University of Cincinnati

Date: 7/14/2014

I, Amr Ahmadain , hereby submit this original work as part of the requirements for the degree of Master of Science in Physics.

It is entitled:

Electric-Magnetic Duality-Symmetric Effective Actions in Harmonic Superspace

Student's name: **Amr Ahmadain**

This work and its defense approved by:

Committee chair: Philip Argyres, Ph.D.

Committee member: F Paul Esposito, Ph.D.

Committee member: L.C.R. Wijewardhana, Ph.D.

UNIVERSITY OF Cincinnati

10461

Electric-Magnetic Duality-Symmetric Effective Actions in Harmonic Superspace

A Thesis submitted to
the Graduate School
in partial fulfillment of the
requirements for the Degree of

Master of Science

in the Department of
Physics
of the McMicken College of Arts and Sciences
August 2014
by

Amr Ahmadain

MS Computer Science, University of Louisville, 1998
BS Computer Information Systems, Sadat Academy for Management Sciences, 1998

Committee Chairperson: Philip Argyres, PhD

Abstract

In this thesis, a strategy for constructing electric-magnetic (EM) duality-symmetric $N = 2$ supersymmetric infrared effective actions (IREAs) is presented using *harmonic superspace*. Our aim is to elevate the EM duality from being equivalent descriptions of distinct IREAs to a symmetry of a single IREA under $Sp(2r, \mathbb{Z})$ transformations. Our strategy is to build the IREA out of geometric objects which are manifestly $Sp(2r, \mathbb{Z})$ invariant. We conjecture that a manifestly EM duality-symmetric action can be constructed in this way on harmonic superspace. The main invariant geometric object is the *total space*, $\mathcal{X}$, of the Coulomb branch moduli space of the IREA, which has a natural hyperkähler structure, and is thus a suitable manifold to act as the target space of an $N = 2$ supersymmetric nonlinear σ-model (nlsm). We build the IREA as a nlsm with target space the *twistor space* of $\mathcal{X}$. The twistor space is a fiber bundle with base space the projective line, $\mathbb{CP}^1$, and $\mathcal{X}$ as fiber. The nlsm action is formed by pulling back the invariant holomorphic two-form on twistor space by the hypermultiplet superfield in harmonic superspace, with the base $\mathbb{CP}^1$ identified with the internal $\mathbb{CP}^1$ of harmonic superspace. We also conjecture, but do not prove, that the pullback approach introduced in this thesis for constructing the hypermultiplet nlsm is equivalent to using the standard harmonic superspace procedure of constructing the nlsm action using a harmonic-analytic potential for the hypermultiplet superfields.

Acknowledgements

First and foremost, I have to thank my advisor Prof. Philip Argyres. Professor Argyres, I'm not aware there does exist words in any language that can eloquently describe and fully express my feelings as I'm writing this. I'm very grateful to you for believing in me when I myself did not and for supporting me with your very encouraging words at times when you were one of very few who could have done that. Throughout the last five years that I have known you, besides being my advisor, you have also been my mentor and my friend. I have to say that without you and the support I received from the entire physics department at UC, my whole physics career would have probably been shattered. I owe you most of what I have learned in physics and I'm very proud of that. I hope I could have given you more in terms of physics but I'm sure that sometime later I will. Your teaching style, and way of reasoning, and clarity of thought are only three qualities among many others that have deeply affected me. This thesis would have never seen the light if it weren't for your dedication and guidance. I will miss our very engaging and often illuminating discussions about physics in your office and at Starbucks. Thank you professor.

I will also never forget how much support Prof. Rohana Wijewardhana and Prof. Paul Esposito gave me during the last five years especially right after my PhD qualifying examination. This was a moment when your heartening words meant a lot to me. You taught me a lot of physics and I'm very grateful for that. I'm really fortunate to have you as my physics professors. Thank you.

Finally, I can never forget to acknowledge my dad for the infinite emotional support he provided me with over the last ten years. I promise to make you very proud of me dad. My mom, who died exactly five years ago today, would have also been very proud of me now. I miss you so much mom. My wife, over the last couple of years has literally been my sweetheart. She had to put up with my very bad temper along many other challenges that we both had to go through. I love you Menna.

Contents

Chapter 1

Introduction

The idea of electric-magnetic (EM) duality probably goes back to Dirac who observed that the source-free Maxwell equations are symmetric under the exchange of the electric and magnetic fields. More precisely, the symmetry is $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$, or

$$F_{\mu\nu} \to \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{1.1}$$

(Here $\epsilon_{\mu\nu\rho\sigma}$ is the flat-space antisymmetric ϵ-tensor with $\epsilon^{0123} = +1$ and the Minkowski metric $\eta_{\mu\nu}$ has signature $-{+}{+}{+}$.) To maintain this symmetry in the presence of sources, Dirac introduced in an ad hoc way magnetic monopoles with magnetic charges q_m in addition to the electric charges q_e, and showed that consistency of the quantum theory requires a charge quantization condition $q_m q_e = 2\pi n$ with integer n. Hence the minimal charges obey $q_m = 2\pi/q_e$. EM duality exchanges q_e and q_m, i.e. q_e and $2\pi/q_e$. Now recall that the electric charge q_e also is the coupling constant. So EM duality exchanges the coupling constant with its inverse (up to the factor of 2π), hence exchanging strong and weak coupling. This is one reason why EM duality is of so much interest to the field theory community: the hope is to learn about strong-coupling physics from the weak-coupling physics of a dual formulation of the theory. Of course, in classical Maxwell theory we know all we may want to know, but this is no longer true in quantum electrodynamics if both the electric and magnetic sources are dynamical.

In this thesis we are interested in the application of EM duality to $N = 2$ supersymmetric gauge theories. These theories are characterized by having a "Coulomb branch" moduli space of vacua on which there is generically an infrared effective action (IREA) consisting of r copies of Maxwell electromagnetism, for some integer r which is also the complex dimension of the Coulomb branch. This low energy electromagnetism is just a free theory of r photons since it has only massive (non-dynamical) sources. The strong-weak aspect of EM duality therefore plays a limited role in this problem. Instead, EM duality is important for more subtle reasons to do with the global structure of the Coulomb branch, to be explained below.

Following the seminal work of Seiberg and Witten [1, 2], it turns out that understanding EM duality on Coulomb branches is a powerful tool for understanding the phase structure of $N = 2$ supersymmetric gauge theories. The study of $N = 2$ supersymmetric quantum field theories in four dimensions has been a fertile field for theoretical physics for twenty years. These theories always have non-chiral matter representations, and therefore can never be directly relevant for describing the real world. That said, the existence of two sets of supersymmetries allows us to study their properties in much greater detail than both non-supersymmetric theories and $N = 1$ supersymmetric theories. Being able to do so is quite fun in itself, and hopefully the general lessons thus learned concerning $N = 2$ supersymmetric theories might be useful when we study the dynamics of theories with less supersymmetry. At least, the physical properties of $N = 2$ theories have been successfully used to point mathematicians to a number of new mathematical phenomena previously unknown to them.

A major difficulty in understanding EM duality on Coulomb branches is that there are no satisfactory action principles for (super-)electromagnetism which make their EM duality-invariance manifest. This is the problem this thesis will address.

1.1 Overview of the problem

Constructing actions that simultaneously combine manifest EM duality symmetry with manifest Lorentz invariance has long been known to be problematic [3, 4, 6, 7, 8, 14, 15, 16, 12, 13, 17, 18, 25, 19, 20, 21, 22, 23]. The problem is that the Maxwell action is written in terms of the gauge potentials, and although EM duality acts on the field strength tensor in a Lorentz covariant way as in (1.1), it does not act on the gauge potentials in any local, Lorentz-covariant way. Many attempts during the last four decades have attempted to build actions that have duality symmetry and Lorentz invariance using different approaches and tricks. In doing so, the action is usually augmented with other auxiliary fields on which constraints are imposed so that the Maxwell action that contains the correct number of propagating degrees of freedom is later recovered. In Chapter 2, these different methods and approaches will be discussed in greater detail.

Adding $N = 2$ supersymmetry to the duality symmetry problem described in the previous paragraph makes it even more challenging [26, 27, 29, 30, 31, 32, 33, 34, 35]. The basic problem is that $N = 2$ supersymmetry relates electromagnetic fields to scalars and so EM duality transformations need to be extended to transformations of the scalars as well if $N = 2$ supercovariance is to be maintained. However, the scalar fields can be thought of as coordinates on an associated Riemannian manifold, $\mathcal{M}_V$, (as we will explain below) and the coordinates in which the EM duality transformations are linear do not linearize the geometry of $\mathcal{M}_V$. It is this tension that makes formulating manifestly EM duality-symmetric and $N = 2$ supersymmetric actions so difficult.

The manifold $\mathcal{M}_V$ mentioned above is the *moduli space of vacua* of the $N = 2$ supersymmetric field theory. This continuous family of inequivalent vacua are labelled by the vacuum expectation values of the massless scalar fields in the theory, thus the values (vevs) of these scalars are the coordinates on $\mathcal{M}_V$. $N = 2$ field theories always have such moduli spaces, and, in fact, their moduli spaces often have many components. We are interested in the component, called the *Coulomb branch*, in which the scalars with vevs are the superpartners of $U(1)$ gauge fields (Maxwell fields). The leading terms in a low energy or *infrared effective action* (IREA) on the Coulomb branch will be the kinetic terms of the scalars and their photon superpartners (as well as of their fermionic superpartners). The scalar field terms in such an IREA are called a *nonlinear σ-model* (nlsm), and have the general form $g_{mn}(\varphi)\partial_\mu\varphi^m\partial^\mu\varphi^n$ where the φ^m label the scalars. Then the kinetic couplings $g_{mn}(\varphi)$ can be interpreted as a metric on $\mathcal{M}_V$, thus giving $\mathcal{M}_V$ a Riemannian structure. (The kinetic terms for the associated photons give other structures on $\mathcal{M}_V$ which give it the structure of a *special Kähler* manifold, and will be discussed at great length later.)

So far, EM duality transformations have not been realized as *symmetries* of the IREA, but only as equivalences of apparently different free $U(1)^r$ field theories coupled to classical massive sources under symplectic $Sp(2r,\mathbb{Z})$ redefinitions of electric and magnetic charges. The importance of this redundancy in the lagrangian description of IREAs becomes apparent when there is a moduli space $\mathcal{M}_V$ of inequivalent vacua. In that case, upon traversing a closed loop in $\mathcal{M}_V$ the physics must, by definition, be the same at the beginning and end of the loop, but the Lagrangian description need not because it may have suffered an EM duality transformation.

A central problem of $N = 2$ field theories is the construction of their IREAs. The main technique that is used is analytic continuation of the IREA Lagrangian on the

Coulomb branch. The multi-valuedness of the Lagrangian under EM duality transformations is the main technical hurdle in carrying out this continuation procedure. A formulation of the IREA which is EM duality-symmetric would greatly simplify this problem, as the object being analytically continued would be single-valued on the Coulomb branch (and thus would be much easier to determine in terms of its boundary values).

1.2 Thesis objective

The key goal of this thesis is to develop a strategy for constructing EM duality-symmetric $N = 2$ IREAs. In other words, we want to construct $N = 2$ nlsm actions where EM duality transformations are a *manifest symmetry* as opposed to a map between different lagrangians. The key ingredient in achieving this goal is to reformulate the nlsm in terms of mathematical objects which are manifestly EM-duality invariant. The chief such object is the *total space*, $\mathcal{X}$, of the Coulomb branch (and *not* the Coulomb branch itself), and its inherent symplectic-invariant geometric data (the complex structure, Hodge form, and Donagi-Witten two-form). It is thus natural to use $\mathcal{X}$ as the target space of the nlsm and its geometric data to construct its Lagrangian.

Heuristically, we want to to build the EM duality-symmetric $N = 2$ nlsm action by using superfields on *harmonic superspace* to pull back those invariant geometric structures on the target space which can naturally be integrated over harmonic superspace [36, 37]. We call this strategy the *pullback approach* to constructing EM duality-symmetric actions.

To make this strategy more concrete, we need to describe in more detail what is the geometry of the target space, $\mathcal{X}$, and what is the geometry of harmonic superspace. We will now give a very brief outline of these geometries, highlightling those geometrical objects that will play central roles in what follows. Later chapters in the thesis will be devoted to fleshing out the details of these geometries.

It is well known [1, 38, 39] that a Coulomb branch is a complex manifold with a *special Kähler* structure. A consequence of this [39] is that its total space, $\mathcal{X}$, has a natural *hyperkähler structure* which encodes the special Kähler geometry of the Coulomb branch. Very succinctly, a hyperkähler space $\mathcal{X}$ supports three complex structures J_1, J_2, J_3 obeying the quaternion algebra: $J_1 J_2 = -J_2 J_1 = J_3$, plus cyclic permutations [40]. Any real linear combination of these, $J(\vec{r}) := \sum_{a=1}^{3} r^a J_a$, is again a complex structure on $\mathcal{X}$ if $\sum_a (r^a)^2 = 1$, *i.e.*, if $\vec{r}$ lies on a two-sphere $S^2 \simeq \mathbb{CP}^1$. Furthermore, each complex structure has an associated Kähler form — a real non-degenerate closed 2-form which is of type (1,1) with respect to its complex structure. It is traditional to denote the Kähler form associated to J_a by ω_a.

It will be convenient for us to use not the total space $\mathcal{X}$ as the target space, but instead to use the equivalent *twistor space*, $\mathcal{Z}$, associated to $\mathcal{X}$ as the target space. The twistor space $\mathcal{Z}$ of a hyperkähler space $\mathcal{X}$ is a fiber bundle over the two-sphere of complex structures of $\mathcal{X}$ with $\mathcal{X}$ as the fiber [41, 40]. Thus $\mathcal{Z} \simeq \mathcal{X} \times \mathbb{CP}^1$ as a topological space, but it is endowed with a special choice of complex structure, $J^{\mathcal{Z}}$, given by choosing the complex structure $J(\vec{r})$ on the fiber over each point $\vec{r} \in \mathbb{CP}^1$. Furthermore, it has a holomorphic (2,0)-form, Ω, with respect to this complex structure, given by $\Omega := (\omega_1 + i\omega_3) + \zeta\omega_2 - \zeta^2(\omega_1 + i\omega_3)$, where ζ is a complex coordinate on the $\mathbb{CP}^1$. It is a theorem [41] that given a complex space $\mathcal{Z}$ with such a two-form Ω, one can uniquely reconstruct the hyperkähler space $\mathcal{X}$.

Harmonic superspace [37] is one way of dealing with supersymmetric theories with $N = 2$ supersymmetry (SUSY) in four dimensions in a manifestly covariant manner. 4-dimensional $N = 2$ SUSY has eight hermitian spin-1/2 generators which can be organized into a pair of complex 2-component (Weyl) spinors transforming in the fundamental (doublet) representation of $SU(2)_R$. Here $SU(2)_R$ is the internal symmetry group of the $N = 2$ SUSY algebra. The main feature of harmonic superspace is that it makes $N = 2$ SUSY manifest in field theories by using *superfields* which are functions not only of space-time coordinates, but also of eight anti-commuting spinor coordinates corresponding to the SUSY generators, as well as further "internal" (commuting) coordinates on a 2-sphere $S^2 \simeq \mathbb{CP}^1$.

Two points will be important in what follows. First, we see from the short description given above that both the harmonic superspace and twistor space have an extra 2-sphere $S^2 \simeq \mathbb{CP}^1$, a fact that will be key in the success of the strategy of pulling back the $Sp(2r, \mathbb{Z})$-invariant geometric data from the twistor space to the harmonic superspace. Second, the uniqueness of the complex structure $J^{\mathcal{Z}}$ of the twistor space plays a significant role in the success of the pullback approach as we will see in chapters 4, 5 and 6 when we describe the mathematical details of the pullback approach.

1.3 New findings/results

In this section, we briefly summarize our key new findings:

- We give a recipe for how to construct EM duality-symmetric $N = 2$ IREAs using the pullback approach. This construction is the main result of our investigations, and is nearly uniquely specified by the invariant geometric pullback approach. However, we have not completed a proof that it gives the correct set of propagating degrees of freedom.

- We conjecture the equivalence of our pullback approach for constructing hyperkähler nlsm harmonic superspace actions to the traditional harmonic superspace method described in [37]. This conjectured equivalence gives a simple geometric picture of harmonic superspace nlsms in terms of the twistor space construction of hyperkähler manifolds.

- Finally, along the way we discover an extra geometric structure in the Donagi-Witten formulation [38] of the special Kähler geometry of the Coulomb branches of $N = 2$ supersymmetric gauge theories. This extra structure may have interesting implications for the low-energy physics on the Coulomb branch.

1.4 Thesis layout

In the rest of this thesis we proceed to flesh out this brief description of our problem and solution approach.

Chapter 2 will review some of the literature related to constructing EM duality-invariant nonsupersymmetric actions and the different methods and approaches devised to restore the manifestly broken Lorentz invariance of the action. Along the way we describe the $SL(2, \mathbb{Z})$ group of EM duality equivalences of a quantum theory of a single photon.

Chapter 3 defines the central problem of why $N = 2$ IREAs on the Coulomb branch in its current formulation are *not* EM duality-symmetric. This leads to a concrete description of the *special Kähler* structure of the Coulomb branch in terms of the low energy physics of r photons and their $N = 2$ superpartners. Also the action of the $Sp(2r, \mathbb{Z})$ group EM duality transformations on this structure is described.

Chapter 4 describes in some detail three nearly equivalent descriptions of special Kähler geometry which are manifestly $Sp(2r, \mathbb{Z})$-invariant. These are: (1) the "total space" geometry described as a bundle of abelian varieties fibered over the Coulomb branch; (2) the same space viewed as a hyperkähler manifold; and (3) the "twistor space" of the hyperkähler space fibered over an S^2 base space.

Chapter 5 briefly introduces *harmonic superspace* which is a superspace suited for writing manifestly $N = 2$ SUSY-invariant actions.

With these mathematical results in hand, we can then describe in Chapter 6 in concrete terms our strategy for constructing actions symmetric under both EM-duality and $N = 2$ SUSY transformations. This is done by pulling back geometric structures of twistor space to harmonic superspace by hypermultiplet superfields, and then gauging the isometries of twistor space using vector multiplet superfields.

Chapter 7 concludes the thesis with a brief recap of our results and an outlook for future work.

Chapter 2

Background and Related Work

In this chapter we will review two major topics:

- attempts to construct non-supersymmetric EM duality-invariant actions, and
- attempts to construct EM duality-invariant actions with $N = 1$ supersymmetry.

We will then briefly critically evaluate the suitability of these formulations for constructing $N = 2$ supersymmetric EM duality-invariant actions. For a general introduction to EM duality, pleaser refer to [42].

2.1 EM duality-symmetric actions

Attempts to build EM duality-symmetric actions go back to Dirac who originally wrote down a non-local, Lorentz invariant Lagrangian [43]. In this formulation, the magnetic current does not couple directly to the gauge field. Instead it only couples through the Dirac string attached to each monopole, which makes calculations very difficult.

2.1.1 Zwanziger's action and a first look at EM duality

Later, Zwanziger [4] reformulated the theory in terms of a local, but non-Lorentz invariant lagrangian with two gauge potentials A_μ and B_μ. Even though there are two gauge potentials, the form of the non-Lorentz invariant kinetic mixing ensures that the are only two on-shell degrees of freedom for the gauge fields. The advantage of having two gauge potentials is that one, A_μ, has a local coupling to electric currents, while B_μ has a local coupling to magnetic currents. Although *manifest* Lorentz invariance is lost in this formulation, after the Dirac quantization conditions are imposed, Lorentz invariance is recovered at the level of the equations of motion.

Zwanziger's goal was to construct an action leading to Maxwell's equations,

$$\partial_\mu F^{\mu\nu} = e^2 j_e^\nu, \qquad \partial_\mu \star F^{\mu\nu} = 4\pi j_g^\nu, \tag{2.1}$$

in the presence of electric and magnetic currents j_e and j_g, which are separately conserved,

$$\partial_\nu j_e^\nu = \partial_\nu j_g^\nu = 0. \tag{2.2}$$

Here $e^2/(4\pi)$ is the fine structure constant. The normalization of the electric and magnetic currents has been chosen so that electric and magnetic charges,

$$q := \int d^3x\, j_e^0, \qquad g := \int d^3x\, j_g^0, \tag{2.3}$$

satisfy the Dirac quantization condition [4, 43] in the form

$$q_i g_j - q_j g_i = \frac{n}{2}, \tag{2.4}$$

where n is an integer and (q_i, g_i) is the electric and magnetic charge of the ith dyon. ("Dyon" is the general name for a particle which may carry either or both electric and magnetic charges.)

The dual field strength, $\star F$, appearing in (2.1) is defined by

$$\star F_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F^{\kappa\lambda}. \tag{2.5}$$

It is convenient to introduce an indexless notation in which index contraction is denoted by a dot, and index antisymmetrization by a wedge: $p \cdot q := p_\mu q^\mu$ and $p \wedge q := p_\mu q_\nu - p_\nu q_\mu$.

Zwanziger substituted the following general parameterization of F into (2.1)

$$F = \frac{1}{n^2}\Big[\{n \wedge [n \cdot (\partial \wedge A)]\} - \star\{n \wedge [n \cdot (\partial \wedge B)]\}\Big], \tag{2.6}$$

to obtain equations for the two A_μ and B_μ. Here n_μ is an arbitrary fixed space-like four-vector (thus breaking manifest Lorentz invariance), which can be thought of as the (arbitrarily chosen) space-like direction of the "Dirac string" singularity in A and B emanating from any point magnetic or electric charges. The Lagrangian density that generates the resulting equations of motion for A and B is

$$\begin{aligned}\mathcal{L} = -\frac{1}{2e^2n^2}\Big\{&[n \cdot (\partial \wedge A)] \cdot [n \cdot \star(\partial \wedge B)] - [n \cdot (\partial \wedge B)] \cdot [n \cdot \star(\partial \wedge A)] \\ &+ [n \cdot (\partial \wedge A)]^2 + [n \cdot (\partial \wedge B)]^2\Big\} - j_e \cdot A - \frac{4\pi}{e^2} j_g \cdot B.\end{aligned} \tag{2.7}$$

This Lagrangian is invariant under the EM duality transformation

$$\begin{pmatrix} B \\ A \end{pmatrix} \to \frac{4\pi}{e^2}\begin{pmatrix} A \\ -B \end{pmatrix}, \qquad \begin{pmatrix} j_g \\ j_e \end{pmatrix} \to \begin{pmatrix} j_e \\ -j_g \end{pmatrix}, \qquad e^2 \to \frac{(4\pi)^2}{e^2}, \tag{2.8}$$

which inverts the coupling and exchanges electric and magnetic charges. Note that in the absence of sources, the coupling can be absorbed in a rescaling of A and B, in which case the EM duality transformation would be a manifest symmetry of the lagrangian. This is the sense in which (2.7) is manifestly EM duality-symmetric. Note that for the theory with the coupling to sources present, the EM duality transformation (2.8) is *not* a symmetry since it acts not only on the fields, but also on the coupling constant, thus changing the theory itself.

Recently Csaki, Shirman and Terning [44] generalized this lagrangian to include a θ-angle parameter which plays a nontrivial role in the quantum theory with both electric and magnetic sources. In particular, their action incorporates the "Witten effect" [45] which states that the electric charges of dyons are effectively shifted by $\theta/(2\pi)$ times their magnetic charges. Their lagrangian is

$$\begin{aligned}\mathcal{L} = -\mathrm{Im}\frac{\tau}{8\pi n^2}&\{[n\cdot\partial\wedge(A+iB)]\cdot[n\cdot\partial\wedge(A-iB)]\}\\ &-\mathrm{Re}\frac{\tau}{8\pi n^2}\{[n\cdot\partial\wedge(A+iB)]\cdot[n\cdot\star\partial\wedge(A-iB)]\}\\ &-j_e\cdot A-\frac{4\pi}{e^2}j_g\cdot B,\end{aligned} \tag{2.9}$$

where the fine structure constant and the θ-angle are combined into a complex coupling

$$\tau := \frac{\theta}{2\pi}+i\frac{4\pi}{e^2}. \tag{2.10}$$

This lagrangian is invariant under a discrete $Sp(2,\mathbb{Z}) \simeq SL(2,\mathbb{Z})$ group of transformations which acts on the fields and coupling as

$$A+iB\to\frac{1}{a-c\overline{\tau}}(A+iB),\qquad \begin{pmatrix} j_g \\ j_e \end{pmatrix}\to\begin{pmatrix} aj_g+cj_e \\ bj_g+dj_e \end{pmatrix},\qquad \tau\to\frac{d\tau-b}{a-c\tau}, \tag{2.11}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, i.e., a, b, c, d are arbitrary integers satisfying $ad - bc = 1$. The integrality of a, b, c, d is necessary to preserve the Dirac quantization condition (2.4). This infinite discrete $Sp(2,\mathbb{Z})$ invariance is the full quantum-mechanical EM duality group of Maxwell theory. Note that for $a = d = 0$ and $-b = c = 1$ (2.11) reduces to the classical EM duality transformation (2.8).

2.1.2 Actions with an extra gauge invariance

The Zwanziger lagrangian (and its θ-angle extension) suffers both from lack of manifest Lorentz invariance, and from the obscure way in which the correct counting of propagating degrees of freedom comes about. The latter problem was solved independently in [6] and [13] who proposed an apparently different non-Lorentz-invariant but manifestly EM duality-invariant action. The new feature of this action is that it has an added gauge invariance that makes the counting of the degrees of freedom more obvious. (From now on, for simplicity, we give actions with no electric or magnetic sources or θ angles. We will also rescale the fields to remove the coupling. Adding the sources and couplings back in as in (2.9) is straight forward.)

Henneaux and Teitelboim [10] and Schwarz and Sen [13] put two Abelian gauge fields, A^a_μ ($a = 1, 2$), on an equal footing in the action

$$S = -\frac{1}{2}\int d^4x (B^{i\,a}\epsilon^{ab}E^b_i + B^{i\,a}B^a_i), \tag{2.12}$$

where $E^{i\,a} := F^a_{0i}$ and $B^{i\,a} := \frac{1}{2}\epsilon^{ijk}F^a_{jk}$ with $F^a := \partial \wedge A^a$, are the usual electric and magnetic fields derived from each potential. Here $i, j, k = 1, 2, 3$ are spatial indices, and ϵ^{ab} is the antisymmetric unit matrix with $\epsilon^{12} = 1$. Note that (2.12) is not only invariant under the usual Abelian gauge transformations of A^a_μ, but also under the additional local transformations,

$$A_0^a \to A_0^a + \Psi^a(x) \tag{2.13}$$

for arbitrary scalar functions Ψ^a. This extra gauge invariance is responsible for reducing the number of propagating degrees of freedom from those of two Maxwell fields to only one Maxwell field.

The action (2.12) is also manifestly invariant under global $SO(2)$ transformations mixing A_μ^1 and A_μ^2, which contains the EM duality symmetry as a discrete subgroup:

$$A_\mu^a \to e^{ab} A_\mu^b. \tag{2.14}$$

Using the above local symmetries and the A_μ^a equations of motion, one can eliminate one of the gauge fields and get the conventional Maxwell theory for the other one. The EM duality symmetry is then reduced to the duality between the electric and magnetic field strengths of the remaining Maxwell field.

The action (2.12) can be slightly generalized [20] by using a constant vector n^μ subject to the constraint, $n \cdot n \neq 0$, to rewrite (2.12) as

$$S = -\frac{1}{4} \int d^4x \left\{ -\frac{1}{2} \operatorname{tr}(F^a \cdot F^a) + \frac{1}{n \cdot n} n \cdot \mathcal{F}^a \cdot \mathcal{F}^a \cdot n \right\}, \tag{2.15}$$

where

$$\mathcal{F}^a := \epsilon^{ab} F^b - \star F^a. \tag{2.16}$$

In (2.15), the transformation in (2.13) takes the form

$$A_\mu^a \to A_\mu^a + n_\mu \Psi^a(x). \tag{2.17}$$

If we take $n_\mu = \delta^0_\mu$, i.e. a unit time-like vector, it is easy to check that (2.15) reduces to (2.12).

It is also easy to see that (2.15) is *almost* the same as the Zwanziger action (2.7) if we identify n_μ, A_μ and B_μ in the Zwanziger action with n_μ, A^2_μ, and A^1_μ in (2.15), respectively. The only difference between the two actions is that the Zwanziger action is (2.15) but with a change in the relative sign between the two terms. This sign difference is crucial since the Zwanziger action does *not* have the extra gauge symmetry (2.17).

2.1.3 Sorokin et. al.'s actions

Pasti, Sorokin and Tonin (PST) proposed [17, 18] to generalize (2.15) by turning the constant vector n_μ into by a dynamical x-dependent vector field. The problem with making n_μ a dynamical field is that it violates the local symmetry (2.17) which was crucial to ensure the correct number of propagating degrees of freedom. So the action in (2.15) must be modified to restore this symmetry. PST found the generalized action

$$S = -\frac{1}{4}\int d^4x \left(-\frac{1}{2}\,\mathrm{tr}(F^a \cdot F^a) + \frac{1}{n\cdot n} n \cdot \mathcal{F}^a \cdot \mathcal{F}^a \cdot n - \mathrm{tr}[\star\Lambda \cdot (\partial \wedge n)] \right). \qquad (2.18)$$

The last term ensures the invariance of the action with respect to (2.17) provided the auxiliary antisymmetric 2-index field $\Lambda_{\mu\nu}$ transforms as

$$\Lambda \to \Lambda + n \wedge (n \cdot \mathcal{F}^a)\, \Psi^a. \qquad (2.19)$$

Note that for constant normalized n_μ, the PST action (2.18) reduces to (2.15).

As before, one can eliminate one of the gauge fields (for example A^2) from the PST action using its equations of motion. This reduces (2.18) to the ordinary Maxwell action plus a term which contains the decoupled auxiliary field Λ. Thus, PST constructed a manifestly Lorentz-invariant version of the EM duality-symmetric action (2.15) which contains two abelian gauge potentials, and an additional decoupled redundant field.

One disadvantage of the PST action is the somewhat complicated form of the Lagrange multiplier and of its transformation (2.19) under the auxiliary gauge symmetry. PST noted, however, that the equation of motion for the Λ field is solved in general by

$$n_\mu(x) = \partial_\mu \varphi(x) \tag{2.20}$$

for an arbitrary scalar field φ. Substituting this back into the PST action gives the equivalent action [20]

$$S = -\frac{1}{4}\int d^4x \left(-\frac{1}{2}\,\mathrm{tr}(F^a \cdot F^a) + \frac{1}{\partial\varphi\cdot\partial\varphi}\partial\varphi\cdot\mathcal{F}^a\cdot\mathcal{F}^a\cdot\partial\varphi\right), \tag{2.21}$$

which has only a single scalar Lagrange multiplier field, φ.

Also, Maznytsia, Preitschopf and Sorokin showed explicitly [20] how (2.21) is equivalent to the original Zwanziger action (2.7) through a procedure involving dualizing φ to a 2-form auxiliary gauge field.

2.1.4 Siegel's self-dual actions

The idea of PST to use Lagrange multiplier fields to make both Lorentz and EM duality invariance manifest following Siegel, who was the first to use that idea in duality-symmetric actions. In [14] Siegel proposed manifestly Lorentz-invariant actions for *self-dual* antisymmetric tensor gauge fields by using Lagrange multiplier fields to eliminate half of the propagating degrees of freedom.

Self-dual fields are rank-p antisymmetric tensor gauge potentials in $D = 2(p+1)$ space-time dimensions with p an even integer. (Thus the space-time dimension is 2 mod 4.) Denote such a p-form field by A and its gauge-invariant $p+1$-form field strength tensor by $F := \partial \wedge A$. Then self-dual fields satisfy, in addition to the usual free Maxwell-type equations, the additional constraint that $F = \star F$, where $\star$ generalizes (2.5) to D dimensions. (This constraint is only consistent for p even in Minkowski space-times, and cannot be imposed when p is odd.) Siegel's action is then

$$S = -\frac{1}{2}\int d^D x \left\{ \frac{1}{(p+1)!} F_{\mu_1 \cdots \mu_{p+1}} F^{\mu_1 \cdots \mu_{p+1}} - \lambda^{\alpha}{}_{\beta} \mathcal{F}_{\alpha \nu_1 \cdot \nu_p} \mathcal{F}^{\beta \nu_1 \cdot \nu_p} \right\}, \tag{2.22}$$

where

$$\mathcal{F} := F - \star F. \tag{2.23}$$

$\lambda^{\alpha\beta}(x)$ is a rank-2 symmetric tensor Lagrange multiplier field. Although this is a simple-looking action, the λ and A equations of motion enforce the self-duality constraint together with the free Maxwell equations in a non-trivial way. These actions were generalized in [17, 18, 23].

2.1.5 Miao et. al.'s actions

Miao et. al. [23] unified the actions (2.21) and (2.22) in a single series of Lorentz-invariant actions in $D = 2(p+1)$ dimensions for abelian p-form gauge fields with manifest EM-duality (for p odd) or self-duality (for p even). Their action has the Siegel-type form but uses one or two q-form fields, Y^a, as Lagrange multipliers,

$$S = -\frac{1}{2\Delta}\int d^D x \left\{ \frac{F^a_{\mu_1 \cdots \mu_{p+1}} F^{a\,\mu_1 \cdots \mu_{p+1}}}{(p+1)!} - \frac{T^{b\mu\sigma_1 \cdots \sigma_q} T^b_{\nu\sigma_1 \cdots \sigma_q}}{T^2} \mathcal{F}^a_{\mu\mu_1 \cdots \mu_p} \mathcal{F}^{a\,\nu\mu_1 \cdots \mu_p} \right\}, \tag{2.24}$$

where

$$F^a := \partial \wedge A^a, \qquad \mathcal{F}^a := \Gamma^{ab} F^b - \star F^a,$$
$$T^a := \partial \wedge Y^a, \qquad T^2 := T^a_{\mu_1 \cdots \mu_{q+1}} T^{a\,\mu_1 \cdots \mu_{q+1}}. \tag{2.25}$$

Thus F^a is the $(p+1)$-form field strength for the p-form gauge potentials A^a, and T^a is a $(q+1)$-form "field strength" for the q-form "gauge potential" lagrange multipliers Y^a. The explicit definition of the Hodge star is

$$\star F^a_{\mu_1 \cdots \mu_{p+1}} := \frac{1}{(p+1)!} \epsilon_{\mu_1 \cdots \mu_{p+1} \nu_1 \cdots \nu_{p+1}} F^{a\,\nu_1 \cdots \nu_{p+1}}, \tag{2.26}$$

which satisfies $\star^2 = (-)^p$. Furthermore the constants and a, b indices are defined by

$$\Delta = \begin{cases} 1 & \text{for even } p \\ 2 & \text{for odd } p \end{cases}, \qquad \Gamma^{ab} = \begin{cases} \delta^{ab} & (a,b \in \{1\}) & \text{for even } p \\ \epsilon^{ab} & (a,b \in \{1,2\}) & \text{for odd } p \end{cases}. \tag{2.27}$$

Note that $q \in \{0, 1, \cdots, p\}$ can be chosen arbitrarily. In particular, one can choose $q = 0$, in which case the Lagrange multipliers Y^a are scalar fields. For p even (i.e., space-time dimensions 2 mod 4) we retrieve Siegel's action (2.22) with his symmetric tensor Lagrange multiplier replaced essentially by the square of $T := \partial \wedge Y$. For $p = 1$ (i.e., space-time dimension 4) we retrieve the PST EM duality-invariant action (2.21) but with two scalar auxiliary fields, Y^a. It is easy to see that one of the Y^a can be eliminated by its equations of motion, after which the remaining one can be identified with φ.

2.2 Supersymmetric EM duality-invariant actions

Relatively recently, Bunster and Henneaux in [46] provided an $N = 1$ supersymmetric extension of the two-potential formulation of Maxwell's theory of [10, 13] described above. Their action for a photon and photino is manifestly invariant under EM duality transformations at the price of manifest Lorentz invariance.

In order to properly implement EM duality together with supersymmetry it was necessary for Bunster and Henneaux to define EM duality as acting chirally on the spinors. Their starting point was the two-potential action (2.12) given in [6]. Their supersymmetric action for the photon and its superpartner the photino is

$$S_{SUSY} = S_{\text{bose}} + S_{\text{fermi}}, \tag{2.28}$$

where S_{bose} is the action (2.12) for the photon, and

$$S_{\text{fermi}} = -\frac{i}{2} \int d^4x \, \overline{\psi} \gamma^\mu \partial_\mu \psi. \tag{2.29}$$

Here ψ is an anticommuting Majorana spinor. S_{fermi} is invariant under the chirality transformation, $\psi \to e^{\beta\gamma_5}\psi$, which is an $SO(2)$-rotation because $(\gamma_5)^2 = -I$. They then showed that supersymmetry and EM duality transformations only commute if the latter are defined as the simultaneous transformation of *both* the vectors $\vec{A}^a$ and the spinor ψ given by

$$\vec{A} \; \to \; \exp\{\alpha\epsilon\} \, \vec{A}, \qquad \psi \; \to \; \exp\{\alpha\gamma_5\} \, \psi. \tag{2.30}$$

2.3 Towards N=2 supersymmetric EM duality

Now that we have briefly summarized the key efforts to date for constructing EM duality-symmetric actions, we will move on in chapter 3 to precisely define the problem of constructing EM duality-symmetric $N = 2$ supersymmetric nlsm actions, which is the main topic of this thesis. However, it will be useful to first make a preliminary evaluation of the utility of the above approaches to manifest EM duality invariance for $N = 2$ nlsms.

As outlined in the last chapter, the new feature of EM duality in $N = 2$ nlsms is that the effective gauge couplings of the IREA vary continuously on the Coulomb branch, and upon traversing closed loops in the Coulomb branch may return to their original values only up to a discrete EM duality transformation. Thus $N = 2$ nlsms must necessarily carry an explicit dependence on the gauge couplings: the couplings cannot be (continuously) reabsorbed in a rescaling of the Maxwell fields everywhere on the Coulomb branch.

This should be contrasted with all the EM duality-invariant actions described above. There we saw in the discussion of Zwanziger's action that true EM duality-symmetry is not attained if the coupling is made explicit, since the coupling changes under duality transformations. It was only when the coupling was absorbed into the gauge fields by a rescaling that EM duality transformations acted as true symmetries.

Thus it is clear that the EM duality-invariant formulations reviewed above are not capable of giving a duality-symmetric formulation of $N = 2$ nlsms. Clearly a new idea is needed.

Chapter 3

Problem Definition

The main purpose of this chapter is to provide a clear and precise definition of the problem we are trying to address in this thesis without clouding its simplicity with intricate mathematical details. The mathematical details of our work will be presented in chapters 4, 5, and 6. We will begin this chapter with a very quick review of the $N = 2$ supersymmetry algebra before we present the general form of an $N = 2$ IREA and define the vacuum structure of the theory. In my presentation, I closely follow the discussion and sequence in [32]. Other excellent references on the dynamics of $N = 2$ supersymmetric gauge theories are [26, 28, 27, 29, 33, 34, 35].

3.1 N=2 supersymmetric nonlinear sigma-models

The basic $N = 2$ *supertranslation algebra* (i.e., the $N = 2$ supersymmetry algebra after dropping the Lorentz generators and any central charges) is, in a notation suppressing space-time indices,

$$\{Q^i, \overline{Q}^j\} = \delta^{ij} P, \qquad \{Q^i, Q^j\} = 0, \qquad i, j \in \{1, 2\}, \tag{3.1}$$

where Q^i are two Weyl spinor supercharges, and P is the energy-momentum 4-vector. Q^i transforms as a doublet under the $SU(2)_R$ group of automorphisms.

On shell irreducible field representations of (3.1) are easy to construct. There are two solutions with no spins greater than one: the *hypermultiplet*, containing two propagating complex scalars, ϕ^i, as well as two Weyl fermions ψ^i; and a *vector multiplet*, made from one complex scalar a, two Weyl spinors λ^i, and a vector field A_μ. An important distinguishing factor of the hypermultiplet is that its scalars form a complex $SU(2)_R$ doublet. The bosonic degrees of freedom of the vector multiplet are a single complex scalar and a vector field, both transforming in the adjoint representation of the gauge group, and both are singlets under $SU(2)_R$. In particular, in the case of a $U(1)^r$ gauge group, which we are interested in for describing IREAs, the vector multiplet scalars are necessarily neutral. More generally, all the massless scalars, whether they are in vector multiplets or hypermultiplets, whose vevs parameterize the moduli space of the theory must be neutral because, when a charged scalar gets a vev, it Higgses the $U(1)$ it is charged under and thereby gets a mass.

The leading (2-derivative) bosonic terms of an IREA action on a moduli space $\mathcal{M}$ with an abelian gauge group and neutral hypermultiplets, a priori has an action of the following form

$$\mathcal{L} = g_{mn}(\varphi)\,\partial\varphi^m \cdot \partial\varphi^n + \mathrm{Im}[\tau_{IJ}(\varphi)\,\mathcal{F}^I \cdot \mathcal{F}^J], \tag{3.2}$$

where the dots denote contraction of space-time indices. The φ^m are the real scalar fields in both the vector and hypermultiplets, and the kinetic coefficient function g_{mn} is real, symmetric, and positive definite, and can be interpreted as a metric on $\mathcal{M}$. (No potential term is allowed since, by hypothesis, the scalar vevs parameterize the vacuum manifold, $\mathcal{M}$.) The second term in (3.2) is a generalized Maxwell term for the $U(1)$ field strengths $F^I_{\mu\nu} := \partial_\mu A^I_\nu - \partial_\nu A^I_\mu$ where $I, J \in \{1, \ldots, r\}$ run over the r $U(1)$ gauge groups, and we have defined the complex self-dual field strength,

$$\mathcal{F}^I := F^I - i \star F^I, \tag{3.3}$$

which satisfies $\star\mathcal{F}^I = i\mathcal{F}^I$. The gauge kinetic coefficient, τ_{IJ} — the central object of our study — is a complex function of the scalars φ^m, symmetric in I and J and whose imaginary part is positive definite by unitarity. For if we define the real and imaginary parts of the coupling by

$$\tau_{IJ} := \frac{\theta_{IJ}}{2\pi} + i\frac{4\pi}{(e^2)^{IJ}}, \tag{3.4}$$

then the generalized Maxwell term can be expanded as

$$\mathcal{L}_{U(1)^r} = \frac{4\pi}{(e^2)^{IJ}} F^I \cdot F^J + \frac{\theta_{IJ}}{2\pi} F^I \cdot \star F^J \tag{3.5}$$

which shows that the imaginary part of τ_{IJ} is a matrix of couplings and the real part are theta angles.

However, compatibility with $N = 2$ supersymmetry tightly constrains this action; see, for example, [32]. The result is that the general $N = 2$ IREA with gauge group $U(1)^r$ and n_f neutral hypermultiplets (labeled by indices $m, n \in \{1, \ldots, n_f\}$) has the following form

$$\mathcal{L} = g_{mn}(\phi, \overline{\phi}) \partial\phi^{im} \cdot \partial\overline{\phi}^n_i + \mathrm{Im}\ \tau_{IJ}(a) \left(\partial a^I \cdot \partial\overline{a}^J + \mathcal{F}^I \cdot \mathcal{F}^J\right). \tag{3.6}$$

Here ϕ^{im} are the complex scalars of the hypermultiplets, a^I the complex scalars of the vector multiplets, and $\mathcal{F}^I$ are the complex self-dual $U(1)$ field strengths of the vector multiplets. The $U(1)^r$ couplings, τ_{IJ}, are locally holomorphic functions of the

a^I which satisfy the *special Kähler condition*

$$\partial_{[I}\tau_{J]K} = 0, \tag{3.7}$$

where $\partial_I := \partial/\partial a^I$ and square brackets denote antisymmetrization. Globally τ_{IJ} can be a multi-valued function of the a^I because of EM duality identifications; as we will describe below they are more properly holomorphic sections of an $Sp(2r, \mathbb{Z})$ bundle over the moduli space. The special Kähler condition can be locally integrated to give

$$\tau_{JK} = \frac{\partial b_J}{\partial a^K} \tag{3.8}$$

for some holomorphic functions $b_I(a)$. A set $\{a^I, b_I\}$ of holomorphic functions on the moduli space satisfying (3.8) are called a basis of *special coordinates*.

$N = 2$ SUSY implies that there are no kinetic cross terms between the vector and hypermultiplet scalars, implying in turn that the moduli space has a natural (local) product structure $\mathcal{M} = \mathcal{M}_{\mathcal{H}} \times \mathcal{M}_{\mathcal{V}}$, where $\mathcal{M}_H$ is the subspace of $\mathcal{M}$ along which only the hypermultiplet vevs vary while the vector multiplet vevs remain fixed, and vice versa for $\mathcal{M}_V$. In cases where $\mathcal{M}_V$ is a point, $\mathcal{M} = \mathcal{M}_H$ is called a *Higgs branch* of the moduli space; when $\mathcal{M}_H$ is trivial $\mathcal{M}_V$ is called the Coulomb branch (since there are always the massless $U(1)$ vector bosons from the vector multiplets). Cases where both $\mathcal{M}_H$ and $\mathcal{M}_V$ are non-trivial are called mixed branches. We are interested in the Coulomb branch, $\mathcal{M}_V$, since that is the part of the moduli space involving $U(1)$ gauge fields, and so is the only part of the IREA where EM duality plays a role. So from now on we will be concerned only with the vector multiplet part of the general IREA:

$$\mathcal{L}_{CB} = \text{Im}\ \tau_{IJ}(a) \left(\partial a^I \cdot \partial \overline{a}^J + \mathcal{F}^I \cdot \mathcal{F}^J\right). \tag{3.9}$$

3.2 EM duality transformations

The most important point of this chapter is the following. The IR free low energy $U(1)^r$ *physics* is invariant under EM duality transformations, but the *lagrangian* (3.9) describing this physics is not. As explained in the last chapter for a single $U(1)$ gauge group, EM duality transformations relabel the fields, interchanges electric and magnetic charges, and inverts the couplings. Also, 2π shifts of the theta angles shift electric charges by multiples of the magnetic charges, and makes integer shifts of the real part of τ. When applied to the matrix τ_{IJ} of couplings these transformations generate the infinite discrete $Sp(2r,\mathbb{Z})$ group of duality transformations:

$$\tau_{IJ} \to \left(A_I^L \tau_{LM} + B_{IM}\right)\left(C^{JN}\tau_{NM} + D_M^J\right)^{-1} \tag{3.10}$$

where

$$M := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2r,\mathbb{Z}). \tag{3.11}$$

The conditions on the $r\times r$ integer matrices, $A,\ B,\ C$, and D for M to be in $Sp(2r,\mathbb{Z})$ are

$$\begin{aligned} AB^T = B^T A, &\qquad B^T D = D^T B, \\ A^T C = C^T A, &\qquad D^T C = C D^T, \\ A^T D - C^T B &\ = \ AD^T - BC^T = 1, \end{aligned} \tag{3.12}$$

where T denotes the transpose. These imply that

$$M^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}. \tag{3.13}$$

The action of an EM duality transformation on the $2r$-component row vector of magnetic and electric charges $(n_m^I \; m_{e,J})$ of massive states is

$$(n_m \; m_e) \to (n_m \; n_e) M^{-1}. \tag{3.14}$$

Also, from (3.10) and (3.8) it follows that the column vector of special coordinates $(b_I, a^I)^T$ transforms as

$$\begin{pmatrix} b \\ a \end{pmatrix} \to M \begin{pmatrix} b \\ a \end{pmatrix}. \tag{3.15}$$

The fact that the coupling matrix τ_{IJ} transforms under $Sp(2r, \mathbb{Z})$ means that EM duality transformations are *not* symmetry transformations of the IREA since they change the couplings of the theory. Instead, they simply express the equivalence of free $U(1)^r$ field theories coupled to classical massive sources under symplectic $Sp(2r, \mathbb{Z})$ redefinitions of electric magnetic and magnetic charges. The importance of this redundancy in the Lagrangian description of IREAs becomes apparent upon traversing a closed loop in $\mathcal{M}_V$. The physics must, by definition, be the same at the beginning and end of the loop, but the lagrangian description need not because it may have suffered an EM duality transformation. This possibility is often expressed by saying that the coupling matrix τ_{IJ}, in addition to being symmetric and having positive definite imaginary part, is also a section of a (flat) $Sp(2r, \mathbb{Z})$ bundle over $\mathcal{M}_V$ with action given by (3.10).

The key goal of this thesis is to develop a strategy for constructing EM duality-symmetric $N = 2$ IREAs. In other words, we want to construct $N = 2$ nlsm actions where EM duality transformations are a *manifest symmetry* as opposed to a map between different lagrangians. The key ingredient in achieving this goal is to reformulate the nlsm in terms of mathematical objects which are manifestly EM duality-invariant. The chief such object is the *total space*, $\mathcal{X}$, of the Coulomb branch (and *not* the Coulomb branch itself), and its inherent symplectic-invariant geometric data (the complex structure, Hodge form, and Donagi-Witten two-form). It is thus natural to use $\mathcal{X}$ as the target space of the nlsm and its geometric data to construct its Lagrangian.

Heuristically, we want to to build the EM duality-symmetric $N = 2$ nlsm action by using the *harmonic superspace* superfields to pull back those invariant geometric structures on the target space which can naturally be integrated over harmonic superspace. We call this strategy the *pullback approach* to constructing EM duality-symmetric actions.

To make this strategy more concrete, we need to describe in more detail what the geometry of the target space is, $\mathcal{X}$, and what the geometry of harmonic superspace is. The next two chapters will be devoted to fleshing out the details of these geometries. In chapter 4, we will give an in-depth presentation of the target space geometry where we will describe it in terms of *special Kähler* and *hyperkähler* manifolds as well the *twistor space* construction of the latter. In chapter 5 we will briefly introduce harmonic superspace.

Chapter 4

Geometry of the Coulomb Branch

In this chapter we describe several nearly equivalent formulations of the special Kähler (SK) structure of the Coulomb branch (CB), $\mathcal{M}_V$. In particular, we show how the SK structure is encoded in the Donagi-Witten (DW) geometry of the total space, $\mathcal{X}$, of the CB. Then we show how DW geometry gives rise to a hyperkähler (HK) structure on $\mathcal{X}$. Then we describe how an HK structure on $\mathcal{X}$ is encoded in the complex geometry of the twistor space, $\mathcal{Z}$, associated to $\mathcal{X}$.

We will illustrate all these constructions by computing the relevant geometrical objects (complex structures, metrics, symplectic forms, etc.) explicitly in coordinates. Along the way we will see that the DW geometry has extra structure compared to that of the SK, HK, or twistor geometries.

4.1 Special Kähler structures

In the last chapter we found the following basic structures in the IREA on the CB, $\mathcal{M}_V$:[1]

1.) $\mathcal{M}_V$ is an r-dimensional complex manifold.

[1]We change notation slightly from that used in chapter 3: we now use lower case roman indices $i, j \in \{1, \ldots, r\}$ to label the complex coordinates and tangent space directions.

2.) There exists a symmetric rank-2 tensor, τ_{ij}, which is a holomorphic section of an $Sp(2r,\mathbb{Z})$ bundle over $\mathcal{M}_V$ transforming as in (3.10).

3.) $\mathrm{Im}\,\tau$ is positive definite.

4.) τ satisfies the special Kähler condition (3.7).

From these it followed that:

- The special Kähler condition can be locally integrated as (3.8) to give special coordinates $\mathbf{c} := (a^i, b_i)$ which form a holomorphic section of an $Sp(2r,\mathbb{Z})$ bundle transforming as

$$\mathbf{c} \to \mathbf{c}\, M^T \tag{4.1}$$

  for $M \in Sp(2r,\mathbb{Z})$.

- In a special coordinate basis the metric components on $\mathcal{M}_V$ are given by $g_{i\bar{\jmath}} = \mathrm{Im}\,\tau_{ij}$.

Though this determines a geometry, it leaves out two additional closely related structures on $\mathcal{M}_V$ which are part of the IR physics on the CB and so also of the definition of an SK structure:

5.) the Schwinger product of dyon charges, and

6.) the central charge of the $N = 2$ SUSY algebra.

The rest of this section will define these last two SK structures.

If the $2r$-component row vector of magnetic and electric charges of a dyon is $\mathbf{z} = (p^i, q_i)$, then the Schwinger product of the charges of two dyons is

$$\langle \mathbf{z}_1, \mathbf{z}_2 \rangle := \mathbf{z}_1 J \mathbf{z}_2^T, \tag{4.2}$$

where J is the symplectic form given by $J = \mathbb{I}_r \otimes \epsilon$ where $\mathbb{I}_r$ is the $r \times r$ identity matrix and $\epsilon = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Since the charges transform under $Sp(2r, \mathbb{Z})$ duality transformations as in (3.14), that is as

$$\mathbf{z} \to \mathbf{z}\, M^{-1} \tag{4.3}$$

for $M^{-1} \in Sp(2r, \mathbb{Z})$, and since by definition elements of $Sp(2r, \mathbb{Z})$ leave J invariant, $MJM^T = J$, it follows that the Schwinger product is also $Sp(2r, \mathbb{Z})$ invariant. Physically the Schwinger product measures the angular momentum (in units of $\hbar$) carried by the electromagnetic fields in the presence of two static dyons.

The central charge, $Z_{\mathbf{z}}$, is a complex linear combination of the dyon charge vector $\mathbf{z}$ and the vector of special coordinates, $\mathbf{c} := (b_i, a^i)$. Thus

$$Z_{\mathbf{z}} = \mathbf{z}\, \mathbf{c}^T. \tag{4.4}$$

From (4.1) and (4.3) we see that $Z_{\mathbf{z}}$ is invariant under $Sp(2r, \mathbb{Z})$ transformations. Also, since $\mathbf{c}$ is a holomorphic section on $\mathcal{M}_V$, it follows that $Z_{\mathbf{z}}$ is a holomorphic function on $\mathcal{M}_V$. Physically, the norm of $Z_{\mathbf{z}}$ is the BPS mass of dyons of charge $\mathbf{z}$.[2]

SK structure is described here in an explicitly $Sp(2r, \mathbb{Z})$ non-invariant way. This is unavoidable since the basic structures, $\{\tau, \mathbf{c}, \mathbf{z}\}$, all transform under $Sp(2r, \mathbb{Z})$. So this is not an EM duality-invariant description. We should note that Freed [39] has proposed an $Sp(2r, \mathbb{Z})$-invariant definition of SK geometry which is intrinsic to the CB manifold, $\mathcal{M}_V$. But this proposal leaves out the dyon charge vectors $\mathbf{z}$, the Schwinger inner product, and the central charge structures. An $Sp(2r, \mathbb{Z})$-invariant description of all the SK structures requires enlarging the geometry beyond that of the CB, $\mathcal{M}_V$. This is the subject of the next section.

[2]There can be other contributions to the central charge coming from other global $U(1)$ charges that states might carry. We suppress them here for simplicity.

4.2 Donagi-Witten geometry

Donagi and Witten [38], following on earlier work of [47, 2], pointed out that a natural EM duality-invariant geometric object encoding the SK structure of the Coulomb branch, $\mathcal{M}_V$, is the total space, $\mathcal{X}$, of a fiber bundle of abelian varieties over $\mathcal{M}_V$.

The total space, $\mathcal{X}$, of an SK manifold, $\mathcal{M}_V$, is defined [1, 2, 38, 47] to be a $2r$-$\text{dim}_{\mathbb{C}}$ complex manifold with three additional geometric structures (π, t_u, L). They are:

- $\pi : \mathcal{X} \to \mathcal{M}_V$ is a holomorphic surjection whose fibers, $\mathcal{X}_u := \pi^{-1}(u)$, are r-$\text{dim}_{\mathbb{C}}$ abelian varieties;
- t_u is a Hodge form on $\mathcal{X}_u$ which varies continuously with $u \in \mathcal{M}_V$;
- L is a holomorphic symplectic form having the $\mathcal{X}_u$ as lagrangian submanifolds.

We will call such a space a *Donagi-Witten (DW) geometry*. The rest of this section explains the definitions of these structures by writing them out explicitly in coordinates. We then show how they encode an SK structure on $\mathcal{M}_V$.

4.2.1 Fiber bundle structure

The projection map π makes $\mathcal{X}$ a fiber bundle over $\mathcal{M}_V$ with the fiber $\mathcal{X}_u$ over a point $u \in \mathcal{M}_V$. Since $\mathcal{X}_u$ is r-$\text{dim}_{\mathbb{C}}$, so must $\mathcal{M}_V$ be. From now on we will only work locally in $\mathcal{M}_V$, so we may think of $\mathcal{M}_V$ concretely as a polydisk in $\mathbb{C}^r$ with complex coordinates $\widetilde{u}^i$, $i \in \{1, \dots, r\}$.

The fibers $\mathcal{X}_u$ are r-$\text{dim}_{\mathbb{C}}$ abelian varieties. An abelian variety is a complex torus with additional properties described below. Any complex torus T^{2r} is equivalent to $\mathbb{C}^r/\Lambda$ with Λ a rank-$2r$ lattice in $\mathbb{R}^{2r}$. We choose a basis of the lattice, $\{e_\alpha\}$, and a basis of dual coordinates, $\{x^\alpha\}$, $\alpha \in \{1, \dots, 2r\}$, such that $\int_{e_\beta} dx^\alpha = \delta^\alpha_\beta$. This implies that the x^α are identified up to integer shifts, $x^\alpha \sim x^\alpha + 1$.

These coordinates trivialize the fiber bundle. That is, the projection map is simply $\pi : (x^\alpha, \widetilde{u}^i) \mapsto \widetilde{u}^i$.

Choose a complex basis $\{f_i\}$ of $\mathbb{C}^r$, and a dual basis of complex coordinates z^i, such that $\int_{f_i} dz^j = \delta_i^j$. By an appropriate $GL(r, \mathbb{C})$ transformation we can always rotate $\{f_i\}$ so that $z^i = \Delta^i_j x^j + \tau^{ij} x^{r+j} + \mu^i$ for any given invertible real $r \times r$ matrix Δ. Here t^{ij} is some $r \times r$ complex matrix and μ^i a vector of complex constants. So rename $\widehat{x}_i := x^{r+i}$ for $i \in \{1, \ldots, r\}$, so that

$$z^i := \Delta^i_j x^j + \tau^{ij} \widehat{x}_j + \mu^i. \tag{4.5}$$

The identification of the x's by integer shifts implies that the z's are similarly identified under constant shifts $z^i \sim z^i + \Delta^i_j n^j + \tau^{ij} m_j$ with $n^i, m_i \in \mathbb{Z}$.

An *abelian variety* can be defined[3] as a complex torus for which there exists a basis of complex coordinates as in (4.5) such that: 1) $\Delta^i_j = \delta_i \delta^i_j$ is a diagonal matrix whose entries are positive integers $\delta_i \in \mathbb{Z}^+$ satisfying the divisibility condtions $\delta_i \mid \delta_{i+1}$; and, 2) τ^{ij} is a symmetric matrix, $\tau = \tau^T$, with positive definite imaginary part, $\operatorname{Im} \tau > 0$.

Since $\mathcal{X}$ is a holomorphic fiber bundle (i.e., π is a holomorphic map), the z^i must depend holomorphically on the $\widetilde{u}$ coordinates on $\mathcal{M}_V$, i.e., $\tau^{ij} = \tau^{ij}(\widetilde{u})$, and $\mu^i = \mu^i(\widetilde{u})$ in (4.5).

Mathematically, $\tau^{ij}(\widetilde{u})$ is called the complex modulus of the abelian variety $\mathcal{X}_u$. We will identify τ^{ij} with the matrix of complex $U(1)^r$ couplings that appeared in the description of the SK structure on $\mathcal{M}_V$. We saw in the last paragraph that the symmetry and positive-definiteness of τ are built into the definition of an abelian variety. The condition that τ^{ij} is a section of an $Sp(2r, \mathbb{Z})$ bundle is also built into it, as we will see below when we discuss polarizations on abelian varieties.

[3] Abelian varieties also have a coordinate-independent definition as complex tori which can be embedded in projective space by polynomial equations.

We now deduce some basic properties of the complex coordinates we have introduced on $\mathcal{X}$. The important point is that while the real coordinates, $\{x, \widehat{x}\}$, trivialized the fiber bundle structure of $\mathcal{X}$, the complex coordinates $\{z, \overline{z}\}$ defined by (4.5) do not because they depend on the coordinates $\{\widetilde{u}, \overline{\widetilde{u}}\}$ of the base. To make the change of variables from the real to the complex coordinates on the fibers, introduce the two sets of coordinates $\{z, \overline{z}, u, \overline{u}\}$ and $\{x, \widehat{x}, \widetilde{u}, \overline{\widetilde{u}}\}$. They are related by

$$
\begin{aligned}
z &= \Delta x + \tau \widehat{x} + \mu, & u &= \widetilde{u}, \\
\overline{z} &= \Delta x + \overline{\tau} \widehat{x} + \overline{\mu}, & \overline{u} &= \overline{\widetilde{u}},
\end{aligned} \tag{4.6}
$$

with inverses

$$
\begin{aligned}
x &= \frac{1}{\Delta} \left(\tau \frac{1}{\tau_-} (\overline{z} - \overline{\mu}) - \overline{\tau} \frac{1}{\tau_-} (z - \mu) \right), & \widetilde{u} &= u, \\
\widehat{x} &= \frac{1}{\tau_-} (z - \overline{z} - \mu + \overline{\mu}), & \overline{\widetilde{u}} &= \overline{u}.
\end{aligned} \tag{4.7}
$$

Here we are using a vector notation where the coordinates and μ are all r-component column vectors, and τ and Δ are $r \times r$ matrices. Also, we have defined $\tau_\pm := \tau \pm \overline{\tau}$; thus, $\tau_+ = 2 \operatorname{Re} \tau$, and $\tau_- = 2i \operatorname{Im} \tau$.

Since x and $\widehat{x}$ are identified by constant (integer) shifts, it follows that $\{\partial_x, \partial_{\widehat{x}}, \partial_{\widetilde{u}}, \partial_{\overline{\widetilde{u}}}\}$ form a basis of globally defined vector fields on $\mathcal{X}$, and that $\{dx, d\widehat{x}, d\widetilde{u}, d\overline{\widetilde{u}}\}$ form a dual basis of globally defined one-forms on $\mathcal{X}$. But this is *not* true of the coordinate vector fields and one-forms in the $\{z, \overline{z}, u, \overline{u}\}$ basis. For, by the chain rule, we compute that

$$
\begin{aligned}
\partial_z &= -\frac{1}{\tau_-} \overline{\tau} \frac{1}{\Delta} \partial_x + \frac{1}{\tau_-} \partial_{\widehat{x}}, & \partial_u &= \partial_{\widetilde{u}} - (\widehat{x}^T \partial_u \tau + \partial_u \mu) \partial_z, \\
\partial_{\overline{z}} &= +\frac{1}{\tau_-} \tau \frac{1}{\Delta} \partial_x - \frac{1}{\tau_-} \partial_{\widehat{x}}, & \partial_{\overline{u}} &= \partial_{\overline{\widetilde{u}}} - (\widehat{x}^T \partial_{\overline{u}} \overline{\tau} + \partial_{\overline{u}} \overline{\mu}) \partial_{\overline{z}}.
\end{aligned} \tag{4.8}
$$

Thus ∂_z and $\partial_{\overline{z}}$ are globally defined vector fields, but ∂_u and $\partial_{\overline{u}}$ are not because of their explicit dependence on x and $\widehat{x}$ which are not single-valued on the fibers. (For later use, note that

$$\partial_x = \Delta\partial_z + \Delta\partial_{\overline{z}}, \qquad \partial_{\widehat{x}} = \tau\partial_z + \overline{\tau}\partial_{\overline{z}}, \tag{4.9}$$

invert the relations (4.8).) Similarly, we compute from (4.8) the one-forms

$$dz = \Delta dx + \tau d\widehat{x} + d_u\tau \cdot \widehat{x} + d_u\mu, \qquad du = d\widetilde{u}, \tag{4.10}$$

and their complex conjugates, in a notation where $d_uX := \partial_{u^i}X\, du^i$. This implies that du and $d\overline{u}$ are global one-forms, but dz and $d\overline{z}$ are not because of their explicit $\widehat{x}$-dependence.

So, we define the following, globally defined, vector fields and one-forms in the $\{z, u\}$ coordinate system:

$$\begin{aligned} U &:= \partial_{\widetilde{u}}, & \theta &:= \Delta dx + \tau d\widehat{x}, \\ \overline{U} &:= \partial_{\overline{\widetilde{u}}}, & \overline{\theta} &:= \Delta dx + \overline{\tau} d\widehat{x} \end{aligned} \tag{4.11}$$

(with

$$dx = \frac{1}{\Delta}\left(\tau\frac{1}{\tau_-}\overline{\theta} - \overline{\tau}\frac{1}{\tau_-}\theta\right), \qquad d\widehat{x} = \frac{1}{\tau_-}(\theta - \overline{\theta}) \tag{4.12}$$

as useful inverse relations). Then $\{\partial_z, U\}$ is a basis of global holomorphic vector fields on $\mathcal{X}$ and $\{\theta, du\}$ is a dual basis of global (1,0)-forms on $\mathcal{X}$.

It is important to realize that, since θ is not a coordinate differential, it is neither closed nor holomorphic! Indeed, from their definition, (4.11), it follows that $d\theta = d_u\tau\, d\widehat{x} = d\tau\frac{1}{\tau_-}(\theta - \overline{\theta})$. Decomposing the exterior derivative into the Dolbeault operators, $d = \partial + \overline{\partial}$, this implies

$$\partial\theta = \frac{1}{\tau_-}\theta, \qquad\qquad \overline{\partial}\theta = \frac{1}{\tau_-}\overline{\theta}, \tag{4.13}$$

and complex conjugated relations for $\overline{\theta}$.

4.2.2 Polarization

A *Hodge form*, t_u, on an abelian variety, $\mathcal{X}_u$, is defined as a positive-definite, integral, (1,1)-form on $\mathcal{X}_u$. In fact, the existence of such a form on a complex torus can be taken as the definition of an abelian variety. In the coordinate definition of abelian variety we gave above, such a form is manifest. In the real basis for the fibers, it is simply

$$t_u := \delta_i \, dx^i \wedge d\widehat{x}_i = dx^T \, \Delta \, d\widehat{x} \tag{4.14}$$

where matrix multiplication and wedge products are understood in the last expression.

A given abelian variety typically admits many different Hodge forms. A choice of a Hodge form on an abelian variety is called a *polarization*. We will take (4.14) as our choice of polarization. If the integers $\delta_i = 1$ for all i, then the polarization is called a *principal polarization*.

A basis $\{e_\alpha\}$ of the torus fiber lattice Λ such that t_u has above the above form is called a *canonical basis*. There is a freedom in choosing a canonical basis. A general Λ basis change is a linear map $G \in GL(2r, \mathbb{Z})$. To preserve the form of t_u we need $GZG^T = Z$ where $Z := \left(\begin{smallmatrix} 0 & \Delta \\ -\Delta & 0 \end{smallmatrix}\right)$. This defines a subgroup of $Sp(2r, \mathbb{Z})$; in the case of a principal polarization, $\Delta = \mathbb{I}_r$, it is just $Sp(2r, \mathbb{Z})$ itself.

Note that such a change of canonical basis preserves Δ, but changes τ by a fractional linear transformation as in (3.10). Thus, in the case of a principal polarization, we see that τ is indeed a holomorphic section of an $Sp(2r, \mathbb{Z})$ bundle.

(The case of a non-principal polarization corresponds to a restriction on the allowed transformations of τ.)

Once t_u is chosen as in (4.14) for one $u \in \mathcal{M}_V$, continuity in u fixes it for all other u. This is because t_u depends on discrete data (a choice of lattice basis and the integers δ_i) and so must be constant if it is continuous. Thus in our $\{x, \widehat{x}, \widetilde{u}, \overline{\widetilde{u}}\}$ coordinate system, t_u is independent of u. It thus defines a unique closed, real (1,1)-form, t, on all of $\mathcal{X}$ by the same formula as for t_u, (4.14). (In coordinate invariant language, t is the unique closed 2-form on $\mathcal{X}$ which coincides with t_u when restricted to $\mathcal{X}_u$, and which has rank $2r$ at every point, i.e., there is a $2r$-$\dim_{\mathbb{R}}$ subspace of the tangent space to $\mathcal{X}$ at each point on which t vanishes.)

In the complex coordinate basis $\{z, u\}$, a short computation using (4.12) shows that

$$t = -\theta^T \frac{1}{\tau_-} \overline{\theta}. \tag{4.15}$$

4.2.3 Holomorphic symplectic structure

The last structure of a DW geometry is the holomorphic symplectic form, L, which we will call a "DW form" for short.

A holomorphic symplectic form is a closed, non-degenerate, (2,0)-form on $\mathcal{X}$. The condition that the fibers, $\mathcal{X}_u$, are lagrangian means that, when acting on pairs of vectors in the tangent bundle to the fibers, L vanishes. Being a (2,0)-form with vanishing restriction to the fiber, implies L has the general form in the $\{u, z\}$ coordinate system

$$L = \lambda_i^j \theta^i du_j + \mu^{ij} du_i du_j, \tag{4.16}$$

with λ_i^j and μ^{ij} holomorphic functions of z and u. L being non-degenerate means that $0 \neq \det \left(\begin{smallmatrix} 0 & \lambda \\ -\lambda^T & \mu \end{smallmatrix} \right) = \det^2 \lambda$ which implies $\det \lambda \neq 0$. Being a closed (2,0)-form means that $dL = 0$. This leads to a system of differential equations which can be shown to imply that τ^{ij}, λ_i^j, and μ^{ij} are holomorphic functions of u alone, and satisfy

$$\begin{aligned} 0 &= \partial^{[i}\tau^{j]k}, \\ 0 &= \partial^{[i}\mu^{jk]}, \\ 0 &= \partial^{[j}\lambda_i^{k]}, \end{aligned} \tag{4.17}$$

where $\partial^i := \partial/\partial u_i$.

The first equation in (4.17) is precisely the SK condition on τ, (3.7), found in the last chapter. Thus we see that DW geometry encodes the first four SK conditions listed in section 4.1. We will see how the last two SK conditions are encoded in the next subsection.

The second equation in 4.17 implies that $\mu^{jk} = \partial^{[j}A^{k]}$ for some holomorphic functions $A^k(u)$ on $\mathcal{M}_V$. Note that A^k is only determined by this equation up to "gauge" equivalences of the form $A^k \sim A^k + \partial^k\Lambda$, for an arbitrary holomorphic function $\Lambda(u)$ on $\mathcal{M}_V$. Define a holomorphic 1-form on $\mathcal{M}_V$ by $A := A^i(u)du_i$, then the above gauge equivalence is

$$A \sim A + d\Lambda. \tag{4.18}$$

Thus A is a holomorphic connection on a $U(1)^r_{\mathbb{C}}$-bundle over $\mathcal{M}_V$.

The third equation in 4.17 together with $\det\lambda \neq 0$ implies that there exists a holomorphic change of variables to new coordinates $u'_k = u'_k(u)$ such that $du'_k = \lambda^j_k du_j$. In these coordinates (dropping the primes on u') the DW 2-form takes the simple form

$$L = \theta^T du + dA. \tag{4.19}$$

To summarize, we have derived coordinate expressions for the Hodge form t in (4.15) and for the DW form L in (4.19). They depend on a holomorphic matrix τ which satisfies all the conditions that the matrix of $U(1)^r$ couplings τ does in an SK geometry. Thus DW geometry encodes these SK conditions in an $Sp(2r,\mathbb{Z})$-invariant way.

One surprise is that there is an extra structure — namely, the dA term in (4.19) — in the DW geometry that does not appear in the SK structure. The existence of this structure is implicitly recognized in the work of Donagi [47] and Freed [39], but they simply set it to zero as an extra condition on DW geometry. While this can always be done locally on $\mathcal{M}_V$, it may not be consistent to do so globally. We speculate on the possible physical interpretation of this extra structure in chapter 7.

Another surprise, pointed out by Donagi and Witten [38], is that since L is a symplectic form and it has tori as Lagrangian submanifolds, it endows the total space $\mathcal{X}$ with the structure of a phase space of a (complex) classical integrable dynamical system. The physical interpretation of this classical integrable system is unclear. We will see at the end of chapter 6 that another complex classical dynamical system enters in the harmonic superspace description of $N = 2$ supersymmetric hypermultiplet nlsms. We speculate on the possible relation of these two classical dynamical systems in chapter 7.

4.2.4 Recovery of SK structures from DW geometry

So far we have recovered the SK conditions on the matrix τ of complex $U(1)^r$ couplings from DW geometry. But we have not seen how the Schwinger product of dyon charges, (4.2), and the central charge, (4.4), are encoded in DW geometry.

The first thing we need to locate in DW geometry are the dyon charge vectors, $\mathbf{z} \in \mathbb{Z}^{2r}$. The first homology group of the fiber is $H_1(\mathcal{X}_u) \simeq \mathbb{Z}^{2r}$ since $\mathcal{X}_u$ is a T^{2r} torus. We will thus identify $H_1(\mathcal{X}_u)$ with the lattice of EM charges. Therefore a dyon charge vector, $\mathbf{z}$, is identified with the homology class, $[\gamma_\mathbf{z}] \in H_1(\mathcal{X}_u)$, of some 1-cycle (closed curve) $\gamma_\mathbf{z}$ wrapping certain cycles of the $\mathcal{X}_u$ torus.

The Schwinger product is an alternating, integer-valued product of two charges, $\mathbf{y}$ and $\mathbf{z}$. It is naturally identified with the period of the Hodge form over the 2-cycle in $\mathcal{X}_u$ given by the Pontryagin product of the two 1-cycles, $\gamma_\mathbf{y}$ and $\gamma_\mathbf{z}$, representing the charges $\mathbf{y}$ and $\mathbf{z}$. That is

$$\langle \mathbf{y}, \mathbf{z} \rangle = \int_{\gamma_\mathbf{y} * \gamma_\mathbf{z}} t_u. \tag{4.20}$$

Here $*$ denotes the Pontryagin product, which can be thought of as forming the 2-cycle defined by the surface swept out by translating the first curve along the second in the product. The Pontryagin product is alternating since reversing the order reverses the orientation. Since the Hodge form is closed, the value of the integral only depends on the homology classes of the various cycles. Since the Hodge form is integral, the value of the integral will be an integer. Thus (4.20) has all the necessary properties to define the Schwinger product.

It can be easily evaluated in the $\{x, \widehat{x}\}$ coordinate system. Denote the basis of the torus' lattice Λ dual to these coordinates by $\{e^i, \widehat{e}_i\}$, which therefore specify a basis $[\mathbf{e}] := \{[e^i], [\widehat{e}_i]\}$ of $H_1(\mathcal{X}_u)$. Then any two homology classes, $[\gamma_\mathbf{y}]$ and $[\gamma_\mathbf{z}]$, can be expressed in this basis as $[\gamma_\mathbf{y}] = \mathbf{y}^T[\mathbf{e}]$ and $[\gamma_\mathbf{z}] = \mathbf{z}^T[\mathbf{e}]$. It then immediately follows from (4.14) that $\oint_{\gamma_\mathbf{y} * \gamma_\mathbf{z}} t_u = \mathbf{y}\, J\mathbf{z}^T$ where $J = \Delta \otimes \epsilon$. In case the polarization is principal, $\Delta = \mathbb{I}_r$, then this is precisely the Schwinger product (4.2). In case the polarization is not principal, this gives a generalization of the Schwinger product where there can be different units of charge quantization with respect to the r different $U(1)$ gauge groups. This is a physically sensible generalization, and gives a physical interpretation of the Δ matrix appearing in the Hodge form.

The central charge, $Z_{\mathbf{z}}(u)$, is a holomorphic function which depends linearly on the charges $\mathbf{z}$. Because the DW 2-form, L, has vanishing restriction to the fibers of $\mathcal{X}$, it is well-defined to integrate L over 1-cycles in $\mathcal{X}_u$ to get a 1-form on $\mathcal{M}_V$. Thus $\oint_{\gamma_{\mathbf{z}}} L$ is a 1-form which is holomorphic on $\mathcal{M}_V$ and linear in $\mathbf{z}$. It thus has the natural intepretation as the differential of the central charge:

$$\oint_{\gamma_{\mathbf{z}}} L = dZ_{\mathbf{z}}, \tag{4.21}$$

where the "d" on the right side is the exterior derivative on $\mathcal{M}_V$. Since L is closed, the right side is indeed exact. Since L is (2,0), the right side is (1,0), so $Z_{\mathbf{z}}$ is holomorphic. Thus the DW 2-form determines the central charge (up to some integration constants) by (4.21). This completes the determination of all the basic SK structures from $Sp(2r,\mathbb{Z})$-invariant geometrical objects on the total space $\mathcal{X}$.

For later use, it will be convenient to also derive explicit coordinate expressions for the Kähler form, ω, and hermitian metric, g, on $\mathcal{M}_V$, from the DW geometry data. The Kähler form is given by [DW95]

$$\omega(u) := \int_{\mathcal{X}_u} t_u^{r-1} L\overline{L}, \tag{4.22}$$

which, after some algebraic manipulation using the expressions (4.15) and (4.19) for t and L, can be written as

$$\omega = -(\det \Delta)\, du^T\, \tau_-\, d\overline{u}. \tag{4.23}$$

The metric on $\mathcal{M}_V$ can be derived from the Kähler form as

$$g(\partial_u, \partial_{\overline{u}}) := \omega(i\partial_u, \partial_{\overline{u}}) = -i(\det \Delta)\, \tau_- = 2(\det \Delta)\ \mathrm{Im}\, \tau. \tag{4.24}$$

Since $\mathrm{Im}\,\tau > 0$, it follows that the metric is also positive definite.

4.3 Hyperkähler structure

In this section we will follow a construction of Freed's [39] to find explicit coordinate expressions for a canonical HK structure on $\mathcal{X}$. But first we give a quick overview of the basic definition [40] of a HK structure.

A HK manifold $\mathcal{X}$ has three complex structures J_a, $a \in \{1,2,3\}$, obeying the quaternion algebra, $J_a J_b = \epsilon_{abc} J_c - \delta_{ab} 1$. Any real linear combination of these,

$$J(\vec{r}) := r^a J_a, \tag{4.25}$$

is again a complex structure on $\mathcal{X}$ if $\sum_a (r^a)^2 = 1$, *i.e.*, if $\vec{r}$ lies on a two-sphere $S^2 \simeq \mathbb{CP}^1$. Furthermore, each complex structure has an associated Kähler form — a real non-degenerate closed 2-form of type (1,1) with respect to its complex structure. We denote the Kähler form associated to J_a by ω_a. Then the HK metric is given by $g(\cdot,\cdot) := \omega_a(J_a\cdot,\cdot)$ independent of a. So to specify a HK structure we need give only the Kähler forms, ω_a, and one complex structure, J_1, since the metric, g, can then be deduced from ω_1 and J_1, and the remaining complex structures can be deduced from g and $\omega_{2,3}$.

According to a theorem by Cecotti, Ferrara, and Girardello [48], the cotangent bundle $T^*\mathcal{M}_V$ of a special Kähler (SK) manifold $\mathcal{M}_V$ carries a canonical HK structure given by identifying J_1 with the natural complex structure of $T^*\mathcal{M}_V$, and defining the Kähler forms by

$$\begin{aligned} \omega_1(q_1 \oplus p_1,\, q_2 \oplus p_2) &:= \omega(q_1,q_2) + \omega^{-1}(p_1,p_2), \\ [\omega_2 + i\omega_3](q_1 \oplus p_1,\, q_2 \oplus p_2) &:= \frac{1}{2}\left[p_1([1-iJ_1]q_2) - p_2([1-iJ_1]q_1)\right], \end{aligned} \tag{4.26}$$

for any $q_i \in T\mathcal{M}_V$ and $p_i \in T^*\mathcal{M}_V$. Here ω is the Kähler form on $\mathcal{M}_V$ found above (4.23), and ω^{-1} is its inverse (which is therefore a rank-2 antisymmetric tensor field, and so naturally acts on pairs of one-forms).

Freed [39] shows that the DW 2-form on $\mathcal{X}$ provides an isomorphism between the cotangent space $T^*_u\mathcal{M}_V$ and the tangent space to the fiber $T_p\mathcal{X}_u$ at any point $p \in \mathcal{X}_u$. Since $\mathcal{X}_u \simeq \mathbb{R}^{2r}/\Lambda$ is a torus, it is translation invariant, and so $T_p\mathcal{X}_u \simeq T_{p'}\mathcal{X}_u$ for any points $p, p' \in \mathcal{X}_u$. Denote $\mathcal{V}_u := T_p\mathcal{X}_u$, and form the "vertical" fiber bundle $\mathcal{V} \to \mathcal{M}_V$ with fibers $\mathcal{V}_u$. Using the total space fiber bundle projection, $\pi : \mathcal{X} \to \mathcal{M}_V$, and the inverse of the DW 2-form, L^{-1}, Freed defines the map $\pi^* L^{-1} : T^*\mathcal{M}_V \to \mathcal{V}$. The non-degeneracy of L implies this map is an isomorphism. Thus $T^*\mathcal{M}_V \simeq \mathcal{V}$, and so $\mathcal{V}$ has a HK structure.

Finally, Freed notes that since $\mathcal{X}_u \simeq \mathbb{R}^{2r}/\Lambda$ for some lattice Λ, it follows that $\mathcal{X}_u \simeq \mathcal{V}_u/\Lambda$, and thus $\mathcal{X} \simeq \mathcal{V}/\Lambda$. Since the HK structure on $\mathcal{V}$ is translation-invariant along the fibers, $\mathcal{X}$ inherits an HK structure from $\mathcal{V}$ simply by modding out by the action of Λ.

After going through the algebra of the three steps given above, we obtain the following coordinate expressions for the HK structures on $\mathcal{X}$:

$$\begin{aligned}
J_1 &= +i\left(\partial_z^T \otimes \theta + U^T \otimes du\right) + \text{c.c.} \\
J_2 &= -i\left(\frac{|\Delta|}{\kappa}\partial_z^T \tau_- \otimes d\overline{u} + \frac{\kappa}{|\Delta|}U^T \frac{1}{\tau_-} \otimes \overline{\theta}\right) + \text{c.c.} \\
J_3 &= +\left(\frac{|\Delta|}{\kappa}\partial_z^T \tau_- \otimes d\overline{u} + \frac{\kappa}{|\Delta|}U^T \frac{1}{\tau_-} \otimes \overline{\theta}\right) + \text{c.c.} \\
\omega_1 &= -|\Delta|\, du^T \tau_- \wedge d\overline{u} + \frac{\kappa^2}{|\Delta|}\theta^T \frac{1}{\tau_-} \wedge \overline{\theta} + \text{c.c.} \\
\omega_2 &= +\kappa\, \theta^T \wedge du + \text{c.c.} \\
\omega_3 &= -i\kappa\, \theta^T \wedge du + \text{c.c.} \\
g &= -i|\Delta|\, du^T \tau_- \otimes d\overline{u} + i\frac{\kappa^2}{|\Delta|}\theta^T \frac{1}{\tau_-} \otimes \overline{\theta} + \text{c.c.}
\end{aligned} \tag{4.27}$$

Here $|\Delta| := \det \Delta$ and κ is an arbitrary positive constant. This arbitrary constant implies that there is actually a one-parameter family of HK structures on $\mathcal{X}$. κ can be interpreted as the arbitrary scale of the HK metric on the torus fibers of $\mathcal{X}$.

Note that the holomorphic $U(1)^r_{\mathbb{C}}$ connection on $\mathcal{M}_V$, dA, that appeared as part of the DW form does not appear in the HK structure of $\mathcal{X}$. The reason is that the projection pullback, π^*, in Freed's cotangent to vertical bundle isomorphism, $\pi^* L^{-1}$, annihilates dA. Also, by a rescaling of κ, the dependence on $|\Delta|$ can be removed from the HK structures in 4.27. Thus the HK structure of $\mathcal{X}$ carries less information than the full DW geometry does.

4.4 Twistor space geometry

We have seen that a HK manifold, $\mathcal{X}$, admits a whole $S^2 \simeq \mathbb{CP}^1$ of complex structures compatible with the metric. The *main idea* of the twistor space approach is to incorporate all these structures into one complex structure on a larger manifold, the *twistor space*, $\mathcal{Z}$. The specification of a holomorphic 2-form, Ω^{++}, on $\mathcal{Z}$ allows one to recover the full HK structure on $\mathcal{X}$. Thus HK manifolds and twistor spaces are effectively equivalent [41, 40].

This is useful because, as we will see in chapter 5, construction of supersymmetric nlsms requires that one chooses a complex structure on the target space. Choosing one complex structure out of the whole 2-sphere of them on a HK target space destroys the symmetry among them. However, the equivalent twistor space has a unique complex structure which combines the S^2 of complex structures of the HK manifold in a symmetric way. Thus it is natural to formulate HK nlsms using the twistor space as target space.

After briefly explaining the key properties of the twistor space associated to a HK manifold, we will quote Hitchin et. al.'s theorem [41] on the equivalence of twistor spaces to HK geometries.

The twistor space $\mathcal{Z}$ of a HK space $\mathcal{X}$ is a fiber bundle over the two-sphere of complex structures of $\mathcal{X}$ with $\mathcal{X}$ as the fiber. Thus $\mathcal{Z} \simeq \mathcal{X} \times \mathbb{CP}^1$ as a topological space, but it is endowed with a special choice of complex structure, $J^{\mathcal{Z}}$, given by choosing the complex structure $J(\vec{r})$, defined in (4.25), on the fiber over each point $\vec{r} \in \mathbb{CP}^1$. Thus there is a holomorphic projection map $p : \mathcal{Z} \to \mathbb{CP}^1$ defining the fiber bundle structure. Choosing a complex coordinate, ζ, on the $\mathbb{CP}^1$ base, we have $p : (m, \zeta) \mapsto \zeta$ where $m \in \mathcal{X}$. In these coordinates, the complex structure $J^{\mathcal{Z}}$ acting on $T_{(m,\zeta)}\mathcal{Z} \simeq T_m\mathcal{X} \oplus T_\zeta\mathbb{CP}^1$ is

$$J^{\mathcal{Z}} = \left(\frac{1-\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}} J_1 + \frac{\zeta+\bar{\zeta}}{1+\zeta\bar{\zeta}} J_2 + i\frac{\zeta-\bar{\zeta}}{1+\zeta\bar{\zeta}} J_3 \right) \oplus J_0, \tag{4.28}$$

where J_0 is the unique complex structure on $\mathbb{CP}^1$ (i.e., $J_0 : \partial_\zeta \mapsto i\partial_\zeta$). Note that the $\bar{\zeta}$-dependence of $J^{\mathcal{Z}}$ means that although ζ is a complex coordinate on the $\mathbb{CP}^1$ base, it is *not* a complex coordinate on the total twistor space, $\mathcal{Z}$.

For each $\zeta \in \mathbb{CP}^1$ we define the 2-form

$$\Omega^{++} := (\omega_1 + i\omega_3) + \zeta\omega_2 - \zeta^2(\omega_1 - i\omega_3), \tag{4.29}$$

built out of the three Kähler forms on $\mathcal{X}$. One can check that Ω^{++} is a symplectic (2,0)-form on the fibers $\mathcal{Z}_\zeta := p^{-1}(\zeta)$ with respect to the $J^{\mathcal{Z}}$ complex structure. The quadratic dependence of Ω^{++} on ζ is signalled by the ++ superscript. In global terms, this dependence means that Ω^{++} is a holomorphic section of an $\mathcal{O}(2)$ complex line bundle over $\mathbb{CP}^1$ (which can be pulled back to an $\mathcal{O}(2)$ bundle over $\mathcal{Z}$ by the projection map). Another way of saying this is that Ω^{++} transforms as a charge +2 field under the $U(1)$ group of phase rotations of the homogeneous projective coordinates describing the $\mathbb{CP}^1$. This $U(1)$ group will play an important role in chapter 6.

A final structure on twistor space is an automorphism, $\sigma : \mathcal{Z} \to \mathcal{Z}$, satisfying $\sigma^2 = 1$ and acting without fixed points. It is given in coordinates simply by the antipodal map on the $\mathbb{CP}^1$ base and the identity map on the fibers,

$$\sigma : (m, \zeta) \mapsto (m, -1/\bar{\zeta}). \tag{4.30}$$

It is easy to check that $J^{\mathcal{Z}} \circ d\sigma = -d\sigma \circ J^{\mathcal{Z}}$, showing that σ is antiholomorphic. It is also true that Ω^{++} is invariant under σ in the sense that $\sigma^* \Omega^{++} = \Omega^{++}$.

This construction of the $(J^{\mathcal{Z}}, \Omega^{++}, \sigma)$ structures on $\mathcal{Z}$ is interesting because of a theorem of Hitchin et. al. [41, 40], which says that from any space $\mathcal{Z}$ with these structures the HK space $\mathcal{X}$ can be reconstructed. More precisely, let $\mathcal{Z}$ be a $(2r+1)$-$\dim_{\mathbb{C}}$ complex manifold such that

- $\mathcal{Z}$ is a holomorphic fiber bundle $p : \mathcal{Z} \to \mathbb{CP}^1$ over the projective line;
- there exists a holomorphic section Ω^{++} of $\Lambda^2 T_F^* \otimes \mathcal{O}(2)$ defining a symplectic form on each fiber, where T_F is the bundle of tangent spaces to the fibers of p;
- $\mathcal{Z}$ has a free antiholomorphic involutive automorphism σ preserving Ω^{++} and inducing the antipodal map on $\mathbb{CP}^1$.

Then the parameter space of holomorphic sections of p invariant under σ is a $4r$-$\dim_{\mathbb{R}}$ manifold, $\mathcal{X}$, with a natural HK structure for which $\mathcal{Z}$ is the twistor space.

Chapter 5

Harmonic Superspace

In this chapter we give a brief overview of the harmonic superpace formalism [36, 37] for making $N = 2$ SUSY manifest in quantum field theory actions. We first review the geometric setup of harmonic superspace, which extends space-time by both additional fermionic coordinates as well as by additional bosonic coordinates parameterizing an internal 2-sphere. We then summarize the properties of superfields which describe hyper- and vector multiplets, and give their action principles. Next we describe how the harmonic superspace action for hypermultiplets is related to the hyperkähler geometry of the nlsm target space. We end with a brief comment on the relation of harmonic superspace to "projective superspace", which gives a slightly different $N = 2$ covariant formalism.

5.1 Geometry of harmonic superspace (HSS)

Harmonic superspace [36, 37] is one way of dealing with supersymmetric theories with 8 real SUSY generators in a manifestly covariant manner.

In four space-time dimensions the 8 real SUSY generators form an $SU(2)_R$ doublet of 2-component (Weyl) spinors, Q^i_α, where $i \in \{1, 2\}$ is the doublet index and $\alpha \in \{1, 2\}$ is the spinor index. $SU(2)_R$ is (part of) the automorphism group of the $N = 2$

SUSY algebra, (3.1). The other generator of the $N = 2$ supertranslation algebra is P^μ, the generator of space-time translations.

Superspace is a space with one coordinate for each supertranslation generator. Thus $N = 2$ superspace has the usual Minkowski space ($\mathbb{R}^{3,1}$) coordinates, x^μ, corresponding to P^μ, and anticommuting (Grassmann) coordinates, θ^i_α, corresponding to Q^i_α (as well as their complex conjugates). This space is denoted $\mathbb{R}^{3,1|8}$, where the "8" superscript refers to the number of real Grassmann coordinates. Scalar superfields, $\Phi(x, \theta, \overline{\theta})$, are maps from superspace to other spaces,

$$\Phi : \mathbb{R}^{3,1|8} \to \mathcal{M}, \tag{5.1}$$

where $\mathcal{M}$ might be $\mathbb{R}$ or $\mathbb{C}$ for a real or complex superfield, or might be a more general manifold, $\mathcal{M}$, to describe a nlsm with target space $\mathcal{M}$.

Manifestly supersymmetric actions can now be written as scalar translation-invariant functionals of superfields, i.e., as integrals over superspace of superlagrangians which are local scalar functions of superfields. The problem with this approach is that superfields on $\mathbb{R}^{3,1|8}$ give highly reducible representations of the SUSY algebra, so the superlagrangians describe too many propagating degrees of freedom, typically of high spins. It is generally a difficult problem to formulate actions that preserve manifest supersymmetry and restrict to the desired set of degrees of freedom (e.g., some number of hypermultiplets and vector multiplets) at the same time.

Harmonic superspace (HSS) solves this problem for $N = 2$ supersymmetric field theories by defining a subspace of $N = 2$ superspace which has half the anticommuting coordinates while still preserving $N = 2$ supersymmetry. However, in order to define this subspace, an extra 2-sphere must be added to the commuting coordinates. Thus (the analytic subspace of) HSS is $\mathbb{R}^{3,1|4} \times S^2$.

In [37] the S^2 is identified with the coset space $SU(2)_S/U(1)_S$, and "harmonic coordinates" $u_i^\pm$, $i \in \{1,2\}$ are introduced on the $SU(2)_S$ group. Note that this $U(1)_S$ has nothing to do with the electromagnetic $U(1)$ gauge group! To make this clear, we have put an "S" — for "sphere" — subscript on the internal $SU(2)_S$ and $U(1)_S$ groups used in the coset construction of the S^2.

In particular, $u_i^\pm$ are complex coordinates transforming as a doublet of $SU(2)_S$, and with charges ± 1 under the $U(1)_S$, and satisfy

$$u^{\pm i} := \epsilon^{ij} u_j^\pm, \qquad \overline{u^{\pm i}} = \pm u_i^\mp, \qquad u^{+j} u_i^- - u^{-j} u_i^+ = \delta_i^j. \tag{5.2}$$

As a result of the last relation, any analytic function of the $u_i^\pm$ can be expanded in a power series in symmetrized products of the $u_i^\pm$. Non-vanishing functions on $S^2 \simeq SU(2)_S/U(1)_S$ are those with vanishing net $U(1)_S$ charge.

$U(1)_S$-covariant differential operators on the S^2 are given by

$$D^{++} := u^{+i} \frac{\partial}{\partial u^{-i}}, \qquad D^{--} := u^{-i} \frac{\partial}{\partial u^{+i}}, \qquad D^0 := u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}}, \tag{5.3}$$

whose commutators satisfy the $SU(2)_S$ algebra. A function, $f^{(q)}$, of definite $U(1)_S$ charge q, satisfies $D^0 f^{(q)} = q f^{(q)}$. The exterior derivative of any function, f, on the S^2 is given by

$$df = (D^{++} f)\omega^{--} + (D^{--} f)\omega^{++}, \tag{5.4}$$

where $\omega^{\pm\pm} := \mp 2i u^{\pm j} du_j^\pm$ are covariant one-forms on the S^2.

Invariant integration, "$\int du$", over the S^2 is very simple in harmonic coordinates, giving convenient relations such as

$$\int du \, 1 \; = \; 1, \qquad \int du \, u^+_{(i_1} \cdots u^+_{i_n} u^-_{j_1} \cdots u^-_{j_m)} \; = \; 0, \tag{5.5}$$

and many others [37].

A final important property of the S^2 is that it has a fixed-point-free orientation-reversing $\mathbb{Z}_2$ automorphism, the *antipodal map*, $\tau' : S^2 \to S^2$, which is inversion through the origin if the S^2 is realized as the unit sphere in $\mathbb{R}^3$. τ' acts on the $u_i^{\pm}$ coordinates by

$$\tau' : u_i^{\pm} \mapsto \pm u_i^{\mp}. \tag{5.6}$$

Note that $(\tau')^2 = -1$ on the $u_i^{\pm}$, reflecting the fact that they are double-valued coordinates on S^2.

Introduce the new spinor and space-time coordinates

$$\theta_\alpha^{\pm} := u_i^{\pm}\theta_\alpha^i, \qquad \overline{\theta}_{\dot\alpha}^{\pm} := u_i^{\pm}\overline{\theta}_{\dot\alpha}^i, \qquad x_A^\mu := x^\mu - i(\theta^+\sigma^\mu\overline{\theta}^- + \theta^-\sigma^\mu\overline{\theta}^+), \tag{5.7}$$

where the $\sigma^\mu_{\alpha\dot\alpha}$ are the usual Weyl spinor σ-matrices [49]. Note in particular that the $\overline{\theta}^{\pm}$ are *not* the complex conjugates of the $\theta^{\pm}$; rather

$$\overline{(\overline{\theta}^{\pm})} = \pm\theta^{\mp}. \tag{5.8}$$

Then the *analytic subspace of* HSS is the space $\simeq \mathbb{R}^{3,1|4} \times S^2$ described by the subset of coordinates

$$\{x_A^\mu, \theta_\alpha^+, \overline{\theta}_{\dot\alpha}^+, u_i^{\pm}\}. \tag{5.9}$$

Thus *analytic superfields* are just local functions of these coordinates. Note that the combination of complex conjugation and the antipodal map on the S^2 preserves the analytic subspace of HSS. Denoting this combination of conjugations by a tilde, we have

$$\widetilde{(u_i^{\pm})} = u^{\pm i}, \quad \widetilde{(u^{\pm i})} = -u_i^{\pm}, \quad \widetilde{(\theta^{\pm})} = \overline{\theta}^{\pm}, \quad \widetilde{(\overline{\theta}^{\pm})} = -\theta^{\pm}, \quad \widetilde{(x_A^\mu)} = x_A^\mu. \tag{5.10}$$

Note also that in this “analytic basis” (5.9), expressions like (5.3) are no longer valid, and have to be modified by the usual change of basis manipulations.

It is remarkable that $N = 2$ SUSY transformations preserve the analytic subspace of HSS. Superfields defined on the analytic subspace are called *analytic*, and their components therefore form representations of the $N = 2$ SUSY algebra. This analytic subspace of HSS plays in $N = 2$ SUSY a role similar to that played by the *chiral* subspace in $N = 1$ SUSY. It is evident that any analytic superfield contains the same number of anticommuting coordinates as a general (non-chiral) $N = 1$ superfield. This leads to reducing the number of independent components in comparison with general $N = 2$ superfields. However, all component fields depend now on extra bosonic coordinates $u_i^{\pm}$. Therefore, any analytic superfield contains an *infinite* number of component fields from the point of view of conventional field theory. Nevertheless, we will see that actions can be formulated for which all but a finite number of these component fields are non-propagating: they are “auxiliary” fields which can be completely eliminated in favor of the propagating fields by solving their algebraic equations of motion.

Differentiation with respect to the spinor coordinates is most conveniently expressed in terms of the standard $N = 2$ supercovariant spinor derivatives, D_α^i, $\overline{D}_{\dot\alpha}^i$ in $N = 2$ superspace [49] by forming the combinations

$$D_\alpha^{\pm} = u_i^{\pm} D_\alpha^i, \qquad \overline{D}_{\dot\alpha}^{\pm} = u_i^{\pm} \overline{D}_{\dot\alpha}^i. \tag{5.11}$$

Thus, for example, D_α^+ is a supercovariant version of $\partial/\partial\theta^{-\alpha}$. One can show that an analytic superfield, Φ, satisfies

$$D_\alpha^+ \Phi = \overline{D}_{\dot\alpha}^+ \Phi = 0, \tag{5.12}$$

and, in fact, these conditions completely define the analytic subspace of HSS. The algebra of the supercovariant derivatives (suppressing space-time indices and irrelevant normalization constants) is

$$
\begin{aligned}
&0 = \{D^{\pm}, D^{\pm}\} = \{\overline{D}^{\pm}, \overline{D}^{\pm}\} = \{D^{\pm}, \overline{D}^{\pm}\}, \qquad \partial \propto \{D^{\pm}, \overline{D}^{\mp}\}, \\
&0 = \{\partial, D^{\pm}\} = \{\partial, \overline{D}^{\pm}\} = \{\partial, D^{\pm\pm}\}, \\
&0 = [D^{\pm}, D^{\pm\pm}] = [\overline{D}^{\pm}, D^{\pm\pm}], \qquad D^{\mp} \propto [D^{\pm}, D^{\mp\mp}], \qquad \overline{D}^{\mp} \propto [\overline{D}^{\pm}, \overline{D}^{\mp\mp}], \\
&0 = [D^{\pm\pm}, D^{\pm\pm}], \qquad D^{0} \propto [D^{\pm\pm}, D^{\mp\mp}], \qquad qX^{(q)} = [D^{0}, X^{(q)}],
\end{aligned}
\tag{5.13}
$$

where in the last commutator, $X^{(q)}$ is any operator carrying $U(1)_S$ charge q. Thus $\{\partial, D^{+}, \overline{D}^{+}, D^{++}\}$ is a maximal (anti)commuting subset of the supercovariant algebra, showing that (5.12) is integrable, and, furthermore, that if Φ is analytic, then so is any local functional of $\partial^{m}(D^{++})^{n}\Phi$, $\partial^{m}(D^{++})^{n}\widetilde{\Phi}$, and the harmonic coordinates $u^{\pm}$.

Integration over the anticommuting coordinates is, as usual, essentially the same as differentiation. Thus the invariant measure on the analytic subspace of HSS is

$$
\int d\theta^{(-4)} := \int d^{2}\theta^{+} d^{2}\overline{\theta}^{+} := (D^{-})^{2}(\overline{D}^{-})^{2}\Big|_{\theta=0}, \tag{5.14}
$$

where the vertical line in the last expression means that it should be evauated at $\theta^{\pm} = \overline{\theta}^{\pm} = 0$ after differentiation. Note that this measure carries $U(1)_S$ charge -4, therefore only integrands with $U(1)_S$ charge $+4$ can give non-vanishing answers upon integration.

5.2 Hyper- and vector multiplet superfields

5.2.1 Hypermultiplets

The most general action for hypermultiplets can be written in terms of a complex scalar analytic superfield, q^+ (or a set of such superfields in the case of several hypermultiplets) (Chapter 5 of [37]). It carries $U(1)_S$ charge +1, and its lowest component has mass scaling dimension 0. Thus the leading (2-derivative) terms of a general $N = 2$ IREA action for hypermultiplets is

$$S_H = \int d^4x\, du\, d\theta^{(-4)} \mathcal{L}^{(+4)}(q^+, \widetilde{q^+}, u^\pm, D^{++}). \tag{5.15}$$

Here $\mathcal{L}^{(+4)}$ is a general analytic local real superfield functional of q^+ with $U(1)_S$ charge +4. As such it is a functional of q^+, $\widetilde{q^+}$, and their D^{++} derivatives. Reality is with respect to tilde-conjugation introduced in (5.10). This action describes a hypermultiplet nlsm once (infinitely many) auxiliary component fields are eliminated by their equations of motion.

There is a great deal of redundancy — different lagrangians describing the same physics — in (5.15). It turns out (Chapter 11 of [37]) that the most general hypermultiplet nlsm can be written in the form

$$S_H = \int d^4x\, du\, d\theta^{(-4)} \left[-\widetilde{q^+} D^{++} q^+ + \frac{1}{2} L^{(+4)}(q^+, \widetilde{q^+}, u^-) \right] \tag{5.16}$$

which depends on only a single D^{++} derivative, and has no explicit dependence on u^+. The first term describes a free hypermultiplet while the second encodes interactions (the nonlinear terms in the nlsm). The equations of motion for q^+ and $\widetilde{q^+}$ following from (5.16) are

$$D^{++}q^{+} = +\frac{1}{2}\frac{\partial L^{(+4)}}{\partial \widetilde{q^{+}}}, \qquad D^{++}\widetilde{q^{+}} = -\frac{1}{2}\frac{\partial L^{(+4)}}{\partial q^{+}}. \tag{5.17}$$

(Invariance of $L^{(+4)}$ under tilde conjugation implies that the second equation is just the tilde conjugate of the first.) Expanding this equation in powers of $q^{\pm}$, θ^{+}, and $\overline{\theta}^{+}$ gives an infinite set of coupled, nonlinear, 2nd order differential equations. In the case of vanishing interaction term it is easy to eliminate an infinite number of auxiliary fields, leaving the free equations of motion of a massless hypermultiplet. (Hypermultiplet masses can only be described in a slight generalization of HSS which includes central charges [37]; this will not be needed for our purposes.) In the case of non-vanishing interactions elimination of the auxiliary fields is considerably more complicated, and will essentially be the subject of section 5.3, where we will also see the straight forward generalization of the above action to multiple hypermultiplet superfields.

5.2.2 Vector multiplets

General $N = 2$ IREAs for $U(1)$ vector multiplets in HSS are considerably more complicated to describe. (There exists a simple description in *chiral* $N = 2$ *superspace*, but it does not seem to be able to be made EM duality-invariant, so we will not describe it here.) However our pullback approach to constructing EM duality-symmetric $N = 2$ actions will only require coupling $U(1)$ vector multiplets to a hypermultiplet nlsm by "gauging $U(1)$ isometries" of the target space. This can achieved in HSS in a way that is closely analogous to the usual minimal coupling of gauge fields (i.e., the operation of "gauging global symmetries") in non-supersymmetric quantum field theory (chapter 7 of [37]).

In particular, to construct a hypermultiplet nlsm with gauged $U(1)$ isometries, one simply replaces the D^{++} derivative on the S^2 everywhere by its gauge-covariant version,

$$\nabla^{++} := D^{++} + iV^{++}, \tag{5.18}$$

where V^{++} is a real analytic superfield with $U(1)_S$ charge $+2$. Reality means that $\widetilde{V^{++}} = V^{++}$. This gauge field transforms under $U(1)$ gauge transformations as

$$\delta V^{++} = D^{++}\lambda \tag{5.19}$$

where λ is an arbitrary real analytic superfield with vanishing $U(1)_S$ charge. λ can be used to gauge away all but the leading term of the harmonic expansion of V^{++} so that after gauge fixing,

$$V^{++}_{\text{gauge-fixed}} = u_i^+ u_j^+ \; V^{(ij)}(x_A, \theta^+, \overline{\theta}^+). \tag{5.20}$$

Thus there are only finitely many propagating component fields, and it is not difficult to show that they correspond those of a $U(1)$ vector multiplet.

The kinetic (Maxwell) action for V^{++} is [GIOS, ch 7]

$$S_V = \frac{1}{4e^2} \int d^4x \, d^8\theta \int du_1 du_2 \, \frac{V^{++}(x,\theta,u_1) V^{++}(x,\theta,u_2)}{(u_1^+ u_2^+)^2}, \tag{5.21}$$

where $(u_1^+ u_2^+) := u_1^{+i} u_{2i}^+$ and e^2 is the coupling. Note that this action is given as an integral over the whole of $N=2$ superspace (not just the analytic subspace) and is non-local on the S^2. An integration by parts together with the identity $D_1^{++}(u_1^+ u_2^+)^2 = D_1^{--}\delta(u_1 - u_2)$ shows that the gauge variation of this action with respect to (5.19) is proportional to $\int d^8\theta V^{++} D^{--}\lambda \propto \int d\theta^{(-4)} V^{++}(D^+)^4 D^{--}\lambda = 0$ using the analyticity of V^{++} and of λ in the last 2 steps. Thus S_V is indeed gauge-invariant.

The generalization of the above action to multiple vector multiplet superfields is straight forward.

5.3 HSS construction of N=2 nlsm

As mentioned in the last section, it is a difficult problem to eliminate the auxiliary component fields in the hypermultiplet nlsm action (5.16). Thus the connection between the HSS hypermultiplet "potential" $L^{(+4)}$ and the IREA for the propagating fields is obscure. As explained in chapter 3, the nlsm IREA is completely determined by the target space geometry, $\mathcal{X}$. A hypermultiplet nlsm has a hyperkähler target space. So the question is, what is the connection between a given hyperkähler metric, g_{MN}, and the HSS potential $L^{(+4)}$? We will now briefly summarize the answer, following [37], by showing how to calculate $L^{(+4)}$ given g_{MN}.

First, we introduce a convenient labelling for the r hypermultiplet superfields, q^{+i}, $i \in \{1, \ldots, r\}$, and their tilde-conjugates, $\widetilde{q^{+i}}$. We combine them into a $2r$-component vector of superfields, $q^{+\alpha}$, $\alpha \in \{1, \ldots, 2r\}$, satisfying the reality condition

$$\widetilde{q^{+\alpha}} := q^+_\alpha = \Omega_{\alpha\beta} q^{+\beta}, \tag{5.22}$$

where $\Omega_{\alpha\beta}$ is the symplectic form $\Omega = \mathbb{I}_r \otimes \epsilon$ where $\mathbb{I}_r$ is the $r \times r$ identity matrix, and ϵ is the 2-index antisymmetric tensor normalized by $\epsilon_{12} = -\epsilon^{12} = 1$. Note that $\Omega^2 = -\mathbb{I}_{2r}$, so $\widetilde{q^+_\alpha} = -q^{+\alpha}$.

Now start with an r quaternionic-dimensional hyperkähler manifold $\mathcal{X}$ with given metric g_{MN} and complex structures $(J^a)_M^N$, $a \in \{1, 2, 3\}$, in a coordinate system $\widetilde{\xi}^M$ where $M, N \in \{1, \ldots, 4r\}$.

1. Change coordinates to $\xi^{\mu i}$ with $\mu, \nu \in \{1, \ldots, 2r\}$ and $i, j \in \{1, 2\}$ such that the complex structures have the simple form

$$(J^a)_{\mu i}^{\nu j} = i\delta_\mu^\nu (\tau^a)_i^j, \tag{5.23}$$

   where τ^a are the Pauli matrices. Then,

2. Express the metric in these coordinates in terms of vielbeins,

$$g_{\mu i\ \nu j} = E_{\mu i}^{\alpha k}\, \Omega_{\alpha\beta}\epsilon_{k\ell}\, E_{\nu j}^{\beta\ell}, \tag{5.24}$$

   and their inverses, $e_{\alpha i}^{\mu j}$, defined by $e_{\alpha i}^{\mu j} E_{\mu j}^{\beta k} = \delta_\alpha^\beta \delta_i^k$.

3. Define $\xi^{\mu\pm} := \xi^{\mu i} u_i^\pm$ and $e_\alpha^{+\mu\pm} := u^{i+} e_{\alpha i}^{\mu j} u_j^\pm$ with $u_i^\pm$ harmonic coordinates on an S^2. Then solve the linear first order system of PDEs,

$$e_\alpha^{+\mu+}(\delta_\mu^\nu + \partial_{\mu+} v^{\nu+}) \ + \ e_\alpha^{+\mu-}\partial_{\mu-} v^{\nu+} \ = \ 0, \tag{5.25}$$

   for the "harmonic bridge", $v^{\mu+}(\xi^{\nu i}, u_i^\pm)$, where $\partial_{\mu\pm} := \partial/\partial\xi^{\mu\pm}$.

4. Define "analytic coordinates" by

$$\xi_A^{\mu+} := \xi^{\mu+} + v^{\mu+}(\xi^{\nu i}, u_i^\pm), \tag{5.26}$$

   and then define functions $H^{++\mu+}$ of them by

$$H^{++\mu+}(\xi_A^{\nu+}, u_i^\pm) := D^{++} v^{\mu+}(\xi^{\nu i}, u_i^\pm), \tag{5.27}$$

   where D^{++} is given in (5.3). Note that (5.27) requires one to invert (5.26) to find $\xi^{\mu+} = \xi^{\mu+}(\xi_A^{\nu+}, u_i^\pm)$.

5. Finally one inverts $H^{++\mu+} = \frac{1}{2}\Omega^{\mu\nu}\partial^A_{\nu+}L^{(+4)}$ to find $L^{(+4)}$, where $\partial^A_{\nu+} := \partial/\partial\xi^{\nu+}_A$. That is,

$$L^{(+4)}(q^{+a}, u) = 2\int_*^{q^{+a}} d\zeta^\mu \, \Omega_{\mu\nu}H^{++\nu+}(\zeta, u). \tag{5.28}$$

The fact that each of these steps can be carried out in principle follows non-trivially from properties of hyperkähler geometry. All of these steps, except for the last one, are difficult to carry out analytically except in a few special cases. Note that there are infinitely many solutions to the PDEs (5.25), but that a unique one exists for which $H^{++\mu}$ defined by (5.27) is a function only of $\xi^{\nu+}_A$ and u^-_i (i.e., has no explicit u^+_i-dependence).

These steps can be reversed to calculate g_{MN} from $L^{(+4)}$. In this case, (5.27) becomes a nonlinear differential equation for the bridge $v^{\mu+}$. Once a bridge is found, the metric $g_{\mu i\ \nu j}$ can be reconstructed in a straight forward manner as outlined in chapter 11 of [37].

5.4 Projective superspace

An alternative formalism to HSS for writing $N = 2$ covariant actions is "projective superspace", introduced in [50, 51, 52]. The projective superspace and HSS formalisms are closely related. The geometry of projective superspace is the same as that of HSS: both take place on the same supermanifold $\mathbb{R}^{3,1|8} \times \mathbb{CP}^1$. In projective superspace, different coordinates are used to parameterize the $\mathbb{CP}^1$, namely a pair of complex "isotwistor" coordinates, v^i, which are simply the usual homogeneous coordinates realizing $\mathbb{CP}^1$ as $\mathbb{C}^2/\mathbb{C}^*$. They are related to the harmonic coordinates by

$$u^{i+} = \frac{v^i}{\sqrt{v^k\overline{v}_k}}\ , \qquad u^-_i = \frac{\overline{v}_i}{\sqrt{v^k\overline{v}_k}}\ . \tag{5.29}$$

In projective superspace, complex analyticity with respect to the isotwistor coordinates plays the role of harmonic analyticity in HSS. This leads to projective superspace superfields which are allowed to have singularities (poles) on $\mathbb{CP}^1$, in contrast to the HSS superfields which are well-defined on the whole $S^2 \simeq \mathbb{CP}^1$. Finally, projective superspace actions are similar to those in HSS except that instead of integrating lagrangians over the whole $\mathbb{CP}^1$ internal space, in projective superspace one only integrates over a closed path (avoiding the poles) in the $\mathbb{CP}^1$.

Recently, Jain and Siegel [53] and Butter [54] have shown that the projective superspace formalism can be derived from the HSS one in a simple way. They perform a kind of analytic continuation of the 2-dimensional $\int du$ integration of HSS to show that one of the integrations can be reduced to boundary terms, thus leaving a one-dimensional integral along a path in $\mathbb{CP}^1$.

Since the two formalisms are equivalent, we have not restricted ourselves by focussing on the HSS formalism.

Chapter 6

Proposed Solution

After having described all the necessary ingredients of our solution, we can finally present a recipe combining them to create EM duality-symmetric $N = 2$ IREA actions. We start by presenting a summary of the key ideas that are instrumental in making our strategy successful. We then outline the four specific steps to be followed in implementing it. Finally, we provide some details for step number 2.

6.1 Key ideas

The first key idea we use in our construction of the HSS $N = 2$ nlsm is to use the total space, $\mathcal{X}$, of DW geometry as the target space. In the traditional formulation of the $N = 2$ IREA, the vector multiplet scalar fields are maps from space-time $\mathbb{R}^{3,1}$ into the Coulomb branch (CB), $\mathcal{M}_V$, which is an r-dimensional special Kähler space. However, as we saw in chapter 3, the coupling matrix τ_{IJ} transforms under the action of the $Sp(2r, \mathbb{Z})$ EM duality group, thus breaking the EM duality-invariance of the action. In contrast, the total space, $\mathcal{X}$, of the CB is invariant with respect to $Sp(2r, \mathbb{Z})$ transformations, thus making $\mathcal{X}$ the natural geometrical object to use to construct a duality-symmetric formulation of the IREA.

The next major idea in our recipe is to use the facts: (1) that the target space of an $N = 2$ hypermultiplet nlsm is necessarily a hyperkähler manifold [57]; and (2) that the total space $\mathcal{X}$ has a natural hyperkähler structure, as described in chapter 4. Thus the natural $N = 2$ supersymmetric nlsm on $\mathcal{X}$ is a hypermultiplet nlsm.

The third idea we use is the harmonic superspace (HSS) construction of hypermultiplet nlsms [36, 37]. This gives a manifestly $N = 2$ supersymmetric formulation of our theory. However, it is important that we construct this nlsm in a way that keeps the invariant geometric structures of $\mathcal{X}$ manifest. The existing HSS constructions, reviewed in the last chapter, do not do this. The reason is that $\mathcal{X}$ as a hyperkähler manifold has a whole 2-sphere of complex structures, and choosing one of them to pull back with a HSS hypermultiplet analytic superfield destroys the symmetry between these complex structures.

Our fourth idea is to use the twistor space $\mathcal{Z}$ formulation of $\mathcal{X}$ as the target space of the HSS nlsm. As we have explained in chapter 4, the twistor space is the Cartesian product $\mathcal{Z} = \mathcal{X} \times \mathbb{CP}^1$ endowed with a *unique* complex structure. Thus we can naturally identify HSS analyticity with complex analyticity in $\mathcal{Z}$ by pulling back geometrical objects on $\mathcal{Z}$ with HSS hypermultiplet analytic superfields. Furthermore, the extra $\mathbb{CP}^1 \simeq S^2$ dimension of $\mathcal{Z}$ can be naturally identified with the internal S^2 of HSS by pulling back with the identity map.

Fifth, we notice that there is a natural invariant geometric object on $\mathcal{Z}$, namely the holomorphic (2,0) form Ω^{++} described in chapter 4, which when pulled back as described above, can be integrated over HSS. It is thus the natural candidate for the EM duality symmetric Lagrangian of our $N = 2$ nlsm. We will detail this step in more detail in section 6.3 below. We conjecture that it is equivalent to the traditional HSS procedure for constructing nlsm lagrangian reviewed in section 5.3; the difference is simply that we avoid making coordinate choices which obscure the invariant geometric structures of the target space.

Finally, the hyperkahler nlsm constructed in this way has no propagating vector multiplets. The solution to this is well known: vector multiplets can be naturally (and $N = 2$ super-covariantly) coupled to a hypermultiplet nlsm by gauging isometries of the target space. $\mathcal{X}$ has $2r$ isometries corresponding to translations along the $2r$ independent cycles of the T^{2r} fibers. Thus we can couple $2r$ vector multiplets, corresponding to r "electric" and r "magnetic" gauge potentials. This is reminiscent of the original trick of Zwanziger, reviewed in chapter 2, of doubling the number of gauge potentials to achieve EM duality invariance. Gauging the isometries of $\mathcal{X}$ lifts (higgses) the flat directions, reducing the nlsm target space to r complex dimensions, the correct dimension to describe the degrees of freedom of the CB.

6.2 Recipe for constructing the solution

We now give an outline of the key steps that need to be completed in order to implement the above ideas to construct $N = 2$ *HSS EM duality-symmetric IREAs.*

1. Construct the hyperkähler structures on $\mathcal{X}$, and the complex structure on $\mathcal{Z}$ from the CB data. This was completed in chapter 4.

2. Write a hypermultiplet nlsm on $\mathcal{X}$ in HSS using the pullback of the holomorphic $(2, 0)$-form Ω^{++} on the twistor space by the hypermultiplet superfield q^{+}. This will be described in more detail in section 6.3 below.

3. Construct coordinate-invariant expressions for the *Killing vectors* generating the isometries of $\mathcal{X}$.

4. Gauge the isometries by coupling the Killing vectors to $2r$ HSS vector superfields, V^{++} : HSS $\to \mathbb{R}^{2r} \simeq T^*_p\mathcal{M}_V$. This is simply done by replacing D^{++} derivatives with $U(1)$-covariant ∇^{++} derivatives given by (5.18) in the HSS hypermultiplet nlsm.

5. Count the massless propagating degrees of freedom remaining after the Higgs mechanism takes place. If there are more photons than are physically required, then we will need to additionally couple a Siegel-type term to a Lagrange multiplier field (as reviewed in chapter 2) to eliminate additional degrees of freedom.

In the following section, we will show how our pullback method by HSS hypermultiplet superfields is supposed to actually work. We will also conjecture the equivalence of our pullback approach for constructing hyperkähler nlsm harmonic superspace actions to the method described in [36, 37] and reviewed in section 5.3 above. This conjectured equivalence gives a simple geometric picture of harmonic superspace nlsms in terms of the twistor space construction of hyperkähler manifolds.

We will not address the last three steps of the above recipe in this thesis.

6.3 Pullback method for constructing nlsm action

We propose the following simple geometrical picture of harmonic superspace hypermultiplet nlsms in terms of the twistor space description of the hyperkähler target space $\mathcal{X}$.

It is important to point out that Lindstrom and Roček [55] proposed a way of writing an $N = 2$ lagrangian in projective superspace using the twistor space 2-form. This construction is almost certainly equivalent to our approach in harmonic superspace, though the details of how the two approaches are related have not been worked out yet in concrete terms.

Recall that the twistor space $\mathcal{Z} \simeq \mathcal{X} \times \mathbb{CP}^1$ has a natural holomorphic closed (2,0)-form Ω^{++} which is a section of an $\mathcal{O}(2)$ bundle over the $\mathbb{CP}^1$ base of $\mathcal{Z}$. This latter simply means that Ω^{++} carries charge $+2$ with respect to the $U(1)$ symmetry that acts by phase rotations of the complex coordinates on the $\mathbb{CP}^1$. Closure means that $d\Omega^{++} = 0$, where d is the exterior derivative on $\mathcal{Z}$. Thus locally

$$\Omega^{++} = d\Theta^{++} \tag{6.1}$$

for some holomorphic (1,0)-form Θ^{++} carrying $U(1)$ charge +2.

Recall also that hypermultiplet superfields in HSS are maps

$$q^+ : \text{HSS} \to \mathcal{X}, \tag{6.2}$$

carrying $U(1)_S$ charge +1. If we identify the $\mathbb{CP}^1$ base of $\mathcal{Z} \simeq \mathcal{X} \times \mathbb{CP}^1$ with the "internal" $\mathbb{CP}^1 \simeq S^2$ of HSS, then q^+ can be extended by this identification to a map, $q^+ : \text{HSS} \to \mathcal{Z}$, which acts as the identity on the $\mathbb{CP}^1$ factors on the two sides. $U(1)_S$ charges on HSS are then identified with the $U(1)$ charges on $\mathcal{Z}$. Thus q^+ is really a map to a section of an $\mathcal{O}(1)$ bundle over the $\mathbb{CP}^1$ base of $\mathcal{Z}$:

$$q^+ : \text{HSS} \to \mathcal{O}(1). \tag{6.3}$$

Finally, $\mathbb{CP}^1$ has a unique holomorphic (1,0)-form of $U(1)_S$ charge +2, namely ω^{++} introduced below eq. (5.4). Thus one can naturally integrate 1-forms such as Θ^{++} over $\mathbb{CP}^1$ to give the coordinate invariant quantity $\int_{\mathbb{CP}^1} \omega^{++} \wedge \Theta^{++}$. This quantity is uninteresting on $\mathcal{Z}$ because it vanishes identically there, since the two 1-forms in the integrand are both (1,0)-forms. However, upon pulling Θ^{++} back to HSS using q^+ it becomes a linear combination of (1,0)- and (0,1)-forms, and so can give a non-vanishing answer.

In particular, complex coordinates, $\xi^{+\mu}$, on the $\mathcal{O}(1)$ bundle over $\mathcal{Z}$, are maps

$$\xi^{+\mu} : \mathcal{O}(1) \to \mathbb{C}, \tag{6.4}$$

so the pullback of these coordinate functions by q^+ are complex valued functions on HSS:

$$q^{+*}(\xi^{+\mu}) := q^{+\mu} = \xi^{+\mu} \circ q^+ : \mathrm{HSS} \to \mathbb{C}, \tag{6.5}$$

which we identify with hypermultiplet HSS superfields. In these coordinates, the holomorphic (1,0)-form Θ^{++} on $\mathcal{Z}$ has the form

$$\Theta^{++} = \Theta^+_\mu(\xi^{+\nu}, u_i^\pm)\, d\xi^{+\mu} + \Theta(\xi^{+\nu}, u_i^\pm)\, \omega^{++} + \Theta^{(+4)}(\xi^{+\nu}, u_i^\pm)\, \omega^{--}. \tag{6.6}$$

Therefore, by definition of the pullback,

$$\begin{aligned} q^{+*}\Theta^{++} &= (\Theta^+_\mu \circ q^+)\, dq^{+\mu} + (\Theta \circ q^+)\, \omega^{++} + (\Theta^{(+4)} \circ q^+)\, \omega^{--} \\ &= \Theta^+_\mu(q^{+\nu}, u_i^\pm)\, dq^{+\mu} + \Theta(q^{+\nu}, u_i^\pm)\, \omega^{++} + \Theta^{(+4)}(q^{+\nu}, u_i^\pm)\, \omega^{--}. \end{aligned} \tag{6.7}$$

Now, by (5.4), $dq^{+\mu} = (D^{++}q^{+\mu})\omega^{--} + (D^{--}q^{+\mu})\omega^{++}$, and, using the fact that $\int \omega^{++} \wedge \omega^{--} = \int du$, we get

$$\int_{\mathbb{CP}^1} \omega^{++} \wedge q^{+*}\Theta^{++} = \int du\ \left[\Theta^+_\mu(q^{+\nu}, u_i^\pm)\, D^{++}q^{+\mu} + \Theta^{(+4)}(q^{+\nu}, u_i^\pm)\right]. \tag{6.8}$$

This has net $U(1)_S$ charge +4, so is suitable to integrate over the analytic subspace of HSS. Also, the invariance of Ω^{++} under the involutive automorphism of $\mathcal{Z}$ pulls back to the reality of the above integrand with respect to tilde conjugation on HSS. Thus, a natural, coordinate invariant, candidate for the hypermultiplet nlsm action is

$$S_H = \int_{\mathbb{R}^{3,1|4}} d^4x\, d\theta^{(-4)} \int_{\mathbb{CP}^1} \omega^{++} \wedge q^{+*}\Theta^{++}. \tag{6.9}$$

(6.9) is our main result. It gives a coordinate-invariant construction of the harmonic superspace action for a hypermultiplet nlsm. Note that its expression in coordinates, (6.8), is very similar to the traditional HSS form of the hypermultiplet nlsm action, given in (5.16). The main difference is that in place of $\widetilde{q^+}$ in (5.16), (6.8) has the more complicated function $\Theta^{(}q^+, u^\pm)$. This is presumably a reflection of the fact that in writing (6.8) we did not make the special coordinate choices that were made in arriving at (5.16).

Note that (6.9) involves only the (1,0)-form Θ^{++} on twistor space, related to the (2,0)-form Ω^{++} by (6.1). We saw in chapter 4 that Ω^{++} encodes the hyperkähler structures on $\mathcal{X}$. Thus, as long as this action gives the right counting of the propagating degrees of freedom, it can only be the hypermultiplet nlsm with target space $\mathcal{X}$. To actually prove this, though, we will have to expand our pullback action in component fields and eliminate the auxiliary fields.

Chapter 7

Conclusions and Outlook

7.1 What we have learned

Here is a list of the key new lessons we have learned in this thesis

- We gave a recipe for how to construct EM duality-symmetric $N = 2$ IREAs using the pullback approach. This recipe is the main result of our investigations, and is nearly uniquely specified by the invariant geometric pullback approach. However, we have not completed a proof that it gives the correct set of propagating degrees of freedom.

- We conjectured the equivalence of our pullback approach for constructing hyperkähler nlsm harmonic superspace actions to the traditional method described in [37]. This conjectured equivalence gives a simple geometric picture of harmonic superspace nlsms in terms of the twistor space construction of hyperkähler manifolds.

- Finally, along the way we discovered an extra geometric structure in the Donagi-Witten formulation [38] of the special Kähler geometry of the Coulomb branches of $N = 2$ supersymmetric gauge theories. The possible interpretation and implications of this extra structure is discussed below.

7.2 What is left to do

Here are the tasks that need to be completed in the scenario outlined back in chapter 4:

1. Write a hypermultiplet nlsm on $\mathcal{X}$ in HSS using the pullback of the holomorphic $(2,0)$-form Ω^{++} on the twistor space by the hypermultiplet superfield q^+.

2. Construct coordinate-invariant expressions for the Killing vectors generating the isometries of $\mathcal{X}$.

3. Gauge the isometries by coupling the Killing vectors to $2r$ HSS vector superfields.

4. Count the massless propagating degrees of freedom remaining after the Higgs mechanism takes place. If there are more photons than what is physically required, then we may need to additionally couple a Siegel-type term to a Lagrange multiplier to eliminate additional degrees of freedom.

Task 1 is a computation using coordinate expressions derived in chapter 4. However, the main difficulty is to find coordinates on the twistor space, $\mathcal{Z}$, which are complex with respect to the complex structure $J^{\mathcal{Z}}$. The coordinates introduced in chapter 4 do not satisfy this constraint. We may need to solve a coupled set of partial differential equations to find those holomorphic coordinates, a task that may or may not be as difficult as solving the differential equations (5.25) for the bridge functions in HSS as explained in chapter 5.

Task 2 involves first a straightforward computation using the coordinate expressions found in chapter 4 to find the Killing vectors. The next step then is to rewrite these expressions in terms of the holomorphic twistor coordinates found in task 1.

Task 3 is then simply the substitution $D^{++} \to \nabla^{++}$ in the Lagrangian, as described in chapter 5.

Task 4 is a potentially difficult one. One possible difficulty is the need to find a possibly nonlinear Lagrange multiplier superfield. Another difficulty is the need to find the appropriate term to couple it to in the Lagrangian so that it may correctly eliminate the correct number of propagating degrees of freedom using its equation of motion.

7.3 Interesting questions

7.3.1 Extra structure in the DW formulation of SK geometry

We have seen that there exist coordinates, $(x, \widehat{x}, u)$, on the total space $\mathcal{X}$, in terms of which local complex coordinates are given by $z = \Delta x + \tau(u)\,\widehat{x}$ and u. We have also defined $\theta = \Delta dx + \tau(u) d\widehat{x}$ to be a global (1,0)-form and computed that $\partial\theta = d\tau \frac{1}{\tau_-}\theta$ and $\partial\overline{\theta} = d\tau \frac{1}{\tau_-}\overline{\theta}$. This implies that the set of 1-forms $\{\theta, \overline{\theta}\}$ is "in involution", which, by Frobenius' theorem, implies that the subbundle $H \subset T\mathcal{X}$ annihilated by θ and $\overline{\theta}$ is integrable. That is to say that there exists a submanifold $\Gamma \subset \mathcal{X}$ whose tangent space at a point p is H_p and is annihilated by $\{\theta, \overline{\theta}\}$, *i.e.*, $\theta(v) = \overline{\theta}(v) = 0$ for all $v \in H_p$.

Γ is actually a section of $\pi : \mathcal{X} \to \mathcal{M}_V$ simply because the tangent space to the fiber at p is the kernel of $d\pi$: $T_p\mathcal{X}_u = \ker(d\pi) = \{\partial_z, \partial_{\overline{z}}\}$ which further implies that $T_p\mathcal{X}_u \bigcap H_p = \{0\}$. Hence, the projection map π is an isomorphism from H_p into the tangent space to $\mathcal{M}_V$. We will denote this isomorphism by $s : H_p \leftrightarrow T_{\pi(p)}\mathcal{M}_V$.

Donagi and Markman [47] and Freed [39] put an extra condition on Γ such that it is *lagrangian* with respect to the DW form L, *i.e.*, $L = d\theta^T du$ without any terms proportional to $du^i du^j$. This basically means that they set by hand $dA = 0$ in (4.19).

However there is nothing in the low energy physics on the CB that requires this condition.

There are two interesting questions that one can ask in this context:

(1) Is the dA part of L observable *mathematically*? In other words, is there a coordinate-invariant way of describing it?

(2) Is the dA part of L observable *physically*, i.e., in the low energy Coulomb branch physics?

The answer to the first question is simply, yes. The dA terms in L define a (2,0)-form that lives in the cotangent bundle of the Coulomb branch, *i.e.*, $dA \in \Lambda^2 T^* \mathcal{M}_V$, defined by $dA(v, w) := L(s^{-1}v, s^{-1}w)$ for any two vectors v and $w \in T\mathcal{M}_V$. This gives a coordinate-invariant definition of dA. To justify calling it dA, we need to show that it is closed. But since $dL = 0$, $d(dA) = 0$ immediately follows, and implies that dA is locally exact.

The answer to the second question may potentially have very interesting implications. We have seen from the discussion at the end of section 4.2.3 that dA does not appear in the 2-derivative terms of IREA on the Coulomb branch. But how about in the (i) central charge and (ii) higher-derivative terms on the Coulomb branch?

For the central charge, we mentioned before that $\oint_\gamma L = dZ_\gamma$ which implies that $Z_\gamma \sim \int_\Sigma$ where $\Sigma \in H_2(\mathcal{X})$ is a 2-cycle satisfying appropriate boundary conditions. If $\mathcal{M}_V \sim \mathbb{C}^r$ with $r \geq 2$, there exists a non-trivial $\Sigma \in H_2(\mathcal{M}_V)$ homologous to a 2-torus which winds the transverse intersection of two complex codimension 1 singularities of $\mathcal{M}_V$. So perhaps the integral of the DW 2-form along this non-trivial cycle, $\oint_\Sigma L = \oint_\Sigma dA$ computes some global property of Z_γ on $\mathcal{M}_V$.

Also, note the absence of a *local* observable of dA on the Coulomb branch. This follows because for any 2-surface Σ with boundary $\partial\Sigma = C$, $\int_\Sigma dA = \oint_C A$. Since $A(u)$ is a *holomorphic* 1-form, its integral along the boundary vanishes, $\oint_C A = 0$, by Cauchy's theorem. However, if there exists a co-dim-1 singularity linking C, then $\int_\Sigma dA = \oint_C A \neq 0$ if $A(u)$ has a pole as u approaches the singularity. Thus poles in A at singularities on the Coulomb branch could also lead to contributions to the central charge.

As for the physical possible relationship of dA to 4-derivative terms, it has been shown in [56] that there exists 4-derivative terms on $r \geq 2$ Coulomb branches that are holomorphically protected. Thus these terms could in principle be calculated in terms of holomorphic structures on the Coulomb branch. Could this structure possibly be related to dA?

7.3.2 Relation of the DW 2-form to the twistor 2-form

Two a priori different 2-forms have played an important role in our program. The DW 2-form L and the twistor space 2-form Ω^{++}, both described in chapter 4, are closed (2,0) forms. The DW 2-form is a symplectic form on $\mathcal{X}$, while Ω^{++} restricts to a symplectic form on the fibers of twistor space, which are isomorphic ot $\mathcal{X}$. As symplectic forms they each give $\mathcal{X}$ the structure of a phase space of a complex classical dynamical system (which is integrable, to boot). What is the relation between these two auxiliary dynamical systems? After coupling in the vector superfields, do these two systems become equivalent? That's one question we hope to find an answer to.

Bibliography

[1] N. Seiberg and E. Witten, Nucl. Phys. B **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)] [hep-th/9407087].

[2] N. Seiberg and E. Witten, Nucl. Phys. B **431**, 484 (1994) [hep-th/9408099].

[3] D. Zwanziger, Phys. Rev. **176**, 1489 (1968).

[4] D. Zwanziger, Phys. Rev. D **D 3**, 880 (1971).

[5] R. Banerjee and C. Wotzasek, Phys. Rev. D **63**, 045005 (2001).

[6] S. Deser and C. Teitelboim, Phys. Rev. D **13**, 1592 (1976).

[7] S. Deser, J. Phys. A **15**, 1053 (1982).

[8] N. Marcus and J.H. Schwarz, Phys. Lett. B **115**, 111 (1982).

[9] R. Floreanini and R. Jackiw, Phys. Rev. Lett. **59**, 1873 (1987).

[10] M. Henneaux and C. Teitelboim, Phys. Lett. B **206**, 650 (1988).

[11] A. Tseytlin, Phys. Lett. B **242**, 163 (1990).

[12] A. Tseytlin, Nucl. Phys. B **350**, 395 (1991).

[13] J.H. Schwarz and A. Sen, Nucl. Phys. B **411**, 35 (1994).

[14] W. Siegel, Nucl. Phys. B **238**, 307 (1984).

[15] P.P. Srivastava, Phys. Rev. Lett. **63**, 2791 (1989).

[16] B. McClain, Y.S. Wu and F. Yu, Nucl. Phys. B **343**, 689 (1990).

[17] P. Pasti, D. Sorokin and M. Tonin, Phys. Lett. B **352**, 59 (1995).

[18] P. Pasti, D. Sorokin and M. Tonin, Phys. Rev. D **52**, R4277 (1995).

[19] P. Pasti, D. Sorokin and M. Tonin, Phys. Rev. D **55**, 6292 (1997).

[20] A. Maznytsia, C.R. Preitschopf and D. Sorokin, Nucl. Phys. B **539**, 438 (1999); [hep-th/9808049].

[21] Y.-G. Miao and H.J.W. Müller-Kirsten, Phys. Rev. D **62**, 045014 (2000).

[22] Y.-G. Miao, R. Manvelyan and H.J.W. Müller-Kirsten, Phys. Lett. B **482**, 264 (2000).

[23] Y.-G. Miao, H.J.W. Muller-Kirsten and D.K. Park, Nucl. Phys. B **612**, 215 (2001) [hep-th/0106197].

[24] Y.-G. Miao, H.J.W. Müller-Kirsten and D.K. Park, [hep-th/0105029].

[25] A. Khoudeir and N. Pantoja, Phys. Rev. D **53**, 5974 (1996).

[26] A. Bilal, [hep-th/9601007].

[27] W. Lerche, Nucl. Phys. Proc. Suppl. **83** (1997) [hep-th/9611190].

[28] L. Álvarez-Gaumé and S. F. Hassan, Fortsch. Phys. **159** (1997) [hep-th/9701069].

[29] A. Klemm, [hep-th/9705131].

[30] M. E. Peskin, [hep-th/9702094].

[31] P. Di Vecchia, [hep-th/9803026].

[32] P. Argyres, "Non-perturbative dynamics of four-dimensional supersymmetric field theories," in *Conformal Field Theory: New Non-Perturbative Methods in String and Field Theory*. Perseus, 2000. `http://www.physics.uc.edu/~argyres/661/`.

[33] P. Argyres, "Non-perturbative dynamics of four-dimensional supersymmetric field theories," `http://www.physics.uc.edu/~argyres/661/`.

[34] A. Bilal, [hep-th/0101055].

[35] Y. Tachikawa, [arXiv:1312.2684].

[36] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. **1**, 469 (1984).

[37] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, "Harmonic superspace," Cambridge Univ. Press (2001) 306 pp.

[38] R. Donagi and E. Witten, Nucl. Phys. B **460**, 299 (1996) [hep-th/9510101].

[39] D. S. Freed, Commun. Math. Phys. **203**, 31 (1999) [hep-th/9712042].

[40] D. Joyce, "Compact Manifolds with Special Holonomy," Oxford University Press. (2000) 448 pp.

[41] N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, Commun. Math. Phys. **108**, 535 (1987). [hep-th/0009157].

[42] D. Olive, P. West, eds., "Duality and Supersymmetric Theories," Cambridge University Press. (2000) 484 pp.

[43] P. A. M. Dirac, Phys. Rev. **74**, 817 (1948).

[44] C. Csaki, Y. Shirman and J. Terning, Phys. Rev. D **81**, 125028 (2010) [arXiv:1003.0448 [hep-th]].

[45] E. Witten, Phys. Lett. B **86**, 283 (1979).

[46] C. Bunster and M. Henneaux, Phys. Rev. D **86**, 065018 (2012) [arXiv:1207.1761].

[47] R. Donagi and E. Markman arXiv:alg-geom/9408004

[48] S. Cecotti, S. Ferrara, L. Girardello, Int. J. Mod. Phys. A **4**, 2475 (1989).

[49] J. Wess and J. Bagger, Princeton, USA: Univ. Pr. (1992) 259 p

[50] A. Karlhede, U. Lindström and M. Roček, Phys. Lett. B **147**, 297 (1984).

[51] U. Lindström and M. Roček, Commun. Math. Phys. **115**, 21 (1988).

[52] U. Lindström and M. Roček, Commun. Math. Phys. **128**, 191 (1990).

[53] D. Jain and W. Siegel, Phys. Rev. D **80**, 045024 (2009) [arXiv:0903.3588].

[54] D. Butter, [arXiv:1406.6235].

[55] U. Lindstrom and M. Rocek, Commun. Math. Phys. **293**, 257 (2010) [arXiv:0807.1366].

[56] P.C. Argyres, A.M. Awad, G.A. Braun and F.P. Esposito, JHEP **0307**, 060 (2003) [hep-th/0306118].

[57] L. Alvarez-Gaumé and D. Z. Freedman, Commun. Math. Phys. **80**, 443 (1981)