Entropy and irreversible processes in gravity and cosmology

Llorenç Espinosa-Portalés* and Juan García-Bellido†
Instituto de Física Teórica UAM-CSIC, Universidad Autónoma de Madrid
Cantoblanco, 28049 Madrid, Spain
**E-mail: llorenc.espinosa@uam.es*
†E-mail: juan.garciabellido@uam.es

General Relativity is *a priori* a theory invariant under time reversal. Its integration with the laws of thermodynamics allows for a formulation of non-equilibrium phenomena in gravity and the introduction of an arrow of time, i.e. the the breaking of such invariance. Even though most of the evolution of the universe takes place in local thermal equilibrium, the effects of irreversible processes on the expansion via entropic forces may be phenomenologically relevant. We review our previous work on the covariant formulation of non-equilibrium thermodynamics in General Relativity and the proposal to explain the recent cosmic acceleration from it.

Keywords: General Relativity, thermodynamics, non-equilibrium phenomena, entropic forces, cosmic acceleration.

1. Introduction

General Relativity is an extremely successful physical theory. More than a century after its formulation, its predictions continue to be valid at all probed scales, albeit an extension at short scales will be required in order to obtain a UV-complete quantum theory of gravity and resolve space-time singularities.

Thermodynamics is an even older discipline. Its fundamental laws seem to resist the passage of time and are still of great relevance today. On the one hand, thermodynamics may help in building the bridge between classical and quantum gravity, as the laws of black hole thermodynamics point towards the existence of a microphysical description of gravity yet to be understood.

On the other hand, the second law of thermodynamics, i.e. the growth of entropy, dictates the sign of the arrow of time. Physical laws are usually invariant under time inversion. The increase in entropy with time in out-of-equilibrium phenomena, however, allows one to distinguish the future-directed from the past directed description of a physical process.

There lacks a consistent and rigorous integration between General Relativity and the laws of thermodynamics. Understanding the very notion of the arrow of time is of particular interest for cosmology. In section 2 we argue for the need of going beyond reversible cosmology. This can be achieved for any space-time metric using variational techniques.[1] We review our main results in this new approach to non-equilibrium thermodynamics in General Relativity and present them in sections 3 to 6.

The growth of entropy associated to the causal horizon in open inflation scenarios may explain the current accelerated expansion of the universe within the general relativistic entropic acceleration (GREA) theory.[2] We briefly describe how this mechanism works in section 7 and finish with our conclusions.

2. Reversible cosmology

Let us begin the study of the problem of reversibility in gravity and cosmology by reviewing a prototypic case of reversible gravitational system: a homogeneous and isotropic universe. It is described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2 \right) , \tag{1}$$

where $a(t)$ is the scale factor, $k = -1, 0, 1$ is the curvature parameter corresponding to, respectively, an open, flat and closed universe; and $d\Omega_2^2$ is the solid angle element. This space-time is filled with a perfect fluid, described by the stress-energy tensor

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} , \tag{2}$$

where ρ and p are, respectively, the density and pressure of the fluid. The Einstein field equations for this metric and matter content deliver the dynamics for the scale factor, the well-known Friedmann equations

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho , \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) , \tag{3}$$

where $H = \dot{a}/a$ is the Hubble parameter. There is a constraint on the stress-energy tensor due to the Einstein field equations and the Bianchi identities, namely its covariant conservation $D_\mu T^{\mu\nu} = 0$. From this constraint one can derive the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0 . \tag{4}$$

However, one can also derive this equation from the second law of thermodynamics. Indeed, changes in entropy are related to changes in internal energy and work $TdS = \delta U + \delta W$. If we apply this to a region of fixed comoving volume $a(t)^3$ we get

$$T\frac{dS}{dt} = \frac{d}{dt}\left(\rho a^3\right) + p\frac{d}{dt}\left(a^3\right) . \tag{5}$$

If the expansion of the universe is reversible, we can set the LHS to 0 and recover the continuity equation. However, this is only true in thermodynamical equilibrium and, in general, entropy is a monotonically increasing function of time. Most of the expansion history of the universe is indeed adiabatic. However, it is out-of-equilibrium at certain key points such as (p)reheating, phase transitions or gravitational collapse. Allowing for a time-varying entropy implies the addition of a term in the continuity equation

$$\dot{\rho} + 3H(\rho + p) = \frac{T\dot{S}}{a^3} . \tag{6}$$

Combining this with the first Friedmann equation we obtain a modified, non-equilibrium second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p - \frac{T\dot{S}}{a^3 H}\right) . \tag{7}$$

In principle, this evolution equation does not seem to be compatible with the Einstein field equations. In order to achieve that, the laws of thermodynamics need to be rigorously incorporated into the computation of the equations of motion. This can be achieved by applying the variational formalism of non-equilibrium thermodynamics, developed in another context by Gay-Balmaz and Yoshimura,[3,4] to General Relativity.

3. Entropic forces in mechanics and field theory

Entropic forces emerge naturally in any physical system out of equilibrium. They are a consequence of the coarse-graining of physical degrees of freedom and the laws of thermodynamics, which impose entropy to be a monotonically increasing function of time. This breaks time reversibility.

The dynamics of the coarse-grained degrees of freedom is unknown or ignored and so they do not appear in the action of the physical system. It would seem that a variational treatment of an out-of-equilibrium system is not possible. However, both the extremal-action principle and the second law of thermodynamics can be merged consistently by imposing the latter as a constrain on the variational problem defined by the action.[3,4] On the other hand, the first law of thermodynamics is obtained from the symmetries of the problem.

3.1. *Entropic forces in mechanics*

Let us start by reviewing the emergence of entropic forces in a mechanical system. Consider the action

$$\mathcal{S} = \int dt L(q, \dot{q}, S) , \tag{8}$$

where the Lagrangian depends on the generalized coordinate $q(t)$, its time derivative $\dot{q}(t)$ and the entropy $S(t)$. The variation of the action gives

$$\delta\mathcal{S} = \int dt \left(\frac{\delta L}{\delta q}\delta q + \frac{\partial L}{\partial S}\delta S\right) . \tag{9}$$

Setting $\delta\mathcal{S} = 0$ defines the variational problem. In order to enforce the second law of thermodynamics, we need to impose the variational constraint

$$\frac{\partial L}{\partial S}\delta S = f\delta q , \tag{10}$$

which simply states the relationship between variations of the entropy and the generalized coordinate. If we plug this in the variation of the action, then we can readily obtain the equations of motion

$$\frac{\delta L}{\delta q} = \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = -f\,, \tag{11}$$

which is the Euler-Lagrange equation modified by the addition of the entropic force f. In order to know its precise form, we need to impose as well the phenomenological constraint, which is obtained by formally replacing the variations with time derivatives

$$\frac{\partial L}{\partial S}\dot{S} = f\dot{q}\,. \tag{12}$$

Note that usually temperature can be introduced as

$$T = -\frac{\partial L}{\partial S} > 0 \tag{13}$$

providing a clearer meaning to the variational and phenomenological constraint.

The imposition of the constraints and the emergence of an entropic force break the symmetry under time inversion. First, note that formally the Euler-Lagrange equation is invariant under the $t \to -t$ transformation, even with a non-vanishing entropic force. Next, the positiveness of the temperature and the change of entropy imposes $f\dot{q} < 0$. The entropic force has qualities of a generalized friction, as it opposes the coordinate velocity. Now, if one performs time inversion, this sign constraint becomes $f\dot{q} > 0$, as the temperature remains positive but entropy decreases with time. Before solving the equations of motion, $\dot{q}$ is still a degree of freedom and, thus, one must conclude a flip in the sign of f, which flips the overall sign of the Euler-Lagrange equation. Hence, one concludes that the emergence of an entropic force breaks symmetry under time reversal. The evolution of the system becomes irreversible.

3.2. *Entropic forces in classical field theory*

The extension of the variational formalism of non-equilibrium thermodynamics to the continuum is somewhat involved. We present a short-cut derivation that relies on the introduction of an additional constraint. We refer the reader to the appendix of Ref. 1 to check its equivalence with the full variational derivation originally presented in Ref. 4.

The action of a scalar field on Minkowski space-time contains now a dependency on a scalar function $s(t, \vec{x})$ that encodes information related to coarse-grained degrees of freedom

$$\mathcal{S} = \int d^4x \mathcal{L}\left(\phi, \partial_\mu \phi, s\right)\,. \tag{14}$$

In a similar fashion as before, the extremal-action principle needs to be supplemented by a variational constraint

$$\frac{\partial \mathcal{L}}{\partial s}\delta s = f\delta\phi\,, \tag{15}$$

so that the equation of motion becomes

$$\frac{\delta \mathcal{L}}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = -f\,, \tag{16}$$

which is nothing but the Euler-Lagrange equation of a scalar field with a new term of entropic origin. As before, one can usually introduce a temperature as

$$T = -\frac{\partial \mathcal{L}}{\partial s} > 0 \tag{17}$$

and we would like to interpret s as the entropy density. However, due to spatial fluxes entropy does not need to necessarily increase locally and, thus, the constraint would not have a fixed sign. Before proceeding, we inspect the corresponding phenomenological constraint

$$\frac{\partial \mathcal{L}}{\partial s} \partial_0 s = f \partial_0 \phi\,. \tag{18}$$

Instead, we interpret δs and $\partial_0 s$ as local entropy production and introduce a new function s_{tot} which is the actual entropy density and whose changes δs_{tot} and $\partial_0 s_{\text{tot}}$ are indeed total local changes of the entropy density. Both are related as

$$\partial_0 s = \partial_0 s_{\text{tot}} - \partial_i j_s^i\,, \tag{19}$$

where j_s^i is the entropy flux. This latter equation is an additional constraint we impose for the variational formalism to be consistent. Now one can check that $f\partial_0 \phi < 0$ and time reversibility is broken by the same argument used in the mechanics example.

3.3. *Entropic forces in presence of additional symmetries*

The generalization of the above formalism to higher order tensors or to representations of some internal symmetry group is straightforward. Let us consider a field tensor z of contravariant rank r, which is also in some representation of an internal symmetry labelled by an index A. Then one builds the variational constraint as

$$\frac{\partial \mathcal{L}}{\partial s} \delta s = f_{A;\mu_1,\ldots\mu_r} \delta z_A^{\mu_1,\ldots\mu_r}\,, \tag{20}$$

which delivers the equation of motion

$$\frac{\delta \mathcal{L}}{\delta z_A^{\mu_1,\ldots\mu_r},} = -f_{A;\mu_1,\ldots\mu_r}\,. \tag{21}$$

4. Entropic forces in General Relativity

The previous discussion makes us ready to study entropic forces in General Relativity. However, with the introduction of a dynamical space-time the very notion of time evolution becomes non-trivial. As we will see shortly, it is possible to obtain a modification of entropic origin to Einstein's field equation in the Lagrangian formulation of General Relativity. Its proper interpretation will require, nevertheless, the use of the Hamiltonian formalism.

4.1. *Lagrangian formulation*

Let us consider the Einstein-Hilbert action plus a matter term

$$\mathcal{S} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R + \int d^4x \mathcal{L}_m(g_{\mu\nu}, s)\,, \tag{22}$$

where $\kappa = 8\pi G$ is the gravitational coupling and we allow for the dependence of the matter Lagrangian on a function $s(t, \vec{x})$, which will have a similar interpretation to the one presented in the previous section. The extremal-action principle is supplemented by the variational constraint.

$$\frac{\partial \mathcal{L}_m}{\partial s} = \frac{1}{2} \sqrt{-g} f_{\mu\nu} \delta g^{\mu\nu}\,. \tag{23}$$

From the extremal-action principle and the variational constraint we obtain the modified Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa \left(T_{\mu\nu} - f_{\mu\nu} \right)\,, \tag{24}$$

where there is an additional term $f_{\mu\nu}$ of entropic origin. In order to obtain an expression for this term and to check the breaking of symmetry under time inversion we need to work in the Hamiltonian formulation of General Relativity.

4.2. *Hamiltonian formulation*

General Relativity admits a Hamiltonian formulation in the Arnowitt-Deser-Misner (ADM) formalism. Space-time is foliated in constant time hypersurfaces with normal unit vector n^μ, being the 4-metric split as

$$g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu\,, \tag{25}$$

where $h_{\mu\nu}$ is the 3-metric induced on the hypersurfaces. Analogously, one can parametrize the 4-metric in terms of the 3-metric h_{ij} and the lapse and shift functions N and N^i

$$ds^2 = -(Ndt)^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,. \tag{26}$$

Greek indices run from 0 to 3, while Latin ones do from 1 to 3 and are raised and lowered by h_{ij}. The normal vector can be written as

$$n_\mu = (-N, 0, 0, 0)\,. \tag{27}$$

Note that h_{ij} is the purely spatial part of $h_{\mu\nu}$ and is also the pull-back of $g_{\mu\nu}$ onto the hypersurface.

The Einstein-Hilbert action for this parametrization of the metric is given by the following gravitational Lagrangian:

$$\mathcal{L}_G = \sqrt{-g} R = \frac{1}{2\kappa} N \sqrt{h} \left({}^{(3)}R + K_{ij} K^{ij} - K^2 \right)\,, \tag{28}$$

where K_{ij} is the extrinsic curvature of the 3-hypersurface Σ and is given by the Lie derivative along the normal vector n

$$K_{ij} = \frac{1}{2}\pounds_n h_{ij} = \frac{1}{2N}\left(\partial_0 h_{ij} - \nabla_i N_j - \nabla_j N_i\right) , \tag{29}$$

where ∇ denotes the covariant derivative on Σ with respect to the 3-metric h_{ij}. Its trace and traceless part are:

$$\begin{aligned} K &= h^{ij}K_{ij} = \frac{1}{N}\left(\partial_0 \ln\sqrt{h} - \nabla_i N^i\right) \\ \bar{K}_{ij} &= K_{ij} - \frac{1}{3}K h_{ij} \, . \end{aligned} \tag{30}$$

Unlike the intrinsic curvature, described by the Riemann tensor $R^{\rho}_{\mu\nu\lambda}$ and its contractions, the extrinsic curvature is a quantity that depends on the embedding of a surface in a larger manifold.

We are now ready to introduce the Hamiltonian formulation of the theory. Note that the only quantity whose time derivative appears in the gravitational Lagrangian is the 3-spatial metric h_{ij} and, thus, it is the only dynamical or propagating d.o.f. Correspondingly, one defines its conjugate momentum as:

$$\Pi^{ij} = \frac{\partial \mathcal{L}_G}{\partial \dot{h}_{ij}} = \sqrt{h}\left(K^{ij} - K h^{ij}\right) . \tag{31}$$

With this, the gravitational Lagrangian can be rewritten as

$$\begin{aligned} \mathcal{L}_G &= N\sqrt{h}^{(3)}R - \frac{N}{\sqrt{h}}\left(\Pi_{ij}\Pi^{ij} - \frac{1}{2}\Pi^2\right) - 2\Pi^{ij}\nabla_i N_j \\ &= \Pi^{ij}\dot{h}_{ij} - N\mathcal{H} - N_i\mathcal{H}^i - 2\nabla_i\left(\Pi^{ij}N_j\right) , \end{aligned} \tag{32}$$

where $\Pi = h_{ij}\Pi^{ij}$ and we introduced the functions:

$$\begin{aligned} \mathcal{H} &= -\sqrt{h}^{(3)}R + \frac{1}{\sqrt{h}}\left(\Pi_{ij}\Pi^{ij} - \frac{1}{2}\Pi^2\right) \\ \mathcal{H}^i &= -2\nabla_j\left(h^{-1/2}\Pi^{ij}\right) . \end{aligned} \tag{33}$$

Since N and N_i are not dynamical variables, they merely enter the gravitational Lagrangian as Lagrange multipliers. One defines the gravitational Hamiltonian as:

$$\begin{aligned} \mathcal{H}_G &= \Pi^{ij}\dot{h}_{ij} - \mathcal{L}_G \\ &= N\mathcal{H} + N_i\mathcal{H}^i + \nabla_i\left(\Pi^{ij}N_j\right) , \end{aligned} \tag{34}$$

with the Hamiltonian and momentum constraints:

$$\begin{aligned} \frac{\delta\mathcal{H}_G}{\delta N} &= \mathcal{H} = 0 \\ \frac{\delta\mathcal{H}_G}{\delta N_i} &= \mathcal{H}^i = 0 \, . \end{aligned} \tag{35}$$

The Hamiltonian evolution equations are obtained from the variations of the action with respect to the metric and conjugate momentum

$$\delta\mathcal{S} = \int d^4x \left[\left(-\dot{\Pi}^{ij} - \frac{\delta\mathcal{H}_G}{\delta h_{ij}} - \kappa N\sqrt{h}\tilde{f}^{ij} + 2\kappa\frac{\partial\mathcal{L}_m}{\partial h_{ij}} \right) \delta h_{ij} + \left(\dot{h}_{ij} - \frac{\delta\mathcal{H}_G}{\delta\Pi^{ij}} \right) \delta\Pi^{ij} \right] , \tag{36}$$

where the variational constraint

$$\frac{\partial\mathcal{L}_m}{\partial s}\delta s = -\frac{1}{2}N\sqrt{h}\tilde{f}^{ij}\delta h_{ij} \tag{37}$$

was already implemented. Note that the minus sign arises from

$$\frac{\partial h_{ij}}{\partial h^{kl}} = -\frac{1}{2}\left(h_{ik}h_{jl} + h_{il}h_{jk}\right) \tag{38}$$

and that $\tilde{f}_{ij}$ is the pull-back of $f_{\mu\nu}$ onto the hypersurfaces. By setting the variation to 0 we obtain the two Hamilton equations

$$\begin{aligned} \frac{\delta\mathcal{H}_G}{\delta h_{ij}} &= -\dot{\Pi}^{ij} - \kappa N\sqrt{h}f^{ij} + 2\kappa\frac{\partial\mathcal{L}_m}{\partial h_{ij}} \\ \frac{\delta\mathcal{H}_G}{\delta\Pi^{ij}} &= \dot{h}_{ij} , \end{aligned} \tag{39}$$

which completes the derivation of the entropic modification to the gravitational equations of motion in the Hamiltonian formulation.

The tensor $\tilde{f}_{ij}$ can be obtained from the phenomenological constraint, which can now be stated rigorously. In the ADM formalism, a well-defined notion of time evolution is given by the flow along the normal vector n^μ. Hence, the time derivative is generalized to the Lie derivative along n^μ. Then the phenomenological constraint is given by

$$\frac{\partial\mathcal{L}}{\partial s}\pounds_n s = \frac{1}{2}N\sqrt{h}\tilde{f}_{ij}\pounds_n h^{ij} , \tag{40}$$

where growth in entropy by local processes is related to total entropy density growth by

$$\pounds_n s = \pounds_n s^{tot} - \nabla_i j^i_s . \tag{41}$$

Entropy produced locally is expected to grow over time, i.e. with the flow along the hypersurfaces, in compliance with the second law of thermodynamics. This completes the variational formulation of entropic forces in General Relativity.

4.3. *The Raychauduri equation*

Let us explore an immediate dynamical consequence of the inclusion of entropic forces, namely its effect on a congruence of worldlines with tangent vector n^μ. The congruence is then characterized by the tensor

$$\Theta_{\mu\nu} = D_\nu n_\mu = \frac{1}{3}\Theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} - a_\mu n_\nu , \tag{42}$$

where θ is the expansion rate of the congruence, $\sigma_{\mu\nu}$ is its shear or symmetric traceless part and $\omega_{\mu\nu}$ is its vorticity or antisymmetric part. If the worldline is not a geodesic, then the congruence suffers an acceleration given by:

$$a_\mu = n^\nu D_\nu n_\mu \,. \tag{43}$$

One can compute the Lie derivative of the expansion of the congruence along its tangent vector and find the Raychauduri equation[5]:

$$\pounds_n \Theta = -\frac{1}{3}\Theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}n^\mu n^\nu + D_\mu a^\mu \,. \tag{44}$$

Let us perform the standard analysis of the sign of this equation. It is clear that $\sigma_{\mu\nu}\sigma^{\mu\nu} > 0$ and $\Theta^2 > 0$. On the other hand, if the congruence is chosen to be orthogonal to the spatial hypersurfaces, as we have been considering, then the vorticity vanishes $\omega_{\mu\nu} = 0$. Lastly, it is left to consider the term $R_{\mu\nu}n^\nu n^\nu$, which we can rewrite with the help of the field equations:

$$R_{\mu\nu}n^\mu n^\nu = 8\pi G \left(T_{\mu\nu}n^\mu n^\nu + \frac{1}{2}T - f_{\mu\nu}n^\mu n^\nu - \frac{1}{2}f \right) . \tag{45}$$

If the strong energy condition is satisfied, then

$$T_{\mu\nu}n^\mu n^\nu \geq -\frac{1}{2}T \tag{46}$$

and, in the absence of intrinsic acceleration, $a_\mu = 0$, we can establish the bound:

$$\pounds_n \Theta + \frac{1}{3}\Theta^2 \leq 8\pi G \left(f_{\mu\nu}n^\mu n^\nu + \frac{1}{2}f \right) . \tag{47}$$

For a vanishing entropic force $f_{\mu\nu} = 0$, this means that an expanding congruence cannot indefinitely sustain its divergence and will eventually recollapse. On the contrary, a positive and sufficiently large entropic contribution can avoid such recollapse. This may become relevant for an expanding universe, but also to generic gravitational collapse and the singularity theorems.[6–8]

5. Sources of entropy

A main ingredient in the variational formulation of non-equilibrium thermodynamics in General Relativity is the inclusion of entropy at the Lagrangian level and the derivation of a notion of temperature from it. In this section we present two relevant examples: hydrodynamical matter, which is a prototypical case, and horizons.

5.1. *Entropy from hydrodynamical matter*

A classical fluid is the simplest matter content that can be considered in General Relativity and it is of particular relevance in Cosmology. Without paying attention to microphysical details, the Lagrangian of hydrodynamical matter can be written as

$$\mathcal{L}_m = -\sqrt{-g}\rho(g_{\mu\nu}, s) \,, \tag{48}$$

the temperature being then simply given by

$$T = -\frac{1}{\sqrt{-g}}\frac{\partial \mathcal{L}_m}{\partial s} = \frac{\partial \rho}{\partial s}\,. \tag{49}$$

This is analogous to the case of a mechanical system, where the Lagrangian is generically given by a kinetic and a potential energy

$$L = E_K(q,\dot{q}) - U(q,S) \tag{50}$$

and temperature can be defined as

$$T = \frac{\partial L}{\partial S} = -\frac{\partial U}{\partial S}\,. \tag{51}$$

Thus, the energy density of a fluid can be readily interpreted as the thermodynamic internal energy.

5.2. *Entropy from gravity and horizons*

Gravity itself has thermodynamical features. It is known since the discovery of the laws of black hole mechanics and their promotion to laws of black hole thermodynamics, allowed by the introduction of Bekenstein entropy and Hawking temperature. We propose to include the entropy associated with a horizon $\mathcal{H}$ by extending the Einstein-Hilbert actions with surface terms of Gibbons-Hawking-York (GHY) type

$$S_{GHY} = \frac{1}{8\pi G}\int_{\mathcal{H}} d^3y\sqrt{h}K\,, \tag{52}$$

where h is the determinant of the induced 3-metric on the horizon and K is the trace of its extrinsic curvature. Definitions are analogous to the ones used in the ADM formalism, but we stress that here the hypersurface of interest is a horizon and not constant-time hypersurfaces.

From the thermodynamic point of view, the GHY term contributes to the internal energy of the system. Hence, it can be rewritten as a function of the temperature and entropy of the horizon

$$S_{GHY} = -\int dt\, N(t)\, TS\,. \tag{53}$$

We have kept the lapse function $N(t)$, to indicate that the variation of the total action with respect to it will generate a Hamiltonian constraint with an entropy term together with the ordinary matter/energy terms. In order to illustrate this, let us now compute the GHY for the event horizon of a Schwarzschild black hole.

The space-time of a Schwarzschild black hole of mass M is described by the static metric

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2\,. \tag{54}$$

We foliate it with spherical hypersurfaces, i.e. their intersection with constant time hypersurfaces is a 2-sphere around the origin of coordinates. The corresponding normal vector is

$$n = -\sqrt{1 - \frac{2GM}{r}}\partial_r \,. \tag{55}$$

With this, the trace of the extrinsic curvature for such a sphere scaled by the metric determinant is

$$\sqrt{h}K = (3GM - 2r)\sin\theta \,. \tag{56}$$

Integrating over the angular coordinates and setting the 2-sphere at the event horizon, i.e. $r = 2GM$, and restoring for a moment $\hbar$ and c, the GHY becomes

$$S_{GHY} = -\frac{1}{2}\int dt M c^2 = -\int dt T_{BH} S_{BH} \,, \tag{57}$$

where T_{BH} is the Hawking temperature and S_{BH} is the Bekenstein entropy of the Schwarzschild black hole:

$$T_{BH} = \frac{\hbar c^3}{8\pi GM} \,, \quad S_{BH} = \frac{Ac^3}{4G\hbar} = \frac{4\pi GM^2}{\hbar c} \,. \tag{58}$$

This favors the interpretation of the GHY term of a horizon as a contribution to the internal energy in the thermodynamic sense.

6. Irreversible cosmology

We derived in section 4 a powerful, generic tool to describe non-equilibrium thermodynamic effects in gravity. In the Hamiltonian formulation of General Relativity it is possible to obtain the modified equations of motion and rigorously impose the time-evolution of the entropy as dictated by the second law of thermodynamics.

In section 2 we motivated the study of these phenomena by our interest in understanding the dynamics of irreversible cosmology and justifying its equations of motion. One can obtain them using the Hamilton equations.[1] Here, however, we present a slightly different approach. Due to the symmetries of the FLRW universe, homogeneity and isotropy, there is a preferred slicing and time evolution is well-defined even at the Lagrangian level. Therefore, we can obtain the equations of non-equilibrium cosmology by imposing these symmetries, i.e. making an ansatz for the metric

$$ds^2 = -N(t)^2 dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2\right) \,, \tag{59}$$

where the lapse function $N(t)$ accounts for the freedom in choosing the time coordinate, i.e. the symmetry under $t \to f(t)$. The Ricci scalar associated to this metric is

$$R = \frac{6}{a^2}\left(\frac{a\ddot{a}}{N^2} + \frac{\dot{a}^2}{N^2} + k\right) \,. \tag{60}$$

Let us stress that this result is imposed by symmetry, not dynamics. Without loss of generality, we can restrict the action to a region of comoving volume 1 and write it as

$$S = \int dt L = \frac{3}{8\pi G}\int dt N a\left(\frac{a\ddot{a}}{N^2} + \frac{\dot{a}^2}{N^2} + k\right) + \int dt N a^3 \mathcal{L}_m(N, a, S)\,. \tag{61}$$

Effectively, this action describes a mechanical system, for the scale factor $a(t)$ has no spatial dependency and we got rid of the integral over spatial coordinates. The first term can be rewritten using integration by parts in order to get only terms with at most the first derivative of a

$$S = \frac{3}{8\pi G}\int dt N a\left(-\frac{\dot{a}^2}{N^2} + k\right) + \int dt N a^3 \mathcal{L}_m(N, a, S)\,. \tag{62}$$

The variational constraint is here given by the usual expression for a mechanical system

$$\frac{\partial L}{\partial S}\delta S = f\delta a\,. \tag{63}$$

The Hamiltonian of the system is

$$H = \dot{a}\frac{\partial L}{\partial \dot{a}} - L = \frac{3}{8\pi G}\left(-\frac{\dot{a}^2 a}{N} - kaN\right) - N a^3 \mathcal{L}_m\,. \tag{64}$$

For an arbitrary lapse function $N(t)$ this can be rewritten as

$$H = N^2 a^3 \frac{\partial \mathcal{L}_m}{\partial N}\,, \tag{65}$$

which gives the Hamiltonian constraint of the system. On the other hand, the dynamics is obtained from the equation of motion for a

$$\frac{\delta L}{\delta a} = -f\,. \tag{66}$$

Let us now consider the matter Lagrangian to be that of a perfect fluid, i.e.

$$\mathcal{L}_m = -\rho(a, S)\,. \tag{67}$$

Its stress-energy tensor is given in terms of the density ρ and pressure p by

$$T^{\mu\nu} = (\rho + p)u^{\mu\nu} + p g^{\mu\nu} \tag{68}$$

and $u^m u = (N, 0, 0, 0)$ is the unit vector tangent to a comoving observer. Pressure is then obtained as

$$p = \frac{a^2}{3}T^{ij}\delta_{ij} = -\frac{1}{2a^4}\frac{\partial a^3\rho}{\partial a}\,. \tag{69}$$

Using the expressions for ρ and p and rearranging the terms in the Hamiltonian constraint and the equation of motion for $a(t)$ we arrive at the modified Friedmann equations

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} &= \frac{8\pi G}{3}\rho \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + 3p + \frac{f}{a^2}\right) . \end{aligned} \tag{70}$$

The expression for the entropic force F is obtained from the phenomenological constraint

$$\left(\frac{\partial L}{\partial S}\right)\dot{S} = -T\dot{S} = f\dot{a} < 0 \,, \tag{71}$$

which determines the sign $f < 0$ whenever dealing with an expanding universe $\dot{a} > 0$. We express finally the second Friedmann equation as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p - \frac{T\dot{S}}{a^2\dot{a}}\right) . \tag{72}$$

From this equation we can conclude that entropic forces generally drive an acceleration of the expansion of the universe. Whether this can dominate the dynamics of the scale factor will depend on the particular thermodynamic process. Most of the expansion history of the universe takes place in equilibrium. Out of equilibrium processes, such as (p)reheating, phase transitions or gravitational collapse are short-lived. Should their associated entropic force dominate, we still only expect a short period of accelerated expansion.

Symmetry under time inversion is broken by the same arguments presented in section 3. Hence, the Friedmann equations together with the phenomenological constraint, i.e. the second law of thermodynamics, describe cosmic irreversible dynamics.

We currently live in a universe that is undergoing an accelerated expansion. The possibility of explaining this by means of an entropic force is fascinating. In the next section we review our proposal to achieve this by means of the sustained growth of the entropy associated to a causal horizon.

7. Cosmic acceleration as an entropic force

The growth of entropy associated to the cosmic horizon may be responsible for the current observed accelerated expansion of the universe. The choice of horizon is in principle not unique. The only available one which can be defined locally in time is the cosmic apparent horizon, but it fails to significantly affect the expansion.[2]

There is another option in the framework of eternal inflation, according to which we live in an open universe nucleated by quantum tunneling from a false to a true vacuum. After nucleation the bubble universe undergoes its own inflationary era, which renders the local metric almost flat. However, due to the presence of the bubble wall, the true causal horizon is located at a finite distance. It induces an entropic fluid via GHY term with energy density

$$\rho_H\, a^2 = \frac{T_H S_H}{a} = \frac{x_0}{2G}\,\sinh(2a_0H_0\eta)\,, \quad x_0 \equiv \frac{1-\Omega_0}{\Omega_0} = e^{-2N}\left(\frac{T_{\rm rh}}{T_{\rm eq}}\right)^2(1+z_{\rm eq})\,, \tag{73}$$

where η is the conformal time, Ω_0 is the density parameter, $T_{\rm rh}$ is the reheating temperature, $T_{\rm eq}$ and $z_{\rm eq}$ are, respectively, the temperature and redshift at matter-radiation equality. Introducing $\tau = a_0H_0\eta$ one can write the second Friedmann equation in conformal time as

$$\left(\frac{a'}{a_0}\right)^2 = \Omega_M\left(\frac{a}{a_0}\right) + \Omega_K\left(\frac{a}{a_0}\right)^2 + \frac{4\pi}{3}\Omega_K\left(\frac{a}{a_0}\right)^2\sinh(2\tau)\,, \tag{74}$$

where Ω_M is the matter density paremeter and Ω_K is the curvature parameter. We call this the general relatvistic entropic acceleration (GREA) theory.

By solving this equation with cosmological parameters consistent with the CMB values (Planck 2018: $\Omega_M \simeq 0.31$, $\Omega_K \simeq 0.0006$, $h_0 \simeq 0.68$) and initial conditions deep in the matter era, $a_i(\tau) = a_0\,\Omega_M\tau^2/4$, we find generic accelerating behaviour beyond the scale factor $a \sim 1/2$ (*i.e.* $z \sim 1$), see Fig. 1. This is consistent with the current observed acceleration of the universe and may even resolve the Hubble tension,[9] providing a way to obtain from the CMB a present value of H_0 that is consistent with late-universe observations, see Fig. 2.

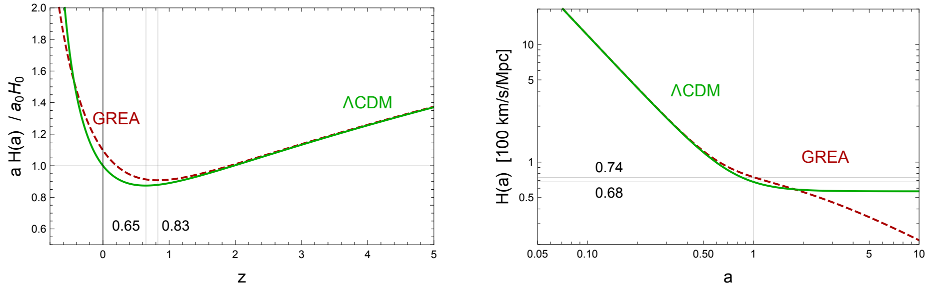

Fig. 1. The left plot shows the evolution of the inverse comoving horizon with the coasting point for each model, at $z \simeq 0.65$ for ΛCDM (in green) and $z \simeq 0.83$ for GREA (in red). The right plot shows the evolution of the rate of expansion. For GREA the present rate of expansion is approximately 74 km/s/Mpc, compared with the value of 68 km/s/Mpc predicted by ΛCDM, in agreement with the asymptotic value at the CMB.

8. Conclusions

The consistent inclusion of non-equilibrium phenomena in General Relativity leads to the modification of the Einstein field equations, as can be checked both in the Lagrangian and Hamiltonian formulations of the theory. This breaks symmetry under time inversion and allows for the introduction of an arrow of time.

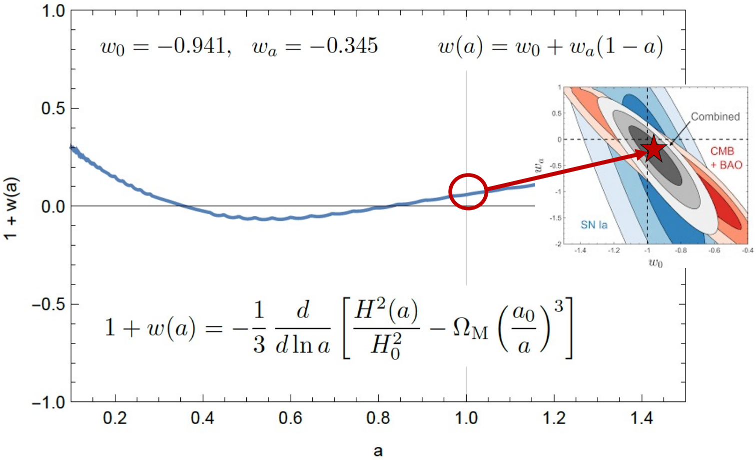

Fig. 2. The effective equation of state of the non-matter component of the GREA theory, as a function of thee sale factor. Note that the predicted effective PCL parameters (w_o, w_a) agree remarkably well with present observations.

In cosmology this implies the appearance of a term of entropic origin in the second Friedmann equation, which tends to accelerate the expansion of the universe as a result of the increase in entropy. Some physical processes such as (p)reheating, phase transitions or gravitational collapse may lead to phenomenologically relevant applications of this formalism. We look forward to further developments.

The sustained entropy growth associated to a causal horizon in the open universe scenario leads to an acceleration consistent with current observations and it may even solve the H_0 tension. Further research will be required to establish the full viability of the GREA theory.

Acknowledgments

The authors acknowledge support from the Spanish Research Project PGC2018-094773-B-C32 (MINECO-FEDER) and the Centro de Excelencia Severo Ochoa Program SEV-2016-0597. The work of LEP is funded by a fellowship from "La Caixa" Foundation (ID 100010434) with fellowship code LCF/BQ/IN18/11660041 and the European Union Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 713673.

References

1. L. Espinosa-Portales and J. Garcia-Bellido, Covariant formulation of non-equilibrium thermodynamics in General Relativity, *Phys. Dark Univ.* **34**, p. 100893 (2021).
2. J. García-Bellido and L. Espinosa-Portales, Cosmic acceleration from first principles, *Phys. Dark Univ.* **34**, p. 100892 (2021).

3. F. Gay-Balmaz and H. Yoshimura, A lagrangian variational formulation for nonequilibrium thermodynamics. part i: Discrete systems, *Journal of Geometry and Physics* **111**, 169 (2017).
4. F. Gay-Balmaz and H. Yoshimura, A lagrangian variational formulation for nonequilibrium thermodynamics. part ii: Continuum systems, *Journal of Geometry and Physics* **111**, 194 (2017).
5. R. M. Wald, *General Relativity* (Chicago Univ. Pr., Chicago, USA, 1984).
6. R. Penrose, Gravitational collapse and space-time singularities, *Phys. Rev. Lett.* **14**, 57 (1965).
7. R. Penrose, Gravitational collapse: The role of general relativity, *Riv. Nuovo Cim.* **1**, 252 (1969).
8. S. W. Hawking and R. Penrose, The Singularities of gravitational collapse and cosmology, *Proc. Roy. Soc. Lond. A* **314**, 529 (1970).
9. A. G. Riess, The Expansion of the Universe is Faster than Expected, *Nature Rev. Phys.* **2**, 10 (2019).