DIGA, and fits to lattice results [4]) could, therefore, well be an artifact of the assumed diluteness of the topological charge carriers, a neglect of their intrinsic T dependence (instantons are not periodic in the Euclidean time coordinate x_4), and the above-mentioned, perturbatively motivated, T-dependent constraint on the instanton scale on the modeling side. As mentioned above, lattice simulations [3,4] suffer from severe finite-spatial-volume constraints; see also [9] for an insightful discussion.

The purpose of the present paper is to compute the T-dependence of χ for deconfining SU(2) Yang–Mills thermodynamics, and to discuss some implications for a modeling of the cosmological dark sector, based on the T-dependence of the axion mass derived from the right-hand side of the Veneziano–Witten formula (VWF). Our work is organized as follows: In Section 2 we briefly review the VWF for the axion mass, and comment on its interpretation in the realm of cosmology. The computation of the topological susceptibility, χ, by far dominated by the spatial center of a HS caloron or anticaloron, relevant for the emergence of the thermal ground state of deconfining SU(2) Yang–Mills thermodynamics, is carried out in Section 3. Here, we will show that, in contrast to lattice and DIGA/IILA modeling, and as a result of the dense spatial packing of (anti)caloron centers in the deconfining thermal ground state, as well as the rapid, power-law rise of the caloron scale parameter with T, the topological susceptibility χ evolves as $\chi = a\,T^4$ for $T \gg T_c$. This can be expected from dimensional analysis and, motivated by perturbative asymptotic freedom at $T = 0$ [29,30], the irrelevance of the Yang–Mills scale Λ at large T [9]. Confirming this expectation, we compute the factor of proportionality to be $a = 2\pi^2/3$ for an SU(2) Yang–Mills theory in terms of an effective contact term in the topological charge correlation. We also show that the (negative) contribution of lower spatial resolutions to the right-hand side of VWF, which can be computed in the effective theory [23] in terms of a non-contact term, are negligible. In Section 4 we indicate cosmological implications for very high redshifts in a model of the dark sector where the present dark matter and dark energy are related to selfgravitating depercolated and global condensates of ultralight axions, respectively. More specifically, cosmological dark matter requires such condensates to be spatially confined to subgalactic scales [11,16,31] (fuzzy dark matter), while dark energy requires a superhorizon-sized axion condensate [31]. Finally, we summarize our results, and present an outlook on future research in Section 5.

Throughout the paper, we work in supernatural units: $\hbar = k_B = c = 1$, where $\hbar$ denotes Planck's (reduced) quantum of action, k_B is Boltzmann's constant, and c refers to the speed of light in vacuum.

2. Axion Mass Squared via the Veneziano–Witten Formula at a Finite Temperature

The VWF for the mass of the $\mathrm{U}(1)_A$ Goldstone field A in QCD, defined on $\mathbf{R}^4$ and subject to gauge group SU(N_c), as well as N_f massless fundamentally charged fermion flavors, reads [32,33]

$$\lim_{N_c\to\infty} \frac{m_A^2 f^2}{2N_f} = \lim_{N_c\to\infty} \int d^4x \langle Q(x)Q(0)\rangle_T \equiv \chi\,, \tag{1}$$

where the topological charge density $Q(x)$ is given as

$$Q(x) \equiv \frac{1}{64\pi^2}\,\epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \equiv \frac{1}{32\pi^2}\,F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}\,. \tag{2}$$

Here, $\langle\cdots\rangle_T$ denotes the canonical ensemble average in *pure* SU(N_c) Yang–Mills thermodynamics, and Greek indices take on values 4, 1, 2, 3. In the present work, we assume, per [1–9,26], the validity of Equation (1) in the deconfinement phase of Yang–Mills thermodynamics at finite temperature ($T > 0$).

In computing the T-dependence of the axion mass m_A, the large N_c limit is relevant to those $\mathrm{SU}(2)_i$ or $\mathrm{SU}(3)_j$ subgroups of SU(N_c) only (associated with a Yang–Mills scale Λ_i or Λ_j) which are in their deconfinement phases. We will show in Section 3.2 that for such Yang–Mills theories, practically only the *thermal ground state* [23] contributes to the right-hand side of Equation (1). In applying SU(2) Yang–Mills thermodynamics to cosmology, the simulated evolution of the Cosmic Microwave Background (CMB) [34], initialized at a redshift of $z \sim 10^9$ and terminating today ($z = 0$), requires a single such

SU(2) theory only [21,23]: $SU(2)_{CMB}$ with $\Lambda_{CMB} \sim 10^{-4}$ eV [35]. According to [20] we may therefore fix in Equation (1) $\lim_{N_c \to \infty} f^2/(2N_f) \sim M_P^2$ for $N_f \gg 1$, where M_P denotes the Planck mass $M_P = 1.22 \times 10^{19}$ GeV.

3. The Veneziano–Witten Integral for Deconfining SU(2) Yang–Mills Thermodynamics

3.1. The Topological Susceptibility χ on a Harrington–Shepard (HS) (Anti)Caloron

In this section, we compute the short-distance contribution to the axion mass squared, relying on Equation (1), which represents a contact term in the effective theory [23]. In the deconfinement phase, the topological susceptibility χ of Equation (1) originates from an HS (anti)caloron center [27] in the thermal ground state (a thermal quantum vacuum [23]. Thereby, HS (anti)caloron centers are densely packed spatially, subject to overlapping peripheries. There is no winding in these peripheries. Thus, the ground-state portion of $\langle Q(x)Q(0)\rangle_T$ is computed on a single caloron center.

At a finite temperature, the spacetime average of $Q(x)$ is over the Euclidean cylinder $S_1 \times \mathbf{R}^3$, $x_4 \equiv \tau, x_i = r\hat{x}_i$ $(i = 1, 2, 3)$, where $\hat{x}$ is a unit vector in $\mathbf{R}^3$ and $r =\mid \mathbf{x} \mid \leq \mid \phi \mid^{-1}$; see Figure 2.

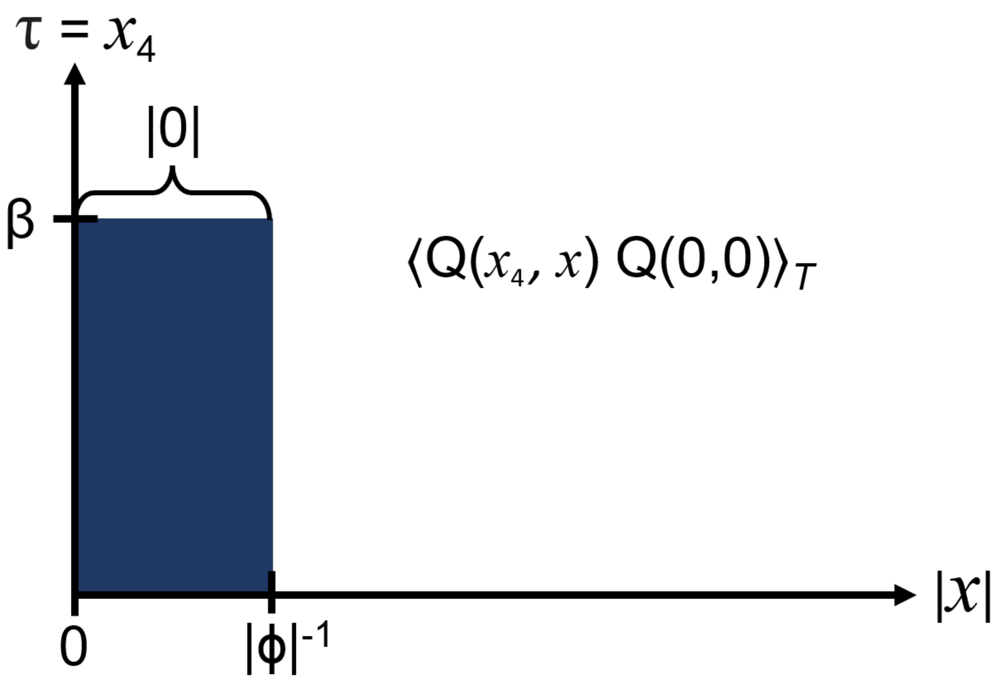

Figure 2. Spatially coarse-graining a fundamental-theory caloron center (integrating the correlator $\langle Q(x_4, \mathbf{x})Q(0, \mathbf{0})\rangle_T$ from $r =\mid \mathbf{x} \mid= 0$ to $r =\mid \mathbf{x} \mid=\mid \phi \mid^{-1}$ and over angles) generates an effective, spatial contact term if resolution is limited as $r =\mid \mathbf{x} \mid\geq\mid \phi \mid^{-1}$. This reduces the shaded region to the line segment $0 \leq x_4 \leq \beta$ in the effective theory. The integral over $0 \leq x_4 \leq \beta$ and $\vec{x} \in \mathbf{R}^3$ of the portion of $\langle Q(x_4, \mathbf{x})Q(0, \mathbf{0} \mid)\rangle_T$, which is due to effective (quasiparticle) fluctuations, is strongly suppressed compared to the integral over $0 \leq x_4 \leq \beta$ of this spatial contact term; see Section 3.2.

Since the right-hand side of Equation (1) is quadratic in Q, and since we *average* over the contribution of an HS caloron and an HS anticaloron [23], the thermal ground-state part of $\langle Q(x)Q(0)\rangle_T$ is represented by an HS caloron which, due to spatial isotropy, spatially is centered at $r = 0$ (peak position of the HS caloron's action density).

When expressing the field strength $F^a_{\mu\nu}$, expanded into a basis t^a $(a = 1, 2, 3)$ of the Lie algebra SU(2) such that tr $t^a t^b = \frac{1}{2}\,\delta^{ab}$, in terms of $O(4)$ scalar and tensor valued functionals $P(x)$ and $P_{\mu\nu}(x)$ of the caloron prepotential $\Pi(x)$, we follow the convention of [36]. Namely, in a singular gauge one has for the gauge field of an HS caloron with topological charge $k = 1$,

$$A^a_\mu = -\bar{\eta}^a_{\mu\kappa}\frac{\partial_\kappa \Pi}{\Pi}\,, \tag{3}$$

where $\bar{\eta}^a_{\mu\kappa}$ denotes the (anti-selfdual) 't Hooft symbol, and Π is the prepotential given as [27]

$$\Pi(\tau, r) = 1 + \frac{\rho^2}{\beta r}\frac{\sinh\frac{2\pi r}{\beta}}{\cosh\frac{2\pi r}{\beta} - \cos\frac{2\pi\tau}{\beta}}\,. \tag{4}$$

Here, β denotes the inverse of temperature T, $\beta \equiv \frac{1}{T}$, and ρ is the associated instanton scale parameter [37–39]. Note that the second term on the right-hand side of Equation (4) originates from summing the instanton prepotential in singular gauge over equidistantly shifted values of its time coordinate τ. This renders $\Pi(\tau, r)$ periodic in τ, which can be seen by the dependence on $\cos\frac{2\pi\tau}{\beta}$. For the field strength, this implies staticity for sufficiently large values of r. The field strength $F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^b_\mu A^c_\nu$ is expressed as

$$F^a_{\mu\nu} = \bar{\eta}^a_{\mu\nu}P - \bar{\eta}^a_{\nu\kappa}P_{\mu\kappa} + \bar{\eta}^a_{\mu\kappa}P_{\nu\kappa} \tag{5}$$

if we define

$$P \equiv \frac{(\partial_\kappa\Pi)(\partial_\kappa\Pi)}{\Pi^2}\,, \quad P_{\mu\nu} \equiv \frac{\Pi\partial_\mu\partial_\nu\Pi - 2(\partial_\mu\Pi)(\partial_\nu\Pi)}{\Pi^2}\,. \tag{6}$$

Here, ϵ^{abc} denotes the 3D totally antisymmetric tensor with $\epsilon^{123} = 1$. Note that

$$P_{\mu\nu} = P_{\nu\mu}\,, \quad P_{\mu\mu} = -2P\,. \tag{7}$$

Therefore, one has, for a selfdual caloron field[2],

$$Q \equiv \frac{1}{32\pi^2}F^a_{\mu\nu}F^a_{\mu\nu} = \frac{1}{8\pi^2}\left(P^2_{\mu\nu} - P^2\right). \tag{8}$$

Considering that $\Pi = \Pi(\tau, r)$, we have

$$\begin{aligned} F^a_{\mu\nu}F^a_{\mu\nu} \;=\;& 4\Bigg(3\left(\frac{\partial_r\Pi}{\Pi}\right)^4 + \left(\frac{\partial_r^2\Pi}{\Pi}\right)^2 + 2\left(\frac{\partial_r\Pi}{r\Pi}\right)^2 + 3\left(\frac{\partial_\tau\Pi}{\Pi}\right)^4 + \\ & \left(\frac{\partial_r\Pi}{\Pi}\right)^2\left(-4\frac{\partial_r^2\Pi}{\Pi} + 6\left(\frac{\partial_\tau\Pi}{\Pi}\right)^2\right) - 4\left(\frac{\partial_\tau\Pi}{\Pi}\right)^2\frac{\partial_\tau^2\Pi}{\Pi} + \left(\frac{\partial_\tau^2\Pi}{\Pi}\right)^2 - \\ & 8\frac{\partial_r\Pi}{\Pi}\frac{\partial_\tau\Pi}{\Pi}\frac{\partial_r\partial_\tau\Pi}{\Pi} + 2\left(\frac{\partial_r\partial_\tau\Pi}{\Pi}\right)^2\Bigg)\,. \end{aligned} \tag{9}$$

Introducing rescaled variables, $y \equiv \frac{\tau}{\beta}$, $x \equiv \frac{r}{\beta}$, the dimensionless temperature $\lambda = \frac{2\pi T}{\Lambda}$, Λ denoting the SU(2) Yang–Mills scale, and setting $\rho = |\,\phi\,|^{-1} = \sqrt{\frac{2\pi T}{\Lambda^3}}$ to be the dominating value of the scale parameter in the thermal ground state [23], Equation (4) is cast into

$$\Pi(\tau, r, T) = \bar{\Pi}(y, x, \lambda) = 1 + \frac{\lambda^3}{4\pi x}\frac{\sinh 2\pi x}{\cosh 2\pi x - \cos 2\pi y}\,. \tag{10}$$

Here, we have introduced the general dimensionless function $\bar{f}(y, x, \lambda) = T^{-n}f(\tau, r, T)$, where n is the mass dimension of function f. Substituting Equation (10) into Equation (9) and using Equation (8) yields

$$\lim_{r\to 0} Q(\tau, r, T) = T^4 \lim_{x\to 0}\bar{Q}(y, x, \lambda) = 2\pi^2 T^4\lambda^6\frac{(8 + \lambda^3 + 4\cos 2\pi y)^2}{3(2 + \lambda^3 - 2\cos 2\pi y)^4}\,. \tag{11}$$

According to Equation (11), the y and the λ dependencies of $\bar{Q}(y, x = 0, \lambda)$ are very weak for $\lambda \geq \lambda_c = 13.87$ [23]. At finite x, this behavior of $\bar{Q}(y, x, \lambda)$ with respect to variations in y and λ persists, which is demonstrated in Figure 3 for $x = 0, 0.5, 1.0$ and $\lambda = \lambda_c, 3\,\lambda_c, 10\,\lambda_c$.

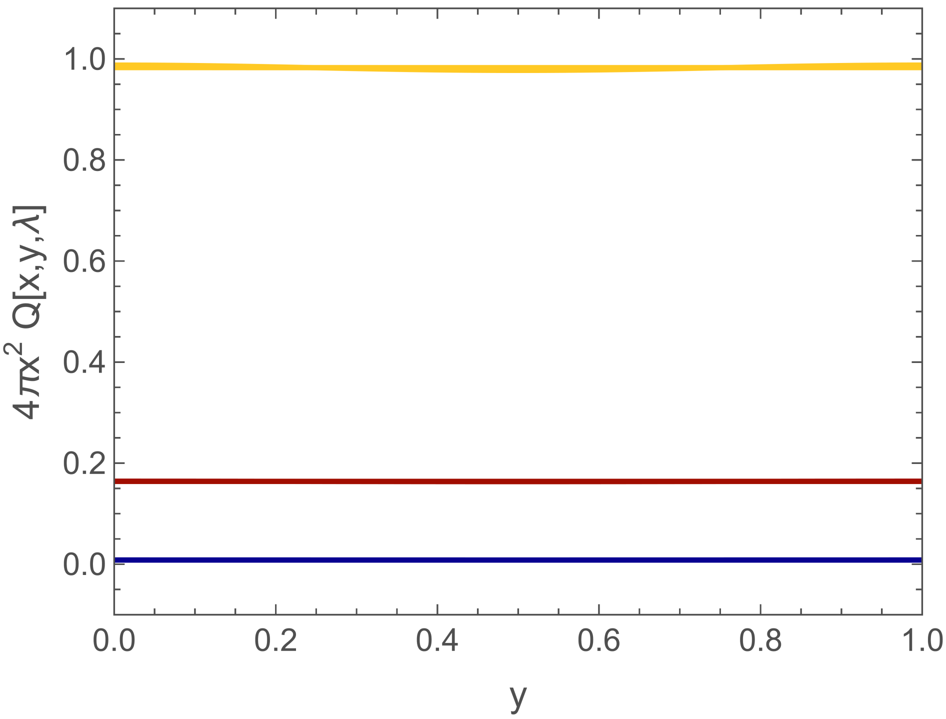

Figure 3. The weak y and λ dependencies of function $4\pi x^2 \bar{Q}(x, y, \lambda)$: $\lambda = \lambda_c, 3\,\lambda_c, 10\,\lambda_c$ and $x = 0$ (lowest set), $x = 0.5$ (uppermost set), and $x = 1$ (intermediate set).

For $\lambda \geq \lambda_c = 13.87$, the topological charge $k \equiv 4\pi \int_0^1 dy \int_0^\infty dx\, x^2 \bar{Q}(y, x, \lambda) = 1$, thus, to a very good accuracy, can be computed as

$$k = 4\pi \int_0^\infty dx\, x^2 \bar{Q}(\frac{1}{2}, x, \lambda)\,. \tag{12}$$

If we truncate the integration over x in Equation (12) at the scale $x_{\max} = \frac{\rho}{\beta} = \frac{|\phi|^{-1}}{\beta} = \frac{\lambda^{3/2}}{2\pi}$,

$$k = 4\pi \int_0^{\frac{\lambda^{3/2}}{2\pi}} dx\, x^2 \bar{Q}(\frac{1}{2}, x, \lambda)\,, \tag{13}$$

then we numerically obtain $k = 0.9768, 0.9927, 0.9938$ for $\lambda = \lambda_c, 2\,\lambda_c, 2.5\,\lambda_c$, respectively. This shows that, for practically all λ, the topological charge of an HS (anti)caloron resides within its center. The (dimensionless) radius $x_{\max}$ of this center, however, grows $\propto \lambda^{3/2}$ with dimensionless temperature λ. This explains why lattice simulations, essentially operating at a fixed spatial volume at varying T, fail to capture large portions of the topological charge of a relevant (anti)caloron in $\chi(T)$; compare this with Figure 1.

The short-distance correlation of the thermal ground state of an SU(2) Yang–Mills theory are mediated by the centers of an HS caloron or anticaloron [23]. As confirmed above, the (anti)caloron action, representing a nontrivial topological charge of this gauge-field configuration, is localized within the (anti)caloron center. As a consequence, the ground-state contribution to the ensemble average on the right-hand side of Equation (1) represents the contribution of an (anti)caloron center—a contact term within the effective, deconfining Minkowskian quantum thermodynamics [23]; see also Figure 2. In contrast to calorons, which are action minimizers of the 4D defining, classical, and Euclidean Yang–Mills theory, and are infinitely resolved, the effective theory describes the quantum dynamics of gauge fields in (3+1)D Minkowski space at spatial distances larger than $|\,\phi\,|^{-1}$. By matching the effective theory with the fundamental one at this resolution limit, it was argued in [40] that the caloron action coincides with Planck's quantum of action $\hbar$. Light-cone localized, propagating solutions to the source-free Minkowskian Yang–Mills equations—photons—, which could be initiated by sufficiently resolving, time-periodic probes of (anti)caloron centers, were constructed recently by A. Rabinowitch [41]. Given these localized solutions, a probe's circular frequency ω could thus provoke propagating quanta of energy $\hbar\omega$.

We are now in a position to compute the thermal ground state contribution of the right-hand side of Equation (1) as

$$\begin{aligned} \chi_{\text{cal. cen.}} &= \int_{\text{cal. cen.}} d\tau d\Omega dr\, r^2\, Q(\tau,r,T)Q(0,0,T) \\ &= 4\pi T^4\, \bar{Q}(0,0,\lambda) \int_0^1 dy \int_0^{\frac{\lambda^{3/2}}{2\pi}} dx\, x^2 \bar{Q}(y,x,\lambda) \\ &\simeq 4\pi T^4\, \bar{Q}(0,0,\lambda) \int_0^{\frac{\lambda^{3/2}}{2\pi}} dx\, x^2 \bar{Q}(\frac{1}{2},x,\lambda) \\ &\simeq \frac{2}{3}\frac{(12+\lambda^3)^2\pi^2}{\lambda^6}\, T^4 \\ &\stackrel{\lambda\gg\lambda_c}{\longrightarrow} \frac{2}{3}\pi^2\, T^4\,, \end{aligned} \tag{14}$$

where $d\Omega$ denotes the angular part of the measure in the spatial integration. Therefore,

$$m_A = \sqrt{\frac{2}{3}}\pi\frac{T^2}{M_P}\,, \quad (T \gg T_c)\,. \tag{15}$$

Note that, due to the integral of Equation (13) being close to unity and due to the function on the right-hand side in Equation (11) being near to $2\pi^2 T^4/3$ already at $\lambda = \lambda_c = 13.87$, the T-dependence of m_A, as in Equation (15), is already an excellent approximation for $T \sim T_c$.

3.2. Do Thermal Quasiparticle Fluctuations Contribute to χ?

Equation (14) associates with a contact term of the correlator $\langle Q(x_4,\mathbf{x})Q(0,\mathbf{0})\rangle_T$ in the effective theory of the deconfining phase of SU(2) Yang–Mills thermodynamics [23]. In [42], it is suggested that precisely such a contact term saturates the right-hand side of Equation (1) at $T = 0$; see also an associated argument for the Schwinger model in [43]. In Appendix A, we show that, in the effective theory for the deconfining phase of SU(2) Yang–Mills thermodynamics [23], a negative contribution to χ arises from massless and massive, effectively free thermal quasi-particles. Judging by the smallness of two- and three-loop corrections to the free quasi-particle approximation of thermodynamic quantities, this should be precise to the sub-percent level [23]. As computed in Appendix A, free quasi-particles in $\chi(T)$ generate a positive coefficient in front of $-T^4$, which is much smaller than unity. Therefore, we can consider the result of Equation (14), a reliable estimate of the entire topological susceptibility χ: the expression for the axion mass m_A in Equation (15), which emerges from the axial anomaly invoked by SU(2) (anti)calorons, should be accurate to the sub-percent level. In [10], $\chi(T)$ was computed in the model of the stochastic vacuum. For the nonperturbative part of the correlator $\langle Q(x_4,\mathbf{x})Q(0,\mathbf{0})\rangle_T$, a factorized negative contribution was shown to be overcompensated by a positive nonfactorized contribution. Both are expressed in terms of nonperturbative vacuum parameters measured at zero temperature, which were extracted from lattice simulations. As a result, a T^4-dependence with a positive coefficient also was found in [10].

4. Cosmological Implications: Bose Condensation of Axion Particles

As a function of increasing cosmological redshift z axion masses for four different species are shown in Figure 4. In the confining phase of the associated SU(2) Yang–Mills theory, the axion mass is constant in z. For the three species concerned at the present temperature of the CMB, $T_0 = 2.725\,\mathrm{K}$, and up to $z \sim 10^8$, the lowest axion mass can be extracted phenomenologically from rotation curves of low-surface-brightness galaxies [31]. The other two masses follow from Yang–Mills scaling relations (subject to a common Peccei–Quinn scale, M_P) in the fuzzy dark matter model. Such scaling relies on a link to lepton families introduced in [31], and explicated for the first lepton family in [21]. As soon as a SU(2) Yang–Mills theory transitions to its deconfinement phase, the associated axion mass starts to depend on z via Equation (15), and is subject to the T–z relation in the deconfinement phase [44]. At high redshifts ($z > 10^{12}$) all axion species exhibit the same dependence of their mass on redshift z. Note that the validity of our axion–mass computation in an expanding Friedmann–Lemaître–Robertson–Walker universe hinges on temperature T being larger than the Hubble expansion rate H. In a radiation dominated universe we estimate $H/T = O(10)\,T/M_P$. Therefore, we demand $T/M_P \ll O(10^{-1})$, which allows us to address most of the universe is expansion history.

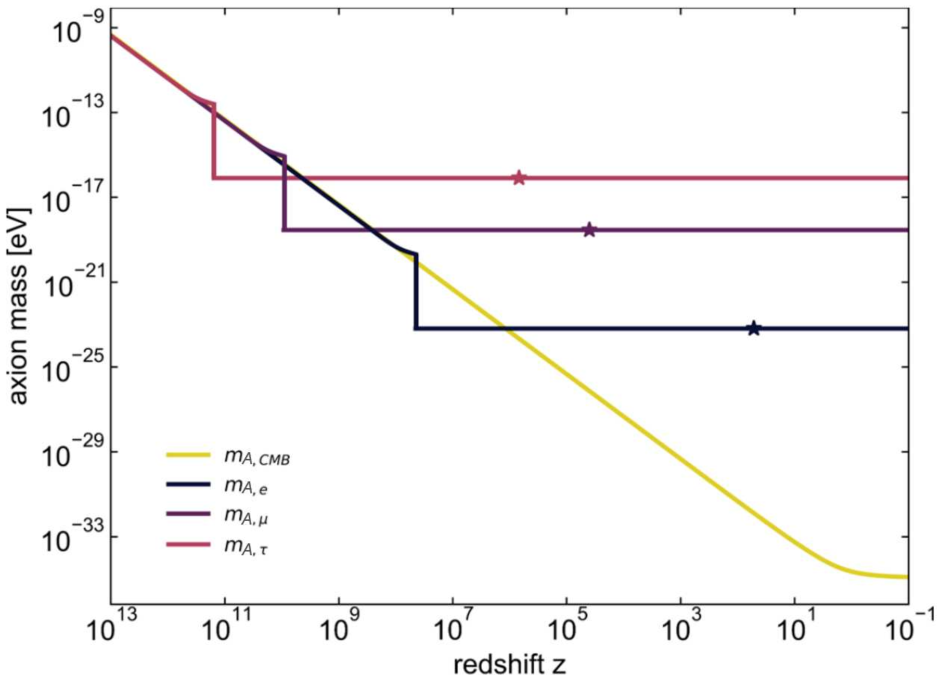

Figure 4. Axions have a temperature dependent mass $m_A \propto T^2$ in deconfinement phases of SU(2) Yang–Mills thermodynamics (see Equation (15)); in the confining phase, the axion mass is constant (see [31]). According to [31] and Equation (15), the temperature-dependence of m_A is shown for four axion species [31]: An axion associated with the Yang–Mills scale potentially implying the third lepton family possesses a mass $m_{A,\tau}(z=0) \approx 8 \times 10^{-17}$ eV (ultramarine). Accordingly, for the second lepton family, one would have $m_{A,\mu}(z=0) \approx 3 \times 10^{-19}$ eV (orange). An axion that associates with the first lepton family carries a mass $m_{A,e}(z=0) \approx 7 \times 10^{-24}$ eV (green), and an axion that couples to a Yang–Mills theory potentially describing the CMB would possess a mass $m_{A,\mathrm{CMB}}(z=0) \approx \times 10^{-35}$ eV (yellow). At the respective deconfinement phase transitions, the axion mass, which is supported by a reduced topological susceptibility in the confining phase (round-point center-vortex loops being the topological charge carriers [23]), as compared to the deconfinement phase (see the derivation of Equation (14)), jumps up to a higher value. To convert the T-dependence of the axion mass in Equation (15) into a z-dependence, the temperature–redshift relation $T(z) = \mathcal{S}(z) \cdot (z+1) \cdot T_0$ was used [44]. Here, $\mathcal{S}(z)$ derives from a numerical evaluation of the SU(2) deconfinement entropy density. It is fitted as $\mathcal{S}(z) = \exp(-1 - 1.7\,z) + 1/4^{1/3}$.

Let us now investigate the implications of Equation (15) for the critical temperature $T_{c,B}$ of Bose condensation involving non-relativistic axion particles of mass m_A and number density[3] n_A. One has [50]

$$T_{c,B} = \frac{2\pi}{m_A}\left(\frac{n_A}{\zeta\left(\frac{3}{2}\right)}\right)^{\frac{2}{3}}, \tag{16}$$

where $\zeta(x)$ denotes the Riemann zeta function ($\zeta\left(\frac{3}{2}\right) = 2.6124$). If we assume that, due to the long-range quantum correlations facilitated by a super-horizon (reduced) Compton wave length[4] $m_{A,i}^{-1}$, the energy density $\rho_{\Lambda_i} = m_{A,i}n_{A,i}$ of a global axion condensate exhibits negligible redshift (or T)-dependence then the dependence of the axion mass on temperature of Equation (15) predicts

$$T_{c,B,i} = \left(\frac{486}{\pi^4}\right)^{1/26}\left(\frac{\rho_{\Lambda_i}}{\zeta\left(\frac{3}{2}\right)}\right)^{2/13} M_P^{5/13}. \tag{17}$$

Setting $\rho_{\Lambda_{\rm CMB}} = (10^{-3}\,\mathrm{eV})^4$ and $M_P = 1.22\times 10^{28}$ eV yields $T_{c,B,\rm CMB} = 8.3\times 10^8$ eV, which associates with a redshift $z > 10^{12}$. This is beyond the initial redshift $z \sim 10^9$ assumed in CMB simulations [34]. Higher vacuum energy densities, $\rho_{\Lambda_i} > \rho_{\Lambda_{\rm CMB}}$, yield higher values of $T_{c,B,i}$, even though their spread is small due to the small power of $2/13$ in Equation (17).

Due to the large hierarchy between $T_{c,B,i}$ and M_P, and by virtue of Equation (15), thermal axions constitute radiation for all temperatures T within the regime $M_P \gg T > T_{c,B,i}$. However, in a radiation dominated universe, they only represent a small fraction of *radiation* energy density. This is because radiation domination arises due the Stefan–Boltzmann limit of Yang–Mills thermodynamics. For $T \sim M_P$, where we expect Einstein gravity to profoundly develop quantum behavior, thermal axions would become nonrelativistic again.

5. Summary and Outlook

In this paper, we have revisited the computation of axion mass according to the Veneziano–Witten formula applied to the deconfining phase of an SU(2) Yang–Mills theory. Our result for the T-dependence of the topological susceptibility χ deviates from what is found in the literature: a T^4-dependence, which is expected from dimensional analysis [9], and indeed shown to emerge in our computation involving the center of a Harrington–Shepard caloron to represent the thermal ground state locally [23], contrasts with power laws of large negative exponents [1–8]. In Section 1, we have pointed out various possible reasons for why such a discrepancy arises. We also have shown that the contributions to χ stemming from massless and massive effective excitations of the thermal ground state of a given SU(2) Yang–Mills theory yield an extremely suppressed negative correction (reflection positivity) to the positive caloron contribution. Finally, we have investigated the consequences of this result for cosmological model building. Such models consider the dark sector as axial-anomaly induced by virtue of SU(2) Yang–Mills theories subject to a common Peccei–Quinn scale M_P [20]. Such gauge theories possibly relate to the three lepton families of the Standard Model of Particle Physics [21,31]. A fourth SU(2) theory of scale $\Lambda_{\rm CMB} \sim 10^{-4}$ eV (and Peccei–Quinn scale M_P)—SU(2)$_{\rm CMB}$—may generate a present axion mass of $\sim 10^{-35}$ eV; therefore, a super-horizon Compton wavelength could, thus, relate to dark energy. It is worth asking at what temperatures these theories Bose condense their axion particles in a cooling universe. We find that the associated critical temperatures depend on the respective energy densities of the condensate through a power law of fractional power $2/13$. This suggests, in heating the very early universe through these transitions, that for *all* axion species, the epoch of dissolving dark energy into radiation is occurring at nearly the same temperature.

The results obtained here could be useful for cosmological model building in a framework where the dark sector of the universe is due to the selfgravitating quantum dynamics of ultralight axions [11,14–17,51].

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Appendix A

Here, we evaluate the contribution to the topological susceptibility χ on the right-hand side of Equation (1) arising from quasi-particle fluctuations in the effective theory of deconfining SU(N_c) Yang–Mills thermodynamics [23]. Throughout Appendix A, we denote by x or y spacetime vectors in $\mathbf{R}^4$.

Appendix A.1. Factorized $\chi(T)$ in the Model of the Stochastic Yang–Mills Vacuum at $T > T_c$

Here, in contrast to Sections 2 and 3, we work with the perturbative definition, where the coupling is not absorbed into the gauge field. Let us then consider the expression $\varepsilon_{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}$, where one of the indices can be equal to 4, and $F_{\mu\nu}$ is the antisymmetric field-strength tensor, defined either in terms of the fundamental gauge field A_μ or effective gauge fields a_μ. In the present section, both kinds of gauge fields are identified and subjected to the coupling g. In Appendix A.2, we refer to effective gauge fields and set $g = e$. Thus, $F_{\mu\nu} = F^a_{\mu\nu}t^a$, where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$, t^a's are SU(N_c)-generators in the fundamental representation, normalized as $\mathrm{tr}\, t^a t^b = \delta^{ab}/2$, $a = 1, \ldots, N_c^2 - 1$, and f^{abc} are the structure constants of the Lie algebra su(N_c). One readily obtains that $\varepsilon_{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} = 4\,\varepsilon_{4ijk}F_{4i}F_{jk}$. Moreover, one has

$$\varepsilon_{4ijk}\varepsilon_{4lmn}F_{4i}(x)F_{jk}(x)F_{4l}(0)F_{mn}(0) = 2F_{4i}(x)F_{jk}(x)\Big[F_{4i}(0)F_{jk}(0) + 2F_{4k}(0)F_{ij}(0)\Big]\,. \tag{A1}$$

For latter usage, we define $x = (\mathbf{x}, x_4) = (x_1, x_2, x_3, x_4)$, $\mathbf{x}^2 = x_1^2 + x_2^2 + x_3^2$, and $|x|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$.

The local density of topological charge is defined as

$$Q(x) = \frac{g^2}{32\pi^2}\varepsilon_{\mu\nu\lambda\rho}\,\mathrm{tr}(F_{\mu\nu}(x)F_{\lambda\rho}(x))\,. \tag{A2}$$

From Equations (A1) and (A2), it is straightforward to obtain

$$\langle Q(x)Q(0)\rangle = \frac{g^4}{128\pi^4}\Big[\langle F^a_{4i}(x)F^a_{jk}(x)F^b_{4i}(0)F^b_{jk}(0)\rangle + 2\langle F^a_{4i}(x)F^a_{jk}(x)F^b_{4k}(0)F^b_{ij}(0)\rangle\Big]\,. \tag{A3}$$

Let us now consider the factorized part of Equation (A3), which consists of six pairwise products of the two-point correlation functions of the field strengths. As indicated by the lattice data in [52], the deconfinement phase transition at $T = T_c$ is associated with the disappearance of the chromo-electric condensate $\langle (gE_i^a)^2\rangle_T$, and hence also of $\langle F_{4i}^a(x)F_{4j}^b(y)\rangle_T$. Accordingly, Equation (A3) yields $\langle F_{4i}^a(x)F_{4j}^b(0)\rangle_T = 0$ in the stochastic Yang–Mills vacuum at $T > T_c$ [52]. Thus, one has $\langle Q(x)Q(0)\rangle_{T,\text{factorized}} = \frac{1}{128\pi^4}\Big[\langle g^2F_{4i}^a(0)F_{jk}^a(0)\rangle_T^2 + \langle g^2F_{4i}^a(x)F_{jk}^b(0)\rangle_T^2 +$

$$+2\langle g^2F_{4i}^a(0)F_{jk}^a(0)\rangle_T\langle g^2F_{4k}^b(0)F_{ij}^b(0)\rangle_T + 2\langle g^2F_{4i}^a(x)F_{ij}^b(0)\rangle_T\langle g^2F_{4k}^a(x)F_{jk}^b(0)\rangle_T\Big]. \tag{A4}$$

We thus see that $\langle Q(x)Q(0)\rangle_{T,\text{factorized}}$ is fully expressed in terms of the correlation function $\langle g^2E_i^a(x)B_k^b(0)\rangle_T$, where $E_i^a = F_{i4}^a$ denotes the chromo-electric field and $B_k^b = 1/2\varepsilon_{kij}F_{ij}^b$ is the chromo-magnetic field. This correlation function can be parameterized through a scalar function $f(x)$ as

$$\langle g^2E_i^a(x)B_k^b(0)\rangle_T = \delta^{ab}\varepsilon_{ikn}x_n f(x)\,. \tag{A5}$$

If one multiplies this equation by t^at^b and takes the trace, one obtains

$$\text{tr}\,\langle g^2E_i^a(x)t^aB_k^b(0)t^b\rangle_T = \frac{N_c^2-1}{2}\,\varepsilon_{ikn}x_n f(x).$$

This parameterization can now be compared with the one adopted in Equation (2.9) of [52], which reads

$$\text{tr}\,\langle g^2E_i^a(x)t^aB_k^b(0)t^b\rangle_T = -\frac{1}{2}\varepsilon_{ikn}x_n\frac{\partial D_1^{\text{BE}}}{\partial x_4}\,,$$

where the lattice values for the function $D_1^{\text{BE}}(x)$ can also be found in [52].

Therefore, $f(x)$ is unambiguously related to the function $D_1^{\text{BE}}(x)$ as

$$f(x) = -\frac{1}{N_c^2-1}\frac{\partial D_1^{\text{BE}}}{\partial x_4} \tag{A6}$$

Using the aforementioned definitions of the chromo-electric and the chromo-magnetic fields, one has

$$\langle g^2F_{4i}^a(x)F_{jk}^b(0)\rangle_T = \delta^{ab}\left(\delta_{ij}\delta_{kn} - \delta_{ik}\delta_{jn}\right)x_n f(x)\,. \tag{A7}$$

Equations (A4) and (A7) finally yield

$$\begin{aligned}\langle Q(x)Q(0)\rangle_{T,\text{factorized}} &= \frac{1}{128\pi^4}\Big[\langle g^2F_{4i}^a(x)F_{jk}^b(0)\rangle_T^2 + 2\langle g^2F_{4i}^a(x)F_{ij}^b(0)\rangle_T\langle g^2F_{4k}^a(x)F_{jk}^b(0)\rangle_T\Big]\\ &= -\frac{N_c^2-1}{32\pi^4}\mathbf{x}^2f^2\,.\end{aligned} \tag{A8}$$

Note the negative sign of this contribution to $\langle Q(x)Q(0)\rangle_T$, in accordance with the reflection-positivity property and the pseudoscalar nature of the topological charge [42,53].

Appendix A.2. Contributions to Topological Susceptibility Due to Massless and Massive Quasi-Particles

Let us now calculate contributions produced to the correlation function (A8) by free massless and massive quasi-particles in the effective theory of SU(N_c) Yang–Mills thermodynamics ($N_c = 2, 3$) for the deconfinement phase [23]. These contributions are denoted by $\langle Q(x)Q(0)\rangle_{T,\text{factorized, free}}$ in what follows. We first assume all the N_c^2-1 effective gauge fields to be massless, and subsequently show how the corresponding contribution of massless quasi-particles dominates over that of the massive ones. For the case of free massless quasi-particles, we need to consider the Abelian part of the field-strength tensor, $f^a_{\mu\nu} = \partial_\mu a^a_\nu - \partial_\nu a^a_\mu$. Setting $g = e$ yields

$$\langle e^2 E^a_i(x) B^b_k(y) \rangle_{T=0,\mathrm{free}} = \frac{1}{2}\varepsilon_{klm} \langle e^2 f^a_{i4}(x) f^b_{lm}(y) \rangle_{T=0} = \varepsilon_{klm} \partial^x_4 \partial^y_m \langle e^2 a^a_i(x) a^b_l(y) \rangle_{T=0} \,,$$

where $\langle e^2 a^a_i(x) a^b_l(y) \rangle_{T=0} = \frac{e^2}{4\pi^2} \frac{\delta^{ab}\delta_{il}}{(x-y)^2}$ (Feynman gauge), and we continue to work in Euclidean spacetime. Hence,

$$\langle e^2 E^a_i(x) B^b_k(0) \rangle_{T=0,\mathrm{free}} = e^2 \delta^{ab} \varepsilon_{ikn} \partial_4 \partial_n \frac{1}{4\pi^2 x^2} \,. \tag{A9}$$

At finite temperature $T \equiv 1/\beta$, the propagator of massless gauge modes, $\frac{1}{4\pi^2 x^2}$, can be represented as an integral over the Schwinger proper time s as

$$\int_0^\infty \frac{\mathrm{d}s}{(4\pi s)^2} \sum_{n=-\infty}^{+\infty} \exp\left[-\frac{\mathbf{x}^2 + (x_4 + \beta n)^2}{4s} \right] . \tag{A10}$$

Poisson resummation casts the sum over winding modes into the sum over Matsubara frequencies as

$$\sum_{n=-\infty}^{+\infty} \exp\left[-\frac{(x_4 + \beta n)^2}{4s} \right] = 2T\sqrt{\pi s} \sum_{k=-\infty}^{+\infty} \exp(-\omega_k^2 s + i\omega_k x_4) \,, \tag{A11}$$

where $\omega_k = 2\pi T k$ is the k-th Matsubara frequency. Therefore, the x_4-differentiation in Equation (A9) returns the prefactor of $i\omega_k$ which vanishes for $k = 0$. Hence, we approximate the sum over Matsubara frequencies by only keeping terms with $k = \pm 1$. The finite-temperature generalization of $\partial_4 \partial_n \frac{1}{4\pi^2 x^2}$ is thus approximated as

$$\frac{T^2 x_n}{4\sqrt{\pi}} \int_0^\infty \frac{\mathrm{d}s}{s^{5/2}} \, \mathrm{e}^{-(2\pi T)^2 s - \frac{\mathbf{x}^2}{4s}} \sin(2\pi T x_4) \,.$$

The s-integration in this expression, along with Equation (A5), yields

$$f_{\mathrm{free}}(x) \simeq \frac{2\pi e^2 T^3}{\mathbf{x}^2} \left(1 + \frac{1}{2\pi T|\mathbf{x}|} \right) \mathrm{e}^{-2\pi T|\mathbf{x}|} \sin(2\pi T x_4) \,.$$

Thus, by employing Equation (A8), we obtain

$$\langle Q(x) Q(0) \rangle_{T,\mathrm{factorized,\,free}} \simeq -\frac{N_c^2 - 1}{8\pi^2} \frac{(e^2 T^3)^2}{\mathbf{x}^2} \left(1 + \frac{1}{2\pi T|\mathbf{x}|} \right)^2 \mathrm{e}^{-4\pi T|\mathbf{x}|} \sin^2(2\pi T x_4) \tag{A12}$$

By using Equation (A12), we now calculate $\chi_{T,\mathrm{factorized,\,free}}$, given as

$$\begin{aligned} &\int \mathrm{d}^3 x \int_0^\beta \mathrm{d}x_4 \langle Q(x) Q(0) \rangle_{T,\mathrm{factorized,\,free}} \\ = \; & -\frac{N_c^2 - 1}{8\pi^2} (e^2 T^3)^2 \frac{1}{2T} \cdot 4\pi \int_{|\phi|^{-1}}^\infty \mathrm{d}|\mathbf{x}| \left(1 + \frac{1}{2\pi T|\mathbf{x}|} \right)^2 \mathrm{e}^{-4\pi T|\mathbf{x}|} \,, \end{aligned} \tag{A13}$$

where the prefactor of $\frac{1}{2T}$ stems from $\int_0^\beta \mathrm{d}x_4 \sin^2(2\pi T x_4)$, $|\phi|^{-1} = \frac{\lambda^{3/2}}{2\pi T}$, $\lambda \geq \lambda_c = 13.87$ [23] and $e \sim \sqrt{8}\pi$ for $T \gg T_c$. Therefore, we have

$$|\, \chi_{T,\mathrm{factorized,\,free}} \,| \simeq \frac{N_c^2 - 1}{16\pi^2} (eT)^4 \, \mathrm{e}^{-2\lambda^{3/2}} \leq 4 \times 10^{-39} T^4 \,, \quad (\lambda \geq \lambda_c, N_c = 2) \,. \tag{A14}$$

The above calculation of the contribution $\chi_{T,\text{factorized, free}}$ of massless gauge fields to $\chi(T)$ can readily be generalized to the case of massive vector bosons of common mass m. Here, $1/(4\pi^2 x^2)$ in Equation (A9) just needs to be replaced by $m\,K_1(m|x|)/(4\pi^2|x|)$, where K_1 is the Macdonald function. This amounts to adding the term $-m^2 s$ into the exponentials in Equations (A10) and (A11), and introducing a factor of 3/2 for the additional polarization state. We obtain (cf. Equation (A13))

$$\begin{aligned}\chi_{T,\text{factorized, free, massive}} &\simeq -\frac{3}{2}\frac{N_c^2-1}{32\pi^4}(eT)^4\frac{1}{2T}\left[(2\pi T)^2+m^2\right]\times \\ &\quad 4\pi\int_{|\phi|^{-1}}^{\infty}\mathrm{d}|\mathbf{x}|\,\mathrm{e}^{-2\sqrt{(2\pi T)^2+m^2}\,|\mathbf{x}|} \\ &= -3\frac{N_c^2-1}{64\pi^3}\,e^4T^3\sqrt{(2\pi T)^2+m^2}\,\mathrm{e}^{-2\frac{\sqrt{(2\pi T)^2+m^2}}{2\pi T}\lambda^{3/2}}\,. \qquad \text{(A15)}\end{aligned}$$

This expression indicates an even stronger exponential suppression than that of Equation (A14). In the effective theory for the deconfinement phase of SU(2) Yang–Mills thermodynamics, the adjoint Higgs mechanism leaves one gauge mode massless, but generates a common mass $m = 4\pi e T\lambda^{-3/2}$ [23] for the other two gauge modes. Here, $e = \sqrt{8}\pi$ for $T \gg T_c$. For SU(3) two different masses emerge which, however, exhibit the same $T^{-1/2}$ power-law fall-off at high temperatures. This shows that, modulo a smaller number of polarization states (six vs. eight for SU(2), 16 vs. 22 for SU(3)), the hypothetic contribution to $\chi_{T,\text{factorized, free}}$ of massless gauge fields dominates among the contributions of all effective excitations. Since the modulus of $\chi_{T,\text{factorized, free}}$ is ridiculously small (see Equation (A14)), we justify the omission of (negative) contributions to χ that arise from quasi-particle modes in the effective theory.

Finally, let us discuss an important property to be respected by the full correlation function $\langle Q(x)Q(0)\rangle_T$: it should be negative for all $x \neq 0$, while yielding a positive $\chi(T)$ [42,53] to define a positive axion mass squared according to Equation (1). For the contribution $\langle Q(x)Q(0)\rangle_{T,\text{factorized, free}}$ in Equation (A12) this is the case, albeit subject to an exponentially strong suppression of the modulus as compared to the positive caloron contact contribution; compare this with Equation (11).

Notes

1. Since the correlator of Q is computed on one field configuration only using a Harrington–Shepard caloron, it naturally factorizes into a local and a nonlocal part. Note that the contribution and weight of the anticaloron is identical and, therefore, the average over both caloron and anticaloron reduces to the caloron contribution.
2. For useful relations on the contractions of bilinears in $\bar{\eta}^a_{\mu\nu}$ see [36].
3. In cosmology, axion particles can be shown to be nonrelativistic [11]: Roughly speaking, their speed $v_{A,i}$ is bounded from above by $v_{A,i}(z) = \frac{M(z)m_{A,i}(z)}{M_P^2}$ where $M(z)$ denotes the entire mass of the instantaneously gravitating system at redshift z after virialization [11]. To axions that presently form a super-horizon sized, self-gravitating Bose condensate to represent dark energy we associate the gauge group SU(2)$_{\text{CMB}}$ [20,31,45]. In ΛCDM this system presently gravitates subject to a mass $M_{\Lambda\text{CDM}}(z=0)$ made from the axion condensate, dark matter, and baryons. For a spatial ball of present Hubble radius $r_{H_0} = H_0^{-1}$—the causally connected, selfgravitating region to which a typical axion particle virializes—we have $M_{\Lambda\text{CDM}}(z=0) \sim \frac{4\pi}{3}H_0^{-3}H_0^2 M_P^2\frac{3}{8\pi} = \frac{1}{2}M_P^2/H_0$. Using $H_0 = 1.34\times10^{-32}\,h\,\text{eV}$ and $h = 0.74$ [46–48] yields $M_{\Lambda\text{CDM}}(z=0) \sim 7.5\times10^{87}\,\text{eV}$. Setting $m_{A,\text{CMB},0} = 10^{-35}\,\text{eV}$ [35], one thus arrives at $v_{A,\text{CMB}}(z=0) \sim 5.04\times10^{-4} \ll 1$. In a spatially flat, matter-dominated universe, $M(z)$ evolves as $M(z=0)\,(z+1)^{-3/2}$. With $m_{A,i}(z) = (1/4)^{2/3}\,m_{A,i,0}\,(z+1)^2$ [49], this produces $v_{A,i} = v_{A,i,0}(1/4)^{2/3}\,(z+1)^{1/2}$. In a radiation-dominated universe, $v_{A,i}$ does not evolve in z. Ignoring a small regime in redshift of dark-energy domination and taking the redshift of radiation-matter equality as $z \sim 3000$, the axion-particle speed $v_{A,\text{CMB}}$ thus increases from its present value by a factor of $\sim (1/4)^{2/3}(3000+1)^{1/2} = 21.7$. Therefore, axions that associate with SU(2)$_{\text{CMB}}$ are non-relativistic throughout the entire expansion history. Axion particles of species that are associated with fuzzy dark matter receive their redshift independent masses due to the Veneziano–Witten formula applied to confining phases of SU(2) Yang–Mills theories for redshifts $z < 10^8$ [31], where topological charge is carried by round-point center-vortex loops [23]. Presently, they are similarly slow to the axions of SU(2)$_{\text{CMB}}$ [11,31]. With increasing z, and for $z < 10^8$, the velocities of these axion particles remains constant until the percolation of their selfgravitating lumps sets in at $z_{p,i}$. For $z > z_{p,i}$, axion velocities either decay with $(z+1)^{-3/2}$ (matter domination) or

with $(z+1)^{-2}$ (radiation domination) as z increases, and so are guaranteed to be non-relativistic. In the deconfining phases of the associated SU(2) Yang–Mills theories, i.e., in the very early, radiation dominated universe, axion velocities do not evolve.

4 One can readily see that the horizon size r_H remains smaller than the Compton wavelength $m_{A_i}^{-1}$ in a radiation dominated universe. Namely, $r_H m_{A_i} = O(1)/O(10)$. This shows that no gravitational back reaction occurs while maintaining the condensed state of axion particles. Thus, the condition to determine the critical temperature for condensation is solely Equation (16).

References

1. Gross, D.J.; Pisarski, R.D.; Yaffe, L.G. QCD and instantons at finite temperature. *Rev. Mod. Phys.* **1981**, *53*, 43–80. [CrossRef]
2. Schäfer, T.; Shuryak, E.V. Instantons in QCD. *Rev. Mod. Phys.* **1998**, *70*, 323–425. [CrossRef]
3. Berkowitz, E.; Buchoff, M.I.; Rinaldi, E. Lattice QCD input for axion cosmology. *Phys. Rev. D* **2015**, *92*, 034507. [CrossRef]
4. Borsanyi, S.; Dierigl, M.; Fodor, Z.; Katz, S.D.; Mages, S.W.; Nogradi, D.; Redondo, J.; Ringwald, A.; Szabo, K.K. Axion cosmology, lattice QCD and the dilute instanton gas. *Phys. Lett. B* **2016**, *752*, 175–181. [CrossRef]
5. Gattringer, C.; Göckeler, M.; Rakow, P.; Schaefer, S.; Schäfer, A. A comprehensive picture of topological excitations in finite temperature lattice QCD. *Nucl. Phys. B* **2001**, *618*, 205–240. [CrossRef]
6. Gattringer, C.; Hoffmann, R.; Schaefer, S. The topological susceptibility of SU(3) gauge theory near Tc. *Phys. Lett. B* **2002**, *535*, 358–362. [CrossRef]
7. Jahn, P.T.; Moore, G.D.; Robaina, D. $\chi_{\mathrm{top}}(T \gg T_{\mathrm{c}})$ in pure-glue QCD through reweighting. *Phys. Rev. D* **2018**, *98*, 054512. [CrossRef]
8. Burger, F.; Ilgenfritz, E.M.; Lombardo, M.P.; Müller-Preussker, M.; Trunin, A. Topology (and axion's properties) from lattice QCD with a dynamical charm. *Nucl. Phys. A* **2017**, *967*, 880–883. [CrossRef]
9. Giusti, L.; Lüscher, M. Topological susceptibility at $T > T_{\mathrm{c}}$ from master-field simulations of the SU(3) gauge theory. *Eur. Phys. J. C* **2019**, *79*, 207. [CrossRef]
10. Antonov, D. Topological Susceptibility of the Gluon Plasma in the Stochastic-Vacuum Approach. *Universe* **2024**, *10*, 377. [CrossRef]
11. Sin, S.J. Late time cosmological phase transition and galactic halo as Bose liquid. *Phys. Rev. D* **1994**, *50*, 3650–3654. [CrossRef] [PubMed]
12. Suárez, A.; Robles, V.H.; Matos, T. A Review on the Scalar Field/Bose-Einstein Condensate Dark Matter Model. *Astrophys. Space Sci. Proc.* **2014**, *38*, 107–142. [CrossRef]
13. Magana, J.; Matos, T. A brief Review of the Scalar Field Dark Matter model. *J. Phys. Conf. Ser.* **2012**, *378*, 012012. [CrossRef]
14. Schive, H.Y.; Chiueh, T.; Broadhurst, T. Cosmic Structure as the Quantum Interference of a Coherent Dark Wave. *Nat. Phys.* **2014**, *10*, 496–499. [CrossRef]
15. Martinez-Medina, L.A.; Robles, V.H.; Matos, T. Dwarf galaxies in multistate scalar field dark matter halos. *Phys. Rev. D* **2015**, *91*, 023519. [CrossRef]
16. Hui, L.; Ostriker, J.P.; Tremaine, S.; Witten, E. Ultralight scalars as cosmological dark matter. *Phys. Rev. D* **2017**, *95*, 043541. [CrossRef]
17. Niemeyer, J.C. Small-scale structure of fuzzy and axion-like dark matter. *Prog. Part. Nucl. Phys.* **2020**, *113*, 103787. [CrossRef]
18. Perez, A.; Rovelli, C. Physical effects of the Immirzi parameter in loop quantum gravity. *Phys. Rev. D* **2006**, *73*, 044013. [CrossRef]
19. Candelas, P.; Raine, D.J. General-relativistic quantum field theory: An exactly soluble model. *Phys. Rev. D* **1975**, *12*, 965. [CrossRef]
20. Giacosa, F.; Hofmann, R.; Neubert, M. A model for the very early Universe. *JHEP* **2008**, *02*, 077. [CrossRef]
21. Meinert, J.; Hofmann, R. Electroweak Parameters from Mixed SU(2) Yang–Mills Thermodynamics. *Symmetry* **2024**, *16*, 1587. [CrossRef]
22. Trombetti, T.; Burigana, C. Semi-analytical description of clumping factor and cosmic microwave background free–free distortions from reionization. *Mon. Not. R. Astron. Soc.* **2013**, *437*, 2507–2520. [CrossRef]
23. Hofmann, R. *The Thermodynamics of Quantum Yang-Mills Theory: Theory and Applications*, 2nd ed.; World Scientific Publishing Co. Pte Ltd.: Singapore, 2016.
24. Peccei, R.D.; Quinn, H.R. CP Conservation in the Presence of Pseudoparticles. *Phys. Rev. Lett.* **1977**, *38*, 1440–1443. [CrossRef]
25. Peccei, R.; Quinn, H.R. Constraints Imposed by CP Conservation in the Presence of Instantons. *Phys. Rev. D* **1977**, *16*, 1791–1797. [CrossRef]
26. Wantz, O.; Shellard, E.P.S. Axion cosmology revisited. *Phys. Rev. D* **2010**, *82*, 123508. [CrossRef]
27. Harrington, B.J.; Shepard, H.K. Periodic Euclidean solutions and the finite-temperature Yang-Mills gas. *Phys. Rev. D* **1978**, *17*, 2122–2125. [CrossRef]
28. Hofmann, R. Nonperturbative approach to Yang-Mills thermodynamics. *Int. J. Mod. Phys. A* **2005**, *20*, 4123–4216. [CrossRef]
29. Gross, D.J.; Wilczek, F. Ultraviolet Behavior of Nonabelian Gauge Theories. *Phys. Rev. Lett.* **1973**, *30*, 1343–1346. [CrossRef]
30. Politzer, H.D. Reliable Perturbative Results for Strong Interactions? *Phys. Rev. Lett.* **1973**, *30*, 1346–1349. [CrossRef]
31. Meinert, J.; Hofmann, R. Axial Anomaly in Galaxies and the Dark Universe. *Universe* **2021**, *7*, 198. [CrossRef]
32. Veneziano, G. U(1) without instantons. *Nucl. Phys. B* **1979**, *159*, 213–224. [CrossRef]
33. Witten, E. Current algebra theorems for the U(1) "Goldstone boson". *Nucl. Phys. B* **1979**, *156*, 269–283. [CrossRef]
34. Ma, C.P.; Bertschinger, E. Cosmological Perturbation Theory in the Synchronous and Conformal Newtonian Gauges. *Astrophys. J.* **1995**, *455*, 7. [CrossRef]

35. Hofmann, R. Low-frequency line temperatures of the CMB. *Ann. Phys.* **2009**, *18*, 634. [CrossRef]
36. Herbst, U. The Deconfining Phase of SU(2) Yang-Mills Thermodynamics. *arXiv* **2005**, arXiv:hep-th/0506004.
37. Kraan, T.C.; van Baal, P. Exact T duality between calorons and Taub - NUT spaces. *Phys. Lett. B* **1998**, *428*, 268–276. [CrossRef]
38. Kraan, T.C.; van Baal, P. Periodic instantons with nontrivial holonomy. *Nucl. Phys. B* **1998**, *533*, 627–659. [CrossRef]
39. Kraan, T.C.; van Baal, P. Monopole constituents inside SU(n) calorons. *Phys. Lett. B* **1998**, *435*, 389–395. [CrossRef]
40. Hofmann, R.; Kaviani, D. The quantum of action and finiteness of radiative corrections: Deconfining SU(2) Yang-Mills thermodynamics. *Quant. Matt.* **2012**, *1*, 41–52. [CrossRef]
41. Rabinowitch, A.S. On a new class of progressive waves in Yang–Mills fields with SU(2) symmetry. *Nucl. Phys. B* **2024**, *1001*, 116505. [CrossRef]
42. Seiler, E. Some more remarks on the Witten–Veneziano formula for the η' mass. *Phys. Lett. B* **2002**, *525*, 355–359. [CrossRef]
43. Azakov, S.; Joos, H.; Wipf, A. Witten–Veneziano relation for the Schwinger model. *Phys. Lett. B* **2000**, *479*, 245–248. [CrossRef]
44. Hahn, S.; Hofmann, R. Exact determination of asymptotic CMB temperature-redshift relation. *Mod. Phys. Lett. A* **2018**, *2016*, 1850029. [CrossRef]
45. Frieman, J.A.; Hill, C.T.; Stebbins, A.; Waga, I. Cosmology with ultralight pseudo Nambu-Goldstone bosons. *Phys. Rev. Lett.* **1995**, *75*, 2077–2080. [CrossRef] [PubMed]
46. Riess, A.G.; Macri, L.M.; Hoffmann, S.L.; Scolnic, D.; Casertano, S.; Filippenko, A.V.; Tucker, B.E.; Reid, M.J.; Jones, D.O.; Silverman, J.M.; et al. A 2.4% Determination of The Local Value of the Hubble Constant. *Astrophys. J.* **2016**, *826*, 56. [CrossRef]
47. Riess, A.G.; Anand, G.S.; Yuan, W.; Casertano, S.; Dolphin, A.; Macri, L.M.; Breuval, L.; Scolnic, D.; Perrin, M.; Anderson, I.R. JWST Observations Reject Unrecognized Crowding of Cepheid Photometry as an Explanation for the Hubble Tension at 8σ Confidence. *Astrophys. J. Lett.* **2024**, *962*, L17. [CrossRef]
48. Riess, A.G.; Yuan, W.; Macri, L.M.; Scolnic, D.; Brout, D.; Casertano, S.; Jones, D.O.; Murakami, Y.; Breuval, L.; Brink, T.G.; et al. A Comprehensive Measurement of the Local Value of the Hubble Constant with 1 km/s/Mpc Uncertainty from the Hubble Space Telescope and the SH0ES Team. *arXiv* **2021**, arXiv:2112.04510. [CrossRef]
49. Hofmann, R. SU(2) Yang-Mills Theory: Waves, Particles, and Quantum Thermodynamics. *Entropy* **2016**, *18*, 310. [CrossRef]
50. Einstein, A. Quantentheorie des einatomigen idealen Gases—Zweite Abhandlung. In *Sitzungsberichte der Preussischen Akademie der Wissenschaften*; Wiley: Hoboken, NJ, USA, 1925; p. 3.
51. Matos, T.; Guzman, F.S. Scalar fields as dark matter in spiral galaxies. *Class. Quant. Grav.* **2000**, *17*, L9–L16. [CrossRef]
52. D'Elia, M.; Di Giacomo, A.; Meggiolaro, E. Gauge invariant field strength correlators in pure Yang-Mills and full QCD at finite temperature. *Phys. Rev. D* **2003**, *67*, 114504. [CrossRef]
53. Vicari, E.; Panagopoulos, H. θ dependence of SU(N) gauge theories in the presence of a topological term. *Phys. Rep.* **2009**, *470*, 93–150. [CrossRef]