Tales of 2D CFTs and moonshine

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Tales of 2D CFTs and Moonshine

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Publications

This thesis is based on the following publications:

[1] Anagiannis V, Cheng MC, "TASI lectures on Moonshine", *Volume 305 - Theoretical Advanced Study Institute Summer School 2017 "Physics at the Fundamental Frontier" (TASI2017) - Weeks 1-4, TASI Lectures on Moonshine*, `arXiv:1807.00723`.

Presented in Chapter **1**.

[2] Anagiannis V, Cheng MC, Harrison SM, "K3 elliptic genus and an umbral moonshine module.", *Communications in Mathematical Physics 366.2 (2019): 647-680*, `arXiv:1709.01952`.

Presented in Chapter **2**.

[3] Anagiannis V, Cheng MC, Duncan J, Volpato R, "Vertex operator superalgebra/sigma model correspondences: The four-torus case", 2020, `arXiv:2009.00186`.

Presented in Chapter **3**.

Other publications by the author:

[4] Cheng MC, Anagiannis V, Weiler M, de Haan P, Cohen TS, & Welling M (2019). "Covariance in Physics and Convolutional Neural Networks", `arXiv:1906.02481`.

- Anagiannis V, Cheng MC, "Entangled q-Convolutional Neural Nets", to appear 2021

Contribution of the author to the publications:

The author participated in the conceptual discussions, writing and editing of the all the publications used in this thesis. The order of authors in these publications is purely alphabetical. More detailed accounts for the author's contributions follow below.

In [1], the author contributed to the writing of sections 1, 3, 4, and 7, while participating in the editing of the whole manuscript.

In [2], the author contributed to the writing of section 2.2, to the writing and computations entailed in sections 3, 4.2 and appendices B,C, as well as to the editing of the whole manuscript.

In [3], the author contributed to the writing and computations entailed in all sections apart from appendices B,C, while participating in the editing of the whole manuscript.

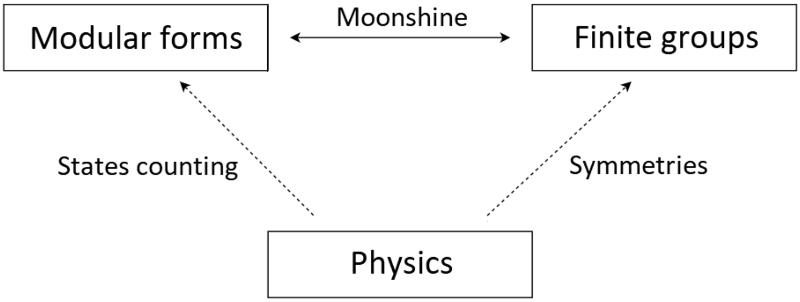
Contents

1 Introduction

Throughout its history, fundamental research on theoretical high-energy physics has had an undoubtedly essential constituent; mathematics. Not only does it enter the playing field as a strong driving force enabling new results, but new mathematics is also often born by drawing inspiration from physics. This is perhaps most vividly encountered in the field of string theory, where many advances have been made thanks to that dynamic. Another field where mathematics and physics frequently come together in fruitful ways holds the imaginative name "moonshine". It is this field that all the topics discussed in this thesis revolve around.

Moonshine prominently features an important aspect of contemporary research: connections and correspondences between different or loosely related fields and topics. Mathematically speaking, it comprises a - not yet fully understood - relationship between representations of finite groups and modular objects of various flavours. These two mathematical structures, finite groups and modular objects, have a priori little conceptual overlap with each other; moonshine provides the glue that seems to be tightly tying them together. Furthermore, both facets of this relationship attain a manifestation in physics; groups appear naturally as the mathematical structure that describes symmetries, whereas modular objects, such as modular forms, appear frequently when one is counting the states of certain two-dimensional conformal field theories, often arising in the context of string theory. It is mostly through the eyes of such two-dimensional conformal field theories that we will be exploring aspects related to moonshine in this thesis.

A natural research direction in moonshine is to explicitly construct realizations of such relations, known as moonshine modules. Whereas proving their existence is more of a mathematical endeavour, their construction can sometimes be facilitated by drawing inspiration from physics. Chapter 3 of this thesis is focused on constructing one such module for a specific case of moonshine, by using elements, ideas and language that are more readily available in the toolkit of a physicist.



Looking at this interplay between physics and mathematics from the other direction, constructions born out of moonshine may inspire results that are independently useful for physics. An example of this is detailed in Chapter 4 of this thesis, where a correspondence between a vertex operator algebra, a structure originating from moonshine, and a whole family of sigma models, theories that appear in string theory, is established. This result is a product of inspiration from the world of moonshine, demonstrating how physics and mathematics can interact in productive ways.

Being written by a physicist, this thesis does not always use the language that a mathematician would when describing moonshine-related topics; it instead opts for one more familiar to physicists. For the sake of clarity, and for the sanity of the author, attempts have been made to make this thesis as self-contained as possible, to the benefit of the reader. This is the goal of Chapter 2, which attempts to capture the depth at which most relevant topics and tools, regardless of their complexity, are utilized in the later chapters. A much more detailed introduction to the main pillar of this thesis, moonshine, is also part of that chapter.

Outline of this thesis

Chapter 2 contains background material on finite groups, modular objects and two-dimensional conformal field theories. It also contains an introduction to moonshine both as a general topic, but also focusing more specifically on the type of moonshine, umbral moonshine, that is encountered in the later chapters.

Chapter 3 focuses on the construction of a module for the $D_4^{\oplus 6}$ case of umbral moonshine. All the steps taken to arrive at the final construction are detailed, together with some motivation material and a discussion on possible future directions.

Chapter 4 describes a correspondence between a vertex operator superalgebra (one based on the E_8 lattice) and the family of sigma models with target space consisting of complex four-dimensional tori (T^4). The nature of this correspondence is made precise, and connections with previous works are discussed.

Finally, five appendices are also included, handling miscellanea topics and holding tabular data tied to the main material.

2 Background

The goal of this chapter is to introduce the main concepts and tools used in the rest of the thesis. As we aim for a self-contained approach as much as possible, in this chapter we include both basic and advanced topics that the reader can refer back to when needed. Most of the material is presented as in the lectures notes [1] co-written by the author.

2.1 Finite groups and representations

Finite groups do not only play a central role in moonshine, but also capture special symmetries of several CFTs that appear throughout this thesis. Here we establish some notations and go over some basic and more specialized concepts, pertaining to the role that finite groups take for us. For more complete expositions see [5–7] (see also [8]).

2.1.1 Basics

A group is a set G, together with a "multiplication" operation $\bullet : G \times G \to G$, formally denoted as $(G, \bullet)$. The symbol for this operation is usually implicit, and we often write ab for $a \bullet b$. A group must satisfy the following axioms:

1. **Closure:** $ab = c \in G$ for any $a, b \in G$.
2. **Associativity:** $(ab)c = a(bc)$ for any $a, b, c \in G$.
3. **Identity:** There exists a unique identity element $e \in G$, such that $eg = ge = g$ for any $g \in G$.
4. **Inverses:** For every $g \in G$, there exists a unique inverse element $g^{-1} \in G$, such that $gg^{-1} = g^{-1}g = e$. We also have that $e^{-1} = e$.

Notice that $ab \neq ba$ in general. In the case that $ab = ba$ for every $a, b \in G$, i.e. the group operation is commutative, the group is called *Abelian*. The number of elements of G is called the *order* of G, and it can be either finite or infinite. We also define the *order of an element*, $|g|$, to be the minimum number of times we need to multiply it with itself in order to reach the identity, i.e. $g^{|g|} = e$ (the order can also be infinite).

Next we give a brief summary of a few important notions of group theory.

Group homomorphisms. We say that a map $\phi : G \to F$ between two groups $(G, \bullet)$ and $(F, \star)$ is a *group homomorphism* if it preserves the group structure of G. In other words, ϕ must satisfy

$$\phi(a \bullet b) = \phi(a) \star \phi(b) \ , \tag{2.1.1}$$

for all $a, b \in G$. If there also exists an inverse homomorphic map $\phi^{-1} : F \to G$, then G and F are isomorphic; such groups are abstractly the same, but they may still have different realisations. An isomorphism $G \to G$ is called *automorphism*, and is often called a symmetry of G. The set of all automorphisms of G, denoted $\mathrm{Aut}\, G$, forms a group.

Conjugacy classes. Two group elements $a, b \in G$ are said to be conjugate to each other if there exists an element $g \in G$ such that $gag^{-1} = b$. In this case, we symbolically write $a \sim b$. Conjugation is an equivalence relation, since it is reflective ($a \sim a$), symmetric ($a \sim b$ iff $b \sim a$) and transitive (if $a \sim b$ and $b \sim c$ then $a \sim c$). Such a relation implies that G can be split into disjoint subsets $[a] \subset G$, called *conjugacy classes*, each containing all elements that are conjugate to each other:

$$[a] = \left\{ b \in G \mid gag^{-1} = b \ \text{ for some } \ g \in G \right\} \ . \tag{2.1.2}$$

Obviously, a conjugacy class can be represented by any one of its elements, i.e. $[a] = [b]$ for all $b \sim a$. The number of distinct conjugacy classes is referred to as the *class number* of G, denoted here as $\mathrm{Cl}(G)$. All elements of a class have the same order. It is easy to see that an element constitutes a conjugacy class of its own if it commutes with all other elements of the group. As a result, in an Abelian group each class contains only one element and the class number equals the order of the group.

A common notation for conjugacy classes is to write the order of its elements, followed by an alphabetical letter. For example, $4A$ denotes a class of order four, $4B$ a different class of order four, $6A$ a class of order six, and so on. The identity is always a class of its own, namely $1A$, the unique class of order one.

Subgroups. A subgroup H is a subset $H \subset G$ which is itself a group, with the group structure inherited from G. Note that the identity element e always forms a subgroup $\{e\}$ of order 1, called the trivial subgroup. Subgroups H other than the trivial subgroup and G itself are called *proper subgroups* of G, and the notation $H < G$ is used for them (we use the notation $H \leq G$ if we can have $H = G$).

A *normal subgroup* N, also denoted as $N \triangleleft G$, is a subgroup of G that is invariant under conjugation by all elements of G:

$$N \triangleleft G \;\Leftrightarrow\; gNg^{-1} = N \text{ for all } g \in G \; . \tag{2.1.3}$$

As such, N is necessarily a union of conjugacy classes. A *maximal normal subgroup* of G is a normal subgroup which is not contained in any other normal subgroup of G, apart from G itself. Normal subgroups play a prominent role in quotient groups and group extensions (see below).

The centre $Z(G)$ of a group G is the set of all elements that commute with every other element, i.e.

$$Z(G) := \{a \in G \,|\, ab = ba \;\text{ for all } b \in G\} \; . \tag{2.1.4}$$

The centre is always a normal subgroup of G. The *centralizer* of an element $g \in G$ is similarly defined by

$$C_G(g) = \{a \in G \,|\, ag = ga\}, \tag{2.1.5}$$

being the set of all elements that commute with g. Clearly, the centralizer of an element is always a subgroup of G.

Cosets. Let H be a subgroup of G, and take $g \in G$. We define the *left coset* of H in G with respect to g as the subset

$$gH = \{gh \mid h \in H\} \; . \tag{2.1.6}$$

The set of all left cosets of H in G is denoted by $G/H := \{gH \mid g \in G\}$. Similarly, the *right coset* of H in G with respect to g is defined as

$$Hg := \{hg \mid h \in H\} \; , \tag{2.1.7}$$

and the set of all right cosets of H in G is denoted by $H\backslash G := \{Hg \mid g \in G\}$.

One can more intuitively define left cosets in terms of an equivalence relation on G (not to be confused with conjugation); namely, for $a, b \in G$ we set $a \sim b$ iff $ah = b$ for some $h \in H$, i.e. a and b are related by multiplication of an element in H to the right. Then a, b represent the same equivalence class, which is exactly the coset

$aH = bH$. All such classes make up G/H, which is viewed as a disjoint partition of G, as a set. The corresponding statements also hold for right cosets. Some useful facts about cosets include:

- The number of left cosets is always equal to the number of right cosets, and is known as the *index* of H in G, denoted by $[G : H]$.
- If G is a finite group, then Lagrange's theorem states that the index equals the quotient of the order of G over the order of H, i.e. $[G : H]|H| = |G|$. This is indicatory of how G is partitioned under the coset equivalence relation associated with H.
- The left and the right cosets of H have the same number of elements, which is equal to the order of H.
- The left and right cosets of a normal subgroup coincide, as can be easily seen from its definition.

Quotient groups and group extensions. Cosets, like conjugacy classes, are in general not subgroups. However, given a normal subgroup N, the set G/N of right cosets (which coincides with the set of left cosets) inherits the group structure of G, and is called the *quotient group*. This can be seen from $(aN)(bN) = (ab)N$. The normal subgroup N can then be viewed as the kernel of the homomorphism $\psi : G \to G/N$. Note that in general G/N is not isomorphic to any subgroup of G. Moreover, the order of G/N is equal to the index $[G : N] = |G|/|N|$.

Consider now a *short exact sequence* of groups

$$1 \to M \xrightarrow{\phi} G \xrightarrow{\psi} Q \to 1 \ . \tag{2.1.8}$$

This means that $\phi(M)$, the embedding of M inside G, by ϕ is the kernel of the homomorphism ψ; in other words M is isomorphic to a normal subgroup $N \triangleleft G$, and $Q \cong G/N$. We then say that G is an *extension* of Q by M. An extension, as well as the corresponding sequence, is called *split* if there exists a homomorphism (embedding) $\widetilde{\psi} : Q \to G$ such that $\psi \circ \widetilde{\psi} = \mathrm{id}_Q$, the identity map on Q. We use the semi-direct product to denote such a split extension, $G = N \rtimes Q$. Otherwise, the extension is called *non-split*, and we write $G = N.Q$.

2.1.2 Classification of finite groups

Finite groups are, as the name suggests, groups with finite number of elements. The problem of classifying them can be reduced to the classification of finite *simple* groups, i.e. the ones that have no proper normal subgroups. If a finite group G is not simple, then it can always be "decomposed" into a series of smaller groups, by considering quotients by maximal normal subgroups. To be precise, one can consider the *composition series*, which has the form

$$1 \lhd N_1 \lhd N_2 \lhd \cdots \lhd N_{n-1} \lhd N_n = G \ . \tag{2.1.9}$$

Here 1 denotes the trivial group, and every step of the series involves a maximal normal subgroup N_{i-1} of N_i, as well as the implied quotient group N_i/N_{i-1}. It can be shown that all the resulting quotient groups are simple, and the Jordan–Hölder theorem guarantees that for given G, two different composition series lead to the same simple groups. As a result, studying finite simple groups is to a large extent sufficient to understand general finite groups.

After heroic efforts spanning over half a century and involving many dozens of mathematicians and thousands of pages of proofs, all finite simple groups have been classified (see [9, 10] for historical remarks). Each one belongs to one of the following four categories: cyclic groups $\mathbb{Z}_p$ for prime p, alternating groups $\mathcal{A}_n$ $(n \geq 5)$, 16 families of Lie type and 26 *sporadic groups*. Unlike the rest of finite simple groups, the 26 sporadic groups appear "sporadically" and are not part of infinite families. We next discuss these special sporadic groups in more detail, since they are of instrumental importance for moonshine.

2.1.3 Sporadic groups and lattices

The largest sporadic group is the Fischer–Griess Monster group $\mathbb{M}$, which gets its name from its enormous size. The number of its element is

$$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8 \times 10^{53} \ ,$$

which is roughly the same as the number of atoms in the solar system! The Monster contains 20 of the 26 sporadic groups as its subgroups or quotients of subgroups, and these 20 is said to form 3 generations of a *happy family* by Robert Griess. In particular, the happy family includes the five Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$. They are all subgroups of M_{24}, which is in turn a subgroup of the permutation group S_{24}, and are the the first sporadic groups that were discovered. The rest 6 which are not related to the Monster are called the *pariahs* of sporadic groups.

The sporadic nature of the sporadic groups makes their existence somewhat mysterious and one might wonder what their "natural" representations are. An important hint is that many of the sporadic groups, especially those connected to the Monster, arise as subgroups of quotients of the automorphic groups of various special lattices. The appearance of moonshine involving sporadic groups sheds important light on the question, and the construction of moonshine often relies on the existence of these special lattices. As a result, in what follows we will briefly review the definition of lattices and their root systems, and introduce the special lattices relevant for moonshine.

Let V be a finite-dimensional real vector space of dimension r, equipped with an inner product $\langle\cdot,\cdot\rangle$. A finite subset $X \subset V$ of non-zero vectors is called a *root system of rank* r, if the following conditions are satisfied

- X spans V.
- X is closed under reflections. Namely, $\beta - 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha \in X$ for all $\alpha, \beta \in X$.
- The only multiples of $\alpha \in X$ that belong to X are α and $-\alpha$.
- For all $\alpha, \beta \in X$, we have $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$.

The elements $\alpha \in X$ of a root system are called *roots*. A root system X is said to be irreducible if it cannot be partitioned into proper orthogonal subsets $X = X_1 \cup X_2$. It turns out that the roots of such a system can have at most two possible lengths. If all roots have the same length, then the irreducible root system is called *simply-laced*. One can choose a subset Φ of roots $f_i \in X$ with $i = 1, \ldots, r$, such that each root can be written as an integral combination of $f_i \in \Phi$ with either all negative or all positive coefficients. Such a subset is called a set of *simple roots*, and is unique up to the action of the group generated by reflections with respect to all roots, called the *Weyl group* of X and denoted by $\mathrm{Weyl}(X)$.

To each irreducible root system we can attach a connected *Dynkin diagram*. Each simple root is associated with a node, and nodes associated to two distinct simple roots f_i, f_j are connected with N_{ij} lines, where

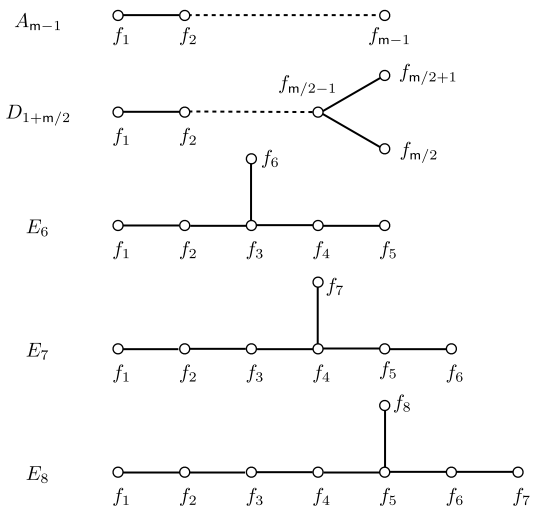
$$N_{ij} = \frac{2\langle f_i, f_j\rangle}{\langle f_i, f_i\rangle}\frac{2\langle f_j, f_i\rangle}{\langle f_j, f_j\rangle} \in \{0,1,2,3\}\ . \tag{2.1.10}$$

For simply-laced root systems we only have $N_{ij} \in \{0,1\}$. These correspond to the Dynkin diagrams of type A_n, D_n, E_6, E_7, E_8 with the subscript denoting the rank of the associated root system, as shown in figure 2.1.

Each irreducible root system contains a unique *highest root* θ with respect to a given set Φ of simple roots, whose decomposition

$$\theta = \sum_{i=1}^{r} a_i f_i \tag{2.1.11}$$

Figure 2.1: The ADE Dynkin diagrams.



maximizes the sum $\sum a_i$. The *Coxeter number* of X is then defined by

$$\mathrm{Cox}(X) := 1 + \sum_{i=1}^{r} a_i \ . \tag{2.1.12}$$

The Coxeter number can also be defined in terms of Weyl(X). The product of reflections with respect to all simple roots $w = r_{f_1} r_{f_2} \cdots r_{f_r} \in W^X$ is called *Coxeter element*, and its order equals the Coxeter number m.

A *lattice* L of rank n is a free Abelian group isomorphic to the additive group $\mathbb{Z}^n$, equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$. Embedding L into $\mathbb{R}^n$ gives the picture of a set of points inside the vector space $\mathbb{R}^n$. A few properties some lattices have that will be useful for us include the following:

- *Positive-definite*: the bilinear form induces a positive-definite inner product on $\mathbb{R}^n$.
- *Integral*: $\langle \lambda, \mu \rangle \in \mathbb{Z}$ for all $\lambda, \mu \in L$.
- *Even*: $\langle \lambda, \lambda \rangle \in 2\mathbb{Z}$ for all $\lambda \in L$.
- *Unimodular*: the dual lattice, defined by $L^* := \{\lambda \in L \otimes_{\mathbb{Z}} \mathbb{R} |\ \langle \lambda, L \rangle \subset \mathbb{Z}\}$, is isomorphic to the lattice itself.

All elements $\lambda \in L$ such that $\langle \lambda, \lambda \rangle = 2$ are called the *roots* of L.

Even, unimodular, positive-definite lattices in 24 dimensions play a distinguished role in several instances of moonshine, as we will discuss later in this chapter. It was proven by H. V. Niemeier in 1973 that there are only 24 inequivalent such lattices [11]. One of them, first discovered by J. Leech in 1967 and named the *Leech lattice*, is the only one of the 24 that has no root vectors [12–14]. The other 23, which we refer to as the *Niemeier lattices*, have non-trivial root systems. In fact, one useful construction of the Niemeier lattices is by combining the root lattices with the appropriate "glue vectors" [15]. It turns out that the 23 Niemeier lattices are uniquely labelled by the root systems X, called the *Niemeier root systems*, which are precisely one of the 23 unions of simply-laced (ADE) root systems $X = \cup_i Y_i$ satisfying the following conditions: 1) All components have the same Coxeter number, $\mathrm{Cox}(Y_i) = \mathrm{Cox}(Y_j)$; 2) the total rank equals the rank of the lattice $\sum_i \mathrm{rk}(Y_i) = 24$. Some examples out of the 23 include $A_1^{\oplus 24}$, $A_{12}^{\oplus 2}$, $E_8^{\oplus 3}$ and $D_{16}E_8$ (where $X^{\oplus n}$ denotes a direct sum of n copies of X).

For each of these 24 even, unimodular, positive-definite lattices of rank 24 N we define a finite group

$$G_N := \mathrm{Aut}(N)/\mathrm{Weyl}(N), \tag{2.1.13}$$

where $\mathrm{Weyl}(N)$ denotes the Weyl group of the root system of N. In particular, when $N = \Lambda$ is the Leech lattice, the Weyl group $\mathrm{Weyl}(\Lambda)$ is the trivial group and $G_\Lambda \cong Co_0$ is the Conway group Co_0. By considering the quotients of this group and subgroups stabilising various structures we can obtain many of the sporadic groups. For instance, the sporadic simple group Co_1 is given by the quotient by the centre $Co_1 \cong Co_0 / \{\pm 1\}$, and the Mathieu group M_{23} arises as the subgroup fixing a specific rank-2 sublattice. See Chapter 10 of [15] for a detailed discussion. If instead we choose N to be the Niemeier lattice with root system $A_1^{\oplus 24}$, for instance, the finite group $G_N \cong M_{24}$ is given by the largest Mathieu group. For the Niemeier lattice with root system $A_2^{\oplus 12}$, the finite group is $2.M_{12}$, the non-trivial extension of the Mathieu group M_{12}. These groups play an important role in several instances moonshine (cf. §2.4).

2.1.4 Representations and characters

Groups offer the mathematical tool that is best suited to describe symmetries. In order to have concrete descriptions we furthermore need the concept of representations of groups. In what follows we limit our discussion to *complex representations*; namely we consider the group action on a complex vector space V. More precisely, consider the group homomorphism $\rho : G \to GL(V)$. We can think of the images $\rho(g)$ as invertible $n \times n$ complex matrices. In particular we have $\rho\left(g^{-1}\right) = (\rho(g))^{-1}$. The vector space together with the map (V, ρ) is called a *representation* of dimension n. Often one refers to either V or ρ as the representation, while implicitly referring to the full data. The vector space V is also called a G-module in this context, and is said to carry a

G-action. We say that the G-action is *faithful*, if no two distinct elements $g, g' \in G$ lead to $\rho(g) = \rho(g')$ (the corresponding representation is also called faithful).

Irreducible representations and dual representations. Given two representations (V, ρ) and (V', ρ') one can define their direct sum and their tensor product in a straightforward way, which leads to new representations $V \oplus V'$ and $V \otimes V'$.

Two representations ρ, ρ' are *equivalent* if there exists an invertible $n \times n$ matrix M such that $M\rho'(g) = \rho(g)M$ for all $g \in G$. A *subrepresentation* of a representation (V, ρ) is a representation (U, ρ'), where U is a subspace $U \subset V$ that is preserved by the action of G, and ρ' is the restriction of ρ to U. A representation V is said to be *irreducible* if it does not contain any proper subrepresentation, and *indecomposable* if it cannot be written as a direct sum of two (or more) non-zero subrepresentations. For finite groups, these two notions coincide. A representation is called *completely reducible* if it is a direct sum of finitely many irreducible representations, i.e. if it can be fully decomposed into irreducible pieces. An irreducible representation of G can become reducible if we restrict to a subgroup $H < G$, and its decomposition into irreducible representations of H is given by the so-called *branching rules*.

Maschke's theorem states that all (finite-dimensional) representations of a finite group are always completely reducible. There are two steps for proving this. First we show that a unitary representation is always completely reducible, by using the fact that given an inner product $\{\cdot, \cdot\} : V \times V \to \mathbb{C}$, the orthogonal complement of U in V is also a subrepresentation if U itself is a subrepresentation of V. Next we show that any representation is unitary with respect to the group-invariant inner product

$$\{v, w\} := \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle, \quad v, w \in V \ , \tag{2.1.14}$$

which then completes the proof.

We also mention the *dual representation* ρ^* of a representation ρ, defined by

$$\rho^*(g) := \left(\rho\left(g^{-1}\right)\right)^T \ , \quad g \in G \ , \tag{2.1.15}$$

which is the natural group action on the dual space $V^* = \mathrm{End}(V, \mathbb{C})$. Taking ρ to be unitary, we have $\rho^*(g) = \overline{\rho(g)}$. In other words, the dual representation is equivalent to the complex conjugate representation.

Characters. The character χ_ρ of a representation (V, ρ), with V a vector space over $\mathbb{C}$, is a map $G \to \mathbb{C}$ defined by the trace of the representation matrices,

$$\chi_\rho(g) := \mathrm{Tr}(\rho(g)) \ , \quad g \in G. \tag{2.1.16}$$

We will also often denote this trace by $\mathrm{Tr}_V\, g$. If ρ is irreducible, χ_ρ is called an irreducible character. Some properties of characters (for finite groups) are summarized below:

- The character is a *class function*, i.e. $\chi_\rho\left(hgh^{-1}\right) = \chi_\rho(g) \quad \forall\ g, h \in G$. This follows directly from the cyclic property of the trace.
- Two complex representations for a finite group have the same characters if and only if they are equivalent, which can be shown using the orthogonality property discussed below.
- The restriction of a character of G to a subgroup $H < G$ is a character of H.
- $\chi_\rho\left(g^{-1}\right) = \overline{\chi_\rho(g)}$, as follows from the fact that all eigenvalues of $\rho(g)$ are $|g|$-th roots of unity.
- For two representations ρ, ρ' of G and $g \in G$, we have:

$$\chi_{\rho\oplus\rho'}(g) = \chi_\rho(g) + \chi_{\rho'}(g)\ ,\ \ \chi_{\rho\otimes\rho'}(g) = \chi_\rho(g)\chi_{\rho'}(g)\ ,\ \ \chi_{\rho^*}(g) = \overline{\chi_\rho(g)}\ . \quad (2.1.17)$$

  This means that the characters form a commutative and associative algebra.

Characters are extremely important for theories with finite symmetry groups, as well as moonshine; they provide a way to "count" states, thus providing the building blocks for calculating various indices.

Orthogonality. Due to Schur's orthogonality relations (e.g. §4 of [6]), characters of unitary representations are equipped with a Hermitian inner product,

$$\langle \chi_\rho, \chi_{\rho'} \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_\rho(g)\overline{\chi_{\rho'}(g)}\ . \quad (2.1.18)$$

When ρ and ρ' are irreducible representations, one can show that $\langle \chi_\rho, \chi_{\rho'} \rangle = 1$ if the two irreducible representations are equivalent, and it vanishes otherwise. As a result, characters of irreducible representations are orthonormal vectors in the space of class functions. In fact, it is possible to show that they span this space, from which one can conclude the important fact that the number of (inequivalent) irreducible representations equals the number of conjugacy classes (see for instance §3-7 of [5]). Moreover, one can show that there is another orthonormality property,

$$\sum_\rho \chi_\rho(g)\ \overline{\chi_\rho(h)} = \begin{cases} |C_G(g)|, & h \in [g] \\ 0, & \text{otherwise} \end{cases}\ , \quad (2.1.19)$$

where the sum is over inequivalent irreducible representations, and $|C_G(g)|$ denotes the order of the centralizer of $g \in G$, which is equal to the order of the group divided

		1A	2A	3A	3B	4A	5A	5B
	2P	1A	1A	3A	3B	2A	5B	5A
	3P	1A	2A	1A	1A	4A	5B	5A
	5P	1A	2A	3A	3B	4A	1A	1A
χ_1		1	1	1	1	1	1	1
χ_2		5	1	2	-1	-1	0	0
χ_3		5	1	-1	2	-1	0	0
χ_4		8	0	-1	-1	0	A	$^*A$
χ_5		8	0	-1	-1	0	$^*A$	A
χ_6		9	1	0	0	1	-1	-1
χ_7		10	-2	1	1	0	0	0

Table 2.1: Character table for $\mathcal{A}_6$, where $A = \frac{1-\sqrt{5}}{2}$ and $^*A = \frac{1+\sqrt{5}}{2}$.

by the number of elements in the conjugacy class $[g]$.

Character table. We have already mentioned that the number of irreducible representations of a finite group G is equal to the number of conjugacy classes of G. We can group all characters of G into its *character table*, which is a square table of size $\mathrm{Cl}(G) \times \mathrm{Cl}(G)$, with rows labelling the different irreducible representations and columns labelling the different conjugacy classes. In other words, the (i, j) component of the character table is the character $\chi_i(g)$ of the i-th irreducible representation, evaluated at any g in the j-th conjugacy class . As an example, the character table for the alternating group $\mathcal{A}_6$ is displayed in Table 2.1. Note that there is an additional piece of information in the above table, the so-called power map. The row starting with sP gives the conjugacy classes $[g^s]$. Character tables are a very useful tool to visualize the characters of finite groups, and come in handy in all studies that focus on moonshine and their modules (see for example §C).

Supermodules. We say that a G-module on a superspace ($\mathbb{Z}_2$-graded vector space) is a G-supermodule. Explicitly, if V is a G-supermodule it has the structure

$$V = V^{(+)} \oplus V^{(-)} \tag{2.1.20}$$

where $V^{(+)}$ and $V^{(-)}$ are both G-modules. We will sometimes refer to V as a *virtual representation* of G. The supertrace Str is defined to act with a minus sign on the odd subspaces: $\mathrm{Str}_V g := \mathrm{Tr}_{V^{(+)}}\, g - \mathrm{Tr}_{V^{(-)}}\, g$. Not surprisingly, supermodules become relevant when there is some amount supersymmetry involved, which is often the case in both moonshine and CFTs.

Cycle shapes and Frame shapes. As the name suggests, an N-dimensional *permutation representation* ρ_p of a group G has as its representation matrices $N \times N$

permutation matrices (all elements zero, apart from a single entry of 1 in each row and column). Given such a representation, to each conjugacy class in such a representation we can associate a *cycle shape*, which encodes the number and type of permutation cycles that elements of this class correspond to. A cycle shape has the general form

$$n_1^{\ell_1} n_2^{\ell_2} \cdots n_r^{\ell_r} \ , \quad \sum_{s=1}^{r} \ell_s n_s = N \ , \tag{2.1.21}$$

where n_s, ℓ_s are all positive integers, and n denotes an n-*cycle*, i.e. it represents a permutation of n elements. The exponents ℓ_s count the number of n_s-cycles. Clearly, an order k element can only have cycles of size n_s which divides k. Note that the cycle shapes can be read directly off the character table, including the power map.

More generally, we can define the *Frame shape* of $g \in G$ given any representation ρ, provided that all characters of ρ are rational numbers. Their rationality ensures that if λ is an eigenvalue of g, then λ^k is also an eigenvalue when k is co-prime to $|g|$ [16]. Denoting by $\lambda_1, \lambda_2, \ldots, \lambda_N$ the g-eigenvalues, then there exists a set of positive integers $i_1, i_2, \ldots, i_s$ and a set of non-zero integers $\ell_1, \ell_2, \ldots, \ell_s$ with the same cardinality such that

$$\det(\mathbf{1} - t\rho(g)) = \prod_{i=1}^{N} (1 - t\lambda_i) = \prod_{r=1}^{s} (1 - t^{i_r})^{\ell_r} \ . \tag{2.1.22}$$

Clearly one must have $\sum_{s=1}^{r} \ell_s i_s = N$ and we call $i_1^{\ell_1} i_2^{\ell_2} \cdots i_r^{\ell_r}$ the Frame shape of the conjugacy class $[g]$ for the representation ρ.

We will see these shapes appear prominently when discussing umbral moonshine §2.4.2.

2.2 Modular objects

Here we introduce the concept of modular forms and their extensions, including mock modular forms, Jacobi forms, and mock Jacobi forms, all of which appear in various instances of moonshine.

2.2.1 Modular forms

One of the standard references on modular forms, which we partially follow here, is [17]. We start by considering the well-known fact that $SL_2(\mathbb{R})$ acts on the upper-half plane $\mathbb{H} := \{\tau \in \mathbb{C} |\ \Im(\tau) > 0\}$ by a fractional linear (Möbius) transformation:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{H} \to \mathbb{H}\ , \quad \tau \mapsto \gamma\tau := \frac{a\tau + b}{c\tau + d}\ . \tag{2.2.1}$$

In order to define various modular forms we also need to consider discrete subgroups of $SL_2(\mathbb{R})$, an important example of which is the modular group $SL_2(\mathbb{Z})$. It is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ , \tag{2.2.2}$$

which satisfy $(ST)^3 = 1$ and $S^2 = -1$. We will often work with $PSL_2(\mathbb{Z}) \cong SL_2(\mathbb{Z})/\{\pm 1\}$, which is also the mapping class group of the torus (cf. §2.3). Moreover, we will often consider the upper-half plane extended by adding the *cusps* $\{i\infty\} \cup \mathbb{Q}$, on which $SL_2(\mathbb{Z})$ acts transitively, as is evident from the fact that $\gamma\infty = \frac{a}{c}$.

We first define weight zero modular forms on the *modular group* $SL_2(\mathbb{Z})$, which are simply holomorphic functions on $\mathbb{H}$ that are invariant under the action of $SL_2(\mathbb{Z})$:

$$f(\tau) = f(\gamma\tau) \quad \forall\ \gamma \in SL_2(\mathbb{Z}). \tag{2.2.3}$$

In particular, f has to be holomorphic as τ approaches the boundary of $\mathbb{H}$ at the cusps $\{i\infty\} \cup \mathbb{Q}$. But this turns out to be too restrictive: basic complex analysis tells us that constants are the only such functions. As a result, we would like to further generalise the above definition in the following directions:

1. *Analyticity*: the function is allowed to have exponential growth near the cusps. Such functions are said to be *weakly holomorphic modular forms.*
2. *Weights*: one allows for a scaling factor in the transformation rule. See (2.2.6).
3. *Other Groups*: one replaces $SL_2(\mathbb{Z})$ by a general $\Gamma < SL_2(\mathbb{R})$ in the transformation property (2.2.3).
4. *Multipliers*: one modifies the transformation rule (2.2.3) by allowing for a non-trivial character $\psi : SL_2(\mathbb{Z}) \to \mathbb{C}^*$. See (2.2.6).
5. *Vector-Valued*: instead of $f : \mathbb{H} \to \mathbb{C}$ we consider a vector-valued function $f : \mathbb{H} \to \mathbb{C}^n$ with n components.

Of course, the above generalisations can be combined. For instance one can consider a vector-valued modular form with multipliers for a subgroup Γ of $SL_2(\mathbb{R})$. In the vector-valued case the character ψ is, of course, no longer a phase but a matrix. Also, the above concepts are not entirely independent. For instance, a component of a vector-valued modular form for $SL_2(\mathbb{Z})$ can be considered as a (single-valued) modular form for a subgroup of $SL_2(\mathbb{Z})$, and vice versa.

Let's start with the first generalisation and introduce the concept of *modular functions*. We say that $f : \mathbb{H} \to \mathbb{C}$ is a modular function if f is meromorphic in $\mathbb{H}$, satisfies the transformation rule (2.2.3), and grows like $e^{2\pi i\tau m}$ for some $m > -\infty$. In fact, modular functions form a function field with a single generator, called the *Hauptmodul* or *principal modulus*. This is because the fundamental domain $SL_2(\mathbb{Z})\backslash\mathbb{H}$ is a genus zero Riemann surface when finitely many points are added. Writing the upper-half plane with the cusps attached as $\widehat{\mathbb{H}} = \mathbb{H} \cup \{i\infty\} \cup \mathbb{Q}$, the Hauptmodul has the property that it is an isomorphism between the two spheres $SL_2(\mathbb{Z})\backslash\widehat{\mathbb{H}}$ and $\widehat{\mathbb{C}}$. Such a Hauptmodul is unique up to Möbius transformations, or the choice of three points on the sphere. As a result, there is a unique Hauptmodul with the expansion

$$J(\tau) = q^{-1} + O(q) \tag{2.2.4}$$

near $\tau \to i\infty$. Here and in what follows we will write $q := \mathrm{e}(\tau)$, where $\mathrm{e}(x) := e^{2\pi i x}$ for $x \in \mathbb{C}$. In terms of the Eisenstein series and Dedekind eta function (cf. (2.2.7) and (2.2.19)), the J-function is given by

$$J(\tau) = j(\tau) - 744 = \frac{E_4^3(\tau)}{\eta^{24}(\tau)} - 744. \tag{2.2.5}$$

In general, a Hauptmodul can be defined as the generator of the field of modular functions for $\Gamma \leq SL_2(\mathbb{R})$ whenever $\Gamma\backslash\widehat{\mathbb{H}}$ is genus zero. These Hauptmoduls play an important role in moonshine.

Apart from the definition given above, there are three other equivalent ways of viewing modular functions. First, due to (2.2.3) we can view f as a function from the suitably compactified fundamental domain $SL_2(\mathbb{Z})\backslash\mathbb{H}$ to the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Second, due to the relation between $SL_2(\mathbb{Z})$ and rank two lattices we can associate to each τ a complex lattice $\Lambda_\tau := 1\cdot\mathbb{Z} + \tau\cdot\mathbb{Z}$, and identify f as a function that associates to each such lattice Λ_τ a complex function $f(\tau)$, which is moreover invariant under a rescaling of the lattice. The third way, which plays an important role in the a relation between modular forms and 2-dimensional conformal field theories, stems from the interpretation of $SL_2(\mathbb{Z})\backslash\mathbb{H}$ as the complex structure moduli space of a Riemann surface of genus one. This can be easily understood from the fact that a torus can be described as the complex plane modulo a rank two lattice, and is therefore up to a scale given by $\mathbb{C}/\Lambda_\tau$ for some $\tau \in \mathbb{H}$. The modular function can then be thought of as associating to each torus a complex number which only depends on its complex structure modulus τ. In this context, the group $PSL_2(\mathbb{Z}) := SL_2(\mathbb{Z})/\{\mathbb{1}, -\mathbb{1}\}$ plays the role of the mapping class group of a torus (cf. §2.3), where the $\mathbb{Z}_2 = \{\mathbb{1}, -\mathbb{1}\}$ central subgroup acts trivially on $\mathbb{H}$.

Next we turn to the second generalisation and introduce modular forms on the modular group $SL_2(\mathbb{Z})$ of a general weight k. They are defined as holomorphic functions on $\mathbb{H}$ that transform under the action of $SL_2(\mathbb{Z})$ as:

$$f(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ . \tag{2.2.6}$$

From the lattice point of view, we consider complex functions f associated to a lattice Λ that scale like $f \mapsto \lambda^{-k} f$ under a rescaling $\Lambda \mapsto \lambda\Lambda$, $\lambda \in \mathbb{C}$, of the lattice. We will consider integral and half-integral weight k.[1]

With this definition we start to get some non-trivial examples, even when holomorphicity at the cusps is required. For instance the following Eisenstein series

$$\begin{aligned} E_4(\tau) &= 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} = 1 + 240\, q + 2160\, q^2 + \dots \\ E_6(\tau) &= 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} = 1 - 504\, q - 16632\, q^2 + \dots \end{aligned} \tag{2.2.7}$$

are examples of modular forms of weight 4 and weight 6, respectively. But the definition is still somewhat too restrictive as these two Eisenstein series are all there is: the ring of modular forms on $SL_2(\mathbb{Z})$ is generated freely by E_4 and E_6. Namely, any modular form of integral weight k can be written (uniquely) as a sum of monomials $E_4^\alpha E_6^\beta$ with $k = 4\alpha + 6\beta$. We denote the space of modular forms of weight k for group Γ by $M_k(\Gamma)$. Among modular forms, the so-called cusp forms are often of special interest. We say that a modular form f of weight k is a cusp form if $y^{k/2} f(x + iy)$ is bounded as $y \to \infty$. This condition guarantees that f has vanishing constants in its Fourier coefficients at all cusps.

In the third type of generalisation, we often encounter the $SL_2(\mathbb{Z})$ subgroups defined by the following congruences. For a positive integer N, we define

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ \mid c \equiv 0 \bmod N \right\} \ . \tag{2.2.8}$$

Below we will illustrate the generalisations above with some examples, which will also come in handy in the next chapters.

First we consider the Jacobi theta functions. Consider a 1-dimensional lattice with bilinear form $\langle x, x \rangle = x^2$. The associated theta function is

$$\theta_3(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2/2} \, . \tag{2.2.9}$$

[1] Clearly, special care needs to be taken when k is half-integral. Strictly speaking, one should work with the metaplectic double cover of $SL_2(\mathbb{Z})$. However we will avoid discussing the subtleties here as they will not cause any difficulty for us. We will refer the reader to [18] for more details.

This simple function turns out to admit an expression in terms of infinite products

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2})^2 \ , \tag{2.2.10}$$

and has nice modular properties. To describe the modular properties, it is most natural to introduce another two theta functions,

$$\begin{aligned} \theta_2(\tau) &= \sum_{n+\frac{1}{2}\in\mathbb{Z}} q^{n^2/2} = 2q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2 \ , \\ \theta_4(\tau) &= \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1/2})^2 \ . \end{aligned} \tag{2.2.11}$$

It turns out that they are the three components of a vector-valued modular form for $SL_2(\mathbb{Z})$

$$\Theta(\tau) := \begin{pmatrix}\theta_2(\tau)\\ \theta_3(\tau)\\ \theta_4(\tau)\end{pmatrix}, \quad \Theta(\tau) = \sqrt{\frac{i}{\tau}}\mathcal{S}\,\Theta\left(-\frac{1}{\tau}\right) = \mathcal{T}\,\Theta(\tau+1)\,, \tag{2.2.12}$$

where

$$\mathcal{S} = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix}e(-\frac{1}{8})&0&0\\0&0&1\\0&1&0\end{pmatrix}. \tag{2.2.13}$$

To illustrate the relation between vector-valued modular forms and modular forms for a congruence subgroup, consider $\theta(\tau) := \theta_3(2\tau)$. This transforms in the following way as a weight 1/2 modular form for $\Gamma_0(4)$ with a non-trivial multiplier:

$$\theta(\tau) = \left(\frac{c}{d}\right)\epsilon_d\ (c\tau+d)^{-\frac{1}{2}}\ \theta(\gamma\tau) \tag{2.2.14}$$

for all $\gamma\in\Gamma_0(4)$, where

$$\epsilon_d := \begin{cases} 1\ , & d\equiv 1 \bmod 4\\ i\ , & d\equiv 3 \bmod 4\end{cases}$$

and the Legendre symbol used above is defined as[2]

$$\left(\frac{\kappa}{\lambda}\right) := \begin{cases} +1\ , & \text{if}\ \ \kappa\not\equiv 0 \bmod \lambda \text{ and } \kappa \text{ is a quadratic residue modulo } \lambda\\ -1\ , & \text{if}\ \ \kappa\not\equiv 0 \bmod \lambda \text{ and } \kappa \text{ is not a quadratic residue modulo } \lambda\\ 0\ , & \text{if}\ \ \kappa\equiv 0 \bmod \lambda\end{cases} .$$

[2] κ is said to be a quadratic residue modulo λ if $\exists\ x\in\mathbb{Z}$ such that $x^2\equiv\kappa \bmod \lambda$.

Later we will see that these theta functions can be naturally considered as the specialisation at $z = 0$ of the two-variable Jacobi theta functions, defined either as infinite sums or infinite products:

$$\begin{aligned}
\theta_1(\tau,z) &= -i \sum_{n+\frac{1}{2}\in\mathbb{Z}} (-1)^{n-\frac{1}{2}} y^n q^{n^2/2} \\
&= -iq^{1/8}(y^{1/2}-y^{-1/2}) \prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n)\,, \\
\theta_2(\tau,z) &= \sum_{n+\frac{1}{2}\in\mathbb{Z}} y^n q^{n^2/2} \\
&= (y^{1/2}+y^{-1/2})q^{1/8} \prod_{n=1}^{\infty}(1-q^n)(1+yq^n)(1+y^{-1}q^n)\,, \\
\theta_3(\tau,z) &= \sum_{n\in\mathbb{Z}} y^n q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1+yq^{n-1/2})(1+y^{-1}q^{n-1/2})\,, \\
\theta_4(\tau,z) &= \sum_{n\in\mathbb{Z}} (-1)^n y^n q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1-yq^{n-1/2})(1-y^{-1}q^{n-1/2})\;.
\end{aligned} \tag{2.2.15}$$

They transform in the following way. Let

$$\Theta(\tau,z) := \begin{pmatrix}\theta_1(\tau,z)\\ \theta_2(\tau,z)\\ \theta_3(\tau,z)\\ \theta_4(\tau,z)\end{pmatrix}\;. \tag{2.2.16}$$

Then we have (cf. §2.2.2),

$$\Theta(\tau,z) = \sqrt{\frac{i}{\tau}}\, e\big(-\frac{z^2}{2\tau}\big)\, \mathcal{S}'\,\Theta\big(-\frac{1}{\tau},\frac{z}{\tau}\big) = \mathcal{T}'\,\Theta(\tau+1,z)\,, \tag{2.2.17}$$

where

$$\mathcal{S}' = \begin{pmatrix} i & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}, \quad \mathcal{T}' = \begin{pmatrix} e(-1/8) & 0 & 0 & 0\\ 0 & e(-1/8) & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}. \tag{2.2.18}$$

Another modular form one frequently encounters is the *Dedekind eta function*

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\;. \tag{2.2.19}$$

It is a weight 1/2 modular form with a non-trivial multiplier, satisfying

$$\eta(\tau) = \sqrt{\frac{i}{\tau}}\eta\big(-\frac{1}{\tau}\big), \quad \eta(\tau) = e\big(-\frac{1}{24}\big)\,\eta(\tau+1). \tag{2.2.20}$$

It is related to the theta functions by

$$\eta(\tau)^3 = \frac{1}{2}\theta_2(\tau)\theta_3(\tau)\theta_4(\tau). \tag{2.2.21}$$

Its 24-th power $\Delta := \eta^{24}$ is the familiar weight 12 cusp form for the modular group $SL_2(\mathbb{Z})$.

2.2.2 Skew-holomorphic modular forms

Here we collect the definitions of (skew-)holomorphic Jacobi forms. These types of objects play a crucial role in moonshine and its connection to physics, especially for chapter §3. This subsection, consisting mostly of definitions, follows §3.1 of [19] very closely.

We first define elliptic forms [20]. For m an integer define the index m *elliptic action* of the group $\mathbb{Z}^2$ on functions $\phi : \mathbb{H}\times\mathbb{C}\to\mathbb{C}$ by setting

$$(\phi|_m(\lambda,\mu))(\tau,z) := \mathrm{e}(m\lambda^2\tau + 2m\lambda z)\,\phi(\tau, z+\lambda\tau+\mu) \tag{2.2.22}$$

for $(\lambda,\mu)\in\mathbb{Z}^2$. Say that a smooth function $\phi : \mathbb{H}\times\mathbb{C}\to\mathbb{C}$ is an *elliptic form* of index m if $z\mapsto\phi(\tau,z)$ is holomorphic. Denote by $\mathcal{E}_m$ the space of elliptic forms of index m. Observe that any elliptic form $\phi\in\mathcal{E}_m$ admits a *theta-decomposition*

$$\phi(\tau,z) = \sum_{r \bmod 2m} h_r(\tau)\theta_{m,r}(\tau,z)\ , \tag{2.2.23}$$

where the theta series are given by

$$\theta_{m,r}(\tau,z) := \sum_{\ell = r \bmod 2m} q^{\ell^2/4m}y^\ell\ , \tag{2.2.24}$$

for some $2m$ smooth functions $h_r : \mathbb{H}\to\mathbb{C}$. To see this, note from $\phi(\tau,z)=\phi(\tau,z+1)$ that we have $\phi(\tau,z)=\sum_{\ell\in\mathbb{Z}}c_\ell(\tau)y^\ell$ for some $c_\ell : \mathbb{H}\to\mathbb{C}$. Then the identity $\phi|_m(1,0)=\phi$ implies that $c_r(\tau)q^{-r^2/4m}$ depends only on $r \bmod 2m$. The $2m$ functions $h_r(\tau) := c_r(\tau)q^{-r^2/4m}$ are precisely the theta-coefficients of ϕ in the theta-decomposition.

It will be convenient to regard h_r and $\theta_{m,r}$ in (2.2.23) as defining $2m$-vector-valued functions $h := (h_r)_{r \bmod 2m}$ and $\theta_m := (\theta_{m,r})_{r \bmod 2m}$. Then the theta-decomposition

(2.2.23) may be more succinctly written as $\phi = h^t\theta_m$, where the superscript t denotes matrix transpose. It follows from the Poisson summation formula that the vector-valued function $\theta_m = (\theta_{m,r})$ has the following behaviour under $SL_2(\mathbb{Z})$:

$$\theta_m\left(-\frac{1}{\tau}, \frac{z}{\tau}\right)\frac{1}{\sqrt{\tau}}\,\mathrm{e}\left(-\frac{mz^2}{\tau}\right) = \mathcal{S}\theta_m(\tau,z), \quad \theta_m(\tau+1,z) = \mathcal{T}\theta_m(\tau,z), \tag{2.2.25}$$

where $\mathcal{S} = (\mathcal{S}_{rr'})$ and $\mathcal{T} = (\mathcal{T}_{rr'})$ are unitary matrices defined for a fixed positive integer m, given by $\mathcal{S}_{rr'} := \frac{1}{\sqrt{2m}}\,\mathrm{e}\left(-\frac{1}{8} - \frac{rr'}{2m}\right)$ and $\mathcal{T}_{rr'} := \mathrm{e}\left(\frac{r^2}{4m}\right)\delta_{r,r'}$. (Cf. e.g. §5 of [21].) This suggests that we obtain elliptic forms $\phi = h^t\theta_m \in \mathcal{E}_m$ with good modular transformation properties $SL_2(\mathbb{Z})$ by requiring suitable conditions on h.

To formulate these notions precisely, define the weight k *modular*, and *skew-modular* actions of $SL_2(\mathbb{Z})$ on $\mathcal{E}_m$, for k and m integers, by setting

$$\begin{aligned}(\phi|_{k,m}\gamma)(\tau,z) &:= \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)\frac{1}{(c\tau+d)^k}\,\mathrm{e}\left(-\frac{cmz^2}{c\tau+d}\right)\\ (\phi|^{\mathrm{sk}}_{k,m}\gamma)(\tau,z) &:= \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)\frac{1}{(c\bar\tau+d)^k}\frac{c\bar\tau+d}{|c\tau+d|}\,\mathrm{e}\left(-\frac{cmz^2}{c\tau+d}\right)\ ,\end{aligned} \tag{2.2.26}$$

for $\phi \in \mathcal{E}_m$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.

Roughly speaking, a Jacobi form of weight k and index m is an elliptic form $\phi(\tau,z)$, holomorphic in the τ-variable, which is moreover invariant under $|_{k,m}\gamma$ for all $\gamma \in SL_2(\mathbb{Z})$. Note that $\phi|_{k,m}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right) = \phi$ implies the expansion

$$\phi(\tau,z) = \sum_{\substack{D,\ell\in\mathbb{Z}\\ D=\ell^2 \bmod 4m}} C_\phi(D,\ell)q^{-D/4m}q^{\ell^2/4m}y^\ell\ , \tag{2.2.27}$$

where $C_\phi(D,\ell)$ depends only on $\ell \bmod 2m$, corresponding to the theta decomposition

$$h_r(\tau) = \sum_{\substack{D\in\mathbb{Z}\\ D=r^2 \bmod 4m}} C_\phi(D,r)q^{-D/4m}\ . \tag{2.2.28}$$

The invariance under $SL_2(\mathbb{Z})$ of $\phi = h^t\theta_m$ leads to the modularity of the vector-valued function $h = (h_r)$. In other words, $h = (h_r)$ transforms as a vector-valued modular form and contains precisely the same information as the Jacobi form. To complete the definition, we also need to specify the growth behaviour of $h(\tau)$ near the cusp. We say that $\phi \in \mathcal{E}_m$, invariant under $|_{k,m}\gamma$ for all $\gamma \in SL_2(\mathbb{Z})$, is a *weak holomorphic/holomorphic/cuspidal holomorphic Jacobi form* if the Fourier coefficients satisfy $C_\phi(D,r) = 0$ unless $-D + r'^2 \geq 0$ for all $r' = r \bmod 2m$, $C_\phi(D,r) = 0$ for

$D > 0$, or $C_\phi(D,r) = 0$ for $D \geq 0$, respectively. We denote the space of weak holomorphic Jacobi forms of weight k and index m by $J^{\text{wk}}_{k,m}$. Notice that, at odd weight, applying (2.2.26) to the case $\gamma = -\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ shows that the Jacobi form must be odd under $z \leftrightarrow -z$. It will therefore be convenient to introduce

$$\tilde{\theta}_{m,r} = \theta_{m,r} - \theta_{m,-r} \ . \tag{2.2.29}$$

We now turn to the closely related skew-holomorphic Jacobi forms. An elliptic form $\phi \in \mathcal{E}_m$ is called a *weak skew-holomorphic Jacobi form* if it meets the following conditions. First, its theta-coefficients are anti-holomorphic functions on $\mathbb{H}$; second, it is invariant for the weight k skew-modular action (2.2.26), so that $\phi|^{\text{sk}}_{k,m}\gamma = \phi$ for all $\gamma \in SL_2(\mathbb{Z})$; finally, $\tau \mapsto \phi(\tau, z)$ remains bounded as $\Im(\tau) \to \infty$ for fixed $z \in \mathbb{C}$. Thus a weak skew-holomorphic Jacobi form admits a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{D,\ell \in \mathbb{Z} \\ D = \ell^2 \bmod 4m}} C_\phi(D,\ell)\, \bar{q}^{D/4m} q^{\ell^2/4m} y^\ell \ , \tag{2.2.30}$$

for some $2m$ functions $D \mapsto C_\phi(D,r)$, and we recover its theta-coefficients by writing

$$h_r(\tau) = \sum_{\substack{D \in \mathbb{Z} \\ D = r^2 \bmod 4m}} C_\phi(D,r) \bar{q}^{D/4m} \ . \tag{2.2.31}$$

A weak skew-holomorphic Jacobi form ϕ is called a *skew-holomorphic Jacobi form*, or a *cuspidal skew-holomorphic Jacobi form*, when the Fourier coefficients satisfy $C_\phi(D,r) = 0$ for $D < 0$, or $C_\phi(D,r) = 0$ for $D \leq 0$, respectively.

We will close this subsection with an example that will come in handy later. Define

$$\begin{aligned} \phi_{0,1}(\tau, z) &= 4 \sum_{i=2,3,4} \Big(\frac{\theta_i(\tau,z)}{\theta_i(\tau,0)}\Big)^2 \ , \\ \phi_{-2,1} &= -\frac{\theta_1(\tau,z)^2}{\eta^6(\tau)} \ . \end{aligned} \tag{2.2.32}$$

The ring of weak Jacobi forms of even weight is freely generated by $\phi_{0,1}$ and $\phi_{-2,1}$ over the ring of modular forms for $SL_2(\mathbb{Z})$:

$$J^{\text{wk}}_{2k,m} = \sum_{j=0}^{m} M_{2k+2j}(SL_2(\mathbb{Z}))\, \phi^j_{-2,1} \phi^{m-j}_{0,1} \ . \tag{2.2.33}$$

The function $\phi_{0,1}$ plays an important role in Mathieu and umbral moonshine, since $2\phi_{0,1}$ coincides with the $K3$ elliptic genus $\mathbf{EG}(\tau, z; K3)$. See §2.3.3 for a definition of the elliptic genus.

2.2.3 Mock modular forms

Here we introduce mock modular forms and the closely related concept of mock Jacobi forms. We follow the treatment of §7.1 of [20] and §3.2 of [19] closely. The subject, initiated by the legendary mathematician Srinivasa Ramanujan, has a fascinating history. We recommend [22] for a short account of it.

Let $w \in \frac{1}{2}\mathbb{Z}$ and let h be a holomorphic function on $\mathbb{H}$ with at most exponential growth at all cusps. We say that h is a (weakly holomorphic) mock modular form of weight w for a discrete subgroup $\Gamma \leq SL_2(\mathbb{R})$ if there is a modular form g of weight $2-w$ such that the sum $\hat{h} := h + g^*$ transforms like a holomorphic modular form of weigh w for Γ. Moreover, we say that g is the *shadow* of the mock modular form h and $\hat{h}$ is its *completion*. In the above we have used the following definition of g^*. Writing the Fourier expansion of g as $g(\tau) = \sum_{n\geq 0} c_g(n)q^n$, then

$$g^*(\tau) := \overline{c_g(0)}\frac{(-\Im(\tau))^{1-w}}{w-1} + \sum_{n>0}(-4\pi n)^{w-1}\overline{c_g(n)}q^{-n}\Gamma(1-w, 4\pi n\Im(\tau)) \ , \quad (2.2.34)$$

where $\Gamma(1-w,x) = \int_x^\infty e^{-t}t^w dt$ denotes the incomplete gamma function. When $c_g(0) = 0$, the above coincides with the so-called *non-holomorphic weight w Eichler integral* of g, given by

$$g^*(\tau) := (-2)^{w-1}\,\mathrm{e}(\tfrac{w-1}{4})\int_{-\bar{\tau}}^{\infty}(\tau'+\tau)^{-w}\overline{g(-\overline{\tau'})}\mathrm{d}\tau' \ . \quad (2.2.35)$$

Note that

$$-2i\Im(\tau)^w\frac{\partial}{\partial\bar{\tau}}g^*(\tau) = \overline{g(\tau)} \ , \quad (2.2.36)$$

and hence $\hat{h}$ is annihilated by the weight w Laplacian $\Delta_w := \Im(\tau)^{2-w}\partial_\tau\Im(\tau)^w\partial_{\bar{\tau}}$. Such functions are called harmonic Maass forms, and one can identify h as the (uniquely defined) holomorphic part of the harmonic Maass form $\hat{h}$. Finally, note that from (2.2.34) it is obvious that the harmonic Maass form $\hat{h}$ transforms with a multiplier which is the inverse of that of the modular form g.

Just as in the case of usual modular forms, one can generalise the above definition of mock modular forms in various directions, including incorporating non-trivial multiplier systems and considering vector-valued mock modular forms. Next we turn our attention to a specific type of vector-valued mock modular forms, namely those arising from the so-called mock Jacobi forms. For integers k and m, we say that an elliptic form $\phi \in \mathcal{E}_m$ is a *weak mock Jacobi form* of weight k and index m if the following is true. Write the theta-decomposition of ϕ as $\phi = \sum_r h_r \theta_{m,r}$. First, $\tau \mapsto \phi(\tau, z)$ is bounded as $\Im(\tau) \to \infty$ for every fixed $z \in \mathbb{C}$; second, all the h_r are holomorphic; finally, there exists a skew-holomorphic Jacobi form $\sigma = \sum_r \overline{g_r} \theta_{m,r} \in S^{\rm sk}_{3-k,m}$, such that $\hat{\phi} := \sum_r \hat{h}_r \theta_{m,r}$ is invariant for the weight k modular action $|_{k,m}$ of $SL_2(\mathbb{Z})$ on $\mathcal{E}_m$ (cf. (2.2.26)) with the definition

$$\hat{h}_r(\tau) := h_r(\tau) + \frac{1}{\sqrt{2m}} g_r^*(\tau) \ . \tag{2.2.37}$$

As was discussed in [23] and analysed carefully in [20, 24], meromorphic Jacobi forms – what one obtains when relaxing the condition on Jacobi forms to allow for poles at torsion points $z \in \mathbb{Q} + \mathbb{Q}\tau$ – naturally give rise to mock Jacobi forms. In particular, all the mock Jacobi forms featured in umbral moonshine can be viewed as arising from meromorphic Jacobi forms.

From a physical point of view, as demonstrated in a series of recent works, the "mockness" of these mock modular objects is often related to the non-compactness of relevant spaces in the theory. See, for instance, [20, 25–28]. Let us take 2d CFTs with a non-compact target space as an example. The non-compactness of the target space often leads to a continuous part of the spectrum. In this case the standard CFT arguments might fail. In particular there could be imperfect pairing between the bosonic and fermionic states in the continuous part of the spectrum and we could end up with a non-holomorphic BPS index, given by the completion of a mock modular object, as a result. See for instance [26, 29–33] for details for some specific examples, and see the remark at the end of §2.3.3 for a more detailed discussion in the context of elliptic genus. Another context in which non-compactness appears and leads to a role for mock modular forms is wall-crossing (when approaching the wall, the distance of the bound black hole centers goes to infinity). The BPS counting of the black hole microstates hence depends on the moduli and correspondingly the countour of integration [34], and the result of the integration is mock modular [20].

Another source of mock modular forms in physics is the characters of supersymmetric infinite algebras, such as the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal algebras mentioned in §2.4.2. Some more examples can be found in for instance [35] and references therein. Interestingly, as we will explain in §3.1, the mockness of the mock modular form in (2.2.42) can be seen as either arising from CFT with non-compact target space or as a result to the mockness of characters of the $\mathcal{N} = 4$ superconformal algebra.

We now provide a couple of examples.

- Ramanujan wrote down the following simple-looking Eulerian series in his 1920 letter to Hardy [36],

$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(1-q^{n+1})(1-q^{n+2})\dots(1-q^{2n})} = 1 + q + q^2 + 2\,q^3 + \dots\ ,$$
$$\chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n}{(1-q^{n+1})(1-q^{n+2})\dots(1-q^{2n})(1-q^{2n+1})} = 1 + 2\,q + 2\,q^2 + 3\,q^3 + \dots\ ,$$

as two of the examples of his *mock theta functions* (of order 5). In fact, they are closely related to mock Jacobi forms.

Define $I^{E_8^{\oplus 3}} = \{1, 7\}$, $A = \{1, 11, 19, 29\}$, and

$$H_1^{E_8^{\oplus 3}}(\tau) = q^{-1/120}(2\chi_0(q) - 4)\ ,\quad H_7^{E_8^{\oplus 3}}(\tau) = 2q^{71/120}\chi_1(q), \tag{2.2.38}$$

then $(H_r^{E_8^{\oplus 3}})_{r\in I^{E_8^{\oplus 3}}}$ is a vector-valued mock modular form of weight 1/2 for the modular group. Its shadow is given by the index 30 theta functions

$$g_r^{E_8^{\oplus 3}} = 3 \sum_{a\in A} \theta^1_{30,a\,r}. \tag{2.2.39}$$

Writing $\widehat{H}_r^{E_8^{\oplus 3}}(\tau) = H_r^{E_8^{\oplus 3}} + (g_r^{E_8^{\oplus 3}})^*$, we have

$$\begin{pmatrix} \widehat{H}_1^{E_8^{\oplus 3}} \\ \widehat{H}_7^{E_8^{\oplus 3}} \end{pmatrix}(\tau+1) = \begin{pmatrix} \mathrm{e}(-\frac{1}{120}) & 0 \\ 0 & \mathrm{e}(-\frac{49}{120}) \end{pmatrix} \begin{pmatrix} \widehat{H}_1^{E_8^{\oplus 3}} \\ \widehat{H}_7^{E_8^{\oplus 3}} \end{pmatrix}(\tau) \tag{2.2.40}$$

and

$$\begin{pmatrix} \widehat{H}_1^{E_8^{\oplus 3}} \\ \widehat{H}_7^{E_8^{\oplus 3}} \end{pmatrix}\left(-\frac{1}{\tau}\right) = \tau^{1/2} i^{3/2} \begin{pmatrix} \frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}} & \frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}} \\ \frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}} & -\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}} \end{pmatrix} \begin{pmatrix} \widehat{H}_1^{E_8^{\oplus 3}} \\ \widehat{H}_7^{E_8^{\oplus 3}} \end{pmatrix}(\tau) \tag{2.2.41}$$

Moreover, $H_i^{E_8^{\oplus 3}}$ can be viewed as arising from the theta composition of the mock Jacobi form $\psi^{E_8^{\oplus 3}}(\tau, z) := \sum_{r\in I^{E_8^{\oplus 3}}} H_r^{E_8^{\oplus 3}} \sum_{a\in A} \tilde{\theta}_{a\,r}^{E_8^{\oplus 3}}$. More specifically, $\hat{\psi}^{E_8^{\oplus 3}}(\tau, z) := \sum_{r\in I^{E_8^{\oplus 3}}} \widehat{H}_r^{E_8^{\oplus 3}} \sum_{a\in A} \tilde{\theta}_{a\,r}^{E_8^{\oplus 3}}$ is non-holomorphic in τ and transforms as a Jacobi form of weight 1 and index 30. As the notation suggests, $H_r^{E_8^{\oplus 3}}$ encodes the graded dimension of the umbral moonshine module underlying the case of umbral moonshine corresponding to Niemeier lattice N with root system $E_8^{\oplus 3}$, as we will discuss in §2.4.2.

- Let $H : \mathbb{H} \to \mathbb{C}$ be given by

$$H(\tau) = \frac{-2E_2(\tau) + 48F_2^{(2)}(\tau)}{\eta(\tau)^3} = 2q^{-\frac{1}{8}}\left(-1 + 45\,q + 231\,q^2 + 770\,q^3 \ldots\right)\ , \tag{2.2.42}$$

where

$$E_2 = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}$$

is the weight two Eisenstein series (which is not a modular form) and

$$F_2^{(2)}(\tau) = \sum_{\substack{r>s>0\\ r-s=1 \bmod 2}} (-1)^r\, s\, q^{rs/2} = q + q^2 - q^3 + q^4 + \ldots\ .$$

Note that the first few Fourier coefficients of $H/2$: 45, 231 770, 2277 , 5796, coincide with dimensions of certain irreducible representations of the sporadic group M_{24}! Indeed, in umbral moonshine $H = H^{A_1^{\oplus 24}}$ plays the role of the graded dimensions of the underlying M_{24}-module. See §2.4.2.

This function is a mock modular form with shadow $24\eta^3(\tau)$ (and therefore with a multiplier given by the inverse of that of $\eta^3(\tau)$). In other words,

$$\widehat{H}(\tau) = H(\tau) + 24\,(4i)^{-1/2}\int_{-\bar{\tau}}^{\infty}(z+\tau)^{-1/2}\overline{\eta(-\bar{z})^3}\mathrm{d}z, \tag{2.2.43}$$

transforms as a weight 1/2 modular form for the modular group $SL_2(\mathbb{Z})$.

Moreover, the two-variable function $\psi^{A_1^{\oplus 24}}(\tau, z) := H(\tau)\tilde{\theta}_{2,1}$ is a mock Jacobi form of weight one and index two. This mock Jacobi form can be seen as arising from a meromorphic Jacobi form by subtracting its "polar part". To see this, consider the weight one index two meromorphic Jacobi form

$$\psi(\tau, z) := -2i\frac{\theta_1(\tau, 2z)\eta^3(\tau)}{\theta_1^2(\tau, z)}\phi_{0,1}(\tau, z) = -i\frac{\theta_1(\tau, 2z)\eta^3(\tau)}{\theta_1^2(\tau, z)}\mathbf{EG}(\tau, z; K3)\ , \tag{2.2.44}$$

(cf. (2.2.32)) which has a simple pole at $z \in \mathbb{Z} + \mathbb{Z}\tau$. Then the following identity holds,

$$\psi(\tau, z) = \psi^{A_1^{\oplus 24}}(\tau, z) - 24\,\mathrm{Av}^{(2)}\left[\frac{y+1}{y-1}\right]\ . \tag{2.2.45}$$

In the above $\mathrm{Av}^{(m)}$ denotes the index-m averaging operator

$$\mathrm{Av}^{(m)}[F(y)] = \sum_{k\in\mathbb{Z}} q^{mk^2}y^{2mk}F(q^k y)\ ,$$

with the elliptic symmetry $\mathrm{Av}^{(m)}[F(y)]|_m(\lambda,\mu) = \mathrm{Av}^{(m)}[F(y)]$ for all $\lambda,\mu \in \mathbb{Z}$, and the second term in (2.2.45) can be interpreted as the canonical "polar part" of the meromorphic Jacobi form φ.

2.3 2d CFT

Here we provide a brief summary of some key ingredients of two-dimensional conformal field theories (CFTs) that we will need. CFTs are relevant for moonshine, since in the cases that are known so far the corresponding moonshine modules feature vertex operator algebra (VOA) structures, which capture the structure of the chiral algebra of a two-dimensional CFT. Instead of the more formal VOA language, we opt for the CFT language which is more familiar to physicists. This section is also useful for the discussion on sigma models in §4, especially with regards to orbifolds. More complete references on the basics of CFT include [37–41].

2.3.1 General structure

A conformal field theory is a quantum field theory with conformal symmetry. Conformal transformations are coordinate transformations that preserve the conformal flatness of the metric. Focusing on Riemannian manifolds of Euclidean signature, a metric is said to be conformally flat if it can be written in the form $ds^2 = e^{\omega(x)}\delta_{\mu\nu}dx^\mu dx^\nu$. Conformal transformations locally preserve the angles but may deform the lengths arbitrarily, so conformal symmetry is typically associated with the absence of an intrinsic length scale. On the conformal compactification (by adding the point at infinity which is necessary for the special conformal transformation to be well-defined) of $\mathbb{R}^{n,0}$ for $n \geq 3$, all conformal transformations are globally well-defined and form a group isomorphic to $SO(n+1,1,\mathbb{R})$. The corresponding local transformations thus form a finite-dimensional Lie algebra isomorphic to $\mathfrak{so}(n+1,1,\mathbb{R})$. In two dimensions, however, the condition of conformal invariance is equivalent to the Cauchy-Riemann equation and any holomorphic function gives rise to an infinitesimal conformal transformation. Using the generators

$$\ell_n = -z^{n+1}\partial_z \ , \quad \bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{2.3.1}$$

for $n \in \mathbb{Z}$, we see that the local conformal transformations form an infinite-dimensional Lie algebra, which contains two commuting copies of the Witt algebra with commutation relations

$$\mathfrak{Witt}: \quad [\ell_m, \ell_n] = (m-n)\ell_{m+n} \ . \tag{2.3.2}$$

It is important to emphasise that most of the conformal generators in 2 dimensions are purely local, i.e. they do not generate globally well-defined transformations. Consider, for example, the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \infty$, i.e. the Riemann surface of genus zero. On $\widehat{\mathbb{C}}$, only $\ell_0, \ell_{\pm 1}$ generate global conformal transformations, which form the Möbius group $SO(3,1,\mathbb{R}) \cong PSL(2,\mathbb{C})$.

The quantisation of a 2-dimensional CFT is typically done on $\mathbb{C}$. The theory on the Riemann sphere determines the theory on any other Riemann surface uniquely, but does not guarantee their consistency, as one must also require crossing symmetry and modular invariance of the torus partition function (see below). To see how to quantise on $\mathbb{C}$, we note that $\mathbb{C}$ with origin removed is conformally equivalent to a cylinder $S^1 \times \mathbb{R}$. Denoting by y and t the coordinates for S^1 and the Euclidean time $\mathbb{R}$, the conformal map $z = e^{t+iy}$ maps the cylinder to $\mathbb{C}\backslash\{0\}$ and in particular maps the infinite past to the origin. The usual time ordering on the cylinder becomes radial ordering on the plane, and the associated space of states is built on radial slices.

Anything that resembles a local field $\phi(z,\bar{z})$ is called a field in CFT. If a field depends only on the holomorphic variable z we call it *chiral field* (or *anti-chiral* if it depends only on $\bar{z}$). Upon quantisation, fields become operator-valued distributions that create states in the space of states $\mathcal{H}$, by acting on the vacuum $|0\rangle \in \mathcal{H}$. This is called the *state-field correspondence*, which maps an field ϕ to a state

$$\phi \mapsto |\phi\rangle := \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle \ , \tag{2.3.3}$$

created at the origin on the plane (or past infinity on the cylinder). A crucial property of a CFT is that the above map is bijective; every state corresponds uniquely to a single local operator, whereas for a typical QFT different fields can produce the same asymptotic state. This can be understood through the fact that under conformal transformation $t \to -\infty$ is mapped to a single local point on $\widehat{\mathbb{C}}$.

The product of two fields inserted at the same point is generically singular. The singularity structure is captured by the so-called *operator product expansion* (OPE)

$$\phi_1(z)\phi_2(z') \sim \sum_{n=0}^{\infty} D_n(z-z')O_n(z') \ , \tag{2.3.4}$$

where $\sim$ means that we only keep the singular terms. Here $O_n(z)$ are fields of the theory and $D_n(z-z')$ are complex-valued functions with polynomial or logarithmic singularities when $z \to z'$. The non-singular part of $\phi_1(z)\phi_2(z')$ is captured by the *normal-ordered product*, which can be defined as

$$:\phi_1(z)\phi_2(z'): := \phi_1(z)\phi_2(z') - \sum_{n=0}^{\infty} D_n(z-z')O_n(z') \ . \tag{2.3.5}$$

When there are only polynomial singularities in $D_n(z-z')$ we say that the fields ϕ_1

and ϕ_2 are local with respect to each other, in the sense that there are no branch cuts and contour integrals are well-defined.

The conserved current associated with the continuous conformal symmetry of a 2d CFT is the *stress-energy tensor*, and we denote $T(z) := T_{zz}(z)$ and $\bar{T}(\bar{z}) := T_{\bar{z}\bar{z}}(\bar{z})$. Classically, these are the only non-vanishing components. Upon quantisation on a generic Riemann surface this is broken to $\langle T^a_{\ a} \rangle = -\frac{c}{12}R$ where R is the Ricci scalar.

Since the treatment of the chiral and anti-chiral parts is identical, we will from now on focus on the former. The holomorphicity of the stress-energy tensor $T(z)$ follows from the fact that the associated conserved charges are precisely the generators of infinitesimal holomorphic conformal transformations (2.3.1). Specifically, we have the mode expansion

$$L_n := \frac{1}{2\pi} \oint T(z) z^{n+1} dz \;\; \Leftrightarrow \;\; T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \; . \tag{2.3.6}$$

The modes L_n however, do not generally satisfy the Witt algebra (2.3.2). This is because the conformal symmetry is typically "softly" broken by quantum effects. The OPE of $T(z)$ with itself,

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \; , \tag{2.3.7}$$

is equivalent via the mode expansions to the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \; . \tag{2.3.8}$$

In the above, the real constant c is called *central charge*, and the new algebra is the *Virasoro algebra* $\mathfrak{Vir}$, which is a central extension of $\mathfrak{Witt}$ by the term containing the central charge. Moreover, the two resulting $\mathfrak{Vir}$ copies commute, i.e. $[L_m, \bar{L}_n] = 0$, and there is a central charge $\bar{c}$ associated with the anti-chiral part, which can in principle be different from c. The central charge captures important information of a CFT and gives a measure for the number of degrees of freedom, but there can exist multiple different CFTs with the same central charge. It is related to a "soft" breaking of the conformal symmetry because it indicates that the stress-energy tensor, which generates conformal transformations, transforms anomalously under conformal mappings. For instance, for the transformation $z = e^w, w = t + iy$ from the cylinder to the Riemann sphere, we have

$$T(z) = z^{-2}\left(T_{\text{cyl}}(w) + \frac{c}{24}\right) \; , \tag{2.3.9}$$

which leads to the following relation between the Hamiltonian $\oint_{\text{time-slice}} dw(T_{\text{cyl}}(w) + \bar{T}_{\text{cyl}}(\bar{w}))$ and $\oint_{\text{radial-slice}} \frac{dz}{z}(T(z) + \bar{T}(\bar{z}))$:

$$H = L_0 + \tilde{L}_0 - \frac{c + \tilde{c}}{24}. \tag{2.3.10}$$

Similarly, we have the momentum (or *spin*)

$$P = L_0 - \tilde{L}_0 - \frac{c - \tilde{c}}{24}. \tag{2.3.11}$$

As a result, the eigenvalues of L_0 plays the role of the chiral part of the energy, and the central charge gives rise to non-vanishing ground state energy. The eigenvalue h under L_0 of an eigenstate $|h\rangle \in \mathcal{H}$, i.e. $L_0|h\rangle = h|h\rangle$, is called the *conformal weight* of $|h\rangle$. If, moreover, $L_n|h\rangle = 0$ for all $n > 0$, then $|h\rangle$ is called a *Virasoro primary state* and the corresponding field called a primary field. This terminology also extends to the corresponding fields ϕ_h via the state-field correspondence. A state of the form $L_{-k_1}L_{-k_2}\cdots L_{-k_n}|h\rangle$ $(k_i > 0)$ is called a *Virasoro descendant* of $|h\rangle$. If $|h\rangle$ is a primary state, then along with all of its descendants they form a so-called *Verma module* for $\mathfrak{Vir}$. The primary state $|h\rangle$ is then called the *highest-weight state* of the module, since it has the lowest (somewhat confusingly) conformal weight among all of its descendants.

Since the states organise themselves into Virasoro representations, one can decompose the space of states of a CFT into a direct sum of $\mathfrak{Vir}$ and $\overline{\mathfrak{Vir}}$ modules. In general, focussing on the chiral part, one can have an enlarged symmetry algebra that contains $\mathfrak{Vir}$. This is called the *chiral algebra* of the CFT, and is denoted here by $\mathcal{V}$. We are mainly interested in *rational conformal field theories* (RCFTs), which contain a finite number of such modules; let $\Phi(\mathcal{V})$, $\overline{\Phi}(\overline{\mathcal{V}})$ denote the sets of these (chiral and anti-chiral respectively). The space of states can then be written as

$$\mathcal{H} \;=\; \bigoplus_{M\in\Phi(\mathcal{V}),\overline{N}\in\overline{\Phi}(\overline{\mathcal{V}})} \mathcal{Z}_{M,\overline{N}}\left(M \otimes \overline{N}\right)\;. \tag{2.3.12}$$

The states in such RCFTs get organized in $\mathcal{V}$-modules, whose highest-weight states correspond to *chiral primaries*, which are not only Virasoro primaries, but also primaries with respect to $\mathcal{V}$. The chiral descendants are generated by acting with $\mathcal{V}$ on the chiral primaries. This means that if $A(z) \in \mathcal{V}$, then $A(z)M \subset M$ for any $M \in \Phi(\mathcal{V})$.

The subspace $V \subset \mathcal{H}$, corresponding to the chiral algebra $\mathcal{V}$ via the state-field correspondence, always forms an irreducible $\mathcal{V}$-module $V := M_1 \in \Phi(\mathcal{V})$, which contains the vacuum and all states corresponding to the Virasoro primaries generating $\mathcal{V}$, also commonly called *currents*, along with their chiral descendants. Modular invariance requires that the eigenvalues of $L_0 - \overline{L}_0$ are integers, which also means that all states in any $M \in \Phi(\mathcal{V})$ have the same weight up to an integer. Specifically, all states in the vacuum module V should have integer weights. However, by dropping modular invariance as an initial requirement, the chiral algebra can possibly contain currents of half-integer weights (fermionic currents), or any rational weight (parafermions). The price to pay is that these currents have non-local OPEs (in the sense discussed previously), with the corresponding branch cuts leading to the introduction of various sectors (for fermionic currents these would be the Ramond and Neveu-Schwarz sectors). The modular invariant theory can then be constructed by a suitable projection.

The chiral algebras themselves are the central objects in the theory of Vertex Operator Algebras (VOAs), where they are discussed in an axiomatic manner. In the context of moonshine, an important property of a chiral algebra is the finite group part of $\mathrm{Aut}(\mathcal{V})$. The most famous example is the Monster CFT $V^\natural$, which is a VOA with $\mathrm{Aut}(V^\natural) = \mathbb{M}$, the Monster group. Furthermore, $V^\natural$ is an example of a *holomorphic VOA*, i.e. a VOA that has a unique irreducible $\mathcal{V}$-module, namely the space V.

In 2d CFT one is interested in calculating correlation functions of fields, inserted at specific points on a Riemann surface Σ. These can be cast in terms of chiral quantities called *chiral blocks*. Writing $\Sigma = \Sigma_{g,n}$ with genus g and n marked points $p_1, \dots, p_n$, a chiral block is a multilinear map from $M_1 \otimes \cdots \otimes M_n$ to a meromorphic function. This notation means that a field in $M_i \in \Phi(\mathcal{V})$ is inserted at the point p_i. In the case of RCFT, they can often be obtained as solutions to certain differential equations [42–44]. The chiral blocks form representations of the *mapping class group* $\Gamma_{g,n}$, which captures the discrete (and almost always infinite) symmetries of $\Sigma_{g,n}$. It can be defined by the quotient $\Gamma_{g,n} \cong \mathrm{Aut}(\Sigma_{g,n}) / \mathrm{Aut}_0(\Sigma_{g,n})$, where $\mathrm{Aut}_0(\Sigma_{g,n})$ is the component of $\mathrm{Aut}(\Sigma_{g,n})$ that is connected to the identity. Hence, $\Gamma_{g,n}$ maps between equivalent Riemann surfaces $\Sigma_{g,n}$, which only differ by a discrete automorphism. As a result, the *moduli space* $\mathcal{M}_{g,n}$, which parametrises the conformally inequivalent Riemann surfaces, has naturally the following quotient form,

$$\mathcal{M}_{g,n} = \mathcal{T}_{g,n} / \Gamma_{g,n} \ , \tag{2.3.13}$$

where $\mathcal{T}_{g,n}$ is the so-called *Teichmüller space*.

Chiral blocks have in general non-trivial monodromy as functions of the moduli space $\mathcal{M}_{g,n}$ (see for example [39] for more details). Chiral blocks will thus generally be multi-valued functions on $\mathcal{M}_{g,n}$, and in order to make them well-defined one should lift them to $\mathcal{T}_{g,n}$. As a result, they will then carry a representation of the mapping class group $\Gamma_{g,n}$. This is one way to understand the origin of the modular properties of torus blocks, and in particular moonshine modules.

To explain this, let us now focus on the case of $\Sigma_{1,1}$, i.e. tori with a single marked point. As discussed in §2.2.1, a torus can be described up to a scale by $\mathbb{C}/\Lambda_\tau$, where Λ_τ is the lattice in $\mathbb{C}$ generated by the vectors 1 and $\tau \in \mathbb{H}$. An $SL_2(\mathbb{Z})$ transformation leaves the lattice invariant and as a result the mapping class group $\Gamma_{1,0} = PSL_2(\mathbb{Z})$ is given by the part of $SL_2(\mathbb{Z})$ that acts non-trivially on the Teichmüller space $\mathbb{H}$. As any point is equivalent to any other point on a torus due to its translation symmetries, we also have $\Gamma_{1,0} = \Gamma_{1,1}$ and $\mathcal{M}_{1,0} = \mathcal{M}_{1,1}$.

Chiral blocks on $\Sigma_{1,1}$, when lifted to $\mathcal{T}_{1,1}$, will consequently be functions of the modular parameter τ. For RCFTs, they form a space of finite dimensions, and the dimension is given by the number of irreducible modules in $\Phi(\mathcal{V})$. They admit a natural basis given by the *graded dimensions*, or *characters*, of the irreducible modules $M \in \Phi(\mathcal{V})$

$$ch_M(\tau) = \mathrm{Tr}_M\, q^{L_0 - c/24} \ , \tag{2.3.14}$$

where $q = e^{2\pi i \tau}$ as before. As discussed previously, the characters furnish a representation of $\Gamma_{1,1} = PSL(2,\mathbb{Z})$, so that the $ch_M(\tau)$ are components of a weakly holomorphic vector-valued modular form for $PSL(2,\mathbb{Z})$. In other words, they mix with each other under the action of the modular group and the way they mix determines their OPE via the Verlinde formula. The modularity of characters of RCFTs is rigorously shown in the context of VOAs by Zhu's Theorem [45].

The *partition function* of a 2d CFT is defined as the 0-point correlation function on the torus, which encodes the spectrum of the theory. In the operator formalism, a torus with modular parameter $\tau = \tau_1 + i\tau_2$ can be obtained from the Riemann sphere by first conformally mapping it to the cylinder $S^1 \times \mathbb{R}$, and then imposing periodic boundary conditions on the Euclidean time direction $\mathbb{R}$. The Hamiltonian and momentum operators H, P then propagate states along both cycles of the torus, so the spectrum is embodied in the trace of the corresponding evolution operator over the space of states,

$$Z(\tau,\bar{\tau}) := \mathrm{Tr}_{\mathcal{H}}\, e^{2\pi i \tau_1 P - 2\pi \tau_2 H} \ . \tag{2.3.15}$$

Using (2.3.10)-(2.3.11), we can rewrite it as

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \ , \tag{2.3.16}$$

making it manifest that it is a generating function of the multiplicities of states at given chiral and anti-chiral conformal weights in $\mathcal{H}$. From (2.3.12) we see that it has the following decomposition in terms of chiral blocks

$$Z(\tau,\bar{\tau}) = \sum_{M\in\Phi(\mathcal{V}),\overline{N}\in\overline{\Phi}(\overline{\mathcal{V}})} \mathcal{Z}_{M,\overline{N}}\ ch_M(\tau)\overline{ch_{\bar{N}}(\tau)} \ . \tag{2.3.17}$$

The partition function (2.3.15) can also be computed using the path integral formalism when a Lagrangian description of the CFT is available. In this language, we have $Z = \int D\phi \; e^{-S[\phi]}$, with the fields having appropriate boundary conditions on the two cycles of the torus. Also from this point of view, it is clear that the partition function should be modular invariant. This invariance imposes severe constraints on the spectrum of 2d CFTs. For instance, modular invariance was used to classify supersymmetric minimal models and further extensions. See [46, 47] and references therein for some of these results. In the context of moonshine, we are mainly interested in the chiral CFT, where the modular properties are not as stringent.

2.3.2 Orbifolds

A special class of CFTs which is of particular interest for moonshine is the so-called *orbifold CFTs* [48]. The orbifold construction essentially entail "gauging" a discrete symmetry group G of the chiral algebra $\mathcal{V}$. More precisely, it builds a theory whose chiral algebra contains the G-invariant subalgebra $\mathcal{V}^G$ of $\mathcal{V}$, by retaining the G-invariant states of the original theory and introducing new "g-twisted" sector states, for every $g \in G$.

There are two important ways orbifold considerations enter the study of moonshine. First, we will see in §2.4.1 explicit constructions of moonshine chiral CFTs obtained by $\mathbb{Z}_2$-orbifolds. Second, the partition functions twined by the finite group symmetries provide the necessary information about the group actions on the moonshine CFT and constitute the modular objects playing a central role in moonshine. Generalising this to the twisted sectors leads to the so-called generalised moonshine, which we will briefly mention in the next section.

Apart from moonshine considerations, we will also encounter orbifolds when discussing sigma models on four-tori and K3 surfaces in §4, where they will play an important role in providing further evidence that the proposed correspondence is a natural construction.

Orbifold chiral algebra. Here we are mainly interested in orbifolds of chiral RCFTs (rational VOAs). We are interested in automorphisms of the operator algebra. If such an automorphism acts trivially on the operator algebra, i.e. without permuting the modules M_i^e, then it is said to be *inner*. In particular it preserves the chiral algebra of the chiral CFT. Let $\mathcal{V}$ denote the chiral algebra, $G \subseteq \mathrm{Aut}(\mathcal{V})$ a finite symmetry group, and $M_1^e, \dots, M_n^e$ its irreducible $\mathcal{V}$-modules. Here $e \in G$ denotes the identity element which will later be generalised to arbitrary $g \in G$. In particular, we have $M_1^e = V$, the vacuum module corresponding to $\mathcal{V}$.

Given such a symmetry, the chiral algebra is decomposed in G-representations as

$$\mathcal{V} = \bigoplus_a \rho_a \otimes \mathcal{V}_a \ , \tag{2.3.18}$$

where the corresponding spaces V_a contain states that transform under the irreducible representations ρ_a of G, and a runs over all of them. The G-invariant subalgebra

$$\mathcal{V}^G := \{\phi \in \mathcal{V} \mid h\phi = \phi \ \ \forall\, h \in G\} \ , \tag{2.3.19}$$

corresponding to the trivial representation ρ_0 of G, is called the *orbifold chiral algebra* in this setup. Note that while V is irreducible as a $\mathcal{V}$-module, it is generically reducible as a $\mathcal{V}^G$-module, as shown in (2.3.18). We instead identify the corresponding space V_a, corresponding to $\mathcal{V}_a$, as the irreducible $\mathcal{V}^G$-modules relevant for the orbifold CFT.

An analogous statement holds for the rest of the $\mathcal{V}$-modules, and we have decompositions

$$M_i^e = \bigoplus_a \rho_a \otimes M_{i,a}^e \ . \tag{2.3.20}$$

An important subtlety is that ρ now runs over all irreducible *projective representations* of G. Projective representations generalise the usual notion of representations introduced in §2.1.4, by allowing them to respect the group operation up to a phase,

$$\rho(h_1 h_2) = c_e(h_1, h_2)\rho(h_1)\rho(h_2) \ , \tag{2.3.21}$$

where $c_e(h_1, h_2)$ is a $U(1)$-valued 2-cocycle, representing a class in the group cohomology $H^2(G, U(1))$ of G. This type of behaviour is allowed in CFT because such a phase cancels when the chiral and the anti-chiral contributions are combined and hence is not in conflict with the modular invariance of the final theory. See [49] for a nice survey on projective representations of finite groups. Also note that the G-invariance of the vacuum implies that the vacuum module V carries true representations in the decomposition $V = \bigoplus_a \rho_a \otimes V_a$.

Twinings. For each $h \in G$, acting as an inner automorphism of the operator algebra, we define the *twined characters*[3]

$$ch_i\left(h \underset{e}{\square} \, ; \tau\right) := \mathrm{Tr}_{M_i^e}\left[h \ q^{L_0 - \frac{c}{24}}\right] \ . \tag{2.3.22}$$

Note that the the special case $h = e$ simply gives the usual character or *graded-dimensions*, of M_i^e. In terms of the decomposition into irreducible $\mathcal{V}^G$-modules, the

[3]Note that throughout this thesis orbifolds will arise frequently, and each time our notation will vary accordingly in order to better accommodate the task at hand.

twined characters are expressed as

$$ch_i\left(h \underset{e}{\square}\,;\tau\right) = \sum_a \chi_a(h)\; ch\left(M^e_{i,a}\right)\;, \tag{2.3.23}$$

where χ_a are projective characters of G, and $ch\left(M^e_{i,a}\right)$ are the graded dimensions of $M^e_{i,a}$. Using the orthogonality of the projective representations analogous to (2.1.18) one can obtain the character $ch_i\left(h \underset{e}{\square}\,;\tau\right)$ from the 2-cycle and the character of the projective representation ρ_a.

In a similar fashion, we define the twined partition function as

$$Z\left(h \underset{e}{\square}\,;\tau,\bar\tau\right) := \mathrm{Tr}_{\mathcal{H}}\left[h\; q^{L_0-\frac{c}{24}}\bar q^{\bar L_0-\frac{\bar c}{24}}\right]\;. \tag{2.3.24}$$

In the path integral language, the twined partition function is obtained by imposing h-twisted boundary condition for the fields on the cycle of the torus which is identified with the "temporal" circle, while the boundary condition along the spatial circle remains unchanged, i.e. $\phi(z+\tau) = h\cdot\phi(z)$ and $\phi(z+1) = \phi(z)$. From this point of view, it is clear that $Z\left(h \underset{e}{\square}\,;\tau,\bar\tau\right)$ should be invariant under a subgroup of $SL_2(\mathbb{Z})$ that preserves the h-twisted boundary condition ($SL_2(\mathbb{Z})$ transforms the boundary conditions on the two independent cycles of the torus as in (2.3.27) below).

Twisted sectors. Provided that $\mathcal{V}$ is sufficiently nice, in the sense that it satisfies the so-called C_2-cofiniteness condition (see [39] for the definition), then for any $g \in \mathrm{Inn}(\mathcal{V})$, the inner automorphisms of $\mathcal{V}$, one can define an irreducible *g-twisted $\mathcal{V}$-module* M_i^g for each $i = 1,\cdots,n$ [50]. In an orbifold theory these modules make up the g-twisted sector of the theory. Clearly, G is no longer a symmetry group for these modules; only the centraliser subgroup $C_G(g)$ (cf. §2.1.1) remains as a symmetry of the g-twisted sector. As a result, for any commuting pair $g,h \in G$, we can analogously define the *twisted-twined characters*

$$ch_i\left(h \underset{g}{\square}\,;\tau\right) := \mathrm{Tr}_{M_i^g}\left[h\; q^{L_0-\frac{c}{24}}\right]\;, \tag{2.3.25}$$

i.e. the twined characters in the twisted sectors (of which (2.3.22) is a special case). In the path integral language, they are obtained by additionally imposing g-boundary conditions for the spacial cycle of the torus, i.e. we have $\phi(z+1) = g\cdot\phi(z)$ as well as $\phi(z+\tau) = h\cdot\phi(z)$. They similarly admit the decomposition

$$M_i^g = \bigoplus_a\; \rho_a \otimes M^g_{i,a}\;, \tag{2.3.26}$$

where the sum now runs over all irreducible projective representations of $C_G(g)$. Accordingly, an obvious generalisation of (2.3.23), obtained by replacing e with g and G with $C_G(g)$, also holds for the twisted sectors.

We have already mentioned that the twined partition functions enjoy modular properties. Similarly the characters (2.3.25) form vector-valued modular forms for some congruence subgroup with certain multiplier systems. This can be understood via the $SL_2(\mathbb{Z})$-action on the boundary conditions: under modular transformations $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ on the torus, the boundary conditions (g,h) on the two cycles change as

$$(g,h) \mapsto \left(h^c g^d, h^a g^b\right) \ . \tag{2.3.27}$$

As a result, the twisted-twined characters transform as

$$ch_i\left(h \underset{g}{\square}\,; \frac{a\tau+b}{c\tau+d}\right) = \sum_{j=1}^{n} \psi(\gamma,g,h)_{ij}\ ch_i\left(h^a g^b \underset{h^c g^d}{\square}\,; \tau\right) \ , \tag{2.3.28}$$

where $\psi(\gamma,g,h)$ is an $n\times n$ matrix with scalar entries. The special case of *holomorphic VOAs* , i.e. those that contain only a single irreducible (untwisted) $\mathcal{V}$-module $\mathcal{H}$, is the easiest to describe. In this case, the chiral partition function coincides with the character of the chiral algebra and (2.3.28) becomes [51]

$$\begin{aligned} Z\left(h \underset{g}{\square}\,; \tau+1\right) &= c_g(g,h) Z\left(gh \underset{g}{\square}\,; \tau\right) \ , \\ Z\left(h \underset{g}{\square}\,; -\frac{1}{\tau}\right) &= \overline{c_h\,(g,g^{-1})} Z\left(g^{-1} \underset{h}{\square}\,; \tau\right) \ , \end{aligned} \tag{2.3.29}$$

where now the phases are given by a 2-cocycle representing a class in $H^2(C_G(g),U(1))$ as in (2.3.21). Moreover, all the phases for all g should descend from a 3-cocycle representing a class in $H^3(G,U(1))$[51, 52].

In a non-chiral CFT, the spectrum consists of the G-invariant parts of all the twisted sectors, leading to the following expression for the partition function

$$Z(\tau,\overline{\tau}) = \frac{1}{|G|} \sum_{gh=hg} Z\left(h \underset{g}{\square}\,; \tau,\bar{\tau}\right) \epsilon(g,h) \tag{2.3.30}$$

where $\epsilon(g,h)$ is a phase called the *discrete torsion*, which is just 1 in the simplest cases of orbifold constructions. As usual, the above partition function is modular invariant in a consistent orbifold CFT.

2.3.3 Elliptic genus

So far we have discussed conformal theories in general, but we often encounter 2d CFTs with supersymmetries. Here we will introduce introducing some necessary background on superconformal algebras and their representations, and in particular explain what an elliptic genus is, first from a physics point of view and then from a geometric point of view.

With supersymmetries, the presence of fermions leads to many new features, stemming from the fact that there is now an extra $\mathbb{Z}_2$ grading on the Hilbert space: $V = V_0 \oplus V_1$. (In the context of moonshine, this leads to supermodules of finite groups, cf. (2.1.20).) For instance, in the context of type II superstrings compactified on Calabi-Yau manifolds, the relevant "internal" CFT is a non-linear sigma model with $\mathcal{N} = 2$ supersymmetry. The Calabi-Yau structure of the target space guarantees that the theory has the $\mathcal{N} = 2$ extension of Virasoro symmetry, given by the so-called $\mathcal{N} = 2$ superconformal algebra (SCA). In particular, superstrings on $K3$ manifolds and the corresponding elliptic genus will play an important role in §3. Moreover, the elliptic genus will be the main tool we use in order to establish the correspondence between sigma models and VOAs in chapter §4.

The terminology "$\mathcal{N} = 2$" refers to the fact that we include 2 fermionic currents in the algebra on top of the bosonic energy-momentum tensor $T(z)$. Furthermore, there's now an extra automorphism, called the R-symmetry, that rotates different fermionic currents onto each other. We denote the two fermionic currents by $G_+(z)$ and $G_-(z)$ and the $U(1)$ R-symmetry current rotating the two by $J(z)$. The algebra reads

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\,\delta_{m+n,0} \\
[J_m, J_n] &= \frac{c}{3}m\,\delta_{m+n,0} \\
[L_n, J_m] &= -m\,J_{m+n} \\
[L_n, G_r^{\pm}] &= (\frac{n}{2}-r)\,G_{r+n}^{\pm} \\
[J_n, G_r^{\pm}] &= \pm G_{r+n}^{\pm} \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\,(r^2-\frac{1}{4})\,\delta_{r+s,0}\ ,
\end{aligned} \tag{2.3.31}$$

and all other (anti-)commutators are zero. As before we have two possible boundary conditions for the fermions

$$\begin{cases} 2r = 0 \bmod 2 & \text{for R sector} \\ \\ 2r = 1 \bmod 2 & \text{for NS sector} \end{cases} . \tag{2.3.32}$$

Two comments about this algebra are in order here. First, we have now two generators, L_0 and J_0, of the Cartan subalgebra. As a result, the representations will now be graded by two "quantum numbers", given by the eigenvalues of the L_0 and J_0 of the highest weight vector. The second new feature is that there is a non-trivial inner automorphism of the algebra, which means that the algebra remains the same under the following redefinition

$$\begin{aligned} & L_n \mapsto L_n + \eta J_n + m\,\eta^2\,\delta_{n,0} \\ \mathbf{SF}_\eta : \quad & J_n \mapsto J_n + 2m\,\eta\,\delta_{n,0} \\ & G_r^\pm \mapsto G^\pm_{r\pm\eta} \end{aligned} \tag{2.3.33}$$

with $\eta \in \mathbb{Z}$. This automorphism is called *spectral flow*, and in the above we have written $m := c/6$. If instead we choose $\eta \in \mathbb{Z}+1/2$ we exchange the Ramond and the Neveu-Schwarz algebra. Note that the only operator (up to trivial rescaling and the addition of central terms, of course) invariant under such a transformation is $4mL_0 - J_0^2$. Recall also that NS sector states give spacetime fermions and Ramond sector states give spacetime bosons in the context of string theory. Hence the spectral flow operator has an intimate relation to spacetime supersymmetries.

Ramond ground states and the Witten index. In what follows we will focus on the Ramond algebra and define the Ramond ground states of $\mathcal{N}=2$ SCFT. As usual, we require the ground states to be annihilated by all the positive modes:

$$L_n|\phi\rangle = J_m|\phi\rangle = G_r^\pm|\phi\rangle = 0 \quad \text{for all} \quad m,n,r>0\ .$$

Moreover, they have to annihilated by the zero modes of the fermionic currents

$$G_0^\pm|\phi\rangle = 0\ .$$

This condition fixes their L_0-eigenvalue to be

$$\frac{1}{2}\{G_0^+,G_0^-\}|\phi\rangle = \Big(L_0 - \frac{c}{24}\Big)\,|\phi\rangle = 0\ .$$

Let's ignore the right-moving part of the spectrum for a moment and consider a chiral Hilbert space V. We define its Witten index as

$$\mathrm{WI}(\tau,V) = \mathrm{Tr}_V\left((-1)^{J_0}q^{L_0-\frac{c}{24}}\right)\ .$$

If a state $|\psi\rangle$ is not annihilated by G_0^+, then the states $|\psi\rangle$ and $G_0^+|\psi\rangle$ together contribute 0 to $\mathrm{WI}(\tau,V)$ since $[L_0,G_0^+]=0$ while $[J_0,G_0^+]=G_0^+$. The same argument holds for G_0^- and we conclude that only Ramond ground states can contribute to the Witten index. As a result, the Witten index WI : $\{\mathcal{N} = 2 \text{ SCFT}\} \to \mathbb{Z}$ is independent of τ and counts (with signs) the number of Ramond ground states in V.

Notice moreover that the Witten index for $\mathcal{N} = 2$ SCFT acquires an interpretation as computing the graded dimension of the cohomology of the G_0^+ operator, satisfying $(G_0^+)^2 = 0$. For $\{G_0^+, (G_0^+)^\dagger\} = \{G_0^+, G_0^-\} = L_0 - \frac{c}{24}$, the Ramond ground states have the interpretation as the harmonic representative in the cohomology. This fact underlies the rigidity property of the Witten index and the elliptic genus which we will define shortly.

The same analysis can be trivially extended when one has a non-chiral theory with both left- and right-moving degrees of freedom: the Witten index

$$\mathrm{WI}(\tau, \bar{\tau}, V) = \mathrm{Tr}_V \left((-1)^{\tilde{J}_0 + J_0} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} q^{L_0 - \frac{c}{24}}\right)$$

counts states that are Ramond ground states for both the left- and the right-moving copy of $\mathcal{N} = 2$ SCA.

The $\mathcal{N} = 2$ elliptic genus. It is fine to be able to compute the graded dimension of a cohomology, but we can go further and compute more interesting properties of this vector space. For instance, we have learned that the representations of $\mathcal{N} = 2$ SCA are labelled by two quantum numbers corresponding to the Cartan generators L_0 and J_0. It will hence be natural to consider the following quantity which computes the dimension of $\tilde{G}_0^+$ cohomology graded by the left-moving quantum numbers L_0, J_0.

The elliptic genus of a $\mathcal{N} = (2, 2)$ SCFT is defined as the following Hilbert space trace

$$\mathbf{EG}(\tau, z) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}} \left((-1)^{J_0 + \bar{J}_0} y^{J_0} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}\right), \quad y = e^{2\pi i z}, \tag{2.3.34}$$

where $\mathcal{H}_{\mathrm{RR}}$ denotes the Hilbert space of states that are in the Ramond sector of the $\mathcal{N} = 2$ SCA both for the left- and right-moving copy of the algebra. From the same argument as that for the Witten index, this quantity will be independent on $\bar{q}$ and will hence be holomorphic as a function of both τ and z.

Note that the elliptic genus can be seen as something between the partition function and the Witten index. While the former counts all states and the latter counts only RR ground states, the elliptic genus counts states that are Ramond ground state on the one side and unconstrained on the other side. It contains a lot more information but still has the rigidity property of the Witten index which makes it possible to compute for many SCFTs, and as such it offers a good balance between information content and computability.

When the theory has a finite group symmetry G which commutes with the superconformal symmetries, one can define the elliptic genus twined by $g \in G$ as

$$\mathbf{EG}_g(\tau, z) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}} \Big(g\,(-1)^{J_0+\bar{J}_0} y^{J_0} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} \Big). \tag{2.3.35}$$

These objects will play an important role in the rest of the thesis.

Modular properties. As in the case of partition functions (cf. §2.3.1), a path integral interpretation of the elliptic genus suggests it has nice transformation property under the torus mapping class group. Moreover, the inner automorphism of the algebra (the spectral flow symmetry) implies that the graded dimension of a L_0-, J_0-eigenspace should only depends on its eigenvalue under the eigenvalue of the combined operator $4mL_0 - J_0^2$ and the charge of J_0 mod $2m$ where $m = c/6$. Hence, the Fourier expansion of the elliptic genus should take the form

$$\mathbf{EG}(\tau, z) = \sum_{n,\ell} q^n y^\ell \, c(4mn - \ell^2, \ell) \ .$$

where $c(D, \ell)$ only depends on D and $\ell \bmod 2m$ (cf. (2.2.27)). From these facts one can deduce that the elliptic genus of an $\mathcal{N} = (2, 2)$ SCFT with central charge $c = 6m$ is a weak Jacobi form of weight zero and index m. Similarly, following the same argument and that in §2.3.2, the twined elliptic genera are also weak Jacobi form of weight zero and the same index, but with the modular group $SL_2(\mathbb{Z})$ in (2.2.26) replaced by a certain subgroup which depends on the twining symmetry g.

The geometric elliptic genus. For a compact complex manifold M with $\dim_{\mathbf{c}} M = d_0$, we can define its elliptic genus as the character-valued Euler characteristic of the infinite-dimensional formal vector bundle [53–57]

$$\mathbf{E}_{q,y} = y^{d/2} {\textstyle\bigwedge}_{-y^{-1}} T_M^* \bigotimes_{n\geq 1} {\textstyle\bigwedge}_{-y^{-1}q^n} T_M^* \bigotimes_{n\geq 1} {\textstyle\bigwedge}_{-yq^n} T_M \bigotimes_{n\geq 0} S_{q^n}(T_M \oplus T_M^*),$$

where T_M and T_M^* are the holomorphic tangent bundle and its dual, and we adopt the notation

$${\textstyle\bigwedge}_q V = 1 + qV + q^2 {\textstyle\bigwedge}^2 V + \ldots, \quad \text{and} \quad S_q V = 1 + qV + q^2 S^2 V \ldots,$$

with $S^k V$ denoting the k-th symmetric power of V. In other words, we have

$$\mathbf{EG}(\tau, z; M) = \int_M ch(\mathbf{E}_{q,y}) \mathrm{Td}(M). \tag{2.3.36}$$

From the above definition we see that this "stringy" topological quantity reduces to the familiar ones: the Euler number, the signature, and the $\hat{A}$ genus of M, when we specialise z to $z = 0, \tau/2, (\tau+1)/2$, respectively.

When M has vanishing first Chern class, in particular when M is a Calabi–Yau manifold, its elliptic genus $\mathbf{EG}(\tau, z; M)$ can be shown to be a weak Jacobi form of weight zero and index $d_0/2$ [57]. Note that the supersymmetric sigma model on a Calabi–Yau manifold flows to a superconformal SCFT in the infrared. The elliptic genus of this $\mathcal{N} = (2,2)$ SCFT, defined as in (2.3.34), then coincides with the geometric elliptic genus defined in (2.3.36) of the Calabi–Yau manifold.

Examples: $K3$ and T^4. There are two topologically distinct Calabi-Yau two-folds: $K3$ and T^4, which are also the main focus of §4. Since both are equipped with a hyper-Kähler structure, extending the Kähler structure of generic Calabi–Yau manifolds, the superconformal symmetry is enhanced from $\mathcal{N} = (2,2)$ to $\mathcal{N} = (4,4)$. From the above argument, we expect their elliptic genus to be weight zero weak Jacobi forms with index 1. Coincidentally, the space of such a form is one-dimensional and is spanned by $\phi_{0,1}(\tau, z)$ (cf. (2.2.32)), and hence we only need one topological invariant of the Calabi-Yau two-folds to fix the whole elliptic genus. From

$$\mathbf{EG}(\tau, z = 0; T^4) = \chi(T^4) = 0 \quad , \quad \mathbf{EG}(\tau, z = 0; K3) = \chi(K3) = 24$$

and

$$\phi_{0,1}(\tau, z = 0) = 12$$

we obtain

$$\mathbf{EG}(\tau, z; T^4) = 0 \quad , \quad \mathcal{Z}(\tau, z; K3) = 2\phi_{0,1}(\tau, z) \ .$$

This clearly demonstrates the power of modularity in gaining extremely non-trivial information about the spectrum of a SCFT.

Remark. The argument for the holomorphicity of the elliptic genus fails in an interesting way for theories whose spectrum contains a continuous part. Due to the possible spectral asymmetry (i.e. non-perfect pairing between bosonic and fermionic states), the elliptic genus, when defined as a trace/integral over the full Hilbert space with continuous spectrum included, of such a theory could develop a non-trivial $\bar{q}$-dependence. For such an object the usual path-integral intuition still holds and the resulting non-holomorphic function transforms as a Jacobi form. Restricting to the discrete part of the spectrum, the analogous trace will be holomorphic but will no longer be modular. In particular, it will be a mock Jacobi form. As a result, in this context the holomorphic part of the elliptic genus is a well-defined notion both from a physical and mathematical point of view. From the physics perspective, the holomorphic part corresponds to the contribution from the discrete part of the spectrum [26, 29, 32, 58]. From the mathematical point of view, the holomorphic part corresponds to the holomorphic part of the harmonic Maass form [59]. We will encounter such a situation in §3.1.1.

2.4 Moonshine

In this section we give an introduction to moonshine, starting from the classic case of monstrous moonshine and moving on to other instances that we will encounter in the next chapters. The different cases are organised in terms of the weights of the modular objects involved.

2.4.1 Moonshine at weight zero: monstrous and Conway

Here we review the two moonshine connections, monstrous and Conway moonshine, that occur at weight zero. They are the moonshine cases that are best understood at the moment, in terms of the specification of the modular objects, the origin of the symmetries, and their physical context. In the case of monstrous moonshine we will provide more details than we actually need for this thesis, with the goal of acclimating the reader to the general concepts that carry on to all instances of moonshine.

Monstrous moonshine

Monstrous moonshine is arguably one of the most fascinating chapters of mathematics in the last century, where finite groups and modular objects were first noticed to be related via physical structures. As the theory of moonshine further develops, we believe that monstrous moonshine will remain the most distinguished example of moonshine phenomenon from various points of view. In this section we briefly describe the features of monstrous moonshine, and we refer to [39, 60, 61] and references therein for other excellent reviews of this beautiful story, in particular the historical aspects of it.

The term *monstrous moonshine*, coined in [62], refers to the unexpected connection between the representation theory of the Monster group $\mathbb{M}$ and the modular form

$$J(\tau) = \sum_{n\geq -1} a_n\, q^n = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \cdots \ , \qquad (2.4.1)$$

which we encountered in (2.2.5). The development of monstrous moonshine was initiated with the key observation, due to McKay, that the coefficient 196884 in the q-expansion of J can be decomposed as $196884 = 1 + 196883$, where the summands are the dimensions of the two smallest irreducible represenations of $\mathbb{M}$. Similar decompositions were observed for the next few coefficients by Thompson in [63]:

$$\begin{aligned} 1 &= 1 \\ 196884 &= 196883 + 1 \\ 21493760 &= 21296876 + 196883 + 1 \\ 864299970 &= 842609326 + 21296876 + 2 \cdot 196883 + 2 \cdot 1 \\ &\cdots \end{aligned} \tag{2.4.2}$$

where 1, 196883, 21296876, and 842609326 are dimensions of certain irreducible representations of $\mathbb{M}$. The observation led to the conjecture of the existence of an infinite-dimensional $\mathbb{Z}$-graded Monster module,

$$V = \bigoplus_{n \geq -1} V_n \ , \tag{2.4.3}$$

such that $\dim V_n = a_n$ for all $n \geq -1$. In other words, J acquires the interpretation as the graded dimension of V

$$J(\tau) = \sum_{n \geq -1} \dim V_n q^n. \tag{2.4.4}$$

Notice that V_0 is empty, corresponding to the vanishing constant coefficient of J.

The conjecture. This conjecture as stated above is not interesting since one could take each V_n to contain $c(n)$ copies of the trivial representation of $\mathbb{M}$ to make (2.4.4) true, given the fact that all a_n are non-negative integers. To access the information on the $\mathbb{M}$-action, Thompson also proposed in [64] to look at the graded characters of V, the so-called *McKay-Thompson series* defined by

$$T_g(\tau) \coloneqq \sum_{n \geq -1} \mathrm{Tr}_{V_n}(g) q^n \ , \tag{2.4.5}$$

for each element $g \in \mathbb{M}$ (with $T_e = J$). Note that the q-series $T_g(\tau)$ must also have vanishing constant term. As is clear from the definition, the T_g are class functions, i.e. $T_g = T_{hgh^{-1}}$. As a result, there are at most 194 distinct T_g as $\mathbb{M}$ has 194 conjugacy classes. In fact, it turns out that T_g only gives rise to 171 distinct functions. The main point of monstrous moonshine lies in the fact that these graded trace functions also exhibit modular properties and are moreover the unique Hauptmoduls with no constant terms (cf. §2.2.1), as stated in the following astonishing conjecture made by Conway and Norton [62]:

Conjecture 1. (Monstrous Moonshine Conjecture)
For each $g \in \mathbb{M}$ the McKay-Thompson series T_g coincides the unique Hauptmodul J_{Γ_g} with expansion $q^{-1} + O(q)$ near $\tau \to i\infty$, for some genus zero subgroup $\Gamma_g \leq SL_2(\mathbb{R})$. Furthermore, each Γ_g contains $\Gamma_0(N)$ as a normal subgroup, for some N dividing the quantity $|g|\gcd(24,|g|)$.

Given the importance of this conjecture, we will pause to make a few comments. Note that Γ_g is often not a subgroup of $SL_2(\mathbb{Z})$; only for some g we have $\Gamma_g = \Gamma_0(N)$ (cf. (2.2.8)), for some N satisfying the conditions mentioned above. In general, Γ_g is a normaliser of $\Gamma_0(N)$ in $SL_2(\mathbb{R})$, which in general involves the so-called Atkin-Lehner involutions. For later purpose we will be particular interested in the groups of the form

$$\Gamma^{N+K} := \left\{ \frac{1}{\sqrt{n}} \begin{pmatrix} an & b \\ cN & dn \end{pmatrix} \mid adn - bcN/n = 1,\ n \in K \right\}, \tag{2.4.6}$$

where $K < \mathrm{Ex}_N$ is a subgroup of the group of exact divisors of N. We say that e is an exact divisor of N if $e|N$ and $(f, \frac{N}{f}) = 1$, and they form a group with multiplication $f * f' = \frac{ff'}{(f,f')^2}$. An especially simple case is when N is a prime number p, and the full normaliser (corresponding to $K = \{1, p\}$) is given by

$$\Gamma_0(p)+ := \left\langle \Gamma_0(p), \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \right\rangle . \tag{2.4.7}$$

A harbinger of monstrous moonshine, predating the observation by McKay, is the following observation made by Ogg [65]. He noted that $\Gamma_0(p)+$ defines a genus zero quotient on the upper-half plane if and only if

$$p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}, \tag{2.4.8}$$

and this is precisely the set of primes dividing the order of the Monster group, and subsequently offered a bottle of Jack Daniel's to anyone who can explain the coincidence [65]. Monstrous moonshine sheds light on this mysterious coincidence through the fact that the Hauptmoduls of all the genus zero $\Gamma_0(p)+$ feature in moonshine as the McKay-Thompson series T_g for a $g \in \mathbb{M}$ of order p. In the case $\Gamma_g \not\subset SL_2(\mathbb{Z})$ the modularity of CFT does not help to explain the appearance of modularity for Γ_g, since in CFT modularity arises from the mapping class group of the torus (cf. §2.3). The crucial genus zero property of monstrous moonshine received a useful paraphrasing [66] as the property that these Hauptmoduls can be obtained (up to a constant) as a *Rademacher sum*, a regularised sum over the images of the polar term (in this case q^{-1}) under the action of the appropriate subgroup of $SL_2(\mathbb{R})$ (in this case Γ_g). This *Rademacher summability* property subsequently played a key role in the discovery of umbral moonshine (cf. §2.4.2). Recently, the genus zero property is explained by noting that Γ_g plays the role of the stringy symmetry group in the string realisation of the Monster theory and by requiring a physical analyticity condition on the supersymmetric index of the theory [67, 68].

The moonshine module. This conjecture was verified numerically by Atkin, Fong and Smith (cf. [69, 70]), following the idea of Thompson (see [39] for references). To be more specific, they showed that, for each $n \geq -1$, the q^n-coefficient of the Hauptmoduls specified in [62] coincide with the characters of a certain virtual representation of $\mathbb{M}$. A constructive verification was later obtained by Frenkel, Lepowsky and Meurman [71, 72], with the explicit construction of a Monster module $V = V^\natural$. This module has the structure of holomorphic VOA, i.e. a VOA with a single irreducible $V^\natural$-module, namely itself.

The starting point for constructing $V^\natural$ is 24 chiral bosons $X^i(z)$, compactified on the 24 dimensional torus $\mathbb{R}^{24}/\Lambda$ defined by the Leech lattice Λ. This results into a VOA $V(\Lambda)$ with central charge $c = 24$, leading to a partition function whose q-expansion starts with $Z_{V(\Lambda)}(\tau) = q^{-1} + \ldots$. This, together with the modular invariance, fixes the function to be the same as $J(\tau)$ up to an additive constant. At the same time, we know what this constant has to be since the Leech lattice has no root vectors and hence $\Theta_\Lambda(\tau) = \sum_{v\in\Lambda} q^{\langle v,v\rangle/2} = 1 + O(q^2)$, leading to

$$Z_{V(\Lambda)}(\tau) = \frac{\Theta_\Lambda(\tau)}{\eta^{24}(\tau)} = J(\tau) + 24\ . \tag{2.4.9}$$

In other words, thanks to the root-free property of the Leech lattice, the lattice vertex operators of the form $e^{ik\cdot\Phi}$ all have weight larger than one, and the only remaining weight one primaries are the 24 fields ∂X^i.

In order to have an exact matching with J we would like to remove these primaries, which can be achieved by a simple $\mathbb{Z}_2$ orbifold of $V(\Lambda)$, acting as $X^i \to -X^i$, which corresponds to the $\{\mathrm{id}, -\mathrm{id}\} \cong \mathbb{Z}_2$ symmetry of Λ, contained in $\mathrm{Aut}(Ł) \cong Co_0$. Indeed, one can easily compute the partition function of the orbifolded theory explicitly as follows. Note that the $\mathbb{Z}_2$-twined partition function of 24 chiral bosons is given by

$$Z\left(-\underset{+}{\square}\,;\tau\right) = \frac{1}{q\prod_{n>0}(1+q^n)^{24}} = \left(\frac{2\eta(\tau)}{\theta_2(\tau)}\right)^{12}. \tag{2.4.10}$$

The orbifold entails that $V^\natural$ is the direct sum of the $\mathbb{Z}_2$-invariant projections of the untwisted and twisted sectors respectively (cf. (2.3.30)). From (2.3.29) and (2.2.12) we have

$$
\begin{aligned}
Z_{V^\natural}(\tau) &= \frac{1}{2}\left(J(\tau) + 24 + \left(\frac{2\eta(\tau)}{\theta_2(\tau)}\right)^{12} - \left(\frac{2\eta(\tau)}{\theta_3(\tau)}\right)^{12} + \left(\frac{2\eta(\tau)}{\theta_4(\tau)}\right)^{12}\right) \\
&= J(\tau).
\end{aligned}
\tag{2.4.11}
$$

It remains to see that $\mathrm{Aut}(V^\natural)$ is the Monster. Note that $\mathrm{Aut}V(\Lambda)$ has a continuous piece T which is a 24-dimensional torus corresponding to the translation symmetry of the chiral bosons and to the 24 weight-one primary fields ∂X^i. The total symmetry is captured by the (non-split) short exact sequence

$$
1 \to T \to \mathrm{AutV}(\Lambda) \to Co_0 \to 1 \; . \tag{2.4.12}
$$

The $\mathbb{Z}_2$-orbifold breaks the automorphism group to its discrete part $2^{24}.\,Co_0$, which preserves the decomposition $V^\natural = V^\natural_+ \oplus V^\natural_-$ and is suggestively similar to a certain maximal subgroup $2^{1+24}.\,Co_1$ of $\mathbb{M}$. It is clear from the contribution to the weight two (and similarly for weight three, four, ...) states in $V^\natural$ from $V^\natural_+$ and $V^\natural_-$ that the Monster must mix them and hence cannot preserve the (un)twisted sector individually. Note that the 196884-dimensionl space of weight two states of $V^\natural$ has the structure of a commutative and non-associative algebra (as is true for any VOA), and can be shown to be precisely the *Griess algebra* constructed in 1980 and used to construct the Monster group itself [73]. From this and the VOA structure of $V^\natural$ one can show that $\mathrm{Aut}(V^\natural)$ is indeed the Monster, and can be obtained by adjoining a certain order two symmetry mixing $V^\natural_\pm$ to the discrete part of $\mathrm{Aut}(V(\Lambda))$.

The proof of monstrous moonshine. To prove that the $V^\natural$ constructed by Frenkel, Lepowsky and Meurman indeed "does the job", one needs to show that

$$
T_g^{V^\natural}(\tau) := \mathrm{Tr}_{V^\natural}\, g\, q^{L_0 - c/24} \tag{2.4.13}
$$

coincides with the corresponding Hauptmodul J_{Γ_g} specified in [62]. It was known that the coefficients of Hauptmoduls satisfy certain recursive formulas and one can determine all coefficients from just a handful of them. In the simplest case the recursive formulas are encoded in the remarkable identity

$$
p^{-1} \prod_{\substack{m>0 \\ n\in\mathbb{Z}}} (1 - p^m q^n)^{a_{mn}} = J(\rho) - J(\tau) \; , \tag{2.4.14}
$$

independently discovered by Zagier, Borcherds and others. Here $p = e^{2\pi i\rho}$, and a_i denotes the q^i coefficients in the q-expansion of J (cf. (2.4.1)). This identity results in infinitely many relations between a_i, which enables one to completely fix all the coefficients from just a_1, a_2, a_3, a_5. Clearly, the proof can be achieved if one can show the existence of the same type of identities among the coefficients of $T_g^{V^\natural}(\tau)$, and just explicitly compare the handful of coefficients that are necessary to fix the whole functions on both sides.

This is precisely what Borcherds did, and he obtained the replication formulas by introducing the notion of a *generalised Kac-Moody algebra*, which can be viewed as a generalisation of Kac–Moody algebras that allows for imaginary simple roots. Subsequently, he constructed a generalised Kac-Moody algebra (also called "Borcherds-Kac-Moody algebra") $\mathfrak{m}$, called the Monster Lie algebra. Roughly speaking, the construction was achieved by studying the cohomology of a BRST-like operator, which acts on $V^\natural \times \Gamma^{1,1}$, where $\Gamma^{1,1}$ is the unique unimodular lattice of signature $(1,1)$. This construction has a natural interpretation in string theory of considering second quantised strings in the background of $V^\natural$ [67, 68].

Borcherds managed to derive the replication formulas (2.4.14) as the *denominator identities* of the Monster Lie algebra $\mathfrak{m}$ that he attached to $V^\natural$. As in usual Kac–Moody algebras, the denominator identity results from applying the Weyl-Kac character formula of a Lie algebra to the trivial representation, and in this case relates an infinite sum to an infinite product, precisely the structure we see in (2.4.14). Moreover, by considering the $\mathbb{M}$-action on $V^\natural$ one can also obtain from $\mathfrak{m}$ the analogous identity

$$p^{-1} \exp\left[-\sum_{k>0} \sum_{\substack{m>0 \\ n\in\mathbb{Z}}} a_{mn}^{g^k} \frac{p^{mk} q^{nk}}{k} \right] = J_{\Gamma_g}(z) - J_{\Gamma_g}(\tau) \; , \tag{2.4.15}$$

satisfied by the other Hauptmoduls, where a_i^g are the q-expansion coefficients of J_{Γ_g}. Combining the above components then proves the validity of $V^\natural$ as the module of monstrous moonshine.

Generalised monstrous moonshine. In [74] Norton proposed a generalisation of monstrous moonshine under the name of *generalised monstrous moonshine*. He suggested that there is a rule to assign to each element $g \in \mathbb{M}$ a graded projective representation $V(g) = \bigoplus_{n\in\mathbb{Q}} V(g)_n$ of the centralizer group $C_{\mathbb{M}}(g)$, and to each pair (g,h) of commuting elements of $\mathbb{M}$ a holomorphic function $T_{(g,h)}$ on the upper half-plane $\mathbb{H}$, which satisfies the following conditions:

(i) $T_{(g^a h^c, g^b h^d)}(\tau) = \gamma \; T_{(g,h)}\left(\frac{a\tau+b}{c\tau+d}\right)$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and γ being a 24th root of unity.

(ii) $T_{(g,h)}(\tau) = T_{(k^{-1}gk, k^{-1}hk)}(\tau)$ with $k \in \mathbb{M}$.

(iii) There is a lift $\tilde{h}$ of h to a linear transformation on $V(g)$ such that

$$T_{(g,h)}(\tau) = \sum_{n\in\mathbb{Q}} \mathrm{Tr}_{V(g)_n}\left(\tilde{h}\ q^{n-1}\right) \ . \tag{2.4.16}$$

(iv) $T_{(g,h)}(\tau)$ is either a constant or a Hauptmodul for some genus-zero congruence subgroup of $SL_2(\mathbb{Z})$.

(v) $T_{(e,h)}(\tau)$ coincide with $T_h(\tau)$, the McKay-Thompson series attached to $h \in \mathbb{M}$ by monstrous moonshine.

As we can see from the discussion in §2.3.2, all of these properties, apart from (iv), can be understood in the framework of holomorpic orbifolds [75], applied to $V^\natural$. In particular, the function $T_{(g,h)}$ can be thought of the h-twined character of the twisted module $V_g^\natural$. The proof of generalised monstrous moonshine was carried out recently in [76], where a generalised Kac–Moody algeba $\mathfrak{m}_g$, generalising the monster Lie algebra $\mathfrak{m}$, is constructed for all $g \in \mathbb{M}$.

Conway moonshine

Conway moonshine establishes the relation between Co_0, related to Conway's sporadic group Co_1 by $Co_1 \cong \mathrm{Co}_0/\{\pm\mathrm{Id}\}$, and Hauptmoduls of certain genus zero subgroups of $SL_2(\mathbb{R})$.

Recall from §2.1.3 that Co_0 is isomoprhic to the automorphism group of the Leech lattice Λ. In this context, hints of Conway moonshine had already appeared in the original montrous moonshine paper [62], where the authors assigned genus zero groups $\Gamma_g < SL_2(\mathbb{R})$ to elements $g \in \mathrm{Co}_0$: let $\left\{\lambda_i, \lambda_i^{-1}\right\}_{i=1}^{12}$ be the 24 eigenvalues of the natural g-action on the Leech lattice $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ (embedded in a complex vector space), then Γ_g is given by the invariance groups of the holomorphic function

$$t_g(\tau) := q^{-1} \prod_{n>0}\prod_{i=1}^{12}\left(1-\lambda_i q^{2n-1}\right)\left(1-\lambda_i^{-1}q^{2n-1}\right) = q^{-1} - \chi_g + \mathcal{O}(q) \ . \tag{2.4.17}$$

Note that $\chi_g = \sum_i(\lambda_i + \lambda_i^{-1})$ is generically non-vanishing, and t_g has non-zero constant terms unlike the monstrous moonshine functions discussed in the previous subsection.

Conway moonshine, on the other hand, introduces a Conway module $V^{s\natural}$ whose McKay–Thompson series coincide with Hauptmoduls with vanishing constant terms. It was developed in [77, 78] (see also [79], [80] and [81] for nice summaries of the construction). The Conway module $V^{s\natural}$ is the unique, up to isomorphisms, vertex operator superalgebra (VOSA) with $c_{V^{s\natural}} = 12$ and $\mathcal{N} = 1$ superconformal structure, which has no states with weight 1/2. It can be constructed as a $\mathbb{Z}_2$ orbifold of the theory with eight bosons on the E_8 torus together with their fermionic superpartners. Alternatively, it can be constructed as a $\mathbb{Z}_2$ orbifold, acting as $k_a \to -k_a$, of 24 free chiral fermions k_a, $a = 1, 2, \ldots, 24$. This is to be compared with the monstrous moonshine module $V^\natural$, where the corresponding Monster module $V^\natural$ is built as a $\mathbb{Z}_2$ orbifold of the Leech lattice VOA (24 chiral bosons compactified on $\mathbb{R}^{24}/\Lambda$), resulting in theory with $c_{V^\natural} = 24$ and no states of weight 1. It turns out that $V^{s\natural}$ has an interesting relation to stringy symmetries of K3 surfaces, as we will see in §3. In what follows we will give more details on the Conway module $V^{s\natural}$.

Consider 24 real chiral fermions k_a and the corresponding complex fermions

$$\psi_j^\pm = \frac{1}{\sqrt{2}} \left(k_{2j-1} \pm i k_{2j}\right) \ , \quad j = 1, \ldots, 12 \ , \tag{2.4.18}$$

with the following non-vanishing OPEs and stress-energy tensor

$$\psi_i^\pm(z)\psi_j^\mp(w) \sim \frac{\delta_{ij}}{z-w} \quad , \qquad L = -\frac{1}{2}\sum_{i=1}^{12} :\psi_i^+ \partial\psi_i^- + \psi_i^- \partial\psi_i^+: \ . \tag{2.4.19}$$

Denote by $\mathfrak{a}$ the 24-dimensional vector space spanned by the fermions. Since fermions allow for both periodic and anti-periodic boundary conditions, there exist two sectors in the theory. The antiperiodic (Neveu-Schwartz) sector contains a single ground state $|0\rangle$ and excitations of half-integer weight, while the periodic (Ramond) sector contains integral-weight excitations and has 2^{12} degenerate ground states. The degeneracy is due to the Clifford algebra satisfied by the zero modes,

$$\left\{\psi_{i,0}^\pm, \psi_{j,0}^\pm\right\} = 0 \ , \quad \left\{\psi_{i,0}^\pm, \psi_{j,0}^\mp\right\} = \delta_{ij} \ , \tag{2.4.20}$$

which moreover commute with L_0. As a result, one can build the Ramond ground states by acting with $\psi_{i,0}^-$ on a ground state $|s\rangle$ satisfying $\psi_{i,0}^+|s\rangle = 0$. Namely, the Ramond ground states are given by the mononomials

$$\psi_{i_1,0}^- \cdots \psi_{i_k,0}^- |s\rangle \ , \tag{2.4.21}$$

which form a spinor in twenty-four dimensions with Euclidean signature.

Next we want to construct an action of Co_0 on the states described above. To do so, recall that $\mathrm{Co}_0 \cong \mathrm{Aut}(\Lambda)$, so the Conway group is isomorphic to a subgroup of $SO(24)$ and we can make the natural identification $\mathfrak{a} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, i.e. let fermions "live" on the Leech lattice. Then consider a group element $g \in \mathrm{Co}_0$ with complex eigenvalues $\lambda_i^{\pm 1}$, and choose the basis of $\mathfrak{a}$ such that the fermions $\psi_i^\pm$ are acted upon as eigenvectors:

$$g\psi_i^{\pm} = \lambda_i^{\pm 1}\psi_i^{\pm} \ , \quad \lambda_i \equiv e^{2\pi i a_i} \ , \quad i = 1, \ldots, 12 \ . \tag{2.4.22}$$

Moreover, since the ground states in the Ramond sector form a representation of the Clifford algebra associated to $\mathfrak{a}$, we should lift $G < SO(24)$ to a subgroup $\widehat{G} < \mathrm{Spin}(\mathfrak{a})$. An element $x \in \mathrm{Spin}(\mathfrak{a})$ has the property $xux^{-1} \in \mathfrak{a}$ for $u \in \mathfrak{a}$. We define the *lift* $\hat{g} \in \widehat{G} < \mathrm{Spin}(\mathfrak{a})$ of $g \in G < SO(\mathfrak{a})$ by requiring that it results in the same action as g when acting on $\mathfrak{a}$,

$$\hat{g}(u) := \hat{g}u\hat{g}^{-1} = gu \ , \quad \forall \ u \in \mathfrak{a} \ . \tag{2.4.23}$$

The map $u \mapsto \hat{g}(u)$ is a linear transformation on $\mathfrak{a}$ belonging to $SO(\mathfrak{a})$, so $\hat{g} \mapsto \hat{g}(\cdot)$ defines a map $\mathrm{Spin}(\mathfrak{a}) \to SO(\mathfrak{a})$ with kernel $\{\pm\mathbf{1}\}$, i.e. $\mathrm{Spin}(\mathfrak{a})$ is a double cover of $SO(\mathfrak{a})$. It turns out that for $G \cong \mathrm{Co}_0$ there exists a unique lift $\widehat{G} \cong Co_0$ (see [78] for more details).

While the NS ground state $|0\rangle$ is invariant under Co_0, the group action on the 2^{12} Ramond ground states turns out to be

$$\hat{g}|s\rangle = \prod_{i=1}^{12} e^{\pi i a_i}|s\rangle = \nu|s\rangle \ , \quad \nu \equiv \prod_{i=1}^{12} \nu_i \ , \quad \nu_i \equiv e^{\pi i a_i} = \lambda_i^{1/2} \ , \tag{2.4.24}$$

where $|s\rangle$ is the ground state described in (2.4.21). Notice that a priori there is a sign ambiguity for ν_i, since it is the square root of λ_i. But actually the choice of sign is unique since the lift of Co_0 is unique. There is a further ambiguity in the definition of the g-action on the fermions, in that we can swap the complex eigenvalues. This translates into setting $\lambda_i \leftrightarrow -\lambda_i^{-1}$ in (2.4.22), and is referred to as a choice of polarisation.

The last step is to consider a $\mathbb{Z}_2 = \{1, \mathfrak{z}\}$ orbifold of the theory described so far, acting as $\mathfrak{z}\psi_i^{\pm} = -\psi_i^{\pm}$ on the fermions. In other words, it acts as $(-1)^F$ where F is the fermion number. Supposing that it acts trivially on both ground states $|0\rangle$ and $|s\rangle$[78], it splits the two sectors into even/odd eigenspaces,

$$\mathrm{NS} = \mathrm{NS}^0 \oplus \mathrm{NS}^1 \ , \quad \mathrm{R} = \mathrm{R}^0 \oplus \mathrm{R}^1 \ , \tag{2.4.25}$$

where the eigenvalues of $\mathrm{NS}^j/\mathrm{R}^j$ are given by $(-1)^j$. From this point on, one can construct two closely related VOSAs. A useful description is by exploiting the fact that NS^0 forms a (bosonic) VOA on its own, that of the lattice D_{12}. Equivalently, it is the VOA associated to the affine Kac-Moody algebra $\widehat{\mathfrak{so}}(24)_1$, at level 1. The latter has four irreducible integrable modules, namely the vacuum module $A \cong \mathrm{NS}^0$, the vector module V, the spinor module S and the conjugate spinor module C. By extending the D_{12} VOA NS^0 by either of the spinor modules, one arrives at two

VOSAs:

$$
\begin{aligned}
V^{f\natural} &= \mathrm{NS}^0 \oplus S \cong \mathrm{NS}^0 \oplus \mathrm{R}^0 \\
V^{s\natural} &= \mathrm{NS}^0 \oplus C \cong \mathrm{NS}^0 \oplus \mathrm{R}^1 \ .
\end{aligned}
\tag{2.4.26}
$$

From the orbifold point of view we have the identifications $S \cong \mathrm{R}^0$ and $C \cong \mathrm{R}^1$ in our notation.

The two VOSAs $V^{f\natural}$ and $V^{s\natural}$ are isomorphic as VOSAs, and are uniquely characterized by their central charge $c_{V^{f\natural}} = c_{V^{s\natural}} = 12$ and the absence of weight 1/2 states. In [77] is was shown that the $\mathcal{N} = 1$ supercurrent of $V^{f\natural}$ is fixed by a subgroup of Spin(24) isomorphic to Co_0, which is identified by the group $\widehat{G}$ in the notation above. In particular, note that $Z(\mathrm{Spin}(24)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ where the first $\mathbb{Z}_2$ factor can be identified with the kernel of Spin(24) $\to SO(24)$ and the latter with the center of $SO(24)$. The centre $Z(\widehat{G}) \cong \mathbb{Z}_2$ can be identified with the second $\mathbb{Z}_2$ in Z(Spin(24)), and has the same action as the $\mathbb{Z}_2$ in the orbifold construction. As a result, it follows immediately from (2.4.26) that Co_0 does not act faithfully on $V^{f\natural}$, since the latter is invariant under the action of the centre $Z(\widehat{G})$. Instead, $V^{f\natural}$ carries a faithful action of the quotient group $\widehat{G}/\mathbb{Z}_2 \cong Co_1$. On the other hand, Co_0 acts faithfully on $V^{s\natural}$, and this is ultimately the reason why we consider $V^{s\natural}$ instead of $V^{f\natural}$ in what follows. Another notable difference between $V^{f\natural}$ and $V^{s\natural}$ is that the $\mathcal{N} = 1$ supercurrent in $V^{f\natural}$ fixed by Co_0 is not contained in $V^{s\natural}$, but rather in $V^{s\natural}_{\mathrm{tw}}$ (inside the R^0 part). A "canonically twisted" (or Ramond sector) module for $V^{s\natural}$ can also be constructed as

$$
V^{s\natural}_{\mathrm{tw}} = \mathrm{NS}^1 \oplus \mathrm{R}^0 \ , \tag{2.4.27}
$$

which is twisted with respect to the $Z(\widehat{G})$ symmetry. The action of Co_0 on this twisted module is also faithful.

In order to formulate the Conway moonshine statement, first define the functions

$$
\begin{aligned}
\eta_{\pm g}(\tau) &:= q \prod_{n=1}^{\infty} \prod_{i=1}^{12} \left(1 \mp \lambda_i^{-1} q^n\right)\left(1 \mp \lambda_i q^n\right) \\
C_{\pm g} &:= \nu \prod_{i=1}^{12} \left(1 \mp \lambda_i^{-1}\right) = \prod_{i=1}^{12} \left(\nu_i \mp \nu_i^{-1}\right) \ .
\end{aligned}
\tag{2.4.28}
$$

The twined partition functions of Conway moonshine are then given by

$$
\begin{aligned}
T^s_g(\tau) &:= \mathrm{str}_{V^{s\natural}}\left[\hat{g} q^{L_0 - 1/2}\right] = \mathrm{tr}_{V^{s\natural}}\left[\mathfrak{z}\hat{g} q^{L_0 - 1/2}\right] = \frac{\eta_g(\tau/2)}{\eta_g(\tau)} + \chi_g \\
T^s_{g,\mathrm{tw}}(\tau) &:= \mathrm{str}_{V^{s\natural}_{\mathrm{tw}}}\left[\hat{g} q^{L_0 - 1/2}\right] = \mathrm{tr}_{V^{s\natural}_{\mathrm{tw}}}\left[\mathfrak{z}\hat{g} q^{L_0 - 1/2}\right] = C_g \eta_g(\tau) - \chi_g \ ,
\end{aligned}
\tag{2.4.29}
$$

where the super-gradings can be defined by inserting $(-1)^F$ into the trace, whose action coincides with that of $\mathfrak{z}$ (recall that we identified $\mathbb{Z}_2 = \{1, \mathfrak{z}\}$ with the centre of Co_0). The main theorem of Conway moonshine states [78]:

Theorem 2. *The function $T^s_g(2\tau) = q^{-1} + O(q)$ is a Hauptmodul for a genus zero group $\Gamma_g < SL_2(\mathbb{R})$ that contains some $\Gamma_0(N)$, for every $g \in Co_0$. If g has a fixed point in its action on Λ, then $T^s_{g,tw}(\tau)$ is equal to the constant $-\chi_g$. Furthermore, if g has no such fixed point, then $T^s_{g,tw}(\tau)$ is also a Hauptmodul for a genus zero subgroup of $SL_2(\mathbb{R})$.*

In §3.1.2 we will see a way that Conway moonshine relates to the symmetries of $K3$ CFTs.

2.4.2 Moonshine at weight one-half: Mathieu and umbral

Somewhat unexpectedly, a wave of moonshine development started in 2010 which led to the discovery of many more examples of moonshine connections. The modern examples share some similarities, but also display important differences with the classical moonshine examples discussed in §2.4.1. The modular objects in these examples are typically mock modular forms, an important and natural generalisation of modular forms introduced in §2.2.3, which furthermore have non-vanishing weights. The first and very fruitful arena that was explored is that of weight 1/2 mock modular forms. Here we aim to describe umbral moonshine, which is the main focus of §3, by first discussing its precursor, Mathieu moonshine. Both are examples of moonshine relating finite groups and weight 1/2 mock modular forms.

Mathieu moonshine

The first example of the new type of moonshine, *Mathieu moonshine*, was initiated with certain observations about the weight 1/2 mock modular form H introduced in (2.2.42), in an analogous fashion as how observations about the classical J function initiated the development of monstrous moonshine. In [82] it was pointed out that the first few Fourier coefficients of H coincide with twice the dimensions of certain irreducible representations of the largest sporadic group M_{24}.

By now we have understood that, from many different points of view, Mathieu moonshine should really be thought of as a component of umbral moonshine, which we will review in the next subsection. However, in many ways Mathieu moonshine stands out among the other cases of umbral moonshine, not just historically but also in terms of its direct relation to the $K3$ elliptic genus. As a result, we will devote a separate subsection to Mathieu moonshine before discussing umbral moonshine.

Recall that the mock modular form H (2.2.42) can be viewed as arising from a meromorphic Jacobi form ψ given in (2.2.45). Using the relation between ψ and the $K3$ elliptic genus (2.2.44), as well as the identity $(\theta_{2,1} - \theta_{2,-1})(\tau, z) = -i\theta_1(\tau, 2z)$, we obtain the following relation between the elliptic genus of $K3$ (cf. §2.3.3) and the mock modular form H:

$$\mathbf{EG}(\tau, z; K3) = \frac{\theta_1^2(\tau, z)}{\eta^3(\tau)} \left(24\, \mu(\tau, z) + H(\tau)\right) , \tag{2.4.30}$$

where

$$\mu(\tau, z) = \frac{i}{\theta_1(\tau, 2z)} \mathrm{Av}^{(2)} \left[\frac{y+1}{y-1}\right] = \frac{-iy^{1/2}}{\theta_1(\tau, z)} \sum_{\ell=-\infty}^{\infty} \frac{(-1)^\ell y^\ell q^{\ell(\ell+1)/2}}{1 - yq^\ell} . \tag{2.4.31}$$

Note that while none of the two summands at the right-hand side of (2.4.30) transforms modularly, their modular anomalies cancel and the left-hand side is a perfectly well-behaved Jacobi form, as discussed in §2.2.1 and §2.3.3. In particular, a simple way to derive the shadow of H is by studying the modular properties of the Appell–Lerch sum $\mu(\tau, z)$ [23, 83]. We will see in §3.1.1 the two interesting physical interpretations of the above splitting (2.4.30) of $\mathbf{EG}(K3)$, one in terms of the characters of $\mathcal{N} = 4$ superconfomal algebra and one in terms of the elliptic genus of du Val singularities.

The aforementioned observation on the conspicuous relation between the first few coefficients of the mock modular form H and certain representations of M_{24} led to the suspicion that there exists a $\mathbb{Z}$-graded, infinite-dimensional M_{24}-module $K = \bigoplus_{n=1}^{\infty} K_n$ underlying H, namely $H(\tau) = q^{-\frac{1}{8}}\big(-2 + \sum_{n=1}^{\infty} q^n\,(\dim(K_n))\big)$. A natural question is thus whether the other summand in the splitting of the Jacobi form $\mathbf{EG}(K3)$ (2.4.30) harbors an action of M_{24} as well. A simple guess arises from the fact that M_{24} is a subgroup of the permutation group S_{24} and as a result has a defining permutation representation $\mathbf{R}$, of dimension 24. A natural proposal for the "twined" version of (2.4.30) is therefore

$$\phi_g(\tau, z) = \frac{\theta_1^2(\tau, z)}{\eta^3(\tau)} \big((\mathrm{Tr}_{\mathbf{R}}\, g)\, \mu(\tau, z) + H_g(\tau)\big) , \tag{2.4.32}$$

where H_g denotes the graded characters of the M_{24} module K. Following the spirit of monstrous moonshine, we say that there is a non-trivial moonshine connection if all such ϕ_g transform nicely as Jacobi forms under some $\Gamma_g \subseteq SL_2(\mathbb{Z})$. Physical considerations reviewed in §2.3.2 moreover suggest that $\Gamma_0(|g|) \subseteq \Gamma_g$.

Fortunately, the possibility for this type of Jacobi forms is very limited and we are constrained to consider

$$\phi = c\,\phi_{0,1} + F\,\phi_{-2,1}\ , \tag{2.4.33}$$

where $\phi_{0,1}$ and $\phi_{-2,1}$ are given in (2.2.32), $c \in \mathbb{C}$, and F is a weight two modular form for Γ_g, possibly with a non-trivial multiplier system when $c = 0$. The dimension of the space of possible F is often small for the $SL_2(\mathbb{Z})$-subgroup Γ_g we are interested in. For instance, when $\Gamma_g = SL_2(\mathbb{Z})$ the only possible weight two form is $F = 0$. Hence, knowing the first few of the Fourier coefficients of ϕ_g, dictated by our guesses for the first few M_{24}-representations, is often sufficient to fix the whole function. As a result, not long after the original observation [82], candidates for the McKay–Thompson series were proposed for all conjugacy classes $[g] \subset M_{24}$ in [84–87], and they take the form

$$\phi'_g(\tau, z) = \frac{\mathrm{Tr}_{\mathbf{R}}\, g}{12}\, \phi_{0,1}(\tau, z) + \tilde{T}_g(\tau)\, \phi_{-2,1}(\tau, z)\ , \tag{2.4.34}$$

where the functions $\tilde{T}_g(\tau)$ are weight 2 modular forms explicitly specified in the references given above and collected in Table 2 of [88]. More precisely, these ϕ_g for any $g \in M_{24}$ are weak Jacobi form of weight zero and index one satisfying the elliptic invariance $\phi_g|_1(\lambda, \mu) = \phi_g$ for all $(\lambda, \mu) \in \mathbb{Z}^2$ (cf. (2.2.22)), and transform as

$$\phi'_g(\tau, z) = \rho_{n_g|h_g}(\gamma)\, e\left(-\frac{cz^2}{c\tau + d}\right) \phi'_g\big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\big)\ ,$$

for $\gamma \in \Gamma_0(|g|)$, where the multiplier $\rho_{n_g|h_g}$ is summarised in [88].

In terms of the weak Jacobi forms (2.4.34), the main statement of Mathieu moonshine is the following.

Conjecture 3. *There exists a naturally defined $\mathbb{Z}$-graded, infinite-dimensional M_{24} module $K = \bigoplus_{n=1}^{\infty} K_n$ such that for any $g \in M_{24}$, the graded character*

$$H_g(\tau) := q^{-\frac{1}{8}}\big(-2 + \sum_{n=1}^{\infty} q^n\, (\mathrm{Tr}_{K_n}\, g)\big) \tag{2.4.35}$$

satisfies $\phi_g = \phi'_g$, where ϕ_g is as defined in (2.4.32) and ϕ'_g is the explicitly given weak Jacobi form (2.4.34). Moreover, the representations K_n are even in the sense that they can all be written in the form $K_n = k_n \oplus k_n^*$ for some M_{24}-representations k_n and their dual representation k_n^*.

A proof of the key fact in the above conjecture, namely the existence of an M_{24}-module $K = \bigoplus_{n=1}^{\infty} K_n$ such that (2.4.37) holds, has been attained in [89]. However, a construction of the module K, analogous to the construction of $V^\natural$ by Frenkel–Lepowsky–Meurman in the case of monstrous moonshine, is still absent. Therefore in no way do we know why K should be "natural". As explicit data, the first few Fourier coefficients of the q-series $H_g(\tau)$ and the corresponding M_{24}-representations are given in [88].

Note that the above implies that there is a M_{24}-supermodule underlying all terms in the q-, y-expansion of the $K3$ elliptic genus. It is hence tempting to endow the McKay–Thompsen series ϕ_g with the physical interpretation as twined elliptic genera of $K3$ CFT (cf. (2.3.35)). This however turns out to not be entirely possible for all $g \in M_{24}$; namely the symmetry group of any individual $K3$ sigma model needs to be a 4-plane preserving subgroup of Co_0 [90, 91] (also see §3), which is not true for M_{24}. Finally, note that the modular form property of ϕ_g, as well as the mock modular property of $\mu(\tau, z)$, immediately lead to the fact that H_g are also mock modular forms. Explicitly, they are given by

$$H_g(\tau) = \frac{\mathrm{Tr}_{\mathbf{R}}\, g}{24} H(\tau) - \frac{\tilde{T}_g(\tau)}{\eta(\tau)^3}\ , \tag{2.4.36}$$

and they are weight 1/2 mock modular forms with shadows given by $(\mathrm{Tr}_{\mathbf{R}}\, g)\eta^3(\tau)$, generalising the mock modular property of $H(\tau)$ discussed around (2.2.43).

Umbral moonshine

A few years after the discovery of Mathieu moonshine, it was realised that it is in fact just one instance of a larger system of moonshine, called "umbral moonshine" [92, 93]. There are in total 23 instances of umbral moonshine, which admit a uniform description (see Figure 2.2). The main statement of umbral moonshine is as follows.

Conjecture 4. *Let G^X be one of the 23 finite groups specified in (2.4.38), m be the corresponding positive integer specified in (2.4.43), and I^X be the specific subset of $\{1, 2, \ldots, m-1\}$ described in (2.4.44). Then there exists a naturally defined bi-graded, infinite-dimensional G^X-module*

$$K^X = \bigoplus_{r\in I^X} \bigoplus_{\substack{D\leq 0\\ D=r^2 \bmod 4m}} K^X_{r,D}$$

such that for any $g \in G^X$ and for any $r \in I^X$, the graded character ("corrected" by a polar term $-2q^{-\frac{1}{4m}}$ as in below) coincides with the component $H^X_{g,r}$ of a vector-valued mock modular forms $H^X_g = (H^X_{g,r})_{r\in I^X}$:

$$H^X_{g,r} = -2q^{-\frac{1}{4m}}\delta_{r,1} \ + \sum_{\substack{D\leq 0\\ D=r^2 \bmod 4m}}^{\infty} q^{-D/4m}\,(\mathrm{Tr}_{K^X_{r,D}}\, g). \tag{2.4.37}$$

In what follows we will briefly describe the specification of the main players, the finite groups G^X and the mock modular forms H^X_g, in the above conjecture. See Figure

2.2.

The starting point of this uniform construction are the 23 Niemeier lattices N^X introduced in §2.1.3. Recall that they are uniquely labelled by their root systems. We will denote by X the root systems, by N^X the corresponding Niemeier lattices, and by G^X the finite groups arising from the automorphisms $\mathrm{Aut}(N^X)$ via (2.1.13):

$$G^X := \mathrm{Aut}(N^X)/\mathrm{Weyl}(X) \tag{2.4.38}$$

These are the finite groups relevant for umbral moonshine and we will refer to them as the *umbral groups*.

On the modular side, we use the root system X to specify certain mock modular forms related to the finite group G^X. To explain how this is done, first recall that the McKay–Thompson series T_g in monstrous moonshine and the mock modular forms H_g in Mathieu moonshine have very special properties. First, once their (mock) modular data (consisting of the group Γ_g, the weight, and the multiplier) are specified, the functions are completely determined by the analyticity property of how they grow near the cusps $i\infty \cup \mathbb{Q}$. Second, they have "optimal growth" in the following sense. These functions 1) are bounded at all cusps that are not Γ_g-equivalent to $i\infty$ and 2) have the slowest possible growth near $i\infty$ that is compatible with the modular data. For instance, in the case of monstrous moonshine it is elementary to see that a modular form satisfying condition 1) for $\Gamma_g \supset \langle T \rangle$ must behave like $q^{-n}(1 + O(q))$ for some integer n near the cusp $i\infty$. As a result, the condition 2) states that $n = 1$, which is indeed the case for the moonshine functions T_g. Another way to state the above is to say that the functions in monstrous and Mathieu moonshine can be written (up to a constant) in terms of Rademacher sums over the minimal polar term in the expansion near $i\infty$ [66, 94]. See also §2.2.3 for a discussion on Rademacher sums.

The functions of umbral moonshine turn out to have analogous uniqueness properties, and the relevant concept here is the notion of optimal mock Jacobi forms. We will first focus on the case $g = e$ and $\Gamma_g = SL_2(\mathbb{Z})$. Let $\psi = \sum_r h_r \theta_{m,r}$ be a mock Jacobi form of weight one and index m. We say it is an optimal mock Jacobi form if

$$h_r(\tau) = O(q^{-\frac{1}{4m}}) \tag{2.4.39}$$

as $\Im(\tau) \to \infty$, for each $r \in \mathbb{Z}/2m$. For instance, the function $\psi^{E_8^{\oplus 3}}$ defined in §2.2.3 is an optimal mock Jacobi since it has index 30 and $H_1^{E_8^{\oplus 3}}(\tau) = -2q^{-\frac{1}{120}}(1 + O(q))$, while $H_7^{E_8^{\oplus 3}}$ vanishes at $\Im(\tau) \to \infty$ (cf. (2.2.38)). Similarly, $\psi^{A_1^{\oplus 24}}$ is an index 2 optimal mock Jacobi form.

At weight one, the space of such optimal mock Jacobi forms turns out to be very restricted: the mock modular transformation property together with the pole structure of the functions near the cusps are sufficient to determine the whole q-series. In particular, they can be obtained as simple Rademacher sums involving only the polar parts as input. Such forms are even more scarce if we want them to have non-transcendental Fourier coefficients. Note that this must be the case for the function to play a role in moonshine, since the graded dimensions are necessarily integers and of course non-transcendental. In [19] it is shown that if ψ is such a form, it must lie in a 34-dimensional space, irrespective of its index. Moreover, inside this 34-dimensional space there are 39 special elements (which span the 34-dimensional space) distinguished by their special symmetries. Recall that Atkin–Lehner symmetries normalising $\Gamma_0(m)$ are specified by a subgroup K of the group of exact divisors Ex_m (cf. (2.4.6)). Given such a pair m and K, we say that an index m mock Jacobi form ψ is *K-symmetric* if

$$\psi = \sum_{r \bmod 2\mathrm{m}} h_r \theta_{m,r} = \sum_{r \bmod 2\mathrm{m}} h_r \theta_{m,a(n)r} \quad \text{for all } n \in K \ , \tag{2.4.40}$$

where, for a given n, we define $a(n)$ to be the unique element in $\mathbb{Z}/2m$ satisfying

$$a(n) = \begin{cases} 1 \bmod 2m/n \\ -1 \bmod 2n \end{cases} .$$

Note that the symmetry is an involution, since $a^2 = 1 \bmod 2m$. For instance, the mock Jacobi form $\psi^{E_8^{\oplus 3}}$ introduced in (2.2.38) is invariant under the action of $K = \{1, 6, 10, 15\} < \mathrm{Ex}_{30}$, corresponding to $a(n) = 1, 11, 19, 29$. The surprising result in [19] then states that a non-vanishing K-symmetric index m optimal mock Jacobi form at weight one has non-transcendental coefficients if and only if the corresponding $SL_2(\mathbb{R})$ subgroup Γ^{m+K} defines a genus zero quotient in the upper-half plane. Recall that these genus zero groups also play an important role in monstrous moonshine. Note that we necessarily need to have $m \notin K$ (referred to as the "non-Fricke" property) for the mock Jacobi form to be non-vanishing, since at weight one has $\sum_r h_r \theta_{m,r} = -\sum_r h_r \theta_{m,-r}$ and $a(m) = -1$. There are just 39 such non-Fricke genus zero groups $\Gamma^{m+K} < SL_2(\mathbb{R})$ and we will denote the corresponding unique optimal mock Jacobi form, with the normalisation

$$h_1 = -2q^{-\frac{1}{4m}} (1 + O(q)) \ , \tag{2.4.41}$$

by ψ^{m+K}. In fact, these 39 distinguished optimal mock Jacobi forms ψ^{m+K} turn out to have Fourier coefficients that are not only non-transcendental, but also integral. Moreover, 23 among the 39 have positive coefficients in the following sense. By writing (cf. (2.2.29))

$$\psi = \sum_{1 \le r \le m-1} h_r \tilde{\theta}_{m,r} \ ,$$

h_r has the expansion

$$h_r = \begin{cases} -2q^{-1/4m} + \sum_{n\geq 0} c_{r,n} q^{n/4m}, & \text{if } r^2 = 1 \bmod 4m \\ \sum_{n\geq 0} c_{r,n} q^{n/4m}, & \text{otherwise} \end{cases} , \tag{2.4.42}$$

with $c_{r,n} \in \mathbb{Z}_{\geq 0}$. This positivity property makes it possible for it to be the graded dimensions of finite group representations[4]. To sum up, for any index, there are 23 special mock Jacobi forms of weight one for $SL_2(\mathbb{Z})$ distinguished by

1. the optimality growth condition (2.4.39),
2. the Atkin–Lehner symmetries (2.4.40),
3. the normalisation (2.4.41),
4. the positivity and integrality of the coefficients (2.4.42).

The interesting observation is that these ψ^{m+K} with positivity properties are in one-to-one correspondence with the 23 Niemeier root systems X! To explain this correspondence, first recall the ADE classification of the modular invariant combination of $\widehat{A_1}$ affine Lie algebras [46], which has led to a classification of $\mathcal{N} = 2$ superconformal minimal models with spectral flow symmetries. Their classification gives rise to a square matrix Ω^Y of size $2m$ for each simply-laced root system Y, where m coincides with the Coxeter number of Y. Moreover, the term $(\Omega^Y_{r,r} - \Omega^Y_{r,-r})$ coincides with the multiplicity of r as a Coxeter exponents (the degrees of the invariant polynomials shifted by one) of Y and takes values in $\{0, 1, 2\}$. The above can be generalised to a union of simply-laced root systems with the same Coxeter number (recall that this is indeed the case for Niemeier root systems) $X = \cup_i Y_i$ by defining $\Omega^X = \sum_i \Omega^{Y_i}$. Then the mock Jacobi form $\psi^X = \psi^{m+K}$, with theta-decomposition $\psi^X = \sum_r H^X_r \theta_{m,r}$, corresponding to the Niemeier root system X, display the following relations to X.

1. The Coxeter number of X coincides with the index of the ψ^X,

$$m = \mathrm{Cox}(X) \ . \tag{2.4.43}$$

2. The matrix Ω^X and the form $\psi^X = \psi^{m+K}$ have the same Atkin–Lehner symmetries: $(\Omega^X)_{r,r'} = (\Omega^X)_{r,a(n)r'}$ for all $n \in K$. Using these symmetries, it is convenient to define a set I^X of the orbits of the Atkin–Lehner symmetry group acting on $\{1, \ldots, m-1\}$ (in $\mathbb{Z}/2m$), labelling the independent components H^X_r of the vector-valued mock modular form (H^X_r) and leading to

$$\psi^X = \sum_{r\in I^X} H^X_r \sum_{n\in K} \tilde{\theta}_{m,a(n)r} \ . \tag{2.4.44}$$

[4]It is believed [19] that the remaining 16 optimal mock Jacobi forms ψ^{m+K} with positive and negative integral coefficients have also umbral type moonshine attached to them, but with additional supermodule structure that accounts for the minus sign.

3. The shadow of ψ^X is determined by Ω^X. More precisely, the completion of ψ^X is specified by the skew-holomorphic Jacobi form $\sigma = \sum_r \overline{\theta^1_{m,r}} \O^X_{r,r'} \theta_{m,r'}$ (cf. (2.2.37) and the preceeding text).

The mock Jacobi form ψ^X then gives us the vector-valued mock modular forms $H^X = (H^X_r)$ which will play the role of the graded dimensions of the module for the umbral group G^X. In other words, we have $H^X_r = H^X_{e,r}$ in Conjecture 4. For the case $X = A_1^{\oplus 24}$, this is the Mathieu moonshine function $\psi^{A_1^{\oplus 24}}$ we discussed in §2.2.1 and §2.4.2. Another simple example is $X = A_2^{\oplus 12}$, where $\psi^X = \sum_{r=1,2} H^X_r \tilde{\theta}_{3,r}$ (so $m = 3$ and $I^X = \{1, 2\}$), with

$$\begin{aligned} H^X_1(\tau) &= 2q^{-1/12}(-1 + 16\,q + 55\,q^2 + 144\,q^3 + \dots) \\ H^X_2(\tau) &= 2q^{8/12}(10 + 44\,q + 110\,q^2 + \dots). \end{aligned} \tag{2.4.45}$$

At the same time, the symmetries of the corresponding Niemeier lattice gives $G^X \cong 2.M_{12}$. The relation between the finite group G^X and the vector-valued mock modular form H^X can be observed from the fact that the group $2.M_{12}$ has irreducible representations of dimensions $16, 55, 144$ as well as $10, 44, 110$.

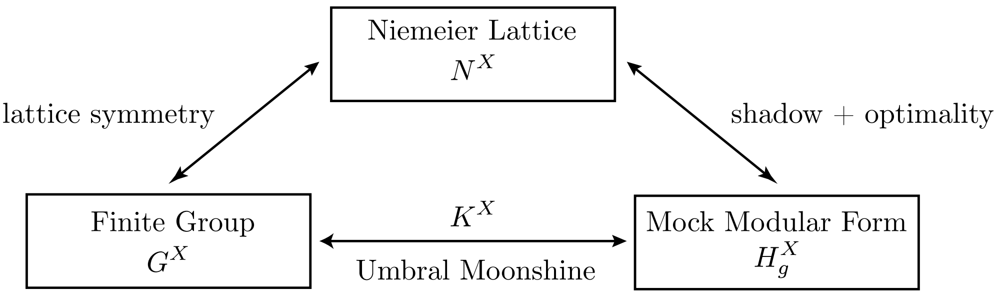


Figure 2.2: The construction of umbral moonshine.

After specifying the mock Jacobi forms for $SL_2(\mathbb{Z})$, in order to describe the moonshine relation we also need a set of mock Jacobi forms $\psi^X_g = \sum_r H^X_{g,r}\theta_{m,r}$, one for each conjugacy class $[g] \subset G^X$, for subgroups of $SL_2(\mathbb{Z})$. The mock modular forms $H^X_g = (H^X_{g,r})$ will then play the role of graded characters of the umbral moonshine module, as described in Conjecture 4. This can be achieved in a way largely analogous to the $SL_2(\mathbb{Z})$ case, though additional subtleties do emerge and extra care needs to be taken. We refer to [95] for more details.

Once the mock modular forms H^X_g are specified, it is trivial to verify the existence of the G^X-module $K^X_{r,D}$ in Conjecture 4 term by term, namely one D at a time. Furthermore, the existence of the whole umbral module $K^X = \bigoplus_r \bigoplus_D K^X_{r,D}$ has been proven mathematically using properties of (mock) modular forms [89, 96]. However, the construction, or even an understanding of the exact nature of K^X, is not yet obtained in general. Construction of K^X has so far only been achieved for certain particularly simple cases of umbral moonshine, corresponding to Niemeier root systems $3E_8$ [97], $A_6^{\oplus 4}$ and $A_{12}^{\oplus 2}$ [98], $D_6^{\oplus 4}$, $D_8^{\oplus 3}$, $D_{12}^{\oplus 2}$ and D_{24} [99], as well as $D_4^{\oplus 6}$ [2], which is the main topic of §3. The construction in [97] relies on special identities satisfied by the mock modular forms $H_{g,r}^{E_8^{\oplus 3}}$ relating it to a lattice-type sum, while in [98, 99] the modules are constructed using the interpretation of the meromorphic Jacobi forms associated to Ψ_g^X as the twined partition function of certain vertex operator algebras (or chiral CFTs). As we will see, in [2] the module for the $D_4^{\oplus 6}$ case of umbral moonshine is constructed by exploiting the relation between the (twined) $K3$ elliptic genus, umbral and Conway moonshine. Note that, so far, this is the only constructed module for which the corresponding umbral group (when embedded in Co_0) does not fix a 4-plane in the 24-dimensional representation of Co_0. The significance of this will be discussed in §3.

Moreover, generalised umbral moonshine, analogous to the generalised monstrous moonshine discussed in §2.4.1, has been established in [100], hinting that some elements of CFT/modular tensor category structure should be present at the umbral moonshine module K^X. Despite these results, it is fair to say that a uniform construction of the umbral module, reflecting the uniform description of umbral moonshine, is currently one of the biggest challenges in the study of moonshine.

We will end our review on umbral moonshine by noting a special property, called *discriminant property*, of umbral moonshine. It relates the discriminants D (the power of individual terms $q^{-D/4m}$ in the q-series H_r^X) and the number field generated by the characters of representations showing up in $K_{r,D}^X$. For instance, in the case $X = A_1^{\oplus 24}$, the M_{24}-representation underlying the $q^{7/8}$ term in $H_g = H_{g,1}^{A_1^{\oplus 24}}$ is $K_{-7}^{A_1^{\oplus 24}} = \rho \oplus \rho^*$, where ρ is a 45-dimensional irreducible representation and ρ^* is the dual representation. At the same time, $\mathrm{Tr}_\rho\, g$ (and hence also $\mathrm{Tr}_{\rho^*}\, g$) generates the field $\mathbb{Q}(\sqrt{-7})$. Analogous relations continue for larger q-power as long as $\mathbb{Q}(D) = \mathbb{Q}(-7)$, and similar properties hold uniformly for all 23 cases of umbral moonshine. At present there is no physical understanding of this surprising and profound-looking property.

3 A module for the $D_4^{\oplus 6}$ case of umbral moonshine

This chapter is dedicated to the construction of an umbral moonshine module for the case $X = D_4^{\oplus 6}$ as in [2], where the umbral group is $G^{D_4^{\oplus 6}} \cong 3.S_6$. Specifically, we will see how to construct three infinite-dimensional bi-graded $G^{D_4^{\oplus 6}}$-supermodules, whose second grading (corresponding to the powers of y) turns out to be trivial and whose odd part turns out to vanish. As a result of this, we will be left with three $\mathbb{Z}$-graded $G^{D_4^{\oplus 6}}$-modules, corresponding to the three components (H_1^X, H_3^X, H_5^X) of the vector-valued mock modular forms for the $X = D_4^{\oplus 6}$ case of umbral moonshine.

The first element of this construction is that the weak Jacobi form ϕ_g^X, which arises from an association between umbral moonshine and elliptic genera of ADE singularities that $K3$ surfaces can develop, coincides in many (but not all) cases with one of the functions $\phi_{\epsilon,g'}$ arising from Conway moonshine (see §3.1). In particular, for the case $X = D_4^{\oplus 6}$, for all $g \in G^{D_4^{\oplus 6}}$ there is some Conway element g' and a certain sign ϵ such that $\phi_g^{D_4^{\oplus 6}} = \phi_{\epsilon,g'}$. However, $G^{D_4^{\oplus 6}} \cong 3.S_6$ is a subgroup of Co_0 which is *not* 4-plane preserving, and as such the (twisted) Conway module cannot be directly used to construct a module for the $X = D_4^{\oplus 6}$ case of umbral moonshine since the corresponding $U(1)$ grading that appears in the characters is not preserved by the umbral group action.

The second ingredient is constructing a chiral conformal field theory $\mathcal{T}$ which strongly resembles the (twisted) Conway module but does not have the aforementioned difficulty related to the $U(1)$ grading. This is achieved by taking an $\mathbb{Z}/2$-orbifold of 24 free chiral fermions and 2 pairs of fermionic and bosonic ghost fields. The theory $\mathcal{T}$ is such that when graded by the charges of the ghost $U(1)$ current in a specific way, its twined partition function coincides with $\phi_{\epsilon,g}$. The crucial difference, however, is that the symmetry of this chiral theory accommodates the full Co_0, and thus also $G^{D_4^{\oplus 6}} \cong 3.S_6$, without the 4-plane preserving constraints. As a result, in a sense $\mathcal{T}$ plays the role of a bridge between Conway and umbral moonshine.

The above two elements together with a special property (cf. §3.3.3 and Conjecture 6.2 of [93]) of the $D_4^{\oplus 6}$ module then leads to a construction of the umbral module. We will also comment on how one can recover all H_g^X for $X = A_5^{\oplus 4}D_4$, $A_7^{\oplus 2}D_5^{\oplus 2}$, $A_{11}D_7E_6$, $A_{17}E_7$, and $D_{10}E_7^{\oplus 2}$, and some of the H_g^X for $X = A_1^{\oplus 24}$, $A_2^{\oplus 12}$, $A_3^{\oplus 8}$, $A_8^{\oplus 3}$, $E_4^{\oplus 6}$ using the same ingredients.

This chapter is organised as follows. In §3.1 we review how umbral and Conway moonshine lead to the weak Jacobi forms ϕ_g^X and $\phi_{\pm,g}$ respectively, for every $g \in G^X$ in the former case and every 4-plane preserving element g of Co_0 in the latter case. In §3.2 we present the construction of the chiral conformal field theory $\mathcal{T}$ and demonstrate that its graded twined partition functions coincide with $\phi_{\pm,g}$ when making a specific choice of chemical potentials for the ghost $U(1)$ currents. In §3.3 we combine these ingredients and explicitly describe the $G^{D_4^{\oplus 6}}$–action on the infinite dimensional $\mathbb{Z}/2$-graded vector space underlying the $D_4^{\oplus 6}$ case of umbral moonshine. In §3.4 we describe how to recover umbral moonshine functions for certain other cases of umbral moonshine from the twined parition functions of $\mathcal{T}$ and the singularity CFTs, and comment on a few open questions.

Furthermore, there are three appendices associated with this chapter. In §A we elaborate more on the ghost ground states appearing in the chiral CFT in §3.2. In §B we comment on the supersymmetry associated with the aforementioned ghosts, while in §C we present the character table of the umbral group $G^{D_4^{\oplus 6}}$ along with some other useful information about it.

3.1 Umbral moonshine and K3 elliptic genus

Here we review the construction developed in [28] and [79] of certain weak Jacobi forms that play the role of twined elliptic genera of $K3$ sigma models [91, 101], originating from umbral and Conway moonshine.

3.1.1 Umbral Twining Genera

As mentioned in the previous chapter, the first case of umbral moonshine, namely Mathieu moonshine which corresponds to the Niemeier root system $X = A_1^{\oplus 24}$, was discovered in the context of the $K3$ elliptic genus. In [28] it was proposed that all 23 cases of umbral moonshine, not just Mathieu moonshine, are relevant for describing the symmetries of $K3$ sigma models. In particular, one can associate in a uniform way, using (twined) elliptic genera of ADE singularities as one of the main ingredients, a weak Jacobi form (of weight 0 and index 1) ϕ_g^X to each of the 23 root systems X and each conjugacy class $[g] \in G^X$. This proposal was further tested in [102] and refined in [91]. This weak Jacobi form ϕ_g^X consists of two parts: one is the (twined) elliptic genus of the SCFT describing the surface singularities corresponding to the root system X, and the second comes from the contribution of the umbral moonshine mock modular forms.

We start by reviewing the first part, the singularity elliptic genus. The type of singularities a $K3$ surface can develop are given by the so-called du Val or Kleinein singularities, which admit an ADE classification. These singularities are isomorphic to $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $SU(2)$ as in the McKay correspondence. Let m denote the Coxeter number of the corresponding root system. Recall that $\mathcal{N}=2$ superconformal minimal models (which have spectral flow symmetries) also admit an ADE classification [46], and the central charge $c := 3\hat{c}$ is given by

$$\hat{c} = 1 - \frac{2}{m}. \tag{3.1.1}$$

The classification of these supersymmetric minimal models stems from the classification of modular invariant combinations of left- and right-moving characters of affine $\mathfrak{sl}_2$, given in terms of a $2m \times 2m$ matrix, which we denote by Ω^Φ for the minimal model corresponding to the simply-laced root system Φ. The explicit expression for Ω^Φ can be found in [46]. In terms of these matrices, the elliptic genus of the super minimal model is given by [56]

$$Z^{\rm Str}_{\rm minimal}(\tau,\zeta) = \sum_{r,r' \in \mathbb{Z}/2m\mathbb{Z}} \Omega^{\rm Str}_{r,r'} \tilde{\chi}^r_{r'}(\tau,\zeta) = {\rm Tr}(\Omega^{\rm Str} \cdot \tilde{\chi}). \tag{3.1.2}$$

In the above, $\tilde{\chi}^r_s(\tau,\zeta)$, with $|s| \leq r-1 < m$ are the corresponding minimal model characters, that are furthermore related to parafermionic characters [103], as reviewed in appendix B of [28].

In [104], a 2d CFT description of type II string theory compactified on $\mathbb{C}^2/\Gamma$ was proposed to be given by an $\mathbb{Z}_m$–orbifold of the corresponding supersymmetric minimal model tensored with a non-compact CFT, and takes the form

$$\left(\mathcal{N}=2 \text{ minimal } \otimes \mathcal{N}=2 \left(\frac{SL(2,\mathbb{R})}{U(1)}\right)_m \text{ coset}\right)/\mathbb{Z}_m, \tag{3.1.3}$$

where the second factor, the $\left(\frac{SL(2,\mathbb{R})}{U(1)}\right)_m$ supercoset model, describes the geometry of a semi-infinite cigar [105] and has central charge

$$\hat{c} = 1 + \frac{2}{m}.$$

The spectrum of the theory contains a discrete part as well as a continuous part; the latter exists due to the fact that the theory is non-compact and gives a non-holomorphic (in the τ-variable) contribution to the elliptic genus of the theory [26].

Both mathematically and physically, there is a well-defined way to isolate the holomorphic part of the elliptic genus, which we denote by Z_{L_m}, corresponding to the contribution from the discrete part of the spectrum. It is given by [26, 29, 32][1]

$$\begin{aligned} Z_{L_m}(\tau,\zeta) &= \frac{1}{2}\sum_{s=1}^{m} \mathrm{Ch}^{(\tilde{R})}_{\text{massless}}(\tau,\zeta;m+2-s) + \mathrm{Ch}^{(\tilde{R})}_{\text{massless}}(\tau,\zeta;s) \\ &= \frac{1}{2}\mu_{m,0}\left(\tau,\frac{\zeta}{m}\right)\frac{i\theta_1(\tau,\zeta)}{\eta(\tau)^3}. \end{aligned} \tag{3.1.4}$$

In the above equation, we make use of the Ramond character (the sum over spectral flow images of the characters of Ramond vacuum representation of $\mathcal{N}=2$ SCA at $\hat{c}=1+\frac{2}{m}$ [29]) graded by $(-1)^F$

$$\mathrm{Ch}^{(\tilde{R})}_{\text{massless}}(\tau,\zeta;s) = \frac{i\theta_1(\tau,\zeta)}{\eta^3(\tau)}\sum_{k\in\mathbb{Z}} y^{2k}q^{mk^2}\frac{(yq^{mk})^{\frac{s-1}{m}}}{1-yq^{mk}}$$

where s encodes the $U(1)$ charge of the highest weight, and the (specialized) Appell–Lerch sum

$$\mu_{m,0}(\tau,\zeta) = -\sum_{k\in\mathbb{Z}} q^{mk^2}y^{2km}\frac{1+yq^k}{1-yq^k} \tag{3.1.5}$$

which is responsible for the mock modularity of Z_{L_m}.

Putting it together using the "orbifoldization formula"[57] , the (holomorphic part of the) elliptic genus of the orbifold theory is given by

$$\mathbf{EG}(\tau,\zeta;\Phi) = \frac{1}{m}\sum_{a,b\in\mathbb{Z}/m\mathbb{Z}} q^{a^2}y^{2a}\, Z^{\text{Str}}_{\text{minimal}}(\tau,\zeta+a\tau+b)Z_{L_m}(\tau,\zeta+a\tau+b). \tag{3.1.6}$$

See also [31, 33, 106]. Here we use the following definition: given $X=\Phi_1\oplus\Phi_2\oplus\dots$ a union of simply-laced root systems Φ_i with the same Coxeter number, we write $\mathbf{EG}(X) := \mathbf{EG}(\Phi_1)+\mathbf{EG}(\Phi_2)+\dots$. For instance, when $X=D_4^{\oplus 6}$, we have $\mathbf{EG}(\tau,\zeta;D_4^{\oplus 6}) = 6\,\mathbf{EG}(\tau,\zeta;D_4)$.

Now we discuss the second part of the weak Jacobi form ϕ^X_g, arising from the contribution of the umbral moonshine mock modular forms. It is shown in [28] that for each of the 23 Niemeier lattices N^X, the following function

$$\phi^X_e(\tau,\zeta) := \mathbf{EG}(\tau,\zeta;X) + \frac{\theta_1^2(\tau,\zeta)}{2\,\eta^6(\tau)}\left(\frac{1}{2\pi i}\frac{\partial}{\partial\omega}\Psi^X_e(\tau,\omega)\right)\Big|_{\omega=0}, \tag{3.1.7}$$

[1] The factor of 1/2 appears just due to certain identities among characters, as explained in footnote 6 of [30]

is always equal to $\mathbf{EG}(\tau,\zeta;K3)$. In other words, the above expression gives us 23 ways to split $\mathbf{EG}(K3)$ into a part given by the singularity elliptic genus and a part given by umbral moonshine mock modular forms. Moreover, for each $g \in G^X$ umbral moonshine gives us a natural analogue for the second part by replacing Ψ^X_e with the graded character Ψ^X_g of the umbral moonshine module. At the same time, the explicit G^X–action on the Niemeier root system X translates into an G^X–action on the singularity CFT which preserves its superconformal structure, and leads to a definition of its twined elliptic genus $\mathbf{EG}_g(X) := \mathrm{Tr}(g \ldots)$. As a result, it is natural to define

$$\phi^X_g(\tau,\zeta) := \mathbf{EG}_g(\tau,\zeta;X) + \frac{\theta_1^2(\tau,\zeta)}{2\,\eta^6(\tau)}\left(\frac{1}{2\pi i}\frac{\partial}{\partial w}\Psi^X_g(\tau,w)\right)\Big|_{w=0}. \tag{3.1.8}$$

It can be shown that for all 23 Niemeier root systems X and $g \in G^X$ the above definition leads to a weak Jacobi form for certain $\Gamma_g \subseteq SL_2(\mathbb{Z})$, possibly with a non-trivial multiplier system. Conjecturally, when g is a 4-plane preserving element, these play the role of twined elliptic genera that are realised at certain points in the moduli space of $K3$ sigma models (see [91]).

3.1.2 Conway Twining Genera

Another moonshine connection to $K3$ sigma model comes from Conway moonshine. In [107], a generalisation of the Mukai theorem states that the physical symmetries relevant for twining the $K3$ elliptic genus are given by 4-plane preserving subgroups of the Conway group Co_0, the automorphism group of the Leech lattice Λ_{Leech}.[2] This classification inspired the interesting construction that associates to each 4-plane preserving conjugacy class $[g] \in Co_0$ two (possibly coinciding) weak Jacobi forms [79], denoted $\phi_{\pm,g}$. Furthermore, it was proposed that they play the role of twined elliptic genera of $K3$ sigma models. Inspired by the above results and relying on various empirical evidence, in [91] (Conjecture 6) it was conjectured that all the weak Jacobi forms arising from (the 4-plane preserving part of) Conway and umbral moonshine are realised as $K3$ elliptic genera twined by a supersymmetry-preserving symmetry of the sigma model at certain points in the moduli space. Conversely, every $K3$ twined elliptic genus (at any point in the moduli space) coincides with one of the moonshine Jacobi forms alluded to above. Physical arguments given in [101] promote this conjecture to a near-theorem.

To describe this in more details, consider the $\frac{1}{2}\mathbb{Z}$-graded infinite-dimensional Co_0-module $V^{s\natural}$ [78], which we reviewed in §2.4.1. Given a fixed n-dimensional subspace

[2] We say that a subgroup of Co_0 or G^X is n-plane preserving if it fixes pointwise an n-dimensional subspace in the natural 24-dimensional representation, given by the corresponding lattice Λ_{Leech} or N^X.

in $\Lambda_{\text{Leech}} \otimes_{\mathbb{Z}} \mathbb{R}$, there are different ways to build $U(1)$ currents from the fermions of $V^{s\natural}$[79, 80]. Here we are interested in the case when a $U(1)$ current J is constructed from fermions associated to a subspace of dimension $n = 4$. Fixing this $U(1)$ together with the compatibility with $\mathcal{N} = 1$ supersymmetry breaks the symmetry of the theory from Co_0 to the subgroup of Co_0 that preserves the given 4-plane. Conversely, given a 4-plane preserving $G \subset Co_0$ one can construct a $U(1)$ current J such that the twisted module $V^{s\natural}_{\text{tw}}$, when equipped with a module structure for J and for the $\mathcal{N} = 1$ superconformal algebra, has symmetry G. Interestingly, the $U(1)$-charged partition function of $V^{s\natural}_{\text{tw}}$ coincides with $\mathbf{EG}(K3)$ (up to a sign) [79]. More generally, one can consider the $U(1)$-graded character of the twisted Conway module twined by a 4-plane preserving element of Co_0:

$$\phi_g := -\operatorname{Tr}_{V^{s\natural}_{\text{tw}}}\left[\mathfrak{z}\ \hat{g}\ y^{J_0} q^{L_0 - \frac{c}{24}}\right], \tag{3.1.9}$$

where J_0 is the zero mode of the aforementioned $U(1)$ current. In the above $\hat{g}$ denotes the *lift* of g from SO(24) to Spin(24), which is necessitated by the fact that the ground states in the Ramond sector form a 4096-dimensional spinor representation (Clifford module) that we denote by $\mathbf{CM}$, and $\mathfrak{z}$ is the lift of $-\text{Id} \in Co_0$. Explicitly, it is given by

$$\begin{aligned}\phi_g(\tau,\zeta) = \frac{1}{2}\Bigg[&\frac{\theta_3(\tau,\zeta)^2}{\theta_3(\tau,0)^2}\frac{\eta_{-g}(\tau/2)}{\eta_{-g}(\tau)} - \frac{\theta_4(\tau,\zeta)^2}{\theta_4(\tau,0)^2}\frac{\eta_g(\tau/2)}{\eta_g(\tau)} \\ &- \frac{\theta_2(\tau,\zeta)^2}{\theta_2(\tau,0)^2}C_{-g}\eta_{-g}(\tau) - \frac{\theta_1(\tau,\zeta)^2}{\eta(\tau)^6}D_g\eta_g(\tau)\Bigg].\end{aligned} \tag{3.1.10}$$

The notation and functions used above follow §2.4.1 and 2.2.15. Note that $C_{-g} = \operatorname{Tr}_{\mathbf{CM}}\hat{g}$ and this is what determines the branch choice of ν_i (see [79] for more details). Furthermore, assume that $g \in Co_0$ fixes at least a 4-plane in the 24-dimensional representation. Then D_g is defined by

$$D_g := \nu \prod_{i=1}^{12}{}' \left(1 - \lambda_i^{-1}\right) = \prod_{i=1}^{12}{}' \left(\nu_i - \nu_i^{-1}\right). \tag{3.1.11}$$

where $\prod'$ skips two pairs of eigenvalues for which $\lambda_i^{\pm 1} = 1$. Notice that D_g is non-vanishing if and only if it fixes exactly a 4-plane. In the latter case, D_g is determined up to a sign by the eigenvalues of g, since we are free to exchange what we call λ_i and λ_i^{-1}. As a result, for exactly four-plane preserving elements there are in fact two choices of ϕ_g depending on the choice of the sign of D_g, and we define

$$\begin{aligned}\phi_{\epsilon,g}(\tau,\zeta) := \frac{1}{2}\bigg[&\frac{\theta_3(\tau,\zeta)^2}{\theta_3(\tau,0)^2}\frac{\eta_{-g}(\tau/2)}{\eta_{-g}(\tau)} - \frac{\theta_4(\tau,\zeta)^2}{\theta_4(\tau,0)^2}\frac{\eta_g(\tau/2)}{\eta_g(\tau)} \\ &-\frac{\theta_2(\tau,\zeta)^2}{\theta_2(\tau,0)^2}C_{-g}\eta_{-g}(\tau) - \epsilon\,\frac{\theta_1(\tau,\zeta)^2}{\eta(\tau)^6}\,|D_g|\,\eta_g(\tau)\bigg].\end{aligned} \tag{3.1.12}$$

where $\epsilon = \pm 1$ encodes the sign ambiguity of D_g. In all cases, it was shown that ϕ_g are Jacobi forms of weight 0 and index 1, at some level, for every $g \in Co_0$ that fixes at least a 4-plane. As mentioned earlier, they play the role of a twined $K3$ elliptic genus in the study of symmetries of $K3$ sigma models. Shortly we will see how they differ from the chiral CFT that we present in the next section, and which plays a crucial role in the construction of the $D_4^{\oplus 6}$ umbral moonshine module.

3.2 Chiral CFT

Here we present the construction of a chiral CFT $\mathcal{T}$, by $\mathbb{Z}_2$-orbifolding a free theory consisting of 12 complex chiral fermions, 2 fermionic and 2 bosonic ghost systems. Its symmetries accommodate the umbral group we are interested in, and its twined partition functions reproduce (among others) the weak Jacobi forms ϕ_g reviewed in §3.1.

3.2.1 The Fermions

The first ingredient to build our chiral theory $\mathcal{T}$ is 24 real chiral fermions $\tilde{\psi}_1, \ldots, \tilde{\psi}_{24}$, similar to the starting point of the Conway module discussed in §2.4.1. Equivalently, this theory, which we call $\mathcal{T}_\psi$, is given by 12 complex chiral fermions, $\psi_i^{\pm} := \frac{1}{\sqrt{2}}(\tilde{\psi}_i \pm i\tilde{\psi}_{i+12})$, with the action

$$S_\psi = \frac{1}{4\pi}\int d^2z \sum_{i=1}^{12}\left(\psi_i^+\bar{\partial}\psi_i^- + \psi_i^-\bar{\partial}\psi_i^+\right). \tag{3.2.1}$$

Their OPEs take the form

$$\psi_i^{\pm}(z)\psi_j^{\pm}(z') \sim \mathcal{O}(z-z'), \quad \psi_i^{\pm}(z)\psi_j^{\mp}(z') \sim \frac{\delta_{ij}}{z-z'}. \tag{3.2.2}$$

The associated Viraroso operator is given by

$$L^\psi = \sum_{n\in\mathbb{Z}} L_n^\psi z^{-n-2} = -\frac{1}{2}\,\circ\!\!\!\circ\left(\psi_i^+\partial\psi_i^- + \psi_i^-\partial\psi_i^+\right)\circ\!\!\!\circ, \tag{3.2.3}$$

with respect to which $\psi_i^{\pm}$ are holomorphic primary fields with weight 1/2. The open dots denote the regular part of the associated expression; we refer to this as the canonical ordering[3]. In terms of modes, it means that the annihilators are always put to the right. By expanding the fields in modes,

$$\psi_i^{\pm}(z) = \sum_r \psi_{i,r}^{\pm} \; z^{-r-\frac{1}{2}}, \tag{3.2.4}$$

the OPEs lead to the standard anti-commutation relations

$$\left\{\psi_{i,r}^{\pm}\psi_{j,s}^{\pm}\right\} = 0 \;, \quad \left\{\psi_{i,r}^{\pm}\psi_{j,s}^{\mp}\right\} = \delta_{ij}\delta_{r+s,0}. \tag{3.2.5}$$

The $SL(2,\mathbb{R})$-invariant vacuum $|0\rangle$ satisfies the usual highest weight condition

$$\psi_{i,r}^{\pm}|0\rangle = 0 \quad \forall \; r > 0. \tag{3.2.6}$$

Note that, in terms of modes, here the canonical ordering coincides with what we usually refer to as normal ordering, where the *positive* modes annihilate the above canonical vacuum.

To compute the characters of the theory, consider the conformal mapping from the complex plane to the cylinder given by $z = e^w$. We denote the Virasoro zero mode on the cylinder by

$$L^{\psi}_{\text{cyl},0} := L_0^{\psi} - \frac{c_\psi}{24} \tag{3.2.7}$$

where $c_\psi = 12$ is the central charge. Consider general boundary conditions parametrized by ρ

$$\psi_i^{\pm}(w+2\pi i) = e^{\mp 2\pi i\rho}\psi_i^{\pm}(w). \tag{3.2.8}$$

The periodic (P, $\rho = 0$) and anti-periodic (A, $\rho = 1/2$) cases correspond to the usual Ramond and NS sectors. Note that the $\psi_i^{\pm}$ must acquire opposite phases as in (3.2.8) so that the Virasoro operator remains periodic.

The A sector Hilbert space $\mathcal{T}_{\psi,\text{A}}$ is built by acting on the ground state $|0\rangle$ with the creation operators $\psi_{i,r}^{\pm}$ with $r \leq -1/2$, and its character is given by

$$\begin{aligned}\chi_\psi^{\text{A}}(\tau) &:= \text{Tr}_{\mathcal{T}_{\psi,\text{A}}}\left[q^{L^{\psi}_{\text{cyl},0}}\right] \\ &= q^{-1/2}\prod_{n=1}^{\infty}\prod_{i=1}^{12}(1+q^{n-1/2})^2 = \left(\frac{\theta_3(\tau,0)}{\eta(\tau)}\right)^{12}.\end{aligned} \tag{3.2.9}$$

We also define an operator $(-1)^F$ which has the property that it anticommutes with

[3]To avoid confusion, note that this is the same as the normal-ordered product that is mentioned in §2.1.1. Here we use the name canonical ordering to differentiate it from the normal ordering in terms of *modes*. See the next subsection for more details on this.

all the fermionic modes, squares to the identity, and acts trivially on $|0\rangle$. Note that $(-1)^F$ commutes with the Virasoro operator and hence we can define the following character,

$$\begin{aligned}\tilde{\chi}^{\mathrm{A}}_{\psi}(\tau) &:= \mathrm{Tr}_{\mathcal{T}_{\psi,\mathrm{A}}}\left[(-1)^F q^{L^{\psi}_{\mathrm{cyl},0}}\right] \\ &= q^{-1/2}\prod_{n=1}^{\infty}\prod_{i=1}^{12}(1-q^{n-1/2})^2 = \left(\frac{\theta_4(\tau,0)}{\eta(\tau)}\right)^{12}.\end{aligned} \tag{3.2.10}$$

In the P sector $\mathcal{T}_{\psi,\mathrm{P}}$, the ground states form a 2^{12}-dimensional representation of the 24-dimensional Clifford algebra. Explicitly, a basis can be given by the mononomials

$$\psi^-_{i_1,0}\cdots\psi^-_{i_k,0}|s\rangle, \tag{3.2.11}$$

where we single out $|s\rangle$ to be the state annihilated by all the $\psi^+_{i,0}$ and we require that $(-1)^F$ acts trivially on $|s\rangle$. The conformal weight of the P ground states is equal to $\frac{24}{16}=\frac{3}{2}$, as each of the complex fermions (along with its complex conjugate) contributes $\frac{2}{16}$ due to the presence of twist fields that interpolate between the A and P sectors. Putting things together, we obtain the following P sector characters.

$$\chi^{\mathrm{P}}_{\psi}(\tau) := \mathrm{Tr}_{\mathcal{T}_{\psi,\mathrm{P}}}\left[q^{L^{\psi}_{\mathrm{cyl},0}}\right] = 2^{12}q\prod_{n=1}^{\infty}\prod_{i=1}^{12}(1+q^n)^2 = \left(\frac{\theta_2(\tau,0)}{\eta(\tau)}\right)^{12}, \tag{3.2.12}$$

$$\tilde{\chi}^{\mathrm{P}}_{\psi}(\tau) := \mathrm{Tr}_{\mathrm{P}}\left[(-1)^F q^{L^{\psi}_{\mathrm{cyl},0}}\right] = \left(\frac{\theta_1(\tau,0)}{\eta(\tau)}\right)^{12} = 0. \tag{3.2.13}$$

The latter vanishes because half of the ground states have -1 eigenvalue under $(-1)^F$, while the rest have $+1$.

Later we will consider an orbifold of $\mathcal{T}_{\psi}$ by a $\mathbb{Z}_2$ generated by ξ, which acts on the fermions by

$$\xi\psi^{\pm}_i = -\psi^{\pm}_i, \tag{3.2.14}$$

and trivially on the A ground state. Note that any state with an odd (resp. even) number of excitations is an eigenstate of ξ with eigenvalue -1 (resp. $+1$), and hence ξ acts in exactly the same way as $(-1)^F$ on the quantum states in both A and P sectors. Therefore we conclude that

$$\mathrm{Tr}_{\mathcal{T}_{\psi,\mathrm{A}}}\left[\xi\ q^{L^{\psi}_{\mathrm{cyl},0}}\right] = \tilde{\chi}^{\mathrm{A}}_{\psi}(\tau) = \left(\frac{\theta_4(\tau,0)}{\eta(\tau)}\right)^{12}, \tag{3.2.15}$$

$$\mathrm{Tr}_{\mathcal{T}_{\psi,\mathrm{P}}}\left[\xi\ q^{L^{\psi}_{\mathrm{cyl},0}}\right] = \tilde{\chi}^{\mathrm{P}}_{\psi}(\tau) = 0. \tag{3.2.16}$$

3.2.2 The Ghosts

The next ingredients we need are the fermionic and bosonic ghost systems (see [108], [109–113] and references therein for related discussions). They are described by the action

$$S_{\text{gh}} = \frac{1}{2\pi} \int d^2z \; \mathbf{b}\bar{\partial}\mathbf{c}, \tag{3.2.17}$$

where $\mathbf{b}$ and $\mathbf{c}$ are holomorphic fields of weights h and $1-h$ respectively. We focus on the cases where $h \in \frac{1}{2}\mathbb{Z}$. Since there are many similarities between the fermionic and bosonic cases, we use the boldface notation to refer to either. When we need to make the distinction, we use b, c to denote the fermionic ghosts and β, γ to denote the bosonic ghosts. We will also use a parameter κ, which equals $+1$ for the fermionic case and -1 for the bosonic case.

The OPEs between the ghost fields have the form

$$\mathbf{b}(z)\mathbf{c}(w) \sim \frac{\kappa}{z-w}, \quad \mathbf{b}(z)\mathbf{b}(w) = \mathbf{c}(z)\mathbf{c}(w) \sim \mathcal{O}(1) \tag{3.2.18}$$

and the Virasoro operator is given by

$$L^{\text{gh}} = (1-h) \,{}^{\circ}_{\circ}(\partial\mathbf{b})\mathbf{c}{}^{\circ}_{\circ} - h \,{}^{\circ}_{\circ}\mathbf{b}(\partial\mathbf{c}){}^{\circ}_{\circ}, \tag{3.2.19}$$

with respect to which $\mathbf{b}$, $\mathbf{c}$ are primary. The central charge of the ghost system is then given by

$$c_{\mathbf{bc}} = \kappa(1-3Q^2), \tag{3.2.20}$$

where we have introduced $Q := \kappa(1-2h)$ for later convenience. The mode expansions on the complex plane are

$$\mathbf{b}(z) = \sum_r \mathbf{b}_r z^{-r-h} \;, \quad \mathbf{c}(z) = \sum_r \mathbf{c}_r z^{-r-(1-h)}, \tag{3.2.21}$$

and canonical quantization leads to the (anti)commutation relations

$$\{\mathbf{b}_r, \mathbf{c}_s\}_\kappa := \mathbf{b}_r\mathbf{c}_s + \kappa\mathbf{c}_s\mathbf{b}_r = \kappa\delta_{r+s,0}. \tag{3.2.22}$$

From (3.2.19) and (3.2.21) we see that the $SL(2,\mathbb{R})$-invariant vacuum $|0\rangle$ is determined by the highest weight condition

$$\begin{aligned} \mathbf{b}_r|0\rangle &= 0 \;\forall\; r \geq 1-h, \\ \mathbf{c}_r|0\rangle &= 0 \;\forall\; r \geq h. \end{aligned} \tag{3.2.23}$$

Consequently, in this case the canonical ordering does not generally coincide with the usual normal ordering, where the positive modes are annihilators (see footnote 3). As before, we consider the ρ-twisted sectors for the ghost systems corresponding to the boundary conditions

$$\mathbf{b}(w+2\pi i)=e^{-2\pi i\rho}\mathbf{b}(w),\quad \mathbf{c}(w+2\pi i)=e^{2\pi i\rho}\mathbf{c}(w), \tag{3.2.24}$$

where w is the natural coordinate on the cylinder and is given by $z=e^{w}$. The periodic case $\rho=0$ corresponds to the P sector, while the anti-periodic case $\rho=1/2$ corresponds to the A sector[4].

The natural ground states on the cylinder are defined as the states annihilated by all positive modes, $\mathbf{b}_r, \mathbf{c}_r$ with $r>0$. Note that, for the ghost systems, these are in general different from the $SL(2,\mathbb{R})$-invariant ground state $|0\rangle$. The corresponding energies, namely the eigenvalues of $L^{\rm gh}_{\rm cyl,0}=L^{\rm gh}_0-c_{\mathbf{bc}}/24$, of the P and A sector ground states are given by $\frac{\kappa}{12}$ and $-\frac{\kappa}{24}$ respectively, as is calculated in §A.

Another important feature of the ghost systems is that they have the following $U(1)$ current

$$J=-\,{}^{\circ}_{\circ}\mathbf{bc}{}^{\circ}_{\circ}\,. \tag{3.2.25}$$

In fact, as we show in §A, the A sector ground states for both fermonic and bosonic ghosts are unique. The P sector has two degenerate ground states for the fermonic ghost system due to the presence of the fermonic zero modes b_0, c_0, while it has a single ground state for the bosonic system. We denote the (unique) A sector ground states for the fermonic (F) and bosonic (B) ghosts by $|\Omega^{\rm F}_{\rm A}\rangle$ and $|\Omega^{\rm B}_{\rm A}\rangle$, respectively. The (unique) P sector ground state for the bosonic ghost is denoted by $|\Omega^{\rm B}_{\rm P}\rangle$, and the two degenerate P sector ground states for the fermionic ghost system are denoted by $|\Omega^{\rm F}_{{\rm P},\pm}\rangle$. They are distinguished by

$$b_0|\Omega^{\rm F}_{{\rm P},-}\rangle=0,\quad b_0|\Omega^{\rm F}_{{\rm P},+}\rangle=|\Omega^{\rm F}_{{\rm P},-}\rangle,\quad c_0|\Omega^{\rm F}_{{\rm P},+}\rangle=0,\quad c_0|\Omega^{\rm F}_{{\rm P},-}\rangle=|\Omega^{\rm F}_{{\rm P},+}\rangle. \tag{3.2.26}$$

Next we derive the characters of the ghost systems, defined by

$$\chi^{\rm S}_a(\tau,\zeta):={\rm Tr}_{\rm S}\left[y^{J_{\rm cyl,0}}\ q^{L^{\rm gh}_{\rm cyl,0}}\right], \tag{3.2.27}$$

where ${\rm S}=\{{\rm P},{\rm A}\}$ denotes the sector and $a=\{{\rm F},{\rm B}\}$ distinguishes between the fermionic and bosonic ghosts, respectively. Note that the other commonly used character, defined by ${\rm Tr}_{\rm S}\left[(-1)^{J_{\rm cyl,0}}y^{J_{\rm cyl,0}}\ q^{L^{\rm gh}_{\rm cyl,0}}\right]$, is simply given by the above by a shift $\zeta\mapsto\zeta+\frac{1}{2}$.

Building on the unique ground state $|\Omega^{F}_{\rm A}\rangle$, the A sector Hilbert space of the fermionic

[4]We introduce both sectors irrespective of the statistics of the fields, since they will both appear when we consider the $\mathbb{Z}_2$ orbifold.

ghost system leads to the character

$$\chi_{\rm F}^{\rm A}(\tau,\zeta) = q^{-1/24}\prod_{n=1}^{\infty}\left(1+yq^{n-1/2}\right)\left(1+y^{-1}q^{n-1/2}\right) = \frac{\theta_3(\tau,\zeta)}{\eta(\tau)}. \tag{3.2.28}$$

Similarly, by taking into account for all possible states in the Fock space created by the negative integral modes of the ghost fields b, c acting on both of the ground states $|\Omega^F_{{\rm P},\pm}\rangle$, we obtain the character

$$\chi_{\rm F}^{\rm P}(\tau,\zeta) = q^{1/12}\left(y^{1/2}+y^{-1/2}\right)\prod_{n=1}^{\infty}\left(1+yq^{n}\right)\left(1+y^{-1}q^{n}\right) = \frac{\theta_2(\tau,\zeta)}{\eta(\tau)}. \tag{3.2.29}$$

For the bosonic ghost system, the A sector character is given similarly by

$$\chi_{\rm B}^{\rm A}(\tau,\zeta) = q^{1/24}\prod_{n=1}^{\infty}\left(1-yq^{n-1/2}\right)^{-1}\left(1-y^{-1}q^{n-1/2}\right)^{-1} = \frac{\eta(\tau)}{\theta_4(\tau,\zeta)}. \tag{3.2.30}$$

In the P sector, care has to be taken due to the presence of the bosonic zero mode γ_0. As we will see in §3.2.4 (also see [111]), the contribution of γ_0 can be regularised and the total character is given by

$$\chi_{\rm B}^{\rm P}(\tau,\zeta) = q^{-1/12}y^{1/2}(1-y)^{-1}\prod_{n=1}^{\infty}\left(1-yq^{n}\right)^{-1}\left(1-y^{-1}q^{n}\right)^{-1} = i\frac{\eta(\tau)}{\theta_1(\tau,\zeta)}. \tag{3.2.31}$$

Finally, we would like to consider a $\mathbb{Z}_2$-orbifold of the ghost systems, where the non-trivial group action is given by $\xi\mathbf{b} = -\mathbf{b}$ and $\xi\mathbf{c} = -\mathbf{c}$. The resulting characters will be related to the characters $\chi_a^{\rm S}(\tau,\zeta+1/2)$ we calculated above, since the action of the corresponding group element ξ corresponds to including the operator $(-1)^{J_{{\rm cyl},0}}$, similarly to the case of the chiral fermions discussed in §3.2.1. The only nontrivial part of this implementation is the sign of the ground state(s) under ξ, which is analysed in §A. The results are given by

$$\tilde\chi_{\rm F}^{\rm S}(\tau,\zeta) := {\rm Tr}_{\rm S}\left[\xi\; y^{J_{{\rm cyl},0}}q^{L^{\rm gh}_{{\rm cyl},0}}\right] = (-1)^{h-\frac{1}{2}}\;\chi_{\rm F}^{\rm S}(\tau,\zeta+1/2) \tag{3.2.32}$$

for the fermionic ghosts, while for the bosonic ghosts we have

$$\tilde\chi_{\rm B}^{\rm S}(\tau,\zeta) := {\rm Tr}_{\rm S}\left[\xi\; y^{J_{{\rm cyl},0}}q^{L^{\rm gh}_{{\rm cyl},0}}\right] = (-1)^{3h+\frac{1}{2}}\;\chi_{\rm B}^{\rm S}(\tau,\zeta+1/2), \tag{3.2.33}$$

where S denotes either of the two sectors. Notice that all the characters we have computed in this section coincide with the standard characters of the usual charged bosons/fermions [111], and do not depend on the central charge of the ghost systems.

However, we will see that by requiring the final CFT to have certain supersymmetry we can completely fix the central charge of the ghosts systems.

3.2.3 The Orbifold Theory

After describing the basic ingredients, we now put them together and construct the chiral CFT that will reproduce the K3 elliptic genus and its twinings. Let $\mathcal{T}_{\rm B}$ denote the theory of 2 copies of the bosonic ghost system and the theory of 2 copies of the fermionic ghost system. We will later see that $\mathcal{T}$ can be equipped with an $\mathcal{N}=4$ superconformal symmetry for certain choices of $h_{\rm B}$ and $h_{\rm F}$. We will let $h_{\rm B}=\frac{1}{2}$ and $h_{\rm F}=1$, corresponding to the total central charge $c_{\mathcal{T}}=-4-2+12=6$ (along with the free fermions).

We want to consider a $\mathbb{Z}_2$ orbifold of the theory

$$\mathcal{T}^{\rm free}=\mathcal{T}_{\rm B}\otimes\mathcal{T}_{\rm F}\otimes\mathcal{T}_{\psi}, \tag{3.2.34}$$

where $\mathbb{Z}_2=\{1,\xi\}$ acts on the individual components of $\mathcal{T}^{\rm free}$ as we have described in the previous sections. Specifically, we want to consider

$$\mathcal{T}=\left(\mathcal{T}^0_{\rm B,A}\otimes\mathcal{T}^0_{\rm F,A}\otimes\mathcal{T}^0_{\psi,\rm A}\right)\oplus\left(\mathcal{T}^1_{\rm B,P}\otimes\mathcal{T}^1_{\rm F,P}\otimes\mathcal{T}^1_{\psi,\rm P}\right). \tag{3.2.35}$$

where the $0,1$ superscripts denote respectively the invariant and anti-invariant part under the orbifold, in the corresponding sector denoted by the subscript. Notice that $\mathcal{T}^0_{\psi,\rm A}\oplus\mathcal{T}^1_{\psi,\rm P}$ is isomorphic (as a VOA) to the Conway module $V^{s\natural}$ (2.4.26), and as such the individual components $\mathcal{T}^0_{\psi,\rm A}$, $\mathcal{T}^1_{\psi,\rm P}$ admit a Spin(24) action as described in §2.4.1 (and hence the umbral group acts on the by automorphisms). Abusing the notation slightly, we will use the symbol $\mathcal{T}$ for the theory as well as its space of states.

Introducing chemical potentials $y_1=e^{2\pi i\zeta_1}$ and $y_2=e^{2\pi i\zeta_2}$ for the bosonic and fermionic ghosts respectively, we now define the following partition function

$$Z(\tau,\zeta_1,\zeta_2):={\rm Tr}_{\mathcal{T}}\left[y_1^{J^{\rm B}_{{\rm cyl},0}}y_2^{J^{\rm F}_{{\rm cyl},0}}q^{L_0^{\rm tot}-\frac{6}{24}}\right], \tag{3.2.36}$$

where $L_0^{\rm tot}$ is the total Virasoro zero mode of the theory, and $J^{\rm B}_{{\rm cyl},0}=J^{\rm B,1}_{{\rm cyl},0}+J^{\rm B,2}_{{\rm cyl},0}$, and $J^{\rm F}_{{\rm cyl},0}=J^{\rm F,1}_{{\rm cyl},0}+J^{\rm F,2}_{{\rm cyl},0}$ are the zero modes of the $U(1)$ currents of the two bosonic and two fermionic ghosts, respectively. Using the results of the previous sections, we compute

$$Z(\tau,\zeta_1,\zeta_2) = \mathrm{Tr}_{\mathcal{T}_{\mathrm{A}}^{\mathrm{free}}}\left[\frac{1}{2}(1+\xi)y_1^{J^{\mathrm{B}}_{\mathrm{cyl},0}}y_2^{J^{\mathrm{F}}_{\mathrm{cyl},0}}q^{L_0^{\mathrm{tot}}-\frac{6}{24}}\right] + \mathrm{Tr}_{\mathcal{T}_{\mathrm{P}}^{\mathrm{free}}}\left[\frac{1}{2}(1-\xi)y_1^{J^{\mathrm{B}}_{\mathrm{cyl},0}}y_2^{J^{\mathrm{F}}_{\mathrm{cyl},0}}q^{L_0^{\mathrm{tot}}-\frac{6}{24}}\right]$$
$$= \frac{1}{2}\left[\frac{\theta_3(\tau,\zeta_2)^2}{\theta_4(\tau,\zeta_1)^2}\left(\frac{\theta_3(\tau,0)}{\eta(\tau)}\right)^{12} - \frac{\theta_4(\tau,\zeta_2)^2}{\theta_3(\tau,\zeta_1)^2}\left(\frac{\theta_4(\tau,0)}{\eta(\tau)}\right)^{12} - \frac{\theta_2(\tau,\zeta_2)^2}{\theta_1(\tau,\zeta_1)^2}\left(\frac{\theta_2(\tau,0)}{\eta(\tau)}\right)^{12}\right]. \tag{3.2.37}$$

We observe that, by specializing to $\zeta_1 = 1/2$ and $\zeta_2 := \zeta$, we retrieve the K3 elliptic genus in the non-standard form presented in [79]

$$\mathbf{EG}(\tau,\zeta;K3) = Z\left(\tau,\frac{1}{2},\zeta\right) = \mathrm{Tr}_{\mathcal{T}}\left[(-1)^{J^{\mathrm{B}}_{\mathrm{cyl},0}}\; y^{J^{\mathrm{F}}_{\mathrm{cyl},0}}\; q^{L_0^{\mathrm{tot}}-\frac{6}{24}}\right]$$
$$= \frac{1}{2}\left[\frac{\theta_3(\tau,\zeta)^2}{\theta_3(\tau,0)^2}\left(\frac{\theta_3(\tau,0)}{\eta(\tau)}\right)^{12} - \frac{\theta_4(\tau,\zeta)^2}{\theta_4(\tau,0)^2}\left(\frac{\theta_4(\tau,0)}{\eta(\tau)}\right)^{12} - \frac{\theta_2(\tau,\zeta)^2}{\theta_2(\tau,0)^2}\left(\frac{\theta_2(\tau,0)}{\eta(\tau)}\right)^{12}\right]. \tag{3.2.38}$$

Note that, while in principle it is possible to consider more elaborate ghost theories and obtain a theory with the same symmetry and having the same partition function as above, our choice corresponds to the most minimal ghost systems with these properties, which moreover have the feature of rendering a CFT with central charge 6. Moreover, it is possible to equip our chosen $\mathcal{T}$ with an $\mathcal{N}=4$ superconformal structure at $c=6$. We will however not make use of this superconformal structure in the rest of the paper, since preserving it would reduce the symmetries of $\mathcal{T}$ that we want to exploit. Especially, different from the construction in [79], the symmetry groups of our theory $\mathcal{T}$ are not restricted to be 4-plane preserving subgroups of Co_0 since we do not require the symmetry to preserve the $U(1)$ current constructed from fermions. Correspondingly, note that in computing the partition function (3.2.36) we only introduce chemical potentials for $U(1)$ of the ghost theories (3.2.25). For completeness we discuss the $\mathcal{N}=4$ superconformal symmetry of $\mathcal{T}$ in §B.

3.2.4 The Twined Characters

We can consider the twined partition function

$$Z_g(\tau,\zeta_1,\zeta_2) := \mathrm{Tr}_{\mathcal{T}}\left[g\; y_1^{J^{\mathrm{B}}_{\mathrm{cyl},0}}y_2^{J^{\mathrm{F}}_{\mathrm{cyl},0}}q^{L_0^{\mathrm{tot}}-\frac{6}{24}}\right], \tag{3.2.39}$$

by an element $g \in \mathrm{Spin}(24)$, which has a manifest action on $\mathcal{T}_\psi$ (as in §2.4.1) and acts trivially on the ghost systems. We now specialize to the case $g \in Co_0 < \mathrm{Spin}(24)$.

The relevant characters are twined as follows

$$\mathrm{Tr}_{\mathcal{T}_{\mathrm{A}}^{\mathrm{free}}}\left[g\ y_1^{J^{\mathrm{B}}_{\mathrm{cyl},0}} y_2^{J^{\mathrm{F}}_{\mathrm{cyl},0}} q^{L_0^{\mathrm{tot}}-\frac{6}{24}}\right] = \frac{\eta_{-g}(\tau/2)}{\eta_{-g}(\tau)} \frac{\theta_3(\tau,\zeta_2)^2}{\theta_3\left(\tau,\zeta_1-\frac{1}{2}\right)^2}, \tag{3.2.40}$$

$$\mathrm{Tr}_{\mathcal{T}_{\mathrm{A}}^{\mathrm{free}}}\left[\xi\ g\ y_1^{J^{\mathrm{B}}_{\mathrm{cyl},0}} y_2^{J^{\mathrm{F}}_{\mathrm{cyl},0}} q^{L_0^{\mathrm{tot}}-\frac{6}{24}}\right] = -\frac{\eta_{g}(\tau/2)}{\eta_{g}(\tau)} \frac{\theta_4(\tau,\zeta_2)^2}{\theta_4\left(\tau,\zeta_1-\frac{1}{2}\right)^2}, \tag{3.2.41}$$

$$\mathrm{Tr}_{\mathcal{T}_{\mathrm{P}}^{\mathrm{free}}}\left[g\ y_1^{J^{\mathrm{B}}_{\mathrm{cyl},0}} y_2^{J^{\mathrm{F}}_{\mathrm{cyl},0}} q^{L_0^{\mathrm{tot}}-\frac{6}{24}}\right] = -C_{-g}\eta_{-g}(\tau) \frac{\theta_2(\tau,\zeta_2)^2}{\theta_2\left(\tau,\zeta_1-\frac{1}{2}\right)^2}, \tag{3.2.42}$$

$$\mathrm{Tr}_{\mathcal{T}_{\mathrm{P}}^{\mathrm{free}}}\left[\xi\ g\ y_1^{J^{\mathrm{B}}_{\mathrm{cyl},0}} y_2^{J^{\mathrm{F}}_{\mathrm{cyl},0}} q^{L_0^{\mathrm{tot}}-\frac{6}{24}}\right] = q\nu \prod_{n=1}^{\infty}\prod_{i=1}^{12}\left(1-\lambda_i q^n\right)\left(1-\lambda_i^{-1}q^{n-1}\right) \frac{\theta_1(\tau,\zeta_2)^2}{\theta_1\left(\tau,\zeta_1-\frac{1}{2}\right)^2}. \tag{3.2.43}$$

where the factors $\theta_i(\tau,\zeta_2)^2/\theta_i(\tau,\zeta_1-1/2)^2$ originate from the ghosts contribution.

In order to make contact with K3 and the umbral module discussed in the next section, we further specialize to a subgroup G of Co_0, such that each $g \in G$ generates a 4-plane preserving subgroup of Co_0. Note that by requiring that $g \in G$ is 4-plane preserving does not imply in general that G is 4-plane preserving. For instance, in the case $G \cong 3.S_6$ that is of special interest for us, different $g \in 3.S_6$ do not in general fix the same 4-plane, and thus $3.S_6$ does not preserve a 4-plane.

Finally, we specialise the fugacities of the ghost currents to the values $\zeta_2 = \zeta$ and $\zeta_1 = 1/2$. Note that care has to be taken when taking the $\zeta_1 \to 1/2$ limit in (3.2.43). On the one hand, the degeneracy of A ground states in $\mathcal{T}_\psi$ and the fact that at least two of the twelve pairs of g-eigenvalues are given by unity leads to a zero in the numerator. On the other hand, the infinite degeneracy of bosonic ghost ground states requires regularisation when taking $\zeta_1 \to 1/2$. As a result, we regularise the partition function by introducing an adiabatic shift in boundary condition given by a small positive parameter η. We consider the boundary conditions $\rho = 0+\eta$ and $\rho = 1/2+\eta$ as in (3.2.8), (3.2.24), and compute the $\eta \to 0^+$ limit of the partition function $Z_g^\eta\left(\tau,\frac{1}{2},\zeta\right)$ with the regulator η present. This is straightforward for all the terms except for (3.2.43), which receives the following contributions

$$\begin{aligned}
\tilde{\chi}_{\mathrm{B}}^{\mathrm{P},\eta}(\tau,1/2)^2 &= q^{-2/12}(1-q^{\eta})^{-2}\prod_{n=1}^{\infty}\left(1-q^{n+\eta}\right)^{-2}\left(1-q^{n-\eta}\right)^{-2},\\
\tilde{\chi}_{\mathrm{F}}^{\mathrm{P},\eta}(\tau,\zeta)^2 &= -q^{2/12}\left(iy^{1/2}q^{\eta}-iy^{-1/2}\right)^2\prod_{n=1}^{\infty}\left(1-yq^{n+\eta}\right)^2\left(1-y^{-1}q^{n-\eta}\right)^2,\\
\tilde{\chi}_{\psi}^{\mathrm{P},\eta}(\tau) &= q\nu\prod_{n=1}^{\infty}\left(1-q^{n-\eta}\right)^2\left(1-q^{n-1+\eta}\right)^2\prod_{i=1}^{10}\left(1-\lambda_i q^{n-\eta}\right)\left(1-\lambda_i^{-1}q^{n-1+\eta}\right).
\end{aligned} \tag{3.2.44}$$

We see that, upon multiplying the above expressions, the potentially problematic factors $(1-q^{\eta})^{\pm 2}$ drop out and we get

$$\begin{aligned}
\lim_{\eta\to 0^+}\mathrm{Tr}_{\mathcal{T}_{\mathrm{P}}^{\mathrm{free}}}\left[\xi\ g\ y_1^{J^{\mathrm{B}}_{\mathrm{cyl},0}}y_2^{J^{\mathrm{F}}_{\mathrm{cyl},0}}q^{L_0^{\mathrm{tot}}-\frac{6}{24}}\right] &= \lim_{\eta\to 0^+}\tilde{\chi}_{\mathrm{B}}^{\mathrm{P},\eta}(\tau,1/2)^2\ \tilde{\chi}_{\mathrm{F}}^{\mathrm{P},\eta}(\tau,\zeta)^2\ \tilde{\chi}_{\psi}^{\mathrm{P},\eta}(\tau)\\
&= \frac{\theta_1(\tau,\zeta)^2}{\eta(\tau)^6}D_g\eta_g.
\end{aligned} \tag{3.2.45}$$

Putting everything together, we get

$$\begin{aligned}
\lim_{\eta\to 0^+}Z_g^{\eta}\left(\tau,\frac{1}{2},\zeta\right) = \frac{1}{2}\Bigg[&\frac{\theta_3(\tau,\zeta)^2}{\theta_3(\tau,0)^2}\frac{\eta_{-g}(\tau/2)}{\eta_{-g}(\tau)}-\frac{\theta_4(\tau,\zeta)^2}{\theta_4(\tau,0)^2}\frac{\eta_g(\tau/2)}{\eta_g(\tau)}\\
&-\frac{\theta_2(\tau,\zeta)^2}{\theta_2(\tau,0)^2}C_{-g}\eta_{-g}(\tau)-\frac{\theta_1(\tau,\zeta)^2}{\eta(\tau)^6}D_g\eta_g(\tau)\Bigg].
\end{aligned} \tag{3.2.46}$$

We observe that this equals with $\phi_{\epsilon,g}$ as defined in (3.1.10) and (3.1.12), the Conway twining graded by a $k=1$ $U(1)$ current. In particular, note that the same sign ambiguity in D_g in the twining of the Conway CFT described in §2.4.1 is also present here, leading to the sign ϵ in the definition of the twining functions. However, the crucial difference, and what we are aiming for, is that the $U(1)$ grading in $\mathcal{T}$ is preserved by the G-action since it is constructed out of the ghost fields, instead of the free fermions, which the group acts trivially on. We also stress that this equality holds for any 4-plane preserving Conway element and is not restricted to the specific groups we consider in this paper.

3.3 Module

Here we explain how the ingredients of the previous sections lead to a $\mathbb{Z}_2$-graded infinite dimensional vector space admitting a $G^{D_4^{\oplus 6}}$–action that underlies the $D_4^{\oplus 6}$

case of umbral moonshine. In particular, we will describe how the umbral mock modular forms $H_g^{D_4^{\oplus 6}}$ for all elements g of the umbral group $G^{D_4^{\oplus 6}}$ are recovered from the twined partition functions of the chiral CFT $\mathcal{T}$. In §3.3.1 we describe an explicit construction of the group. In §3.3.2 we explain the action of the group on the BPS states of 6 copies of the CFT describing a singularity of D_4 type. In §3.3.3 we combine the ingredients and give expressions for $H_g^{D_4^{\oplus 6}}$ in terms of them. As this section is completely devoted to the $D_4^{\oplus 6}$ case of umbral moonshine, we will denote by $\mathfrak{G}$ the umbral group $G^{D_4^{\oplus 6}} \cong 3.S_6$. Similarly, we will denote the mock modular forms $H_g^{D_4^{\oplus 6}} = (H_{g,r}^{D_4^{\oplus 6}})$ simply by $H_g = (H_{g,r})$, and denote the weak Jacobi forms $\phi_g^{D_4^{\oplus 6}}$ simply by ϕ_g.

3.3.1 The Group

For completeness, we describe a concrete realization of the group $3.S_6$, following [114]. The hexacode is the unique three-dimensional code of length 6 over $\mathbb{F}_4$ that is Hermitian and self-dual. It is the glue code of the Niemeier lattice $N^{D_4^{\oplus 6}}$ with root system $D_4^{\oplus 6}$ [15], and for this reason it plays a significant role in the case of umbral moonshine corresponding to $N^{D_4^{\oplus 6}}$. Moreover, this code also plays an important role in the construction of the largest Mathieu group M_{24}. Its automorphism is given by $3.A_6$, which can be explicitly constructed in the following way. Write $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$ with

$$\omega^2 = \bar{\omega}, \quad \bar{\omega}^2 = \omega, \quad \omega^3 = 1.$$

The triple cover of the alternating group A_6 can be generated by the permutations $(1,2)(3,4)$, $(1,2)(5,6)$, $(3,4)(5,6)$, $(1,3,5)(2,4,6)$, $(1,3)(2,4)$, as well as the composition of the permutation and multiplication $(1,2,3)\mathrm{diag}(1,1,1,1,\bar{\omega},\omega)$. This group acts on the 6 coordinates and in particular induces all even permutations. It also contains the element corresponding to scalar multiplication by ω and by $\bar{\omega}$. Hence we have constructed a group with centre $3 \cong \mathbb{Z}/3$ and we will call z the generator of the center corresponding to $\mathrm{diag}(\omega,\omega,\omega,\omega,\omega,\omega)$, the scalar multiplication by ω.

The group $3.A_6$ can be enlarged to our umbral group $\mathfrak{G} = 3.S_6$ by adjoining an extra generator which acts on a vector in $\mathbb{F}_4^6$ by permuting the last two coordinates followed by a complex conjugation: $\omega \leftrightarrow \bar{\omega}$. This group is often referred to as the semi-automorphism group of the hexacode, since it leaves the code invariant but does not act linearly on it.

From the above description, we can define a representation for the group $\mathfrak{G}$ given by the group homomorphism $\epsilon : \mathfrak{G} \to \{1, -1\}$, where $\epsilon_g = 1$ (-1) when g induces an even (odd) permutation on the 6 coordinates. In the notation in Table C.1, this is given by the irreducible character χ_2. This representation will play an important role in describing the umbral module.

More generally, the action of $\mathfrak{G}$ on $\mathbb{F}_4^6$ determines the umbral moonshine module for the $D_4^{\oplus 6}$ case of umbral moonshine. For later use we will now describe this action in more detail. Writing the natural basis of $\mathbb{F}_4^6$ as given by $\mathbf{e}_0^i$, $\mathbf{e}_1^i$, $\mathbf{e}_\omega^i$ and $\mathbf{e}_{\bar{\omega}}^i$ for $i = 1, \ldots, 6$, we obtain a 24-dimensional permutation representation of $\mathfrak{G}$. The corresponding 24-dimensional cycle shape is denoted by $\widetilde{\Pi}_g$ in Table C.2. Furthermore, from the above construction of $\mathfrak{G}$ it is clear that the action of $\mathfrak{G}$ does not mix $\mathbf{e}_0^i$ with $\mathbf{e}_1^i$, $\mathbf{e}_\omega^i$ and $\mathbf{e}_{\bar{\omega}}^i$ and hence we arrive at a six-dimensional representation of $\mathfrak{G}$. The corresponding 6-dimensional cycle shape is denoted by $\bar{\Pi}_g$ in Table C.2, and the corresponding character denoted by $\bar{\chi}$. One has $\bar{\chi} = \chi_1 + \chi_3$ in terms of the irreducible representations (cf. Table C.1). Alternatively, one might think of the 6-dimensional representation as spanned by the 6 vectors of the form $\mathbf{e}_1^i + \mathbf{e}_\omega^i + \mathbf{e}_{\bar{\omega}}^i$. Similarly, we also define another character χ by $\chi_g = \bar{\chi}_g \epsilon_g$. One has $\chi = \chi_2 + \chi_4$ in terms of the irreducible representations. Finally, we have the 12-dimensional representation with basis $\mathbf{e}_1^i - \mathbf{e}_\omega^i$ and $\mathbf{e}_1^i - \mathbf{e}_{\bar{\omega}}^i$ for $i = 1, \ldots, 6$. We denote the corresponding character by $\check{\chi}$, given by $\check{\chi} = \chi_{14}$ in terms of the irreducible characters. Moreover, we denote by $\bar{R}, R, \check{R}$ the representations corresponding to the characters $\bar{\chi}, \chi, \check{\chi}$.

One can translate the above description of the group action on the hexacode into an action on the root systems $D_4^{\oplus 6}$ in a straightforward way. First one identifies each copy of $\mathbb{F}_4$ with a copy of D_4, $\mathbf{e}_0$ with the central node of the dynkin diagram, and $\mathbf{e}_1$, $\mathbf{e}_\omega$, $\mathbf{e}_{\bar{\omega}}$ with the three nodes connected to the central node.

3.3.2 The Singularities

As reviewed in §3.1.1, there are 23 different natural ways to decompose the K3 elliptic genus (and twinings thereof) into two parts, corresponding to the 23 Niemeier root systems X. The first part is given by the elliptic genus of the CFTs that describes the singularities associated with X. The second part is the contribution from the umbral moonshine vector-valued mock modular forms H^X. Since the umbral group G^X naturally acts on the singularities X as well as on the umbral moonshine module, we can generalise the construction and define a g-twined weak Jacobi form ϕ_g^X as in (3.1.8).

In this subsection we describe the construction of the twined singularity elliptic genus $\mathbf{EG}_g(\tau, \zeta; X)$ for $X = D_4^{\oplus 6}$ explicitly, for all $g \in \mathfrak{G} = G^{D_4^{\oplus 6}} \cong 3.S_6$ This is expressed via (3.1.2) in terms of the elliptic genus of the D_4 supersymmetric minimal model, given by

$$Z_{\text{minimal}}^{D_4}(\tau, \zeta) = \frac{1}{2} \operatorname{Tr}\left(\Omega^{D_4} \cdot \tilde{\chi}(\tau, \zeta)\right) = \frac{\theta_1^2\left(\tau, \frac{2}{3}\zeta\right)}{\theta_1^2\left(\tau, \frac{1}{3}\zeta\right)}, \tag{3.3.1}$$

where the Cappelli–Itzykson–Zuber [46] omega matrix Ω^{D_4} is given by

$$\left(\Omega^{D_4}\right)_{r,r'} = \begin{cases} 2\delta_{r,r'(12)} & , \quad r = 0,3 \mod 6 \\ \delta_{r,r'(12)} + \delta_{r,-r'(6)}\delta_{r,r'(4)}, & r = 1,5 \mod 6 \\ \delta_{r,r'(12)} + \delta_{r,-r'(6)}\delta_{r,r'(4)}, & r = 2,4 \mod 6 \end{cases} , \tag{3.3.2}$$

and the indices r, r' take values in $\mathbb{Z}/12$. Using the property $\tilde{\chi}^r_s(\tau,\zeta) = -\tilde{\chi}^{-r}_s(\tau,\zeta)$ of the parafermion characters, we can rewrite (3.3.1) as

$$Z^{D_4}_{\text{minimal}} = \text{Tr}_* \left(\widehat{\Omega}^{D_4} \cdot \tilde{\chi}\right) \tag{3.3.3}$$

where $(\widehat{\Omega}^{D_4})_{r,s} = (\Omega^{D_4})_{r,s} - (\Omega^{D_4})_{r,-s}$ and is expicitly given by

$$\widehat{\Omega}^{D_4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} , \tag{3.3.4}$$

and we have used the notation Tr_* to denote tracing over the indices $\{1, 2, 3, 4, 5\}$. Then from (3.1.6) this gives the corresponding singularity elliptic genus

$$\mathbf{EG}(\tau, \zeta; D_4) = \text{Tr}_* \left(\widehat{\Omega}^{D_4} \cdot \Xi(\tau,\zeta)\right) \tag{3.3.5}$$

where we have defined

$$\Xi^r_s(\tau,\zeta) := \frac{1}{6} \sum_{a,b \in \mathbb{Z}/6\mathbb{Z}} q^{a^2} y^{2a}\, \tilde{\chi}^r_s(\tau, \zeta + a\tau + b) Z_{L_m}(\tau, \zeta + a\tau + b) , \tag{3.3.6}$$

which has integer coefficients in the q, y expansions. This can be understood from the fact that it is the graded dimension of an infinite-dimensional vector space, which we denote by V^r_s. Specifically, the space V^r_s, which has

$$\text{sdim}^{(q,y)} V^r_s = \Xi^r_s(\tau,\zeta) ,$$

can be constructed explicitly from the above $\mathbb{Z}/6$ orbifold projection, the parafermionic construction of the $\mathcal{N} = 2$ minimal model characters $\tilde{\chi}^r_s$ (cf. (3.1.2)), and the construction of Z_{L_m} in terms of the $(-1)^F$-graded Ramond characters (cf. (3.1.4)). In the above, we have introduced the graded super-dimension for a bi-graded vector space

$$V = \bigoplus_{\substack{\varepsilon\in\{+,-\}\\ n,\ell\in\mathbb{Z}}} (V^{(\varepsilon)})_{n,\ell}$$

by defining

$$\mathrm{sdim}^{(q,y)} V := \sum_{n,\ell\in\mathbb{Z}} q^n y^\ell \left(\dim(V^{(+)})_{n,\ell} - \dim(V^{(-)})_{n,\ell}\right). \tag{3.3.7}$$

Similarly, if V admits an action by a finite group G, we define the corresponding graded super-character as

$$\mathrm{str}_V^{(q,y)} g := \sum_{n,\ell\in\mathbb{Z}} q^n y^\ell \left(\mathrm{tr}_{(V^{(+)})_{n,\ell}} g - \mathrm{tr}_{(V^{(-)})_{n,\ell}} g\right) \tag{3.3.8}$$

for $g \in G$.

Recall that the automorphism group of the D_4 root system is generated by an order 2 element g_2 and an order 3 element g_3. The corresponding action on the minimal model is then captured by $Z^{D_4}_{\mathrm{minimal},g_{2,3}} = \mathrm{Tr}_* \left(\widehat{\Omega}^{D_4}_{g_{2,3}} \cdot \tilde{\chi}\right)$, where the so-called twined Omega matrices for D_4 are given by

$$\widehat{\Omega}^{D_4}_{g_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix} , \quad \widehat{\Omega}^{D_4}_{g_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.3.9}$$

From this and the explicit description of the group action of $\mathfrak{G}$ on the root system $D_4^{\oplus 6}$ given in §3.3.1, we can construct a bi-graded supermodule of $\mathfrak{G}$:

$$V_{\mathrm{sing}} = \left(\bar{R} \otimes V_5^1\right) \oplus \left(R \otimes V_1^5\right) \oplus \left(\check{R} \otimes V_3^3\right) , \tag{3.3.10}$$

with the property that

$$\mathbf{EG}_g(\tau,\zeta; D_4^{\oplus 6}) = \mathrm{str}^{(q,y)}_{V_{\mathrm{sing}}} g , \quad g \in \mathfrak{G} \tag{3.3.11}$$

where $\bar{R}, R, \check{R}$ are the specific $\mathfrak{G}$-representations described by the end of §3.3.1 and $\mathfrak{G}$ acts trivially on V_s^r.

Explicitly, we have

$$\mathbf{EG}_g(\tau,\zeta; D_4^{\oplus 6}) = \mathrm{Tr}_* \left(\widehat{\Omega}_g^{D_4^{\oplus 6}} \cdot \Xi\right), \tag{3.3.12}$$

where the g-twined omega matrix $\Omega_g^{D_4^{\oplus 6}}$ is given by the group charaters χ, $\bar{\chi}$, $\check{\chi}$ discussed in §3.3.1 as

$$\widehat{\Omega}_g^{D_4^{\oplus 6}} = \begin{pmatrix} \bar{\chi}_g & 0 & 0 & 0 & \chi_g \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \check{\chi}_g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \chi_g & 0 & 0 & 0 & \bar{\chi}_g \end{pmatrix} \tag{3.3.13}$$

for $g \in 3.S_6$.

3.3.3 The Module

Recall that for the $X = D_4^{\oplus 6}$ case of umbral moonshine, we have three non-vanishing mock modular forms $H_{g,r}^X$, for $r = 1, 3, 5$ (cf. (2.4.37)). In this subsection we will tie together the different elements discussed thus far and construct three $\mathfrak{G}$-modules K^r with $r = 1, 3, 5$, with infinite dimensions, a bi-grading, and an additional (a priori) $\mathbb{Z}_2$ grading, such that

$$\mathrm{str}_{K^r}^{(q,y)} g = H_{g,r}(\tau), \quad \text{for all } g \in \mathfrak{G}. \tag{3.3.14}$$

As mentioned before, our main strategy is to employ the relation between the moonshine mock Jacobi forms $\Psi_g^X(\tau,\zeta) := \sum_{r\in\mathbb{Z}/2m} H_{g,r}^X(\tau)\theta_{m,r}(\tau,\zeta)$, the weak Jacobi forms ϕ_g^X, and the singularity elliptic genus $\mathbf{EG}(X)$, summarised in (3.1.8), for the specific case $X = D_4^{\oplus 6}$. Explicitly, in this case we have

$$H_{g,1}(\tau) = \epsilon_g H_{g,5}(\tau) \tag{3.3.15}$$

and (3.1.8) gives

$$H_{g,1}(\tau)\left(\theta_{6,1}^1(\tau) + \epsilon_g\theta_{6,5}^1(\tau)\right) + H_{g,3}(\tau)\theta_{6,3}^1(\tau) = \frac{\eta^6(\tau)}{\theta_1^2(\tau,\zeta)}\left(\phi_g - \mathbf{EG}_g(\tau,\zeta;D_4^{\oplus 6})\right) , \tag{3.3.16}$$

where $\theta_{m,r}^1$ are the unary theta functions defined by (cf. 2.2.24)

$$\theta_{m,r}^1(\tau) := \left(\frac{1}{2\pi i}\frac{\partial}{\partial\zeta}\theta_{m,r}(\tau,\zeta)\right)\Big|_{\zeta=0} . \tag{3.3.17}$$

We will do this in a few steps.

Step 1: Obtaining the weak Jacobi forms ϕ_g

It was shown in appendix D of [91] that, for any embedding $\iota : \mathfrak{G} \to Co_0$ we have

$$\phi_{\epsilon_g,\iota(g)} = \phi_g, \tag{3.3.18}$$

where we are using definition (3.1.12) with ϵ given by $\epsilon_g : \mathfrak{G} \to \{1, -1\}$, the character defined in §3.3.1. Applying the result of §3.2.4 and letting $\mathfrak{G}$ act on $\mathcal{T}$ according to the above embedding into Co_0, we obtain the above functions as twining partition functions of $\mathcal{T}$. This results in the structure of $\mathcal{T}$ as a bi-graded $\mathfrak{G}$- supermodule with graded super-characters given by

$$\mathrm{str}_{\mathcal{T}}^{(q,y)} g = \phi_g^{D_4^{\oplus 6}}, \quad \text{for all } g \in \mathfrak{G}. \tag{3.3.19}$$

Step 2: Obtaining the mock Jacobi forms $(\phi_g - \mathbf{EG}_g(\tau, \zeta; D_4^{\oplus 6}))$

Combining $\mathcal{T}$ with the $\mathfrak{G}$-supermodule V_{sing} constructed in §3.3.2, we arrive at a bi-graded $\mathfrak{G}$-supermodule

$$W = \mathcal{T} \ominus V_{\mathrm{sing}}, \tag{3.3.20}$$

where we use $\ominus$ to denote the following operations on vector spaces with super-structures (see (2.1.20)),

$$(V_1 \ominus V_2)^{(\pm)} = V_1^{(\pm)} \oplus V_2^{(\mp)}. \tag{3.3.21}$$

From (3.3.19) and (3.3.11), it follows immediately that

$$\mathrm{Str}_W^{(q,y)} g = \phi_{\epsilon_g,g}(\tau, \zeta) - \mathbf{EG}_g(\tau, \zeta; D_4^{\oplus 6}) \tag{3.3.22}$$

for all elements g of $\mathfrak{G}$.

Step 3: Introducing the auxiliary spaces

In the next step of the construction, we define the following "auxiliary" (in the sense of not arising directly from the two CFTs discussed in previous sections) bi-graded supermodule $\mathcal{H}_{\mathrm{aux}}^r$ with a simple $\mathfrak{G}$ action, satisfying

$$\mathrm{Str}_{\mathcal{H}_{\mathrm{aux}}^r}^{(q,y)} g = \begin{cases} \frac{\eta^6(\tau)}{\theta_1^2(\tau,\zeta)} \left(\theta_{6,1}^1(\tau) + \epsilon_g \theta_{6,5}^1(\tau)\right)^{-1} &, \quad r = 1 \\ \frac{\eta^6(\tau)}{\theta_1^2(\tau,\zeta)} \left(\epsilon_g \theta_{6,1}^1(\tau) + \theta_{6,5}^1(\tau)\right)^{-1} &, \quad r = 5 \\ \frac{\eta^6(\tau)}{\theta_1^2(\tau,\zeta)} \left(\theta_{6,3}^1(\tau)\right)^{-1} &, \quad r = 3 \end{cases}. \tag{3.3.23}$$

Step 4: The projection

In the final step, we recall that there are two types of irreducible representations for $\mathfrak{G}$: those that are faithful and those that factor through $S_6 \subset \mathfrak{G} \cong 3.S_6$. This distinction is featured prominently in the present case of umbral moonshine due to the property that each component of the umbral mock modular form receives only contributions from one type of the irreducible representations (see Conjecture 6.2 of [93], proven in [96]). To implement this, we introduce the corresponding projection operator P: we write P (resp. $\mathbf{1}-P$) to denote the projection operators that project out the irreducible representations which are faithful (resp. factor through S_6.) Explicitly, let $n_i \in \mathbb{Z}$ and denote by V_i the irreducible representation corresponding to the character χ_i in Table C.1. Then P acts on a virtual representation $V = \sum_{i=1}^{16} n_i V_i$ of $\mathfrak{G}$ by $V|P = \sum_{i=1}^{11} n_i V_i$, and similarly $\mathbf{1}-P$ acts as $V|(\mathbf{1}-P) = \sum_{i=12}^{16} n_i V_i$. At the level of characters, one has

$$\mathrm{Tr}_{V|P}\, g = \frac{1}{3}\left(\mathrm{Tr}_V(g) + \mathrm{Tr}_V(zg) + \mathrm{Tr}_V(z^2 g)\right) \ , \tag{3.3.24}$$

where z is a generator of the center subgroup of $\mathfrak{G}$.

Finally, we define

$$K^r = \mathcal{H}^r_{\mathrm{aux}} \otimes \begin{cases} W|P & r=1,5 \\ W|(\mathbf{1}-P) & r=3 \end{cases} \ , \tag{3.3.25}$$

where under the tensor product the bi-gradings are additive. Putting (4.18) and (4.19) together, and employing the aforementioned property of the umbral moonshine module, we arrive at (3.3.14) and thereby complete the construction of the relevant $\mathfrak{G}$-supermodule. We will next discuss features of this construction further.

3.4 Discussion

We have described the construction of a module for the $D_4^{\oplus 6}$ case of umbral moonshine, as it appeared in [2]. This is the first time that the module is constructed for a case of umbral moonshine with a non-4-plane preserving umbral group, which is moreover significantly more sizeable compared to the previously constructed cases (with $|G^{D_4^{\oplus 6}}| \sim 10^3$, this group is larger than the cases discussed in [97–99], where the groups have order dividing 24). This is also the first construction of the umbral module which utilises the connection to symmetries of K3 string theory. At the same time, there are clearly important open questions remaining. In the following we discuss a few of them.

- Note that our construction naturally leads to a super-module for $G^{D_4^{\oplus 6}}$. However, apart from the virtual representation corresponding to the leading polar term (cf. (2.4.37)), the umbral module is known to constitute the even part. Moreover, our construction gives a priori a bi-graded super-module (3.3.14), whose components corresponding to a non-trivial y-grading happens to be empty. In other words, the characters we computed via (3.3.14) is a priori a q, y-series that happens to be just a q-series. It would be nice to make the positivity and y-independence manifest.

- What is the physical or geometric meaning of the chiral CFT $\mathcal{T}$? The relation between the Conway CFT, which is closely related to $\mathcal{T}$, and a specific K3 sigma model has been elucidated in [79, 115, 116]. It would be interesting to understand the physical role played by the ghost systems.

- An obvious question is whether one can employ a similar construction for the other cases of umbral moonshine. Note that the chiral CFT $\mathcal{T}$ has Spin(24) symmetry which preserves the fermionic and bosonic $U(1)$ ghost currents. It is hence possible to define the regularised twined partition function $\lim_{\eta\to 0^+} Z_g^\eta(\tau, \zeta_1, \zeta_2)$ (cf. §3.2.4) for any element of any of the 23 umbral groups. To make contact with weak Jacobi forms of the type of K3 elliptic genus, one has to specialise the fugacity to $\zeta_1 = \frac{1}{2}$. However, this leads to a finite answer only when taking $\eta \to 0^+$ if g is 4-plane preserving. To construct umbral moonshine modules for cases where not all group elements are 4-plane preserving ($X = A_1^{\oplus 24}$, $A_2^{\oplus 12}$, and $6A_4$), one needs a construction that works with the two-elliptic-variable functions $Z_g(\tau, \zeta_1, \zeta_2)$ directly.

- Note that the contribution of the vector-valued umbral moonshine mock modular forms $(H_{g,r}^X)$ to the twined partition function of the theory $\mathcal{T}$ is basically given by a single q-series $\frac{1}{2\pi i}\frac{\partial}{\partial\omega}\Psi_e^X(\tau, w)$. See (3.1.8). What allows us to recover from it the individual components $H_{g,r}^X$ of the mock modular forms is the following two facts. First, there are just two independent components in the case $X = D_4^{\oplus 6}$, which can be taken to be $H_{g,1}^X$ and $H_{g,3}^X$. Second, the representations underlying the 1st resp. 3rd component have the feature that they factor through S_6 resp. are faithful representations. As a result, it is possible to use the projection operator (3.3.24) to isolate the contributions from the two independent components from the twined partition function of $\mathcal{T}$. A similar projection property also holds for other 14 cases of umbral moonshine (cf. Conjecture 6.3 in [93]).

  In view of this, another challenge when attempting to generalise the current construction to other cases of umbral moonshine is how to disentangle the contributions from different components of the vector-valued umbral moonshine mock modular forms $(H_{g,r}^X)$ in the twined partition functions for the cases of X with many independent components. Recall that an important feature of umbral moonshine is the "multiplicative relations" relating $H_{g'}^{X'}$ and H_g^X for specific pairs of Niemeier root systems (X, X') and group elements $g \in G^X$ and $g' \in G^{X'}$ (cf. §5.3 of [93]). As we will see in more detail below, these relations together with the projection property enable us to disentangle different components in the vector-valued functions H_g^X in various cases.

Finally, we point out that for the cases that g is a 4-plane preserving group element of a umbral group G^X, many mock modular forms H_g^X for many different X and g can be obtained in a similar way as discussed in §3.3.

$A_1^{\oplus 24}$ *:*

Exactly the same procedure as discussed in the main part of this paper can be used to obtain a supermodule for the group $M_{22} < G^{A_1^{\oplus 24}}$ that is compatible with the M_{24} moonshine, or equivalently the $X = A_1^{\oplus 24}$ case of umbral moonshine.

$A_2^{\oplus 12}$*:*

An analogous procedure, using a projection operator projecting out representations factoring through $M_{12} < G^{A_2^{\oplus 12}} \cong 2.M_{12}$, recover from twined parition functions (3.2.39) the mock modular forms $H_g^{A_2^{\oplus 12}}$ for $g \in 2.M_{12}$ that are not in the conjugacy classes $11AB, 12A, 20AB, 22AB$. (Here and below we use the same naming of the conjugacy classes as in [93].) As a result, one can construct modules for $\tilde{G} < 2.M_{12}$ compatible with the corresponding case of umbral moonshine, for three of the maximal subgroups of $2.M_{12}$. For completeness we list the explicit generators of these three maximal subgroups in terms of permutation groups on 24 objects:

$$
\begin{aligned}
G_1 = \langle &(1,18,5,9,24,16)(2,6,8,11,17,20)(3,12,23,13,22,14)(4,10,19,15,21,7),\\
&(1,9)(2,19)(3,13(4,10)(5,24)(6,17)(7,11)(8,23)(12,22)(14,20)(15,21)(16,18)\rangle\\
G_2 = \langle &(1,13)(2,19)(3,9)(4,18)(5,21)(6,17)(7,11)(8,20)(10,16)(12,22)(14,23)(15,24),\\
&(1,5,11)(2,9,16)(3,18,8)(4,17,7)(6,19,15)(10,12,23)(13,24,20)(14,21,22),\\
&(1,9)(2,11)(3,13)(4,15)(5,16)(6,17)(7,19)(8,20)(10,21)(12,22)(14,23)(18,24)\rangle\\
G_3 = \langle &(1,12,18)(3,15,6,21,16,14)(4,17,10,5,23,13)(7,20)(8,19)(9,22,24),\\
&(1,11,22,24)(2,12,18,9)(3,20,10,19)(4,5)(6,23)(7,13,8,21)(14,17)(15,16)\rangle
\end{aligned}
$$

$A_3^{\oplus 8}$*,* $A_8^{\oplus 3}$*,* $E_6^{\oplus 4}$*:*

Using similar analysis as above, one can recover H_g^X for all elements of $g \in \tilde{G} < G^X$, for $X = A_3^{\oplus 8}$ and $A_8^{\oplus 3}$. In particular, in the $X = A_3^{\oplus 8}$ case we also make use of the multiplicative relations between H_g^X and $H_{g'}^{X'}$, where $X' = A_1^{\oplus 24}$ and $g' \in G^{X'}$, and thereby obtain all H_g^X except for $g \in [8A]$. In the $X = A_8^{\oplus 3}$ we make use of the multiplicative relations between H_g^X and $H_{g'}^{X'}$, where $X' = A_2^{\oplus 12}$ and $g' \in G^{X'}$, and thereby obtain all H_g^X except for $g \in [3A]$ and $g \in [6A]$. In the $X = E_6^{\oplus 4}$ we make use of the multiplicative relations between H_g^X and $H_{g'}^{X'}$, where $X' = A_2^{\oplus 12}$ and $g' \in G^{X'}$, and thereby obtain all H_g^X except for $g \in [8A]$ and $g \in [8B]$.

$A_5^{\oplus 4}D_4$*,* $A_7^{\oplus 2}D_5^{\oplus 2}$ *,* $A_{11}D_7E_6$*,* $A_{17}E_7$*,* $D_{10}E_7^{\oplus 2}$*:*

For $X = A_5^{\oplus 4} D_4$ all non-vanishing components of $H_g^X = (H_{g,r}^X)$, $r = 1, 2, \ldots, 5$, can be recovered from the twined partition functions, by relating them to the umbral moonshine mock modular forms for $X' = D_4^{\oplus 6}$ that we constructed in the main part of the paper, and for the $X'' = A_2^{\oplus 12}$ case that we described above. Explicitly, we have

$$H_{g',1}^{X'}(\tau) = H_{g,1}^X(\tau) + H_{g,5}^X(\tau) \tag{3.4.1}$$

for the pairs

$$(g', g) = (1A, 1A/2A), (2A, 2B), (2A, 4A), (3B, 3A/6A), (4A, 8AB), \tag{3.4.2}$$

and

$$H_{g'',1}^{X'}(\tau) = H_{g,1}^X(\tau) - H_{g,5}^X(\tau) \tag{3.4.3}$$

for the pairs $(g'', g) = (2B, 1A/2A), (2B, 2B), (2C, 4A), (6B, 3A/6A), (4B, 8AB)$. For the 3rd component we make use the relation

$$H_{g,3}^X(\tau) = \frac{1}{2} H_{g',3}^{X'}(\tau) \tag{3.4.4}$$

for the same pairs (g', g) as in (3.4.2). The even components satisfy

$$H_{g,2r}^X = -H_{zg,2r}^X, \quad r \in \mathbb{Z}/3 \tag{3.4.5}$$

where z denotes a generator of the center subgroup $\langle z \rangle \cong \mathbb{Z}_2 < G^X$. This forces the even components of the vector-valued mock modular forms H_g^X to vanish for elements g in conjugacy classes $2B, 4A, 8AB$. The rest of $H_{g,2r}^X$ can be recovered by using the relation to $X'' = A_2^{\oplus 12}$ case of umbral moonshine:

$$H_{g,2}^X(\tau) - H_{g,4}^X(\tau) = H_{g',2}^{X''}(\tau). \tag{3.4.6}$$

for the pairs $(g, g') = (1A, 2B), (3A, 6C)$. Note that the two terms on the left-hand side contribute to different powers of q when regarding the whole function as a q-series and the above relation is therefore enough to determine both the 2nd and the 4th components of the mock modular forms H_g^X.

Using similar analysis as above, one can recover all H_g^X for all $g \in G^X$, for $X = A_7^{\oplus 2} D_5^{\oplus 2}$, $A_{11} D_7 E_6$, $A_{17} E_7$, and $D_{10} E_7^{\oplus 2}$. The only things that remains for these cases is to construct an explicit group action which reproduces these functions, something that is beyond the scope of this thesis.

4 Vertex operator superalgebra - sigma model correspondence: the T^4 case

The goal of this chapter is to explain the correspondence between the vertex operator superalgebra (VOSA) $V^f_{E_8}$ (to be defined shortly) and the nonlinear sigma models on complex four-dimensional tori (T^4), as presented in [3]. In order to provide some context for this correspondence, first recall from the previous chapter that the relation between sporadic finite simple groups and symmetries of K3 surfaces and K3 sigma models has attracted a lot of attention since the pioneering work of [117] and [82] (for some instances of this see [84–87, 92, 93, 102, 118–124]). Apart from the Mathieu groups featured in [82, 117], symmetries of $\mathcal{N} = (4,4)$ supersymmetric non-linear sigma models on K3 surfaces have also been related to other groups, including the sporadic simple Conway groups [79, 107, 125], and the groups of umbral moonshine [28, 91]. Moreover, recall that the twined elliptic genera play a critical role in quantifying this relation since they are sensitive to the way that symmetries act on quantum states. Of special interest is the fact that many of the twined elliptic genera of sigma models on K3 surfaces can be reproduced by the vertex operator superalgebra (VOSA) $V^{s\natural}$, which has played a prominent role in Conway moonshine [77–79]. (Here and in the remainder we use *sigma model* as a shorthand for $\mathcal{N} = (4,4)$ supersymmetric non-linear sigma model.)

As a final reminder from the previous chapter, recall that the analysis of [91] indicates that not all the twined K3 elliptic genera can be reproduced by Conway group symmetries of $V^{s\natural}$. It is nonetheless interesting that the single VOSA $V^{s\natural}$ can capture the symmetry properties of a large family of sigma models in the K3 moduli space, especially given that $V^{s\natural}$ is, in physical terms, a chiral theory, with central charge $c = 12$, while the K3 sigma models are non-chiral theories, with $c = \bar{c} = 6$. This novel chiral/non-chiral connection between $V^{s\natural}$ and K3 sigma models has been made precise at a special (orbifold) point in the moduli space, where $V^{s\natural}$ can be retrieved as the image of the corresponding K3 theory under *reflection*: a procedure explored in [79] for the specific case of $V^{s\natural}$ and later formerly investigated in more generality by Taormina–Wendland in [115]. (See also [116] for a complementary approach).

To put this connection in a more structured context let us consider sigma models with target space X within one connected component of the full moduli space $\mathcal{M} = \mathcal{M}(X)$ of sigma models on X, and denote the corresponding sigma models by $\Sigma(X;\mu)$, for μ a point in $\mathcal{M}$. For instance, for $X = T^4$ or $X = K3$ the moduli space consists of a single component, and takes the form

$$\begin{aligned} \mathcal{M}(T^4) &= (SO(4)\times SO(4))\backslash SO^+(4,4)/SO^+(\Gamma^{4,4})\ , \\ \mathcal{M}(K3) &= (SO(4)\times O(20))\backslash O^+(4,20)/O^+(\Gamma^{4,20})\ . \end{aligned} \tag{4.0.1}$$

Here $\Gamma^{a,b}$ denotes an even unimodular lattice of signature (a,b).

The chiral/non-chiral connection between $V^{s\natural}$ and K3 sigma models discussed above now motivates the following question:

> *Are there pairs of VOSA/sigma model family pairs $(V,\mathcal{M}(X))$ such that the following properties hold?*
>
> 1. *The symmetry group of $V = V(X)$ contains the symmetry groups of $\Sigma(X;\mu)$ for every $\mu \in \mathcal{M}(X)$.*
> 2. *The twined partition functions of V capture all the twined elliptic genera arising from the $\Sigma(X;\mu)$ for all $\mu \in \mathcal{M}$.*
> 3. *There exists a particular point $\mu^* \in \mathcal{M}$ such that the reflection procedure maps $\Sigma(X;\mu^*)$ to V.*

We will refer to pairs $(V,\mathcal{M})$ satisfying these 3 properties as *VOSA/sigma model correspondences.*

As we have explained, $(V^{s\natural},\mathcal{M}(K3))$ comes tantalisingly close to being an example of such a VOSA/sigma model correspondence. However, there are (conjecturally) a handful of twined elliptic genera of $\Sigma(X;\mu)$, with μ lying in certain high codimensional subspaces of $\mathcal{M}(X)$, that do not arise from $V^{s\natural}$. See Conjectures 5 and 6, and Table 4 of [91]. As a result, Property 2 above fails to hold for the $(V^{s\natural},\mathcal{M}(K3))$ pair. Our main objective in this work is to illustrate a complete example of the correspondence, where K3 surfaces are replaced by (complex) four-dimensional tori (T^4). The counterpart to $V^{s\natural}$ in this case is the VOSA naturally associated to the E_8 lattice, which we here denote $V^f_{E_8}$ (as in [72, 77]). With the $K3$ case in mind this is perhaps unsurprising, given that $V^{s\natural}$ can be written as a suitable $\mathbb{Z}_2$ orbifold of $V^f_{E_8}$ (see

$$
\begin{array}{ccc}
V_{E_8} & \longleftrightarrow & T^4 \\
\downarrow{\scriptstyle \mathbb{Z}_2} & & \downarrow{\scriptstyle \mathbb{Z}_2} \\
V^{s\natural} & \longleftrightarrow & K3
\end{array}
$$

Figure 4.1: VOSA/sigma model connections and the orbifold procedure.

[72, 77]), while on the orbifold locus of $\mathcal{M}(K3)$, the corresponding sigma models can also be obtained as $\mathbb{Z}_2$ orbifolds of four-torus sigma models (see Figure 4.2.2). In fact, as we will see, the VOSA/sigma model correspondence works better in the four-torus case since it holds for all points in $\mathcal{M}(T^4)$: The twined elliptic genera of *any* $\Sigma(T^4;\mu)$ can be reproduced by the supersymmetry preserving twined partition functions of $V^f_{E_8}$. (See Theorem 6.) So all three properties of our proposed VOSA/sigma model correspondence, including the one which failed for the $(V^{s\natural},\mathcal{M}(K3))$ example, indeed hold in this case. It would be very interesting to understand whether a *complete* realization of the VOSA/sigma model correspondence might exist even for K3 surfaces. Our results can be regarded as encouraging evidence in this direction.

The rest of this chapter is organized as follows. In §4.1 we discuss the supersymmetry-preserving symmetries of $\Sigma(T^4;\mu)$ across the moduli space, as well as the corresponding twined elliptic genera. In §4.2 we summarise important results on the groups arising in §4.1. In §4.3 we discuss the VOSA $V^f_{E_8}$, naturally associated to the E_8 lattice, and show that its supersymmetry-preserving symmetry group contains all the symmetry groups discussed in §4.1. Hence we obtain that Property 1 of VOSA/sigma model correspondences holds for $(V^f_{E_8},\mathcal{M}(T^4))$. We then prove in Theorem 6 that the VOSA $V^f_{E_8}$ recovers all the twined elliptic general of the $\Sigma(T^4;\mu)$, thereby proving Property 2.

In §4.4 we elaborate on the relation between the VOSA/sigma model correspondences for T^4 and the closely related example for $K3$ via orbifolding. In particular, we prove in Proposition 7 that the diagram in Figure 4.1 commutes, for all orbifolding procedures of the theory. Then in §4.5 we demonstrate that $V^f_{E_8}$ can be obtained as the image of $\Sigma(T^4;\mu^*)$ at a particular special point $\mu^* \in \mathcal{M}(T^4)$ under reflection, thus establishing the final VOSA/sigma model correspondence property (Property 3) for $(V^f_{E_8},\mathcal{M}(T^4))$. This is the content of Theorem 8.

There are also two appendices associated with this chapter. In §D we provide further information on the supersymmetry-preserving symmetries of four-torus sigma models. In §E we describe how automorphisms of a lattice lift to automorphisms of a corresponding lattice VOSA, and detail the workings of this in the specific case of $V^f_{E_8}$.

4.1 The Sigma Models

Here we setup some notation and discuss T^4 sigma models and their symmetries, following [126].

4.1.1 Symmetries

A sigma model on T^4 is a supersymmetric conformal field theory defined in terms of four pairs of left- and right-moving bosonic $u(1)$ currents $j^a(z), \tilde{j}^a(\bar{z})$, with $a = 1, \ldots, 4$, four pairs of left- and right-moving free real fermions $\psi^a(z), \tilde{\psi}^a(\bar{z})$, as well as exponential (primary) fields $V_k(z, \bar{z})$ labelled by vectors $k = (k_L, k_R) \in \Gamma^{4,4}_{\mathrm{w-m}}$.

To explain nature of the lattice $\Gamma^{4,4}_{\mathrm{w-m}}$, let $\Gamma^{4,4}$ denote an even unimodular lattice of signature $(4,4)$. The real vector space

$$\Pi = \Gamma^{4,4} \otimes \mathbb{R} \cong \mathbb{R}^{4,4} \tag{4.1.1}$$

admits orthogonal decompositions into positive- and negative-definite subspaces

$$\Pi = \Pi_L \oplus_\perp \Pi_R. \tag{4.1.2}$$

Correspondingly, we decompose $k \in \Pi$ as $k = (k_L, 0) + (0, k_R)$, where the two summands lie in the positive- and negative-definite subspaces respectively. The relative position of Π_L and Π_R uniquely determines each four-torus sigma model, and the corresponding Narain moduli space is as in (4.0.1), where $O(\Gamma^{4,4})$ acts as T-dualities and we restrict to the T-dualities that moreover preserve world-sheet parity (cf. [91]). We use $\Gamma^{4,4}_{\mathrm{w-m}}$ to denote the lattice $\Gamma^{4,4}$ equipped with a choice of an orthogonal decomposition into positive- and negative-definite subspaces. This structure is also known as the winding-momentum or Narain lattice in this context.

The chiral algebra of every four-torus sigma model contains an $\mathfrak{u}(1)^4$ algebra generated by the currents j^a, as well as an $\mathfrak{so}(4)_1$ Kac-Moody algebra generated by $:\psi^a\psi^b:$, with $a, b = 1, \ldots, 4$. It also contains a small $\mathcal{N} = (4,4)$ superconformal algebra at central charge $c = \tilde{c} = 6$, whose holomorphic part is generated by the holomorphic stress tensor $T(z)$, four supercurrents $G^\pm(z), G'^\pm(z)$ of weight $(3/2, 0)$ that consist of linear combinations of terms of the form $:\psi^a j^b:$. In particular, the fermionic $\mathfrak{so}(4)_1$ algebra contains an $\mathfrak{su}(2)_1$ 'R-symmetry' Kac-Moody algebra, generated by currents J^1, J^2, J^3. Since the anti-chiral discussion is completely analogous, from now on we focus just on the chiral part.

To describe the superconformal algebra in detail, it is convenient to define complex fermions

$$\begin{aligned} \chi^1 &:= \frac{1}{\sqrt{2}}(\psi^1 + i\psi^3)\ , \quad \chi^{1^*} := \frac{1}{\sqrt{2}}(\psi^1 - i\psi^3)\ , \\ \chi^2 &:= \frac{1}{\sqrt{2}}(\psi^2 + i\psi^4)\ , \quad \chi^{2^*} := \frac{1}{\sqrt{2}}(\psi^2 - i\psi^4)\ , \end{aligned} \tag{4.1.3}$$

obeying the standard OPEs

$$\chi^i(z)\chi^j(w) \sim \mathcal{O}(z-w)\ , \quad \chi^i(z)\chi^{j^*}(w) \sim \chi^{i^*}(z)\chi^j(w) \sim \frac{\delta_{ij}}{z-w}\ . \tag{4.1.4}$$

In terms of the complex fermions, the stress tensor is given by

$$T = -\sum_{a=1}^{4} :j^a j^a: -\frac{1}{2}\sum_{i=1}^{2}(:\chi^i\partial\chi^{i^*}: + :\chi^{i^*}\partial\chi^i:)\ , \tag{4.1.5}$$

while the R-symmetry currents are given by[1]

$$\begin{aligned} J^1 &= i\left(:\chi^1\chi^2: + :\chi^{1^*}\chi^{2^*}:\right)\ , \quad J^2 = :\chi^1\chi^2: - :\chi^{1^*}\chi^{2^*}:\ , \\ J^3 &= :\chi^1\chi^{1^*}: + :\chi^2\chi^{2^*}:\ . \end{aligned} \tag{4.1.6}$$

The symmetry groups occuring at different points in the moduli space of sigma models on T^4 that preserve the $\mathcal{N} = (4,4)$ superconformal algebra were fully classified in [126]. To describe these groups, let $U(1)^4_L$ and $U(1)^4_R$ be the Lie groups generated by the zero modes j^a_0 and $\tilde{j}^a_0$ respectively. They describe the (independent) translations along the four-torus. Apart from the R-symmetry $\mathfrak{su}(2)_1$ algebra with generators (4.1.6), there is another copy of $\mathfrak{su}(2)_1$ algebra in the fermionic $\mathfrak{so}(4)_1$ algebra, generated by the currents

$$\begin{aligned} A^1 &= i\left(:\chi^1\chi^{2^*}: + :\chi^{1^*}\chi^2:\right)\ , \quad A^2 = :\chi^1\chi^{2^*}: - :\chi^{1^*}\chi^2:\ , \\ A^3 &= :\chi^1\chi^{1^*}: - :\chi^2\chi^{2^*}:\ . \end{aligned} \tag{4.1.7}$$

Focussing on the zero modes, we have the relation

$$SO(4)_L \cong (SU(2)^J_L \times SU(2)^A_L)/(-1)^{A^3_0+J^3_0}, \tag{4.1.8}$$

where $(-1)^{A^3_0/J^3_0}$ is the non-trivial central element of $SU(2)^{A/J}_L$, and similarly for the right-moving side. Preserving the $\mathcal{N} = 4$ superconformal algebra restricts us to the subgroup $SU(2)^A_L$ which commutes with the R-symmetry $SU(2)^J_L$. Moreover,

[1] Note that this normalisation for the currents, while convenient and common in the physics literature, differs by a factor of $\frac{1}{2}$ from the normalisation that is common in the Kac–Moody algebra context.

identifying $SO(4)_L$ with $SO(\Pi_L)$, we need to consider subgroups that induce an automorphism of $\Gamma^{4,4}_{\rm w-m}$[2].

These considerations lead to the following specification of the symmetry groups of the four-torus sigma models. They take the form

$$G = (U(1)^4_L \times U(1)^4_R).G_0\ . \tag{4.1.9}$$

The group G_0 here is given by the intersection

$$G_0 = \left(SU(2)^A_L \times SU(2)^A_R\right) \cap O\left(\Gamma^{4,4}_{\rm w-m}\right)\ , \tag{4.1.10}$$

where the above identification is understood.

Notice that the groups G_0 defined in (4.1.10) manifestly do not mix the spaces Π_L and Π_R, and always contains a central $\mathbb{Z}_2$ subgroup generated by $(-1,-1) \in SU(2)^A_L \times SU(2)^A_R$. Consider the set of all possible groups arising as

$$G_1 := G_0/(-1,-1). \tag{4.1.11}$$

This set turns out to be bijective to the set of subgroups of the group of even-determinant Weyl transformations of E_8, denoted by $W^+(E_8)$, that fix an E_8-sublattice of rank at least 4. See [126] for a complete and descriptive list of all the possible groups G_0. We note here that the groups G_0 and G_1 are interesting finite groups only at certain special points in the moduli space $\mathcal{M}(T^4)$ of sigma models on T^4. Generically, G_0 is isomorphic to $\mathbb{Z}_2$ and G_1 is trivial.

4.1.2 Twined Genera

The elliptic genus of an $\mathcal{N} = (4,4)$ superconformal theory is defined in terms of the superconformal algebra generators as the following trace over the RR sector (see §2.3.3),

$$\phi(\tau,z) = \mathrm{Tr}_{\rm RR}\left[(-1)^F y^{J^3_0}\, q^{L_0-\frac{c}{24}}\, \bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}\right]\ ,\quad q := e^{2\pi i\tau}\ ,\quad y := e^{2\pi i z}\ , \tag{4.1.12}$$

where L_0 is the zero mode of the stress energy tensor T, and the fermion number operator $(-1)^F$ will be discussed in more detail later. It receives non-vanishing contributions only from right-moving BPS states and thus does not depend on $\bar{\tau}$. For the $\mathcal{N} = (4,4)$ theories that we are considering, it is also a weak Jacobi form of weight 0 and index 1, and does not depend on the moduli. For four-torus sigma models,

[2]The identification between $SO(4)_L$ with $SO(\Pi_L)$ is given by the choice of the $\mathcal{N} = 1$ supercurrent such that its generator is proportional to $\sum_{a=1}^4 :\psi^a j^a:$. Different choices of the $\mathcal{N} = 1$ supercharge lead to different isomorphisms that are related to each other by R-symmetry transformations in $SU(2)^J_L$.

we have $c = \tilde{c} = 6$ and the elliptic genus is in fact identically zero due to cancelling contributions from the BPS states, which form an even-dimensional representation of the Clifford algebra of the right-moving fermionic zero modes $\tilde{\chi}_0^i, \tilde{\chi}_0^{i^*}$. When the theory has additional symmetries G preserving the superconformal algebra (i.e. at special points in the moduli space), we can also consider the elliptic genus twined by an element $g \in G$ acting on the RR states,

$$\phi_g^G(\tau, z) = \mathrm{Tr}_{\mathrm{RR}} \left[g \ (-1)^F \ y^{J_0^3} \ q^{L_0 - \frac{c}{24}} \ \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \right] \ , \tag{4.1.13}$$

where the upper-script in the notation serves to remind us about moduli dependence[3]. The twined genus ϕ_g^G depends only on the conjugacy class of g in G and is a weak Jacobi form of weight 0 and index 1 for some congruence subgroup $\Gamma_g \subseteq SL_2(\mathbb{Z})$. Note that the normal subgroup $U(1)_L^4 \times U(1)_R^4$ of G (4.1.9) acts trivially on all oscillators. For this reason we will first focus on the G_0 part when computing the twined elliptic genera.

To compute the elliptic genus twined by $g \in G_0 \subset SU(2)_L^A \times SU(2)_R^A$, let us first describe the Fock space representation of the RR states in the present theory. This is built from all possible combinations of the free fermionic χ_n^i, χ_n^{i*}, $\tilde{\chi}_n^i$, $\tilde{\chi}_n^{i*}$ and bosonic oscillators j_n^a, $\tilde{j}_n^a$, with $a = 1, \ldots, 4$, $i = 1, 2$ and $n \in \mathbb{Z}_{\leq -1}$, acting on the Fock space ground states. The latter has a convenient basis given by

$$|k_L, k_R; s\rangle \ , \quad s = (s_1, s_2; \tilde{s}_1, \tilde{s}_2) \ , \quad s_1, s_2, \tilde{s}_1, \tilde{s}_2 \in \left\{ \frac{1}{2}, -\frac{1}{2} \right\} \ . \tag{4.1.14}$$

Here s is an index for the 2^4-dimensional representation of the eight-dimensional Clifford algebra generated by the fermionic zero modes χ_0^i, χ_0^{i*}, $\tilde{\chi}_0^i$, $\tilde{\chi}_0^{i*}$, which correspond to the fermionic RR ground states $|s\rangle := |0, 0; s\rangle$. The indices k_L and k_R label points in the winding-momentum lattice, $k = (k_L, k_R) \in \Gamma^{4,4}_{\mathrm{w-m}}$. In terms of the primary operators $V_k(z, \bar{z})$, the ground states in (4.1.14) are given by $|k_L, k_R; s\rangle := V_k(0, 0)|s\rangle$.

In this basis, the eigenvalues of the fermionic ground states under the operators J_0^3 and $\tilde{J}_0^3$ are given by

$$J_0^3|s\rangle = (s_1 + s_2)|s\rangle \ , \quad \tilde{J}_0^3|s\rangle = (\tilde{s}_1 + \tilde{s}_2)|s\rangle \ , \tag{4.1.15}$$

and similarly

$$A_0^3|s\rangle = (s_1 - s_2)|s\rangle \ , \quad \tilde{A}_0^3|s\rangle = (\tilde{s}_1 - \tilde{s}_2)|s\rangle \ , \tag{4.1.16}$$

[3] Specifically, here G (see (4.1.9)) is not viewed as an abstract group, but rather as the specific symmetry group at a single point in the moduli space, equipped with the representation of the RR spectrum.

while the J^3 charges of the fields are given by

$$\begin{array}{c|c|c} \chi^i & \chi_n^{i*} & j_n^a \\ \hline +1 & -1 & 0 \end{array} \tag{4.1.17}$$

and similarly for the right-movers. In these terms, the fermion number operator is defined as $(-1)^F := (-1)^{J_0^3+\tilde{J}_0^3}$.

Let ρ_ψ denote the 8-dimensional representation of G_0 on the space spanned by $\psi^1,\ldots,\psi^4$ and $\tilde{\psi}^1,\ldots,\tilde{\psi}^4$. For a given element $g \in G_0$, choose the parametrisation of the complex fermions such that g acts as (cf. Table D.1)

$$\rho_\psi(g)\chi^1 = \zeta_L\chi^1, \qquad \rho_\psi(g)\tilde{\chi}^1 = \zeta_R\tilde{\chi}^1. \tag{4.1.18}$$

Since $g \in SU(2)^A_L \times SU(2)^A_R$, it follows that g acts on the eight-dimensional representation ρ_ψ as

$$\begin{aligned} &\rho_\psi(g)\chi^1 = \zeta_L\chi^1\ ,\ \ \rho_\psi(g)\chi^{1^*} = \zeta_L^{-1}\chi^{1^*}\ ,\ \ \rho_\psi(g)\tilde{\chi}^1 = \zeta_R\tilde{\chi}^1\ ,\ \ \rho_\psi(g)\tilde{\chi}^{1^*} = \zeta_R^{-1}\tilde{\chi}^{1^*} \\ &\rho_\psi(g)\chi^2 = \zeta_L^{-1}\chi^2\ ,\ \ \rho_\psi(g)\chi^{2^*} = \zeta_L\chi^{2^*}\ ,\ \ \rho_\psi(g)\tilde{\chi}^2 = \zeta_R^{-1}\tilde{\chi}^2\ ,\ \ \rho_\psi(g)\tilde{\chi}^{2^*} = \zeta_R\tilde{\chi}^{2^*}\ , \end{aligned} \tag{4.1.19}$$

and similarly on the bosonic currents since the superconformal algebra is preserved. Note that the choice of parametrisation in (4.1.18) is always possible, since by conjugations in $SU(2)^A_L \times SU(2)^A_R$ we can let g to be contained in the Cartan subgroup generated by A_0^3 and $\tilde{A}_0^3$.

From the preceding discussion we conclude that the twined elliptic genus of the four-torus sigma model factors as

$$\phi_g^G(\tau,z) = \phi_g^{\rm osc}(\tau,z)\phi_g^{\rm gs}(z)\phi_g^{\rm w-m}(\tau)\ , \tag{4.1.20}$$

where the three factors capture the contributions from the oscillators, the fermionic ground states, and winding-momentum (i.e. primaries V_k), respectively. In what follows we will discuss them separately.

The action of $g \in G_0$ on the ground states is given by

$$g|s\rangle = \zeta_L^{A_0^3}\zeta_R^{\tilde{A}_0^3}|s\rangle = \zeta_L^{s_1-s_2}\zeta_R^{\tilde{s}_1-\tilde{s}_2}|s\rangle\ . \tag{4.1.21}$$

Summing over the 2^4 ground states $|s\rangle$ we arrive at

$$\begin{aligned} \phi_g^{\rm gs}(z) &= y^{-1}(1-\zeta_L y)(1-\zeta_L^{-1}y)(1-\zeta_R)(1-\zeta_R^{-1}) \\ &= 2(1-\Re(\zeta_R))(y^{-1}+y-2\Re(\zeta_L))\ , \end{aligned} \tag{4.1.22}$$

where $\Re(z)$ denotes the real part of z. From (4.1.19), we compute that the total contribution from the fermionic and bosonic oscillators is

$$\phi_g^{\rm osc}(\tau,z)=\prod_{n=1}^{\infty}\frac{(1-\zeta_L y q^n)(1-\zeta_L^{-1} y q^n)(1-\zeta_L y^{-1} q^n)(1-\zeta_L^{-1} y^{-1} q^n)}{(1-\zeta_L q^n)^2(1-\zeta_L^{-1} q^n)^2}\ . \tag{4.1.23}$$

Notice that the contribution from the right-moving oscillators, and thus the $\bar\tau$ dependence, cancels out completely.

Finally, the contribution from winding-momentum is given by

$$\phi_g^{\rm w\text{–}m}(\tau)=\sum_{k=(k_L,k_R)\in\left(\Gamma^{4,4}_{\rm w\text{–}m}\right)^g}\xi_g(k_L,k_R)\ q^{\frac{k_L^2}{2}}\ \bar q^{\frac{k_R^2}{2}}\ . \tag{4.1.24}$$

Here $\left(\Gamma^{4,4}_{\rm w\text{–}m}\right)^g$ is the g-fixed sublattice of $\Gamma^{4,4}_{\rm w\text{–}m}$, and $\xi_g\,(k_L,k_R)$ are suitable phases that depend on the choice of the lift of g from G_0 to G. As discussed in §E one can always choose the standard lift, where the phases $\xi_g(k_L,k_R)$ are trivial for all $(k_L,k_R)\in\left(\Gamma^{4,4}_{\rm w\text{–}m}\right)^g$.

Notice that if g acts trivially on the right-movers, then $\zeta_R=1$ and $\phi_g^{\rm gs}$, and therefore ϕ_g^G vanishes. On the other hand, if both ζ_R and ζ_L are different from one, then $\left(\Gamma^{4,4}_{\rm w\text{–}m}\right)^g=\{0\}$ and $\phi_g^{\rm w\text{–}m}=1$. Thus, determining $\phi_g^{\rm w\text{-}m}$ is nontrivial only when $\zeta_R\neq 1$ and $\zeta_L=1$. As a result, we can rewrite

$$\phi_g^{\rm w\text{–}m}(\tau)=\sum_{k=(k_L,0)\in\left(\Gamma^{4,4}_{\rm w\text{–}m}\right)^g}\xi_g(k_L,0)\ q^{\frac{k_L^2}{2}} \tag{4.1.25}$$

which is indeed holomorphic in τ as required.

4.2 The Symmetry Groups

In this section we summarise important results on the groups that we will make use of later. In particular, we will show that the G_0, related to the total symmetry groups of the four-torus sigma models via (4.1.9), are all subgroups of $W^+(E_8)$, the group of even-determinant Weyl transformations of E_8. This fact will be crucial in §4.3, as it makes it possible to equate the twined elliptic genera of the four-torus sigma models and the twined traces of the E_8 lattice VOSA.

By definition, $W^+(E_8)$ has a natural action on the E_8 lattice via its unique eight-dimensional irreducible representation, and as such is a subgroup of $SO(8)$. Under the inclusion map $W^+(E_8)\hookrightarrow SO(8)$, the center of $W^+(E_8)$ is mapped to the central $\mathbb{Z}_2$ subgroup of $SO(8)$, acting as $-id$ in the eight-dimensional vector representation of $SO(8)$ in the former case, and in the eight-dimensional non-trivial representation of $W^+(E_8)$ in the latter case. We denote by ι_v the generator of this latter central subgroup $\langle\iota_v\rangle\cong\mathbb{Z}_2<W^+(E_8)$. The corresponding central quotient is isomorphic to the finite simple group $O_8^+(2)$, the group of linear transformations of the vector space $\mathbb{F}_2^8$ preserving a certain quadratic form. (See e.g.

[127] for a discussion of this.) In other words, we have

$$W^+(E_8) \cong \langle \iota_v \rangle . O_8^+(2) \ .$$

Recall that G_1, related to G_0 as in (4.1.11), can be identified with subgroups of $W^+(E_8)$ that fix an E_8 sublattice of rank at least 4 [126]. Since ι_v does not preserve any subspace in the eight-dimensional vector representation of $W^+(E_8)$, we conclude that $\iota_v \notin G_1$, and by combining the inclusion $G_1 \hookrightarrow W^+(E_8)$ and the projection $W^+(E_8) \xrightarrow{\pi'} O_8^+(2)$ we obtain an *injective* homomorphism $G_1 \to O_8^+(2)$. As a consequence, the group G_1 is always isomorphic to a subgroup of $O_8^+(2)$.

To show that the discrete part of the sigma model symmetry group G_0 is always a subgroup of $W^+(E_8)$, it will be useful to consider the group Spin(8). The kernel of the spin covering map $\mathrm{Spin}(8) \xrightarrow{\pi} SO(8)$ is an involution $\langle \iota_s \rangle \cong \mathbb{Z}_2$. Considering $W^+(E_8) < SO(8)$, the preimage of the spin covering map is $\langle \iota_s \rangle . W^+(E_8) < \mathrm{Spin}(8)$. Its center can be identified with the center of Spin(8), given by $\langle \iota_s, \iota_v \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We thus have that

$$\langle \iota_s \rangle . W^+(E_8) \cong \langle \iota_s, \iota_v \rangle . O_8^+(2) \ .$$

The kernel of the spin covering map $\mathrm{Spin}(8) \xrightarrow{\pi} SO(8)$ is naturally identified with the kernel of the quotient map $G_0 \to G_1$ (cf. (4.1.11), Table 4.2.2). Indeed, the preimage of $G_1 < W^+(E_8) < SO(8)$ in $\langle \iota_s \rangle . W^+(E_8) < \mathrm{Spin}(8)$ is precisely the group $G_0 \cong \langle \iota_s \rangle . G_1$. As we have seen in §4.1.2, in the sigma models ι_s acts by flipping the sign of all the fermions in the representation ρ_ψ (cf. (4.1.19)).

At this point it is crucial to recall that Spin(8) has a triality symmetry, i.e. an S_3 outer automorphism group. Also, it has one vector and two spinor eight-dimensional irreducible representations, which we will denote by ρ^s_ψ, ρ^s_{e} and ρ^s_{o} respectively, and the action of triality on the group Spin(8) extends to an S_3 permutation action on the three representations ρ^s_ψ, ρ^s_{e} and ρ^s_{o}. This S_3 group also permutes the three non-trivial generators ι_v, ι_s, $\iota_v\iota_s$ of the center of Spin(8), and in each of the three aforementioned eight-dimensional representations one of these generators acts trivially. Triality for Spin(8) induces an S_3 group of outer automorphisms of $\langle \iota_s \rangle . W^+(E_8) \cong \langle \iota_s, \iota_v \rangle . O_8^+(2)$.

As a result, the G_0 subgroup of $\langle \iota_s \rangle . W^+(E_8)$ has three representations, which we denote ρ_ψ, ρ_{e} and ρ_{o}, corresponding to three eight-dimensional representations of Spin(8), that are permuted by the outer automorphisms of $\langle \iota_s \rangle . W^+(E_8)$. As we have seen in (4.1.19), in the sigma model the representation ρ_ψ captures the action of the symmetry group G_0 on the eight (left- and right-moving) NS-NS fermions $\chi^i, \chi^{i^*} \tilde{\chi}^i, \tilde{\chi}^{i^*}$. The other two representations, ρ_{e} resp. ρ_{o}, capture the action of G_0 on the Ramond-Ramond sector quantum states with even resp. odd fermion numbers. As mentioned before, in the representation ρ_ψ the central involution ι_s acts by flipping the signs of all fermions as well as all bosons (which has to be the case since G_0 preserves the superconformal algebra). On the other hand, in the representation ρ_{e} the central element of G_0 acts trivially, so that only the quotient G_1 acts faithfully on the RR ground states of even fermion numbers. This is also the representation where G_1 fixes a 4-dimensional subspace (cf. Table D.1).

Now the S_3 outer automorphisms of $\langle \iota_s, \iota_v \rangle.O_8^+(2)$ guarantee that the quotient by any of the three generators of the central subgroup $\langle \iota_s, \iota_v \rangle$ is a group isomorphic to $W^+(E_8)$. In particular, since $\iota_v \notin G_1$ and hence $G_0 \cong \langle \iota_s \rangle.G_1 < \langle \iota_s \rangle.W^+(E_8)$ does not contain the central involution ι_v, the homomorphism $G_0 \to W^+(E_8)$ induced by the projection

$$\langle \iota_s \rangle.W^+(E_8) \cong \langle \iota_s, \iota_v \rangle.O_8^+(2) \ \to \ \left(\langle \iota_s, \iota_v \rangle.O_8^+(2)\right)/\langle \iota_v \rangle \cong W^+(E_8) \tag{4.2.1}$$

is injective. Thus we have proved the following result.

Proposition 5. *For any four-torus sigma model the corresponding group G_0 is isomorphic to a subgroup of $W^+(E_8)$.*

The discussion of this section is visually summarized in the following diagram.

$$\begin{array}{ccc}
\mathrm{Spin}(8) & \xrightarrow{\pi} & SO(8) \\
\uparrow & & \uparrow \\
\langle \iota_s \rangle.W^+(E_8) \cong \langle \iota_v, \iota_s \rangle.O_8^+(2) & \longrightarrow & W^+(E_8) \cong \langle \iota_v \rangle.O_8^+(2) \\
\downarrow & & \downarrow \pi' \\
\langle \iota_s \rangle.O_8^+(2) \cong W^+(E_8) & \xrightarrow{\pi''} & O_8^+(2) \\
\uparrow & & \uparrow \\
G_0 & \longrightarrow & G_1
\end{array} \tag{4.2.2}$$

4.3 The VOSA

In this section we discuss the VOSA side of the VOSA/sigma model correspondence; in this case the E_8 lattice VOSA $V^f_{E_8}$. In §4.3.1 we introduce the theory, and in §4.3.2 we outline the computation of the twined traces of this VOSA and prove the main theorem (Theorem 6) of this chapter.

4.3.1 The Theory

The VOSA $V^f_{E_8}$ is a $c = 12$ chiral superconformal field theory (SCFT) with eight free chiral fermions $\beta^a(z)$ and eight free chiral bosons $Y^a(z)$, with $a = 1, \ldots, 8$. Moreover, it has chiral vertex operators $V_\lambda(z) = c(\lambda) : e^{\lambda \cdot Y} :$ corresponding to the E_8 lattice. In the above, we have $\lambda \in E_8$ and $c(\lambda)$ is the standard operator needed for locality [128, 129]. The stress tensor is given by

$$T = -\sum_{a=1}^{8} : \partial Y^a \ \partial Y^a : -\frac{1}{4}\sum_{a=1}^{8} : \beta^a \partial \beta^a :, \tag{4.3.1}$$

and an $\mathcal{N} = 1$ structure is provided by the supercurrent Q, proportional to the combination

$$\sum_{a=1}^{8} : \beta^a \partial Y^a : \ . \tag{4.3.2}$$

The 8 currents ∂Y^b form a $\mathfrak{u}(1)^8$ bosonic algebra, while the 28 currents $:\beta^a\beta^b:$ generate a fermionic Kac-Moody algebra $\mathfrak{so}(8)_1$. Let F be the eight-dimensional real vector space spanned by the fermions β^a. To facilitate the comparison with the sigma models, we split F into two four-dimensional subspaces $F = X \oplus \bar{X}$ such that X is spanned by β^a for $a = 1, \dots, 4$ and $\bar{X}$ is spanned by β^b for $b = 5, \dots, 8$. As usual, it is convenient to work with the complex fermions

$$\begin{aligned} \gamma^i &:= \frac{1}{\sqrt{2}}(\beta^i + i\beta^{i+2}) , \\ \bar{\gamma}^i &:= \frac{1}{\sqrt{2}}(\beta^{i+4} + i\beta^{i+6}) , \end{aligned} \tag{4.3.3}$$

for $i = 1, 2$. The splitting of F leads to the subalgebra $\mathfrak{so}(4)_1 \oplus \mathfrak{so}(4)_1$ of the fermionic Kac-Moody algebra $\mathfrak{so}(8)_1$. Focussing on the first $\mathfrak{so}(4)_1 \cong \mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1$, corresponding to $X \subset F$, the two factors of $\mathfrak{su}(2)_1$ are generated by $J_X^{1,2,3}$ and $A_X^{1,2,3}$ respectively, completely analogous to the sigma model case ((4.1.6) and (4.1.7)) upon replacing the χs with γs.

At the level of the zero-modes, we have

$$\begin{aligned} SO(X) &= (SU(2)^A_X \times SU(2)^J_X)/(-1,-1) \cong SO(4) , \\ SO(\bar{X}) &= (SU(2)^A_{\bar{X}} \times SU(2)^J_{\bar{X}})/(-1,-1) \cong SO(4) . \end{aligned} \tag{4.3.4}$$

Note that all four $SU(2)$s above preserve the $\mathcal{N} = 1$ superconformal algebra.

Next we discuss the quantum states of the above model. We will sometimes refer to the space of states of this VOSA as an NS sector, since the chiral fermions satisfy the antiperiodic boundary condition. One can also construct a canonically twisted module for this VOSA, i.e. a Ramond sector with periodic boundary conditions for the fermions. The Ramond sector contains $2^{8/2} = 16$ ground states, forming a representation of the Clifford algebra of the fermionic zero modes. A convenient basis for these ground states may be denoted

$$|r\rangle := |r_1, r_2, r_3, r_4\rangle , \quad r_1, r_2, r_3, r_4 \in \left\{\frac{1}{2}, -\frac{1}{2}\right\} . \tag{4.3.5}$$

Similar to the case of the sigma models (4.1.14), the Fock space ground states are then given by $|\lambda; r\rangle := V_\lambda(0)|r\rangle$, where $\lambda \in E_8$.

With the sigma model elliptic genus (4.1.12) in mind we define the following twisted module trace,

$$Z(\tau, z) := \mathrm{Tr}_{\mathrm{tw}} \left[(-1)^F \, y^{J_0^{X,3}} \, q^{L_0 - \frac{c}{24}} \right] . \tag{4.3.6}$$

The action of the operator $J_0^{X,3}$ on the oscillators and the ground states is completely analogous to its counterpart in the sigma models. Namely, it acts as a number operator for the fermionic oscillators, counting γ_n^j excitations (with $n \leq -1$) as $+1$ and $\gamma_n^{j^*}$ excitations as -1, for $j = 1, 2$, while on the ground states (4.3.5) it acts as

$$J_0^{X,3}|r\rangle = (r_1 + r_2)|r\rangle . \tag{4.3.7}$$

Similarly, the fermion number operator is defined as $(-1)^F := (-1)^{J_0^{X,3}+J_0^{\bar{X},3}}$, and acts on the ground states as

$$(-1)^F|r\rangle = (-1)^{J_0^{X,3}+J_0^{\bar{X},3}}|r\rangle = (-1)^{r_1+r_2+r_3+r_4}|r\rangle \ . \tag{4.3.8}$$

From this it follows immediately that states built on the ground states $|r\rangle$ with opposite signs of r_3 (or r_4) lead to opposite contributions to the trace $Z(\tau,z)$ and hence the trace vanishes. In the next subsection we will see that, similar to the sigma models, the trace is generically not vanishing when twined by a symmetry.

4.3.2 Twined Traces

Recall (Proposition 5) that the symmetry groups G_0 of the T^4 sigma models may be regarded as subgroups of $W^+(E_8)$. We may thus identify them with symmetry groups of $V^f_{E_8}$ which act on the E_8 lattice by even-determinant Weyl automorphisms, according to the vector representation ρ_ψ. The lattice E_8 is naturally contained in F, the 8-dimensional real vector space spanned by the fermions β^a, so we have $G_0 < W^+(E_8) < SO(F)$. As discussed in §4.1.1, the groups G_0 are contained in an $SU(2)_L \times SU(2)_R$ subgroup of $SO(4)_L \times SO(4)_R \subset SO(8)$, and thus they do not mix the spaces Π_L and Π_R. We can further identify the vector spaces $X = \Pi_L$ and $\bar{X} = \Pi_R$, so that G_0 is contained in $SU(2)^A_X \times SU(2)^A_{\bar{X}}$ (and commutes with $SU(2)^J_X$ and $SU(2)^J_{\bar{X}}$) when acting on the E_8 lattice of the VOSA. The action of G_0 is then lifted to automorphisms of the E_8 VOSA that preserve the $\mathcal{N}=1$ supercurrent Q. (One may choose lifts where all phases are trivial. Consult §E for details.) As a result, for each $g \in G_0$ we may define the following g-twined trace in the twisted module for the E_8 VOSA

$$Z_g(\tau,z) := \mathrm{Tr}_{\mathrm{tw}}\left[g\ (-1)^F\ y^{J_0^{X,3}}\ q^{L_0-\frac{c}{24}}\right]\ , \tag{4.3.9}$$

generalising (4.3.6).

Analogous to the sigma models (4.1.20), the above g-twined trace naturally decomposes into three factors,

$$Z_g(\tau,z) = Z_g^{\mathrm{osc}}(\tau,z) Z_g^{\mathrm{gs}}(z) Z_g^{E_8}(\tau)\ , \tag{4.3.10}$$

capturing the contribution from the oscillators, the fermionic ground states, and the E_8 lattice chiral operators, respectively.

Choosing a convenient basis for the fermions we observe that the action of g is precisely the same as in (4.1.19), with χ^i replaced by γ^i and χ^{i^*} replaced by γ^{i^*}, $\tilde{\chi}^i$ replaced by $\bar{\gamma}^i$ and $\tilde{\chi}^{i^*}$ replaced by $\bar{\gamma}^{i^*}$, for $i=1,2$. As a result, the oscillators give a factor of

$$\begin{aligned} Z_g^{\mathrm{osc}}(\tau,z) &= \prod_{n=1}^{\infty} \frac{(1-\zeta_L y q^n)(1-\zeta_L^{-1} y q^n)(1-\zeta_L y^{-1} q^n)(1-\zeta_L^{-1} y^{-1} q^n)(1-\zeta_R q^n)^2(1-\zeta_R^{-1} q^n)^2}{(1-\zeta_L q^n)^2(1-\zeta_L^{-1} q^n)^2(1-\zeta_R q^n)^2(1-\zeta_R^{-1} q^n)^2} \\ &= \prod_{n=1}^{\infty} \frac{(1-\zeta_L y q^n)(1-\zeta_L^{-1} y q^n)(1-\zeta_L y^{-1} q^n)(1-\zeta_L^{-1} y^{-1} q^n)}{(1-\zeta_L q^n)^2(1-\zeta_L^{-1} q^n)^2}\ . \end{aligned} \tag{4.3.11}$$

Similarly, the group action on the fermionic ground states is given by

$$g|r\rangle = \zeta_L^{A_0^{X,3}} \zeta_R^{A_0^{\bar{X},3}} |r\rangle = \zeta_L^{r_1-r_2}\zeta_R^{r_3-r_4}|r\rangle \ , \tag{4.3.12}$$

leading to the contribution

$$Z_g^{\rm gs}(\tau,z) = y^{-1}(1-\zeta_L y)(1-\zeta_L^{-1}y)(1-\zeta_R)(1-\zeta_R^{-1}) = 2(1-\Re(\zeta_R))(y^{-1}+y-2\Re(\zeta_L)) \ . \tag{4.3.13}$$

Finally, the contribution from the E_8 lattice is

$$Z_g^{E_8}(\tau) = \sum_{\lambda\in(E_8)^{\rho_\psi(g)}} \xi_g(\lambda)\ q^{\frac{\lambda^2}{2}} \ , \tag{4.3.14}$$

where $(E_8)^{\rho_\psi(g)}$ is the sublattice of E_8 fixed by g (which acts on the lattice according to the ρ_ψ representation of G_0), and $\xi_g(\lambda)$ are phases analogous to those in the sigma models (4.1.24) that can be chosen to be trivial.

We now state and prove the main result.

Theorem 6. *For every $g\in G_0$ for any of the possible groups G_0 we have*

$$Z_g(\tau,z) = \phi_g^G(\tau,z) \ . \tag{4.3.15}$$

Proof. To begin we note that, from the preceeding discussion, it is evident that for each $g\in G_0$ we have

$$Z_g^{\rm osc} = \phi_g^{\rm osc} \ , \quad Z_g^{\rm gs} = \phi_g^{\rm gs} \ . \tag{4.3.16}$$

So we require (see (4.1.20), (4.3.10)) to show that $Z_g^{E_8} = \phi_g^{\text{w-m}}$. Since we have $Z_g^{\rm gs} = \phi_g^{\rm gs} = 0$ whenever $\zeta_R = 1$, we may focus solely on the case where $\zeta_R \neq 1$. Moreover, if both $\zeta_L, \zeta_R \neq 1$ then $Z_g^{E_8} = \phi_g^{\text{w-m}} = 1$, as in this case both lattices $(E_8)^{\rho_\psi(g)}$ and $\left(\Gamma^{4,4}_{\text{w–m}}\right)^g$ are empty. Therefore, we only need to prove that whenever $\zeta_L = 1$ and $\zeta_R \neq 1$, the fixed sublattice $(E_8)^{\rho_\psi(g)}$ is isomorphic to $\left(\Gamma^{4,4}_{\text{w–m}}\right)^g$. We will achieve this by performing a case-by-case analysis. There are only four conjugacy classes in ρ_ψ with $\zeta_L = 1$ and $\zeta_R \neq 1$. In the notation explained in §D, they are 2A, 2E, 3E, 4A (see Table D.1).

To proceed we note that by inspecting the character table of $W^+(E_8)$ we may deduce that the aforementioned classes are necessarily fixed by the action of any outer automorphism. Since the representations ρ_ψ and ρ_e are related by such triality outer automorphisms (cf. §4.2), we deduce that for these classes we have $(E_8)^{\rho_\psi(g)} \cong (E_8)^{\rho_e(g)}$, the latter being the lattice fixed by $\langle g\rangle \subseteq G_0$ in the representation ρ_e. In §4 of [126], both lattices $(E_8)^{\rho_e(g)}$ and $\left(\Gamma^{4,4}_{\text{w–m}}\right)^g$ were described in detail. In particular, it was shown that they are as follows:

(4.3.17)

	$2A$	$2E$	$3E$	$4A$
$(E_8)^{\rho_e(g)}\ \left(\cong (E_8)^{\rho_\psi(g)}\right)$	D_4	A_1^4	A_2^2	D_4
$\left(\Gamma^{4,4}_{\text{w–m}}\right)^g$	D_4	A_1^4	A_2^2	D_4

We thus see that the fixed sublattice of the winding-momentum lattice of the T^4 sigma model and the fixed sublattice of the E_8 lattice are isomorphic in each case. This completes the proof. □

4.4 Orbifolds

In this section we investigate the extent to which the diagram Figure 4.1 commutes, or not, with an arbitrary symmetry in place of the specific $\mathbb{Z}_2$ action indicated. We will demonstrate that in fact the diagram commutes for all possible choices, at least if we assume a certain claim about orbifolds of T^4 sigma models. We regard this result—Proposition 7—as further evidence that the VOSA/sigma model correspondence for T^4 sigma models proposed herein represents a natural structure.

The claim about orbifold sigma models we will require to assume is the statement that:

> *The orbifold of a T^4 sigma model by a discrete supersymmetry-preserving symmetry is either a sigma model with T^4 target or a sigma model with K3 target.*

This claim follows, for example, from the conjecture that the only $\mathcal{N} = (4,4)$ SCFTs with four spectral flow generators, central charge $c = \bar{c} = 6$ and discrete spectrum come from sigma models with T^4 or K3 target space. This conjecture is widely believed to be true (see e.g. [130]) and was implicitly assumed in early string theory literature. Here we refer to it as the *uniqueness conjecture*.

Alternatively, the above claim on four-torus sigma model orbifolds is supported by the following heuristic argument which is independent of the uniqueness conjecture. Call a symmetry g of a sigma model $\mathcal{T}$ with target X *geometric* if it is lifted (cf. §E) from a symmetry $\bar{g}$ of the target space X. Then the orbifold of $\mathcal{T}$ by g should be a sigma model on the orbifold of X by $\bar{g}$. Any orbifold of a four-torus is a singular limit of K3 surfaces, so the claim about orbifolds should hold at least for geometric symmetries.

For more general symmetries note that it can be shown, independently of the uniqueness conjecture (see e.g. [130]), that the elliptic genus of an $\mathcal{N} = (4,4)$ SCFT with four spectral flow generators and $c = \bar{c} = 6$ is either 0 or coincides with the K3 elliptic genus. Furthermore, if the elliptic genus is 0 then the corresponding sigma model has T^4 target[130]. So, if the elliptic genus of an orbifold is 0, there is no doubt that it is a sigma model on T^4.

To handle the case that the elliptic genus of the orbifold is non-vanishing we recall the *reverse orbifold construction*: If $\mathcal{T}$ is a sigma model and g is a discrete supersymmetry preserving symmetry of $\mathcal{T}$ then the orbifold $\mathcal{T}'$ of $\mathcal{T}$ by g has a distinguished symmetry g' with the property that the orbifold of $\mathcal{T}'$ by g' is $\mathcal{T}$. (See e.g. [131] for an analysis of this in the VOA setting.)

The supersymmetry preserving symmetries of sigma models with K3 target have been classified in [107], and this allows us to determine the pairs $(\mathcal{T}', g')$, with $\mathcal{T}'$ a K3 sigma model and g' a symmetry of $\mathcal{T}'$, for which the orbifold of $\mathcal{T}'$ by g' is a sigma model on T^4. (One just checks if the elliptic genus of the orbifold vanishes or not.) So by the reverse orbifold construction we obtain a corresponding set of pairs $(\mathcal{T}, g)$, with $\mathcal{T}$ a sigma model on T^4 and g a symmetry of $\mathcal{T}$, for which $\mathcal{T}$ is an orbifold of a K3 sigma model $\mathcal{T}'$ and g is the corresponding distinguished symmetry such that the orbifold of $\mathcal{T}$ by g is $\mathcal{T}'$. Finally, we can check case-by-case that every non-geometric four-torus sigma model symmetry, for which the corresponding orbifold elliptic genus is non-vanishing, occurs in such a pair. So there are simply no candidates for four-torus sigma model orbifolds by non-geometric symmetries with non-vanishing elliptic genus except for K3 sigma models.

Note that the claim above on T^4 sigma model orbifolds has a rigorous counterpart for VOSAs. Namely, if $\hat{g} \in \mathrm{Aut}(V^f_{E_8})$ is the standard lift (cf. §E) of a T^4 sigma model symmetry $g \in W^+(E_8)$, then the orbifold of $V^f_{E_8}$ by $\hat{g}$ is either isomorphic to $V^f_{E_8}$ or to the Conway moonshine module $V^{s\natural}$ (see §2.4.1). We will establish this in the course of proving our next result, Proposition 7. Note that a more general orbifolding of $V^f_{E_8}$ might result in the VOSA that describes 24 free fermions (cf. e.g. [116]).

We now prove the main result of this section. For the formulation of this we assume the notation of (4.1.9).

Proposition 7. *Let $\mathcal{T}$ be a T^4 sigma model and let $g \in G_0 < W^+(E_8)$ be a symmetry of $\mathcal{T}$ that preserves the $\mathcal{N} = 4$ superconformal algebra. Let $\hat{g}$ denote the standard lift of $g < W^+(E_8)$ to a symmetry of the VOSA $V^f_{E_8}$ as described in §E. If we assume that any orbifold of a T^4 sigma model by a discrete supersymmetry-preserving symmetry is either a sigma model on T^4 or a sigma model on $K3$, then the orbifold of $V^f_{E_8}$ by $\hat{g}$ is isomorphic to $V^f_{E_8}$ or $V^{s\natural}$ according to whether the orbifold of $\mathcal{T}$ by g is a sigma model on T^4 or a sigma model on $K3$, respectively.*

Proof. The orbifold of $V^f_{E_8}$ by $\hat{g}$ can be $V^f_{E_8}$, $V^{s\natural}$, or the VOSA associated to 24 free fermions according to Theorem 3.1 of [116]. To tell the three possibilities apart we can simply compute the partition function $Z_{\hat{g}\text{-orb}}(\tau)$ of the orbifold theory. It will develop that either $Z_{\hat{g}\text{-orb}}(\tau) = Z(V^f_{E_8}; \tau)$ or $Z_{\hat{g}\text{-orb}}(\tau) = Z(V^{s\natural}; \tau)$, where $Z(V; \tau)$ is the partition function of V. (In particular, the free fermions model will not arise.)

Let us denote the anti-periodic and periodic boundary conditions for the fermions by A and P, respectively. We are interested in the case where the fermions are in the $[A, A]$ sector. The bosons will always have periodic boundary condition in the current context so we will not explicitly specify the boson boundary condition in what follows.

Let ${}^{\tilde{D}}_{D}Z^g_h(\tau)$ denote the h-twisted, g-twined partition function of $V^f_{E_8}$ in the sector where the fermions have $[D, \tilde{D}]$ boundary conditions, with $D, \tilde{D} \in \{A, P\}$. The orbifold partition function is then given by

$$Z_{\hat{g}\text{-orb}}(\tau) = \frac{1}{|\hat{g}|} \sum_{k,\ell \in \mathbb{Z}/|\hat{g}|} {}^{A}_{A}Z^{\hat{g}^\ell}_{\hat{g}^k}(\tau), \tag{4.4.1}$$

so we need to compute ${}^{\tilde{D}}_{D}Z^{\hat{g}^\ell}_{\hat{g}^k}$ for all $k, \ell \in \mathbb{Z}/|\hat{g}|$.[4] We have $|\hat{g}| = |g|$ for all cases except for when g lies in the class $2E$, in which case $|\widehat{2E}| = 2|2E| = 4$. More details on this can be found in §E.

Recall (see §2.3.3) that modular transformations change the twisting and twining boundary conditions according to

$$\mathrm{PSL}_2(\mathbb{Z}) \ni \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (h, g) \mapsto (g^c h^d, g^a h^b) \ . \tag{4.4.2}$$

Notice that $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ implies that (h, g) and (h^{-1}, g^{-1}) correspond to equal partition functions, since in our case all fields are invariant (self-conjugate) under charge conjugation $C = S^2 = (ST)^3$. Additionally, modular transformations also mix the fermionic sectors $[A, A], [A, P], [P, A]$, while leaving the bosonic sector $[P, P]$ invariant. In particular, for a holomorphic VOSA of central charge c, the partition functions ${}^{A}_{A}Z$, ${}^{P}_{A}Z$, ${}^{A}_{P}Z$ span a 3-dimensional representation $\rho_c : PSL_2(\mathbb{Z}) \to GL(3)$ given by

$$\begin{aligned} \begin{pmatrix} {}^{A}_{A}Z \\ {}^{P}_{A}Z \\ {}^{A}_{P}Z \end{pmatrix} \left(-\frac{1}{\tau}\right) &= \rho_c(S) \begin{pmatrix} {}^{A}_{A}Z \\ {}^{P}_{A}Z \\ {}^{A}_{P}Z \end{pmatrix} (\tau) \ , \qquad \rho_c(S) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ , \\ \begin{pmatrix} {}^{A}_{A}Z \\ {}^{P}_{A}Z \\ {}^{A}_{P}Z \end{pmatrix} (\tau + 1) &= \rho_c(T) \begin{pmatrix} {}^{A}_{A}Z \\ {}^{P}_{A}Z \\ {}^{A}_{P}Z \end{pmatrix} (\tau) \ , \qquad \rho_c(T) = \begin{pmatrix} 0 & \mathrm{e}(-\frac{c}{24}) & 0 \\ \mathrm{e}(-\frac{c}{24}) & 0 & 0 \\ 0 & 0 & \mathrm{e}(\frac{c}{12}) \end{pmatrix} \ . \end{aligned} \tag{4.4.3}$$

Combining the above, we conclude that

$$\begin{pmatrix} {}^{A}_{A}Z^{\hat{g}^n}_{\hat{g}^m} \\ {}^{P}_{A}Z^{\hat{g}^n}_{\hat{g}^m} \\ {}^{A}_{P}Z^{\hat{g}^n}_{\hat{g}^m} \end{pmatrix} (\tau) = \mathrm{e}(\alpha) \ \rho_c^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)\right) \ \begin{pmatrix} {}^{A}_{A}Z^{g'}_{1} \\ {}^{P}_{A}Z^{g'}_{1} \\ {}^{A}_{P}Z^{g'}_{1} \end{pmatrix} \left(\frac{a\tau + b}{c\tau + d}\right) , \tag{4.4.4}$$

for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{PSL}_2(\mathbb{Z})$ that can be determined from (4.4.2), some $g' \in \langle \hat{g} \rangle$ and some phase $\mathrm{e}(\alpha) := e^{2\pi i \alpha}$.

Let us use the fact that the VOSA $V^f_{E_8}$ is the product of a (bosonic) holomorphic lattice VOA based on the E_8 lattice, and the VOSA generated by 8 real (or four complex) free fermions, and that the symmetry $\hat{g}$ acts independently on these two algebras. As a consequence, the twisted-twined partition functions ${}^{A}_{A}Z^{\hat{g}^\ell}_{\hat{g}^k}$ factorize as

$${}^{A}_{A}Z^{\hat{g}^\ell}_{\hat{g}^k} = {}^{A}_{A}F^{\hat{g}^\ell}_{\hat{g}^k} B^{\hat{g}^\ell}_{\hat{g}^k} \tag{4.4.5}$$

into the product of the twisted-twined partition functions ${}^{A}_{A}F^{\hat{g}^\ell}_{\hat{g}^k}$ and $B^{\hat{g}^\ell}_{\hat{g}^k}$ of the fermionic VOSA (with $[A, A]$ boundary conditions) and the bosonic VOA, respectively.

We will consider the fermion and boson contributions separately, and then combine the results. Consider first the four free complex fermions, with $c_F = 4$. Let us denote the

[4]The notation $k \in \mathbb{Z}/|\hat{g}|$ means that $k = 0, \ldots, |\hat{g}| - 1$.

partition function in sector $[D,\widetilde{D}]$ by ${}^{\widetilde{D}}_{D}F$. Then we have

$$\begin{aligned} {}^{A}_{A}F(\tau) &= \frac{\theta_3^4(\tau)}{\eta^4(\tau)}\ , \\ {}^{P}_{A}F(\tau) &= \frac{\theta_4^4(\tau)}{\eta^4(\tau)}\ , \\ {}^{A}_{P}F(\tau) &= \frac{\theta_2^4(\tau)}{\eta^4(\tau)}\ , \\ {}^{P}_{P}F(\tau) &= \frac{\theta_1^4(\tau)}{\eta^4(\tau)} = 0\ , \end{aligned} \tag{4.4.6}$$

where we write $\theta_i(\tau,z)$ for the usual Jacobi theta functions (see §2.2.15) and set $\theta_i(\tau) := \theta_i(\tau,0)$. The sectors ${}^{A}_{A}F(\tau), {}^{P}_{A}F(\tau) {}^{A}_{P}F(\tau)$ transform as in (4.4.3) under $PSL_2(\mathbb{Z})$, with $c=4$.

Now consider a symmetry $\hat{g}$ acting on the fermions, with eigenvalues determined by the representation ρ_ψ, and denoted $\zeta_L = \mathrm{e}(\alpha_L)$ and $\zeta_R = \mathrm{e}(\alpha_R)$, where ζ_L and ζ_R are as in (4.1.19). Then the $\hat{g}^k$-twisted $\hat{g}^\ell$-twined partition function in the four sectors is given by

$$\begin{aligned} {}^{A}_{A}F^{\hat{g}^\ell}_{\hat{g}^k}(\tau) &= q^{(\hat{\alpha}_L^2+\hat{\alpha}_R^2)k^2}\,\frac{\theta_3^2(\tau,\hat{\alpha}_L(k\tau+\ell))\,\theta_3^2(\tau,\hat{\alpha}_R(k\tau+\ell))}{\eta^4(\tau)}\ , \\ {}^{P}_{A}F^{\hat{g}^\ell}_{\hat{g}^k}(\tau) &= q^{(\hat{\alpha}_L^2+\hat{\alpha}_R^2)k^2}\,\frac{\theta_4^2(\tau,\hat{\alpha}_L(k\tau+\ell))\,\theta_4^2(\tau,\hat{\alpha}_R(k\tau+\ell))}{\eta^4(\tau)}\ , \\ {}^{A}_{P}F^{\hat{g}^\ell}_{\hat{g}^k}(\tau) &= q^{(\hat{\alpha}_L^2+\hat{\alpha}_R^2)k^2}\,\frac{\theta_2^2(\tau,\hat{\alpha}_L(k\tau+\ell))\,\theta_2^2(\tau,\hat{\alpha}_R(k\tau+\ell))}{\eta^4(\tau)}\ , \\ {}^{P}_{P}F^{\hat{g}^\ell}_{\hat{g}^k}(\tau) &= q^{(\hat{\alpha}_L^2+\hat{\alpha}_R^2)k^2}\,\frac{\theta_1^2(\tau,\hat{\alpha}_L(k\tau+\ell))\,\theta_1^2(\tau,\hat{\alpha}_R(k\tau+\ell))}{\eta^4(\tau)}\ , \end{aligned} \tag{4.4.7}$$

where $0 \le k,\ell < N$, and $\hat{\alpha}_{L,R} \equiv \alpha_{L,R}(k)$ are rational numbers such that $e(\hat{\alpha}_{L,R}) = \zeta_{L,R}$ and $-\frac{1}{2} < \hat{\alpha}_L k, \hat{\alpha}_R k \le \frac{1}{2}$. Up to a possible redefinition $\zeta_L \leftrightarrow \zeta_L^{-1}$ or $\zeta_R \leftrightarrow \zeta_R^{-1}$, one can restrict $0 \le \hat{\alpha}_L k, \hat{\alpha}_R k \le \frac{1}{2}$. Notice that the expressions (4.4.7) are in general not invariant under $k \to k+N$ and $\ell \to \ell+N$, but they can change by a multiplicative constant phase (an N-th root of unity). This phenomenon reflects an ambiguity in the definition of the phases of ${}^{D}_{D}F^{\hat{g}^\ell}_{\hat{g}^k}$, that depend on the choice of the action of $\langle\hat{g}\rangle$ on the $\hat{g}^k$-twisted module.

Next we consider four free complex bosons on the E_8 torus, with $c_B = 8$. The bosons naturally have periodic boundary conditions on both cycles of the torus. The corresponding partition function is

$$B(\tau) := \frac{\Theta_{E_8}(\tau)}{\eta(\tau)^8}\ , \tag{4.4.8}$$

where

$$\Theta_{E_8}(\tau) = \frac{1}{2}\left(\theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8\right) = E_4(\tau) \tag{4.4.9}$$

is the theta series of the E_8 lattice, equal to the Eisenstein series of weight 4. Under modular transformations the partition function transforms according to

$$B(-\tfrac{1}{\tau}) = B(\tau)\ ,\quad B(\tau+1) = \mathrm{e}(-\tfrac{1}{3})B(\tau)\ . \tag{4.4.10}$$

A symmetry $\hat{g}$ acts on the four complex bosons in the same way as for the fermions, leaving invariant the supersymmetry of the E_8 VOSA. The corresponding untwisted $\hat{g}^n$-twined partition function is thus given by

$$B_1^{\hat{g}^n}(\tau) = q^{-\frac{1}{3}} \prod_{k=1}^{\infty} \left(\frac{1}{1-\zeta_L^n q^k}\right)^2 \left(\frac{1}{1-\zeta_L^{-n} q^k}\right)^2 \left(\frac{1}{1-\zeta_R^n q^k}\right)^2 \left(\frac{1}{1-\zeta_R^{-n} q^k}\right)^2 \Theta_{\Lambda^{\hat{g}^n}}(\tau) \ , \tag{4.4.11}$$

where $\Theta_{\Lambda^{\hat{g}^n}}(\tau)$ is the theta series of the sublattice fixed by $\hat{g}^n$ (except for the case that g is of class $2E$, and $n = 2$, wherein $\Theta_{\Lambda^{\hat{g}^n}}$ takes a slightly different meaning, as explained below). When $\zeta_L^n, \zeta_R^n \neq 1$, one has $\Theta_{\Lambda^{\hat{g}^n}} = 1$ and the above may be conveniently written as

$$B_1^{\hat{g}^n}(\tau) = (\zeta_L^{\frac{n}{2}} - \zeta_L^{-\frac{n}{2}})^2 (\zeta_R^{\frac{n}{2}} - \zeta_R^{-\frac{n}{2}})^2 \frac{\eta(\tau)^4}{\theta_1^2(\tau, n\alpha_L)\theta_1^2(\tau, n\alpha_R)} . \tag{4.4.12}$$

The cases for which $\Theta_{\Lambda^{\hat{g}^n}}(\tau)$ is not identically 1 are summarized in Table 4.1, so that $\Theta_{\Lambda^{\hat{g}^n}}$ is the theta series of the D_4 lattice, for example, when g is of class $2A$ or $4A$ and $n = 1$. As hinted above, the case that g belongs to $2E$ and $n = 2$ is a bit more subtle. This is because $\hat{g}^2$ is non-trivial, even though g has order 2. We have

$$\hat{g}^2\left(V_\lambda\right) = (-1)^{(\lambda, g(\lambda))} V_\lambda \ , \tag{4.4.13}$$

and the result of this is that $\Theta_{\Lambda^{\hat{g}^2}}$ should be interpreted as $\Theta_{\widetilde{E_8}}(\tau) := \theta_3^4(\tau)\theta_4^4(\tau)$, rather than just the theta series (4.4.9) of E_8, when g is of class $2E$.

	$\widehat{2A}$	$\widehat{2E}$	$\widehat{3E}$	$\widehat{4A}$	$\widehat{-4A}$	$\widehat{-3E}$	$\widehat{6BC}$
$\hat{g}$	D_4	A_1^4	A_2^2	D_4	—	—	—
$\hat{g}^2$		$\widetilde{E_8}$	A_2^2	D_4	D_4	A_2^2	—
$\hat{g}^3$						—	D_4

Table 4.1: Fixed sublattices of E_8 in ρ_ψ, by powers of conjugacy classes of $W^+(E_8)$

The whole set of bosonic twisted-twined partition functions $B_{\hat{g}^k}^{\hat{g}^\ell}$ can be recovered from the untwisted ones $B_1^{\hat{g}^n}$ using the analog of (4.4.4) for the bosonic case, namely

$$B_{\hat{g}^m}^{\hat{g}^n}(\tau) = \mathrm{e}(\alpha_B) B_1^{g'}\left(\frac{a\tau + b}{c\tau + d}\right) \ . \tag{4.4.14}$$

for some phases $\mathrm{e}(\alpha_B) := e^{2\pi i \alpha_B}$.

We need to have some control over the phases $\mathrm{e}(\alpha_B)$ in (4.4.14). For orbifolds of holomorphic VOAs by cyclic groups, these phases were discussed in [132]. More precisely, if V is a simple, rational, C_2-cofinite, self-contragredient vertex operator algebra and g is an automorphism of V of order N then the phases are governed by a 2-cocycle representing a class in $H^2(\mathbb{Z}_N, \mathbb{Z}_N) \cong \mathbb{Z}_N$. According to Proposition 5.10 of [132], the cohomology class depends on $2N^2\rho_1 \mod N$, where ρ_1 is the conformal weight of the irreducible g-twisted V-modules

$V(g)$. Different cocycles in the same class correspond to different choices for the action of $\langle g\rangle$ on the twisted sectors.

It turns out that, upon combining the fermions and bosons into the full twisted twined partition functions ${}^{D}_{D}Z^{\hat{g}^\ell}_{\hat{g}^k} = {}^{D}_{D}F^{\hat{g}^\ell}_{\hat{g}^k} B^{\hat{g}^\ell}_{\hat{g}^k}$, the phases $\mathrm{e}(\alpha_B)$ always cancel against the analogous phases for the fermionic contribution, so that the phases $\mathrm{e}(\alpha)$ in (4.4.4) are trivial.

For example, when $\zeta_L^n, \zeta_R^n \neq 1$, where $n = \gcd(k, \ell)$, one obtains

$$B^{\hat{g}^\ell}_{\hat{g}^k}(\tau) = (\zeta_L^{\frac{n}{2}} - \zeta_L^{-\frac{n}{2}})^2(\zeta_R^{\frac{n}{2}} - \zeta_R^{-\frac{n}{2}})^2 q^{-(\hat{\alpha}_L^2+\hat{\alpha}_R^2)k^2} \frac{\eta(\tau)^4}{\theta_1^2(\tau, \hat{\alpha}_L(k\tau+\ell))\theta_1^2(\tau, \hat{\alpha}_R(k\tau+\ell))} \tag{4.4.15}$$

where $0 \leq \hat{\alpha}_L, \hat{\alpha}_R \leq 1/2$, so that, combining the fermions and bosons, we obtain

$$\begin{aligned}
{}^{A}_{A}Z^{\hat{g}^\ell}_{\hat{g}^k} &= (\zeta_L^{\frac{n}{2}} - \zeta_L^{-\frac{n}{2}})^2(\zeta_R^{\frac{n}{2}} - \zeta_R^{-\frac{n}{2}})^2 \frac{\theta_3^2(\tau, \alpha_L(k\tau+\ell))\theta_3^2(\tau, \alpha_R(k\tau+\ell))}{\theta_1^2(\tau, \alpha_L(k\tau+\ell))\theta_1^2(\tau, \alpha_R(k\tau+\ell))}\ , \\
{}^{P}_{A}Z^{\hat{g}^\ell}_{\hat{g}^k} &= (\zeta_L^{\frac{n}{2}} - \zeta_L^{-\frac{n}{2}})^2(\zeta_R^{\frac{n}{2}} - \zeta_R^{-\frac{n}{2}})^2 \frac{\theta_4^2(\tau, \alpha_L(k\tau+\ell))\theta_4^2(\tau, \alpha_R(k\tau+\ell))}{\theta_1^2(\tau, \alpha_L(k\tau+\ell))\theta_1^2(\tau, \alpha_R(k\tau+\ell))}\ , \\
{}^{A}_{P}Z^{\hat{g}^\ell}_{\hat{g}^k} &= (\zeta_L^{\frac{n}{2}} - \zeta_L^{-\frac{n}{2}})^2(\zeta_R^{\frac{n}{2}} - \zeta_R^{-\frac{n}{2}})^2 \frac{\theta_2^2(\tau, \alpha_L(k\tau+\ell))\theta_2^2(\tau, \alpha_R(k\tau+\ell))}{\theta_1^2(\tau, \alpha_L(k\tau+\ell))\theta_1^2(\tau, \alpha_R(k\tau+\ell))}\ .
\end{aligned} \tag{4.4.16}$$

Using the modular properties of Jacobi theta functions, it is easy to verify that (4.4.4) holds with ρ_c given by (4.4.3) with $c = 12$ and with trivial phases $\mathrm{e}(\alpha)$. An analogous result holds when $\zeta_L^n = 1$ or $\zeta_R^n = 1$, $n = \gcd(k, \ell)$, although the formulae (4.4.16) are not valid in this case.

Combining the above we may verify case-by-case that $Z_{\hat{g}\text{-orb}}(\tau) = Z(V^f_{E_8}; \tau)$ whenever the g-orbifold of the four-torus sigma model is again a four-torus sigma model, and $Z_{\hat{g}\text{-orb}}(\tau) = Z(V^{s\natural}; \tau)$ whenever the g-orbifold of the four-torus sigma model is a K3 sigma model, which is what we required to show. □

4.5 Reflection

The procedure of reflection on a non-chiral theory entails mapping all right-movers to left-movers, resulting in a holomorphic theory that may or may not be consistent. In [115] such a procedure was used to show that the K3 sigma model with $\mathbb{Z}_2^8 : \mathbb{M}_{20}$ symmetry can be consistently reflected to give the Conway moonshine module VOSA $V^{s\natural}$. Moreover, the necessary and sufficient conditions that allow for reflection in a general theory were studied in detail.

In this section we demonstrate that a similar reflection relation holds between a specific T^4 sigma model and the VOSA $V^f_{E_8}$. In other words, we verify that Property 3 of VOSA/sigma model correspondences holds for $V^f_{E_8}$ and T^4 sigma models. To formulate this result precisely we first note that, according to [126], there exists a unique point $\mu^* \in \mathcal{M}(T^4)$ such that the corresponding sigma model $\Sigma(T^4; \mu^*)$ has $G_0 \cong \mathcal{T}_{24} \times_{C_3} \mathcal{T}_{24}$. Now we may state the main result of this section.

Theorem 8. *The image of* $\Sigma(T^4;\mu^*)$ *under the reflection operation is a VOSA isomorphic to* $V^f_{E_8}$.

For the proof of Theorem 8 it will be convenient to use a quaternionic description of the relevant lattices. Let $\mathbb{H}$ be the space of quaternions, and write $\mathbf{i},\mathbf{j},\mathbf{k}$ for the imaginary units satisfying the usual quaternionic multiplication rule. Then $q\in\mathbb{H}$ can be written as $q=q_1+q_2\mathbf{i}+q_3\mathbf{j}+q_4\mathbf{k}$, where $q_1,q_2,q_3,q_4\in\mathbb{R}$. We will often denote an element $q\in\mathbb{H}$ in terms of its components $(q_1,q_2,q_3,q_4)\in\mathbb{R}^4$, and write $q=(q_1,q_2,q_3,q_4)$. We use the following norm on $\mathbb{H}$:

$$||q||^2 := \sum_{i=1}^4 q_i^2 \ , \tag{4.5.1}$$

and the following notation for elements of $\mathbb{H}^2$ and $\mathbb{H}^{1,1}$

$$\begin{aligned} &\mathbb{H}^2 \ni (p|q) := (p_1,p_2,p_3,p_4|q_1,q_2,q_3,q_4) \ , \\ &\mathbb{H}^{1,1} \ni (p;q) := (p_1,p_2,p_3,p_4;q_1,q_2,q_3,q_4) \ , \end{aligned} \tag{4.5.2}$$

where the corresponding norms are given by

$$||(p|q)||^2 := \sum_{i=1}^4 p_i^2+q_i^2 \ , \quad ||(p;q)||^2 := \sum_{i=1}^4 p_i^2-q_i^2 \ . \tag{4.5.3}$$

The following lemma details a quaternionic realisation of the E_8 lattice.

Lemma 9. *The eight-dimensional lattice defined by*

$$\Gamma^8_{w\text{-}m} = \left\{ \frac{1}{\sqrt{2}}(a|b) \mid a_i,b_i\in\mathbb{Z},\ \sum_{i=1}^4 b_i\in 2\mathbb{Z},\ a_i-b_i\equiv a_j-b_j \mod 2 \ \ \forall \ \ i,j\in\{1,2,3,4\} \right\} \tag{4.5.4}$$

is a copy of the E_8 *lattice.*

Proof. Recall that the Hurwitz quaternions are defined by

$$\mathcal{H} = \left\{ q\in\mathbb{H} \mid (q_1,q_2,q_3,q_4)\in\mathbb{Z}^4\cup\left(\mathbb{Z}+\frac{1}{2}\right)^4 \right\} \subset \mathbb{H} \ . \tag{4.5.5}$$

Then, according to §2.6 of [15], for example, we obtain a copy of the E_8 lattice in $\mathbb{H}^2$ by considering

$$\Lambda_{E_8} \cong \left\{ \frac{p}{\sqrt{2}}(2|0) + \frac{q}{\sqrt{2}}(1-\mathbf{i}|1-\mathbf{i}) \ \mid \ p,q\in\mathcal{H} \right\} , \tag{4.5.6}$$

where we write $q'(p|q) := (q'p|q'q)$. In this realisation the 240 roots of E_8 are expressed as follows,

$$\begin{aligned} &16 \;\text{ roots of the form } \;\frac{1}{\sqrt{2}}(\pm 2,0,0,0|0,0,0,0)\;,\\ &32 \;\text{ roots of the form } \;\frac{1}{\sqrt{2}}(\pm 1,\pm 1,\pm 1,\pm 1|0,0,0,0)\;,\\ &192 \;\text{ roots of the form } \;\frac{1}{\sqrt{2}}(\pm 1,\pm 1,0,0|\pm 1,\pm 1,0,0)\;, \end{aligned} \tag{4.5.7}$$

where at the first line the ± 2 can be in any position, at the second line the four factors of ± 1 can be either all at the left or all at the right, and at the last line the pair of ± 1 at the right can either be at the same positions as the pair at the left or at complementary positions.

We claim that the sets defined by (4.5.6) and (4.5.4) are the same. For this note that in terms of components we have

$$\begin{aligned} p(2|0)+q(1-\mathbf{i}|1-\mathbf{i}) &= 2(p_1,p_2,p_3,p_4|0)\\ &+(q_1+q_2,-q_1+q_2,q_3-q_4,q_3+q_4|q_1+q_2,-q_1+q_2,q_3-q_4,q_3+q_4), \end{aligned} \tag{4.5.8}$$

and it follows that $\Lambda_{E_8}\subseteq\Gamma^{8}_{\text{w-m}}$. To check that $\Gamma^{8}_{\text{w-m}}\subseteq\Lambda_{E_8}$, we define, for every $\frac{1}{\sqrt{2}}(a|b)\in\Gamma^{8}_{\text{w-m}}$,

$$p_i := a_i - b_i,\;\; i\in\{1,2,3,4\} \tag{4.5.9}$$

and

$$q_{2i-1} := \frac{1}{2}(b_{2i-1}-b_{2i}),\;\; q_{2i} := \frac{1}{2}(b_{2i-1}+b_{2i}),\quad i\in\{1,2\}. \tag{4.5.10}$$

Then the condition $a_i-b_i\equiv a_j-b_j \mod 2$ guarantees that $(p_1,p_2,p_3,p_4)\in\mathbb{Z}^4\cup\left(\mathbb{Z}+\frac{1}{2}\right)^4$, and the condition $\sum_{i=1}^4 b_i\in 2\mathbb{Z}$ guarantees that $(q_1,q_2,q_3,q_4)\in\mathbb{Z}^4\cup\left(\mathbb{Z}+\frac{1}{2}\right)^4$. This finishes the proof. □

Now we are ready to prove Theorem 8.

Proof of Theorem 8. Recall that $\Sigma(T^4;\mu)$ has a simple description in terms of Fock space oscillators and vertex operators based on the winding-momentum lattice $\Gamma_{\text{w}-\text{m}}(\mu)$ corresponding to the point $\mu\in\mathcal{M}(T^4)$. Since all right-moving oscillators are straightforwardly reflected to left-moving ones, the only non-trivial part of the proof is to show that the reflection of the winding-momentum lattice $\Gamma_{\text{w}-\text{m}}(\mu^*)$ is isomorphic to the E_8 lattice.

At the moduli point μ^* of four-torus sigma model labelled by Λ_{D_4}, where the symmetry group is given by $G_0=\mathcal{T}_{24}\times_{C_3}\mathcal{T}_{24}$ in the notation of [126], the even unimodular winding-momentum lattice is given in quaternionic language by

$$\Gamma^{4,4}_{\text{w-m}}=\left\{\frac{1}{\sqrt{2}}(a;b)\mid a_i,b_i\in\mathbb{Z},\;\sum_{i=1}^4 a_i\in 2\mathbb{Z},\;a_i-b_i\equiv a_j-b_j \mod 2\;\;\forall\;\; i,j\in\{1,2,3,4\}\right\}. \tag{4.5.11}$$

Reflecting $\Gamma^{4,4}_{\text{w-m}}$ amounts to changing the signature from $(4,4)$ to $(8,0)$, by sending $(a;b)\to(a|b)$ for all lattice vectors. This results precisely in the lattice $\Gamma^{8}_{\text{w-m}}$ which according to

Lemma 9 is simply the E_8 lattice. This finishes the proof. $\square$

A More on Ghosts

In this appendix we discuss the ground states of the ghost systems in both the P and A sectors, as well as the action of the $\mathbb{Z}_2$ orbifold on them, as a complement to §3.2.2.

The ghost ground states

The first thing to note is that for the ghost systems the ordering prescription generally changes when we go from the complex plane, where we use canonical ordering, to the cylinder, where it is natural to use normal ordering. By expressing the Virasoro zero mode in terms of the normal ordering, a constant B will appear in the following way (see [133, 134]),

$$L_0^{\text{gh}} = \sum_n (-n) \,\mathring{\circ}\, \mathbf{b}_n \mathbf{c}_{-n} \,\mathring{\circ} = \sum_n (-n) :\mathbf{b}_n \mathbf{c}_{-n}: + B. \tag{A.1}$$

We define a ground state on the cylinder as a state that is annihilated by the normal ordered term above, so it will still have weight B. Thus, this state will not generally be the $SL(2,\mathbb{R})$-invariant vacuum $|0\rangle$. In order to treat both cases together, denote the ground state(s) in the A and P sectors of either ghost system by $|\Omega_{\text{A}}\rangle$ and $|\Omega_{\text{P}}\rangle$ respectively. The constant B depends on the central charge and the sector as

$$B_{\text{A}} = -\frac{\kappa}{8} Q^2 \ , \quad B_{\text{P}} = \frac{\kappa}{8}(1 - Q^2). \tag{A.2}$$

We can also compute the eigenvalues of the ground states under the Virasoro zero mode on the cylinder $L^{\text{gh}}_{\text{cyl},0} = L_0^{\text{gh}} - c_{\mathbf{bc}}/24$. We notice that the Q-dependence cancels and we get

$$L^{\text{gh}}_{\text{cyl},0} |\Omega_{\text{A}}\rangle = -\frac{\kappa}{24} |\Omega_{\text{A}}\rangle, \quad L^{\text{gh}}_{\text{cyl},0} |\Omega_{\text{P}}\rangle = \frac{\kappa}{12} |\Omega_{\text{P}}\rangle, \tag{A.3}$$

for any value of $c_{\mathbf{bc}}$. Note that all the characters we use in this paper are defined in terms of the canonically ordered operators, rather than the normal-ordered ones.

As we have mentioned in §3.2.2, the ghost systems possess a $U(1)$ current $J = -\,\mathring{\circ}\,\mathbf{bc}\,\mathring{\circ}$. Note that J is not a primary field, and the failure to be so is captured by the quantity Q:

$$L^{\mathrm{gh}}(z)J(w) \sim \frac{Q}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}. \tag{A.4}$$

Accordingly, upon going to the cylinder the charge operator is shifted as $J_{\mathrm{cyl},0} = J_0 + \frac{Q}{2}$. This $U(1)$ current can be used to define an infinite family of primary operators [108], which will eventually be related to the cylinder ground states. First we introduce a new (bosonic) field ϕ of zero weight so that $J(z) = \kappa\partial\phi(z)$, which results in the OPE $\phi(z)\phi(w) \sim \kappa \ln(z-w)$. We then define a vertex operator by $V_q(z) := {}^{\circ}_{\circ} e^{q\phi(z)} {}^{\circ}_{\circ}$, which is primary and obeys the OPEs

$$L^{\mathrm{gh}}(z)V_q(w) \sim \left[\frac{\frac{1}{2}\kappa q(q+Q)}{(z-w)^2} + \frac{\partial}{z-w}\right] V_q(w), \quad J(z)V_q(w) \sim \frac{q}{z-w}\, V_q(w). \tag{A.5}$$

Acting on $|0\rangle$, this operator defines the state

$$|q\rangle := V_q(0)|0\rangle, \tag{A.6}$$

which has the following weight and $U(1)$ charge

$$L_0^{\mathrm{gh}}|q\rangle = \frac{1}{2}\kappa q(q+Q)|q\rangle \ , \quad J_0|q\rangle = q|q\rangle. \tag{A.7}$$

These states can be regarded as vacuum states (for the Fock space) on the plane. Note that each such q-vacuum is annihilated by a different set of modes of the ghost fields, depending on the eigenvalue q. In particular, we have

$$\begin{aligned} \mathbf{b}_r|q\rangle &= 0 \ \ \forall\, r \geq \kappa q + 1 - h, \\ \mathbf{c}_r|q\rangle &= 0 \ \ \forall\, r \geq -\kappa q + h. \end{aligned} \tag{A.8}$$

These vacua belong to the periodic sector on the plane if $q \in \mathbb{Z}$, and to the anti-periodic sector if $q \in \left(\mathbb{Z} + \frac{1}{2}\right)$. One way to see this is from (A.8), since $r \pm h$ is should always be an integer in the periodic sector on the plane, and half-integer in the antiperiodic sector. Also note that the vertex operators $V_{\pm 1/2}$ interpolate between the above two sectors, and hence can be regarded as twist fields.

The space of states on the cylinder is built by acting with the creation operators on the ground states $|\Omega_{\mathrm{S}}\rangle$, where $\mathrm{S} = \{\mathrm{P}, \mathrm{A}\}$ denotes the two sectors. These are equal to some of the q-vacua described above. By equating the weight B_{S} and the weight of the q-vacua (A.7), we find that they have the following eigenvalues under J_0:

$$q_{\Omega_{\mathrm{A}}} = -\frac{1}{2}Q, \quad q_{\Omega_{\mathrm{P}}} = -\frac{1}{2}(Q \mp \kappa). \tag{A.9}$$

Accounting for the shift $J_{\mathrm{cyl},0} = J_0 + \frac{Q}{2}$, the corresponding U(1) charges on the cylinder are

$$
\begin{aligned}
J_{\mathrm{cyl},0}|\Omega_{\mathrm{A}}\rangle &= 0, \\
J_{\mathrm{cyl},0}|\Omega_{\mathrm{P}}\rangle &= \pm\frac{\kappa}{2}|\Omega_{\mathrm{P}}\rangle.
\end{aligned}
\tag{A.10}
$$

We see that in the A sector we have a single ground state, denoted $|\Omega_{\mathrm{A}}^{\mathrm{F}}\rangle$ and $|\Omega_{\mathrm{A}}^{\mathrm{B}}\rangle$ for the fermionic and bosonic system respectively. In the P sector of the fermionic ghosts the zero modes b_0, c_0 form a Clifford algebra, which results in two degenerate ground states with opposite $U(1)$ charges, which we denote by $|\Omega_{\mathrm{P},\pm}^{\mathrm{F}}\rangle$. They obey

$$
b_0|\Omega_{\mathrm{P},-}^{\mathrm{F}}\rangle = 0, \quad b_0|\Omega_{\mathrm{P},+}^{\mathrm{F}}\rangle = |\Omega_{\mathrm{P},-}^{\mathrm{F}}\rangle, \quad c_0|\Omega_{\mathrm{P},+}^{\mathrm{F}}\rangle = 0, \quad c_0|\Omega_{\mathrm{P},-}^{\mathrm{F}}\rangle = |\Omega_{\mathrm{P},+}^{\mathrm{F}}\rangle. \tag{A.11}
$$

In the bosonic case, we have to single out one of the two possible ground states in the P sector since they do not belong in the same representation of the β, γ algebra. In other words, the zero modes β_0, γ_0 do not form an analogue of the Clifford algebra of their fermionic counterparts, so one of them must be an annihilator. We choose to use the ground state with positive charge for the torus characters that will follow, which corresponds to γ_0 being a creation and β_0 being an annihilation operator.

The ghost orbifold

We now treat the $\mathbb{Z}_2$ orbifold for both fermionic and bosonic ghosts, generated by $\xi\mathbf{b} = -\mathbf{b}$ and $\xi\mathbf{c} = -\mathbf{c}$. In terms of the field ϕ introduced earlier, the ghost fields are expressed as

$$
\begin{aligned}
b(z) &= {}^\circ_\circ e^{-\phi(z)} {}^\circ_\circ, & c(z) &= {}^\circ_\circ e^{\phi(z)} {}^\circ_\circ, \\
\beta(z) &= {}^\circ_\circ e^{-\phi(z)} {}^\circ_\circ\, \partial\lambda(z), & \mathfrak{g}(z) &= {}^\circ_\circ e^{\phi(z)} {}^\circ_\circ\, \eta(z).
\end{aligned}
\tag{A.12}
$$

where we introduced two auxiliary fields η, λ. These form a free fermionic ghost system by themselves, with $h_{\eta\lambda} = 1$ and central charge $c_{\eta\lambda} = -2$. They need to be introduced because the Virasoro operator that is build out of J,

$$
T_J = -\left(\frac{1}{2}\, {}^\circ_\circ JJ {}^\circ_\circ - \frac{1}{2}Q\partial J\right), \tag{A.13}
$$

is not enough to describe the bosonic ghosts [108]. In particular, there is a “residual” Virasoro operator T_{-2} that needs to be added, so that the total ghost Virasoro operator is given by $L = T_J + T_{-2}$, where T_{-2} is precisely the Virasoro operator of the fermionic ghosts η, λ.

One can implement the action of ξ on the “bosonized” form of the ghost fields (A.12) in both cases by setting

$$
\xi\phi = \phi + (2k+1)\pi i, \quad k \in \mathbb{Z}, \qquad \xi\eta = \eta, \qquad \xi\lambda = \lambda, \tag{A.14}
$$

Consequently, for the vacuum state $|q\rangle$ with $q \in \mathbb{Z}$ we get

$$\xi|q\rangle = \xi \,\circ\!\!\!\circ\, e^{q\phi(0)} \,\circ\!\!\!\circ\, |0\rangle = \begin{cases} +|q\rangle, & q \text{ even} \\ -|q\rangle, & q \text{ odd} \end{cases}, \tag{A.15}$$

while for $q \in \mathbb{Z} + \frac{1}{2}$ we have

$$\xi|q\rangle = \xi \,\circ\!\!\!\circ\, e^{q\phi(0)} \,\circ\!\!\!\circ\, |0\rangle = \begin{cases} +i(-1)^k|q\rangle, & (q-1/2) \;\text{ even} \\ -i(-1)^k|q\rangle, & (q-1/2) \;\text{ odd} \end{cases}. \tag{A.16}$$

For convenience, we will choose $k = 0$ without loss of generality.

As we have seen, the ground states on the cylinder correspond to certain states $|q\rangle$ on the plane, with charge q under J_0. Specifically, for the ground states $|\Omega^{\mathrm{F}}_{\mathrm{A}}\rangle$ and $|\Omega^{\mathrm{F}}_{\mathrm{P},\pm}\rangle$ in the corresponding sectors of the fermionic system, we have already seen that

$$|\Omega^{\mathrm{F}}_{\mathrm{A}}\rangle = \left|-\frac{Q}{2}\right\rangle, \quad |\Omega^{\mathrm{F}}_{\mathrm{P},\pm}\rangle = \left|-\frac{1}{2}(Q\mp 1)\right\rangle. \tag{A.17}$$

From (A.15) and (A.16) we have that

$$\xi|\Omega^{\mathrm{F}}_{\mathrm{A}}\rangle = e^{\pi i\left(h-\frac{1}{2}\right)}|\Omega^{\mathrm{F}}_{\mathrm{A}}\rangle, \qquad \xi|\Omega^{\mathrm{F}}_{\mathrm{P},\pm}\rangle = \pm e^{\pi i h}|\Omega^{\mathrm{F}}_{\mathrm{P},\pm}\rangle. \tag{A.18}$$

Combining the ξ action on the ground states and the oscillators, we get

$$\begin{aligned} \tilde{\chi}^{\mathrm{A}}_{\mathrm{F}}(\tau,\zeta) &:= \mathrm{Tr}_{\mathrm{A}}\left[\xi\; y^{J_{\mathrm{cyl},0}} q^{L^{\mathrm{gh}}_{\mathrm{cyl},0}}\right] \\ &= e^{\pi i\left(h-\frac{1}{2}\right)} q^{-\frac{1}{24}} \prod_{n=1}^{\infty}\left(1-yq^{n-1/2}\right)\left(1-y^{-1}q^{n-1/2}\right) \\ &= (-1)^{h-\frac{1}{2}}\;\chi^{\mathrm{A}}_{\mathrm{F}}(\tau,\zeta+1/2), \end{aligned} \tag{A.19}$$

$$\begin{aligned} \tilde{\chi}^{\mathrm{P}}_{\mathrm{F}}(\tau,\zeta) &:= \mathrm{Tr}_{\mathrm{P}}\left[\xi\; y^{J_{\mathrm{cyl},0}} q^{L^{\mathrm{gh}}_{\mathrm{cyl},0}}\right] \\ &= e^{\pi i h} q^{\frac{1}{12}}\left(y^{1/2}-y^{-1/2}\right) \prod_{n=1}^{\infty}\left(1-yq^{n}\right)\left(1-y^{-1}q^{n}\right) \\ &= (-1)^{h-\frac{1}{2}}\;\chi^{\mathrm{P}}_{\mathrm{F}}(\tau,\zeta+1/2), \end{aligned} \tag{A.20}$$

Similarly, on the bosonic system ground states

$$|\Omega^{\mathrm{B}}_{\mathrm{A}}\rangle = \left|-\frac{Q}{2}\right\rangle, \quad |\Omega^{\mathrm{B}}_{\mathrm{P}}\rangle = \left|-\frac{1}{2}(Q-1)\right\rangle, \tag{A.21}$$

the orbifold acts as

$$\xi|\Omega^{\mathrm{B}}_{\mathrm{A}}\rangle = e^{\pi i\left(-h+\frac{1}{2}\right)}|\Omega^{\mathrm{B}}_{\mathrm{A}}\rangle, \qquad \xi|\Omega^{\mathrm{B}}_{\mathrm{P}}\rangle = e^{\pi i(-h+1)}|\Omega^{\mathrm{B}}_{\mathrm{P}}\rangle. \tag{A.22}$$

Thus we calculate

$$\begin{aligned}\tilde{\chi}^{\mathrm{A}}_{\mathrm{B}}(\tau,\zeta) &:= \mathrm{Tr}_{\mathrm{A}}\left[\xi\; y^{J_{\mathrm{cyl},0}} q^{L^{\mathrm{gh}}_{\mathrm{cyl},0}}\right] \\ &= e^{\pi i\left(-h+\frac{1}{2}\right)} q^{1/24} \prod_{n=1}^{\infty}\left(1+yq^{n-1/2}\right)^{-1}\left(1+y^{-1}q^{n-1/2}\right)^{-1} \\ &= (-1)^{3h+\frac{1}{2}}\;\chi^{\mathrm{A}}_{\mathrm{B}}(\tau,\zeta+1/2),\end{aligned} \tag{A.23}$$

$$\begin{aligned}\tilde{\chi}^{\mathrm{P}}_{\mathrm{B}}(\tau,\zeta) &:= \mathrm{Tr}_{\mathrm{P}}\left[\xi\; y^{J_{\mathrm{cyl},0}} q^{L^{\mathrm{gh}}_{\mathrm{cyl},0}}\right] \\ &= e^{\pi i(-h+1)} q^{-1/12} y^{1/2}(1+y)^{-1} \prod_{n=1}^{\infty}(1+yq^{n})^{-1}\left(1+y^{-1}q^{n}\right)^{-1} \\ &= (-1)^{3h+\frac{1}{2}}\;\chi^{\mathrm{P}}_{\mathrm{B}}(\tau,\zeta+1/2).\end{aligned} \tag{A.24}$$

Also note that all traces over the ghost Hilbert space involve defining a dual Fock space: corresponding to each in-state $|x\rangle = \prod_i \mathbf{c}_{-r_i} \prod_j \mathbf{b}_{-s_j}|q\rangle$ there us an out-state $\langle y| = \langle q'| \left(\prod_i \mathbf{c}_{-r_i} \prod_j \mathbf{b}_{-s_j}\right)^t$ such that their inner product is $\langle y|x\rangle = 1$. Due to the charge asymmetry [108], namely $J_0^t = -J_0 - Q$, the dual to the vertex state $|q\rangle$ is $\langle -q-Q| := \langle 0|\; {}^{\circ}_{\circ} e^{(-q-Q)\phi(z)} {}^{\circ}_{\circ}$, while the duals of the oscillator modes are $\mathbf{c}^t_{-r} = \mathbf{b}_r$ and $\mathbf{b}^t_{-r} = \mathbf{c}_r$. The latter is compatible with transposing the (anti-) commutation relations $\{\mathbf{b}_r, \mathbf{c}_{-r}\}_\kappa = \kappa$, and we have have $L_0^t = L_0$.

B Superconformal Structure of $\mathcal{T}$

In this appendix we discuss the supersymmetry of $\mathcal{T}$. We will see that an $\mathcal{N}=4$ superconformal structure is possible when we choose the ghost central charge such that the total theory is $c_{\mathcal{T}}=6$. In the following sections we will not make use of this $\mathcal{N}=4$ structure. First, one can equip the theory $\mathcal{T}_\psi$ of 12 complex fermions with the structure of $\mathcal{N}=1$ superconformal field theory [77]. To preserve this $\mathcal{N}=1$ superconformal symmetry, the manifest Spin(24) symmetry is broken to Co_0. Moreover, it is also possible equip $\mathcal{T}_\psi$ with an $\mathcal{N}=4$ structure (at $c=12$), which breaks Co_0 symmetry to its 3-plane preserving subgroups depending on the choice of the $SU(2)$ current [80]. One can also equip the whole theory $\mathcal{T}$, for certain choices of the ghost conformal weights, with an $\mathcal{N}=4$ superconformal symmetry by combining the $\mathcal{N}=4$ structure of $\mathcal{T}_\psi$ with an $\mathcal{N}=4$ structure of the ghost theory. In particular, for our choice $h_{\mathrm{B}}=1/2$ and $h_{\mathrm{F}}=1$ the total theory has an $\mathcal{N}=4$ superconformal algebra at $c=12-6=6$, precisely the superconformal symmetry of $K3$ non-linear sigma models. However, since we do not wish to break the Conway symmetry and in particular the compatibility with the action of the umbral group $3.S_6$, we will not make use of the $\mathcal{N}=4$ structure in the following.

If the conformal weights of a pairs of $bc-\beta\gamma$ ghost system satisfy $h_{\mathrm{F}}=h_{\mathrm{B}}+\frac{1}{2}$, there exists an $\mathcal{N}=1$ current of weight 3/2. In our case with two pairs of $bc-\beta\gamma$ ghosts, it is given by

$$G=\sum_{j=1}^{2}\left(-\frac{1}{2}\partial\beta_j c_j+\frac{2h_{\mathrm{F}}-1}{2}\partial(\beta_j c_j)-2b_j\gamma_j\right). \tag{B.1}$$

To enhance this to $\mathcal{N}=4$, we need an $SU(2)$ subalgebra generated by J_i with $i=1,2,3$. One can show that such currents are given by

$$J_1=\frac{i}{2}(\beta_1\gamma_2-\beta_2\gamma_1)\ ,\quad J_2=\frac{1}{2}(\beta_1\gamma_2+\beta_2\gamma_1)\ ,\quad J_3=\frac{1}{2}\ {}^{\circ}_{\circ}\beta_1\gamma_1-\beta_2\gamma_2{}^{\circ}_{\circ}\ . \tag{B.2}$$

From now on we will choose

$$h_{\mathrm{F}}=1\ ,\quad h_{\mathrm{B}}=\frac{1}{2}, \tag{B.3}$$

so that $SU(2)$ cuurent algebra is given by

$$J_i(z)J_j(w) \sim \frac{\delta_{ij}\ k_{\rm gh}/2}{(z-w)^2} + \frac{i\epsilon_{ijk}J_k(w)}{z-w}. \tag{B.4}$$

with $k_{\rm gh} = -1$. Acting with the generators J_i on G (with $h_{\rm F} = 1$) to construct the rest of the supercurrents, we get

$$J_i(z)G(w) \sim -\frac{i}{2}\frac{1}{z-w}G_i(w). \tag{B.5}$$

One can check that, together with the Virasoro field

$$L^{\rm gh} = \sum_{i=1}^{2} \,\circ\!\!\!\circ\, -b_i\partial c_i + \frac{1}{2}\partial\beta_i\gamma_i - \frac{1}{2}\beta_i\partial\gamma_i \,\circ\!\!\!\circ\,, \tag{B.6}$$

the fields G, G_i and J_i indeed form an $\mathcal{N}=4$ SCA with central charge $c_{\rm gh} = -6$ and level $k_{\rm gh} = -1$. As in [80] we can further define

$$G_1^{\pm} := \frac{1}{\sqrt{2}}(G \pm iG_3), \quad G_2^{\pm} := \pm\frac{i}{\sqrt{2}}(G_1 \pm iG_2), \tag{B.7}$$

which transform in the representation $\mathbf{2}+\bar{\mathbf{2}}$ of $SU(2)$, and reproduce the standard small $\mathcal{N}=4$ SCA. We note that the supercurrents, as well as the J_i, survive the orbifold, since they are all bilinears in the ghost fields.

C Character Tables

Table C.1: Character table of $G^X \simeq 3.S_6$, $X = D_4^6$

$[g]$	FS	1A	3A	2A	6A	3B	3C	4A	12A	5A	15A	15B	2B	2C	4B	6B	6C
$[g^2]$		1A	3A	1A	3A	3B	3C	2A	6A	5A	15A	15B	1A	1A	2A	3B	3C
$[g^3]$		1A	1A	2A	2A	1A	1A	4A	4A	5A	5A	5A	2B	2C	4B	2B	2C
$[g^5]$		1A	3A	2A	6A	3B	3C	4A	12A	1A	3A	3A	2B	2C	4B	6B	6C
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1
χ_3	+	5	5	1	1	2	−1	−1	−1	0	0	0	3	−1	1	0	−1
χ_4	+	5	5	1	1	2	−1	−1	−1	0	0	0	−3	1	−1	0	1
χ_5	+	5	5	1	1	−1	2	−1	−1	0	0	0	−1	3	1	−1	0
χ_6	+	5	5	1	1	−1	2	−1	−1	0	0	0	1	−3	−1	1	0
χ_7	+	16	16	0	0	−2	−2	0	0	1	1	1	0	0	0	0	0
χ_8	+	9	9	1	1	0	0	1	1	−1	−1	−1	3	3	−1	0	0
χ_9	+	9	9	1	1	0	0	1	1	−1	−1	−1	−3	−3	1	0	0
χ_{10}	+	10	10	−2	−2	1	1	0	0	0	0	0	2	−2	0	−1	1
χ_{11}	+	10	10	−2	−2	1	1	0	0	0	0	0	−2	2	0	1	−1
χ_{12}	∘	6	−3	−2	1	0	0	2	−1	1	b_{15}	$\overline{b_{15}}$	0	0	0	0	0
χ_{13}	∘	6	−3	−2	1	0	0	2	−1	1	$\overline{b_{15}}$	b_{15}	0	0	0	0	0
χ_{14}	+	12	−6	4	−2	0	0	0	0	2	−1	−1	0	0	0	0	0
χ_{15}	+	18	−9	2	−1	0	0	2	−1	−2	1	1	0	0	0	0	0
χ_{16}	+	30	−15	−2	1	0	0	−2	1	0	0	0	0	0	0	0	0

Table C.2: Twisted Euler characters and Frame shapes at $\ell = 6 + 3$, $X = D_4^6$

$[g]$	1A	3A	2A	6A	3B	3C	4A	12A	5A	15AB	2B	2C	4B	6B	6C
$n_g\|h_g$	1|1	1|3	2|1	2|3	3|1	3|3	4|2	4|6	5|1	5|3	2|1	2|2	4|1	6|1	6|6
$\bar{\chi}_g$	6	6	2	2	3	0	0	0	1	1	4	0	2	1	0
χ_g	6	6	2	2	3	0	0	0	1	1	−4	0	−2	−1	0
$\check{\chi}_g$	12	−6	4	−2	0	0	0	0	2	−1	0	0	0	0	0
$\bar{\Pi}_g$	1^6	1^6	$1^2 2^2$	$1^2 2^2$	$1^3 3^1$	3^2	$2^1 4^1$	$2^1 4^1$	$1^1 5^1$	$1^1 5^1$	$1^4 2^1$	2^3	$1^2 4^1$	$1^1 2^1 3^1$	6^1
$\widetilde{\Pi}_g$	1^{24}	$1^6 3^6$	$1^8 2^8$	$1^2 2^2 3^2 6^2$	$1^6 3^6$	3^8	$2^4 4^4$	$2^1 4^1 6^1 12^1$	$1^4 5^4$	$1^1 3^1 5^1 15^1$	$1^2 2^8$	2^{12}	$1^4 2^2 4^4$	$1^2 2^2 3^2 6^2$	6^4

D Sigma Model Symmetries

In this appendix we record the cyclic symmetry subgroups of four-torus sigma models. Given that $G_1 < O_8^+(2)$ and $G_0 < W^+(E_8)$, we require to consider the lifts of relevant classes X of $O_8^+(2)$ to $W^+(E_8)$. See (4.2.2). If there are two classes in the lift, they are denoted $\pm X$. We use the notation $2.2C$ to refer to the lift of the class $2C \subset O_8^+(2)$ to $W^+(E_8)$, which is a single class of order 4 rather than two classes $\pm 2C$. We follow [127] for the naming of the classes.

Class ρ_e	Non-trivial eigenv. in ρ_e				Class ρ_ψ	Eigenv. in ρ_ψ (twice each)				$o(g)$	orb	$(E_8)^{\rho_e(g)}$
$1A$	-	-	-	-	$1A$	1	1	1	1	1	T^4	rk>4
$-1A$	-	-	-	-	$-1A$	-1	-1	-1	-1	2	$K3$	rk>4
$2B$	-	-	-1	-1	$2.2C$	e($\frac{1}{4}$)	e($\frac{3}{4}$)	e($\frac{1}{4}$)	e($\frac{3}{4}$)	4	$K3$	rk>4
$3A$	-	-	e($\frac{1}{3}$)	e($\frac{2}{3}$)	$3BC$	e($\frac{1}{3}$)	e($\frac{2}{3}$)	e($\frac{1}{3}$)	e($\frac{2}{3}$)	3	$K3$	rk>4
$-3A$	-	-	e($\frac{1}{3}$)	e($\frac{2}{3}$)	$-3BC$	e($\frac{1}{6}$)	e($\frac{5}{6}$)	e($\frac{1}{6}$)	e($\frac{5}{6}$)	6	$K3$	rk>4
$2A$	-1	-1	-1	-1	$2A$	1	1	-1	-1	2	T^4	D_4
$-2A$	-1	-1	-1	-1	$2A'$	-1	-1	1	1	2	T^4	D_4
$2E$	-1	-1	-1	-1	$2E$	1	1	-1	-1	2	T^4	A_1^4
$-2E$	-1	-1	-1	-1	$2E'$	-1	-1	1	1	2	T^4	A_1^4
$3E$	e($\frac{1}{3}$)	e($\frac{2}{3}$)	e($\frac{1}{3}$)	e($\frac{2}{3}$)	$3E$	1	1	e($\frac{1}{3}$)	e($\frac{2}{3}$)	3	T^4	A_2^2
$3E'$	e($\frac{1}{3}$)	e($\frac{2}{3}$)	e($\frac{1}{3}$)	e($\frac{2}{3}$)	$3E'$	e($\frac{1}{3}$)	e($\frac{2}{3}$)	1	1	3	T^4	A_2^2
$-3E$	e($\frac{1}{3}$)	e($\frac{2}{3}$)	e($\frac{1}{3}$)	e($\frac{2}{3}$)	$-3E$	-1	-1	e($\frac{1}{6}$)	e($\frac{5}{6}$)	6	$K3$	A_2^2
$-3E'$	e($\frac{1}{3}$)	e($\frac{2}{3}$)	e($\frac{1}{3}$)	e($\frac{2}{3}$)	$-3E'$	e($\frac{1}{6}$)	e($\frac{5}{6}$)	-1	-1	6	$K3$	A_2^2
$4A$	e($\frac{1}{4}$)	e($\frac{3}{4}$)	e($\frac{1}{4}$)	e($\frac{3}{4}$)	$4A$	1	1	e($\frac{1}{4}$)	e($\frac{3}{4}$)	4	T^4	D_4
$4A'$	e($\frac{1}{4}$)	e($\frac{3}{4}$)	e($\frac{1}{4}$)	e($\frac{3}{4}$)	$4A'$	e($\frac{1}{4}$)	e($\frac{3}{4}$)	1	1	4	T^4	D_4
$-4A$	e($\frac{1}{4}$)	e($\frac{3}{4}$)	e($\frac{1}{4}$)	e($\frac{3}{4}$)	$-4A$	-1	-1	e($\frac{1}{4}$)	e($\frac{3}{4}$)	4	$K3$	D_4
$-4A'$	e($\frac{1}{4}$)	e($\frac{3}{4}$)	e($\frac{1}{4}$)	e($\frac{3}{4}$)	$-4A'$	e($\frac{1}{4}$)	e($\frac{3}{4}$)	-1	-1	4	$K3$	D_4
$4C$	e($\frac{1}{4}$)	e($\frac{3}{4}$)	-1	-1	$8A$	e($\frac{1}{8}$)	e($\frac{7}{8}$)	e($\frac{3}{8}$)	e($\frac{5}{8}$)	8	$K3$	A_1A_3
$-4C$	e($\frac{1}{4}$)	e($\frac{3}{4}$)	-1	-1	$-8A$	e($\frac{3}{8}$)	e($\frac{5}{8}$)	e($\frac{1}{8}$)	e($\frac{7}{8}$)	8	$K3$	A_1A_3
$5A$	e($\frac{1}{5}$)	e($\frac{4}{5}$)	e($\frac{2}{5}$)	e($\frac{3}{5}$)	$5BC$	e($\frac{1}{5}$)	e($\frac{4}{5}$)	e($\frac{2}{5}$)	e($\frac{3}{5}$)	5	$K3$	A_4
$5A'$	e($\frac{1}{5}$)	e($\frac{4}{5}$)	e($\frac{2}{5}$)	e($\frac{3}{5}$)	$5BC'$	e($\frac{2}{5}$)	e($\frac{3}{5}$)	e($\frac{1}{5}$)	e($\frac{4}{5}$)	5	$K3$	A_4
$-5A$	e($\frac{1}{5}$)	e($\frac{4}{5}$)	e($\frac{2}{5}$)	e($\frac{3}{5}$)	$-5BC$	e($\frac{3}{10}$)	e($\frac{7}{10}$)	e($\frac{1}{10}$)	e($\frac{9}{10}$)	10	$K3$	A_4
$-5A$	e($\frac{1}{5}$)	e($\frac{4}{5}$)	e($\frac{2}{5}$)	e($\frac{3}{5}$)	$-5BC'$	e($\frac{1}{10}$)	e($\frac{9}{10}$)	e($\frac{3}{10}$)	e($\frac{7}{10}$)	10	$K3$	A_4
$6A$	e($\frac{1}{6}$)	e($\frac{5}{6}$)	-1	-1	$6BC$	e($\frac{1}{3}$)	e($\frac{2}{3}$)	e($\frac{1}{6}$)	e($\frac{5}{6}$)	6	$K3$	D_4
$-6A$	e($\frac{1}{6}$)	e($\frac{5}{6}$)	-1	-1	$6BC'$	e($\frac{1}{6}$)	e($\frac{5}{6}$)	e($\frac{1}{3}$)	e($\frac{2}{3}$)	6	$K3$	D_4
$6D$	e($\frac{1}{3}$)	e($\frac{2}{3}$)	-1	-1	$12BC$	e($\frac{1}{12}$)	e($\frac{11}{12}$)	e($\frac{5}{12}$)	e($\frac{7}{12}$)	12	$K3$	$A_1^2A_2$
$-6D$	e($\frac{1}{3}$)	e($\frac{2}{3}$)	-1	-1	$-12BC'$	e($\frac{5}{12}$)	e($\frac{7}{12}$)	e($\frac{1}{12}$)	e($\frac{11}{12}$)	12	$K3$	$A_1^2A_2$

Table D.1

Note that the set of possible G_1 is bijective to the set of subgroups of $W^+(E_8)$ which fix an E_8-sublattice of rank at least four, since there is always a rank four subspace in the representation ρ_e in G_0. The column "non-trivial eigenvalues in ρ_e" records the non-trivial eigenvalues in each case. Correspondingly, the $W^+(E_8)$ classes $\pm X$ in the columns "Class ρ_e" denotes the preimage of the class $X \subset O_8^+(2)$ under the projection π' of (4.2.2).

In §4.2 we have learned that this is not the only way to obtain a lift of a class of $O_8^+(2)$ in the context of four-torus sigma models. In the column "Class ρ_ψ" we record the preimage of the class $X \subset O_8^+(2)$ under the projection π'' in (4.2.2). Note that the "Class ρ_ψ" and "Class ρ_e", are of course related by a triality transformation which exchanges ι_s and ι_v, and correspondingly ρ_ψ and ρ_e. By (4.1.19), each eigenvalue appears twice in ρ_ψ and we therefore group the eight eigenvalues in four pairs (of identical values) and record just representative eigenvalues for each of these pairs. In the notation of (4.1.19), the first two eigenvalues are ζ_L and ζ_L^{-1} while the last two are ζ_R and ζ_R^{-1}. The notation $\pm X'$ is a reminder that, the same $W^+(E_8)$ class can act differently on a four-torus sigma model by exchanging left- and right-movers.

In the last part of Table D.1 we write $o(g)$ for the order of the element in G_0 (i.e. in the faithful representation ρ_ψ), while the order in $G_1 = G_0/\mathbb{Z}_2$ (i.e. in the unfaithful representation ρ_e,) can be read off from the symbol of the class, since $G_1 < O_8^+(2)$. We also indicate whether the orbifold by g is a sigma model on T^4 or $K3$. Finally, we indicate the $\rho_e(g)$-fixed sublattice of E_8 if it has rank four, in which case the symmetry g is non-geometric and appears only at a single point in the moduli space characterized by the fixed sublattice, which we record. If the rank is larger than four then the symmetry is geometric and it occurs in some family of models.

E Cocycles and Lifts

In this appendix, we review some well-known results about the OPE of vertex operators in toroidal sigma models and in lattice vertex operator algebras, with a particular focus on the so called 'cocycle factors'. Some early references on the subject are [128, 135] in the VOA literature and [136] in string theory; further references include [129, 132, 137]. In this section, we adopt the language of two dimensional conformal field theory: the lattice VOA version of our statements can be easily derived from the particular case of chiral CFTs.

Let us consider a (bosonic) toroidal conformal field theory, describing d_+ chiral and d_- anti-chiral compact free bosons, whose discrete winding-momentum (Narain) lattice is an even unimodular lattice L of dimension $d = d_+ + d_-$, whose bilinear form $(\cdot,\cdot) : L \times L \to \mathbb{Z}$ has signature (d_+, d_-). Note that such a lattice exists only when $d_+ - d_- \equiv 0 \mod 8$. If $d_- = 0$, then the conformal field theory is chiral, and it can be described as a lattice vertex operator algebra based on the even unimodular lattice L. On the opposite extreme, if $d_+ = d_- = d/2$, the CFT can be interpreted as a sigma model on a torus $T^{d/2}$. The supersymmetric versions of these models are obtained by adjoining d^+ chiral and d^- anti-chiral free fermions. The properties we are going to discuss do not depend on whether the toroidal CFT is bosonic or supersymmetric, so we will focus on the bosonic case for simplicity. As discussed in §4.1.1, for a given unimodular lattice L, there is a whole moduli space of toroidal models based on L, whose points correspond to different decompositions $L\otimes\mathbb{R} = \Pi_L \oplus \Pi_R$ into a positive definite subspace Π_L and a negative definite one Π_R. Every vector $v \in L \otimes \mathbb{R}$ can be decomposed accordingly as $v = (v_L, v_R)$. We can define positive definite scalar products on Π_L and on Π_R, that are uniquely determined by the condition

$$(\lambda, \mu) = \lambda_L \cdot \mu_L - \lambda_R \cdot \mu_R \ , \tag{E.1}$$

for all $\lambda, \mu \in L \otimes \mathbb{R}$.

The CFT contains the vertex operators $V_\lambda(z, \bar{z})$, for each $\lambda \in L$, with OPE satisfying

$$V_\lambda(z, \bar{z})V_\mu(w, \bar{w}) = \epsilon(\lambda, \mu)(z - w)^{\lambda_L\cdot\mu_L}(\bar{z} - \bar{w})^{\lambda_R\cdot\mu_R}V_{\lambda+\mu}(w, \bar{w}) + \dots \tag{E.2}$$

where $\dots$ are subleading (but potentially still singular) terms. In the chiral ($d_- = 0$) case, one can simply set $\lambda_L = \lambda$ and $\lambda_R = 0$ and similarly with μ. Here, $\epsilon : L \times L \to U(1)$ must satisfy

$$\epsilon(\lambda,\mu) = (-1)^{(\lambda,\mu)}\epsilon(\mu,\lambda) \tag{E.3}$$

$$\epsilon(\lambda,\mu)\epsilon(\lambda+\mu,\nu) = \epsilon(\lambda,\mu+\nu)\epsilon(\mu,\nu) \qquad \text{(cocycle condition)} \tag{E.4}$$

in order for the OPE to be local and associative. Given a solution $\epsilon(\lambda,\mu)$ to these conditions, any other solution is given by

$$\tilde{\epsilon}(\lambda,\mu) = \epsilon(\lambda,\mu)\frac{b(\lambda)b(\mu)}{b(\lambda+\mu)}\ , \tag{E.5}$$

for an arbitrary $b : L \to U(1)$. This change corresponds to a redefinition of the fields V_λ: if $V_\lambda(z,\bar{z})$ obey the OPE (E.2) with cocyle ϵ, then the operators $\tilde{V}_\lambda(z,\bar{z}) = b(\lambda)V_\lambda(z,\bar{z})$ obey (E.2) with the cocycle $\tilde{\epsilon}$. Notice that if $b(\lambda+\mu) = b(\lambda)b(\mu)$ for all $\lambda,\mu \in L$ (i.e. if $b : L \to U(1)$ is a homomorphism of abelian groups), then ϵ is unchanged, and the transformation $V_\lambda(z,\bar{z}) \to b(\lambda)V_\lambda(z,\bar{z})$ is a symmetry of the CFT, which is part of the $U(1)^d$ group generated by the zero modes of the currents.

One can show that $\epsilon(\lambda,\mu)$ satisfying the conditions (E.3) and (E.4) can be chosen to take values in $\{\pm 1\}$. Furthermore, one can use the freedom in redefining V_λ to set

$$\epsilon(0,\lambda) = \epsilon(\lambda,0) = 1, \qquad \forall\lambda \in L, \tag{E.6}$$

so that $V_0(z,\bar{z}) = 1$. Cocycles satisfying this condition are sometimes called normalized. Finally, one can choose ϵ such that[1]

$$\epsilon(\lambda+2\nu,\mu) = \epsilon(\lambda,\mu+2\nu) = \epsilon(\lambda,\mu)\ , \qquad \forall\lambda,\mu,\nu \in L\ . \tag{E.7}$$

If we require all these conditions, then ϵ determines a well defined function $L/2L \times L/2L \to \{\pm 1\}$.

More formally (see for example [128]), the cocycle ϵ represents a class in the cohomology group $H^2(L, \mathbb{Z}/2\mathbb{Z})$, where the lattice L is simply regarded as an abelian group. These cohomology classes are in one to one correspondence with isomorphism classes of central extensions

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \hat{L} \to L \to 1\ ,$$

of the abelian group L by $\mathbb{Z}/2\mathbb{Z}$. The specific cohomology class that is relevant for the toroidal CFT is uniquely determined by the condition (E.3). Using this formalism, the CFT can alternatively be defined by introducing a vertex operator $V_{\hat{\lambda}}$ for each element $\hat{\lambda} \in \hat{L}$ in this central extension. Then, the OPE of $V_{\hat{\lambda}}(z,\bar{z})V_{\hat{\mu}}(w,\bar{w})$ is analogous to (E.2), with $\epsilon(\lambda,\mu)V_{\lambda+\mu}$ replaced by $V_{\hat{\lambda}\cdot\hat{\mu}}$ (here, $\hat{\lambda}\cdot\hat{\mu}$ denotes the composition law in the extension $\hat{L}$, which is possibly non-abelian). Our previous description of the CFT can be recovered by choosing a section $e : L \to \hat{L}$ and defining the vertex operators $V_\lambda := V_{e(\lambda)}$ for each $\lambda \in L$. This leads to the OPE (E.2), where the particular cocycle representative ϵ depends on the

[1] One further condition that is usually imposed is $\epsilon(-\lambda,\lambda) = 1$ for all $\lambda \in L$. With this choice the general relation $(V_\lambda)^\dagger = \epsilon(\lambda,-\lambda)V_{-\lambda}$ simplifies as $(V_\lambda)^\dagger = V_{-\lambda}$. Another common choice is $\epsilon(-\lambda,\lambda) = (-1)^{\lambda^2/2}$. We will not impose any of these conditions.

choice of the section e via $e(\lambda)e(\mu) = \epsilon(\lambda,\mu)e(\lambda+\mu)$.

An automorphism $g \in O(L)$ can be lifted (non-uniquely) to a symmetry $\hat{g}$ of the CFT such that

$$\hat{g}(V_\lambda(z,\bar{z})) = \xi_g(\lambda)V_{g(\lambda)}(z,\bar{z}) \ , \tag{E.8}$$

where $\xi_g : L \to U(1)$ must satisfy

$$\frac{\xi_g(\lambda)\xi_g(\mu)}{\xi_g(\lambda+\mu)} = \frac{\epsilon(\lambda,\mu)}{\epsilon(g(\lambda),g(\mu))} \ . \tag{E.9}$$

As shown below, ξ_g satisfying this condition always exists, and any two such $\xi_g, \tilde{\xi}_g$ are related by $\tilde{\xi}_g(\lambda) = \rho(g)\xi_g(\lambda)$, where $\rho : L \to U(1)$ is a homomorphism. Furthermore, one can always find ξ_g taking values in $\{\pm 1\}$ and such that

$$\xi_g(0) = 1 \tag{E.10}$$

$$\xi_g(\lambda+2\mu) = \xi_g(\lambda) \qquad \forall \lambda,\mu \in L \ . \tag{E.11}$$

With these condition, ξ_g induces a well-defined map $\xi_g : L/2L \to \{\pm 1\}$.

A constructive proof of these statements is as follows (see [129]). Choose a basis $e_1,\ldots,e_d$ for L. Define an algebra of operators $\gamma_i \equiv \gamma_{e_i}$, $i = 1,\ldots,d$, satisfying[2]

$$\gamma_i^2 = 1 \qquad \gamma_i\gamma_j = (-1)^{(e_i,e_j)}\gamma_j\gamma_i \ , \tag{E.12}$$

and for every $\lambda = \sum_{i=1}^d a_i e_i \in L$, set

$$\gamma_\lambda := \gamma_1^{a_1}\cdots\gamma_d^{a_d} \ . \tag{E.13}$$

Then, the following properties hold:

$$\gamma_0 = 1 \qquad \gamma_{\lambda+2\mu} = \gamma_\lambda \qquad \gamma_\lambda\gamma_\mu = (-1)^{(\lambda,\mu)}\gamma_\mu\gamma_\lambda \ . \tag{E.14}$$

Define $\epsilon : L \times L \to \{\pm 1\}$ by

$$\gamma_\lambda\gamma_\mu = \epsilon(\lambda,\mu)\gamma_{\lambda+\mu} \ , \tag{E.15}$$

and, for every $g \in O(L)$, define $\xi_g : L \to \{\pm 1\}$ by

$$\gamma_{g(\lambda)} = \xi_g(\lambda)\gamma_{g(e_1)}^{a_1}\cdots\gamma_{g(e_d)}^{a_d} \ . \tag{E.16}$$

It is easy to verify that ϵ and ξ_g satisfy all the properties mentioned above. In particular, this choice of ξ_g is such that $\xi_g(e_i) = 1$ for all the basis elements e_i. It is clear that γ_λ, and therefore also ϵ and ξ_g, depend on λ only mod $2L$.

The constraints that we imposed on ξ_g still leave some freedom in the choice of the lift. There are two further conditions that one might want to impose:

[2] A slightly modified definition sets $\gamma_i^2 = (-1)^{e_i^2/2}$. With the latter choice, one obtains $\epsilon(\lambda,-\lambda) = (-1)^{\lambda^2/2}$ for all $\lambda \in L$, and γ_λ depends on $\lambda \mod 4L$ rather than $2L$. However, both ϵ and ξ_g are still well defined on $L/2L$.

(A) One might require $\hat{g}$ to have the same order $N = |g| < \infty$ as g. Notice that if $\hat{g}$ is a lift of a g of order N, then

$$\hat{g}^N(V_\lambda) = \xi_g(\lambda)\xi_g(g(\lambda))\cdots\xi_g(g^{N-1}(\lambda))V_\lambda \ , \tag{E.17}$$

so that $\hat{g}^N = 1$ if and only if

$$\xi_g(\lambda)\xi_g(g(\lambda))\cdots\xi_g(g^{N-1}(\lambda)) = 1 \qquad \forall\lambda \in L \ . \tag{E.18}$$

(B) Alternatively, one might want $\xi_g(\lambda)$ to be trivial whenever λ is g-fixed

$$\xi_g(\lambda) = 1 \qquad \forall\lambda \in L^g \ , \tag{E.19}$$

or, equivalently,

$$\hat{g}(V_\lambda) = V_\lambda \qquad \forall\lambda \in L^g \ . \tag{E.20}$$

Lifts satisfying this property are usually called *standard lifts*.

Proposition 10. *Every $g \in O(L)$ admits a* standard lift *$\hat{g}$, i.e. such that $\hat{g}(V_\lambda) = V_\lambda$ for all $\lambda \in L^g$.*

Proof. For all $\lambda, \mu \in L^g$, one has obviously $\frac{\epsilon(g(\lambda),g(\mu))}{\epsilon(\lambda,\mu)} = 1$. Therefore, the restriction of ξ_g to L^g is a homomorphism $L^g \to \{\pm 1\}$, and it is trivial if and only if it is trivial on all elements of a basis of L^g. By the construction described above, one can always find a lift $\hat{g}$ such that ξ_g is trivial for all the elements of a given basis of L. Choose a basis of L^g; since L^g is primitive in L, this can be completed to a basis of L. By choosing ξ_g to be trivial on the elements of this basis, we obtain a lift $\hat{g}$ satisfying condition (B). □

Standard lifts are not unique, but they are all conjugate to one each other within the symmetry group of the CFT, as the following proposition shows. (The following two propositions are proved in [132].)

Proposition 11. *Let $g \in O(L)$ and $\hat{g}$, $\hat{g}'$ be two lifts of g with associated functions $\xi_g, \xi'_g : L \to \{\pm 1\}$. Suppose $\xi_g = \xi'_g$ on the fixed-point sublattice L^g. Then $\hat{g}$ and $\hat{g}'$ are conjugate in the group of symmetries of the CFT.*

Since the order and the twined genus of a lift $\hat{g}$ depends only on its conjugacy class within the group of symmetries, this proposition then tells us that these quantities only depend on the restriction of ξ_g on the fixed sublattice L^g. In particular, when g fixes no sublattice of L, all its lifts $\hat{g}$ are conjugate to each other.

The following result gives, for the standard lifts (i.e. for $\xi_g = 1$ on L^g), the order of $\hat{g}$ and the action of every power $\hat{g}^k$ on the corresponding g^k-fixed sublattice L^{g^k}

Proposition 12. *Let $g \in O(L)$ and $\hat{g}$ be a standard lift (i.e. $\xi_g(\lambda) = 1$ for all $\lambda \in L^g$). Then:*

1. *If g has odd order N, then $\hat{g}^k(V_\lambda) = V_\lambda$ for all $\lambda \in L^{g^k}$. In particular $\hat{g}$ has order N.*

2. If g has even order N, then for all $\lambda \in L^{g^k}$,

$$\hat{g}^k(V_\lambda) = \begin{cases} V_\lambda & \text{for } k \text{ odd,} \\ (-1)^{(\lambda, g^{k/2}(\lambda))} V_\lambda & \text{for } k \text{ even.} \end{cases} \tag{E.21}$$

In particular $\hat{g}$ has order N if $(\lambda, g^{N/2}(\lambda))$ is even for all $\lambda \in L$ and order $2N$ otherwise.

For practical applications of this proposition it is important to have an easy way to determine if $(\lambda, g^{N/2}(\lambda))$ is even for all $\lambda \in L$. Consider g of order 2 (these are the important cases, since $g^{N/2}$ is always of order 2). One has

$$(\lambda, g(\lambda)) \equiv \frac{1}{2}(\lambda + g(\lambda))^2 \equiv 2\Big(\frac{1+g}{2}(\lambda)\Big)^2 \mod 2 \ . \tag{E.22}$$

Since $\frac{1+g}{2}$ is the projector onto the g-invariant subspace $L^g \otimes \mathbb{R}$ of $L \otimes \mathbb{R}$, by self-duality of L, one has $\frac{1+g}{2}(L) = (L^g)^*$. Therefore, the existence of $\lambda \in L$ with $(\lambda, g(\lambda))$ odd is equivalent to the existence of $v \in (L^g)^*$ with half-integral square norm $v^2 \in \frac{1}{2} + \mathbb{Z}$. This condition is quite easy to check, once the lattice L^g is known. When the fixed sublattice L^g is positive definite, the order of the standard lift can also be related to properties of the lattice theta series $\theta_{L^g}(\tau) = \sum_{\lambda \in L^g} q^{\lambda^2/2}$. This is well known to be a modular form of weight $r/2$, where r is the rank of L^g, for a congruence subgroup of $SL_2(\mathbb{Z})$. Its S-transform $\theta_{L^g}(-1/\tau)$ is proportional to the theta series $\theta_{(L^g)^*}(\tau)$ of the dual lattice $(L^g)^*$. If $(L^g)^*$ contains a vector v with half-integral square norm $v^2 \in \frac{1}{2} + \mathbb{Z}$, then the q-series of $\theta_{(L^g)^*}(\tau) = \sum_{v \in (L^g)^*} q^{\frac{v^2}{2}}$ contains some powers q^n with $n \in \frac{1}{4}\mathbb{Z}$. As a consequence, the standard lift of g of order 2 has order 2 if and only if the theta series $\theta_{L^g}(\tau)$ is a modular form for a subgroup of level 2, while it has order 4 if it is only modular under a subgroup of $SL_2(\mathbb{Z})$ of level 4.

When g has even order N and its standard lift $\hat{g}$ has order $2N$, it is sometimes convenient to choose a non-standard lift $\hat{g}$ with the same order N as g. The next proposition shows that for $N = 2$ such a lift always exists.

Proposition 13. *Let $g \in O(L)$ have order 2. Then, there is a lift $\hat{g}$ of g of order 2.*

Proof. Let $\hat{g}'$ be a standard lift of g. If $(\lambda, g(\lambda))$ is even for all $\lambda \in L$, then by the previous proposition $\hat{g}'$ has order 2 and we can just set $\hat{g} = \hat{g}'$. Suppose that $(\lambda, g(\lambda))$ is odd for some $\lambda \in L$. One has $(-1)^{(\lambda, g(\lambda))} = (-1)^{\frac{(\lambda+g(\lambda))^2}{2}}$, and the map $\lambda + g(\lambda) \mapsto (-1)^{\frac{(\lambda+g(\lambda))^2}{2}}$ is a homomorphism $(1+g)L \to \{\pm 1\}$. Thus, there is $w \in ((1+g)L)^*$ such that $(-1)^{\frac{(\lambda+g(\lambda))^2}{2}} = (-1)^{w\cdot(\lambda+g(\lambda))}$ for all $\lambda \in L$. Notice that $(1+g)L \subseteq L^g$, so that $(L^g)^* \subseteq ((1+g)L)^*$. On the other hand, it is easy to see that $w \in (L^g)^*$, i.e. that $(v, w) \in \mathbb{Z}$ for all $v \in L^g$. Indeed, if $v \in L^g$, then either $v \in (1+g)L$ (in which case, $(v, w) \in \mathbb{Z}$ is obvious) or $2v \in (1+g)L$ (because $2v = v + g(v)$ for $v \in L^g$). In the latter case. one has $(-1)^{(2v,w)} = (-1)^{\frac{(2v)^2}{2}} = 1$, so that $(2v, w)$ must be even, and therefore $(v, w) \in \mathbb{Z}$. Finally, by self-duality of L, for every $w \in (L^g)^*$ there always exist $\tilde{w} \in L$ such that $(\tilde{w}, v) = (w, v)$ for all $v \in L^g$. In particular, $(-1)^{(\tilde{w}, \lambda+g(\lambda))} = (-1)^{(\lambda, g(\lambda))}$ for all $\lambda \in L$. Then, we can define the lift $\hat{g}$ by

$\xi_g(\lambda) = \xi'_g(\lambda)(-1)^{(\tilde{w},\lambda)}$, where ξ'_g is the function corresponding to a standard lift. Thus, for all $\lambda \in L$,

$$\hat{g}^2(V_\lambda) = \xi_g(\lambda)\xi_g(g(\lambda))V_\lambda = \xi_g(\lambda + g(\lambda))\frac{\epsilon(\lambda, g(\lambda))}{\epsilon(g(\lambda), \lambda)}V_\lambda \tag{E.23}$$

$$= \xi'_g(\lambda + g(\lambda))(-1)^{(\tilde{w},\lambda+g(\lambda))}(-1)^{(\lambda,g(\lambda))}V_\lambda = V_\lambda \ , \tag{E.24}$$

where we used the condition (E.9), and the fact that $\xi'_g(\lambda + g(\lambda)) = 1$, since $\lambda + g(\lambda) \in L^g$ and $\hat{g}'$ is a standard lift. We conclude that $\hat{g}$ has order 2. □

Applications

Let us now apply the results described in the previous section to the cases we are interested in, namely the sigma model on T^4 and the SVOA based on the E_8 lattice. As explained in the article, there is a correspondence between automorphisms g of the lattice $\Gamma^{4,4}$ lifting to symmetries that preserve the $\mathcal{N} = (4, 4)$ superconformal algebra, and certain automorphisms of the lattice E_8. One needs to choose a lift of these lattice automorphisms to symmetries of the corresponding conformal field theory or SVOA. As explained above, a lift is determined, up to conjugation by CFT symmetries, by the restriction of the function ξ_g to the g-fixed sublattice. The most obvious choice is to consider the standard lift both for the sigma model and for the SVOA, so that ξ_g is trivial on the fixed sublattices. In general, the order of the standard lift is either the same or twice the order of the lattice automorphism. Therefore, it is not obvious a priori that the standard lifts in the sigma model and in the SVOA have the same order; we will show now that this is always true in the present the case.

Let g be an automorphism of the lattice $\Gamma^{4,4}$. We denote any such automorphism by the class of ρ_ψ, as in Table D.1. Using Propositions 11 and 12, the orders of the standard lifts are as follows.

- Classes of odd order N (1A, 3BC, 3E, 3E', 5BC, 5BC'): since N is odd, the standard lift has also order N. This conclusion holds also for the lift of the corresponding automorphisms of the E_8 lattice.
- Class -1A: an automorphism g in this class flips the sign of all vectors in $\Gamma^{4,4}$. Therefore, it acts trivially on $\Gamma^{4,4}/2\Gamma^{4,4}$, so that one can set $\xi_g(\lambda) = 1$ for all $\lambda \in \Gamma^{4,4}$, and this lift has obviously order 2. Since g fixes no sublattice, any other lift of g is conjugate to the lift above and has order 2. This also implies that any lift $\hat{g}$ of a lattice automorphism g of even order N, and such that $g^{N/2}$ is in class -1A, has order N. Indeed, $\hat{g}^{N/2}$ is a lift of a symmetry in class -1A, so that it must have order 2. This argument applies to all g in the classes 2.2C, -3BC, -3E, -3E', 8A, -8A, -5BC, -5BC', 12BC, -12BC'. An analogous reasoning holds for the automorphism of the lattice E_8 corresponding to class -1A, which flips the sign of all vectors in E_8. This automorphism has no fixed sublattice and acts trivially on $E_8/2E_8$, so that one can take ξ_g to be trivial. The same reasoning as for the sigma model case shows that all lifts of this symmetry are conjugate to each other and have order $N = 2$. More generally, all automorphisms of E_8 in the classes 2.2C, -3BC, -3E, -3E', 8A, -8A, -5BC, -5BC',

12BC, -12BC' lift to symmetries of the SVOA of the same order.

- Classes 2A and 2A': the fixed sublattice is isomorphic to the root lattice D_4, and its dual D_4^* is an integral lattice. In particular, D_4^* contains no vector of half-integral square norm, and therefore the standard lift has order 2. Furthermore, for any g of even order N such that $g^{N/2}$ is in class 2A or 2A', one has that $(\lambda, g^{N/2}(\lambda))$ is even for all λ, so that a standard lift has the same order N. This applies to all g in the classes 4A, 4A', -4A, -4A', 6BC, 6BC'. For automorphisms of the E_8 lattice in classes 2A and 2A', the fixed sublattice is also isomorphic to D_4, so the standard lift has the same order $N = 2$. The same reasoning holds for the standard lifts of automorphisms in the classes 4A, 4A', -4A, -4A', 6BC, 6BC'.
- Classes 2E and 2E': the fixed sublattice is A_1^4, and its dual $(A_1^4)^*$ contains vectors of square length 1/2. Thus, the standard lift has order $2N = 4$. The corresponding automorphism of the E_8 lattice also fixes a sublattice isomorphic to A_1^4, so its standard lift has order 4.

The conclusion of this analysis is that, both for toroidal sigma models and for the E_8 SVOA, the only case where the standard lift has twice the order of the corresponding lattice automorphism is for the class 2E.

If g is in class 2E, the twined genus for the standard lift (which has order 4) involves the theta series of the A_1^4 lattice

$$\Theta_{A_1^4}(\tau) = \theta_3(2\tau)^4 \ . \tag{E.25}$$

This theta series (and the corresponding twined genus) is a modular form of level 4. This is consistent with the analysis above.

One can also focus on a (non-standard) lift of order 2, with $\xi_g(\lambda) = (-1)^{\lambda^2}/2$ for all $\lambda \in (1+g)\Gamma^{4,4}$. For any g of order 2, one has $(1+g)\Gamma^{4,4} = 2((\Gamma^{4,4})^g)^*$; in particular, for g in class 2E or 2E', one has $(\Gamma^{4,4})^g \cong A_1^4$, so that $(1+g)\Gamma^{4,4} \cong 2(A_1^4)^* \cong A_1^4 \cong \Gamma^g$. For this lift, the twining genus involves the theta series with characteristics

$$\Theta_{A_1^4,\xi_g}(\tau) = \sum_{\lambda\in A_1^4} q^{\lambda^2/2}(-1)^{\lambda^2/2} = \Theta_{A_1^4}(\tau+\frac{1}{2}) = \theta_3(2\tau+1)^4 = \theta_4(2\tau)^4 \tag{E.26}$$

which is modular (with multipliers) for $\Gamma_0(2)$ (its S-transform is proportional to $\theta_4(\tau/2)^4$). As for the E_8 SVOA, since the sublattice fixed by the automorphism is also isomorphic to A_4, one can choose an analogous (non-standard) lift with the same ξ_g on the fixed sublattice, which is also of order 2.

For a general class, it is difficult to define a reasonable correspondence between non standard lifts in the sigma model and the E_8 SVOA, since the fixed sublattices are, in general, not isomorphic.

Summary / Samenvatting

Mathematics lies at the heart of theoretical physics. Whether used as a strong basis, as an inspiration, or merely as a tool, it has always played a part in shaping our theoretical understanding of not only what nature does, but also what it can or cannot do. This is especially relevant for fields like string theory that can not easily get an experimental treatment, where theories are less restricted and mathematics is essential in exploring many possibilities that may ultimately be of physical significance. And even if they end up not having such significance, simply interacting with ideas from physics has the potential to boost mathematical research significantly as well.

In this thesis we have discussed some examples that lie in the intersection between mathematics and physics, admittedly mostly through the lens of the latter. We have seen how one can use chiral conformal field theories along with careful orbifolding and projecting, tools familiar to a physicist that is occupied with two-dimensional conformal field theories in general, in order to “engineer” an umbral moonshine module, something that one could say is in itself of pure mathematical interest. We have also seen how a relatively simple chiral conformal field theory, or vertex operator algebra, contains information about a whole family of sigma models, non-chiral conformal field theories from the realm of string theory, giving rise to an interesting correspondence that draws inspiration and resources from both sides.

As is evident, chiral conformal field theories play an important role in this thesis. They constitute a prime example of a tool that is brought into the spotlight thanks to the interaction between moonshine and string theory. Being essential ingredients of the former, representations of finite groups and modular objects have also featured prominently in most of our discussions. The (twined) partition function in particular, is a central object that ties in all of the above areas and provides us with a tangible way to build new constructions and complete many associated proofs. Together with several others we have used, all of the aforementioned fields and tools come together to form an exciting and still ongoing chapter in the conjoined story of mathematics and theoretical physics.

Wiskunde vormt de kern van de theoretische natuurkunde. Of het nu fungeert als sterke basis, als inspiratiebron, of slechts als hulpmiddel, het heeft altijd een rol gespeeld in niet alleen het het begrijpen van de natuur, maar ook in wat wel of niet zou kunnen. Dit laatste is vooral relevant in vakgebieden die niet gemakkelijk kunnen worden geverifieerd met behulp van experimenten, zoals snaartheorie, waar theorieën minder begrenst zijn en wiskunde essentieel is bij het verkennen van de vele mogelijkheden die uiteindelijk van natuurkundig belang kunnen zijn. En zelfs als blijkt dat zo'n betekenis afwezig is, kan de connectie met ideeën uit de natuurkunde mogelijk het wiskundig onderzoek ook aanzienlijk stimuleren.

In dit proefschrift worden enkele voorbeelden besproken die op het grensvlak tussen wiskunde en natuurkunde liggen, weliswaar meer bekeken vanuit het oogpunt van de tweede. We hebben gezien hoe men chirale conforme veldentheorieën kan gebruiken in combinatie met zorgvuldig orbifolden en projecteren, hulpmiddelen die bekend zijn bij een fysicus die zich in het algemeen bezighoudt met tweedimensionale conforme veldentheorieën,om een umbral moonshine module te "maken", iets wat op zichzelf een puur wiskundige belang heeft. We hebben ook gezien hoe een relatief eenvoudige chirale conforme veldentheorie, of vertexoperatoralgebra, veel informatie bevat over een gehele familie van sigma modellen, niet-chirale conforme veldentheorieën uit de wereld van de snaartheorie, dit geeft een interessante wisselwerking tussen beide kanten.

Het is duidelijk dat chirale conforme veldentheorieën een belangrijke rol spelen in dit proefschrift. Ze vormen een belangrijk voorbeeld van een hulpmiddel dat in de schijnwerpers is gezet door de interactie tussen moonshine en snaartheorie. Representaties van eindige groepen en modulaire objecten zijn ook prominent aanwezig in de meeste discussies, aangezien ze een essentieel rol spelen in moonshine. Met name de (twined) partitiefunctie is een centraal object dat alle bovengenoemde gebieden met elkaar verbindt en biedt ons een tastbare manier om nieuwe constructies te maken en veel bijbehorende bewijzen te voltooien. Alle bovengenoemde vakgebieden en hulpmiddelen, en nog een aantal andere die onbenoemd zijn gebleven, komen samen in een opwindend en voortdurend hoofdstuk in het verstrengelde verhaal van de wiskunde en de theoretische natuurkunde.

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