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Causal Stepanov–Bogoliubov's interacting fields of light-front quantum electrodynamics

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Abstract We explore the properties of Stepanov–Bogoliubov's interacting fields in the framework of null-plane causal perturbation theory. Considering light-front quantum electrodynamics, that have been previously studied in this framework, and extending the results from there, we show that, contrary to what occurs in instant dynamics, in light-front dynamics the interacting equations of motion are satisfied only in the adiabatic limit for the fields whose Feynman's propagators acquire an instantaneous term in the splitting of their (anti-)commutation distribution. The null-plane gauge condition, on the other hand, can be imposed on the interacting electromagnetic field for every switching function, at all orders of perturbation theory. We also show that the interacting fields satisfy finite Källén–Yang–Feldman's equations that are formally independent of the normalization of the retarded distributions.

1 Introduction

In a recent paper [1], the authors have established the axiomatic structure for the construction of local dynamical variables in the framework of instant dynamics causal perturbation theory (CPT), with an emphasis on the interacting fields. This is an important issue because CPT works exclusively with free fields in order to construct the scattering operator as a map in Fock's space. There is a reason for that: free fields are well-defined operator-valued distributions and do not conflict with Haag's theorem [2,3]. As a consequence, there are no divergences in CPT and no regularization scheme is needed. This is the reason why CPT is sometimes invoked in order to solve controversies that could arise when different regularization schemes give different answers for a physical problem [4,5].

CPT has been established in light-front dynamics [6,7] precisely in order to solve some ambiguities which appear in the light-front literature, as it is common in this dynamical form that the propagators acquire some instantaneous terms which are more singular – and hence more difficult to regularize – than the usual propagators in instant dynamics. Consequently, the difficulties in performing calculations raised some doubt about the equivalence between light-front and instant dynamics quantum field theory, from the beginning of its formulation [8–11] until the recent years [12–14] – also, a theoretical analysis of such equivalence problem from an axiomatic non-perturbative perspective can be found in Ref. [15]. For this reason, null-plane CPT was used to study light-front Yukawa's model [16] and quantum electrodynamics (QED) [17,18], unambiguously showing the equivalence with instant-form field theory in the perturbative regime, and thus proving that the developed framework can be satisfactorily used, not only for the deduction of technical information about light-front field theory, but also in practical perturbative calculations.

It is well-known, however, that the properties of free fields and those of the interacting ones are very different [19]. One main difference in the case of QED is that the compatibility between the null-plane gauge condition and the Lorenz's one, which holds in the free case, fails in the interacting field one. The usual equations of motion of the interacting fields are, however, also divergent, and therefore it is at least dubious to conclude technical details of the theory from them. Hence, it is a physically relevant task to obtain well-defined interacting equations of motion as well as their regime of validity, from which technical information can be confidently derived. In

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this manner, we want to establish some basic facts about the possibility of extending the framework of null-plane CPT to the non-perturbative regime, that can help the analysis of bound states in the future. We hope that learning how to write, for example, correct QCD Schwinger–Dyson's equations, will simplify actual calculations by avoiding complications related to regularization. In this paper, we start this program by the construction of interacting fields in null-plane dynamics, focusing on QED both in the non-covariant formulation [17] and in the covariant one [18].

This paper consists of the following parts. In Sect. 2, we will summarize the basic ideas of null-plane CPT in order to familiarize the reader with that theory. Section 3 is devoted to the establishment of the axioms over which the theory of dynamical variables is constructed with its perturbative realization. In Sect. 4 we start our study of the interacting fields of light-front quantum electrodynamics, focusing on the covariant formulation. The non-covariant formulation with the electromagnetic field quantized in the null-plane gauge is then studied in Sect. 5. With these results, we show in Sec. 6 that the interacting fields in light-front dynamics satisfy some finite Källén–Yang–Feldman's equations. The final Sect. 7 contains our conclusions.

2 Main ideas of null-plane CPT

In CPT one uses the adiabatic switching of the interaction [20], which means that its coupling constant is multiplied by a function $g \in \mathscr{S}(\mathbb{R}^4) : \mathbb{R}^4 \to \mathbb{R}$. The scattering operator $S(g)$ is now a functional of the switching function and must respect Bogoliubov–Medvedev–Polivanov's axioms [20–22]: (i) translation invariance, (ii) causality – now referred to the x^+ null-plane time –, (iii) unitarity, (iv) Lorentz's invariance and (v) stability of vacuum and one-particle sectors. Only axioms (i) and (ii) are used for the construction of null-plane CPT; the remaining ones are physical conditions used for normalization. CPT constructs the $S(g)$ operator as a formal series:

$$S(g) = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} \int dX T_n(X) g(X), \tag{1}$$

with $X := \{x_j \in \mathbb{M} \mid j = 1, \dots, n\}$ a set of points in space-time, $T_n(X)$ stands for $T_n(x_1; \cdots; x_n) \in \mathscr{S}'(\mathbb{R}^{4n})$, which is the n-point transition distribution, $g(X) \equiv g(x_1) \cdots g(x_n)$ and $dX \equiv d^4x_1 \cdots d^4x_n$.

The inverse operator $S(g)^{-1}$ is given as a perturbation series as well:

$$S(g)^{-1} = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} \int dX \widetilde{T}_n(X) g(X);$$

$$\widetilde{T}_n(X) = \sum_{r=1}^{n} (-1)^r \sum_{\substack{X_1,\dots,X_r \neq \emptyset \\ X_1 \cup \cdots \cup X_r = X \\ X_j \cap X_k = \emptyset, \forall j \neq k}} T_{n_1}(X_1) \cdots T_{n_r}(X_r). \tag{2}$$

The causality axiom implies that the transition distributions are chronologically ordered, in that $T_n(X) = T_m(X_2) T_{n-m}(X_1)$ for $X_1 < X_2$ and $[T_n(X_1); T_m(X_2)] = 0$ for $X_1 \sim X_2$. To construct them inductively, Stepanov [23] invented a procedure, then refined by Epstein and Glaser [24], based on the axiom of causality: Define the advanced distribution of order n as the following distribution:

$$A_n(Y; x_n) = \sum_{\substack{X \cup X' = Y \\ X \cap X' = \emptyset}} \widetilde{T}_m(X) T_{n-m}(X' \cup \{x_n\}), \tag{3}$$

and the retarded distribution of order n as

$$R_n(Y; x_n) = \sum_{\substack{X \cup X' = Y \\ X \cap X' = \emptyset}} T_{n-m}(X' \cup \{x_n\}) \widetilde{T}_m(X). \tag{4}$$

In these sums, the n-point distribution appears once. Separating it from the other terms:

$$A_n(Y; x_n) = T_n(Y \cup \{x_n\}) + A'_n(Y; x_n), \tag{5}$$

$$R_n(Y; x_n) = T_n(Y \cup \{x_n\}) + R'_n(Y; x_n). \tag{6}$$

The remaining terms A'_n and R'_n are called the advanced subsidiary distribution and the retarded subsidiary distribution, respectively. They do not contain T_n, but only the transition distributions T_m with $m \le n - 1$. The transition distribution of order n can then be obtained as

$$\begin{aligned} T_n(Y \cup \{x_n\}) &= A_n(Y; x_n) - A'_n(Y; x_n) \\ &= R_n(Y; x_n) - R'_n(Y; x_n). \end{aligned} \tag{7}$$

Thus, the n-point distribution can be found if we know the distributions T_m with $m \le n-1$ and the advanced or retarded distribution of order n, which can be found by splitting the causal distribution of order n,

$$\begin{aligned} D_n(Y; x_n) :&= R_n(Y; x_n) - A_n(Y; x_n) \\ &= R'_n(Y; x_n) - A'_n(Y; x_n). \end{aligned} \tag{8}$$

To obtain this causal distribution, therefore, only requires to know the subsidiary distributions. Suppose, then, that the causal distribution of order n was already constructed by means of the inductive procedure. In general, it will have the following form:

$$D_n(x_1; \cdots; x_n) = \sum_k d_n^k(x_1; \cdots; x_n) \, \mathbf{:} C_k(u^A) \mathbf{:}\, , \tag{9}$$

with $d_n^k(x_1; \cdots; x_n)$ a numerical distribution and $\mathbf{:} C_k(u^A) \mathbf{:}$ a Wick's monomial of the different quantized free operator fields u^A. These operator fields do not restrict the support of the distribution, hence it is sufficient to consider the splitting of the numerical distribution d_n^k, that has causal support. The advanced and retarded distributions will maintain the structure of the causal distribution:

$$A_n(x_1; \cdots ; x_n) = \sum_k a_n^k(x_1; \cdots ; x_n) \;{:}C_k(u^A){:}\,, \quad (10)$$

$$R_n(x_1; \cdots ; x_n) = \sum_k r_n^k(x_1; \cdots ; x_n) \;{:}C_k(u^A){:}\,, \quad (11)$$

with a_n^k and r_n^k the advanced and retarded parts, respectively, of the numerical distribution d_n^k. Using translation invariance, define the numerical distribution $d \in \mathscr{S}'(\mathbb{R}^{4n-4})$ as:

$$d(x) := d_n^k(x_1 - x_n; \cdots ; x_{n-1} - x_n; 0); \quad (12)$$

$$\mathrm{supp}(d) \subseteq \Gamma_{n-1}^+(0) \cup \Gamma_{n-1}^-(0), \quad (13)$$

which will be split as:

$$d = r - a;\ \mathrm{supp}(r) \subseteq \Gamma_{n-1}^+(0),\ \mathrm{supp}(a) \subseteq \Gamma_{n-1}^-(0). \quad (14)$$

We are using Schwartz's multi-index notation and denoting:

$$\Gamma_n^+(0) := \Big\{(x_1; \cdots ; x_n) \in \mathbb{M}^n \;\Big|\; \forall j \in \{1, \ldots , n\} : x_j^+ \geq 0 \wedge \Big(\exists x_k \in \overline{V^+}(0)(k \neq j) : x_j \in \widetilde{V}^+(x_k)\Big)\Big\},$$

with $V^{\pm}(x)$ the interior of the future or past, respectively, light-cone with vertex at the point x, $\overline{V^{\pm}}(x)$ its closure and $\widetilde{V}^{\pm}(x)$ the union of its closure and the x^- axis. An analogous definition holds for $\Gamma_n^-(0)$.

In order to perform the splitting, it is very important to know how the distribution behaves near the splitting region. In null-plane dynamics, since the planes of constant x^+ intersect the light-cone along the entire x^--axis, it is the behaviour of the distribution $d(x)$ near the x^--axis that is essential for the splitting procedure.

Definition 1 Let $d \in \mathscr{S}'(\mathbb{R}^m)$ be a distribution, and let ρ be a continuous positive function. If the limit

$$\lim_{s\to 0^+} \rho(s) s^{3m/4} d\left(sx^+; sx^\perp; x^-\right) = d_-(x) \quad (15)$$

exists in $\mathscr{S}'(\mathbb{R}^m)$ and is non-null, then the distribution d_- is called the *quasi-asymptotics* of d at the x^--axis, with regard to the function ρ.

The function $\rho(s)$ can be shown to be an auto-model function [25]: $\forall a > 0$: $\lim_{s\to 0^+} \rho(as)/\rho(s) = a^{\omega_-}$, for some $\omega_- \in \mathbb{R}$, called the singular order of the distribution d at the x^--axis. It determines the space of test functions on which the retarded distribution can be defined in principle: (1) For $\omega_- < 0$, it is the entire Schwartz's space $\mathscr{S}$, and the retarded distribution can be obtained, in momentum space, as

$$\hat{r}(p) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\hat{d}(p_+ - k; \boldsymbol{p})}{k + i0^+} dk. \quad (16)$$

(2) For $\omega_- \geq 0$, the retarded distribution can only be defined (in principle) on the space of test functions for which the first ω_- derivatives at the x^--axis vanish. It can then be extended to the whole $\mathscr{S}$, resulting in the retarded distribution with normalization line $(q_+; q_\perp; p_-)$:

$$\hat{r}_q(p) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{dk}{k + i0^+} \Big\{\hat{d}(p_+ - k; \boldsymbol{p}) - \sum_{|c|=0}^{\lfloor\omega_-\rfloor} \frac{1}{c!} (p_{+,\alpha} - q_{+,\alpha})^c D_{+,\alpha}^c \hat{d}(q_+ - k; q_\perp; p_-)\Big\}. \quad (17)$$

Finally, if (r_1, a_1) and (r_2, a_2) are two solutions of the splitting problem, then by Eq. (8) we have that $r_1 - a_1 = r_2 - a_2$, so that $r_1 - r_2 = a_1 - a_2$. Since the left-hand side of this equation has support on Γ^+, while its right-hand side has support in Γ^-, the difference $r_1 - r_2$ can only have support on $\Gamma^+ \cap \Gamma^-$, which is the x^--axis. Hence, r_1 and r_2 can differ only by normalization terms of the form

$$\hat{r}_1(p) - \hat{r}_2(p) = \sum_{|b|=0}^{M} \widehat{C}_b\,(p_-)\, p_{+,\perp}^b, \quad (18)$$

with $\widehat{C}_b\,(p_-)$ some distributions of the variable p_-. The procedure of determining these unknown distributions by the imposition of physical requirements is called the normalization process.

3 Construction of local dynamical variables in light-front dynamics

In this section we will follow the path of Ref. [1], adapting it to light-front dynamics. We denote by $B(x)$ a generic dynamical variable (Wick's monomial) of the quantized free fields, and by $\mathbf{B}(x; g)$ its interacting version. The axioms that define it are:

Axiom 1 (Initial condition) The dynamical variable $\mathbf{B}(x; g)$ satisfies the *initial condition*

$$\mathbf{B}(x; 0) = B(x). \quad (19)$$

Axiom 2 (Locality) The *retarded dynamical variable*, $\mathbf{B}_{\mathrm{ret}}(x; g)$ and the *advanced dynamical variable*, $\mathbf{B}_{\mathrm{av}}(x; g)$, are local functionals, in that

$$y \gtrsim x \;\Rightarrow\; \frac{\delta \mathbf{B}_{\mathrm{ret}}(x; g)}{\delta g(y)} = 0; \quad (20)$$

$$y \lesssim x \;\Rightarrow\; \frac{\delta \mathbf{B}_{\rm av}(x;g)}{\delta g(y)} = 0. \tag{21}$$

The names "advanced" and "retarded" are justified as the "current" $g(x)$ only affects them in the future or in the past, respectively. The necessity to distinguish these two cases arises because the free dynamical variable can be considered as the one before the interaction or the one after it, since in both cases the free fields generate the asymptotic Fock's spaces.

The expression for the interacting dynamical variables in function of the scattering operator $S(g)$ can be understood by casting the causality condition in (functional) differential form. Following Scharf [25] – see also Ref. [1] –, let $g_1, g_2, f_1, f_2 \in \mathscr{S}\left(\mathbb{R}^4\right)$ be two switching functions such that:

$$\mathrm{supp}(g_1),\ \mathrm{supp}(f_1) \subset \left]-\infty; x^+\right[,$$
$$\mathrm{supp}(g_2),\ \mathrm{supp}(f_2) \subset \left]x^+; +\infty\right[.$$

The causality axiom and the pseudo-unitarity[1] of $S(g)$ imply that – we denote $g := g_1 + g_2$ –:

$$\begin{aligned} &S(g+\varepsilon_1 f_1+\varepsilon_2 f_2)S(g+\varepsilon_1 f_1)^K \\ &\quad = S(g_2+\varepsilon_2 f_2)S(g_1+\varepsilon_1 f_1)S(g_1+\varepsilon_1 f_1)^K S(g_2)^K \\ &\quad = S(g_2+\varepsilon_2 f_2)S(g_2)^K \end{aligned} \tag{22}$$

does not depend on ε_1, from which it follows that:

$$\frac{\partial}{\partial \varepsilon_1}\left[\frac{\partial}{\partial \varepsilon_2} S(g+\varepsilon_1 f_1+\varepsilon_2 f_2)\Big|_{\varepsilon_2=0} S(g+\varepsilon_1 f_1)^K\right]\Bigg|_{\varepsilon_1=0} = 0. \tag{23}$$

In functional differential language this is written as:

$$\int d^4x d^4y f_1(x) f_2(y) \frac{\delta}{\delta g(x)}\left(\frac{\delta S(g)}{\delta g(y)} S(g)^K\right) = 0, \tag{24}$$

which implies, since the functions f_1 and f_2 are arbitrary (within their established domain) and by the fundamental lemma of variational calculus [26], that:

$$x \lesssim y \;\Rightarrow\; \frac{\delta}{\delta g(x)}\left(\frac{\delta S(g)}{\delta g(y)} S(g)^K\right) = 0. \tag{25}$$

Similarly, starting with the product $S(g+\varepsilon_2 f_2)^K S(g+\varepsilon_1 f_1+\varepsilon_2 f_2)$ it can be shown that the causality axiom and the condition of pseudo-unitarity of the scattering operator imply that:

$$x \gtrsim y \;\Rightarrow\; \frac{\delta}{\delta g(x)}\left(S(g)^K \frac{\delta S(g)}{\delta g(y)}\right) = 0. \tag{26}$$

Extrapolating this idea to the construction of general dynamical variables [20,27], we define the "extended scattering operator", $\mathsf{S}(g;g_B)$, as the operator, constructed via the causal method, from the first-order term:

$$\mathsf{S}_1(g;g_B) := \int d^4x \left(T_1(x)g(x) - iB(x)g_B(x)\right), \tag{27}$$

with g_B an auxiliary switching function belonging to some appropriate Schwartz's space, also called "classical current for the interacting dynamical variable $\mathbf{B}$". It must have the necessary characteristics such that the product $B(x)g_B(x)$ is a scalar, commutative and anti-pseudo-self-adjoint distribution. On the other hand, in order that the causal construction to be possible, the operator $\mathsf{S}(g;g_B)$ must satisfy an extended causality axiom, that we state in differential form – see Eqs. (25) and (26) –:

Axiom 3 (*Extended causality*) The extended scattering operator $\mathsf{S}(g;g_B)$ is causal in the extended sense, in that the following relations hold for all $g_j, g_k \in \{g; g_B\}$:

$$x \lesssim y \;\Rightarrow\; \frac{\delta}{\delta g_j(x)}\left(\frac{\delta \mathsf{S}(g;g_B)}{\delta g_k(y)} \mathsf{S}(g;g_B)^K\right) = 0; \tag{28}$$

$$x \gtrsim y \;\Rightarrow\; \frac{\delta}{\delta g_j(x)}\left(\mathsf{S}(g;g_B)^K \frac{\delta \mathsf{S}(g;g_B)}{\delta g_k(y)}\right) = 0. \tag{29}$$

Accordingly, Stepanov proposed that in the general case the interacting dynamical variables are given by the *formulae*

$$\begin{aligned} \mathbf{B}_{\rm av}(x;g) &= i\, \frac{\delta \mathsf{S}(g;g_B)}{\delta g_B(x)} \mathsf{S}(g)^K \Bigg|_{g_B=0}, \\ \mathbf{B}_{\rm ret}(x;g) &= i\, \mathsf{S}(g)^K \frac{\delta \mathsf{S}(g;g_B)}{\delta g_B(x)} \Bigg|_{g_B=0}, \end{aligned} \tag{30}$$

that immediately satisfy the locality axiom. Additionally, using that $\mathsf{S}(0;0) = 1$, from Eqs. (27) and (30) it follows that $\mathbf{B}_{\substack{\rm av\\ \rm ret}}(x;g)$ satisfies the initial condition:

$$\mathbf{B}_{\substack{\rm av\\ \rm ret}}(x;0) = B(x). \tag{31}$$

These interacting dynamical variables satisfy the following properties, whose proof in light-front dynamics is identical to the one in instant dynamics [1].

Theorem 4 *The dynamical variable* $\mathbf{B}_{\substack{av\\ ret}}(x;g)$ *defined by Eqs.* (30) *verifies the micro-causality condition*:

$$x \sim y \;\Rightarrow\; \left[\mathbf{B}_{\substack{av\\ ret}}(x;g); \mathbf{B}_{\substack{av\\ ret}}(y;g)\right]_\mp = 0, \tag{32}$$

in which the commutator holds if $\mathbf{B}_{\substack{av\\ ret}}(x;g)$ *is of boson character, while the anti-commutator holds if it is of fermion character.*

[1] In these equations, the involution K is the one according to which, in the most general case, $S(g)$ is pseudo-unitary. This condition allows to maintain the generality and the possibility of using the technique in the covariant gauge theory [18]. On the other hand, since the K involution is equal to the usual † in the physical subspace, to consider that instead of this has no physical consequences.

Theorem 5 *The dynamical variable* $\mathbf{B}_{\substack{av\\ret}}(x; g)$, *defined by Eq.* (30), *once applied to a test function belonging to the same space as* g_B, *defines a pseudo-self-adjoint operator*:

$$\mathbf{B}_{\substack{av\\ret}}(f; g)^K = \mathbf{B}_{\substack{av\\ret}}(f; g)\ ;$$
$$\mathbf{B}_{\substack{av\\ret}}(f; g) := \int d^4x f(x)\mathbf{B}_{\substack{av\\ret}}(x; g). \tag{33}$$

These two properties mean that the $\mathbf{B}_{\substack{\text{av}\\\text{ret}}}(f; g)$ operators are well-qualified to describe physical observables.

Now we turn to the perturbative construction of the interacting dynamical variables. They will be obtained as a formal series:

$$\mathbf{B}_{\substack{\text{av}\\\text{ret}}}(x; g) = B(x) + \sum_{n\in\mathbb{N}} \frac{1}{n!}\int B_n^{\substack{\text{av}\\\text{ret}}}(x; Y_n)g(Y_n)dY_n, \tag{34}$$

with $Y_n := \{y_1; \cdots ; y_n \mid \forall j \in I_n : y_j \in \mathbb{M}\}$ a set of n points in space-time. Here we also use, as for the $S(g)$ series, the following notations: $B_n^{\substack{\text{av}\\\text{ret}}}(x; Y_n) \equiv B_n^{\substack{\text{av}\\\text{ret}}}(x; y_1; \cdots ; y_n)$, $g(Y_n) \equiv g(y_1)\cdots g(y_n)$, $dY_n \equiv d^4y_1 \cdots d^4y_n$.

Note that the initial condition for the dynamic variable is respected by the formal series in Eq. (34), since by putting $g \equiv 0$ we obtain directly that $\mathbf{B}_{\substack{\text{av}\\\text{ret}}}(x; 0) = B(x)$. On the other hand, the locality axiom [Eqs. (20) and (21)] requires the first functional derivative of the dynamical variable, which according to Eq. (34) is:

$$i\frac{\delta\mathbf{B}_{\substack{\text{av}\\\text{ret}}}(x; g)}{\delta g(y)} = \sum_{n\in\mathbb{N}_0} \frac{1}{n!}\int B_{n+1}^{\substack{\text{av}\\\text{ret}}}(x; y; Y_n)g(Y_n)dY_n. \tag{35}$$

Substituting this expression into Eqs. (20) and (21), we find that the locality axiom translates into the conditions:

$$\exists y_j \in Y_n :\ y_j \lesssim x \ \Rightarrow\ B_n^{\text{av}}(x; Y_n) = 0,$$
$$\exists y_j \in Y_n :\ y_j \gtrsim x \ \Rightarrow\ B_n^{\text{ret}}(x; Y_n) = 0. \tag{36}$$

Or, in other words:

$$\operatorname{supp}\left(B_n^{\text{av}}(x; Y_n)\right) \subseteq \Gamma_n^+(x),$$
$$\operatorname{supp}\left(B_n^{\text{ret}}(x; Y_n)\right) \subseteq \Gamma_n^-(x). \tag{37}$$

This is to say, the distributions $B_n^{\text{av}}(x; Y_n)$ and $B_n^{\text{ret}}(x; Y_n)$ are, respectively, advanced and retarded with respect to the set Y_n.

In order to perform explicit calculations, it will be necessary to obtain a general practical formula for the distributions $B_n^{\substack{\text{av}\\\text{ret}}}(x; Y_n)$. Because of Eq. (30), to calculate the dynamical variables we only need the linear dependence of $\mathsf{S}(g; g_B)$ on g_B. Accordingly, we have that

$$\mathsf{S}(g; g_B) = \sum_{n\in\mathbb{N}_0} \frac{1}{n!}\sum_{l=0}^{n}\int \Big(T_n^{-iB,l}(X_n)g\left(X_n \setminus \{x_l\}\right) g_B(x_l)$$
$$+\mathscr{O}(g_B^2)\Big)\, dX_n, \tag{38}$$

with

$$T_n^{-iB,l}(X_n) := \mathfrak{T}_+\{T_1(x_1)\cdots$$
$$\cdots T_1(x_{l-1})\,(-iB(x_l))\,T_1(x_{l+1})\cdots T_1(x_n)\} \tag{39}$$

the chronological product of $n-1$ T_1 distributions and one $-iB$ distribution at the point x_l. The required functional derivative is then

$$i\left.\frac{\delta\mathsf{S}(g; g_B)}{\delta g_B(x)}\right|_{g_B=0} = \sum_{n\in\mathbb{N}_0} \frac{1}{n!}\int T_{n+1}^B(Y_n; x)g(Y_n)dY_n. \tag{40}$$

We have used here the equality of all the distributions $T_n^{-iB,l}(X_n)$, $l \in I_n$, which holds because of the symmetry of the transition distributions. Moreover, since there is only one $-iB$ distribution, the $-i$ factor multiplies the i factor that appears in the definition of the dynamical variable; this is equivalent to have a B term in the chronological product, as is manifest in our notation: $T_{n+1}^B(Y_n; x)$. Multiplying by $\mathsf{S}(g; 0)^K = S(g)^{-1}$ and grouping the terms:

$$\mathbf{B}_{\text{av}}(x; g) = \sum_{n\in\mathbb{N}_0} \frac{1}{n!}\int dY_n g(Y_n)$$
$$\times \sum_{\substack{Y_j\cup Y_k=Y_n\\ Y_j\cap Y_k=\emptyset}} T_{j+1}^B(Y_j; x)\widetilde{T}_k(Y_k), \tag{41}$$

$$\mathbf{B}_{\text{ret}}(x; g) = \sum_{n\in\mathbb{N}_0} \frac{1}{n!}\int dY_n g(Y_n)$$
$$\times \sum_{\substack{Y_j\cup Y_k=Y_n\\ Y_j\cap Y_k=\emptyset}} \widetilde{T}_k(Y_k)T_{j+1}^B(Y_j; x). \tag{42}$$

From these relations we can conclude that:

$$B_n^{\text{av}}(x; Y_n) = R_{n+1}^B(Y_n; x), \tag{43}$$
$$B_n^{\text{ret}}(x; Y_n) = A_{n+1}^B(Y_n; x), \tag{44}$$

in accordance with Eq. (37).

It is worth noting that the above equations are also valid for $n = 0$ if we define $R_1^B(x) := B(x)$ and $A_1^B(x) := B(x)$, hence $D_1^B(x) = 0$. We convey in defining $R'^B_1(x) := 0$ and $A'^B_1(x) := 0$ and to extend the regime of validity of the relation $D_n = R_n - A_n = R'_n - A'_n$ to $n = 1$. In summary, we adopt by convention the following equalities:

$$A_1,\, R_1 := T_1 = -\widetilde{T}_1,\;\; A'_1,\, R'_1 := 0,\;\; D_1 := R_1 - A_1 = 0. \tag{45}$$

Equations (43) and (44) inductively determine all the terms in the interacting dynamical variable once the distributions $T_1(x)$ and $B(x)$ are known. Such construction is unique modulo normalization terms which arise in the obtaining of the retarded distributions by splitting, similarly as in Sect. 2; those normalization terms have support on the x^--axis.

4 Covariant interacting electromagnetic field

From now on we will focus our attention on those dynamical variables defined by the initial condition of being equal to the free quantized field, $B(x) = u(x)$. They are called "interacting fields". In the case of quantum electrodynamics, however, we met a duplicity in its description, as it can be non-covariant, if Fock's space consists only of physical states when the electromagnetic field is subjected to the null-plane gauge condition [17], or covariant, if Fock's space is extended, by the quantum gauge invariance technique, to contain non-physical states as well (non-physical polarizations and ghost field states) [18]. In both cases, however, the transition distribution of the first order is

$$T_1(x) = i j_a(x) A^a(x),$$

with $j_a(x)$ the free current of the material fields. Note that we are not writing the coupling constant (electric charge) here; it will be absorbed, to lighten the notation, into the adiabatic switching function: $e(x) := eg(x)$. The second order causal distribution for the construction of the interacting field $\mathbf{A}^a_{\substack{\text{av}\\ \text{ret}}}(x; e)$ is

$$D_2^{Aa}(y; x) = \left[T_1(y); A^a(x)\right] = j_b(y) D^{ab}(x - y), \tag{46}$$

and its retarded and advanced parts, with respect to the point x, are

$$\begin{aligned} R_2^{Aa}(y; x) &= -j_b(y)(D^{ab})^{\text{av}}(x - y),\\ A_2^{Aa}(y; x) &= -j_b(y)(D^{ab})^{\text{ret}}(x - y). \end{aligned} \tag{47}$$

From this it follows that the interacting electromagnetic field, up to first order corrections, will be given by:

$$\mathbf{A}^a_{\substack{\text{av}\\ \text{ret}}}(x; e) = A^a(x) - \int e(y) j_b(y) (D^{ab})^{\substack{\text{av}\\ \text{ret}}}(x - y) d^4y + \mathscr{O}(e^2). \tag{48}$$

The difference between the covariant and non-covariant treatments lies in the different advanced and retarded distributions of those two formulations. Hence the interacting field in them will be different, and therefore it is expected that they satisfy different equations of motion. This obliges us to study the two formulations separately. In the present section we will treat the covariant formulation, which is simpler because of its similarity with the instant dynamics case, studied in Ref. [1]. The non-covariant theory will be exposed in the next section.

4.1 Equations of motion

We establish first that Theorem 8 in Ref. [1] is also valid in light-front dynamics. The proof of this fact can be found in Appendix A.

Theorem 6 *Let $\boldsymbol{\mathcal{U}}_{\substack{av\\ ret}}(x; g)$ be an interacting field defined by the initial condition of being equal to the quantized free field operator $u(x)$ satisfying the free equation of motion*

$$\Delta_x u(x) = 0, \tag{49}$$

with Δ_x a differential operator. Let $j_u(x)$ be the free current for the field $u(x)$, in that the first-order transition distribution is $T_1(x) = i\;\; :j_u(x)u(x): $. If the advanced and retarded parts of the (anti-)commutation distribution of the free field $u(x)$, $d_{av,ret}$, are normalized in such a way that $\Delta_x d_{av,ret}(x) = \delta(x)$, and if $\forall F \in \{R; A; T\}$:

$$\Delta_x F_2^u(y; x) = F_1^{j_u}(x)\delta(y - x) = j_u(x)\delta(y - x), \tag{50}$$

then the dynamical variables $\boldsymbol{\mathcal{U}}_{\substack{av\\ ret}}(x; g)$ and $\boldsymbol{\mathcal{J}}^{\substack{av\\ ret}}_{\boldsymbol{\mathcal{U}}}(x; g)$, this last one defined by the initial condition $\boldsymbol{\mathcal{J}}^{\substack{av\\ ret}}_{\boldsymbol{\mathcal{U}}}(x; 0) = j_u(x)$, satisfy the interacting equation of motion

$$\Delta_x \boldsymbol{\mathcal{U}}_{\substack{av\\ ret}}(x; g) = g(x) \boldsymbol{\mathcal{J}}^{\substack{av\\ ret}}_{\boldsymbol{\mathcal{U}}}(x; g). \tag{51}$$

This theorem will be applicable to the covariant formulation of quantum electrodynamics. Note, in first place, that the free electromagnetic field operator, $A^a(x)$, satisfies Klein–Gordon–Fock's equation,

$$\Box A^a(x) = 0, \tag{52}$$

and by no means the equation $(g_{ab}\Box - \partial_a\partial_b)\, A^a(x) = 0$, which is the equation for the classical electromagnetic field when it is not submitted to Lorenz's gauge condition. This is the case even though the mentioned condition, $\partial_a A^a(x) = 0$, is not an identity for the quantized free field, but only for its matrix elements between elements of the physical Fock's space, $\mathcal{F}_{\text{phys}}$. We then expect that the interacting electromagnetic field satisfy the equation[2]:

[2] Note also that it is this equation the one solved by Källén [28] in his perturbation theory in Heisenberg's picture. However, as emphasized in Ref. [1], Källén's currents are not well defined, as they are defined by products of interacting fields, which are also divergent. In CPT, the (finite) current is a dynamical variable on its own, defined by the initial condition of being equal to the free current, and it is constructed by the already explained causal method.

$$\Box \mathbf{A}^a_{\substack{\text{av}\\ \text{ret}}}(x;e) = -e(x)\mathbf{J}^a_{\substack{\text{av}\\ \text{ret}}}(x;e) \quad , \quad \mathbf{J}^a_{\substack{\text{av}\\ \text{ret}}}(x;0) := j^a(x). \tag{53}$$

That expectation is satisfied as a consequence of Theorem 6, because for the covariant theory it holds that $(D^{ab})^{\substack{\text{av}\\ \text{ret}}}(x-y) = g^{ab} D_0^{\substack{\text{av}\\ \text{ret}}}(x-y)$, and $\Box_x D_0^{\substack{\text{av}\\ \text{ret}}}(x-y) = \delta(x-y)$. Together with Eq. (47) this yields:

$$\begin{aligned}\Box_x R_2^{Aa}(y;x) &= -j_b(y) g^{ab}\delta(x-y) \\ &= -\delta(x-y) j^a(y) \\ &= -\delta(x-y) R_1^{ja}(x),\end{aligned} \tag{54}$$

and the same is true for the advanced distribution. Such equations and Eq. (52) are sufficient to establish Eq. (53) at all orders of perturbation theory.

Consider now, for definiteness, that the matter current is due to a Dirac's field ψ:

$$j_a(x) = :\overline{\psi}(x)\gamma_a\psi(x): \; . \tag{55}$$

The equation of motion for the corresponding interacting field $\Psi_{\substack{\text{av}\\ \text{ret}}}$ is

$$(i\partial\!\!\!/ - m)\Psi_{\substack{\text{av}\\ \text{ret}}}(x;e) = -e(x)\left(\mathbf{A}^a\gamma_a\Psi\right)_{\substack{\text{av}\\ \text{ret}}}(x;e). \tag{56}$$

To show this requires, by Theorem 6, to show only that

$$(i\partial\!\!\!/_x - m) R_2^{\psi}(y;x) = -\delta(x-y) R_1^{A^a\gamma_a\psi}(y) \tag{57}$$

and a similar formula for the advanced distributions. The validity of this equation is obtained by direct construction: The required causal distribution is

$$D_2^{\psi}(y;x) = [T_1(y);\psi(x)] = -S(x-y)A^a(y)\gamma_a\psi(y). \tag{58}$$

Its retarded and advanced parts with respect to the point x are

$$\begin{aligned} R_2^{\psi}(y;x) &= S^{\text{av}}(x-y)A^a(y)\gamma_a\psi(y), \\ A_2^{\psi}(y;x) &= S^{\text{ret}}(x-y)A^a(y)\gamma_a\psi(y). \end{aligned} \tag{59}$$

However, as we have seen in Ref. [17], they are not unique, but admit a normalization term of singular order $\omega_- = 0$, because the splitting formula leads to the appearance of an instantaneous term of that singular order in the retarded distribution – and, consequently, also in the advanced one, in order that the equation $S^{\text{ret}} - S^{\text{av}} = S$ remains valid –. Here it is crucial to take, as required by the hypothesis of Theorem 6, S^{av} and S^{ret} so as to satisfy the equations $(i\partial\!\!\!/ - m)S^{\substack{\text{av}\\ \text{ret}}} = -\delta$, i.e., to choose the normalization term to cancel the instantaneous one. We have seen that it is precisely this choice of the normalized solution the one that establishes the equivalence with instant dynamics. Being that way, Eq. (57) is satisfied and the equation of motion [Eq. (56)] is established. The expression of Dirac's field at first order in the coupling constant is

$$\Psi_{\substack{\text{av}\\ \text{ret}}}(x;g) = \psi(x) + \int S^{\substack{\text{av}\\ \text{ret}}}(x-y)A^a(y)\gamma_a\psi(y)e(y)d^4y + \mathscr{O}(e^2). \tag{60}$$

In summary, the interacting fields satisfy the equations of motion if and only if the normalization used is the one in which there are no instantaneous terms in the retarded (and advanced) distributions of the fermion field.

The equations of motion in the light-front literature, nonetheless, are customarily expressed separately for the so-called dynamical variables and for the non-dynamical ones – see Refs. [7,29] for their definition –. To obtain them in our theory, multiply Eq. (56) by the left by $\Lambda_{(+)}\gamma^0$. Thus:

$$\begin{aligned} i\sqrt{2}\partial_+\Psi^{\substack{\text{av}\\ \text{ret}}}_{(+)}(x;e) &= \gamma^0\left(m_1 - i\gamma^{\perp}\partial_{\perp}\right)\Psi^{\substack{\text{av}\\ \text{ret}}}_{(-)}(x;e) \\ &\quad - e(x)\gamma^0\left(\mathbf{A}^a\gamma_a\Psi\right)^{\substack{\text{av}\\ \text{ret}}}_{(-)}(x;e), \end{aligned} \tag{61}$$

with the definitions:

$$\begin{aligned} \Psi^{\substack{\text{av}\\ \text{ret}}}_{(\pm)}(x;e) &:= \Lambda_{(\pm)}\Psi_{\substack{\text{av}\\ \text{ret}}}(x;e), \\ \left(\mathbf{A}^a\gamma_a\Psi\right)^{\substack{\text{av}\\ \text{ret}}}_{(\pm)}(x;e) &:= \Lambda_{(\pm)}\left(\mathbf{A}^a\gamma_a\Psi\right)_{\substack{\text{av}\\ \text{ret}}}(x;e). \end{aligned} \tag{62}$$

Similarly, multiplying Eq. (56) by the left by $\Lambda_{(-)}\gamma^0$:

$$\begin{aligned} i\sqrt{2}\partial_-\Psi^{\substack{\text{av}\\ \text{ret}}}_{(-)}(x;e) &= \gamma^0\left(m_1 - i\gamma^{\perp}\partial_{\perp}\right)\Psi^{\substack{\text{av}\\ \text{ret}}}_{(+)}(x;e) \\ &\quad - e(x)\gamma^0\left(\mathbf{A}^a\gamma_a\Psi\right)^{\substack{\text{av}\\ \text{ret}}}_{(+)}(x;e). \end{aligned} \tag{63}$$

Our Eqs. (61) and (63) differ from those present in the literature – see, for example, Refs. [19,30] – in that here all quantities are well-defined fields by construction. Indeed, the passage from the description by $\Psi_{\substack{\text{av}\\ \text{ret}}}(x;e)$ to the description by $\Psi^{\substack{\text{av}\\ \text{ret}}}_{(\pm)}(x;e)$ is merely algebraic; using Eq. (60), we obtain the interacting fields $\Psi^{\substack{\text{av}\\ \text{ret}}}_{(\pm)}(x;e)$ to first order in the electric charge:

$$\begin{aligned} \Psi^{\substack{\text{av}\\ \text{ret}}}_{(\pm)}(x;e) &= \psi_{(\pm)}(x) + \int d^4y\, e(y)\Big\{ i\partial_{\mp} D_m^{\substack{\text{av}\\ \text{ret}}}(x-y) \\ &\times \left[2A^{\mp}(y)\psi_{(\pm)}(y) + \sqrt{2}\gamma^0 A^{\perp}(y)\gamma_{\perp}\psi_{(\mp)}(y)\right] \\ &+ \left(i\gamma^{\perp}\partial_{\perp} + m\right) D_m^{\substack{\text{av}\\ \text{ret}}}(x-y) \\ &\times \left[\sqrt{2}\gamma^0 A^{\pm}(y)\psi_{(\mp)}(y) + A^{\perp}(y)\gamma_{\perp}\psi_{(\pm)}(y)\right]\Big\} + \mathscr{O}(e^2). \end{aligned} \tag{64}$$

It is clear in this expression that the interacting field $\Psi^{\overset{\text{av}}{\text{ret}}}_{(\pm)}(x;e)$ has as initial condition $\Psi^{\overset{\text{av}}{\text{ret}}}_{(\pm)}(x;0)=\psi_{(\pm)}(x)=\Lambda_{(\pm)}\psi(x)$, and similarly the other dynamical variable defined in Eq. (62) is exactly the one that one would obtain by initializing with $A^a(x)\gamma_a\psi_{(\pm)}(x)$.

On the other hand, the current $(\mathbf{A}^a\gamma_a\Psi)^{\overset{\text{av}}{\text{ret}}}(x;e)$ is the well-defined version of the point-wise product of interacting fields $\mathbf{A}^a_{\underset{\text{ret}}{\text{av}}}(x)\gamma_a\Psi^{\overset{\text{av}}{\text{ret}}}(x)$ of the conventional treatments, which is divergent. In order to construct the first correction to $A^a(x)\gamma_a\psi(x)$, we need the second order causal distribution

$$\begin{aligned}D_2^{A^a\gamma_a\psi}(y;x)&=[T_1(y);A^a(x)\gamma_a\psi(x)]\\&=D^{ab}(x-y)\gamma_a\ :\psi(x)\overline{\psi}(y)\gamma_b\psi(y):\\&\quad-\gamma_a S(x-y)\gamma_b\ :A^a(x)A^b(y):\\&\quad-i\gamma_a\Big[D_+^{ab}(x-y)S_+(x-y)\\&\quad-D_-^{ab}(x-y)S_-(x-y)\Big]\gamma_b\psi(y).\qquad(65)\end{aligned}$$

We recognize in the last two lines of the above equation the numerical causal distribution of the electron's self-energy, calculated in Ref. [18]. Therefore, the retarded and advanced (with respect to the point x) distributions can be immediately written, and the required interacting dynamical variable up to first order in the electric charge is

$$\begin{aligned}(\mathbf{A}^a\gamma_a\Psi)^{\overset{\text{av}}{\text{ret}}}(x;e)&=A^a(x)\gamma_a\psi(x)\\&-\int\Big(D^{ab}\Big)^{\overset{\text{av}}{\text{ret}}}(x-y)\gamma_a\ :\psi(x)\overline{\psi}(y)\gamma_b\psi(y):\ g(y)d^4y\\&+\int\ :A^a(x)\gamma_a S^{\overset{\text{av}}{\text{ret}}}(x-y)A^b(y)\gamma_b:\ \psi(y)g(y)d^4y\\&-\int\Sigma_{\underset{\text{ret}}{\text{av}}}(x-y)\psi(y)g(y)d^4y+\mathscr{O}(e^2).\qquad(66)\end{aligned}$$

If, in particular, one needs the projections $(\mathbf{A}^a\gamma_a\Psi)^{\overset{\text{av}}{\text{ret}}}_{(\pm)}(x;e)$, one must simply multiply by $\Lambda_{(\pm)}$ as specified in Eq. (62).

4.2 Lorenz's gauge condition

We shall study now the validity of Lorenz's gauge condition for the interacting electromagnetic field. We will expose the argument using the advanced interacting field, although it is evident that the same reasoning – by exchanging only the retarded distributions by the advanced ones – can be applied to the retarded one. Taking the divergence of the interacting field:

$$\partial_a\mathbf{A}^a_{\text{av}}(x;e)=\partial_a A^a(x)+\sum_{n\in\mathbb{N}}\frac{1}{n!}\int\partial_a^x R^{Aa}_{n+1}(Y_n;x)e(Y_n)dY_n,\qquad(67)$$

with

$$R^{Aa}_{n+1}(Y_n;x)=\sum_{\substack{I\cup J=Y_n\\I\cap J=\emptyset}}T^{Aa}(I;x)\widetilde{T}(J).\qquad(68)$$

As always, the distribution $R^{Aa}_{n+1}(Y_n;x)$ is normally ordered by Wick's theorem. But, since $\operatorname{supp}\big(R^{Aa}_{n+1}(Y_n;x)\big)$ must be a subset of $\Gamma_n^+(x)$, while at the point x there is one field operator only, $A^a(x)$, it is necessary that it be contracted. Hence, the retarded distribution must have the form

$$R^{Aa}_{n+1}(Y_n;x)=\sum_{j\in I_n}R_b(Y_n\setminus\{y_j\};y_j)(D^{ab})^{\text{ret}}(y_j-x);\qquad(69)$$

$$\operatorname{supp}\big(R_b(Y_n\setminus\{y_j\};y_j)\big)\subseteq\Gamma^+_{n-1}(y_j).$$

Substituting into Eq. (67) and performing an integration by parts, we obtain:

$$\begin{aligned}\partial_a\mathbf{A}^a_{\text{av}}(x;e)&=\partial_a A^a(x)+\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{j\in I_n}\int(D^{ab})^{\text{ret}}(y_j-x)\\&\times\Big[\partial_a^j R_b(Y_n\setminus\{y_j\};y_j)e(y_j)+R_b(Y_n\setminus\{y_j\};y_j)\partial_a^j e(y_j)\Big]\\&\times e(Y_n\setminus\{y_j\})dY_n.\end{aligned}\qquad(70)$$

But for the covariant theory the following identity holds: $(D^{ab})^{\text{ret}}(y_j-x)=g^{ab}D_0^{\text{ret}}(y_j-x)$, so

$$\begin{aligned}\partial_a\mathbf{A}^a_{\text{av}}(x;e)&=\partial_a A^a(x)+\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{j\in I_n}\int D_0^{\text{ret}}(y_j-x)\\&\times\Big[\partial_a^j R^a(Y_n\setminus\{y_j\};y_j)e(y_j)+R^a(Y_n\setminus\{y_j\};y_j)\partial_a^j e(y_j)\Big]\\&\times e(Y_n\setminus\{y_j\})dY_n.\end{aligned}\qquad(71)$$

Since the distribution at the point x is the only one which is different from T_1, while at the other points the distribution is always $T_1(y_j)$, the distribution $R^a(Y_n\setminus\{y_j\};y_j)$, which is independent of x, is a retarded distribution from the principal theory (for the construction of the physical scattering operator). It is equal to the sum of those terms in $-iR_n(Y_n\setminus\{y_j\};y_j)$ that have an external field $A_a(y_j)$, but with that field removed. More clearly, if we write $R_n(Y_n\setminus\{y_j\};y_j)=R_n^a(Y_n\setminus\{y_j\};y_j)A_a(y_j)$, then

$$R^a(Y_n\setminus\{y_j\};y_j)=-iR_n^a(Y_n\setminus\{y_j\};y_j).\qquad(72)$$

On the other hand, for the terms in $R_n(Y_n\setminus\{y_j\};y_j)$ with external field $A_a(y_j)$ it holds the Cg-identity – see Ref. [18] –

$$\partial_a^j R_n^a(Y_n \setminus \{y_j\}; y_j) = 0 \;\Rightarrow\; \partial_a^j R^a(Y_n \setminus \{y_j\}; y_j) = 0, \tag{73}$$

and Eq. (71) reduces to

$$\partial_a \mathbf{A}^a_{\text{av}}(x; e) = \partial_a A^a(x) + \sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{j\in I_n} \int D_0^{\text{ret}}(y_j - x) \times R^a(Y_n \setminus \{y_j\}; y_j)\partial_a^j e(y_j) e(Y_n \setminus \{y_j\}) dY_n. \tag{74}$$

Therefore, if it were possible to take the adiabatic limit $e(x) \to e$ constant, all terms in the sum would be null. In that case – we write the result also valid for the retarded interacting field –,

$$\text{w-}\lim_{e(x)\to e} \partial_a \mathbf{A}^a_{\substack{\text{av}\\ \text{ret}}}(x; e) = \partial_a A^a(x). \tag{75}$$

Here the adiabatic limit is weak because, presumably, it only exists as matrix elements between certain elements of Fock's space, for example, the vacuum state. In any case, between general elements of $\mathcal{F}_{\text{phys}}$, the matrix elements of the divergence of the interacting field reduces to terms that can be made arbitrarily small, by extending more and more the region in which the adiabatic switching function is constant. Finally, note that we need to take the adiabatic limit in order to make the gauge condition to hold; this is not a surprise, since in the very formulation of the covariant theory the condition of perturbative gauge invariance was established under the same condition [18].

5 Non-covariant interacting electromagnetic field

This case contains an additional difficulty, since now the retarded and advanced parts of the commutation distribution of the electromagnetic field cannot be normalized so as to satisfy the hypothesis of Theorem 6. We need to approach this problem from a different perspective.

As was shown in Ref. [17] – see also Ref. [29] –, the quantized free electromagnetic field under the null-plane gauge condition contains in its expression the transverse polarization vectors only, which implies that Lorenz's gauge condition holds simultaneously. In a word, we have in this situation:

$$A^+(x) = 0 \quad \wedge \quad \partial_a A^a(x) = 0. \tag{76}$$

We want to see if these two conditions are also valid for the interacting field. Firstly, from the perturbative expansion of the interacting field, for example the advanced one,

$$\mathbf{A}^a_{\text{av}}(x; e) = A^a(x) + \sum_{n\in\mathbb{N}} \frac{1}{n!} \int R^{Aa}_{n+1}(Y_n; x) e(Y_n) dY_n, \tag{77}$$

we observe that, since the distributions $R^{Aa}_{n+1}(Y_n; x)$ are constructed with n distributions $T_1(y_j)$ at the points $y_j \in Y_n$ and one distribution $A^a(x)$ at the point x, the index a of the interacting field comes from the index a of the free field operator $A^a(x)$. However, since the null-plane gauge condition is in use, $A^+(x) = 0$, we will have that $R^{A+}_{n+1}(Y_n; x) = 0$ for all $n \in \mathbb{N}$. In Eq. (77), this implies that the abovementioned gauge condition also holds for the interacting field, independently of what the adiabatic switching function $e(x)$ is:

$$\mathbf{A}^+_{\substack{\text{av}\\ \text{ret}}}(x; e) = 0. \tag{78}$$

With regard to Lorenz's gauge condition, it is clear that Eqs. (67)–(70) are still valid. However, since now $(D^{ab})^{\text{ret}}$ is not proportional to g^{ab} but has additional terms, the passage to Eq. (71) is not possible. If, instead, we undo the integration by parts that lead us to the final result in Eq. (70), we will have that – using already that for the free field it is $\partial_a A^a(x) = 0$ –

$$\partial_a \mathbf{A}^a_{\text{av}}(x; e) = -\sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{j\in I_n} \int \partial_a^j (D^{ab})^{\text{ret}}(y_j - x) \times R_b(Y_n \setminus \{y_j\}; y_j) e(Y_n) dY_n. \tag{79}$$

Remembering that in momentum space it is

$$\widehat{(D^{ab})^{\text{ret}}}(p) = \frac{-(2\pi)^{-2}}{p^2 + ip_- 0^+} \times \left\{ g^{ab} - \frac{p^a\eta^b + \eta^a p^b}{p_-} + C\frac{p^2}{p_-^2}\eta^a\eta^b \right\}, \tag{80}$$

with the vector $(\eta^a) = (0; 0^\perp; 1)$ and $C \in \mathbb{R}$ a normalization coefficient, we will have that $\partial_a^x (D^{ab})^{\text{ret}}(y_j - x) = 0$ if and only if we use the doubly-transversal retarded distribution, i.e., if we take $C = 1$; in those circumstances, Lorenz's gauge condition would be an identity for the interacting electromagnetic field. Although such an alternative seems to be attractive, the theory developed with that chose is not equivalent to the covariant one (and to the instant dynamics one), since it was a condition for that equivalence to normalize the distributions to not contain instantaneous terms – see Ref. [17]. For this reason, we abandon the possibility of having $C = 1$ and take $C = 0$ from now on.

Applying a derivative with respect to x^- to Eq. (79):

$$\partial_- \partial_a \mathbf{A}^a_{\text{av}}(x; e) = \sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{j\in I_n} \int \partial_-^j \partial_a^j (D^{ab})^{\text{ret}}(y_j - x) \times R_b(Y_n \setminus \{y_j\}; y_j) e(Y_n) dY_n. \tag{81}$$

From Eq. (80) with $C = 0$:

$$\partial_-^j \partial_a^j (D^{ab})^{\text{ret}}(y_j - x) = -\delta(y_j - x)\eta^b, \tag{82}$$

and, therefore,

$$\partial_- \partial_a \mathbf{A}^a_{\rm av}(x;e) = -\sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{j\in I_n} \int \delta(y_j - x) \times R^+(Y_n \setminus \{y_j\}; y_j) e(Y_n) dY_n = -e(x) \sum_{n\in\mathbb{N}_0} \frac{1}{n!} \int R^+(Y_n; x) e(Y_n) dY_n. \tag{83}$$

We have said – see the commentaries after Eq. (71) – that $R^a(Y_n; x)$ is equal to the sum of the terms of the distribution $-iR_n(Y_n; x)$ with an external field $A_a(x)$ with this last field removed. See, however, that to maintain the field $A_a(x)$ as an external field to be removed at the end is equivalent to construct the distribution with $ij^a(x)$ only, at the point x. In other words,

$$R^a(Y_n; x) = -iR^{(ij)a}_{n+1}(Y_n; x) = R^{ja}_{n+1}(Y_n; x). \tag{84}$$

Substituting this into Eq. (83):

$$\partial_- \partial_a \mathbf{A}^a_{\rm av}(x;e) = -e(x) \sum_{n\in\mathbb{N}_0} \frac{1}{n!} \int R^{j+}_{n+1}(Y_n; x) e(Y_n) dY_n. \tag{85}$$

Identifying the right hand side of this equation with the perturbation series for the temporal component of the interacting current, we can establish that (we write the result for the retarded interacting field also):

$$\partial_- \partial_a \underset{\rm ret}{\mathbf{A}^a_{\rm av}}(x;e) = -e(x) \underset{\rm ret}{\mathbf{J}^+_{\rm av}}(x;e). \tag{86}$$

Or, by considering Eq. (78) – the indices from the beginning of the Greek alphabet denote transversal components, $\alpha, \beta, \dots \in I_2$ –:

$$\partial_- \left(\partial_\alpha \underset{\rm ret}{\mathbf{A}^\alpha_{\rm av}}(x;e) + \partial_- \underset{\rm ret}{\mathbf{A}^-_{\rm av}}(x;e) \right) = -e(x) \underset{\rm ret}{\mathbf{J}^+_{\rm av}}(x;e). \tag{87}$$

This is one of the commonly written equations for the interacting field in the Lagrangian approach – see, for example, Ref. [31] –, used for obtaining $\underset{\rm ret}{\mathbf{A}^-_{\rm av}}(x;e)$ as a function of the transverse (dynamical) components of the interacting field and of the temporal component of the current. We have shown here that such equation holds at all orders of perturbation theory, even before taking the adiabatic limit, establishing also the well-known result that the null-plane gauge condition and the Lorenz's one are incompatible in the interacting theory. It is very important to stress, nonetheless, that this is not a problem for the treatment of gauge theory in light-front dynamics *in perturbation theory*, in which the quantized field operators are the free ones and satisfy both gauge conditions simultaneously.

We want to obtain also the equations of motion for the transversal (dynamical) components of the interacting electromagnetic field. We start by writing, using already Eq. (84),

$$\mathbf{A}^\alpha_{\rm av}(x;e) = A^\alpha(x) + \sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{k\in I_n} \int (D^{\alpha b})^{\rm ret}(y_k - x) \times (R^j_n)_b(Y_n \setminus \{y_k\}; y_k) e(Y_n) dY_n. \tag{88}$$

From Eq. (80) with $C = 0$, on the other hand,

$$\partial^k_- \Box_k (D^{\alpha b})^{\rm ret}(y_k - x) = (g^{\alpha b}\partial^k_- - \partial^\alpha_k \eta^b)\delta(y_k - x), \tag{89}$$

so that, replacing this into Eq. (88) and using that for the free field it holds $\Box A^\alpha(x) = 0$,

$$\partial_- \Box \mathbf{A}^\alpha_{\rm av}(x;e) = \sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{k\in I_n} \int \Big[\partial^k_- \delta(y_k - x)(R^j_n)^\alpha(Y_n \setminus \{y_k\}; y_k) + \partial^\alpha_k \delta(y_k - x)(R^j_n)^+(Y_n \setminus \{y_k\}; y_k) \Big] e(Y_n) dY_n. \tag{90}$$

Integrating by parts to eliminate the derivatives that act into Dirac's delta distribution:

$$\partial_- \Box \mathbf{A}^\alpha_{\rm av}(x;e) = \sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{k\in I_n} \int \delta(y_k - x) \times \Big\{ \Big[\partial^k_- (R^j_n)^\alpha(Y_n \setminus \{y_k\}; y_k) e(y_k) + (R^j_n)^\alpha(Y_n \setminus \{y_k\}; y_k) \partial^k_- e(y_k) \Big] + \Big[\partial^\alpha_k (R^j_n)^+(Y_n \setminus \{y_k\}; y_k) e(y_k) + (R^j_n)^+(Y_n \setminus \{y_k\}; y_k) \partial^\alpha_k e(y_k) \Big] \Big\} e(Y_n \setminus \{y_k\}) dY_n. \tag{91}$$

Finally, integrating in the variables y_k and recognizing the series for the interacting current,

$$\mathbf{J}^a_{\rm av}(x;e) = \sum_{n\in\mathbb{N}_0} \frac{1}{n!} \int (R^j_{n+1})^a(Y_n; x) e(Y_n) dY_n, \tag{92}$$

we obtain:

$$\partial_- \Box \mathbf{A}^\alpha_{\rm av}(x;e) = -e(x) \left[\partial_- \mathbf{J}^\alpha_{\rm av}(x;e) + \partial^\alpha \mathbf{J}^+_{\rm av}(x;e) \right] - \partial_- e(x) \mathbf{J}^\alpha_{\rm av}(x;e) - \partial^\alpha e(x) \mathbf{J}^+_{\rm av}(x;e). \tag{93}$$

In this way, the dynamical equation that governs the evolution of the transversal components of the interacting electromagnetic field reduces to its (Lagrangian-derived) known form only in the adiabatic limit, in which the last two terms of Eq. (93) vanish:

$$\text{w-}\lim_{e(x)\to e}\Big(\partial_-\Box\mathbf{A}^{\alpha}_{\substack{\text{av}\\\text{ret}}}(x;e)$$
$$+e(x)\Big[\partial_-\mathbf{J}^{\alpha}_{\substack{\text{av}\\\text{ret}}}(x;e)+\partial^{\alpha}\mathbf{J}^{+}_{\substack{\text{av}\\\text{ret}}}(x;e)\Big]\Big)=0. \tag{94}$$

The results that we have presented in this section point to the following generalization of Theorem 6, that includes the case in which the retarded and advanced parts of the (anti-)commutation distribution of the free field operator contain instantaneous terms that can be cancelled by the application of a sufficient (finite) number of derivatives with respect to the coordinate x^-, as in the studied case. Indeed, Eq. (93) can be considered as a dynamical equation for $\partial_-\mathbf{A}^{\alpha}_{\substack{\text{av}\\\text{ret}}}(x;e)$ better than for $\mathbf{A}^{\alpha}_{\substack{\text{av}\\\text{ret}}}(x;e)$ –.

Theorem 7 *Let $\boldsymbol{U}_{\substack{av\\ret}}(x;g)$ be an interacting field defined by the initial condition of being equal to the free field operator $u(x)$satisfying the free equation of motion*

$$\Delta_x u(x)=0, \tag{95}$$

with Δ_x a differential operator. Let $j_u(x)$ be a free current for $u(x)$, in that the one-point distribution of the theory is $T_1(x)=i\ :j_u(x)u(x):$. If the advanced and retarded distributions, $d_{av,ret}$, of the free field u are normalized in such a way that $\Delta_x d_{av,ret}(y-x)=P(\partial_x)\delta(y-x)$, with $P(\partial_x)$ some polynomial of the derivatives with respect to x, and if $\forall F\in\{R;A;T\}$:

$$\Delta_x F_2^u(y;x)=F_1^{j_u}(x)P(\partial_x)\delta(y-x)$$
$$=j_u(x)P(\partial_x)\delta(y-x), \tag{96}$$

then the dynamical variables $\boldsymbol{U}_{\substack{av\\ret}}(x;g)$ and $\boldsymbol{J}_{\boldsymbol{U}}^{\substack{av\\ret}}(x;g)$, this one defined by the initial condition $\boldsymbol{J}_{\boldsymbol{U}}^{\substack{av\\ret}}(x;0)=j_u(x)$, satisfy the equation of motion

$$\Delta_x\boldsymbol{U}_{\substack{av\\ret}}(x;g)=P(\partial_x)\left(g(x)\boldsymbol{J}_{\boldsymbol{U}}^{\substack{av\\ret}}(x;g)\right), \tag{97}$$

so that the following "weak equation of motion" holds in the adiabatic limit:

$$w\text{-}\lim_{g(x)\to g}\left(\Delta_x\boldsymbol{U}_{\substack{av\\ret}}(x;g)-g(x)P(\partial_x)\boldsymbol{J}_{\boldsymbol{U}}^{\substack{av\\ret}}(x;g)\right)=0. \tag{98}$$

The proof of this theorem is shown in Appendix B.

Equations (93) and (94) can now be considered as consequences of Theorem 7, with $\Delta_x=\partial^x_-\Box_x$ and $P(\partial)=g^{\alpha b}\partial_- - \partial^{\alpha}\eta^b$ [see Eq. (89)]. It could be even possible to write some weak equations of motion for the covariant theory with the doubly transverse gauge field propagator; however, we will not pursue this direction since we want to maintain the equivalence with instant dynamics. Finally, Theorem 6 corresponds to the case $P(\partial)=1$ of Theorem 7, and can hence be regarded as a corollary of it.

6 Light-front finite Källén–Yang–Feldman's equations

The differential equations of motion that we have found in the previous sections can be made into a set of integral equations that can be considered a rigorous version of the ones encountered by Källén [32] and by Yang and Feldman [33] in instant dynamics. They will be a generalization of the so-called "modified Källén–Yang–Feldman's equations" that have been formulated for instant dynamics CPT [1]. The result that we want to establish is the following – the proof is presented in Appendix C –.

Theorem 8 *Let $\boldsymbol{U}_{\substack{av\\ret}}(x;g)$ be the interacting field corresponding to the quantized free field operator $u(x)$, which satisfies the equation of motion*

$$\Delta_x u(x)=0, \tag{99}$$

and let $j_u(x)$ be the free current for this field, in that $T_1(x)=i\ :j_u(x)u(x):$, originating the interacting current $\boldsymbol{J}_{\boldsymbol{U}}^{\substack{av\\ret}}(x;g)$. Let $d_{\substack{av\\ret}}$ be the advanced and retarded parts of the (anti-)commutation distribution of the free field $u(x)$, normalized in such a way that

$$\Delta_x d_{\substack{av\\ret}}(y-x)=P(\partial_x)\delta(y-x), \tag{100}$$

with $P(\partial)$ a polynomial of the derivative operator. The interacting field $\boldsymbol{U}_{\substack{av\\ret}}(x;g)$ satisfies the "finite Källén–Yang–Feldman's equation"

$$\boldsymbol{U}_{\substack{av\\ret}}(x;g)=u(x)+\int d_{\substack{ret\\av}}(y-x)\boldsymbol{J}_{\boldsymbol{U}}^{\substack{av\\ret}}(y;g)g(y)d^4y. \tag{101}$$

Note that we have supposed that the (anti-)commutation distribution of the free field is defined in such a manner that $[u(x);u(y)]_{\mp}=-id(x-y)$. This is valid for the scalar and fermionic fields, but, conventionally, the commutation distribution of the gauge field is defined with an extra minus sign; evidently, this has as a consequence the appearance of a minus sign in the second term at the right hand side of the finite Källén–Yang–Feldman's equation [Eq. (101)]. On the other hand, for the fermionic fields it is not true that $d_{\text{ret}}(x)=d_{\text{av}}(-x)$; the argument leading to Eq. (124), however, remains valid with $d_{\text{av}}(x-y_j)$ instead of $d_{\text{ret}}(y_j-x)$.

It is a remarkable fact that these finite Källén–Yang–Feldman's equations do not depend on what normalization of the retarded distribution of the free field is used, *id est*, it says nothing about what the polynomial $P(\partial)$ in Eq. (100) is. Such polynomial has importance only on the establishment of the differential equation of motion. It is easy to see, moreover, that Eq. (101) is in fact a solution for Eq. (97): Applying the differential operator Δ_x to Eq. (101), then substituting into Eq. (100) and performing an integration by parts, one obtains the mentioned equation.

With the results so obtained, we turn to the finite Källén–Yang–Feldman's equations for light-front quantum electrodynamics.

Källén–Yang–Feldman's equations for light-front QED. As already stressed, these equations do not depend on the polynomial $P(\partial)$ in Eq. (100), hence there is no distinction between the covariant and non-covariant formulations, except for the explicit expression of the retarded part of the commutation distribution of the free radiation field. Being that way, we obtain that Källén–Yang–Feldman's equations for light-front QED are the following:

$$\mathbf{A}^a_{\substack{\text{av}\\\text{ret}}}(x;e) = A^a(x) - \int (D^{ab})^{\substack{\text{ret}\\\text{av}}}(y-x)\mathbf{J}_b^{\substack{\text{av}\\\text{ret}}}(y;e)e(y)d^4y, \tag{102}$$

$$\Psi_{\substack{\text{av}\\\text{ret}}}(x;e) = \psi(x) + \int S^{\substack{\text{av}\\\text{ret}}}(x-y)(\mathbf{A}_a\gamma^a\Psi)_{\substack{\text{av}\\\text{ret}}}(y;e)e(y)d^4y, \tag{103}$$

$$\overline{\Psi}_{\substack{\text{av}\\\text{ret}}}(x;e) = \overline{\psi}(x) - \int (\overline{\Psi}\gamma^a\mathbf{A}_a)_{\substack{\text{av}\\\text{ret}}}(y;e)S^{\substack{\text{ret}\\\text{av}}}(y-x)e(y)d^4y. \tag{104}$$

Note in particular that Eq. (102) does not conflict with Eq. (48), since for the electromagnetic field it holds that $(D^{ab})^{\text{ret}}(x) = (D^{ab})^{\text{av}}(-x)$, both in the covariant and non-covariant formulations.

7 Conclusions

The construction of interacting dynamical variables in light-front dynamics was established axiomatically and then the perturbative construction of them was explored. The principal difficulty of this, with regard to the analogous formulation in instant dynamics, is that the additional normalization freedom that exists in light-front dynamics, by virtue of the possibility of having instantaneous terms, must be taken into account very carefully. Such a care turns out to be especially important in non-covariant formulations like quantum electrodynamics subjected to the null-plane gauge condition.

Of particular interest was the study of the interacting fields, defined by the initial condition of being equal to the quantized free field operator. We showed that, generically, they satisfy, in the adiabatic limit, weak equations of motion similar to the classical ones, in which the coupling constant has been substituted by an adiabatic switching function. The condition of the adiabatic limit can only be relaxed for such fields whose retarded and advanced distributions are normalized in order to not contain instantaneous terms (covariant theories); in the general case, terms involving derivatives of the adiabatic switching function appear. This means that one must be very careful in simply extrapolating the classical equations of motion to the quantum regime. In all cases, the currents for the fields are defined by their initial condition, and are not equal to the formal product of interacting fields, which leads to divergent quantities. This is particularly important in the writing of the equations of motion of the fermion field in terms of its dynamical and non-dynamical components: the current must be constructed first, and only then the projectors $\Lambda_{(\pm)}$ must be applied.

In the covariant formulation of null-plane QED, the equation of motion for the electromagnetic field holds independently of what the switching function is, while Lorenz's gauge condition is satisfied by the interacting field only weakly (in the adiabatic limit). In the non-covariant formulation in which the radiation field is subjected to the null-plane gauge condition, on the other hand, it is this gauge condition which is satisfied by the interacting field for every switching function, as well as the equations of motion (constraints) of the non-dynamical components of the interacting field. The equations of motion for the dynamical components, finally, are satisfied weakly.

These interacting fields are subjected to some finite Källén–Yang–Feldman's equations, which have the interesting property of being (formally) equal independently of the formulation being covariant or non-covariant, a difference that only enters in the explicit expression of the distributions under the integral sign.

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Appendix A: Proof of Theorem 6

The proof of Theorem 6 is similar from that given in instant dynamics in Ref. [1], but differs from it in that now the splitting of the causal distribution requires normalization on the entire x^--axis. In order to make precise this notion, we introduce the notation

$$E_n^- := \big\{(x_1;\cdots;x_{n-1};x_n)\in\Gamma_n^+ \;\big|\; \forall j\in I_{n-1}: x_j\in\{x_n+\mathbb{R}e^-\}\big\}. \tag{105}$$

We proceed by complete mathematical induction. By substituting the formal series of $\mathbf{U}_{\substack{\text{av}\\ \text{ret}}}$ and $\mathbf{J}_{\mathbf{U}}^{\substack{\text{av}\\ \text{ret}}}$ into Eq. (51), we see that what we need to show is that $\forall m\in\mathbb{N}$:

$$\Delta_x F^u_{m+1}(Y_m;x)=\sum_{j\in I_m} F_m^{j_u}(Y_m\setminus\{y_j\};x)\delta(y_j-x) \tag{106}$$

for $F\in\{R;A\}$. This is the case for $F\in\{R;A;T\}$ for $m=1$ by hypothesis of the theorem [Eq. (50)]. Suppose then that Eq. (106) is valid for all $m\in I_{n-1}$; we need to show that it is also valid for $m=n$. We start by constructing the subsidiary distributions

$$R'^u_{n+1}(Y_n;x)=\sum_{\substack{I\cup J=Y_n\\ I\cap J=\emptyset\\ J\neq\emptyset}} T^u(I;x)\widetilde{T}(J), \tag{107}$$

$$A'^u_{n+1}(Y_n;x)=\sum_{\substack{I\cup J=Y_n\\ I\cap J=\emptyset\\ J\neq\emptyset}} \widetilde{T}(J)T^u(I;x). \tag{108}$$

Since $J\neq\emptyset$, all the transition distributions T^u appearing here are of less than $n+1$ points, hence the inductive hypothesis can be applied to them. Applying the differential operator to $R'^u_{n+1}(Y_n;x)$:

$$\begin{aligned}\Delta_x R'^u_{n+1}(Y_n;x)&=\sum_{\substack{I\cup J=Y_n\\ I\cap J=\emptyset\\ J\neq\emptyset}}\Delta_x T^u(I;x)\widetilde{T}(J)\\ &=\sum_{\substack{I\cup J=Y_n\\ I\cap J=\emptyset\\ J\neq\emptyset}}\sum_{y_j\in I} T^{j_u}(I\setminus\{y_j\};x)\delta(y_j-x)\widetilde{T}(J)\\ &=\sum_{y_j\in Y_n}\delta(y_j-x)\sum_{\substack{I\cup J=Y_n\setminus\{y_j\}\\ I\cap J=\emptyset\\ J\neq\emptyset}} T^{j_u}(I;x)\widetilde{T}(J)\\ &=\sum_{j\in I_n} R'^{j_u}_n\big(Y_n\setminus\{y_j\};x\big)\delta(y_j-x).\end{aligned} \tag{109}$$

The same holds for the A'^u_{n+1} distribution. One then concludes that $D^u_{n+1}=R'^u_{n+1}-A'^u_{n+1}$ satisfies

$$\Delta_x D^u_{n+1}(Y_n;x)=\sum_{j\in I_n} D_n^{j_u}\big(Y_n\setminus\{y_j\};x\big)\delta(y_j-x). \tag{110}$$

If this equation also holds for R^u_{n+1} obtained from D^u_{n+1} by distribution splitting, then the proof will be complete, as it would be automatically valid for $A^u_{n+1}=R^u_{n+1}-D^u_{n+1}$ and for $T^u_{n+1}=R^u_{n+1}-R'^u_{n+1}$ [see Eq. (109)]. Note also that $R^u_{n+1}=D^u_{n+1}$ in $\Gamma^+_{n+1}(0)\setminus E^-_{n+1}$. Consequently, the problem is reduced to show that a normalization for which Eq. (110) is satisfied by the retarded distribution do exists.

From Eqs. (107), the causal distribution is given by

$$D^u_{n+1}(Y_n;x)=\sum_{\substack{I\cup J=Y_n\\ I\cap J=\emptyset\\ J\neq\emptyset}}\big[T^u(I;x);\widetilde{T}(J)\big]. \tag{111}$$

The numerical causal distributions are obtained by normally ordering the above expression. Additionally, since the distribution at point x is $u(x)$, it must be contracted; otherwise $D^u_{n+1}(Y_n;x)$ could not have causal support. Moreover, being again a free field, in the terms in which $u(x)$ is already contracted inside $T^u(I;x)$, necessarily it must be a Feynman's propagator, d^F, because no "loop distribution" can be closed at x having this point one free field operator only. On the other hand, in the terms in which $u(x)$ is contracted with $\widetilde{T}(J)$, the positive- and negative-frequency parts of the (anti-)commutation distribution of $u(x)$, $d_\pm$, will appear. If we finally write

$$d^F=\pm(d_++d_{\text{av}})\quad,\quad d_-=d_{\text{ret}}-d_{\text{av}}-d_+, \tag{112}$$

with the $\pm$ sign in the first relation depending on the bosonic or fermionic character of the field u, respectively, then the dependence of the causal distribution with the point x can be explicitly written as

$$\begin{aligned}D^u_{n+1}(Y_n;x)=\sum_{j\in I_n}\big[&d_+(x-y_j)D_+(Y_n\setminus\{y_j\};y_j)\\ &-d_{\text{ret}}(x-y_j)D_{\text{av}}(Y_n\setminus\{y_j\};y_j)\\ &+d_{\text{av}}(x-y_j)D_{\text{ret}}(Y_n\setminus\{y_j\};y_j)\big].\end{aligned} \tag{113}$$

Now, since d_+ has its support not contained in the light-cone, its presence in the causal distribution is contradictory, from which it follows that $D_+(Y_n\setminus\{y_j\};y_j)=0$. Also, $D^u_{n+1}(Y_n;x)$ can be non-null only if $Y_n\in\Gamma^+_n(x)$ or $Y_n\in\Gamma^-_n(x)$ [7]. Since, on the other hand, supp $(d_{\text{ret}})\subseteq\widetilde{V^+}$, while supp $(d_{\text{av}})\subseteq\widetilde{V^-}$, it is also necessary that

$$\begin{aligned}&\text{supp}\,\big(D_{\text{ret}}(Y_n\setminus\{y_j\};y_j)\big)\subseteq\Gamma^+_{n-1}(y_j),\\ &\text{supp}\,\big(D_{\text{av}}(Y_n\setminus\{y_j\};y_j)\big)\subseteq\Gamma^-_{n-1}(y_j).\end{aligned} \tag{114}$$

It is essential to note, for the consistency of the proof, that in spite of the possibility that $d_{\text{ret,av}}$ can contain instantaneous terms, Eq. (114) is perfectly valid, since our definitions of $\Gamma_n^{\pm}$ contain the set E_n^-. In summary,

$$\operatorname{supp}\left(\sum_{j\in I_n} d_{\text{ret}}(x-y_j)D_{\text{av}}(Y_n\setminus\{y_j\};y_j)\right)\subseteq\Gamma_n^-(x), \tag{115}$$

$$\operatorname{supp}\left(\sum_{j\in I_n} d_{\text{av}}(x-y_j)D_{\text{ret}}(Y_n\setminus\{y_j\};y_j)\right)\subseteq\Gamma_n^+(x), \tag{116}$$

and Eq. (113) – with $D_+=0$ – is a solution for the splitting problem for $D_{n+1}^u(Y_n;x)$. If we apply to it the differential operator Δ_x, and using that, by hypothesis, $\Delta_x d_{\text{ret,av}}(x-y_j)=\delta(x-y_j)$, we obtain:

$$\Delta_x D_{n+1}^u(Y_n;x)=\sum_{j\in I_n}\delta(x-y_j)\times\left[D_{\text{ret}}(Y_n\setminus\{y_j\};x)-D_{\text{av}}(Y_n\setminus\{y_j\};x)\right]. \tag{117}$$

By comparison with Eq. (110), this establishes that

$$D_n^{j_u}(Y_n\setminus\{y_j\};x)=D_{\text{ret}}(Y_n\setminus\{y_j\};x)-D_{\text{av}}(Y_n\setminus\{y_j\};x). \tag{118}$$

Simultaneously considering the supports of the involved distributions [see Eq. (114)], this equation is a solution for the splitting problem of $D_n^{j_u}(Y_n\setminus\{y_j\};x)$. Therefore, there are solutions R_{n+1}^u and $R_n^{j_u}$ such that

$$\Delta_x R_{n+1}^u(Y_n;x)=\sum_{j\in I_n}\delta(x-y_j)R_n^{j_u}(Y_n\setminus\{y_j\};x), \tag{119}$$

which completes the proof.

Appendix B: Proof of Theorem 7

We want to show, firstly, that $\forall F\in\{R;A;T\}$: $\forall m\in\mathbb{N}$:

$$\Delta_x F_{m+1}^u(Y_m;x)=\sum_{j\in I_m}F_m^{j_u}(Y_m\setminus\{y_j\};y_j)P(\partial_x)\delta(y_j-x). \tag{120}$$

The case $m=1$ is hypothesis of the theorem [Eq. (96)], so the proof of Eq. (120) can be made by complete mathematical induction; the procedure is identical to the shown for proving Theorem 6, substituting, when necessary, $\delta(y_j-x)$ by $P(\partial_x)\delta(y_j-x)$. Due to Eq. (95), on the other hand, we have that – we argue for the advanced interacting field; similar arguments prove the result for the retarded one –

$$\begin{aligned}\Delta_x\mathbf{U}_{\text{av}}(x;g)&=\sum_{n\in\mathbb{N}}\frac{1}{n!}\int\Delta_x R_{n+1}^u(Y_n;x)g(Y_n)dY_n\\&=\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{j\in I_n}\int R_n^{j_u}(Y_n\setminus\{y_j\};y_j)\\&\quad\times P(\partial_x)\delta(y_j-x)g(Y_n)dY_n\\&=\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{j\in I_n}\int\delta(y_j-x)\\&\quad\times P(\partial_j)\left(R_n^{j_u}(Y_n\setminus\{y_j\};y_j)g(y_j)\right)g(Y_n\setminus\{y_j\})dY_n,\end{aligned} \tag{121}$$

where we have used that $P(\partial_x)\delta(y_j-x)=P(-\partial_j)\delta(y_j-x)$ and made an integration by parts. Integrating the variable y_j in each term of the sum:

$$\begin{aligned}&\Delta_x\mathbf{U}_{\text{av}}(x;g)\\&=\sum_{n\in\mathbb{N}_0}\frac{1}{n!}\int P(\partial_x)\left(R_{n+1}^{j_u}(Y_n;x)g(x)\right)g(Y_n)dY_n\\&=P(\partial_x)\left\{g(x)\sum_{n\in\mathbb{N}_0}\frac{1}{n!}\int R_{n+1}^{j_u}(Y_n;x)g(Y_n)dY_n\right\},\end{aligned} \tag{122}$$

which is exactly Eq. (97), and from which it immediately follows Eq. (98).

Appendix C: Proof of Theorem 8

We start with the perturbative expansion of the interaction field – again, we make the proof for the advanced interacting field –:

$$\mathbf{U}_{\text{av}}(x;g)=u(x)+\sum_{n\in\mathbb{N}}\frac{1}{n!}\int R_{n+1}^u(Y_n;x)g(Y_n)dY_n. \tag{123}$$

As previously argued [see Eq. (69) and the paragraph above it], once the normal ordering is realized, the retarded distribution $R_{n+1}^u(Y_n;x)$ must have the form

$$R_{n+1}^u(Y_n;x)=\sum_{j\in I_n}R(Y_n\setminus\{y_j\};y_j)d_{\text{ret}}(y_j-x). \tag{124}$$

Substituting into Eq. (123),

$$\begin{aligned}\mathbf{U}_{\text{av}}(x;g)&=u(x)+\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{j\in I_n}\int d_{\text{ret}}(y_j-x)\\&\quad\times R(Y_n\setminus\{y_j\};y_j)g(Y_n)dY_n\\&=u(x)+\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{j\in I_n}\int d^4y_j\,g(y_j)d_{\text{ret}}(y_j-x)\end{aligned}$$

$$\times \int R(Y_n \setminus \{y_j\}; y_j) g(Y_n \setminus \{y_j\}) d(Y_n \setminus \{y_j\})$$
$$= u(x) + \int d_{\text{ret}}(y - x)$$
$$\times \left(\sum_{n \in \mathbb{N}_0} \frac{1}{n!} \int R(Y_n; y) g(Y_n) dY_n \right) g(y) d^4 y. \tag{125}$$

The next step stands over the hypothesis that the one-point distribution is $T_1(x) = i \ :j_u(x)u(x):$. Indeed, the term inside the parenthesis in Eq. (125) represents a dynamical variable with initial condition $R(\emptyset; y)$. From Eq. (124), we see that it is defined such that

$$R_2^u(y; x) = R(\emptyset; y) d_{\text{ret}}(y - x). \tag{126}$$

But, by construction, $R_2^u(y; x)$ is the retarded part of the causal distribution $D_2^u(y; x) = [T_1(y); u(x)] = j_u(y) d(y - x)$, hence $R(\emptyset; y) = j_u(y)$. This also holds at higher orders: To construct $R_{n+1}^u(Y_n; x)$ with the field $u(x)$ at the point x, with it necessarily contracted, is equivalent to construct the same distribution letting the field $iu(y_j)$, without contraction, in order to be contracted with $u(x)$ later, in order to obtain $d_{\text{ret}}(y_j - x)$. Therefore, the distribution $R(Y_n \setminus \{y_j\}; y_j)$ in Eq. (124) is constructed only with the current $j_u(y_j)$ at the point y_j. This allows us to write

$$R(Y_n \setminus \{y_j\}; y_j) = R_n^{j_u}(Y_n \setminus \{y_j\}; y_j), \tag{127}$$

and the expression inside the parentheses in Eq. (125) is identified with the interacting current $\mathbf{J}_{\mathbf{U}}^{\text{av}}(y; g)$.

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